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Czech Yield Curve

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Abstract:

We present a yield curve model based on principal component analysis and nonlinear stochastic differential equations. The model is applied everywhere where quantification of interest rate dynamics is needed, i.e. in the Monte Carlo risk management of interest rate instruments or in the evaluation of profitability and risk of interest rate derivatives and different business strategies. Applying presented model, we are also able to price the interest rate derivatives or predict the distribution of interest costs.

Keywords:

interest rate models, yield curve, principal component analysis, nonlinear stochastic differential equations, risk management

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1. INTRODUCTION

The yield curve modeling is very important area of financial risk management. Especially during the last few years, we could observe high volatility of interest rates. Lot of hedge funds, pension funds, investment banks, debt offices and other assets and debt managers were affected by this increasing volatility. The yield of corporate and government bonds have increased significantly during the financial crisis. Due to current debt crisis on the periphery of European monetary union, bond yields remain at high level. These high bond yields are in contrast with historically low money market rates and caused by increasing credit spread of issuers. Thus credit spread of issuer can be expressed by the difference between bond yield and appropriate money market rate. If we want to understand and to manage all interest cost we need to understand the dynamic of money market rates first. Money market rates are also used as reference rate which is further reason for understanding their dynamics.

In this paper, we present a money market yield curve model based on principal component analysis and nonlinear stochastic differential equations. Thus we describe the dynamics of money market rates which can be applied everywhere where the interest rate risk needs to be manage. Applying presented model we are able to price the interest rate derivatives or predict the distribution of interest costs.

We design the presented model for future simulation of yield curves, which can be used, for example, for prediction of yield curves. This paper is based on [1] but updated and more suitable for interest costs estimate. In the model, we add some more information in terms adding daily data. We also include high volatile data from period of financial crisis. Further, we decrease the time step for better description of interest rates dynamics.

1.1. Previous research in the area of modeling the yield curve

In the past we have tried to describe the money market yield curve through short rate models. These models are based on unobservable short rate. Among the famous models of short rate belong Vasicek, CIR (Cox-Ingersoll-Ross), (CKLS Chan-Karolyi-Longstaff-Sanders) etc. The short rate models are primarily designed for pricing of interest rate derivatives. The short rate models describe only the dynamics of short rate and the dynamics of the other observable rates needs to be determined from the short rate dynamics. We have found that these models are not suitable for interest cost estimates since they underestimate the volatility of longer tail of yield curve significantly.

The presented model describes the dynamics of the whole yield curve at once which is desirable and removes the volatility underestimation problem.

2. STOCHASTIC DIFFERENTIAL EQUATIONS AND THEIR APPLICATION IN YIELD CURVE MODELING

The application of stochastic processes in continuous time in financial modeling has become popular. The financial model is usually based on the diffusion process $\{X_t, t \geq 0\}$. The dynamics of this process is determined by following stochastic differential equation (SDE):

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad (1)$$

where $\{W_t, t \geq 0\}$ is the standard Brownian motion. The functions $\mu(\cdot)$ and $\sigma(\cdot)$ are called drift and diffusion of the process, respectively. We assume the following parameterization of SDE:

$$\mu(x) = \mu(x, \theta) \quad \text{and} \quad \sigma^2(x) = \sigma^2(x, \theta), \quad (2)$$

where θ is the parameter vector of parametric space $\Theta \subset R^K$. In this paper we consider only Markov (i) and strictly stationary processes (ii). The Markov process is defined as a process, for which in time t the conditional distribution function of future values $X_s, s > t$ depends only on X_t (i.e. is independent of the historical values $X_u, u < t$). A process is strictly stationary when joint distribution holds for all m , for all time sequences $0 < t_1 < \dots < t_m < \infty$ and for all x_1, \dots, x_m from state space following equation $P(X_{t_1} \leq x_1, \dots, X_{t_m} \leq x_m) = P(X_{t_1+\Delta} \leq x_1, \dots, X_{t_m+\Delta} \leq x_m)$. Drift and diffusion functions fully specify the model. A frequent choice of the drift for interest rate models is a linear function $\mu(x, \theta) = \alpha(\mu - x)$, where the rate x returns to its mean value μ with the speed α . This property is called mean reversion and it is the fundamental postulate of interest rate models. Diffusion function differentiates then among the models. For instance a Vasicek model uses constant diffusion $\sigma^2(x, \theta) = \sigma^2$ the diffusion function of the CIR model is $\sigma^2(x, \theta) = \sigma^2 x$.

The marginal (unconditional) density of a stationary process is 1

$$\pi(x, \theta) = \frac{\xi(\theta)}{\sigma^2(x, \theta)} \exp\left\{ \int_{x_0}^x \frac{2\mu(u, \theta)}{\sigma^2(u, \theta)} du \right\}, \quad (3)$$

¹Marginal density equation comes from Fokker-Planck equation, see [4] or more precisely see [2].

where $\xi(\theta)$ is the normalization constant. The lower bound x_0 of the integral is not unique. The choice of x_0 is neutralized by the normalization constant, which warrants that the integral through the domain of definition of the process is equal to one.

Using the SDE for modeling financial processes, we first need to specify drift and diffusion functions of the SDE and to define the parametric space Θ . Secondly, we need to estimate the parameters of the specified model.

The definition of the parametric space Θ comes from the domain of definition of the process $D = (x, \bar{x})$, from the postulates (i) and (ii) and from specific claims on behavior of the process (e.g. mean reversion).

In this paper we will consider the process of principal components of a yield curve. A process of principal component is defined for all real numbers: $D = (-\infty, \infty)$. The definition of the Principal Component Analysis implies that the expected value of each principal component is equal to zero. We demand mean reversion property of the given process, which is guaranteed if

$$\lim_{x \rightarrow \infty} \mu(x, \theta) < 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \mu(x, \theta) > 0. \quad (4)$$

Following cubic drift and constant diffusion are used in our model:

$$\mu(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \quad \text{and} \quad \sigma^2(x) = \beta_0, \quad (5)$$

where $\theta = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0)$.

