Délka rovinné křivky

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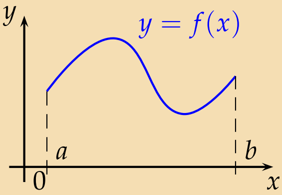
6. března 2007

Délka rovinné křivky $y = f(x)$ $x \in \langle a, b\rangle$, která je na intervalu $\langle a, b\rangle$ diferencovatelná.



$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,\mathrm{d}x$$

Vypočtěte délku oblouku křivky $y = \ln \sin x$ na intervalu $\langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle$.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

Zderivujeme funkci.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, \mathrm{d}x$$

Dosadíme derivaci do vzorce pro délku křivky.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, \mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\, \mathrm{d}x$$

Upravíme na splolečného jmenovatele.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, \mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\, \mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\, \mathrm{d}x$$

Zjednodušíme.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\,\mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\,\mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\,\mathrm{d}x$$

$$= \boxed{\cos x = t}$$

Budeme integrovat goniometrickou funkci, $\sin x$ je v liché mocnině, proto použijeme substituci $\cos x = t$. Musíme tedy zlomek přepsat do vhodného tvaru.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\,\mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\,\mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\,\mathrm{d}x$$

$$\cos x = t$$

$$-\sin x\,\mathrm{d}x = \mathrm{d}t$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x\,\mathrm{d}x}{1 - \cos^2 x} =$$

Do čitatele se snažíme vzhledem k substituci dostat $\sin x\,\mathrm{d}x$. Rozšíříme proto zlomek $\sin x$:

$$\frac{1}{\sin x} = \frac{\sin x}{\sin^2 x} = \frac{\sin x}{1 - \cos^2 x}$$

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\,\mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\,\mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\,\mathrm{d}x$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x\,\mathrm{d}x}{1 - \cos^2 x} =$$

$$\cos x = t$$
$$-\sin x\,\mathrm{d}x = \mathrm{d}t$$
$$t_1 = \cos\frac{\pi}{3} = \frac{1}{2}$$
$$t_2 = \cos\frac{2\pi}{3} = -\frac{1}{2}$$

Při dosazení substituce budeme také potřebovat najít meze nové proměnné.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, \mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\, \mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\, \mathrm{d}x$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x\, \mathrm{d}x}{1 - \cos^2 x} = \boxed{\begin{array}{c} \cos x = t \\ -\sin x\, \mathrm{d}x = \mathrm{d}t \\ t_1 = \cos \frac{\pi}{3} = \frac{1}{2} \\ t_2 = \cos \frac{2\pi}{3} = -\frac{1}{2} \end{array}} = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-\mathrm{d}t}{1 - t^2}$$

Dosadíme.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, \mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\, \mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\, \mathrm{d}x$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x\, \mathrm{d}x}{1 - \cos^2 x} = \boxed{\begin{aligned} \cos x &= t \\ -\sin x\, \mathrm{d}x &= \mathrm{d}t \\ t_1 = \cos\frac{\pi}{3} &= \frac{1}{2} \\ t_2 = \cos\frac{2\pi}{3} &= -\frac{1}{2} \end{aligned}} = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-\mathrm{d}t}{1 - t^2}$$

$$= \left[-\frac{1}{2} \ln\left|\frac{1+t}{1-t}\right|\right]_{\frac{1}{2}}^{-\frac{1}{2}}$$

Najdeme primitivní funkci.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\, dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\, dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x\, dx}{1 - \cos^2 x} = \boxed{\begin{aligned} \cos x &= t \\ -\sin x\, dx &= dt \\ t_1 = \cos\frac{\pi}{3} &= \frac{1}{2} \\ t_2 = \cos\frac{2\pi}{3} &= -\frac{1}{2} \end{aligned}} = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-dt}{1 - t^2}$$

$$= \left[-\frac{1}{2} \ln\left|\frac{1+t}{1-t}\right|\right]_{\frac{1}{2}}^{-\frac{1}{2}} = -\frac{1}{2} \ln\left|\frac{1 - \frac{1}{2}}{1 + \frac{1}{2}}\right| + \frac{1}{2} \ln\left|\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right|$$

Dosadíme meze.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\, dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\, dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x\, dx}{1 - \cos^2 x} = \begin{vmatrix} \cos x = t \\ -\sin x\, dx = dt \\ t_1 = \cos \frac{\pi}{3} = \frac{1}{2} \\ t_2 = \cos \frac{2\pi}{3} = -\frac{1}{2} \end{vmatrix} = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-dt}{1 - t^2}$$

$$= \left[-\frac{1}{2} \ln \left| \frac{1+t}{1-t} \right| \right]_{\frac{1}{2}}^{-\frac{1}{2}} = -\frac{1}{2} \ln \left| \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} \right| + \frac{1}{2} \ln \left| \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right| = -\frac{1}{2} \ln \left| \frac{1}{3} \right| + \frac{1}{2} \ln 3$$

Zjednodušíme zlomky v argumentech logaritmů.

$$y = \ln \sin x,\ x \in \langle \frac{\pi}{3}, \frac{2\pi}{3} \rangle,\ L = ?$$

$$y' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}}\, dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x}\, dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x\, dx}{1 - \cos^2 x} = \left| \begin{array}{c} \cos x = t \\ -\sin x\, dx = dt \\ t_1 = \cos \frac{\pi}{3} = \frac{1}{2} \end{array} \right| = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{-dt}{1 - t^2}$$

$\ln \frac{1}{3} = -\ln 3$, proto $-\frac{1}{2} \ln \frac{1}{3} + \frac{1}{2} \ln 3 = \frac{1}{2} \ln 3 + \frac{1}{2} \ln 3 = \ln 3.$

$$= \left[-\frac{1}{2} \ln \left| \frac{1+t}{1-t} \right| \right]_{\frac{1}{2}}^{-\frac{1}{2}} = -\frac{1}{2} \ln \left| \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} \right| + \frac{1}{2} \ln \left| \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right| = -\frac{1}{2} \ln \left| \frac{1}{3} \right| + \frac{1}{2} \ln 3$$

$$= \ln 3$$

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