Categorical structures for higher-dimensional universal algebra

Habilitation Thesis

Masaryk University

John Bourke

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Contents

Chapter 1

Introduction

1.1 Overview

This thesis gives a snapshot of five different areas in which I have worked to date, including one paper from each area. In this introduction, we will walk through the different topics chronologically, and I will tell the idiosyncratic stories of how these papers came about (which rarely make it to the final version but often form an important part of the research process). At the same time, I will emphasise the many connections between the different areas and point out some of the open questions and future directions to be explored.

The second to fifth of these topics involve categorical structures and settings that arose in my research on higher dimensional structures: in turn, algebraic weak factorisation systems, skew monoidal categories, nervous monads and homotopically enriched category theory. We will see how concrete examples and questions led us to develop the theory of these categorical structures, often leading in new directions beyond the original goals. To begin with, we take a look at a topic that leads naturally to each of the later ones — namely, *two-dimensional universal algebra*.

1.2 Two-dimensional universal algebra

Classical universal algebra describes algebraic structures borne by sets — for instance, monoids, groups, rings, vector spaces and (most of) the usual structures of algebra. To capture this in general, one has a pair (Ω, E) consisting of a signature Ω of operations and a set E of equations — then one can consider (Ω, E)-algebras and their homomorphisms. These form a category (Ω, E)-Alg equipped with a forgetful functor $U : (\Omega, E)\text{-Alg} \to \mathrm{Set}$ to the category of sets.

There are several categorical approaches to the subject, and perhaps the best known is the approach via monads T on Set. In this context, one can speak of a T-algebra — a set X together with a function $x : TX \to X$ satisfying two equations similar to those for a module over a ring — and their homomorphisms $f : (X, x) \to (Y, y)$ which satisfy the commutativity $y \circ Tf = f \circ x$. These form a category Set^T, the Eilenberg-Moore category of algebras, and as before, we have a forgetful functor $U^T : \mathrm{Set}^T \to \mathrm{Set}$, which has a left adjoint.

The appeal of the monad-theoretic approach is partially due to its generality: monads can be considered on any category, not just Set, and so provide a kind of generalised universal algebra. It is also due to the fact that monads arise from adjunctions, the ubiquitous central concept in category theory. Indeed, given an adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend right, "U"'] & \mathcal{B} \arrow[l, bend right, "F"'] \end{tikzcd}$$

with $F \dashv U$ we obtain a monad $T = UF$ on $\mathcal{B}$ and canonical comparison $K : \mathcal{A} \to \mathcal{B}^T$ to the category of algebras such that the triangle

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "K"] \arrow[dr, "U"'] & & \mathcal{B}^T \arrow[dl, "U^T"] \\ & \mathcal{B} & \end{tikzcd}$$

commutes. The fundamental result here is *Beck's monadicity theorem* [3], which famously gives an easily checkable criterion that characterises those adjunctions for which the canonical functor

$$K : \mathcal{A} \to \mathcal{B}^T$$

is an equivalence. In particular, this is the case for the adjunctions of classical universal algebra so that we obtain an equivalence of categories (Ω, E)-Alg $\cong \mathrm{Set}^T$ over Set.[1] Therefore, anything one can prove about categories of the

[1]In fact, the categories of universal algebra are precisely those of the form Set^T for T a filtered colimit preserving (a.k.a. finitary) monad on Set.

form Set^T holds for any variety of universal algebra — this is the categorical approach to universal algebra via monads.

What about two-dimensional universal algebra? Here the objects of interest are categories with algebraic structure, such as monoidal categories. In this context there are four kinds of morphism that naturally arise: the *strict* monoidal functors $F : \mathcal{A} \to \mathcal{B}$ which preserve the tensor product $\otimes$ and unit i on the nose; the *pseudo* (or strong) monoidal functors which involve coherence isomorphisms $Fa \otimes Fb \cong F(a \otimes b)$ and $i \cong Fi$; the *lax* monoidal functors which involve morphisms $Fa \otimes Fb \to F(a \otimes b)$ and $i \to Fi$ and the *colax* ones, in which the orientation of these morphisms is reversed. With monoidal transformations as 2-cells, these assemble into four distinct 2–categories, which are interconnected and sit over Cat as in the diagram below left.

$$\begin{tikzcd}
 & & \mathrm{MonCat}_l \arrow[dddl, dotted, "V_l"] \\
\mathrm{MonCat}_s \arrow[r] \arrow[ddr, "V_s"'] & \mathrm{MonCat}_p \arrow[ur] \arrow[dr] \arrow[dd, "V_p"'] & \\
 & & \mathrm{MonCat}_c \arrow[dl, "V_c"] \\
 & \mathrm{Cat} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \mathrm{T\text{-}Alg}_{\mathrm{l}} \arrow[dddl, dotted, "U_l"] \\
\mathrm{T\text{-}Alg}_{\mathrm{s}} \arrow[r] \arrow[ddr, "U_s"'] & \mathrm{T\text{-}Alg}_p \arrow[ur] \arrow[dr] \arrow[dd, "U_p"'] & \\
 & & \mathrm{T\text{-}Alg}_{\mathrm{c}} \arrow[dl, "U_c"] \\
 & \mathrm{Cat} &
\end{tikzcd}$$

One of the themes of two-dimensional universal algebra is that the 2-categories of strict morphisms, like MonCat_s, are as easily understood as their 1-dimensional counterparts. But the 2-categories of pseudo, lax and colax morphisms are less well behaved and exhibit much subtler behaviour.

How about the about the abstract approach? The approach favoured by 2-category theorists is via 2-monads — just enriched monads — on the 2-category Cat [9]. There are the usual strict algebras[2] and four kinds of morphism — strict, pseudo, lax and colax — much as above. All told, we obtain a diagram of 2-categories as above, labelled by the kind of morphisms they contain, with algebra transformations between them in each case.

Comparing the diagrams left and right above we see that a monadicity theorem in this setting ought to match each 2-category on the left with the

[2]There are also weaker kinds but these are less central to the subject than one might at first guess.

corresponding one on the right in a compatible way. More precisely, there should exist a 2-monad T on Cat and, for each $w \in \{s, p, l, c\}$, an equivalence of 2-categories $E_w : \mathrm{MonCat}_w \to \text{T-Alg}_w$ over Cat, as left below

$$\begin{tikzcd} \mathrm{MonCat}_w \arrow[rr, "E_w"] \arrow[dr, "V_w"'] & & \text{T-Alg}_w \arrow[dl, "U_w"] \\ & \mathrm{Cat} & \end{tikzcd}$$

and these equivalences should be natural in the various inclusions. Establishing a monadicity theorem of this kind was the content of my paper *Two-dimensional monadicity*, included in this collection and published in Advances in Mathematics [11] in 2014.

How should we approach this problem? The starting point is that the world of strict morphisms is easily understood: by the enriched version of Beck's theorem, there exists a 2-monad T and an equivalence $\mathrm{MonCat}_s \simeq \text{T-Alg}_s$ over Cat. Therefore what needs to be done is to extend from strict morphisms to each of the different flavours. Two clearly related questions are

(1) What are the characterising properties of pseudo, lax and colax morphisms in two-dimensional universal algebra?

(2) Are each of the weaker flavours of morphism uniquely determined by their relationship with the strict ones?

What I will do now is explain the answers to these questions, in particular the second of them, from which the main monadicity theorem — Theorem 25 of [11] — easily follows.

The first question was one which had been in the back of my head shortly after completing my Phd. A lightbulb flashed on reading Boardman and Vogt's desiderata of the properties of *homotopy invariant morphisms* in the introduction to their classic book [10], namely:

- if $f : X \to Y$ is a structured map of structured spaces and g homotopic to f then g can be structured;

- if $f : X \to Y$ is a structured map of structured spaces and a homotopy equivalence, then any homotopy inverse to f can be structured.[3]

Certainly 2-categorical analogues of these homotopy invariance properties hold for pseudomorphisms of algebras, and can be expressed as lifting properties of the forgetful 2-functor $U_p : \text{T-Alg}_p \to \text{Cat}$. For lax morphisms, the first property holds much as before whilst I realised that the appropriate version of the second property above should be related to the following fragment of Kelly's *doctrinal adjunction* [40][4]:

- if $f : (X, x) \to (Y, x)$ is a strict morphism of algebras whose underlying functor f has a right adjoint, then the right adjoint can be given the structure of a lax morphism (in a unique way for which the adjunction lifts to T-Alg_l).

The problem with this property is that one needs to quantify over the strict morphisms — so that it is *not* a property of the forgetful 2-functor $U_l : \text{T-Alg}_l \to \text{Cat}$.

The solution was to pass from 2-categories and 2-functors to the recently introduced $\mathbb{F}$-categories and $\mathbb{F}$-functors of Lack and Shulman [43]. These used enriched category theory to formalise the idea, already appreciated in [9], that to study weak morphisms it is useful to consider them in tandem with strict ones. Now an $\mathbb{F}$-enriched category is a 2-category with two kinds of 1-cell, one a subclass of the other so that, for instance, there is an $\mathbb{F}$-category $\text{T-}\mathbb{A}\text{lg}_l$ of algebras, strict and lax morphisms. As $\mathbb{F}$-categories also include 2-categories in a trivial way, there is moreover a *forgetful* $\mathbb{F}$*-functor*

$$U_l : \text{T-}\mathbb{A}\text{lg}_l \to \text{Cat}$$

and we could now express doctrinal adjunction as a lifting property of U_l.

Ultimately, rather than being about lifting general adjunctions, the necessary fragment of doctrinal adjunction turned out to concern only those adjunctions with identity counit. Such adjunctions, or their left adjoints f, are often called *lalis*.[5] The reason that these are sufficient is that each lax

[3]Boardman and Vogt had a third condition regarding homotopy invariance for objects not relevant here.

[4]Doctrinal adjunction, in its most general form, actually states a relationship between colax and lax morphisms.

[5]The paper actually uses the terminology *l-reflection* rather than *lali*.

morphism $f : (X, x) \rightsquigarrow (Y, y)$ could be represented as a span

$$(X, x) \xleftarrow{p_f} (C_f, c_f) \xrightarrow{q_f} (Y, y) \tag{1.2.1}$$

of strict morphisms, where the left leg $p_f : C_f \to X$ is the left adjoint of a lali in Cat, from which the lax morphism f could be recovered using doctrinal adjunction.

This led to the notion of an l-doctrinal $\mathbb{F}$-functor: one, such as U_l : T-$\mathbb{A}$lg$_l \to$ Cat, which had the doctrinal adjunction property for lalis as well as a couple of other related properties. Considering the evident factorisation

$$\begin{tikzcd} \text{T-Alg}_s \arrow[r, "incl"] \arrow[rr, bend right, "U_s"'] & \text{T-}\mathbb{A}\text{lg}_l \arrow[r, "U_l"] & \text{Cat} \end{tikzcd}$$

of the forgetful 2-functor U_s : T-Alg$_s \to$ Cat, we can now express the key point of the paper: namely, the factorisation is an orthogonal decomposition (in the sense of factorisation systems) where the right class consisted of the l-doctrinal $\mathbb{F}$-functors. To see that the left leg belongs to the left class used the covering span of (1.2.1).