The solution (4) for cubic drift (5) can be found out that mean reversion is guaranteed, when $\alpha_3 < 0$.

Our choice of cubic drift for principal component of yield curve modeling support the time series of principal components of Czech and European yield curves. The choice of constant diffusion function makes practical application of the model easier due to analytical solution of the integral in the exponent of equation (3).

3. ESTIMATION OF PARAMETERS OF SDE

We based the parameter estimation of SDE on minimizing the distance between the parametric density of the SDE and its nonparametric estimation (empirical density). Kernel density (see [5]) is used for nonparametric density estimation. We estimate the nonparametric density from time series $\{x_1, \dots, x_n\}$, where observations are recorded in

time interval Δ (i.g. $\Delta = 1$ month). We search for parameter vector θ , which minimize the distance between parametric density of the SDE given by the equation (3) and the nonparametric density estimation. This idea was first introduced by Yacine Ait-Sahalia in [4]. Yacine Ait-Sahalia estimated the so-called General Parametric Model of three months Eurodollar interest rate. General Parametric Model defines the dynamics of interest rate by the following SDE:

$$dr_t = (\alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2 + \alpha_3 / r_t) dt + (\beta_0 + \beta_1 r_t + \beta_2 r_t^{\beta_3}) dW_t$$

As we mentioned above, we estimate the parameters of SDE from observations without discretizing the SDE. The term discretisation refers to the approximation of a continuous process with a discrete process. It is clear that discretisation always brings bias, which increases with the length of the observation interval Δ . If we have high frequency time series, where $\Delta \rightarrow 0$, the bias caused by discretisation is insignificant. For financial time series recorded on daily or even monthly basis, the discretisation can be a too inaccurate approximation of the original, time-continuous process. Under these conditions the bias could be no longer tolerable.

3.1. Nonparametric estimation of marginal density

The nonparametric estimation of marginal density is given by:

$$\hat{\pi}(u) = \frac{1}{n} \sum_{i=1}^n \frac{1}{b_n} K\left(\frac{u - x_i}{b_n}\right), \quad (6)$$

where $K(\cdot)$ is the kernel function and b_n is the smoothing constant (bandwidth). The equation (6) can be interpreted as a "continuous histogram". The choice of b_n is crucially important for the smoothing level of the density - higher b_n causes higher level of smoothing. Setting the optimal value of b_n is difficult for real market data. Some methods for choosing optimal b_n were introduced but these assume independent, identically distributed (i.i.d.) data. The financial data are usually strongly auto correlated. Due to the abovementioned problems, we determine the value of b_n by following empirical rule:

$$b_n = c \sigma n^{-\frac{1}{5}}, \quad (7)$$

where σ is the standard deviation of time series, n is the number of observations and c is a constant. For independent normal distributed

data, the recommended value of c is 1,05. We use the normal kernel function $K(u) = \exp(-u^2/2)/\sqrt{2\pi}$.

3.2. Marginal density of SDE

The easiest practically applicable SDE is the linear drift with constant diffusion (Vasicek model). Marginal density of the corresponding SDE is normally distributed. The marginal distribution for the combination of linear drift and diffusion directly proportional to the value of the process (CIR model) is gamma distribution. The cubic drift and constant diffusion given in (5) has marginal density defined in (3). The marginal density can be adjusted to the form:

$$\pi(x, \theta) = \frac{\xi(\theta)}{\beta_0} \exp\left\{\frac{2\alpha_0 x}{\beta_0} + \frac{\alpha_1 x^2}{\beta_0} + \frac{2\alpha_2 x^3}{3\beta_0} + \frac{\alpha_3^4}{2\beta_0}\right\}, \quad (8)$$

where $\xi(\theta)$ is chosen so that the integral of marginal density (8) over the process domain of definition $D = (x, \bar{x})$ is equal to one (in the case of principal component $D = (-\infty, +\infty)$).

3.3. Objective function of estimation

Estimation is based on the minimization of the distance between the marginal density of SDE (parametric density) given in equation (3) and its nonparametric estimation (empirical density) given in equation (6). We define the objective function of the estimation the same way as in [4] in terms of mean square error (MSE) defined by:

$$\begin{aligned} O &= E\left[(\pi(X, \theta) - \hat{\pi}(X))^2\right] \\ &= \int_{\bar{x}}^{\bar{x}} (\pi(u, \theta) - \hat{\pi}(u))^2 \hat{\pi}(u) du. \end{aligned} \quad (9)$$

The parameters θ are estimated on the basis of observations $\{x_1, \dots, x_n\}$ minimizing the sample objective function:

$$\hat{\theta} = \arg \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n (\pi(x_i, \theta) - \hat{\pi}(x_i))^2. \quad (10)$$

The estimate $\hat{\theta}$ is asymptotically normal:

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \Omega), \quad (11)$$

where Ω and its estimation are described on page 420 in [4]. Practically it is profitable to handle the logarithm of marginal densities, which is easier to optimize. Using the cubic drift and constant diffusion given in (8) we solve

$$\hat{\theta} = \arg \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \left(\log \xi(\theta) - \log \beta_0 + \frac{2\alpha_0 x_i}{\beta_0} + \frac{\alpha_1 x_i^2}{\beta_0} + \frac{2\alpha_2 x_i^3}{3\beta_0} + \frac{\alpha_3 x_i^4}{2\beta_0} - \log \pi(x_i) \right)^2. \quad (12)$$

The domain of integration of constant $\xi(\theta)$ can be determined by extending sufficiently the range of the observations $\{x_1, \dots, x_n\}$. We have identified that adding 1.5times the length of the interval to the maximum and subtracting it from the minimum is mostly sufficient.

