Application of GARCH-Copula Model in Portfolio Optimization

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Abstract: *Although the cornerstone of modern portfolio theory was set by Markowitz in 1952, the portfolio optimization problem is a never-ending research topic for both academics and practitioners. In this problem the future prediction of time series evolution plays an important role. However, it is rarely addressed in research. In the paper we analyze the applicability of the GARCH-copula model. To be more concrete we assume the investor maximizing Sharpe ratio while the future evolution of the time series is simulated by means of the AR(1)-GARCH(1,1) model using the copula modelling approach. The bootstrapping technique is applied as a benchmark. From the empirical results we found out that the GARCH-copula model provides better forecasts of future financial time series evolution than the bootstrapping method. Assuming the investor is maximizing the Sharpe ratio, both the final wealth increases and maximum drawdown decreases when we apply the GARCH-copula model compared to the application of bootstrapping technique.*

Keywords: *portfolio optimization, Sharpe ratio, GARCH, copula function*

JEL codes: *G11, G17*

Introduction

Although the cornerstone of modern portfolio theory was set by Markowitz in 1952, the portfolio optimization problem is a never-ending research topic for both academics and practitioners. In this problem, the prediction of future time series evolution plays an important role. However, when we model the financial time series, there are some characteristic features which have to be taken into consideration.

First, empirically observed returns of a financial time series are characterized by fatter tails compared to the Gaussian (normal) distribution, see e.g. Mandelbrot (1963). Next, empirical volatility of returns is not constant over time, but is rather clustered. Thus, for the same asset, periods with high volatility (high gains/losses) can be seen as well as periods in which volatility is low (the gains/losses are close to zero). This issue can be tackled by volatility modeling. A typical tool which can be utilized is the GARCH model, see Bollerslev (1986). The last issue one has to deal with is dependency within a particular time series. Generally, the returns are not correlated strongly when they are around zero, however in the tails the correlation increases. An appropriate tool for dependency modelling is the copula function, see Sklar (1973). Thus, the sound model of financial returns should be composed of a GARCH model and a copula function accompanied with some heavy-tailed marginal distributions.

While in most papers, see e.g. Farinelli et al. (2008) among others, the parametric stochastic process for financial returns is not assumed and performance ratio is optimized on historically observed returns (bootstrapping technique), in our paper we apply the GARCH-copula model to simulate future returns. The results obtained by means of the bootstrapping method are utilized only as a benchmark.

The aim of this paper is to analyze the applicability of the GARCH-copula model in portfolio optimization. To be more concrete, we assume an investor maximizing Sharpe ratio while the future time series is simulated by means of the GARCH-copula model and by means of the bootstrapping technique. Applying the procedures on a rolling window basis, we compare the values of final wealth at the end of the analyzed period and maximum (percentage) drawdown during the analyzed period for both approaches.

The paper is structured as follows: In the next section, the portfolio optimization model in the Markowitz mean-variance framework is described. Then, in the second section we present the application of the GARCH-copula model. In third and fourth sections the utilized dataset and obtained results are described. In the last section the discussion and conclusion are provided.

1 Portfolio Optimization Problem

The cornerstone of modern portfolio theory was established by pioneer work of Harry Markowitz in 1952 by his well know paper, see Markowitz (1952). Under the proposed assumptions, he assumed the returns to be normally distributed and the investor to be risk-averse, and that the rational investor wants to maximize the portfolio expected return and minimize its variance. However, the relationship between these two characteristics is generally positive – by decreasing the variance also the expected return decreases, see e.g. Lundblad (2007), and thus without the knowledge of the investor's level of risk aversion we can find only the set of (Pareto) efficient portfolios. The portfolio is identified as efficient if and only if there is no other portfolio with a lower risk delivering higher or equal expected return and no other portfolio with the higher expected return and lower or equal risk. For further details of modern portfolio theory see e.g. Elton et al. (2014).