A generalisation of this result, valid not just for 2-monads, appears as Corollary 22 of the paper and the main result on monadicity, Theorem 25, is a straightforward consequence. The monadicity theorem has been used to good effect by Gurski, Johnson and Osorno [34] to understand the lax morphisms of n-fold monoidal categories in their work on spectra.

The following table is intended to express the idea, hinted at for lax morphisms above, that each flavour of weak morphism is parametrised by a particular class of adjunction or equivalence.

Two-dimensional universal algebra	
lax morphisms	lalis
pseudomorphisms	surjective equivalences
colax morphisms	ralis

Table 1.1: Parametrising classes of maps

We will return to this table in the following section on algebraic weak factorisation systems.

1.3 Algebraic weak factorisation systems

Going back to the early days of category theory [48] there have been various notions of factorisation system around. A factorisation system involves two classes of maps $\mathcal{E}$ and $\mathcal{M}$ which are, first of all, orthogonal in the sense that given a solid square as on the outside of

$$\begin{tikzcd} X \arrow[r] \arrow[d, "e\in\mathcal{E}"'] & A \arrow[d, "m\in M"] \\ Y \arrow[r] \arrow[ur, dotted, "\exists!"] & B \end{tikzcd}$$

there exists a *unique* diagonal filler as depicted, and such that each morphism factors as an $\mathcal{E}$ followed by an $\mathcal{M}$. This captures the relationship between, for instance, surjective and injective homomorphisms in algebraic categories. Weak orthogonality, in which one requires only the *existence* but not uniqueness of the diagonal filler, and the associated weak factorisation systems are much more important in homotopy theory — capturing, as they do, the relationship between the cofibrations and trivial fibrations in a model category [54].

The uniqueness condition ensures that factorisations are functorial. However, in the case of weak orthogonality functoriality is not guaranteed. Despite this, Quillen's *small object argument* [54] ensures that most weak factorisation sytems can be equipped with functorial factorisations. Motivated by this gap between theory and practice, Grandis and Tholen introduced *natural weak factorisation systems* in 2006 [31]. Their definition was tweaked and perfected by Garner [30] and these structures are now commonly known as *algebraic weak factorisation systems* (hence awfs'). As is clear with a decade of hindsight, awfs' are the correct categorical structure combining weak orthogonality and functoriality of factorisations.

The definition of an awfs is, by all accounts, rather complicated — it involves a functorial factorisation, with its left and right components respectively extended to comonads and monads L and R on the arrow category, interacting via a distributive law. Given this, one can then consider the coalgebras for the comonad L and algebras for the monad R, which are best thought of as morphisms of $\mathcal{C}$ *equipped with extra structure* witnessing that they belong to the "left and right classes" of the awfs. This point of view was clarified by Garner [30], who modified Quillen's small object argument to show that given a set J of morphisms in a suitable category $\mathcal{C}$, one can *cofibrantly generate* an awfs (L, R) whose R-algebras are maps $f : A \to B$ equipped with a lifting function φ

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "j \in J"'] & A \arrow[d, "f"] \\ Y \arrow[r, "s"'] \arrow[ur, dotted, "\varphi_{l,r,s}" description] & B \end{tikzcd}$$

providing lifts for each lifting problem as above. Of course, once we equip morphisms with a *choice of liftings*, the new possibility emerges of encoding further compatibilities between the liftings — in particular, given a commuting square between morphisms of J, one can ask for the horizontal compatibility

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "j'"'] & X \arrow[r, "r"] \arrow[d, "j"'] & A \arrow[d, "f"] \\ Y' \arrow[r, "v"'] & Y \arrow[r, "s"'] \arrow[ur, dotted, "\varphi_{j,r,s}" description] & B \end{tikzcd} = \begin{tikzcd} X' \arrow[r, "ru"] \arrow[d, "j'"'] & A \arrow[d, "f"] \\ Y' \arrow[r, "sv"'] \arrow[ur, dotted, "\varphi_{j',ru,sv}" description] & B \end{tikzcd} \tag{1.3.1}$$

expressing the equality of the two diagonal fillers $Y' \to B$. This stronger notion was captured by Garner's notion of *cofibrant generation by a category of morphisms.*

Building on these results, Emily Riehl introduced the notion of an algebraic model category [55] — in which the two weak factorisations are made (compatibly) algebraic — and showed that any cofibrantly generated model category could be made algebraic.

For our purposes, a guiding example is the model structure for two-dimensional universal algebra introduced by Steve Lack [42]. Lack showed that for T a suitable 2-monad, the 2-category T-Alg_s of algebras and strict morphisms had a model structure whose trivial fibrations are those morphisms sent by $U : \text{T-Alg}_\text{s} \to \text{Cat}$ to surjective equivalences, henceforth U-surjective equivalences. Lack furthermore showed that the model structure was closely connected to the classical construction of *pseudomorphism classifiers* [9]: the pseudomorphism classifier Q_pA of an algebra A is defined by the property that pseudomorphisms $A \rightsquigarrow B$ correspond to strict morphisms $Q_pA \to B$. In particular, the resulting adjunction induces a comonad Q_p on T-Alg_s and Lack showed that the counit $p_A : Q_pA \to A$ was a cofibrant replacement of A in this model structure.

The correspondence between pseudomorphisms and homotopy theory was further sharpened by Garner [30] by passing from the model structure to the algebraic model structure. To see this, note that each awfs $\mathcal{C}$ has a canonical *cofibrant replacement comonad* Q and so induces a Kleisli category $\mathcal{C}_Q$ whose morphisms are thought of as weak morphisms. In the case of the awfs for U-surjective equivalences on T-Alg_s, Garner showed that the cofibrant replacement comonad coincides with the pseudomorphism classifier so that, in particular, $(\text{T-Alg}_\mathrm{s})_Q = \text{T-Alg}_p$. In other words, the algebraic model structure encodes precisely the correct notion of pseudomorphism in two-dimensional universal algebra.

Bearing in mind Table 1.1 of the previous section, a natural question to ask was whether there is a corresponding story to be told for the *lax morphisms* of T-Alg_l, on switching from surjective equivalences to lalis. To begin with, there is an awfs on Cat whose algebras consist of the lalis — this awfs is, indeed, the dual of one of the early examples from Grandis and Tholen's paper [31]. Thus, the natural goal became to lift this to an awfs on T-Alg_s whose algebras are the U-lalis, and to show that T-Alg_l was the Kleisli 2-category for the associated cofibrant replacement comonad on T-Alg_s.

The problem here is that the usual approach would be to apply the left adjoint $F : \mathrm{Cat} \to \text{T-Alg}_s$ to the generating set or category of cofibrations J and cofibrantly generate the appropriate awfs on T-Alg_s. However the awfs for lalis did not appear to be cofibrantly generated in any sense considered to date. Indeed, whilst all of the awfs arising from classical homotopy theory considered by Riehl are cofibrantly generated by a set, with Garner we realised that this awfs, arising from lax aspects of 2-category theory — namely, adjunctions — was *not* cofibrantly generated even by a small category.

This led us in our 2016 publication [14], included in this collection and published in the JPAA, to extend the notion of cofibrant generation beyond the *horizontal compatibilities* of Diagram (1.3.1) to encode the *vertical compability* that equates the two ways of lifting against a composable pair in J, as depicted below.

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "j"'] & A \arrow[dd, "f"] \\
Y \arrow[d, "k"'] \arrow[ur, dotted, "\varphi(j,u,vk)"] & \\
Z \arrow[r, "v"'] \arrow[uur, dotted, "\varphi(k,\varphi(j,u,vk),v)"] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "j"'] & A \arrow[dd, "f"] \\
Y \arrow[d, "k"'] & \\
Z \arrow[r, "v"'] \arrow[uur, dotted, "\varphi(kj,u,v)"] & B
\end{tikzcd}
$$

This stronger notion can be understood as *cofibrant generation by a double* *category* and, in particular, captures the awfs for lalis. It is also needed to capture other important types of functor involving *universal properties*, such as Grothendieck fibrations. One of the main results — Theorem 25 of [14] — is that each small double category of morphisms on a locally presentable category generates an awfs and each reasonable (accessible) awfs arises in this way — thus, this notion of cofibrant generation is, in a sense, the most general possible.

The second main result — Theorem 6 of [14] — was a *recognition theorem* for awfs comparable to Beck's theorem for monads. The recognition theorem characterises awfs in terms of their double categories of algebras. Just as Beck's theorem enables us to recognise and work with monads via their categories of algebras, so the recognition theorem enables us to recognise awfs far more easily in the wild. Because it shifts attention away from the complex syntactic definition of an awfs to its semantics, it enabled us to give simpler proofs of many old results about awfs — such as the existence of cofibrantly generated awfs — as well as a number of important new ones, such as the existence of projective and injective liftings of awfs along right and left adjoints. These results have been applied to good effect by Hess et al [36] to construct accessible model structures on categories of a coalgebraic flavour.

They were also applied to construct the accessible awfs on T-Alg_s for U-lalis in the followup [15], establishing T-Alg_l as its 2-category of weak maps for the associated cofibrant replacement comonad, as was the original motivation. The same paper also described how weak maps for an awfs (L, R) could be viewed as certain equivalence classes of spans, thus capturing the description of lax morphisms as spans of strict ones in (1.2.1). Similar results about weak maps were given for dg-categories.

The results of these papers have also been influential in the homotopy type theory community, for instance in the work of Gambino and coauthors [29] and most recently in the notion of effective Kan fibration of van-den-Berg and Faber [5], which uses the stronger notion of cofibrant generation by a small double category.

There are several open questions in this area, particularly about *algebraic model categories*. There are various reasons to expect that one might want an *algebraic notion of weak equivalence* to complement the *algebraic fibrations* and *algebraic trivial fibrations* of an algebraic model category. I explored this question of algebraic weak equivalences in *Equipping weak equivalences with algebraic structure* [19], published in Mathematische Zeitschrift. There are surprising results here — for instance, there is a monad such that detects weak homotopy equivalences of topological spaces. Algebraic structure on weak equivalences also naturally arise in homotopy type theory, where each property should be witnessed by structure, and has led to a stronger notion of algebraic model category [61], though it seems to me that stronger definitions may yet emerge. An open question is to understand the analogue of Smith's theorem for accessible model categories, which I anticipate being related to the above questions about algebraic model categories.

