Integrace iracionálních funkcí

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Obsah

Najděte $\int \frac{\sqrt{2x+1}}{x} \, \mathrm{d}x.$

Najděte $\displaystyle\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$.

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$$

Funkce obsahuje odmocninu z lineárního členu,

Najděte $\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$.

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x =$$

$$\boxed{\sqrt{2x+1} = t}$$

proto zavedeme substituci $t = \sqrt{2x+1}$.

Najděte $\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$.

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{aligned} \sqrt{2x+1} &= t \\ 2x+1 &= t^2 \end{aligned}}$$

Umocníme.

Najděte $\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$.

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{aligned} \sqrt{2x+1} &= t \\ 2x+1 &= t^2 \\ x &= \frac{t^2-1}{2} \end{aligned}}$$

Vyjádříme inverzní substituci.

Najděte $\displaystyle\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$.

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{gathered} \sqrt{2x+1} = t \\ 2x+1 = t^2 \\ x = \frac{t^2-1}{2} \\ \mathrm{d}x = t\,\mathrm{d}t \end{gathered}}$$

Diferencujeme.

Najděte $\displaystyle\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x.$

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{gathered} \sqrt{2x+1} = t \\ 2x+1 = t^2 \\ x = \frac{t^2-1}{2} \\ \mathrm{d}x = t\,\mathrm{d}t \end{gathered}} = \int \frac{t}{\quad}$$

Dosadíme.

Najděte $\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x.$

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{gathered}\sqrt{2x+1}=t\\ 2x+1=t^2\\ x=\frac{t^2-1}{2}\\ \mathrm{d}x=t\,\mathrm{d}t\end{gathered}} = \int \frac{t}{\frac{t^2-1}{2}}$$

Dosadíme.

Najděte $\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$.

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{gathered}\sqrt{2x+1}=t\\ 2x+1=t^2\\ x=\frac{t^2-1}{2}\\ \mathrm{d}x=t\,\mathrm{d}t\end{gathered}} = \int \frac{t}{\frac{t^2-1}{2}}\,t\,\mathrm{d}t$$

Dosadíme.

Najděte $\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$.

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{gathered}\sqrt{2x+1}=t\\ 2x+1=t^2\\ x=\frac{t^2-1}{2}\\ \mathrm{d}x=t\,\mathrm{d}t\end{gathered}} = \int \frac{t}{\frac{t^2-1}{2}}t\,\mathrm{d}t = 2\int \frac{t^2}{t^2-1}\,\mathrm{d}t$$

Upravíme složený zlomek na jednoduchý.

Najděte $\displaystyle\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x.$

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{gathered}\sqrt{2x+1} = t\\ 2x+1 = t^2\\ x = \frac{t^2-1}{2}\\ \mathrm{d}x = t\,\mathrm{d}t\end{gathered}} = \int \frac{t}{\frac{t^2-1}{2}}\,t\,\mathrm{d}t = 2\int \frac{t^2}{t^2-1}\,\mathrm{d}t$$

$$= 2\int \frac{t^2-1+1}{t^2-1}\,\mathrm{d}t =$$

Jde o neryze lomenou racionální funkci, proto buď podělíme, nebo upravíme na polynom + ryze lomená funkce. V tomto případě je jednodušší doplnit v čitateli jmenovatel, tj. $-1+1$,

Najděte $\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$.

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{gathered}\sqrt{2x+1}=t\\ 2x+1=t^2\\ x=\frac{t^2-1}{2}\\ \mathrm{d}x=t\,\mathrm{d}t\end{gathered}} = \int \frac{t}{\frac{t^2-1}{2}}t\,\mathrm{d}t = 2\int \frac{t^2}{t^2-1}\,\mathrm{d}t$$

$$= 2\int \frac{t^2-1+1}{t^2-1}\,\mathrm{d}t = 2\int 1\,\mathrm{d}t + 2\int \frac{1}{t^2-1}\,\mathrm{d}t$$

a rozdělit na 2 zlomky.