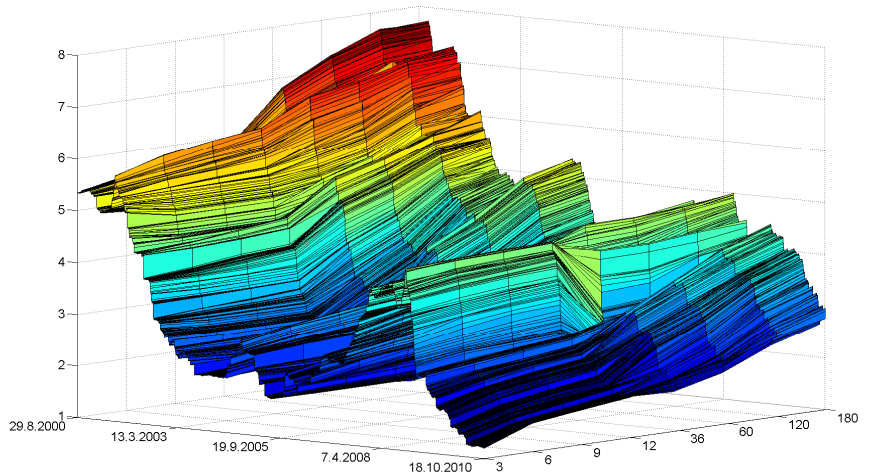
4. PRACTICAL APPLICATION

We apply the cubic drift and constant diffusion SDE on the Czech yield curve (YC). We don't model directly the YC process, but we apply the principal component analysis. We also look at the process behavior with a simulation experiment.

4.1. The data

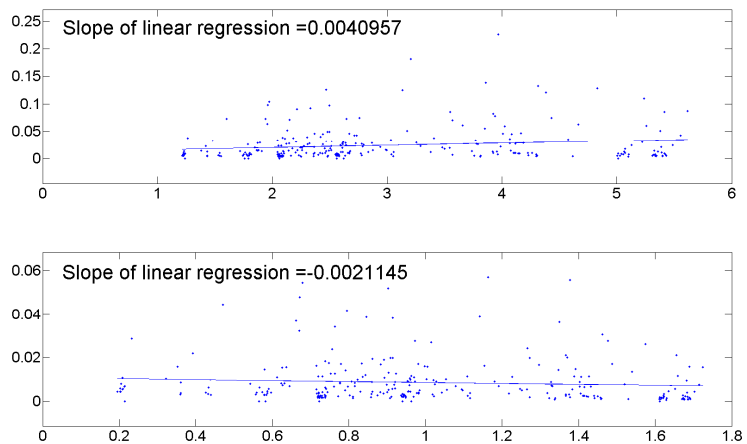
The estimation of parameters is performed on the basis of daily observations. We used maturities of 3M, 6M, 9M, 12M, 3Y, 5Y, 10Y and 15Y Czech zero-coupon yield curve from August 29, 2000 until October 18, 2010. We drop the nontrading days, i.e. for each maturity we have 2554 observations. Maturities shorter than one year are rates of PRIBOR, maturities over one year are rates of swap market. All data are downloaded from Bloomberg. The yield curve is displayed in figure 1.

Figure 1: Czech yield curve since 29.08.2000 till 18.10.2010. Maturities are displayed in months, values 3, 6, 9 a 12 present 3M PRIBOR - 12M PRIBOR, values 36, 60 a 120 present 3Y, 5Y 10Y and 15Y swap. Y - axis presents interest rate in percents.



The variability of the series tends to be dependent on its level, so we performed logarithmic transformation on the series. The relationship between variability and level of interest rates and effectiveness of the logarithmic transformation are shown in figure 2. It is obvious that the logarithmic transformation decreases dependence of the variability of the rates on their level and stabilizes them in term of variance.

Figure 2: The relationship between the variability and the level of interest rates. The time series for all maturities are divided into 9 day periods. For each period we calculate the mean (horizontal axis) and the standard deviation (vertical axis). The upper chart: no transformation; the bottom chart: logarithmic transformation. The line represents the linear trend.