1.4 Skew monoidal categories

One of the cornerstones of modern category theory is the framework of *enriched category theory*. Here the hom objects

$$\mathcal{C}(A, B)$$

of the categories under consideration are not merely sets, but objects of some *monoidal category* $\mathcal{V}$ — for instance, they may be abelian groups, chain complexes, categories or other sorts of structured objects. In his foundational book [41] on the topic, Kelly made the assumption that $\mathcal{V}$ is a *symmetric monoidal closed category* — this generality includes many examples, such as the above ones, and furthermore has the desirable property that the theory of $\mathcal{V}$-enriched categories works much as in the classical case of $\mathcal{V} = \mathrm{Set}$.

That said, this assumption on $\mathcal{V}$ is rather strong. Indeed Foltz, Kelly and Lair [27] showed that algebraic categories often admit few if any monoidal biclosed (and, in particular, symmetric monoidal closed) structures. For instance, Cat admits exactly two — the cartesian and "funny" tensor product, whilst the category of monoids, rings and groups admit none. In a similar direction, with Nick Gurski [12] we showed that the category of *Gray*-categories, the well known model of semi-strict 3-categories [33], admits no monoidal biclosed structure that models weak higher dimensional transformations or which is homotopically well behaved — this rules out the possibility of using monoidal biclosed or symmetric monoidal closed categories as a base of enrichment for a good notion of semistrict 4-category. A long term interest of mine has been to understand what sort of monoidal structure the category of *Gray*-categories does support.

In 2012, motivated by structures such as bialgebroids, Szlachanyi introduced the notion of a skew monoidal category [60]. These involve a tensor product $\otimes$ and I together with coherence constraints of the form

$$(A \otimes B) \otimes C \xrightarrow{\alpha} A \otimes (B \otimes C) \qquad I \otimes A \xrightarrow{l} A \qquad A \xrightarrow{r} A \otimes I$$

satisfying five axioms. When the three constraints are invertible, one obtains the usual monoidal categories. The orientations of the arrows are certainly obscure, but interesting consequences ensue: for instance, tensoring on one side is a monad, on the other a comonad, and there is a distributive law between the two. (In fact, this hints at a curious analogy with algebraic weak factorisation systems.)

The concept came to my attention with the publication of Street's 2013 paper [59], which describes the *closed* correlate of skew monoidal categories: here the basic data are a hom-functor $[-,-] : \mathcal{V}^{op} \times \mathcal{V} \to \mathcal{V}$ and unit object I, together with the relatively strange looking coherence constraints

$$[A,B] \xrightarrow{L} [[C,A],[C,B]] \qquad [I,A] \xrightarrow{i} A \qquad I \xrightarrow{j} [A,A]$$

satisfying 5 axioms. A closed category in the sense of Eilenberg and Kelly is one of these for which

$$i : [I,A] \to A$$

and the induced function

$$\mathcal{V}(A,B) \to \mathcal{V}(I,[A,B]) : f \mapsto [A,f] \circ j_A \tag{1.4.1}$$

is invertible. In the original treatise [26] of Eilenberg and Kelly on enriched category theory, closed categories rather than monoidal ones were taken as the base of enrichment — perhaps because, in practise, the structure of the internal hom is often more easily understood the corresponding tensor product. Against this, the computations, involving iterated contravariance, are harder to parse.

In the presence of a unit I, tensor product $\otimes$ and hom $[-,-]$ related by a natural isomorphism

$$\mathcal{V}(A \otimes B, C) \cong \mathcal{V}(A, [B,C])$$

Street showed that there is a perfect correspondence between skew monoidal structures on $(\mathcal{V}, \otimes, I)$ and skew closed ones on $(\mathcal{V}, [-,-], I)$ — the resulting structures are called skew monoidal closed categories.

Now, whilst the invertibility of the two unit morphisms l and r in a monoidal category correspond to the invertibility of $i : [I, A] \to A$ and (1.4.1) in a closed category, invertibility of the associator $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$ does not appear to correspond to the invertibility of any morphism on the closed side (at least one that is expressible purely in terms of the closed structure). Therefore, in some sense, the correspondence in the skew setting repairs an asymmetry in the classical setting.

With the above in mind, my own interest in the skew story arose on considering the value gained by allowing the function

$$\mathcal{V}(A, B) \to \mathcal{V}(I, [A, B])$$

of (1.4.1) to be non-invertible, or equivalently the morphism $l : I \otimes A \to A$. This function allows us to view morphisms $f : A \to B$ as "elements" of the hom $[A, B]$. In the classical case of a closed category, its invertibility says that elements of $[A, B]$ are precisely the morphisms $A \to B$ but, in the case it is not invertible, it allows the possibility that $[A, B]$ could contain more than simply the morphisms from A to B — for instance, it might contain *weak maps* from A to B, such as the pseudo or lax maps considered in previous sections.

This perspective is developed in my paper *Skew structures in 2-category theory and homotopy theory* [13] published in the Journal of Homotopy and Related Structures, and included in this series. In this paper, I showed that in 2-category theory there are many natural examples of skew monoidal closed categories that arise in this way, and used them to solve an old problem about the construction of monoidal bicategories, which I will illustrate with an example.

A *permutative category* is a symmetric strict monoidal category — here the monoidal structure is strict but the braiding $A \otimes B \cong B \otimes A$ is not. Permutative categories capture all symmetric monoidal categories up to equivalence. Since these structures form a part of 2-dimensional universal algebra, there are 2-categories $Perm_s$, $Perm_p$ and so on, of permutative categories and strict or pseudomorphisms. The idea is that $Perm_p$ should be a (symmetric) monoidal bicategory, categorifying the symmetric monoidal structure on commutative monoids, but the details are complicated — see, for instance [58].

The evident structure here is the internal hom $Perm_p(A, B)$ between permutative categories, whose morphisms are pseudomorphisms, preserving the tensor product and unit up to isomorphism. Given two such $F, G : A \rightrightarrows B$, their pointwise tensor product $(F \cdot G)a = Fa \cdot Ga$ is again pseudo, via the isomorphisms

$$(F{\cdot}G)(a{\cdot}b) = F(a{\cdot}b)G(a{\cdot}b) \cong Fa{\cdot}Fb{\cdot}Ga{\cdot}Gb \cong Fa{\cdot}Ga{\cdot}Fb{\cdot}Gb = (F{\cdot}G)a(F{\cdot}G)b.$$

This is the key in making the category $Perm_p(A, B)$ into a permutative category. Considering the second isomorphism in the above, observe that even if F and G are strict, their pointwise tensor product $F.G$ needn't be — therefore, it is *not possible* to make $Perm_s(A, B)$ into a permutative category.

However, since in the skew setting we have seen that elements of the internal hom can be more general than mere morphisms $A \to B$, it is perfectly possible that $Perm_p(A, B)$ forms the internal hom object of a skew monoidal closed structure on the 2-category of strict morphisms $Perm_s$. Indeed this is the case. Moreover, the skew monoidal closed structure on $Perm_s$ interacts well with the Quillen model structure on $Perm_s$ and so descends to a skew monoidal structure on the homotopy category, which is in fact *genuinely monoidal* — I called skew monoidal categories on Quillen model categories with this property *homotopy monoidal.* In particular on restricting to the cofibrant objects, the constraints $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $l_A : I \otimes A \to A$ and $r_A : A \to A \otimes I$ are all equivalences, and the resulting monoidal bicategory structure is, up to equivalence, the monoidal bicategory structure on $Perm_p$ that we wanted to construct at the start.

This is a good example of the approach championed by Max Kelly, of studying the non-strict world as strictly as possible: this line of thinking, interpreted for monoidal bicategories such as $Perm_p$, leads naturally into the world of skew monoidal categories.

More generally, the above analysis extends to any pseudo-commutative 2-monad T on Cat, thus giving the first detailed proof of a result of Hyland-Power [37] on the construction of monoidal bicategories in this setting. This is Theorem 6.7 of [13]. Other examples of skew structures described in the paper include one for bicategories.

This paper has been influential in a number of ways. Firstly, it established that skew monoidal categories naturally arise in, and have the potential to solve problems in, higher category theory and homotopy theory, thus providing a completely new class of examples to the bialgebroid examples of Szlachanyi [60], which had been the guiding examples until that time. The examples of [13] played an influential role in Campbell's development of skew enriched categories [25]. In Sydney 2016-2017, Steve Lack and I built on ideas in my paper [13] to develop the notion of skew multicategory [18], which has aroused the interest of workers in theoretical computer science [62]. We also introduced the notion of a braiding on a skew monoidal category [21], which curiously involves an isomorphism $(A \otimes B) \otimes C \to (A \otimes C) \otimes B$.

At the point of writing this thesis, skew monoidal categories have naturally appeared in a broad range of settings, and I anticipate that we will see many more examples and applications of them in years to come. A major project that I am involved in at the moment is to use skew monoidal categories as a base of enrichment for semistrict higher categories, moving us beyond the case of Gray-categories in dimension 3.

1.5 Higher groupoids, monads, nerves and theories

In category theory, there are two approaches to classical universal algebra. One of these, as discussed in Section 1.2, is through finitary monads on the category of sets. The other is via *Lawvere theories* [45]. which are identity on objects finite coproduct preserving functors $\mathbb{F} \to \mathbb{T}$ where $\mathbb{F} \hookrightarrow \mathrm{Set}$ is the full subcategory consisting of the finite cardinals $\{0, 1, 2, \ldots..\}$. That the two approaches are equivalent was established by Linton [46], who showed that the categories of finitary monads on Set and of Lawvere theories are equivalent and that the equivalence respects *semantics* — that is, the passage to categories of models.

This story has been generalised in many ways — until recently, the most general of these was via the *monads with arities* of Weber [64] and the theories with arities of [7]. The basic context here is a small dense full subcategory $\mathcal{A} \hookrightarrow \mathcal{E}$ of a category $\mathcal{E}$, generalising the case of $\mathbb{F} \hookrightarrow \mathrm{Set}$ appropriate to classical universal algebra. A guiding example was the full subcategory $\Theta_0 \hookrightarrow [\mathbb{G}^{op}, \mathrm{Set}]$ of globular sets consisting of the *globular cardinals*, of which two are drawn below.

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=50, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M, to=D] \end{tikzcd}$$

Just as the finite cardinals parametrise the shapes of operations of classical universal algebra, so the globular cardinals parametrise the shapes of operations of (globular) higher dimensional category theory. A guiding result here was the *nerve theorem* of Berger [6]. This takes as input a globular operad T (a certain monad on globular sets capturing Batanin's ω-categories [2]) and forms the associated *globular theory* $J_T : \Theta_0 \to \Theta_T$ obtained by taking the (identity on objects,fully faithful)-factorisation

$$K_T \circ J_T : \Theta_0 \to \Theta_T \to \text{T-Alg}$$

of the composite $F^T K_T : \mathcal{A} \hookrightarrow \mathcal{E} \to \text{T-Alg}$ of the inclusion with the free algebra functor to the Eilenberg-Moore category of algebras. Berger's nerve theorem asserts that for such T the *nerve functor*

$$\text{T-Alg}(K_T-, 1) : \text{T-Alg} \to [(\Theta_T)^{op}, \text{Set}]$$

is fully faithful and moreover characterises its essential image as those presheaves satisfying a generalised *Segal condition*. (This could be viewed as a higher dimensional generalisation of the classical nerve theorem which identifies small categories as simplicial sets satisfying the Segal condition.)

