Expenditure systems in the COICOP-classification framework

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Overview

1. ***Motivation***
2. ***Properties of main economic functional types:*** *direct and indirect utility functions, expenditure function, Marshallian and Hicksian demands*
4. ***Review of the classical demand/expenditure systems***
   Linear ES, direct/indirect ADDILOG, Rotterdam, S-branch (SO,S1)
5. ***Some recently developed advanced expenditure systems***
   Quadratic ES, AIDS, Translog, GEF, QUAIDS, Full Laurent model
6. ***Econometric and dynamic specifications for some ES***
7. ***The current COICOP classification of goods and services***
8. ***Estimation of the*** *Rotterdam and AIDS expenditure systems*
9. ***Some interesting conclusions from the demand analysis***

Introduction

The main objectives** followed by the author **were :

- **To bring a short review of classical and modern demand/expenditure systems, which may be effectively employed for research investigations in the consumer demand context.**
- **To summarize the properties adopted for the most important economic functional types (*direct and indirect utility function, expenditure function, and Marshallian and Hicskian demand functions)* if they may be considered as „appropriate“ from the theoretical point of view.**
- **To mention possible (but sometimes even serious) problems with the econometric estimation of the parameters of one/another expenditure system (including eventual the identification problem).**
- **To mention some typical situations, in which just chosen expenditure system can be more effectively used than another.**
- **To carry out calculations of parameters (and accompanying statistical characteristics) of several expenditure systems (these, which do not require to employ the nonlinear regression methods).**
- **To explore, whether current CIOCOP consumption structure already enables to perform reliable econometric demand analysis relative to the data on prices and consumptions in the Czechia.**
- **To bring some suggestions, which might be useful for the on-coming analysis in the future, if they would be based on the similar principles (the COICOP data samples and the demand system econometric analysis)**

Notation commonly used in the demand analysis context

Direct utility function .... $u(x_1, x_2, \ldots, x_n)$

Indirect utility function .... $\Psi(M, p_1, p_2, \ldots, p_n)$

Expenditure function .... $E(u, p_1, p_2, \ldots, p_n)$

Marshallian demand functions .... $g(M, p_1, p_2, \ldots, p_n)$

Hicksian demand functions .... $h(u, p_1, p_2, \ldots, p_n)$

where u is the *utility level*, $p = (p_1, p_2, \ldots, p_n)$ *price vector*, M *consumer's income*

$x = (x_1, x_2, \ldots, x_n)$ *the demanded commodity vector*, $w_i = \frac{p_i x_i}{M}$ *expenditure share of i-th commodity*

Properties of the (direct) utility function

(Direct) utility function $u(x_1, x_2, ..., x_n)$ ***of n commodities***

(U1) $u(x)$ **is real-valued and finite function defined for all** $x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$ **.**

non-negativity $u(x_1, x_2, ..., x_n) \geq 0$ **for all** $x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$ **and** $u(0,0,...,0) = 0$

(U2s) strong monotonicity $u(x)$ **is increasing in each commodity** $x_i \in x$

$$u(x_1, x_2, .., x_k, .., x_n) < u(x_1, x_2, .., x_k^*, .., x_n) \quad \text{for all} \quad x_k < x_k^* \,;\; k \in \{1,2,...,n\} \quad \text{and / or}$$

(U2w) weak monotonicity $u(x)$ **is non-decreasing in any** $x_i \in x$

$$u(x_1, x_2, .., x_k, .., x_n) \leq u(x_1, x_2, .., x_k^*, .., x_n) \quad \text{for all} \quad x_k < x_k^* \,;\; k \in \{1,2,...,n\}$$

(U3) continuity $u(x)$ **is continuous in each commodity** $x = (x_1, x_2, ..., x_n)$

(U4) quasi-concavity $u(x)$ **is quasi-concave in all commodities.**

It means that the following inequality holds for all $x \geq 0, z \geq 0, \mu \in (0,1)$:

$$u(\mu x + (1-\mu) z) \geq \mu . u(x) + (1-\mu) . u(z)$$

(U5) determinateness $u(x)$ **is determined up to increasing continuous transformation** $\varphi(..)$ **.**

Then, the $u(x)$ **and** $\varphi(u(x))$ **represent the same preference ordering.**

Properties of the indirect utility function

Indirect utility function $\Psi(M, p_1, p_2, ..., p_n)$ defined as $\psi(M,p) = Max[u(x); px = M]$

(W1) $\psi(M,p)$ **is real-valued, finite function, defined for** all **positive prices and non-negative income**

non-negativity $\Psi(M,p) \geq 0$ **for all positive prices** $p_i, i = 1,2,...,n$ **and** $\psi(p,0) = 0$ **.**

(W2) monotonicity $\psi(M,p)$ **is increasing in M for any price vector.**

Further, $\psi(M,p)$ **is non-decreasing in each** $p_i\, ; i \in \{1,2,...,n\}$ **for any consumer's income** M

(W3) continuity $\psi(M,p)$ **is continuous in p in for any price vector** $p = (p_1, p_2, ..., p_n)$

and is continuous in M for any fixed level of income/total expenditures M .

(W4) homogeneity $\psi(M,p)$ **is homogeneous of the degree 0** (simultaneously) **in prices and income**

It means that the following inequality holds for any $\lambda > 0$:

$$\psi(\lambda M, \lambda p) = \psi(M,p)$$

(W5) concavity $\psi(M,p)$ **is concave function in prices** $p = (p_1, p_2, ..., p_n)$ **at any income level** M^0

It means that for any price vectors $p > 0, p^* > 0, \mu \in (0,1)$ and every income level holds.

$$\psi(M^0, \mu p + (1-\mu)p^*) \geq \mu.\psi(M^0, p) + (1-\mu).\psi(M^0, p^*)$$

(W6) The Marshallian demand functions are generated by the Roy/Villé's identity:

Properties of the expenditure function

Expenditure function $E(u, p_1, p_2, ..., p_n)$ defined as $E(u^0, p) = \mathrm{Min}[px; u(x) \geq u^0]$

(V1) $E(u^0, p)$ **is real-valued, finite function, defined for** all **positive prices and non-negative utility level**M

non-negativity $E(u^0, p) \geq 0$ **for all positive prices** $p_1 > 0, p_2 > 0, ..., p_n > 0$ **and** .