Sharpe (1966) continued in the framework established by Markowitz and proposed the well-known Sharpe ratio (Sharpe index, the Sharpe measure or the reward-to-variability ratio) which he first defined as the ratio between the excess expected return (i.e. the expected return minus risk-free rate, also known as risk premium) and its volatility,

$$SR\left(\tilde{R}\right)=\frac{E\left(\tilde{R}-R_{RF}\right)}{\sigma_{\tilde{R}-R_{RF}}}=\frac{E\left(\tilde{R}\right)-R_{RF}}{\sigma_{\tilde{R}}}, \tag{1}$$

where $\tilde{R}$ is the observed (or predicted) distribution of returns or equiprobable realizations of this distribution and R_{RF} is risk-free rate. The original ratio was revised by Sharpe (1994) substituting the risk-free rate by an applicable benchmark $\tilde{R}_B$, which can change in time,

$$SR\left(\tilde{R}\right)=\frac{E\left(\tilde{R}-\tilde{R}_B\right)}{\sigma_{\tilde{R}-\tilde{R}_B}}. \tag{2}$$

Further in this paper we assume the original version of Sharpe ratio (1), which in fact is a special case of the revised version (2) in which $\tilde{R}_B=R_{RF}$. The Sharpe ratio defines the profile of an investor who prefers titles with higher expected excess returns for unity of volatility (standard deviation). When comparing two assets versus a common benchmark (in our case risk-free rate R_{RF}), the one with a higher Sharpe ratio provides a better return for the same risk (or, equivalently, the same return for a lower risk).

The Sharpe ratio is closely related to the Markowitz mean-variance framework as it focuses only on the first two moments of the probability distribution. However, as it is known, the empirical distribution of financial asset returns is characterized by heavy-tails and skewness. Thus, many researchers have proposed their own ratios, which take into account the kurtosis and skewness of the probability distribution. Among others, see for instance the Gini ratio (Shalit and Yitzhaki, 1984), mean absolute deviation ratio (Konno and Yamazaki, 1991), mini-max ratio (Young, 1998), Rachev ratio (Biglova et al., 2004) and others. For the summary see e.g. Farinelli et al. (2008).

In this paper we assume the investor maximizes Sharpe ratio, to solve the following portfolio optimization problem, i. e. he solves the following portfolio optimization problem,

$$\begin{cases} \max\limits_{\mathbf{x}} SR\left(\tilde{\mathbf{R}} \times \mathbf{x}\right) \\ \sum\limits_{i=1}^{N} x_i = 1 \\ x_i \geq 0, \quad i = 1, \ldots, N \\ x_i \leq 0.25, \quad i = 1, \ldots, N \end{cases} \tag{3}$$

in which $\mathbf{x}$ represents the vector of weights (portfolio composition) and $\tilde{\mathbf{R}}$ is the matrix of random realizations of returns (rows represent realizations with equal probability and columns represent particular assets the investor can include in the portfolio). The matrix $\tilde{\mathbf{R}}$ contains the random realizations of future returns, and thus is not directly observable but must be simulated. We describe the simulation procedure in the following section. Furthermore, the constraints of the optimization problem bound the weight of each asset between 0% (short selling is not allowed) and 25% (the portfolio is composed of at least four assets).

2 Financial Asset Returns Modelling

The evolution of financial asset returns over time is specific in the following ways, for further details see e.g. Cont (2001). Empirical volatility of returns is not constant over time, but is rather clustered. Thus, for the same asset, periods of high volatility (high gains/losses) can be seen as well as periods in which volatility is low (the gains/losses are low). This issue can be tackled using volatility modeling. In this paper, we apply the GARCH model for this purpose (Bollerslev, 1986). Even after the correction of returns for volatility clustering, the residual time series still exhibits heavy tails. The conditional distribution, however, is less heavy-tailed than the unconditional distribution. In our paper we utilize joint Student distributions for residuals. Due to the estimation and simulation requirements, this joint distribution is decomposed into Student marginal distributions and the Student copula function in line with Sklar's theorem (Sklar, 1973). We address the obtained model as a GARCH-copula model, which was already applied in risk management by Huang et al. (2009), Wang et al. (2010) and others.