Weber in [64] observed that Berger's construction make sense for any monad T on $\mathcal{E}$ equipped with a dense full subcategory $K : \mathcal{A} \hookrightarrow \mathcal{E}$ — thus one can ask whether a given monad satisfies the nerve theorem. He introduced a class of monads called *monads with arities* $\mathcal{A}$, which include the globular operads, and showed that these do satisfy the nerve theorem — this is Weber's nerve theorem. Later, the *theories* corresponding to monads with arities were identified by Berger, Mellies and Weber [7] and called *theories with arities* $\mathcal{A}$. In particular [7] established an equivalence between monads and theories with arities, generalising Linton's equivalence between finitary monads and Lawvere theories. Pleasantly, Weber's nerve theorem becomes the fact that the equivalence respects semantics — that is, algebras for the monad with arities T amount to models of the associated theory $\mathcal{A} \to \mathcal{A}_T$.

My own role in this story began in late 2015 when I read Maltsiniotis' paper [53]. This brought to attention a definition of globular weak ω-groupoid essentially defined by Grothendieck at the beginning of *Pursuing Stacks* [32]. These are based on essentially the same *globular theories* $\Theta_0 \to \mathbb{T}$ as those of Berger, which can be defined as identity on objects functors preserving *globular sums* — certain wide pushouts presenting globular cardinals as colimits of representables. The theories for weak ω-groupoids were singled out as the cellular contractible ones. It struck me that this definition was remarkably simple compared to existing notions of globular weak ω-groupoid. With the aim of justifying this claim, I wrote a very short paper [16] giving a reinterpretation of Garner and van-den-Berg's construction [63] of the globular ∞-groupoid associated to the identity type in homotopy type theory using Grothendieck ∞-groupoids instead of Batanin ∞-groupoids, shortening the argument considerably.

If the reader permits me a final digression on this topic, let me mention that at this time I became somewhat obsessed by these Grothendieck ω-groupoids. Indeed, a significant and long standing open problem is to bridge the gap between globular and simplicial models of ∞-groupoids and ∞-categories. The ∞-groupoid part of this is Grothendieck's homotopy hypothesis (still unproven, in this form.) It struck me that with the relative simple definition of Grothendieck ω-groupoid, it might be possible. To get started, what one would like is to put a Quillen model structure[6] on the category $Mod(\mathbb{T})$ of Grothendieck ω-groupoids. There are obvious notions of fibration and weak equivalence and the problem quickly boils down to proving that the pushout of the generating trivial cofibrations

$$F(\rho_n) : F(\mathbb{G}(-, n)) \to F(\mathbb{G}(-, n+1))$$

are weak equivalences. Here $\rho_n : n \to n+1$ is the inclusion in the globe category $\mathbb{G}$ and $F : [\mathbb{G}^{op}, \mathrm{Set}] \to Mod(\mathbb{T})$ the free algebra functor. After some promising ideas, I spent a week at the State Library in Berlin trying to prove this result, but did not get much further. However, my approach via viewing Grothendieck ω-groupoids as iterated algebraic injectives did enable to me to prove the faithfulness conjecture of Maltsiniotis, resulting ultimately in the publication [20]. Despite further work on these questions by Henry and Lanari amongst others, the problem of constructing the model structure remains very much open.

Returning to the main story, in Sydney in 2016 I continued to think about these higher dimensional groupoids. A key observation was that the globular theories for higher groupoids do *not* have arities Θ_0 in the sense of Berger et al [7]. Therefore, in order to have a notion of theory capturing both the classical notion of Lawvere theory and of globular theory, it was necessary to generalise the notion of a theory with arities.

This is achieved by the simple notion of an $\mathcal{A}$-theory, which appears in my joint paper *Monads and theories* [17] with Richard Garner, included in this series and published in *Advances in Mathematics* in 2019. Having

[6]Or perhaps a semi-model structure.

identified the correct class of $\mathcal{A}$-theories, we quickly realised that the monads which correspond to them are those for which the *conclusion of Weber's nerve theorem [64] holds*, which we hence dubbed the $\mathcal{A}$-nervous monads. In particular we established a semantics respecting equivalence between $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories, showing it to be best possible in a precise sense. All of this works in the enriched setting, and subsumes most if not all monad-theory correspondences considered to date. Apart from this, the main achievement of the paper is that it establishes the excellent properties of $\mathcal{A}$-nervous monads, $\mathcal{A}$-theories and of their enriched categories of models. One of the deeper results in this regard is Theorem 38, which shows that the $\mathcal{A}$-nervous monads are precisely precisely those which can be presented using colimits of free monads on signatures. In particular, if one is interested in *presentations* of general kinds of algebraic structures, this means that the setting of $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories is ideally suited to the purpose.

1.6 Homotopy enriched category theory and higher-dimensional universal algebra

How do we know that algebraic categories, like the categories Grp and Rng of groups and rings, are cocomplete? And how do we know that forgetful functors between algebraic categories, such as the forgetful functor $U : \mathrm{Rng} \to \mathrm{Grp}$ from rings to groups, have left adjoints?

In this section I will describe a project that develops techniques to solve the higher dimensional analogues of these questions in which we, for instance, replace Rng by the 2-category MonCat_p of monoidal categories and pseudomaps, or by the ∞-cosmos $\mathrm{QCat}_{\mathrm{prod}}$ of quasicategories with finite products [38, 47, 57].

There are several approaches to answering these questions for classical algebraic categories, one of which is via the theory of accessible and locally presentable categories [50, 1]. Because filtered colimits are well behaved in such categories, it is not too hard to see that these categories are (finitely) accessible — meaning, suitably generated through filtered colimits by a set of finitely presentable objects — and easy to see that they are complete. Therefore we can obtain cocompleteness via the following well known and important result — see, for instance, [1, Theorem 4.11].

(1) An accessible category is complete if and only if it is cocomplete.

For similar reasons, we can easily apply the following theorem, described in Theorem 1.66 of [1], to obtain the desired left adjoints.

(2) A functor between complete accessible categories has a left adjoint if and only if preserves limits and (sufficiently highly) filtered colimits.

In seeking to extend these results to higher dimensions in a manner applicable to higher dimensional structures and their pseudomaps, one might expect it to be necessary to consider weaker versions of the limits and filtered colimits under consideration, having their universal properties only up to equivalence.

Surprisingly, in many settings, this turns out not to be the case. To begin with, from the work of the Australian school it has long been known that 2-categories such as MonCat_p admit certain genuine weighted limits — namely, flexible (or cofibrantly weighted) limits [8]. Building on work of Makkai [51, 52], in my recent paper [22] I established that such 2-categories are, in fact, genuinely *accessible*. Accessibility in the 2-categorical setting turns out to be closely related to weakness (or cofibrancy) of the categorical structure involved — for instance, whilst the 2-category MonCat_p of monoidal categories is accessible, its full subcategory $\mathrm{SMonCat}_p$ of strict monoidal categories is not.[7] The problem is that the notion of strict monoidal category is too strict.

The infinite dimensional analogue of this question considers ∞-cosmoi — certain simplicially enriched categories — of $(\infty, 1)$-categories with structure, and the structures considered in practise in the ∞-setting are invariably weak. Therefore, by analogy with the two-dimensional case described above, it is natural to conjecture that ∞-cosmoi such as $\mathrm{QCat}_{\mathrm{prod}}$ are genuinely accessible. Indeed this is the case, as is shown for a number of such examples in our 2020 preprint *Adjoint functor theorems for homotopically enriched categories* [23] with Steve Lack and Lukáš Vokřínek, included in this collection, and to be established in much greater generality in a forthcoming paper *Accessible infinity-cosmoi* with Lack [24].

What I would like to describe here is the framework for cocompleteness and adjoint functor theorems developed in [23] with such examples in mind, since it holds in a wide range of contexts and has a broad range of applications.

[7] Indeed — the idempotent completion of $\mathrm{SMonCat}_p$ is MonCat_p.

The basic framework is inspired by the paper *Enriched Weakness* [44] of Lack and Rosický, which introduced the notions of $\mathcal{E}$-weak left adjoint and $\mathcal{E}$-weak colimit relative to a class of morphisms $\mathcal{E}$ in a suitable base for enrichment $\mathcal{V}$: namely, the enriched functor $U : \mathcal{A} \to \mathcal{B}$ is said to have an $\mathcal{E}$-weak left adjoint if for each $X \in \mathcal{B}$ there exists an object $FX \in \mathcal{A}$ and morphism $\eta_X : X \to UFX$ for which the induced morphism

$$\mathcal{A}(FX, Y) \xrightarrow{U} \mathcal{B}(UFX, UY) \xrightarrow{\mathcal{B}(\eta_X, UY)} \mathcal{B}(X, UY)$$

of $\mathcal{V}$ belongs to $\mathcal{E}$. The notion of $\mathcal{E}$-weak weighted colimit is similarly defined. In particular, in the case of $\mathcal{V} = \text{Set}$ with $\mathcal{E}$ the class of isomorphisms one obtains the usual notions of adjoint functor and colimit. For $\mathcal{V} = \text{Set}$ with $\mathcal{E}$ the class of surjections, one obtains the notions of weak left adjoint and weak colimit, which have the existence but not uniqueness part of the appropriate universal property.

Our paper [23] takes the base $\mathcal{V}$ to be a monoidal model category. In this setting, the limits of relevance to our $\mathcal{E}$-weak adjoint functor theorem and $\mathcal{E}$-cocompleteness theorem are powers and cofibrantly weighted limits. The class of morphisms $\mathcal{E}$ is the class of so-called *shrinking* morphisms which are, assuming all objects are cofibrant, certain weak equivalences. The main result of the paper, Theorem 7.8, is our adjoint functor theorem for homotopically enriched categories. From this, the other results involving accessibility and cocompleteness, as discussed above in the classical setting, naturally follow. Rather than describing the most general results in detail, I will only mention a few applications, which are discussed in greater depth in Section 9 of the paper.

- When $\mathcal{V}$ is equipped with the trivial model structure, in which the weak equivalences are the isomorphisms, we obtain the *classical theory*. In particular, we obtain the classical adjoint functor theorem [49] and, when specialised to the accessible setting, the classical results (1) and (2) described at the beginning of this section.