(V2) monotonicity $E(u^0, p)$ **is increasing in** u^0 **for all price vector . is non-decreasing v and increasing at least in one price** $p_i ; i \in \{1,2,...,n\}$ **for any utility level** $u^0 > 0$.

(V3) continuity $E(u^0, p)$ **is continuous in** u^0 **for any price vector .**

Similarly, $E(u^0, p)$ **is continuous in each** $p = (p_1, p_2, ..., p_n)$ **at any utility level.**

(V4) homogeneity $E(u^0, p)$ **is linearly homogeneous in p pro for any utility level.**

It means that the following inequality holds for any $p > 0 ; \lambda > 0$:

$$E(u^0, \lambda p) = \lambda . E(u^0, p)$$

(V5) concavity $E(u^0, p)$ **is concave in prices** $p = (p_1, p_2, ..., p_n)$ **at any utility level.**

It means that for any price vector and every utility level holds $p > 0, p^* > 0, \mu \in (0,1)$

$$E(u^0, \mu p + (1-\mu) p^*) \geq \mu . E(u^0, p) + (1-\mu) . E(u^0, p^*)$$

(W6) The Hicksian demand functions are generated by the Shephard's lemma:

Properties of the Marshallian demand functions

(D1M) Each $x_i^M = g_i(M,p)$ **is real-valued, finite and non-negative function with** $g_i(0,p) = 0$

(D2M) monotonicity $x_i^M = g_i(M,p)$ **is non-increasing in price of the every commodity and is non-decreasing in income .**

(D3M) continuity $x_i^M = g_i(M,p)$ **is continuous in M as well as is continuous in** $p = (p_1, p_2, \ldots, p_n)$ **, too.**

(D4M) homogeneity Marshallian demand functions $x_i^M = g_i(M,p)$ **are homogeneous of degree 0 simultaneously in price s and income. So, the following inequality holds:** $g_i(\lambda M, \lambda p) = g_i(M,p)$

(D5HM) additivity Complete system of the Marshallian demand functions is additive and summable.

It means that $\sum_{i=1}^{n} p_i g_i(M,p) = M$

(D6M) symmetry „Cross" derivatives of the Marshallian demands (according to individual prices) are symmetric, i.e

$$\frac{\partial g_i(M,p)}{\partial p_j} + x_j . \frac{\partial g_i(M,p)}{\partial M} = \frac{\partial g_j(M,p)}{\partial p_i} + x_i . \frac{\partial g_j(M,p)}{\partial M}$$ **for all** $p_i, p_j \in \{p_1, p_2, \ldots, p_n\}$

(D7M) The [n;n] matrix S consisting of the elements $s_{ij} = \frac{\partial g_i(M,p)}{\partial p_j} + x_j . \frac{\partial g_i(M,p)}{\partial M}$ **is negatively semidefinite, so the for each vector** $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ **not identically zero, the quadratic form defined by the matrix S fulfills the condition**

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial g_i(M,p)}{\partial p_j} + x_j . \frac{\partial g_i(M,p)}{\partial M} \right] \cdot \xi_i \cdot \xi_j \leq 0$$

As a result of this, $s_{ii} \leq 0, i = 1, 2, \ldots, n.$

Properties of the Hicksian demand functions

(D1M) Each $x_i^H = h_i(u,p)$ is real-valued, finite and non-negative function with $h_i(0,\mathbf{p}) = 0$

(D2M) monotonicity $x_i^H = h_i(u,p)$ is non-increasing in price of the every commodity p_i and is non-decreasing in any utility level u .

(D3M) continuity $x_i^H = h_i(u,p)$ is continuous in u as well as is continuous in p_i $(i = 1,2,\ldots,n)$, too.

(D4M) homogeneity Marshallian demand functions $x_i^H = h_i(u,p)$ are homogeneous of degree 0 in prices.

Thus, the following inequality holds: $h_i(u,\lambda p) = h_i(u,p)$ for $p = (p_1, p_2, \ldots, p_n)$, $\lambda \in (0,+\infty)$

(D5HM) additivity Complete system of the Hicksian demand functions is additive and summable.

It means that $\sum_{i=1}^{n} p_i h_i(u,p) = M$.

(D6M) symmetry „Cross" derivatives of the Hicksian demands (according to individual prices) are symmetric, i.e

$$\frac{\partial h_i(u,p)}{\partial p_j} = \frac{\partial h_j(u,p)}{\partial p_i} \quad \text{for all} \quad p_i, p_j \in \{p_1, p_2, \ldots, p_n\}$$

(D7M) The $[n \times n]$ matrix S^* consisting of the elements $s_{ij}^* = \dfrac{\partial h_i(u,p)}{\partial p_j}$ is negatively semidefinite,

so the for each vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ not identically zero, the quadratic form defined by the matrix S^* fulfills the condition

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial h_i(u,p)}{\partial p_j}\right]\cdot \xi_i \cdot \xi_j \le 0$$

As a result of this, $s_{ii} \le 0, i = 1,2,\ldots,n.$

$$x_i = h_i(u,p) = h_i(\psi(M,p),p) = g_i(M,p)$$

Roy' s Identity , Shephard' s lemma, C,E agregation conditions

- ***Roy's identity*** $$x_i(M,p) = \frac{-\dfrac{\partial \psi(M,p)}{\partial p_i}}{\dfrac{\partial \psi(M,p)}{\partial M}}$$