4.2. Principal component analysis

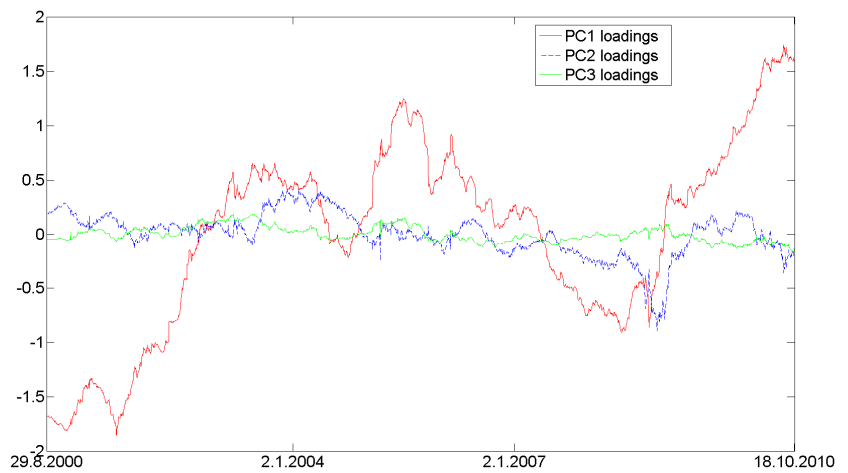
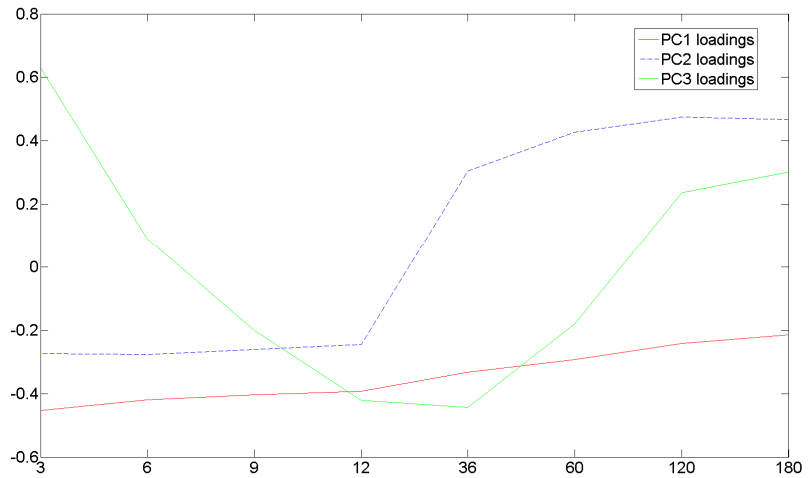
Direct yield curve modeling would represent multivariate stochastic process modeling of highly correlated variables. We solve this problem by applying the principal component analysis. This method is a linear transformation of input variables to new variables - so called components. Input variables are linear combinations of components. The weights of the combination are called component loadings. The aim of the transformation is to reduce the size of the data, while preserving as much information contained in the data as possible. The first three components of the Czech YC logarithms explain over 99.7 % of its variability. The first component explains 93.6 % of the variability of the YC. Another advantage of the principal components analysis is that the components are (following from its definition) independent. That means that we can model the components separately with univariate processes. We denote the first three components with PC_1 , PC_2 and PC_3 .

The upper chart of figure 3 displays the component loadings for different maturities. The component loadings of the first principal component are approximately horizontal. We can interpret the first components as the level factor of the interest rates. Changes in the first component cause a parallel shift of the YC. Component loadings of the second component are increasing and changes in the second component cause rotation of the YC. We can interpret the second component as a slope factor. The component loadings of the third component are "hump" shaped and changes in this component cause change of the shape of the YC. We can interpret this component as a curvature factor. This interpretation of components was first presented in [3].

Figure 3: Component loadings and time series of the first three components of the Czech YC.

Top: The component loadings - maturities in months are on the horizontal axis.

Bottom: Time series of the components from August 29, 2000, until October 18, 2010.



The bottom chart of figure 3 displays the time series of the components from August 29, 2000, until October 18, 2010. The components are calculated from the logarithms of the Czech YC (figure 1). It follows from the definition of the principal components that its expected value is zero.

4.3. The estimate of cubic drift of the principal components

To quantify the dynamics of the first three principal components PC_1 , PC_2 and PC_3 , we estimate parameters of the SDE with cubic drift and constant diffusion (formula (5)). The first step is the nonparametric estimate of the probability distribution function using the kernel density defined in (6). Then we estimate the parameters of the SDE, minimizing the difference between parametric marginal distribution (8) (corresponding to the SDE with cubic drift and constant diffusion) and the nonparametric estimate of the marginal distribution. The objective function is defined in the formula (12). We take the following constraints into consideration: $\alpha_3 < 0$ and $\beta_0 > 0$. The parametric and nonparametric estimates of the marginal distribution function of the first component are displayed in bottom chart of figure 4. The upper left chart displays the cubic drift and the observed daily differences of the time series of PC_1 . The upper right chart displays the volatility and the observed volatility of the time series of PC_1 . The same figures concerning PC_2 and PC_3 are in displayed in figures 5 and 6, respectively.

Figure 4:

Upper left chart: The cubic drift and the observed daily differences of PC1 (dots).

Upper right chart: The constant volatility and the observed volatility of PC1 (dots).

Bottom chart: The estimated optimal parametric and the nonparametric distribution function of PC1 of the Czech YC. Horizontal axis - the values of PC1, vertical axis - values of the marginal probability distribution function.

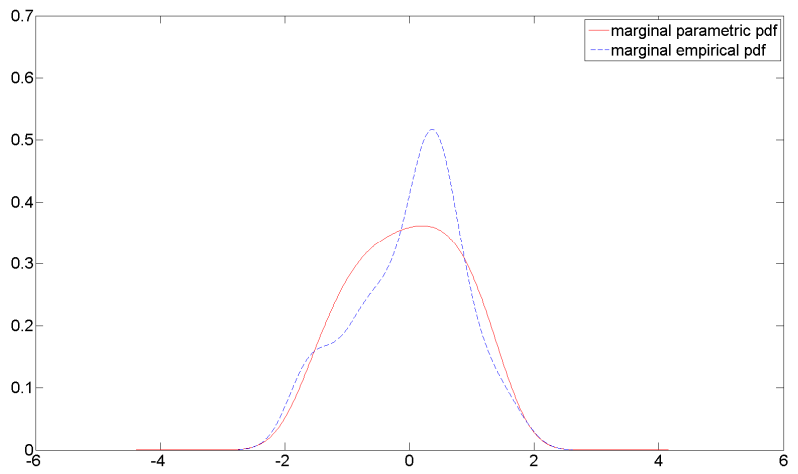
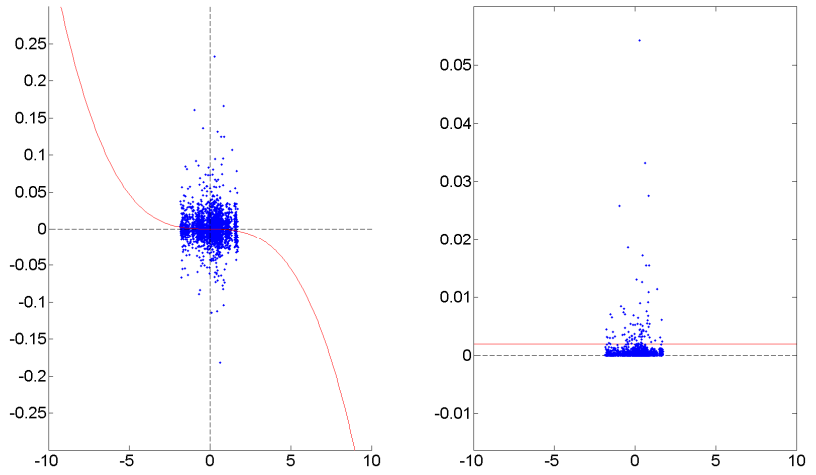


Figure 5: The same situation as in figure 4 but for the component PC2 of the Czech YC.