- When $\mathcal{V}$ is equipped with what we call the split model structure, in which all maps are weak equivalences and fibrations are the split epis, we obtain *weak enriched category theory*. In particular, when $\mathcal{V} = \text{Set}$, we obtain results such as Kainen's weak adjoint functor theorem [39] and that an accessible category has products if and only if it has weak colimits [1, Theorem 4.11].

- When $\mathcal{V} = \mathrm{Cat}$ equipped with the natural model structure, whose weak equivalences are the equivalences of categories, we obtain a new 2-categorical adjoint functor theorem. Since, as described earlier, 2-categories such as MonCat_p are accessible [22] the adjoint functor theorem in this setting applies naturally to forgetful 2-functors such as $U : \mathrm{MonCat}_p \to \mathrm{Cat}$, and more broadly to 2-categories that lie outside the scope of two-dimensional monad theory such as the 2-categories of regular categories, pretopoi and so forth. Our results on cocompleteness also give simple proofs that such 2-categories have (a rather strong form of) bicolimits.

- Our major motivation is where $\mathcal{V} = SSet$ is the category of simplicial sets equipped with the Joyal model structure. In this setting, we obtain a new homotopical adjoint functor theorem applicable to the infinity-cosmoi of Riehl and Verity [56, 57]. This covers examples such as the forgetful cosmological functor $\mathrm{QCat}_{\mathrm{prod}} \to \mathrm{QCat}$ from quasicategories with finite products to quasicategories. Moreover, we obtain the new result that an accessible infinity cosmos, such as $\mathrm{QCat}_{\mathrm{prod}}$, automatically has flexibly weighted homotopy colimits.

The results of this paper are very recent and will be built upon in the forthcoming paper *Accessible Infinity Cosmoi* [24] (where, once again, the *lalis* of Sections' 1.2 and 1.3 play a decisive role!). Beyond this, the goal of this ongoing project is to continue using the interplay between enriched category theory and homotopy theory to develop simple approaches to handling weak higher dimensional structures.

Bibliography

[1] Adámek, J. and Rosický, J. *Locally Presentable and Accessible Categories*, vol. 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1994.

[2] Batanin, Michael. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Advances in Mathematics 136(1)* (1998) 39–103.

[3] Beck, J. Triples, algebras and cohomology. *Rep. Theory Appl. Categ. 2* (2003), 1–59.

[4] van den Berg, Benno and Garner, Richard. Types are weak ω-groupoids. *Proceedings of the London Mathematical Society 2* (2011), 370–394.

[5] van den Berg, Benno and Faber, Eric. Effective Kan fibrations in simplicial sets *Arxiv Preprint 2020,* `https://arxiv.org/abs/2009.12670`*.*

[6] Berger, C. A cellular nerve for higher categories. *Advances in Mathematics 169*, 1 (2002), 118–175.

[7] Berger, C., Melliès, P.-A., and Weber, M. Monads with arities and their associated theories. *Journal of Pure and Applied Algebra 216* (2012), 2029–2048.

[8] Bird, G., Kelly, G. M., Power, A. J. and Street, R. Flexible limits for 2-categories. *Journal of Pure and Applied Algebra 61*, 1 (1989), 1–27.

[9] Blackwell, R., Kelly, G. M., and Power, A. J. Two-dimensional monad theory. *Journal of Pure and Applied Algebra 59*, 1 (1989), 1–41.

[10] Boardman, J.M. and Vogt, R. Homotopy invariant algebraic structures. *Lecture Notes in Mathematics 247*, (1973).

[11] Bourke, John. Two-dimensional monadicity. *Advances in Mathematics 252* (2014), 708–747.

[12] Bourke, J., and Gurski, N. A cocategorical obstruction to tensor products of Gray-categories. *Theory and Applications of Categories*, 11 (2015), 387–409.

[13] Bourke, John. Skew structures in 2-category theory and homotopy theory. *J. Homotopy Relat. Struct.*, 12(1):31–81, 2017.

[14] Bourke, J. and Garner, R. Algebraic weak factorisation systems I: Accessible awfs. *Journal of Pure and Applied Algebra 220* (2016), 108–147.

[15] Bourke, J. and Garner, R. Algebraic weak factorisation systems 2: Categories of weak maps. *Journal of Pure and Applied Algebra 220* (2016), 148–174.

[16] Bourke, John. Note on the construction of globular weak omega-groupoids from types, topological space etc. *Cahiers de topologie et geometrie differentielle categoriques 57*, No 4, 281–294.

[17] Bourke, J. and Garner, R. Monads and theories. *Advances in Mathematics 251* (2019), 1024–1071.

[18] Bourke, J. and Lack, S. Skew monoidal categories and skew multicategories. *Journal of Algebra 506*, (2018), 237–266.

[19] Bourke, John. Equipping weak equivalences with algebraic structure. *Mathematische Zeitschrift 294(3)* (2020), 995-1019.

[20] Bourke, John. Iterated algebraic injectivity and the faithfulness conjecture. *Higher Structures 4(2)*, 2020, 183–210.

[21] Bourke, J. and Lack, S. Braided skew monoidal categories. *Theory and Applications of Categories 35*, (2020), No. 2, 19–63.

[22] Bourke, John. Accessible aspects of 2-category theory, *Journal of Pure and Applied Algebra 225(3)*, (2021).

[23] Bourke, J., Lack, S. and Vokřínek, L. Adjoint functor theorems for homotopically enriched categories. Arxiv Preprint, 2020.

[24] Bourke, J. and Lack, S. Accessible infinity-cosmoi. To appear, Spring 2021.

[25] Campbell, Alexander. Skew enriched categories. *Applied Categorical Structures 26*, (2018), 597–615.

[26] Eilenberg, S., and Kelly, G. M. Closed categories. In *Conference on Categorical Algebra (La Jolla, 1965)*. Springer, New York, 1966, pp. 421–562.

[27] Foltz, F., Lair, C., and Kelly, G. M. Algebraic categories with few monoidal biclosed structures or none. *Journal of Pure and Applied Algebra 17*, 2 (1980), 171–177.

[28] Gabriel, P., and Ulmer, F. *Lokal präsentierbare Kategorien*, vol. 221 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971.

[29] Gambino, N. and Sattler, C. The Frobenius condition, right properness, and uniform fibrations. *Journal of Pure and Applied Algebra 221* (2017), 3027–3068.

[30] Garner, Richard. Understanding the small object argument. *Applied Categorical Structures 17*, 3 (2009), 247–285.

[31] Grandis, M., and Tholen, W. Natural weak factorization systems. *Archivum Mathematicum 42*, 4 (2006), 397–408.

[32] Grothendieck, Alexander. Pursuing Stacks (1983).

[33] Gordon, R., Power, J., and Street, R. Coherence for Tricategories. *Memoirs of the American Mathematical Society 117* (1995).

[34] Gurski, N., Johnson, N. and Osorno, A. Extending homotopy theories across adjunctions. *Homology, Homotopy and Applications 19(2)* (2017), 89–110.

[35] Henry, Simon. Algebraic models of homotopy types and the homotopy hypothesis. Arxiv Preprint 2016, `https://arxiv.org/abs/1609.04622`.

[36] Hess, K., Kedziorek, M., Riehl, E. and Shipley, B. A necessary and sufficient condition for induced model structures. *Journal Of Topology 10* (2017), 324–369

[37] Hyland, M., and Power, J. Pseudo-commutative monads and pseudo-closed 2-categories. *Journal of Pure and Applied Algebra 175* (2002), 141–185.

[38] Joyal, André. Quasi-categories and Kan complexes. *J. Pure Appl. Algebra*, 175(1-3):207–222, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[39] Kainen, P. C. Weak adjoint functors. *Math. Z.*, 122:1–9, 1971.

[40] Kelly, G. M. Doctrinal adjunction. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 257–280.

[41] Kelly, G. M. *Basic concepts of enriched category theory*, vol. 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982.

[42] Lack, Stephen. Homotopy-theoretic aspects of 2-monads. *Journal of Homotopy and Related Structures 7(2)* (2007), 229–260.

[43] Lack, S., and Shulman, M. Enhanced 2-categories and limits for lax morphisms. *Advances in Mathematics 229*, 1 (2011), 294–356.

[44] Lack, S. and Rosický, J. Enriched weakness. *J. Pure Appl. Algebra*, 216(8-9):1807–1822, 2012.

[45] Lawvere, F.W. *Functorial semantics of algebraic theories.* PhD thesis, Columbia University, 1963. Republished as: *Reprints in Theory and Applications of Categories 5* (2004).

[46] Linton, F. E. J. Some aspects of equational categories. In *Conference on Categorical Algebra (La Jolla, 1965)*. Springer, 1966, pp. 84–94.

[47] Lurie, Jacob. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[48] MacLane, Saunders. *Duality for groups. Bulletin of the American Mathematical Society 56*, (1950), no. 6, 485–516.

[49] MacLane, Saunders. *Categories for the working mathematician.* Springer-Verlag, New York, 1971.

[50] Makkai, M., and Paré, R. *Accessible categories: the foundations of categorical model theory*, vol. 104 of *Contemporary Mathematics*. American Mathematical Society, 1989.

[51] Makkai, Michael. Generalized sketches as a framework for completeness theorems. Part I. *Journal of Pure and Applied Algebra 115*, 1 (1997), 49–79.

[52] Makkai, Michael. Generalized sketches as a framework for completeness theorems. Part II. *Journal of Pure and Applied Algebra 115*, 2 (1997), 197–212.

[53] Maltsiniotis, George. Grothendieck ∞-groupoids, and still another definition of ∞-categories. Unpublished, available as `https://arxiv.org/abs/1009.2331`, 2010.

[54] Quillen, Daniel. *Homotopical Algebra*, vol. 43 of *Lecture Notes in Mathematics*. Springer, 1967.

[55] Riehl, Emily. Algebraic model structures. *New York Journal of Mathematics 17* (2011), 173–231.

[56] Riehl, E. and Verity, D. The 2-category theory of quasi-categories. *Adv. Math.*, 280:549–642, 2015.

[57] Riehl, E. and Verity, D. Elements of ∞-category theory. Book draft downloaded 10 June 2020 from `http://www.math.jhu.edu/~eriehl/elements.pdf`.

[58] Schmitt, Vincent. Tensor product for symmetric monoidal categories (2007). Preprint available at http://arxiv.org/abs/0711.0324.

[59] Street, Ross. Skew-closed categories. *Journal of Pure and Applied Algebra 217*, 6 (2013), 973–988.

[60] Szlachányi, Kornél. Skew-monoidal categories and bialgebroids. *Advances in Mathematics 231*, 3-4 (2012), 1694–1730.

[61] Swan, Andrew. Identity Types in Algebraic Model Structures and Cubical Sets. Arxiv Preprint 2018, `https://arxiv.org/abs/1808.00915`.