Derivatives of the budget restriction $\sum_{i=1}^{n} p_i x_i = M$ ***yield***

- ***Engel aggregation condition*** $$\sum_{k=1}^{n} p_k \cdot \frac{\partial g_k(M,p)}{\partial M} = 1$$
- ***Cournot aggregation condition*** $$\sum_{k=1}^{n} p_k \frac{\partial g_k(M,p)}{\partial p_i} + g_i(M,p) = 0$$
- ***Homogeneity*** $\psi(M,p)$ ***of the degree 0 implies*** $$\sum_{k=1}^{n} p_k \cdot \frac{\partial g_k(M,p)}{\partial p_k} + M \cdot \frac{\partial g_k(M,p)}{\partial M} = 0$$
- ***Shephard's lemma*** $$x_i = \frac{\partial E(u^0,p)}{\partial p_i}$$
- ***Symmetry of the Hicksian demands*** $$\frac{\partial x_j(u^0,p)}{\partial p_k} = \frac{\partial E(u^0,p)}{\partial p_j \partial p_k} = \frac{\partial E(u^0,p)}{\partial p_k \partial p_j} = \frac{\partial x_k(u^0,p)}{\partial p_j}$$

Linear expenditure system – LES [Stone Richard 1954, Geary R.Conrad 1949/50]

- ***Direct utility function*** $$u(x) = \beta_0 \cdot \prod_{i=1}^{n} (x_k - \alpha_k)^{\beta_k} \qquad \beta_k > 0, \alpha_k \geq 0, x_k \geq \alpha_k$$
- ***Indirect utility function*** $$\Psi(M,p) = \frac{M - \sum_{k=1}^{n} \alpha_k p_k}{\beta_0 - \prod_{k=1}^{n} p_k^{\beta_k}}$$
- ***Expenditure function*** $$E(u,p) = \sum_{k=1}^{n} \alpha_k p_k + u.\beta_0 \prod_{k=1}^{n} p_k^{\beta_k}$$
- ***Marshallian demand functions*** $$g(x_i) = \alpha_i + \frac{\beta_i}{p_i}\left(M - \sum_{k=1}^{n} \alpha_k p_k\right)_i$$
- ***Hicksian demand functions*** $$h(x_i) = \alpha_i + \frac{\beta_i}{p_i}\left(u.\beta_0 \prod_{k=1}^{n} p_k^{\beta_k}\right)_i$$
- ***Basic characteristics***

suitable for commodities the demand for which displays proportionality to the income.

The α coefficients express the "threshold level"for the utility contribution of i-th commodity.

Classical expenditure systems

Addilog demand system [**Houthakker** Hendrik **1960**]

- ***Direct utility function***

$$u(x) = \sum_{k=1}^{n} \alpha_k x_k^{\beta_k}$$

- ***Indirect utility function***

$$\Psi(M,p) = \sum_{k=1}^{n} \alpha_k \left(\frac{M}{p_k}\right)^{\beta_k}$$

- ***Expenditure function*** ***inexpressible in a closed form***

- ***Marshallian demand functions***

$$p_i x_i = \frac{\alpha_i M^{\beta_i - 1} p_i^{-\beta_i}}{\sum_{k=1}^{n} \alpha_k \beta_k . \left(\frac{M}{p_k}\right)^{\beta_k}}$$

- ***Hicksian demand functions*** ***inexpressible in a closed form***

- ***Basic characteristics***

Suitable for strictly separable utility preferences, not mixing impacts of individual goods. No commodity is essential. A comment: preferences corresponding to the direct and indirect Addilog are not the same.

S-branch expenditure system [Brown Murray, Heien Dale 1972]

- **Direct utility function**

$$u(x_1, x_2, \dots, x_n) = \left\{ \sum_{s=1}^{n} \gamma_s \left[\sum_{j \in s}^{n_s} \beta_{sj} \left(x_{sj} - \alpha_{sj}\right)^{\rho_s} \right]^{\rho / \rho_s} \right\}^{1/\rho}$$

- **Indirect utility function**

$$\Psi(M_1, p_1, \dots, p_n) = \left\{ \sum_{s=1}^{S} \gamma_s \cdot \frac{\left[\sum_{i \in s}^{n_s} \beta_{si} \left(\frac{\beta_{si}}{p_{si}} \right)^{\sigma_s} . w_s \left(M - \sum_{r=1}^{S} \sum_{j \in r}^{n_r} \alpha_{rj} p_{rj} \right))^{\rho_s} \right]^{\frac{\rho}{\rho_s}}}{\left[. \sum_{j \in s}^{n_s} \left(\frac{\beta_{sj}}{p_{sj}} \right)^{\sigma_s} \right]} \right\}^{\frac{1}{\rho}}$$

- **Expenditure function** *inexpressible in a closed form*

- **Marshallian demand functions**

$$x_{si} = \alpha_{si} + \left(\frac{\beta_{si}}{p_{si}} \right)^{\sigma_s} \left[{\gamma_s}^{\sigma} . R_s^{\frac{\sigma - 1}{\sigma_s - 1} - 1} \right] . \left[\sum_{r=1}^{s} {\gamma_r}^{\sigma} R_r^{\frac{\sigma - 1}{\sigma_r - 1}} \right]^{-1} . \left[M - \sum_{r=1}^{s} \sum_{j \in r}^{n_r} \alpha_{1j} p_{rj} \right]$$

$$R_s = \sum_{j \in s}^{n_s} \left(\frac{\beta_{sj}}{p_{sj}} \right)^{\sigma_s} . p_{sj}$$

- **Hicksian demand functions** *inexpressible in a closed form*

- **Basic characteristics**

  suitable for commodity systems containing complements and displaying positive elasticities of any order (S-branch is a generalization of LES)

Classical expenditure systems

S0-branch expenditure system [Brown Murray, Heien Dale 1972]

- ***Direct utility function***

$$u(x_1, x_2, \ldots, x_n) = \left[\sum_{i=1}^{n} \beta_i (x_i - \alpha_i)^{\sigma}\right]^{\frac{1}{\sigma}}$$

- ***Indirect utility function***

$$\Psi(M, p_1, p_2, \ldots, p_n) = \frac{\left[\sum_{i=1}^{n} \frac{\beta_i^{\sigma+1}}{p_i^{\sigma}} \cdot \left(M - \sum_{j=1}^{n} \alpha_j p_j\right)\right]^{\frac{1}{\sigma}}}{\left[\sum_{j=1}^{n} \frac{\beta_j^{\sigma}}{p_j^{\sigma-1}}\right]^{\frac{1}{\sigma}}}$$