Najděte $\int \frac{\sqrt{2x+1}}{x}\,dx$.

$$\int \frac{\sqrt{2x+1}}{x}\,dx = \boxed{\begin{aligned} \sqrt{2x+1} &= t \\ 2x+1 &= t^2 \\ x &= \frac{t^2-1}{2} \\ dx &= t\,dt \end{aligned}} = \int \frac{t}{\frac{t^2-1}{2}} t\,dt = 2\int \frac{t^2}{t^2-1}\,dt$$

$$= 2\int \frac{t^2-1+1}{t^2-1}\,dt = 2\int 1\,dt + 2\int \frac{1}{t^2-1}\,dt$$

a rozdělit na 2 zlomky.

$$\boxed{\text{Najděte } \int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x.}$$

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{gathered}\sqrt{2x+1} = t\\ 2x+1 = t^2\\ x = \frac{t^2-1}{2}\\ \mathrm{d}x = t\,\mathrm{d}t\end{gathered}} = \int \frac{t}{\frac{t^2-1}{2}} t\,\mathrm{d}t = 2\int \frac{t^2}{t^2-1}\,\mathrm{d}t$$

$$= 2\int \frac{t^2-1+1}{t^2-1}\,\mathrm{d}t = 2\int 1\,\mathrm{d}t + 2\int \frac{1}{t^2-1}\,\mathrm{d}t$$

$$= 2t - \frac{2}{2}\ln\left|\frac{1+t}{1-t}\right| + c$$

Integrujeme.

Najděte $\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x$.

$$\int \frac{\sqrt{2x+1}}{x}\,\mathrm{d}x = \boxed{\begin{aligned}\sqrt{2x+1} &= t\\ 2x+1 &= t^2\\ x &= \frac{t^2-1}{2}\\ \mathrm{d}x &= t\,\mathrm{d}t\end{aligned}} = \int \frac{t}{\frac{t^2-1}{2}}t\,\mathrm{d}t = 2\int \frac{t^2}{t^2-1}\,\mathrm{d}t$$

$$= 2\int \frac{t^2-1+1}{t^2-1}\,\mathrm{d}t = 2\int 1\,\mathrm{d}t + 2\int \frac{1}{t^2-1}\,\mathrm{d}t$$

$$= 2t - \frac{2}{2}\ln\left|\frac{1+t}{1-t}\right| + c$$

$$= 2\sqrt{2x+1} - \ln\left|\frac{1+\sqrt{2x+1}}{1-\sqrt{2x+1}}\right| + c$$

Upravíme.

Najděte $\displaystyle\int \frac{1}{\sqrt{x} + x^{1/4}}\,\mathrm{d}x.$

Najděte $\int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x$.

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x$$

Funkce obsahuje druhou a čtvrtou odmocninu z x, proto hledáme jejich nejmenší společný násobek, číslo 4.

Najděte $\displaystyle\int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x.$

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\sqrt[4]{x} = t}$$

Zavedeme substituci $t = \sqrt[4]{x}$.

Najděte $\displaystyle\int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x$.

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{array}{c}\sqrt[4]{x}=t\\ x=t^4\end{array}}$$

Umocníme, čímž vyjádříme inverzní substituci.

Najděte $\int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x$.

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{gathered} \sqrt[4]{x}=t \\ x=t^4 \\ \mathrm{d}x = 4t^3\,\mathrm{d}t \end{gathered}}$$

Diferencujeme.

Najděte $\displaystyle\int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x.$

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{array}{c}\sqrt[4]{x}=t\\ x=t^4\\ \mathrm{d}x=4t^3\,\mathrm{d}t\end{array}} = \int \frac{1}{\sqrt{t^4}}$$

Dosadíme.

Najděte $\displaystyle\int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x.$

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{array}{c}\sqrt[4]{x}=t\\ x=t^4\\ \mathrm{d}x=4t^3\,\mathrm{d}t\end{array}} = \int \frac{1}{\sqrt{t^4}+\sqrt[4]{t^4}}$$

Dosadíme.

Najděte $\int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x$.

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{gathered}\sqrt[4]{x}=t\\ x=t^4\\ \mathrm{d}x=4t^3\,\mathrm{d}t\end{gathered}} = \int \frac{1}{\sqrt{t^4}+\sqrt[4]{t^4}}4t^3\,\mathrm{d}t$$

Dosadíme.