[62] Uustalu, T., Veltri, N. and Zeilberger, N.. Proof theory of partially normal skew monoidal categories. In *Proc. of 3rd Applied Category Theory Conf., ACT 2020 (Cambridge, MA, July 2020)*, Electron. Proc. in Theor. Comput. Sci., Open Publishing Assoc., to appear.

[63] van den Berg, B. and Garner, R.. Types are weak ω-groupoids. *Proceedings of the London Mathematical Society 2* (2011), 370–394.

[64] Weber, Mark. Familial 2-functors and parametric right adjoints. *Theory and Applications of Categories 18* (2007), 665–732.

Chapter 2

Two-dimensional monadicity

This chapter contains the article *Two-dimensional monadicity* by *John Bourke*, published in *Advances in Mathematics 252* (2014) 707–747.

TWO-DIMENSIONAL MONADICITY

JOHN BOURKE

Abstract. The behaviour of limits of weak morphisms in 2-dimensional universal algebra is not 2-categorical in that, to fully express the behaviour that occurs, one needs to be able to quantify over strict morphisms amongst the weaker kinds. $\mathcal{F}$-categories were introduced to express this interplay between strict and weak morphisms. We express doctrinal adjunction as an $\mathcal{F}$-categorical lifting property and use this to give monadicity theorems, expressed using the language of $\mathcal{F}$-categories, that cover each weaker kind of morphism.

1. Introduction

The category of monoids sits over the category of sets via a forgetful functor $U : \mathrm{Mon} \to \mathrm{Set}$. This functor is *monadic* in the sense that it has a left adjoint F and the canonical comparison $E : \mathrm{Mon} \to \mathrm{Set}^T$ to the category of algebras for the induced monad $T = UF$ is an equivalence of categories. So if you are interested in monoids you can set about proving some theorem about algebras for an abstract monad T and be sure it holds for monoids, or any variety of universal algebra for that matter: this is the categorical approach to universal algebra via monads.
Before going down this path one thing must be established – namely, the monadicity of U. To this end the fundamental theorem is *Beck's monadicity theorem* [2] which asserts that a functor $U : \mathcal{A} \to \mathcal{B}$ is monadic just when it admits a left adjoint, is conservative and creates U-absolute coequalisers. What makes the theorem so useful in practice is that the conditions, up to the *existence* of a left adjoint, are cast entirely in terms of the typically simple U – these conditions are clearly met for monoids or indeed any variety (see Section 6.8 of [18]).
Now our interest is not in universal algebra, but in two dimensional universal algebra and 2-monads, and monadicity as appropriate to this setting. What do we mean by the varieties of 2-dimensional universal algebra? Monoidal structure borne by categories provides a basic example: one observes that associated to this notion are several kinds of structure. On the objects front we have at least strict monoidal categories and monoidal categories of interest. Between these are strict, strong, lax and colax monoidal functors all commonly arising, between which we have just one kind of monoidal transformation. Restricting ourselves to just one kind of object, let us take the monoidal categories, we still find that we are presented with four related 2-categories MonCat_w, where $w \in \{s, p, l, c\}$, living over

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the 2-category of categories Cat as on the left below.

$$
\begin{tikzcd}
 & & \mathrm{MonCat}_l \arrow[dddl, dotted, "V_l"] \\
\mathrm{MonCat}_s \arrow[r] \arrow[ddr, "V_s"'] & \mathrm{MonCat}_p \arrow[ur] \arrow[dr] \arrow[dd, "V_p"] & \\
 & & \mathrm{MonCat}_c \arrow[dl, "V_c"] \\
 & \mathrm{Cat} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \text{T-Alg}_l \arrow[dddl, dotted, "U_l"] \\
\text{T-Alg}_s \arrow[r] \arrow[ddr, "U_s"'] & \text{T-Alg}_p \arrow[ur] \arrow[dr] \arrow[dd, "U_p"] & \\
 & & \text{T-Alg}_c \arrow[dl, "U_c"] \\
 & \mathrm{Cat} &
\end{tikzcd}
\tag{1}
$$

The objects in each of these are monoidal categories; the morphisms are respectively strict, strong (or pseudo), lax and colax monoidal functors with monoidal transformations between them in each case. The inclusions witness that strict morphisms can be viewed as pseudomorphisms ($s \leq p$), which can in turn be viewed as lax or colax ($p \leq l$) and ($p \leq c$).

The situation corresponds with that of a 2-monad T based on Cat, associated with which are several kinds of algebra, including strict and pseudo-algebras. We will only ever consider the *strict algebras*: note that even *non-strict* monoidal categories are the *strict* algebras for a 2-monad, as further discussed below. Between strict algebras are strict, pseudo, lax and colax morphisms, with again a single notion of algebra transformation. As before we obtain a diagram of 2-categories[1] and 2-functors over Cat, as on the right above.

Comparing the diagrams left and right above we see that a monadicity theorem in this setting ought to match each 2-category on the left with the corresponding one on the right in a compatible way. More precisely, there should exist a 2-monad T on Cat and, for each $w \in \{s, p, l, c\}$, an equivalence of 2-categories $E_w : \mathrm{MonCat}_w \to \text{T-Alg}_w$ over Cat, as left below

$$
\begin{tikzcd}
\mathrm{MonCat}_w \arrow[rr, "E_w"] \arrow[dr, "V_w"'] & & \text{T-Alg}_w \arrow[dl, "U_w"] \\
 & \mathrm{Cat} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{MonCat}_{w_1} \arrow[r, "E_{w_1}"] \arrow[d] & \text{T-Alg}_{w_1} \arrow[d] \\
\mathrm{MonCat}_{w_2} \arrow[r, "E_{w_2}"'] & \text{T-Alg}_{w_2}
\end{tikzcd}
\tag{2}
$$

and these equivalences should be natural in the inclusions $w_1 \leq w_2$ for $w_1, w_2 \in \{s, p, l, c\}$, as on the right.

Now it is well known that such a 2-monad does indeed exist, with, moreover, each comparison $E_w : \mathrm{MonCat}_w \to \text{T-Alg}_w$ an isomorphism of 2-categories. Likewise many of the other varieties of 2-dimensional universal algebra are monadic in this sense. With such varieties as the primary object of study the subject of 2-dimensional monad theory was developed, notably in [3]. General results were obtained such as those enabling one to deduce that each inclusion $\mathrm{MonCat}_s \to \mathrm{MonCat}_w$ has a left 2-adjoint, or establish the bicategorical completeness and cocompleteness of MonCat_p.

[1]The objects in each of these 2-categories are the strict algebras.

Of course to apply such abstract results one must first establish monadicity. How is this known? Here the subject diverges substantially from the 1-dimensional approach of Beck's theorem. The standard approach is that of colimit presentations [15][11]. Here one explicitly constructs the intended 2-monad T as an iterated colimit of free ones, and then performs lengthy calculations with the universal property of the colimit T to establish monadicity in the sense described for monoidal categories above.

Now although the natural analogue of Beck's theorem has been obtained for *pseudomonads* [17] and pseudoalgebra pseudomorphisms this does not specialise to capture monadicity in the precise sense described above, even when $w = p$.

Our objective in the present paper is to establish 2-dimensional monadicity theorems, in which monadicity is recognised not by using an explicit description of a 2-monad, but by analysing the manner in which the varieties of 2-dimensional universal algebra sit over the base 2-category – as in Diagram 1. Apart from characterising such monadic situations our main results, Theorem 23 through Theorem 25, enable one to establish monadicity when a workable description of a 2-monad is not easily forthcoming.

In seeking to understand monadicity in the above sense the first important observation is that the world of *strict morphisms* is easily understood: one can check that $V_s : \mathrm{MonCat}_s \to \mathrm{Cat}$ has a left 2-adjoint – say, by an adjoint functor theorem – and then apply the enriched version of Beck's monadicity theorem [4] to establish strict monadicity. Having observed this to be the case the natural question to ask is: *Which properties of the commutative triangle*

$$\begin{tikzcd}
\mathrm{MonCat}_s \arrow[rr] \arrow[dr, "V_s"'] & & \mathrm{MonCat}_w \arrow[dl, "V_w"] \\
& \mathrm{Cat} &
\end{tikzcd}$$

ensure that the canonical isomorphism $E : \mathrm{MonCat}_s \to \text{T-Alg}_s$ *extends to an isomorphism* $E_w : \mathrm{MonCat}_w \to \text{T-Alg}_w$ *over the base?* We answer this question in Theorem 25 but not using the language of 2-categories. For it turns out that these determining properties are not 2-categorical in nature: they cannot be expressed as properties of the 2-categories or 2-functors in the above diagram considered individually. Rather, to express these properties we must be able to *single out strict morphisms amongst each weaker kind.* Consequently we treat the inclusion $\mathrm{MonCat}_s \to \mathrm{MonCat}_w$ as a single entity, an $\mathcal{F}$*-category* $\mathbb{M}\mathrm{onCat}_w$, and the above triangle as a single $\mathcal{F}$*-functor* $V : \mathbb{M}\mathrm{onCat}_w \to \mathrm{Cat}$.

Let us now give an overview of the paper and of our line of argument. $\mathcal{F}$-categories were introduced in [16] in order to explain certain relationships between strict and weak morphisms in 2-dimensional universal algebra. We recall some basic facts about $\mathcal{F}$-categories in Section 2, in particular discussing $\mathcal{F}$-categories $\text{T-}\mathbb{A}\mathrm{lg}_w$ of algebras for a 2-monad and $\mathcal{F}$-categories $\mathbb{M}\mathrm{onCat}_w$ of monoidal categories – we will use monoidal categories as our running example throughout the paper.

Given a strict monoidal functor F and an adjunction $(\epsilon, F \dashv G, \eta)$ the right adjoint G obtains a unique lax monoidal structure such that the adjunction becomes a monoidal adjunction. This is an instance of *doctrinal adjunction*, the main topic of Section 3, and the first key $\mathcal{F}$-categorical property that we meet. We express three variants of doctrinal adjunction – w-doctrinal adjunction for $w \in \{l, p, c\}$ – as lifting properties of an $\mathcal{F}$-functor, so that the case just described asserts that the forgetful $\mathcal{F}$-functor $V : \mathbb{M}\mathrm{onCat}_l \to \mathrm{Cat}$ satisfies l-doctrinal adjunction. We define the closely related class w-Doct of *w-doctrinal* $\mathcal{F}$-functors and analyse the relationships between the different notions for $w \in \{l, p, c\}$; each such class of $\mathcal{F}$-functor is shown to be an orthogonality class in the category of $\mathcal{F}$-categories.

In the fourth section we turn to the reason $\mathcal{F}$-categories were introduced in [16] – namely, because of the interplay between strict and weak morphisms, *tight* and *loose*, that occurs when considering limits of weak morphisms in 2-dimensional universal algebra. The crucial limits are $\bar{w}$-limits of loose morphisms for $w \in \{l, p, c\}$ – after defining these we describe the illuminating case of the colax limit of a lax monoidal functor. We then examine how such limits allow one to represent loose morphisms by *tight spans* – the nature of this representation is analysed in detail.