- ***Expenditure function***

$$E(u, p) = u^{\sigma} \cdot \left[\sum_{i=1}^{n} \frac{\beta_i^{\sigma+1}}{p_i^{\sigma}}\right]^{-1} \cdot \left[\sum_{j=1}^{n} \left(\frac{\beta_j^{\sigma}}{p_j^{\sigma-1}}\right)\right] + \left(\sum_{j=1}^{n} \alpha_j p_j\right)$$

- ***Marshallian demand functions***

$$x_i = \alpha_i + \left(\frac{\beta_i}{p_i}\right)^{\sigma} \cdot \left[\sum_{j=1}^{n} \left(\frac{\beta_j}{p_j}\right)^{\sigma} p_j\right]^{-1} \cdot \left(M - \sum_{j=1}^{n} \alpha_j p_j\right)$$

- ***Hicksian demand functions*** *inexpressible in a closed form*
- ***Basic characteristics***

suitable for commodity systems containing only substitutes but displaying positive elasticities of any order between goods (S0-branch is a special case of S-branch, but is a generalization of LES). Specification of SO-branch utility function is a Sató's generalization of the ACMS-function.

Classical expenditure systems

S1-branch expenditure system [**Brown** Murray, **Heien** Dale **1972**]

- ***Direct utility function***

$$u(x_1,x_2,\dots,x_n)=\prod_{s=1}^{S}\left[\sum_{j\in s}^{n_s}\beta_{sj}(x_{sj}-\alpha_{sj})^{\rho_s}\right]^{\frac{w_s}{\rho_s}}$$

- ***Indirect utility function***

$$\Psi(M,p_1,p_2,\dots,p_n)=\prod_{s=1}^{S}\left[\frac{w_s\left(M-\sum_{r=1}^{S}\sum_{k\in r}^{n_r}\alpha_{rk}\cdot p_{rk}\right)}{\left[\sum_{k\in s}^{n_s}\frac{\beta_{sk}{}^{\sigma_s}}{p_{sk}{}^{\sigma_s}}\right]}.\left(\sum_{j\in s}^{n_s}\frac{\beta_{sj}{}^{\frac{\rho_s}{1-\rho_s}}}{p_{sj}{}^{\sigma_s}}\right)^{\frac{1}{\rho_s}}\right]^{w_s}$$

- ***Expenditure function*** ***inexpressible in a closed form***

- ***Marshallian demand functions***

$$x_{si}=\alpha_{si}+\left(\frac{\beta_{si}}{p_{si}}\right)^{\sigma_s}.\left[\sum_{j\in s}^{n_s}\left(\frac{\beta_{sj}}{p_{sj}}\right)^{\sigma_s}\right]^{-1}.w_s\left(M-\sum_{r=1}^{S}\sum_{j\in r}^{n_r}\alpha_{rj}.p_{rj}\right)$$

- ***Hicksian demand functions*** ***inexpressible in a closed form***

- ***Basic characteristics***

suitable for commodity systems containing complements and displaying positiive elasticities of any order among commodity groups. (S1-branch is a generalization of LES and SO-branch and is a special case of the S-branch). Specification of S1-branch utility function is a Uzawa's generalization of the ACMS-function.

Classical expenditure systems

Rotterdam expenditure system [Theil Henri, Barten Anton 1964,1965]

- ***Direct utility function*** — ***unknown/unspecified***
- ***Indirect utility function*** — ***unknown/unspecified***
- ***Expenditure function*** — ***unknown/unspecified***

$$log(x_i) = \beta_i + e_i \cdot log\, M + \sum_{k=1}^{n} e_{ik} \cdot log\, p_k$$

$$x_i = b_i \cdot M^{e_i} \cdot \prod_{k=1}^{n} p_k^{e_{ik}}$$

- ***Marshallian demand functions***

$$w_i d\, log\, x_i = b_i \cdot d\, log\, M^* + \sum_{k=1}^{n} c_{ik} d\, log\, p_k$$

- ***Hicksian demand functions***

$$b_i = w_i e_i = p_i \cdot \frac{dx_i}{dM} \qquad c_{ik} = w_i e_{ik}^* = p_i p_k \frac{s_{ik}}{M}$$

- ***Basic characteristics***

Suitable for commodity bundles with mutually different behavior. The impacts of income and prices influences are strictly separated. The demand for each commodity is written as a differential function of income and individual prices.

TRANSLOG expenditure system [Christensen Laurits, Jorgenson Dale,W.,Lau Lawrence 1973]

- ***Direct utility function*** — ***unknown/unspecified***

- ***Indirect utility function***

$$\log \Psi(M,p) = \beta_0 + \sum_{i=1}^{n} \beta_i \log\left(\frac{p_i}{M}\right) + \sum_{i=1}^{n}\sum_{i=1}^{n} \beta_{ij} \log\left(\frac{p_i}{M}\right).\log\left(\frac{p_j}{M}\right)$$

$$\sum_{i=1}^{n} \beta_i = 1 \;;\; \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} = 0 \;;\; \sum_{i=1}^{n} \beta_{ij} = 0 \;;\; \sum_{j=1}^{n} \beta_{ij} = 0 \;;\; \beta_{ij} = \beta_{ji}$$

- ***Expenditure function*** — ***inexpressible in a closed form***

- ***Marshallian demand functions***

$$w_i^M = \frac{p_i x_i}{M} = \frac{\beta_i + \sum_{j=1}^{n} \beta_{ij}.\log\left(\frac{p_j}{M}\right)}{\sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij}.\log\left(\frac{p_j}{M}\right)}$$

- ***Hicksian demand functions*** — ***inexpressible in a closed form***

- ***Basic characteristics***

suitable for commodity systems exhibiting interactions among (transformed) normalized prices. Its usefulness is limited onto demand structures with a moderate number of commodities (max. 5-8)

Quadratic expenditure system QES [Howe Howard, Pollak Robert, Wales Terence 1979]