Assume that we want to model the future returns of *n* assets. For each asset we assume AR(1)-GARCH(1,1) process, i.e. *i*-th asset returns can be modelled as follows,

$$R_{i,t} = \mu_{i,0} + \mu_{i,1} \cdot r_{i,t-1} + \sigma_{i,t} \cdot \tilde{\varepsilon}_{i,t} \ , \tag{4}$$

$$\tilde{\varepsilon}_{i,t} \sim t_{\nu}(0,1) \ , \tag{5}$$

where $\mu_{i,0}$ and $\mu_{i,1}$ are parameters of the conditional mean equation, $\sigma_{i,t}$ is the standard deviation (volatility) modelled by the GARCH model and $\tilde{\varepsilon}_{i,t}$ is a random number from Student probability distribution $t_\nu(0,1)$ (henceforth filtered residual). The Student distribution is applied for its ability to model fat tails (higher kurtosis) of a probability distribution, which are usually present in financial time series of returns. The volatility is modelled by means of the GARCH model (Bollerslev, 1986), an extension of the ARCH model (Engle, 1982). The applied model takes the following form,

$$\sigma_{i,t}^2 = \alpha_{i,0} + \alpha_{i,1} \cdot \sigma_{i,t-1}^2 + \beta_{i,1} \cdot \varepsilon_{i,t-1}^2 , \tag{6}$$

where $\alpha_{i,0}$, $\alpha_{i,1}$ and $\beta_{i,1}$ are the parameters that must be estimated. Positive variance is assured if all the parameters are equal or greater than zero. The model is stationary if $\alpha_{i,1} + \beta_{i,1} < 1$.

In order to preserve the mutual dependence among the asset returns, the filtered residuals are joined together applying copula function modelling. Copula functions are projections of the dependency among particular distribution functions into $[0,1]$,

$$C : [0,1]^n \to [0,1] \text{ on } R^n, n \in \{2,3,...\}. \tag{7}$$

Basic reference for the theory of copula functions is Nelsen (2006), while Rank (2007) and Cherubini et al. (2004, 2011) target mainly on the application issues in finance. Actually, any copula function can be regarded as a multidimensional distribution function with marginals in the form of a standardized uniform distribution. Following the Sklar's theorem (Sklar, 1959), any joint distribution function, in our case the joint distribution function of filtered residuals $F_{\tilde{\varepsilon}_1,...,\tilde{\varepsilon}_n}(u_1,..., u_n)$, can be decomposed into marginal distributions and a selected copula function,

$$F_{\tilde{\varepsilon}_1,...,\tilde{\varepsilon}_n}(u_1,...,u_n) = C\left(F_{\tilde{\varepsilon}_1}(u_1),...,F_{\tilde{\varepsilon}_n}(u_n)\right). \tag{8}$$

The formulation above should be understood such that the copula function *C* specifies the dependency, nothing less, nothing more. In the paper we apply the Student copula function, which belongs to the family of elliptical copula functions,

$$C_{\nu,\mathbf{Q}}^t(u_1,...,u_n) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu\pi)^n|\mathbf{Q}|}} \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_n)} \left(1 + \frac{\mathbf{z}'\mathbf{Q}\mathbf{z}}{\nu}\right)^{-\frac{\nu+n}{2}} d\mathbf{z} , \tag{9}$$

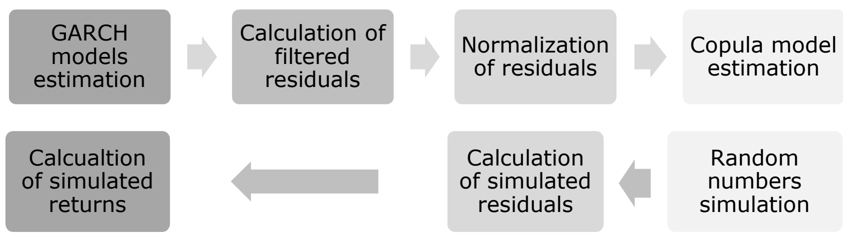
where Γ is gamma function, ν stands for degrees of freedom both in marginals and Student copula function and **Q** is a correlation matrix. Note that by accompanying Student copula function with Student marginals we obtain the joint-Student distribution.