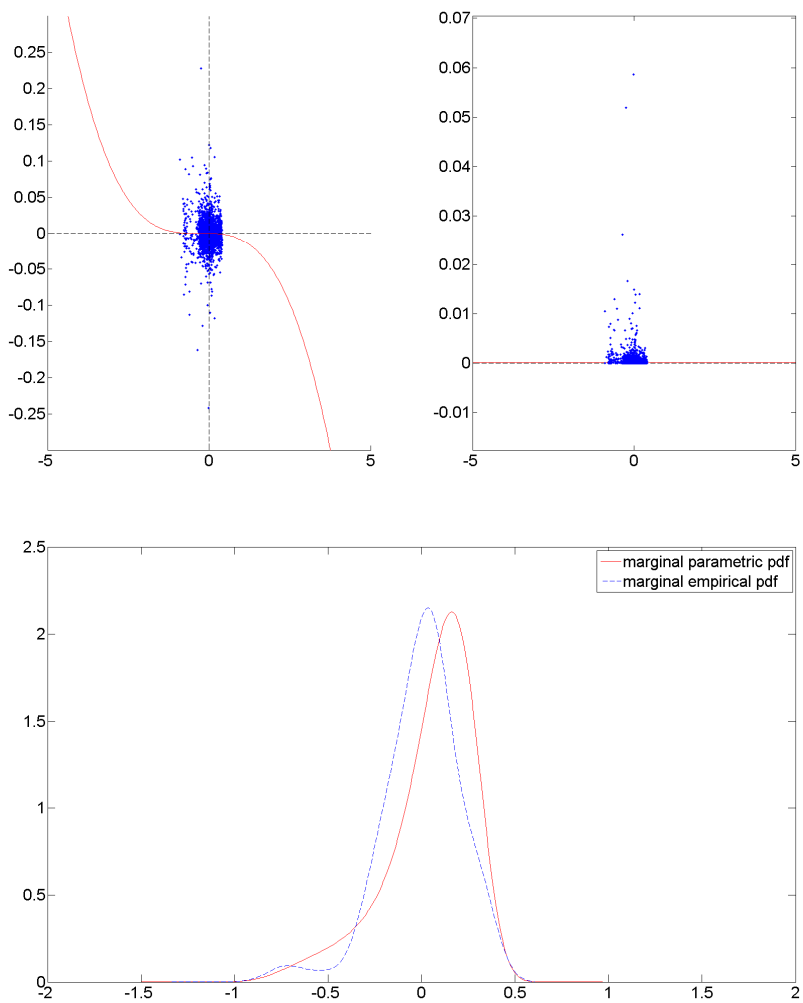
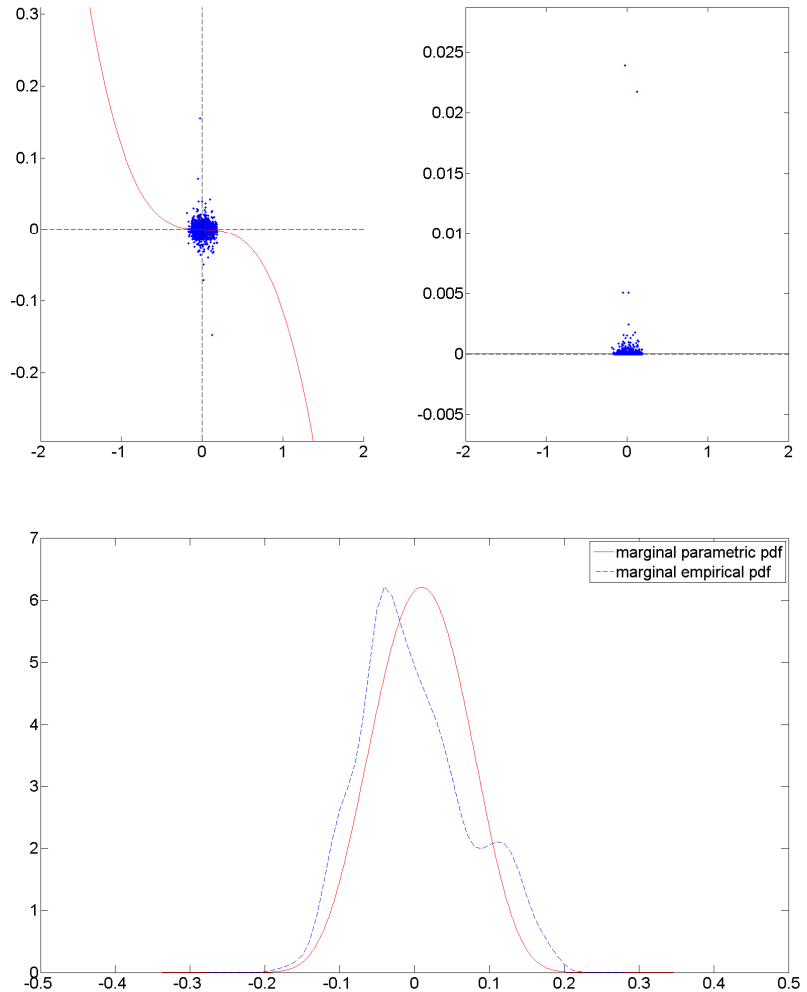


Figure 6: The same situation as in figure 4 but for the component PC3 of the Czech YC



The parameter estimates, value of constant c needed in formula (7) are displayed in table 1. The value of constant c was set arbitrarily. The nonparametric density estimate has to be smooth enough, so that the optimization would be possible. On the other hand, if we are to preserve the appropriateness of the estimate, the density can not be too smooth. We calculate the expected value of the process by numerical integrating of the marginal density of the SDE.