- ***Direct utility function*** — ***unknown/unspecified***
- ***Indirect utility function***

$$\Psi(M,p) = -\frac{\theta(p)}{M-\varphi(p)} - \frac{\chi(p)}{\theta(p)}$$

- ***Expenditure function***

$$E(u,p) = \varphi(p) - \frac{\theta^2(p)}{u.\theta(p) + \chi(p)}$$

- ***Marshallian demand functions***

$$x_i(M,p) = \frac{1}{\theta^2(p)}\left(\chi_i(p) - \frac{\theta_i(p)}{\theta(p)}.\chi(p)\right).(M-\varphi(p))^2 + \frac{\theta_i(p)}{\theta(p)}.(M-\varphi(p)) + \theta_i(p)$$

- ***Hicksian demand functions*** — ***hardly expressible***
- ***Basic characteristics***

suitable for commodities with (nearly) quadratic relation demand to income („moderate“ luxuries)

Almost ideal demand system – AIDS [Deaton Angus, Muellbauer John 1980]

- **Direct utility function** *unknown/unspecified*
- **Indirect utility function** $\Psi(M, p) = \dfrac{\log M - \log P}{\beta_0 . \prod_{i=1}^{n} p_k^{\beta_k}}$
- **Expenditure function** $\log E(u, p) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \log p_k + \frac{1}{2} . \sum_{k=1}^{n} \sum_{j=1}^{n} \gamma_{kj} * \log p_k \log p_j + u . \beta_0 . \prod_{k=1}^{n} p_k^{\beta_k}$
- **Marshallian demand functions** $w_i^M = p_i x_i(M, p) = \alpha_i + \sum_{i=1}^{n} \gamma_{ij} \log p_j + \beta_i . \log\left(\frac{M}{P}\right)$
- **Hicksian demand functions** $w_i^H = p_i x_i(u, p) = \alpha_i + \sum_{i=1}^{n} \gamma_{ij} \log p_j + u . \beta_0 . \beta_i . \prod_{i=1}^{n} p_k^{\beta_k}$
- **Basic characteristics** $\log P = \alpha_0 + \sum_{k=1}^{n} \alpha_k \log p_k + \frac{1}{2} . \sum_{k=1}^{n} \sum_{j=1}^{n} \gamma_{kj} * \log p_k \log p_j$

  suitable for systems with log-linear demand specification and TRANSLOG index price function. Demands are decomposed into "neat "price influences and (by the suitable price index) deflated consumer's income

Advanced recently developed expenditure systems

General exponential form - GEF [Cooper Russel J., McLaren Keith R. 1996]

- **Direct utility function** — unknown/unspecified
- **Indirect utility function**

$$\Psi(M,p)=\frac{\left(\frac{M}{\kappa.P1}\right)^{\mu}-1}{\mu}.\left(\frac{M}{P2}\right)^{\sigma}$$

- **Expenditure function** — inexpressible in a closed form
- **Marshallian demand functions**

$$w_i(M,p)=\frac{p_i x_i^{M}}{M}=\frac{EP1_i.R^{\mu}+\sigma.EP2_i.(\frac{R^{\mu}-1}{\mu})}{R^{\mu}+\sigma\frac{R^{\mu}-1}{\mu}}$$

- **Hicksian demand functions** — inexpressible in a closed form
- **Basic characteristics**

$$R=\frac{M}{\kappa.P1}\qquad EP1_i=\frac{\partial\, ln\, P1}{\partial\, ln\, p_i}\qquad EP2_i=\frac{\partial\, ln\, P2}{\partial\, ln\, p_i}$$

Advanced recently developed **expenditure systems**

Quadratic Almost Ideal Demand System – QUAIDS [Banks J., Blundell R.,Lewbel A.1997]

- ***Direct utility function*** — ***unknown/unspecified***

$$log(b(p)) = \sum_{j=1}^{n} \beta_j \, log(p_j)$$

- ***Indirect utility function***

$$log\Psi(M,p) = \left(\frac{b(p)}{log\left(\frac{M}{a(p)}\right)} - \omega(p) \right)^{-1}$$

$$log(\omega(p)) = \sum_{j=1}^{n} \omega_j \, log(p_j)$$

$$a(p) = \alpha_0 + \sum_{i=1}^{n} \alpha_i . log(p_i) + \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{jk} . log(p_j) . log(p_k)$$

- ***Expenditure function*** — ***inexpressible in a closed form***

- ***Marshallian demand functions***

$$w_i = \alpha_i + \sum_{s=1}^{n} \gamma_{is} \, log \, p_s + \beta_i \, log\left(\frac{M}{a(p)}\right) + \omega_i . \frac{\left(log \frac{M}{a(p)}\right)^2}{b(p)}$$

- ***Hicksian demand functions*** — ***inexpressible in a closed form***

$$\sum_{i=1}^{n} \alpha_i = 1, \sum_{i=1}^{n} \beta_i = 1, \sum_{j=1}^{k} \gamma_{jk} = 0, \sum_{k=1}^{k} \gamma_{jk} = 0, \sum_{j=1}^{n} \omega_j = 0,$$

- ***Basic characteristics***

suitable as an extension of the previous AIDS model, when demands are extended about the quadratic terms comprised** (by the income) **relative prices. The former AIDS is a special case of QUAIDS. Three price index functions** (twice Cobb-Douglas and the TRANSLOG) **are used

Advanced recently developed expenditure systems

Full Laurent Model [Barnett W.A., Lee Y.W. and Wolfe M.D. 1985,1987]

- ***Direct utility function*** ***unknown/unspecified***
- ***Indirect utility function***

$$\Psi\left(\frac{M}{p_1},\frac{M}{p_2},\dots,\frac{M}{p_n}\right)=\alpha_0+2.\sum_{i=1}^{n}\left(\alpha_i\sqrt{\frac{M}{p_i}}-\beta_i\sqrt{\frac{p_i}{M}}\right)+\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\alpha_{jk}\sqrt{\frac{M}{p_j}}.\sqrt{\frac{M}{p_k}}-\beta_{jk}\sqrt{\frac{p_j}{M}}.\sqrt{\frac{p_k}{M}}\right)$$

- ***Expenditure function*** ***inexpressible in a closed form***
- ***Marshallian demand functions***

$$x_i(M,p)=\frac{\dfrac{\alpha_i.M}{p_i^{3/2}}+\dfrac{\beta_i}{(p_i.M)^{1/2}}+\sum_{j=1}^{n}\left[\dfrac{\alpha_{ij}}{p_i^{3/2}p_j^{1/2}}+\dfrac{\beta_{ij}}{M}.\sqrt{\dfrac{p_j}{p_i}}\right]}{\sum_{j=1}^{n}\left[\alpha_j(p_j.M)^{1/2}+\beta_j\left(p_jM^3\right)^{1/2}\right]+\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\alpha_{jk}(p_jp_k)^{-1/2}+\beta_{jk}(p_jp_k)^{1/2}.M^{-2}\right]}$$

- ***Hicksian demand functions*** ***inexpressible in a closed form***
- ***Basic characteristics***

  Suitable for the models extending the classical GL (Generalized Leontieff) flexible form. Interactions among normalized prices influencing demands for individual commodities are included. Not appropriate for large commodity structures.