Najděte $\displaystyle\int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x.$

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{gathered}\sqrt[4]{x}=t\\ x=t^4\\ \mathrm{d}x=4t^3\,\mathrm{d}t\end{gathered}} = \int \frac{1}{\sqrt{t^4}+\sqrt[4]{t^4}}4t^3\,\mathrm{d}t = 4\int \frac{t^3}{t^2+t}\,\mathrm{d}t$$

Upravíme.

$$\boxed{\text{Najděte } \int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x.}$$

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{array}{c} \sqrt[4]{x}=t \\ x=t^4 \\ \mathrm{d}x = 4t^3\,\mathrm{d}t \end{array}} = \int \frac{1}{\sqrt{t^4}+\sqrt[4]{t^4}} 4t^3\,\mathrm{d}t = 4\int \frac{t^3}{t^2+t}\,\mathrm{d}t$$

$$= 4\int \frac{t^2}{t+1}\,\mathrm{d}t$$

Dostáváme neryze lomenou racionální funkci. Před podělením si můžeme všimnout, že lze krátit t (ve jmenovateli t můžeme vytknout).

$$\boxed{\text{Najděte } \int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x.}$$

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{aligned}\sqrt[4]{x} &= t\\ x &= t^4\\ \mathrm{d}x &= 4t^3\,\mathrm{d}t\end{aligned}} = \int \frac{1}{\sqrt{t^4}+\sqrt[4]{t^4}} 4t^3\,\mathrm{d}t = 4\int \frac{t^3}{t^2+t}\,\mathrm{d}t$$

$$= 4\int \frac{t^2}{t+1}\,\mathrm{d}t = 4\int \left(t-1+\frac{1}{t+1}\right)\mathrm{d}t$$

Dělíme:

$$\begin{array}{l} t^2 : (t+1) = t - 1 + \dfrac{1}{t+1} \\ \underline{-(t^2+t)} \\ \quad -t \\ \quad \underline{-(-t-1)} \\ \qquad 1 \end{array}$$

$$\text{Najděte } \int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x.$$

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{aligned} \sqrt[4]{x} &= t \\ x &= t^4 \\ \mathrm{d}x &= 4t^3\,\mathrm{d}t \end{aligned}} = \int \frac{1}{\sqrt{t^4}+\sqrt[4]{t^4}} 4t^3\,\mathrm{d}t = 4\int \frac{t^3}{t^2+t}\,\mathrm{d}t$$

$$= 4\int \frac{t^2}{t+1}\,\mathrm{d}t = 4\int \left(t-1+\frac{1}{t+1}\right)\mathrm{d}t = 4\int t\,\mathrm{d}t - 4\int \mathrm{d}t + 4\int \frac{1}{t+1}\,\mathrm{d}t$$

Sčítance integrujeme každý zvlášť.

$$\boxed{\text{Najděte } \int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x.}$$

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{aligned} \sqrt[4]{x} &= t \\ x &= t^4 \\ \mathrm{d}x &= 4t^3\,\mathrm{d}t \end{aligned}} = \int \frac{1}{\sqrt{t^4}+\sqrt[4]{t^4}} 4t^3\,\mathrm{d}t = 4\int \frac{t^3}{t^2+t}\,\mathrm{d}t$$

$$= 4\int \frac{t^2}{t+1}\,\mathrm{d}t = 4\int \left(t-1+\frac{1}{t+1}\right)\mathrm{d}t = 4\int t\,\mathrm{d}t - 4\int \mathrm{d}t + 4\int \frac{1}{t+1}\,\mathrm{d}t$$

$$= 4\frac{t^2}{2} - 4t + 4\ln|t+1| + c$$

Integrujeme podle vzorců.

Najděte $\displaystyle\int \frac{1}{\sqrt{x}+x^{1/4}}\,\mathrm{d}x.$

$$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,\mathrm{d}x = \boxed{\begin{array}{c}\sqrt[4]{x}=t\\ x=t^4\\ \mathrm{d}x=4t^3\,\mathrm{d}t\end{array}} = \int \frac{1}{\sqrt{t^4}+\sqrt[4]{t^4}}4t^3\,\mathrm{d}t = 4\int \frac{t^3}{t^2+t}\,\mathrm{d}t$$

$$= 4\int \frac{t^2}{t+1}\,\mathrm{d}t = 4\int\left(t-1+\frac{1}{t+1}\right)\mathrm{d}t = 4\int t\,\mathrm{d}t - 4\int \mathrm{d}t + 4\int \frac{1}{t+1}\,\mathrm{d}t$$

$$= 4\frac{t^2}{2} - 4t + 4\ln|t+1| + c = 2\sqrt{x} - 4\sqrt[4]{x} + \ln(\sqrt[4]{x}+1)^4 + c$$

Dosadíme původní proměnnou.

Konec