2.1 Parameters Estimation and Subsequent Returns Simulation

In order to model the future evolution of financial time series by means of GARCH-copula model the following procedure should be undertaken, see Figure 1. First, the parameters of GARCH model are estimated for each particular return time series from past observations. When GARCH models are estimated, the residuals (observed in the past) can be obtained. These are put together into a matrix and the parameters of the copula function are estimated. There exist three main approaches to estimate parameters for copula function based dependency modelling: exact maximum likelihood method (EMLM), inference function for margins (IFM), and canonical maximum likelihood (CML). While for the EMLM all the parameters are estimated within one step, which might be very time consuming (mainly for high dimensional problems or complicated marginal distributions), the other two methods are based on the estimation of the parameters for the marginal distributions and parameters for the copula function separately – marginal distributions are estimated in the first step and the copula function in the second step. Following IFM the estimated marginals are utilized in the second step. For CML instead of estimated marginals the empirical distributions are utilized. In this paper we apply the IFM estimation method as it provides a reasonable trade-off between the accuracy and computational requirements.

For the simulation the sequence is inverse. First, random numbers are simulated, while the dependency among them is maintained by means of the estimated copula function. Then, these simulated random numbers are transformed to the filtered residuals (by inverse distribution function), which are converted to the returns by means of estimated GARCH models. These returns can be then easily utilized for computation of expected portfolio return and/or its risk.

Figure 1 GARCH-Copula model estimation/simulation procedure



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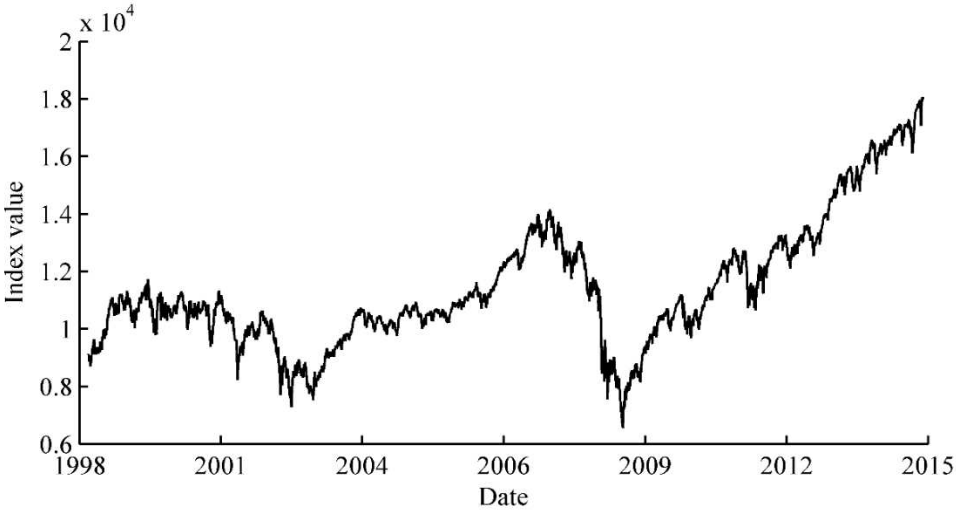
3 Dataset

The utilized dataset consists solely of the stocks incorporated in one of the American stock market indices – Dow Jones Industrial Average (henceforth DJIA). We assumed all the components of the index as of October 6, 2014, except the stocks of The Goldman Sachs Group, Inc. (Yahoo Finance ticker GS) and Visa Inc. (Yahoo Finance ticker V). These two stocks were excluded from the dataset as we were not able to obtain sufficiently long historical data. Thus, the dataset consists of only the remaining 28 stocks.

Historical data of the stocks were obtained from Yahoo Finance website[1] over the period December 1, 1997 until December 31, 2014 (4,298 daily observations for each stock). However, we estimated the parameters from 250 observations, thus the backtesting was performed in the period from November 30, 1998 until December 31, 2014, leaving the first year of data for initial parameter estimation.

The evolution of DJIA price index[2] in the analyzed period is depicted in Figure 2. The index took the value of 9,116.55 on November 30, 1998 and 17,983.07 on December 31, 2014. Thus the average annual return (to be more specific the average return of 250 trading days) in the analysed period was 4.26% whereas the maximum drawdown over the analyzed period was 53.78%.