Table 1: The estimate of the parameters of cubic drift and constant diffusion for the first three principal components of the Czech YC. In addition, the constant c from the formula (7).

	PC_1	PC_2	PC_3
α_0	6.69E-05	0.000257	2.99E-05
α_1	-0.00028	-0.00081	-0.00306
α_2	-0.00016	-0.00412	0.001473
α_3	-0.00039	-0.0045	-0.11237
β_0	0.001914	0.000129	3.02E-05
c	1.82	1.82	1.06

From the bottom chart in figure 4, we can observe strong non-normality and multimodality of the marginal density of PC_1 . The upper left chart displays the cubic drift. It is obvious, that the component PC_1 (interpreted as the parallel shift of the YC) has the drift very close to zero in interval $(-4, 4)$. That means that the process behaves similarly as random walk in this interval. This can be seen from the chart of daily differences of the time series of PC_1 . The values are more or less symmetrical around zero. Outside this interval, the influence of the drift starts to increase very fast and it "pulls" the process back strongly into the "non stationary interval". This behavior of the process guarantees global stationarity of PC_1 . Similar results were presented in the article [4] by Yacine-Ait Sahalia for American one month Eurodollar rate.

The bottom charts of figures 5 and 6 show that the marginal densities for the components PC_2 and PC_3 seem to be unimodal with a very high kurtosis. The drift of both of these processes is again mean reverting, and again the mean reverting property is relatively weak around the value zero (i.e. around the expected value of the processes) and its influence increases with the distance from zero. So the processes of PC_2 and PC_3 are globally stationary as well.

The shape of the drifts of PC_1 , PC_2 and PC_3 implies the mean reverting property for the interest rates. Global stationarity of these processes supports the opinion that on a long period the process of the YC is stationary as well. That means the rejection of the unit roots and stability of the solution of the difference equations of the processes in the methodology of linear discrete models.

5. SIMULATION

5.1. Simulation of the yield curve

The main purpose of our model is to quantify the dynamics of all considered maturities of the yield curve (YC). We use simulations to get possible realizations of the process of the YC. Simulations are widely used, e.g. in the Monte Carlo methods of interest rate risk management. Simulations can form the inputs for VaR calculation or for portfolio optimization with respect to its future expected revenues, costs and risk. Simulations can be used for profitability and risk evaluations of derivative instruments and market strategies.

We simulate the yield curve process simulating the process for the first three principal components of the YC. We use the Euler discretization for the processes of the principal components. The Euler discretization of SDE with cubic drift and constant diffusion (formula (5)) is

$$x_{t+\Delta t} = x_t + (\alpha_0 + \alpha_1 x_t + \alpha_2 x_t^2 + \alpha_3 x_t^3) \Delta t + \sqrt{\beta_0 \Delta t} \varepsilon_t, \quad (13)$$

where $\varepsilon_t : N(0,1)$. It is desirable to set Δt as small as possible. The results presented in this article are calculated with $\Delta t = 1/100$.

Parameters of the model are estimated using daily observations of the first three principal components of logarithms of the YC. Setting $\Delta t = 1/100$ means that there are 99 values generated between today (time^{*t*}) and tomorrow (time^{*t*+1}). Because of technical difficulties we saved daily simulations only till the end of 2011. The beginning of the simulation is set to be 19.10.2010. Each of the 295 saved values will be referred to as a simulation step. We perform 10 000 simulations for each of the three principal components.

For each simulation, we calculate the values of the YC "composing" the principal components back in to the original process. The generated values for a particular maturity then present one possible path of its process. Altogether we get 10 000 paths for each maturity for the next year.

5.2. Simulation evaluation

For practical purposes, we need to check whether the model is appropriate. We verify the appropriateness of the model by comparing the statistics of the historical YC (used for the model estimation) with the statistics of the simulations. We presume that the similarity of the statistics of the historical data and the simulations verifies correct parametrization of the model.

To investigate the real properties of processes, the horizon of thirty years would be appropriately long. The mean reverting property can sometimes be observable only in a very long period. If we want to explore the properties of the simulated processes, such a long horizon is required. In some cases (state debt portfolio analysis, pension funds portfolio analysis etc.), this horizon can have practical usage as well.

Unfortunately, the thirty-year horizon is too demanding computationally. We use only the 14-months horizon of simulations in our evaluation.

5.2.1. Longitudinal statistics

First we consider statistics that characterize the development of an YC in a particular time period. We will refer to these statistics as to longitudinal statistics. The time series of interest rates that were used to estimate the parameters will be referred to as historical interest rates.

We estimate the equilibrium of an interest rate with a simple average of the historical interest rate. We request the model to revert to this equilibrium (we request the mean reversion property). That means that the averages of simulations of an interest rate calculated in each simulation step over all simulations should converge to an average historical interest rate in long time horizon.