This analysis allows us to prove the key result of the paper, Theorem 21 of Section 5, an orthogonality result which has nothing to do with 2-monads at all. Its immediate consequence, Corollary 22, ensures that the decomposition in $\mathcal{F}$-CAT

$$\mathrm{MonCat}_s \xrightarrow{j} \mathbb{M}\mathrm{onCat}_w \xrightarrow{V} \mathrm{Cat} \;=\; \begin{tikzcd} & \mathrm{MonCat}_s \arrow[d, "j"] \arrow[dr, "V_s"] & \\ \mathrm{MonCat}_s \arrow[ur, "1"] \arrow[r, "j"'] & \mathbb{M}\mathrm{onCat}_w \arrow[r, "V_w"'] & \mathrm{Cat} \end{tikzcd}$$

of the forgetful 2-functor $V_s : \mathrm{MonCat}_s \to \mathrm{Cat}$ is an orthogonal $({}^{\perp}w\text{-Doct}, w\text{-Doct})$-decomposition. Likewise for a 2-monad T on Cat the $\mathcal{F}$-category T-$\mathbb{A}\mathrm{lg}_w$ is obtained as a $({}^{\perp}w\text{-Doct}, w\text{-Doct})$-factorisation of $U_s : \text{T-Alg}_s \to \mathrm{Cat}$.

Our monadicity results, given in Section 6, use the uniqueness of $({}^{\perp}w\text{-Doct}, w\text{-Doct})$-decompositions to extend our understanding of monadicity in the strict setting to cover each weaker kind of morphism. For instance, the isomorphism $E : \mathrm{MonCat}_s \to \text{T-Alg}_s$ over Cat induces a commuting diagram as on the outside of

$$\begin{tikzcd} \mathrm{MonCat}_s \arrow[r, "j"] \arrow[d, "E"'] & \mathbb{M}\mathrm{onCat}_w \arrow[r, "V"] \arrow[d, dotted, "E_w"] & \mathrm{Cat} \arrow[d, "1"] \\ \text{T-Alg}_s \arrow[r, "j"'] & \text{T-}\mathbb{A}\mathrm{lg}_w \arrow[r, "U"'] & \mathrm{Cat} \end{tikzcd}$$

with each horizontal path an orthogonal $({}^{\perp}w\text{-Doct}, w\text{-Doct})$-decomposition. The two outer vertical isomorphisms then induce a unique invertible filler $E_w : \mathbb{M}\mathrm{onCat}_w \to \text{T-}\mathbb{A}\mathrm{lg}_w$, so establishing monadicity in each weaker context. This is the idea behind the main monadicity result, Theorem 25. Naturality in different weaknesses (as in Diagram 2 above) is treated in Theorem 24.

In the seventh and final section we describe examples and applications of our results. We begin by completing the example of monoidal categories before moving on to more complex cases. In Theorem 27 we give an example of the kind of monadicity result that cannot be established using techniques, like presentations, that require explicit knowledge of a 2-monad.

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2. $\mathcal{F}$-categories in 2-dimensional universal algebra

In this section we recall the notion of $\mathcal{F}$-category, introduced in [16], and a few basic facts about them.

2.1. **$\mathcal{F}$-categories.** An *$\mathcal{F}$-category* $\mathbb{A}$ is a 2-category with two kinds of 1-cell: those of the 2-category itself which are called *loose* and a subcategory of *tight* morphisms containing all of the identities. A second perspective is that an $\mathcal{F}$-category $\mathbb{A}$ is specified by a pair of 2-categories $\mathcal{A}_\tau$ and $\mathcal{A}_\lambda$ connected by a 2-functor

$$j : \mathcal{A}_\tau \to \mathcal{A}_\lambda$$

which is the *identity on objects, faithful and locally fully faithful.* Here $\mathcal{A}_\lambda$ contains all of the morphisms, which is to say the loose ones, and all 2-cells between them, whilst $\mathcal{A}_\tau$ contains the tight morphisms together with all 2-cells between them in $\mathcal{A}_\lambda$. Loose morphisms in $\mathbb{A}$ are drawn with wavy arrows $A \rightsquigarrow B$ and tight morphisms with straight arrows $A \to B$, so that a typical diagram in $\mathbb{A}$ would be

$$\begin{tikzcd} A \arrow[r, squiggly, "f"] \arrow[dr, squiggly, "h"'] & B \arrow[d, "g"] \arrow[dl, phantom, "\Swarrow\alpha"] \\ & C \end{tikzcd}$$

For each pair of objects $A, B \in \mathbb{A}$ the inclusion of hom categories

$$j_{A,B} : \mathcal{A}_\tau(A, B) \to \mathcal{A}_\lambda(A, B)$$

constitutes an *injective on objects fully faithful functor.* In fact $\mathcal{F}$-categories are precisely categories enriched in $\mathcal{F}$, the full subcategory of the arrow category $\mathrm{Cat}^{\mathbf{2}}$ whose objects are those functors which are both injective on objects and fully faithful. $\mathcal{F}$ is a complete and cocomplete cartesian closed category so that the full theory of enriched categories [8] can be applied to the study of $\mathcal{F}$-categories.

To begin with we have $\mathcal{F}$-CAT, the 2-category of $\mathcal{F}$-categories, $\mathcal{F}$-functors and $\mathcal{F}$-natural transformations. An $\mathcal{F}$-functor $F : \mathbb{A} \to \mathbb{B}$ consists of a pair of 2-functors $F_\tau : \mathcal{A}_\tau \to \mathcal{B}_\tau$ and $F_\lambda : \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ rendering commutative the square

$$\begin{tikzcd} \mathcal{A}_\tau \arrow[r, "j_A"] \arrow[d, "F_\tau"'] & \mathcal{A}_\lambda \arrow[d, "F_\lambda"] \\ \mathcal{B}_\tau \arrow[r, "j_B"'] & \mathcal{B}_\lambda \end{tikzcd}$$

This equally amounts to a 2-functor $F_\lambda : \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ which *preserves tightness*. An $\mathcal{F}$-natural transformation $\eta : F \to G$ is a 2-natural transformation $\eta : F_\lambda \to G_\lambda$ with *tight components*.

2.2. **2-categories as $\mathcal{F}$-categories.** Each 2-category $\mathcal{A}$ may be viewed as an $\mathcal{F}$-category in two extremal ways: as an $\mathcal{F}$-category in which only the identities are tight, or as an $\mathcal{F}$-category in which *all morphisms are tight*, whereupon the induced $\mathcal{F}$-category has the form

$$1 : \mathcal{A} \to \mathcal{A}$$

When we view a 2-category $\mathcal{A}$ as an $\mathcal{F}$-category it will *always be in this second sense* and we again denote it by $\mathcal{A}$.
With this convention established we can treat 2-categories as special kinds of $\mathcal{F}$-categories and unambiguously speak of $\mathcal{F}$-functors $F : \mathcal{A} \to \mathbb{B}$ from 2-categories to $\mathcal{F}$-categories, or from $\mathcal{F}$-categories to 2-categories as in $G : \mathbb{B} \to \mathcal{C}$. These $\mathcal{F}$-functors appear as triangles

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "F_\tau"'] \arrow[dr, "F_\lambda"] & \\ \mathcal{B}_\tau \arrow[rr, "j_B"'] & & \mathcal{B}_\lambda \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B}_\tau \arrow[rr, "j_A"] \arrow[dr, "G_\tau"'] & & \mathcal{B}_\lambda \arrow[dl, "G_\lambda"] \\ & \mathcal{C} & \end{tikzcd}$$

with an $\mathcal{F}$-functor between 2-categories just a 2-functor. Observe that each $\mathcal{F}$-category $\mathbb{A}$ induces an $\mathcal{F}$-functor from its 2-category of tight morphisms

$$j : \mathcal{A}_\tau \to \mathbb{A}$$

which is the identity on tight morphisms and $j : \mathcal{A}_\tau \to \mathcal{A}_\lambda$ on loose ones – we abuse notation by using j in either situation.

2.3. **$\mathcal{F}$-categories of monoidal categories.** In two-dimensional universal algebra one encounters morphisms of four different flavours and so $\mathcal{F}$-categories naturally arise. Here we recall the various $\mathcal{F}$-categories associated to the notion of monoidal structure – we recall the defining equations for monoidal functors as we will use these later on.
The data for a monoidal category consists of a tuple $\overline{A} = (A, \otimes, i^A, \lambda^A, \rho_l^A, \rho_r^A)$ where we use juxtaposition for the tensor product. A lax monoidal functor $(F, f, f_0) : \overline{A} \rightsquigarrow \overline{B}$ consists of a functor $F : A \to B$, coherence constraints $f_{a,b} : Fa \otimes Fb \to F(a \otimes b)$ natural in a and b and a comparison $f_0 : i^B \to Fi^A$, all satisfying the three conditions below

$$\begin{tikzcd} (Fa \otimes Fb) \otimes Fc \arrow[r, "f_{a,b}\otimes 1"] \arrow[d, "\lambda^B_{Fa,Fb,Fc}"'] & F(a \otimes b) \otimes Fc \arrow[r, "f_{a\otimes b,c}"] & F((a \otimes b) \otimes c) \arrow[d, "F\lambda^A_{a,b,c}"] \\ Fa \otimes (Fb \otimes Fc) \arrow[r, "1\otimes f_{b,c}"'] & Fa \otimes F(b \otimes c) \arrow[r, "f_{a,b\otimes c}"'] & F(a \otimes (b \otimes c)) \end{tikzcd}$$

$$\begin{tikzcd} i^B \otimes Fa \arrow[r, "f_0\otimes 1"] \arrow[d, "\rho_l^B"'] & Fi^A \otimes Fa \arrow[r, "f_{i,a}"] & F(i^A \otimes a) \arrow[d, "F\rho_l^A"] \\ Fa \arrow[rr, equal] & & Fa \end{tikzcd} \qquad \begin{tikzcd} (Fa) \otimes i^B \arrow[r, "1\otimes f_0"] \arrow[d, "\rho_r^B"'] & Fa \otimes Fi^A \arrow[r, "f_{a,i}"] & F(a \otimes i^A) \arrow[d, "F\rho_r^A"] \\ Fa \arrow[rr, equal] & & Fa \end{tikzcd}$$

which we call the *associativity, left unit* and *right unit* conditions. We call (F, f, f_0) strong or strict monoidal just when the constraints $f_{a,b}$ and f_0 are invertible or identities respectively; reversing these constraints we obtain the notion of a colax monoidal functor.

$$\begin{tikzcd} Fa \otimes Fb \arrow[r, "\eta_a \otimes \eta_b"] \arrow[d, "f_{a,b}"'] & Ga \otimes Gb \arrow[d, "g_{a,b}"] \\ F(a \otimes b) \arrow[r, "\eta_{a\otimes b}"'] & G(a \otimes b) \end{tikzcd} \qquad \begin{tikzcd} i^B \arrow[r, "f_0"] \arrow[dr, "g_0"'] & Fi^A \arrow[d, "\eta_{i^A}"] \\ & Gi^A \end{tikzcd}$$

Between lax monoidal functors are monoidal transformations $\eta : (F, f, f_0) \to (G, g, g_0)$: these are natural transformations $\eta : F \to G$ satisfying the two conditions above, which we call the *tensor* and *unit* conditions for a monoidal transformation.