Figure 2 Evolution of DJIA index in the analysed period



Source: Author

[1] http://finance.yahoo.com

[2] Price index considers only price movements in the components, dividends are not considered.

4 Empirical Part and Results

In the previous sections we proposed an optimization problem which can be applied to find the optimal portfolio. In this section we apply this portfolio optimization problem on a moving window basis. Starting with the initial wealth $W_0 = 1$ at the beginning of the analyzed period, we can recursively compute the ex-post wealth path W_t over the analyzed period,

$$W_{t+1} = W_t \cdot \left(1 + \sum_{i=1}^{N} r_{i,t} \cdot w_{i,t}\right), \tag{10}$$

where $r_{i,t}$ are ex-post observed returns and $w_{i,t}$ are the weights of particular assets at time *t* (portfolio composition). These weights were obtained by means of maximizing the Sharpe ratio, see problem (3). Under this set-up the matrix of random realizations of future returns ($\tilde{\mathbf{R}}$) was simulated for each day assuming the following two approaches:

1. 100,000 simulated trials (rows) were calculated utilizing GARCH-copula model, which was estimated from 250 days prior to the examined day,
2. taking last 250 observed returns prior to the examined day (bootstrapping method).

Our goal is to analyse the soundness of GARCH-copula model to describe the return time series and moreover to predict their future evolution. The bootstrapping method, which represents a rather naïve model, is analysed only as a benchmark to the proposed GARCH-copula approach.

In this section we present the results obtained by applying two above mentioned approaches for simulation of future returns. In our application we assumed the following values of risk-free rate: 0%, 1%, …, 6% and 10% p.a. for the Sharpe ratio (1). The reason to assume more values of the risk-free rate is to analyze the robustness of the proposed approach.

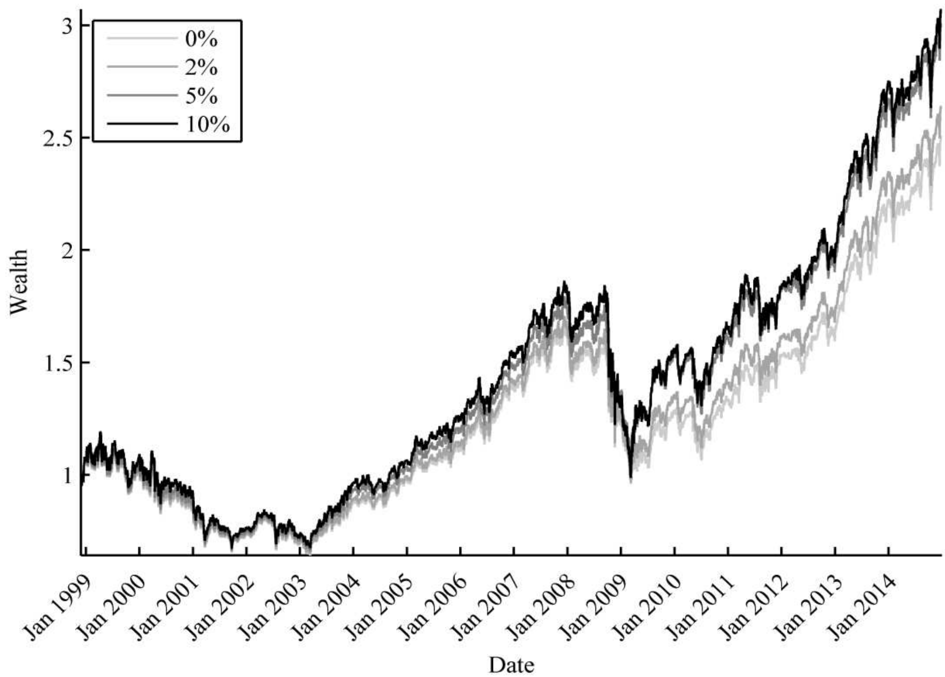
All the computations were performed in Matlab. While doing so we utilized some algorithms already presented in Kresta (2015) while most of them had to be programmed. Nevertheless, all the algorithms, by which the results were obtained, are freely available upon an e-mail request to the author.