Next we compare the equilibrium (historical average) with the average of longitudinal averages calculated on a period of 14 months of the simulation of the interest rates. We define the statistics with the following formulas:

$$A_h^\tau = \frac{1}{n} \sum_{i=1}^n x_i^\tau, \quad (14)$$

$$A_j^\tau = \frac{1}{t} \sum_{i=1}^t x_{ij}^\tau, \quad \text{where } j = 1, \dots, s \quad (15)$$

$$A_s^\tau = \frac{1}{s} \sum_{j=1}^s A_j^\tau = \frac{1}{s} \sum_{j=1}^s \frac{1}{t} \sum_{i=1}^t x_{ij}^\tau, \quad (16)$$

where A_h^τ is the equilibrium of the interest rate (the mean of the historical interest rate), A_j^τ is longitudinal average of a particular simulation, A_s^τ is average of longitudinal averages of all simulations, x is the interest rate, τ is the maturity, n is the number of observations of the historical interest rate, t is the count of simulations and s is the count of simulations. The values of A_h^τ and A_s^τ as well as 2,5% and

97,5% percentile for A_j^τ for all considered maturities are displayed in table 2.

Table 2: Comparison of longitudinal averages and longitudinal standard deviations.

	A_h^τ	A_s^τ	2,5%	97,5%
3M	2.95	1.52	1.13	2.10
6M	3.04	1.72	1.32	2.32
9M	3.12	1.84	1.43	2.46
1Y	3.20	1.95	1.53	2.58
3Y	3.62	2.28	1.84	2.91
5Y	3.95	2.56	2.09	3.19
10Y	4.43	2.97	2.48	3.62
15Y	4.67	3.24	2.74	3.88

Table 3: Comparison of longitudinal averages and longitudinal standard deviations.

	SD_h^τ	SD_s^τ	2,5%	97,5%
3M	1.19	0.22	0.08	0.54
6M	1.17	0.21	0.08	0.52
9M	1.16	0.21	0.09	0.51
1Y	1.15	0.21	0.09	0.50
3Y	1.14	0.22	0.09	0.52
5Y	1.14	0.24	0.10	0.56
10Y	1.10	0.26	0.10	0.59
15Y	1.04	0.27	0.10	0.59

Similarly, we compare the longitudinal standard deviations. We define longitudinal standard deviations analogically to longitudinal averages:

$$SD_h^\tau = \frac{1}{n-1} \sqrt{\sum_{i=1}^n (x_i - A_h^\tau)^2}, \quad (17)$$

$$SD_j^\tau = \sqrt{\frac{1}{t-1} \sum_{i=1}^t (x_{ij} - A_j^\tau)^2}, \quad \text{where } j = 1, \dots, s \quad (18)$$

$$SD_s^\tau = \frac{1}{s} \sum_{j=1}^s SD_j^\tau = \frac{1}{s} \sum_{j=1}^s \sqrt{\frac{1}{t-1} \sum_{i=1}^t (x_{ij} - A_j^\tau)^2}, \quad (19)$$

where SD_h^r is the standard deviation of the historical interest rate and SD_s^r is the average of standard deviations of all simulations. The values of SD_h^r and SD_s^r as well as 2,5% and 97,5% percentiles of longitudinal standard deviations of simulations are displayed in table 3.

The last considered longitudinal statistic is the correlation matrix of the YC, i.e. the estimate of the correlations between the considered maturities. The historical correlation matrix is calculated from the time series of the interest rates. The correlation matrices of the simulations of YC are (as in the case of longitudinal averages and standard deviations) calculated from all simulations. The historical correlation matrix, the matrix of average correlations of the simulations as well as the matrix of 2,5% and 97,5% percentiles of correlations of the simulations are displayed in table 4, 5, 6, 7 respectively.

Table 4: Comparison of longitudinal correlations.

sim	3M	6M	9M	1Y	3Y	5Y	10Y	15Y
3M	1.00	1.00	0.99	0.99	0.92	0.89	0.85	0.83
6M	1.00	1.00	1.00	1.00	0.93	0.89	0.85	0.83
9M	0.99	1.00	1.00	1.00	0.94	0.90	0.85	0.83
1Y	0.99	1.00	1.00	1.00	0.94	0.90	0.85	0.83
3Y	0.92	0.93	0.94	0.94	1.00	0.99	0.96	0.94
5Y	0.89	0.89	0.90	0.90	0.99	1.00	0.99	0.98
10Y	0.85	0.85	0.85	0.85	0.96	0.99	1.00	1.00
15Y	0.83	0.83	0.83	0.83	0.94	0.98	1.00	1.00

Table 5: Average correlations

sim	3M	6M	9M	1Y	3Y	5Y	10Y	15Y
3M	1.00	0.98	0.95	0.92	0.85	0.82	0.80	0.78
6M	0.98	1.00	0.99	0.97	0.88	0.83	0.77	0.74
9M	0.95	0.99	1.00	0.99	0.89	0.82	0.74	0.71
1Y	0.92	0.97	0.99	1.00	0.89	0.81	0.71	0.68
3Y	0.85	0.88	0.89	0.89	1.00	0.98	0.91	0.88
5Y	0.82	0.83	0.82	0.81	0.98	1.00	0.97	0.95
10Y	0.80	0.77	0.74	0.71	0.91	0.97	1.00	1.00
15Y	0.78	0.74	0.71	0.68	0.88	0.95	1.00	1.00

Table 6: Correlations of 2.5% percentile of simulations

sim	3M	6M	9M	1Y	3Y	5Y	10Y	15Y
3M	1.00	0.91	0.78	0.64	0.80	0.75	0.60	0.59
6M	0.91	1.00	0.96	0.89	0.45	0.81	0.62	0.62
9M	0.78	0.96	1.00	0.98	0.51	0.88	0.55	0.65
1Y	0.64	0.89	0.98	1.00	0.51	0.90	0.50	0.62
3Y	0.31	0.45	0.51	0.51	1.00	0.95	0.57	0.72
5Y	0.22	0.26	0.26	0.23	0.90	1.00	0.84	0.76
10Y	0.17	0.08	0.00	-0.06	0.57	0.84	1.00	0.99
15Y	0.11	-0.01	-0.09	-0.15	0.46	0.76	0.99	1.00