Dynamic and stochastic specifications for the

Rotterdam expenditure system [**Theil** Henri, **Barten** Anton **1964,1965, Parks 1969**]

*The **econometric estimation of the Rotterdam model** requires the **transformation of the differential scheme into difference equations** (usually of the first order) generating the model. **The discrete analog** of the basic scheme is*

(*) $$x_t^* - w_{ti}^* = \sum_{i=1}^{n} -\mu_{ni}(\log p_{ti} - \log p_{t-1,k}) + (1-\mu_n).w_{ti}^*(\log x_{t,i} - \log x_{t-1,i}) + \varepsilon_{ni}$$

where $w_{ti}^* = 0{,}5.\left(w_{t,i} + w_{t-1,i}\right)$, $w_{t,i} = p_{ti}x_{ti} / M_{ti}$ *is the **average value share in two successive periods***

,and $x_t^* = \sum_{t=1}^{n} w_{ti}^* x_{ti}$ *is the **value weighted average of the logarithmic differences of the quantity demanded**.*
It is a volume index of the change in total consumption and can be interpreted as a measure of the change in real income.

***The estimation techniques should take account of the covariance singularity** as well as the parameter restriction implied by the **homogeneity, adding-up, and symmetry conditions**. It can be shown that **the restrictions imply** that for each t, **one of the equations is redundant:** adding the first equations (*) gives*

$$x_t^* - w_{ti}^* = \sum_{i=1}^{n} -\mu_{ni}(\log p_{ti} - \log p_{t-1,k}) + (1-\mu_n).w_{ti}^*(\log x_{t,i} - \log x_{t-1,i}) + \varepsilon_{ni}$$

Upon clearing terms and multiplying by -1, this expression becomes the n-th equation (*) [].
Considering this system with the n-th equation deleted, we can impose the homogeneity restriction by deflating prices by the n-th price. Equations (*) then become

$$w_{ti}^*\left(\log x_{t,i} - \log x_{t-1,i}\right) = \mu_i \cdot x_t^* + \sum_{k=1}^{n} \mu_{ik} \cdot (\log p_{tk} - \log p_{t-1,k}) + \varepsilon_{ti} \quad t = 2,\dots,T;\ i = 1,2,\dots,n$$

A comment: ***The unconstrained price coefficients, however, do not satisfy the symmetry condition***
Elimination of the redundancy (the choice of equation to eliminate is arbitrary) reduces the system to be estimated by one equation and also avoids the covariance singularity in the reduced system.

Dynamic and stochastic specifications for the

Almost ideal demand system – AIDS [**Deaton** Angus, **Muellbauer** John **1980**]

The suitable form for econometric estimation of AIDS is the modified version

$$w_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \log p_j + \beta_i . \log\left(\frac{M}{P}\right) \qquad i = 1,2,\ldots,n$$

*When inserting TRANSLOG price index, we get the **dynamized specification for the sample of the length** t = 1,2,…,T* **.**

$$w_{ti} = \left(\alpha_i - \beta_i \alpha_0\right) + \sum_{j=1}^{n} \gamma_{ij} \log p_{tj} + \beta_i \left(\log M_t - \sum_{j=1}^{n} \alpha_j \log p_{ti} - \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{jk} . \log p_{tj} . \log p_{tk} \right) + \varepsilon_{ti}$$

This non-linear system may be estimated by maximum likelihood or another method with and without restrictions

$$\sum_{k=1}^{n} \gamma_{jk} = 0, \qquad \sum_{k=1}^{n} \gamma_{jk} = 0, j = 1,2,\ldots,n$$

Since the data are summed up by construction, the following adding-up conditions are not testable:

$$\sum_{k=1}^{n} \alpha_k = 1 \qquad \gamma_{jk} = \gamma_{kj}, j,k = 1,\ldots,n \qquad \sum_{k=1}^{n} \beta_k = 0$$

Estimation via the maximum likelihood method entails a problem of the identification of parameter α_0 .

*(it can be interpreted as the outlay required for the minimal standard of living when prices are unities). In many cases, it is possible to exploit the collinearity of the prices to yield a simpler estimation technique. **If P were known, the model would be linear in the parameters** α **,** β and γ and **estimation can be done by OLS** which, in this case and given normally distributed errors, is equivalent to ML estimation for the system as a whole. If prices are closely collinear, it may be adequate to approximate **P** as proportional to*

Stone's index $\log P^* = \sum w_k . \log p_k$ *If* $P \cong \lambda . P^*$ *, then the **AIDS model can be estimated as***

$$w_{ti} = \left(\alpha_i - \beta_i \log \lambda\right) + \sum_{j=1}^{n} \gamma_{ij} \log p_{tj} + \beta_i \log\left(\frac{M_t}{P_t^*}\right) + \varepsilon_{ti}$$

The estimation results of the Rotterdam system

[Czech COICOP prices and consumptions quarterly data 1Q2000-4Q2006]