4.1 Bootstrapping Method

The ex-post wealth paths obtained by means of the bootstrapping method are depicted in Figure 3. In order to keep the clarity of the graph we plotted only the wealth paths for selected risk-free rates. As can be seen from the graph, the final wealth ranges between 2.5 (risk-free rate of 0% p.a.) and 3.6 (risk-free rate of

10% p.a.) and there is a clear relationship – the higher the risk-free rate the higher the final wealth. The relationship can be extended to the whole wealth paths – the wealth paths with higher risk-free rates dominate the ones with lower risk-free rates, except the period from December 2008 until April 2009 (in which there was a big drop in portfolio wealth due to the subprime crisis). Note that the risk-free rate is applied only in calculation of the Sharpe ratio, however, the investor is not allowed to invest into a risk-free asset, i.e. he always invests his whole wealth into risky assets, see portfolio optimization problem (3). Due to this reason, the wealth paths dropped in the period 2008-2009 as there was general decline of stocks' prices due to the subprime crisis.

Figure 3 Ex-post wealth paths obtained by means of bootstrapping method



Source: Author

On the other hand, from the figure we can see that the higher the risk-free rate, the higher the volatility of the wealth paths and the drawdowns. However this is true only for the absolute values of the drawdowns. If we compare the maximum drawdowns (which actually took place in 2008-2009) stated relatively, we find out that there is no relationship between the values of risk-free rate and the maximum drawdown (in %), see Table 1.

Table 1 Final wealth and maximum drawdown of particular wealth paths

Risk-free rate	0%	1%	2%	3%	4%	5%	6%	10%
Final wealth	2.50	2.59	2.63	2.75	2.91	3.00	3.03	3.06
Maximum drawdown	44.1%	44.1%	44.0%	43.9%	43.6%	43.4%	43.7%	47.1%

Source: Author

Table 1 summarizes also the values of final wealth for all the risk-free rates assumed. We can see that the above discussed relationship between the values of risk-free rate and final wealth is valid for all analyzed risk-free rates.

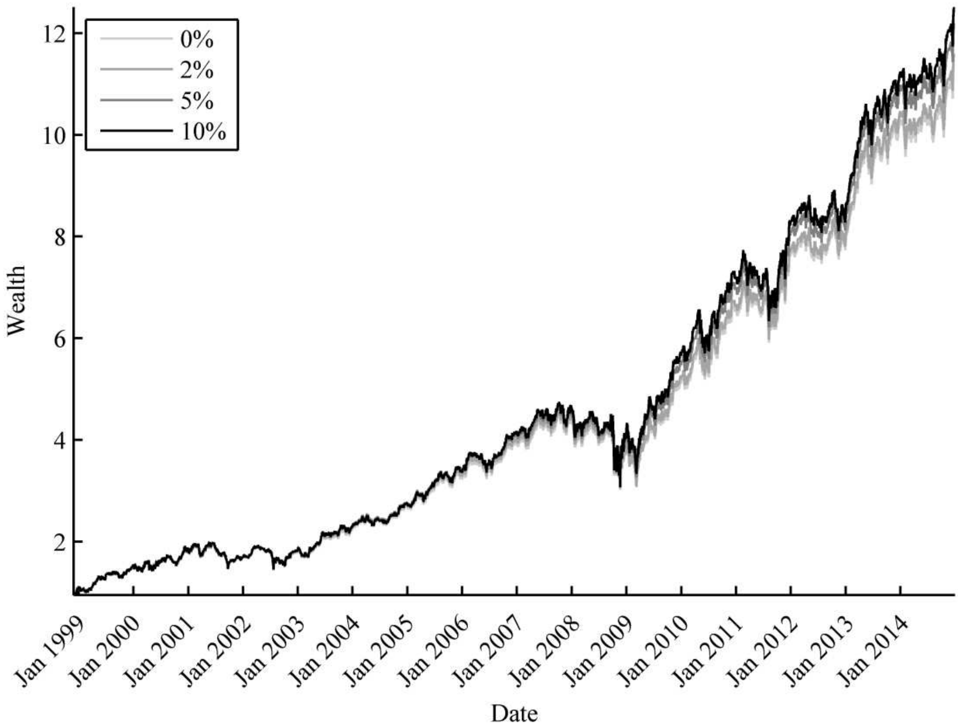
To sum up, we can conclude that the final wealth ranges between 2.5 (average annual return of 5.82%) and 3.06 (average annual return of 7.16%) and maximum drawdown over the analysed period fluctuates around 44%, except for 10% risk-free rate. Although the strategies outperformed the passive investment into Dow Jones Industrial Average index (average annual return of 4.26% and maximum drawdown 53.78%) both in terms of profitability and maximum drawdown, the profitability after accounting for transaction costs is questionable.