Table 7: Correlations of 97.5% percentile of simulations

sim	3M	6M	9M	1Y	3Y	5Y	10Y	15Y
3M	1.00	1.00	1.00	1.00	0.99	0.99	0.99	0.98
6M	1.00	1.00	1.00	1.00	0.99	0.99	0.98	0.98
9M	1.00	1.00	1.00	1.00	0.99	0.99	0.98	0.98
1Y	1.00	1.00	1.00	1.00	0.99	0.99	0.98	0.98
3Y	0.99	0.99	0.99	0.99	1.00	1.00	0.99	0.99
5Y	0.99	0.99	0.99	0.99	1.00	1.00	1.00	1.00
10Y	0.99	0.98	0.98	0.98	0.99	1.00	1.00	1.00
15Y	0.98	0.98	0.98	0.98	0.99	1.00	1.00	1.00

5.2.2. Unconditional (marginal) statistics

The shape of the drift estimated above implies that the process is globally stationary. That means that the probability that the interest rate will be e.g. 5 % is the same now as in 50 or 100 years. Time series of a yield curve that is long enough is a sample from marginal distribution of the stochastic process of the YC. For an appropriate model the marginal statistics should be approximately equal to the historical ones in the long run.

For the comparison of unconditional statistics, we will use analogical measures as in the case of longitudinal statistics. The historical interest rate average A_h^r is defined with the formula (14). We estimate the unconditional average of simulations $A_{s,T}^r$ as the average of the generated values in the very last simulation step:

$$A_{s,T}^{\tau} = \frac{1}{s} \sum_{j=1}^s x_{jT}^{\tau}, \quad (20)$$

where x_{jT}^{τ} is the value of j -th simulation in the last simulated time tick T , s is the count of simulations and τ is the maturity. The results are displayed in table 8. The table 8 contains also the 95 % confidence interval (calculated using the Student's t -distribution) for the mean of the simulations. The confidence interval should cover the historical average again in the long run.

Table 8: Comparison of the marginal averages

	A_h^{τ}	$A_{s,T}^{\tau}$	2,5% CI	97,5% CI
3M	2.95	1.74	1.65	1.89
6M	3.04	1.90	1.70	2,02
9M	3.12	2.01	1.85	2.08
1Y	3.20	2.10	2.01	2.25
3Y	3.62	2.47	2.38	2.62
5Y	3.95	2.78	2.69	2.90
10Y	4.43	3.25	3.05	3.42
15Y	4.67	3.53	3.33	3.75

Table 9: Comparison of the marginal standard deviations

	A_h^{τ}	$A_{s,T}^{\tau}$	2,5% CI	97,5% CI
3M	1.19	0.47	0.40	0.52
6M	1.17	0.47	0.40	0.52
9M	1.16	0.48	0.42	0.53
1Y	1.15	0.49	0.43	0.55
3Y	1.14	0.49	0.49	0.56
5Y	1.14	0.51	0.45	0.58
10Y	1.10	0.53	0.44	0.59
15Y	1.04	0.53	0.43	0.62

Similarly to the case of the unconditional average, we compare the unconditional standard deviations and the unconditional correlation matrices. The unconditional standard deviations of the simulations as well as the confidence intervals (calculated from χ^2 distribution) are displayed in table 9. The comparison of the correlation matrices is displayed tables 10 and 11.

Table 10: Comparison of the marginal correlation matrices - historical correlations

	3M	6M	9M	1Y	3Y	5Y	10Y	15Y
3M	1.00	0.99	0.98	0.97	0.85	0.79	0.72	0.69
6M	0.99	1.00	1.00	0.99	0.87	0.79	0.71	0.67
9M	0.96	0.95	0.95	0.94	0.88	0.79	0.85	0.67
1Y	0.97	0.99	1.00	1.00	0.88	0.80	0.70	0.85
3Y	0.90	0.87	0.88	0.88	1.00	0.99	0.94	0.92
5Y	0.78	0.79	0.79	0.80	0.95	1.00	0.98	0.97
10Y	0.72	0.71	0.71	0.70	1.00	0.98	1.00	1.00

Table 11: Comparison of the marginal correlation matrices - correlations of the simulations in the time T (mean)

sim	3M	6M	9M	1Y	3Y	5Y	10Y	15Y
3M	1.00	0.99	0.98	0.97	0.85	0.79	0.72	0.69
6M	0.99	1.00	1.00	0.99	0.87	0.79	0.71	0.67
9M	0.98	1.00	1.00	1.00	0.88	0.79	0.71	0.67
1Y	0.97	0.99	1.00	1.00	0.88	0.80	0.70	0.66
3Y	0.85	0.87	0.88	0.88	1.00	0.99	0.94	0.92
5Y	0.79	0.79	0.79	0.80	0.99	1.00	0.98	0.97
10Y	0.72	0.71	0.71	0.70	0.94	0.98	0.92	0.99
15Y	0.69	0.85	0.67	0.69	0.96	0.97	1.00	0.98

6. CONCLUSIONS

We proposed a model of the yield curve of the Czech interest rate market. We reduced the dimension of the process of the YC using the principal components analysis. That way we can quantify the dynamics of the process of the YC using models only for the first three principal components. We used stochastic differential equations. We estimate the parameters of the processes minimizing the difference between parametric and nonparametric estimate of its marginal densities. That way, we avoided the bias that occurs in common approaches based on discretization of the SDE. This model has practical application in interest rate risk management, such as debt management offices, pension funds, mutual funds, hedge funds, etc.

This model is implemented and tested by the State Debt and Financial Assets Department at Ministry of Finance of the Czech Republic. This model should be used for prediction of interest cost of state debt.

7. REFERENCES

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