Tab2	Rotterdam model	c.value=2,0930	const.	d_1LP1	d_1LP2	d_1LP3	d_1LP4	d_1LP5	d_1LP7	d_1LP11	SUMWdLQ	R/DW
1	Food and non-alcoholic	parameters	45,363	6,261	106,796	64,002	-14,975	-26,206	-3,634	-105,416	-0,114	0,440
	Beverages	t-statistics	3,765	0,641	2,170	2,783	-0,940	-0,266	-0,384	- 2,280	-1,094	1,293
2	Alcoholic beverages,	parameters	-0,126	-1,320	9,337	8,635	-2,228	8,052	-1,526	-14,502	0,019	0,585
	tobbaco, drugs	t-statistics	-0,054	-0,705	0,989	1,957	-0,729	0,427	-0,840	-1,635	0,948	1,393
3	Clothing and footwear	parameters	-20,455	9,330	-57,501	30,658	-18,900	82,278	-21,657	40,116	0,231	0,854
		t-statistics	- 2,557	1,439	-1,760	2,008	-1,787	1,259	-3,444	1,307	3,348	1,768
4	Housing,water,electricity	parameters	20,965	-6,099	-8,406	-80,675	25,909	85,326	27,182	-5,398	0,064	0,614
	and other fuels	t-statistics	1,628	-0,584	-0,160	- 3,282	1,522	0,811	2,685	-0,109	0,575	1,801
5	Furnishings, household	parameters	-23,524	-2,118	-42,394	-12,047	-3,316	71,415	-15,945	17,351	0,267	0,899
	equip. & maintenance	t-statistics	- 5,640	-0,626	- 2,488	-1,513	-0,601	2,096	- 4,863	1,084	7,408	1,804
6	Health	parameters	2,911	5,827	4,778	5,323	-2,561	-15,195	-2,261	-0,552	-0,019	0,471
		t-statistics	1,541	3,804	0,619	1,476	-1,025	-0,985	-1,522	-0,076	-1,193	1,861
7	Transport	parameters	-34,680	-25,833	78,650	-16,929	46,275	-52,306	53,234	34,877	0,400	0,516
		t-statistics	-1,948	-1,790	1,082	-0,498	1,966	-0,360	3,804	0,511	2,604	1,017
8	Postal services and	parameters	6,496	4,874	-2,969	6,318	-15,548	-92,230	-8,321	-3,833	-0,024	0,556
	Telecommunications	t-statistics	3,577	3,310	-0,400	1,823	- 6,476	- 6,218	- 5,829	-0,550	-1,520	1,886
9	Recreation and culture	parameters	4,737	-12,945	3,028	21,975	0,324	-99,234	6,419	2,596	0,066	0,573
		t-statistics	0,714	- 2,406	0,112	1,734	0,037	-1,831	1,230	0,102	1,156	2,068
10	Education	parameters	3,211	0,351	3,335	-2,251	-0,841	-6,396	-2,357	-9,383	-0,035	0,553
		t-statistics	1,848	0,249	0,470	-0,679	-0,366	-0,451	-1,726	-1,407	- 2,310	1,573
11	Restaurants and hotels	parameters	6,781	-14,348	-55,707	-1,840	10,760	-19,015	15,688	-29,316	-0,020	0,578
		t-statistics	1,057	- 3,003	2,307	-0,163	1,432	-0,393	3,435	-1,265	-0,372	1,532
12	Miscellaneous	parameters	-4,900	21,675	-94,655	-25,007	-14,140	44,497	-31,134	44,144	0,146	0,551
	goods and services	t-statistics	-0,434	2,367	- 2,052	-1,160	-0,947	0,483	- 3,508	1,019	1,495	1,271

The estimation results of the AIDS demand system

[Czech COICOP prices and consumptions quarterly data 1Q2000-4Q2006]

Tab1	AIDS model	c.value=2,0930	const.	$ln(p_1)$	$ln(p_2)$	$ln(p_3)$	$ln(p_4)$	$ln(p_5)$	$ln(p_7)$	$ln(p_{11})$	$ln(M/P)$	R/DW
1	Food and non-alcoholic	parameters	442,919	60,849	98,628	21,874	-11,417	-124,313	-15,812	-101,713	-5,825	0,937
	Beverages	t-statistics	1,245	**5,019**	1,928	0,840	-0,740	-1,406	-1,195	**- 2,424**	-1,321	2,096
2	Alcoholic beverages,	parameters	18,990	-6,780	-2,344	15,656	2,030	-13,472	-0,372	0,196	6,321	0,820
	tobbaco, drugs	t-statistics	0,193	**- 2,024**	-0,166	**2,176**	0,476	-0,552	-0,102	0,017	**5,187**	2,704
3	Clothing and footwear	parameters	-39,940	29,214	-133,284	166,332	30,728	-137,025	-56,542	66,046	80,312	0,851
		t-statistics	-0,057	1,217	-1,316	**3,226**	1,006	-0,783	**- 2,158**	0,795	**9,198**	2,166
4	Housing,water,electricity	parameters	-815,686	-15,335	-270,087	-177,948	26,518	455,032	41,301	180,860	-70,962	0,930
	and other fuels	t-statistics	-1,510	-0,833	**- 3,477**	**- 4,499**	1,132	**3,389**	**2,056**	**2,838**	**-10,595**	2,072
5	Furnishings, household	parameters	68,361	-12,374	-128,691	52,550	21,258	-2,419	-19,610	58,504	37,362	0,808
	equip. & maintenance	t-statistics	0,194	-1,031	**- 2,541**	**2,038**	1,392	-0,028	-1,497	1,408	**8,557**	2,168
6	Health	parameters	226,645	6,393	-27,704	0,216	11,841	-27,203	-16,823	10,250	-6,908	0,700
		t-statistics	1,013	0,838	-0,861	0,013	1,220	-0,489	-2,021	0,388	**- 2,490**	2,335
7	Transport	parameters	497,747	21,043	498,323	-4,080	-72,232	-364,121	123,401	-292,671	-11,199	0,607
		t-statistics	0,443	0,550	**3,086**	-0,050	-1,483	-1,304	**2,954**	**-2,209**	-0,804	2,150
8	Postal services and	parameters	-70,930	-3,241	-12,343	-32,859	-6,228	64,847	-16,102	26,830	-1,453	0,918
	telecommunications	t-statistics	-0,381	-0,510	-0,461	**-2,409**	-0,771	1,400	**-2,323**	1,221	-0,629	1,952
9	Recreation and culture	parameters	-317,300	-31,143	188,465	29,747	-4,802	-54,788	30,438	-82,855	7,112	0,457
		t-statistics	-0,629	-1,811	**2,598**	0,805	-0,219	-0,437	1,622	-1,392	1,137	2,431
10	Education	parameters	82,846	-6,090	9,522	-24,820	1,602	28,582	-5,378	-15,087	-9,541	0,771
		t-statistics	0,664	-1,432	0,531	**- 2,716**	0,296	0,921	-1,158	-1,025	**- 6,165**	2,641
11	Restaurants and hotels	parameters	-433,450	-11,925	110,847	-66,956	-30,755	108,032	43,847	-37,080	-27,655	0,579
		t-statistics	-0,937	-0,756	1,666	-1,977	-1,533	0,939	**2,548**	-0,679	**- 4,821**	2,061
12	Miscellaneous	parameters	237,439	-5,587	-307,593	20,685	54,694	89,332	-86,522	190,723	4,905	0,718
	goods and services	t-statistics	0,326	-0,225	**- 2,937**	0,388	1,731	0,493	**- 3,193**	**2,219**	0,543	1,890