4.2 GARCH-Copula Model

The ex-post wealth paths obtained by means of GARCH-copula model are depicted in Figure 4. From the graph we can observe two findings: the values of final wealth are higher than applying previous approach and the ex-post wealth paths evolved more closely to each other for different risk-free rates. Also in the case of GARCH copula model, the higher the risk-free rate the higher the value of final wealth. It is difficult to analyze the volatility of the wealth paths as they are close to each other.

Table 2 summarizes the values of final wealth and maximum drawdown. We can see that the values of both the final wealth and the maximum drawdown grow as applied risk-free rate increases (this relationship is not strictly true for final wealth, but the general trend is obvious). Concerning the values, we can sum up that the final wealth ranges between 11.33 (average annual return of 16.18%) and 12.48 (average annual return of 16.88%) and maximum drawdown over the analyzed period is in range of 33%–35.3%. As it is obvious, the results are not significantly sensitive to applied values of the risk-free rate and the proposed methodology is robust.

Figure 4 Ex-post wealth paths obtained by means of GARCH-copula model



Source: Author

We can see that when applying the GARCH-copula model, the values of maximum drawdown are smaller and the values of final wealth are higher compared to the bootstrapping method. This approach is thus clearly superior to the previous one. However, note that the results are provided without deduction of transaction costs.

Table 2 Final wealth and maximum drawdown of particular wealth paths

Risk-free rate	0%	1%	2%	3%	4%	5%	6%	10%
Final wealth	11.38	11.33	11.55	11.47	11.61	12.15	12.30	12.48
Maximum drawdown	33.2%	33.4%	33.7%	33.9%	34.2%	34.3%	34.5%	35.3%

Source: Author

Conclusions

The cornerstone of modern portfolio theory was set by Markowitz in 1952 and the portfolio optimization problem is in the constant focus of both academics and practitioners. Although, the prediction of future time series evolution plays an important role in the problem, it is rarely addressed in research. In the paper we analysed the applicability of the GARCH-copula model. To be more concrete we assumed the investor maximizing Sharpe ratio while the future time series are simulated by means of GARCH-copula model and by means of bootstrapping technique.

In our paper we did not subtract the transaction costs. The reason is twofold. Firstly, they differ significantly – although they would represent high fraction of the gains for small private investors, for large institutional investors they would be of smaller values. Secondly, also the considered investments into DJIA index is connected with transaction costs which are caused by the changes in stock prices (and thus also changes in relative weights). We are also aware of another drawback of our analysis when considered for practical purposes – survivorship bias. Loosely speaking, by the survivorship bias we address the situation in which the decision making is influenced by the information which are not known at the moment we are making decision. In our analysis, the problematic point is the selection of dataset – we took the components of DJIA index as of October 6, 2014, however this composition was not known during the whole analysed period (years 1998-2014). We didn't address this feature as we were mainly focused on the analysis of GARCH-copula model applicability for financial time series predictions. We compared the GARCH-copula approach to bootstrapping technique and applied the same dataset for both methods. By doing so, the results can be compared, however it would be tricky to make conclusions about profitability of strategies.

From the empirical results we found out that GARCH-copula model provides better forecasts of future financial time series evolution than bootstrapping method. Assuming the investor, who is maximizing the Sharpe ratio, both the final wealth increased and maximum drawdown decreased.

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