Educational program in Data Analytics

MDA102 – Mathematics II

✍ 5th Set of Problems – MDA102

✎ material of 5th Week

Instructors: Jan Slovák, Ioannis Chrysikos

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Exercises

Instructions: Complete all 5 tasks from one problem, or choose any 5 tasks from both problems. Each task is for 2 points.

Problem 1. (**10 points**) Consider the function $f(x) = e^x \sin(x)$, with domain the real line.

1. Compute the first four derivatives of f at $x = 0$. Next use Sage to confirm the results by appropriately applying the command $\texttt{diff}(\texttt{f}, \texttt{x}, \texttt{k})(\texttt{x} = \texttt{x}_0)$. Recall that this command evaluates the kth-derivative $f^{(k)}(x)$ of f at $x = x_0$. Here is an example using another function, e.g., $g(x) = \ln(x)$ (see also Problem 6.A.2 on page 510 of the BG book).

```
g(x)=ln(x) # introduce the function
dg1=diff(g, x , 2)(x=1) #  compute the second derivative of g at x=1
show(dg1) # display the result
```

2. Find the critical (or stationary) points of f and determine its local minima/maxima. Hint: You can use the fact that the equation $\tan(x) = -1$ has solutions of the form $\frac{3\pi}{4} + k\pi$ with $k = 0, 1, 2, \dots$

3. Present the fourth-degree Taylor expansion of f around $x = 0$ (we denote this expansion by $T_0^4 f(x)$, see Problems 6.A.8 and 6.A.9 on page 513 of the BG book).

4. Use Sage to confirm the Taylor expansion by applying the method presented in Problem 6.A.9 of the BG book.

5. Approximate $f(x)$ near $x = 0.5$ with an accuracy of 5 decimal places, using the third-degree Taylor polynomial centred at $x = 0$, and compare it to the exact value of $f(0.5)$. Next confirm your computations using Sage and include your program in the solutions. Repeat using the fourth-degree Taylor polynomial centred at $x = 0$.

Hints: Refer to Section 6.1.1 and Problem 6.A.1 of the BG book for guidance on higher-order derivatives. For Taylor expansions, see Section 6.1.3 and the related problems outlined in the task instructions.

Problem 2. (**10 points**) Consider the function $h(x) = \ln(1 + x^2)$ with domain the real line.

1. Find the critical points (stationary points) of h and show that this function admits a unique local extreme at $x_0 = 0$ with value $h(0) = 0$. Is this a global minimum of h? Hints: Read the Problem 6.A.20 on the BG book, page 518, and see also Section 6.1.5, page 513.

2. Use the command `find_local_minimum(h, a, b)` in SageMath to numerically approach the local minimum of h at $x_0 = 0$ in the interval $[-1, 1]$. Hints: See Problem 6.A.25 on page 524 of the BG book.

3. Sketch the graph of h for $-3 \leq x \leq 3$ using SageMath and indicate in the figure the local minimum of h. Be sure to include the program in your solution.

4. Are there any asymptotes for h? Determine the intervals over the real line where h is convex or concave. Hints: See Section 6.1.6-6.1.8 on pages 513-516 of the BG book and Problems 6.A.23, 6.A.27. You may also refer to Problems 6.D.11 and 6.D.12 on pages 580-582 of the BG book.

5. Approximate $h(x)$ near $x = 1$ with an accuracy of 6 decimal places, using the fourth-degree Taylor expansion of h centred at 0, i.e., the Taylor polynomial $T_0^4 h(x)$. Next calculate the actual error and use Sage to confirm your answers. Repeat using the fifth-degree Taylor polynomial centred at $x = 0$. Hints: See Problems 6.A.10, 6.A.15 and 6.D.4, 6.D.5 on the BG book.