The COICOP clasification of commodities in the basket

The commodity basket composed for ČR by the Czech statistical office consists since 1999 of

790 (775) **commodities** **structured into**
12 **divisions (oddílů),** **47 groups (skupin)** **and** **117** (105) **cathegories (tříd)**

The 12 groups represents:	*number*	*weights 1993*	*weights 1999*	*weights 2006*
	790	1000,00	1000,00	1000,00
1. *Food and non-alcoholic beverages*	163	260,62	197,57	162,63
2. *Alcoholic beverages, tobacco ,drugs*	16	66,47	79,24	81,72
3. *Clothing and footwear*	80	91,89	56,93	52,43
4. *Housing, water, electricity and other fuels*	58	141,42	236,40	248,29
5. *Furnishings, household equipment and routine maintenance of the house*	96	75,39	67,92	58,05
6. *Health*	39	9,55	14,35	17,86
7. *Transport*	96	100,81	101,41	114,10
8. *Postal services and telecommunications*	21	8,89	22,54	38,73
9. *Recreation and culture*	113	99,58	95,53	98,66
10. *Education*	11	6,16	4,50	6,18
11. *Restaurants and hotels*	47	55,06	74,15	58,39
12. *Miscellaneous goods and services*	50	84,16	49,46	62,96

COICOP:** abbrev. from **Classification of Individual Consumption by Purpose

Conclusions I factual consequences

- **The AIDS model gives moderately good results.The explained variabilities of dependent variables** measured by the **R^2 vary between 0,46 and 0,94 (*groups 9 and 1,4,* resp.). The DW coefficient suggests** none or **slightly negative residual autocorrelations,** partly **due to** apparent **seasonality in consumptions in some commodity groups. The number of significant variables have moved between 2 - 4** (with an exception of 4th group, in which 6 variables were significant). **The price of the 4th group has no** apparent **effect on the consumptions,** at all.
- **Prices of the 2nd divisions influence the relative expenditures within several commodity groups:** On the one hand, **expenditures on this division, are not**, perhaps surprisingly, **influenced by the appropriate price development.** This fact means that **consumption of *tobacco and alcoholic drinks* is intact by its own price.** On the other hand, increased **money spent here are lacking for expenditures on dispensable goods (*divisions 3,5,9,11*), maintaining the consumption of the necessities** (*food, expenditures on housing*) relatively **intact.**
- Deflated **income has** none or only **a small apparent effect on some necessities** such as ***food, transport, posts and telecommunications*, but this is not true for other commodity groups,** in which dispensable goods or services have occurred **(*clothing and footwear, furnishings*). In** some **commodity groups (*housing, water and fuel, health, education*), this term has**, perhaps strikingly, **a negative sign, but this is explainable by the fact that prices** (aggregated in .....) **rose very steeply in these commodity groups.**
- **The Rotterdam model exhibits poorer results than the AIDS model,** as can be seen from the **R^2 value for**, with only two exceptions, **this coefficient lies below 0,62. Only in groups 3 and 5, the fit is quite satisfactory.** As to t-statistics, **the modeling of at least four demand relations** (with none or only one significant variable) **can hardly be considered as successful.**
- **The effects of the most differenced prices onto dependent variables are** quite irregular and **less persuasive than in the AIDS model. The own-price elasticities are seldom significant,** while **some cross-price ones exhibit hardly expectably, despite very high t-values** (particularly in the regressions for *furnishings etc*. and *postal services*). **The most influential price variable is the differenced price of *transport* having an effect on seven demand variables but** in some cases **the appropriate regression coefficients have no expectable signs** (considering the type of goods that can be regarded as substitutes to it)

Conclusions II – methodological comments

- The *Czech COICOP classification* maintained in a quarterly time series enables since 2000 (or some years before) to lead the econometric analysis of the expenditure/demand systems. The annual data are too short for such purpose. The monthly data (albeit available and sufficiently large) are worse to use due to considerable substitution effects in population behavior.
- Some sample data on expenditures and prices exhibit a remarkable seasonality. The seasonal adjustment of the data would bring the clearer view on the behavior of consumers.
- Expenditure systems such as *LES, Rotterdam, Addilog* and *AIDS* can be employed almost readily under the current state of the data samples. For implementation of (globally) flexible functional forms (*TRANSLOG, Generalized Leontief, Minflex Laurent, GEF* etc.), exhibiting many parameters there is not, at present, hope for rational (if the whole COICOP structure is to be captured).
- Only a part of expenditure schemes can be estimated by the means of standard econometric tools: The *LES, Rotterdam* (and with little additional supplement also *AIDS*) models can be estimated by the means of the linear regression analysis for they form a „classical“ SUR model structure. A little more troublesome is, e.g., SO-branch model.
- The quantitative analysis of the most of recently developed expenditure systems can be performed only with the help of advanced estimation techniques requiring, as a rule, a numerical iterative methods (FIML or some other method of nonlinear econometric analysis). Moreover, the up to now analyses have confined themselves usually only to 3-6 commodity groups, so the their settings into COICOP commodity classification is (as far as the author became acquainted) an open question.