For $w \in \{s, l, p, c\}$ we thus have w-monoidal functors (with p-monoidal meaning strong monoidal). Together with monoidal categories and monoidal transformations these form a 2-category MonCat_w which sits over Cat via a forgetful 2-functor $V_w : \mathrm{MonCat}_w \to \mathrm{Cat}$. Now strict monoidal functors are strong ($s \leq p$) and strong monoidal functors can be viewed as lax ($p \leq l$) or colax ($p \leq c$). Whenever $w_1 \leq w_2$ it follows that we have an $\mathcal{F}$-category $\mathbb{M}\mathrm{onCat}_{w_1,w_2}$ of monoidal categories with tight and loose morphisms the w_1 and w_2-monoidal functors respectively, as specified by the inclusion $j : \mathrm{MonCat}_{w_1} \to \mathrm{MonCat}_{w_2}$. Each such $\mathcal{F}$-category comes equipped with a forgetful $\mathcal{F}$-functor $V : \mathbb{M}\mathrm{onCat}_{w_1,w_2} \to \mathrm{Cat}$ where $V_\tau = V_{w_1}$ and $V_\lambda = V_{w_2}$ – see the commuting triangles in Diagram 1 of the Introduction. Of particular importance will be those $\mathcal{F}$-categories $\mathbb{M}\mathrm{onCat}_{s,w}$ for $s \leq w$, which we denote by $\mathbb{M}\mathrm{onCat}_w$.

2.4. **$\mathcal{F}$-categories of algebras.** Of prime importance are those $\mathcal{F}$-categories associated to a 2-monad T on a 2-category $\mathcal{C}$. Each 2-monad has various associated flavours of algebra and morphism. We will only be interested in *strict algebras* and will call them algebras. Between algebras we have strict, pseudo, lax and colax morphisms – as with monoidal functors we specify these using s, p, l and c.

Drawn in turn from left to right below is the data $(f, \overline{f})$ for a strict, pseudo, lax or colax morphism of algebras $(f, \overline{f}) : (A, a) \rightsquigarrow (B, b)$

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{f}\,\cong"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Uparrow\overline{f}"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Thus $\overline{f}$ is an identity 2-cell in the first case, invertible in the second, and points into or out of f in the lax or colax cases; in all cases these 2-cells are required to satisfy two coherence conditions [12].

There is a single notion of 2-cell between any kind of algebra morphisms; for instance given a pair of lax morphisms $(f, \overline{f}), (g, \overline{g}) : (A, a) \rightsquigarrow (B, b)$ an algebra

2-cell $\alpha : (f, \overline{f}) \Rightarrow (g, \overline{g})$ is a 2-cell $\alpha : f \Rightarrow g$ satisfying

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\overline{f}\Downarrow"] & TB \arrow[d, "b"] \\ A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} TA \arrow[r, bend left, "Tf"] \arrow[r, bend right, "Tg"'] \arrow[r, phantom, "T\alpha\Downarrow"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{g}"] & TB \arrow[d, "b"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

whilst the equation in the colax case looks like the lax case with the directions reversed.

Algebras, w-algebra morphisms and transformations live in 2-categories T-Alg_w for $w \in \{s, p, l, c\}$, each of which comes equipped with an evident forgetful 2-functor to the base, which we always denote by $U_w : \text{T-Alg}_w \to \mathcal{C}$. Each strict morphism is a pseudo morphism ($s \leq p$), and each pseudomorphism can be viewed either as lax ($p \leq l$) or colax ($p \leq c$). It follows that for each pair $w_1, w_2 \in \{s, p, l, c\}$ satisfying $w_1 \leq w_2$ we have an $\mathcal{F}$-category T-$\mathbb{A}\mathrm{lg}_{w_1,w_2}$ with tight morphisms the w_1-morphisms, and loose morphisms the w_2-morphisms. Each comes equipped with a forgetful $\mathcal{F}$-functor $U : \text{T-}\mathbb{A}\mathrm{lg}_{w_1,w_2} \to \mathcal{C}$ (so $U_\tau = U_{w_1}$ and $U_\lambda = U_{w_2}$) – see the commuting triangles of Diagram 1 of the Introduction.
Of particular importance are those $\mathcal{F}$-categories T-$\mathbb{A}\mathrm{lg}_{s,w}$ whose tight morphisms are the strict ones, and following [16] we abbreviate these by T-$\mathbb{A}\mathrm{lg}_w$. As well as using $j : \text{T-Alg}_\mathrm{s} \to \text{T-Alg}_w$ for the defining inclusion and $U : \text{T-}\mathbb{A}\mathrm{lg}_w \to \mathcal{C}$ for the forgetful $\mathcal{F}$-functor, we occasionally use j_w or U_w if we are in the presence of multiple j's or U's.

2.5. **Duality between lax and colax morphisms.** Colax algebra morphisms are lax algebra morphisms with 2-cells reversed. This statement can be made precise using the covariant duality 2-functor $(-)^{co} : \text{2-CAT} \to \text{2-CAT}$ which takes a 2-category $\mathcal{C}$ to the 2-category $\mathcal{C}^{co}$ with the same underlying category but with 2-cells reversed. Likewise it takes a 2-monad $T : \mathcal{C} \to \mathcal{C}$ to a 2-monad $T^{co} : \mathcal{C}^{co} \to \mathcal{C}^{co}$ and one then has, as noted in [6], an equality $\mathrm{T}^{co}\text{-Alg}_l = \text{T-Alg}_c^{co}$ which restricts to $\mathrm{T}^{co}\text{-Alg}_s = \text{T-Alg}_s^{co}$. The $(-)^{co}$ duality naturally extends to a 2-functor

$$(-)^{co} : \mathcal{F}\text{-CAT} \to \mathcal{F}\text{-CAT}$$

under whose action we have that $\text{T-}\mathbb{A}\mathrm{lg}_c^{co} = \mathrm{T}^{co}\text{-}\mathbb{A}\mathrm{lg}_l$ and moreover that

$$U^{co} : \text{T-}\mathbb{A}\mathrm{lg}_c^{co} \to \mathcal{C}^{co} \quad \text{equals} \quad U : \mathrm{T}^{co}\text{-}\mathbb{A}\mathrm{lg}_l \to \mathcal{C}^{co} \ .$$

A consequence of this duality is that each theorem about lax morphisms has a dual version concerning colax morphisms. Indeed all of our definitions and results in the colax case will be dual to those in the lax setting – though we will *state* these results for colax morphisms it will always suffice to *prove* results only in the lax setting.

2.6. **Equivalence of $\mathcal{F}$-categories.** Our monadicity theorems in Section 6 will assert that certain $\mathcal{F}$-categories are *equivalent* to $\mathcal{F}$-categories of algebras for a

2-monad. By an equivalence of $\mathcal{F}$-categories we mean an equivalence in the 2-category of $\mathcal{F}$-categories $\mathcal{F}$-CAT, which is to say an equivalence of $\mathcal{V}$-categories for $\mathcal{V} = \mathcal{F}$.

Recall from [8] that a $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ is an equivalence just when it is *essentially surjective on objects* and *fully faithful* in the enriched sense. When $\mathcal{V} = \mathcal{F}$ the first condition amounts to $F_\tau : \mathcal{A}_\tau \to \mathcal{B}_\tau$ being essentially surjective on objects, in the usual sense. This also implies the weaker statement that $F_\lambda : \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ is essentially surjective on objects. Enriched fully faithfulness of F amounts to $F_{\tau A,B} : \mathcal{A}_\tau(A,B) \to \mathcal{B}_\tau(FA,FB)$ and $F_{\lambda A,B} : \mathcal{A}_\lambda(A,B) \to \mathcal{B}_\lambda(FA,FB)$ being isomorphisms of categories for each pair $A, B \in \mathbb{A}$.

We conclude that F is an equivalence of $\mathcal{F}$-categories just when both 2-functors F_τ and F_λ are essentially surjective on objects and 2-fully faithful, which is to say that both F_τ and F_λ are 2-equivalences, or equivalences of 2-categories.

3. Doctrinal adjunction and $\mathcal{F}$-categorical lifting properties

If a strict monoidal functor has a right adjoint that right adjoint admits a unique lax monoidal structure such that the adjunction lifts to a monoidal adjunction. This is an instance of *doctrinal adjunction* – the topic of the present section. We begin by recalling Kelly's treatment of doctrinal adjunction in the setting of 2-monads, recasting the notion in $\mathcal{F}$-categorical terms, so that the above special case becomes the assertion that the forgetful $\mathcal{F}$-functor $V : \mathbb{M}\mathrm{onCat}_l \to \mathrm{Cat}$ satisfies l-doctrinal adjunction – we treat the cases $w \in \{l, p, c\}$. In Section 3.2 we define the closely related notion of a w-doctrinal $\mathcal{F}$-functor before showing that the w-doctrinal $\mathcal{F}$-functors form an orthogonality class in $\mathcal{F}$-CAT.

3.1. **Doctrinal adjunction $\mathcal{F}$-categorically.**

Doctrinal adjunction was first studied in Kelly's paper [6] of the same name. Motivated by the example of adjunctions between monoidal categories amongst others, all known to be describable using 2-monads via clubs [7] or other techniques, he gave his treatment in the setting of 2-dimensional monad theory. Let us now recall the relevant aspects of this. Given T-algebras (A, a) and (B, b) and a morphism $f : A \to B$ together with an adjunction $(\epsilon, f \dashv g, \eta)$ in the base, his Theorem 1.2 asserts that there is a bijection between colax algebra morphisms of the form $(f, \overline{f}) : (A, a) \rightsquigarrow (B, b)$ and lax morphisms of the form $(g, \overline{g}) : (B, b) \rightsquigarrow (A, a)$. The structure 2-cells $\overline{f} : f.a \Rightarrow b.Tf$ and $\overline{g} : a.Tg \Rightarrow g.b$ are expressed in terms of one another as mates as below

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "1"'] \arrow[dr, phantom, "\overset{T\eta}{\Rightarrow}", near start] & TB \arrow[d, "b"] \arrow[dl, "Tg"] \\
TA \arrow[d, "a"'] \arrow[r, phantom, "\overset{\Searrow}{\overline{g}}"] & B \arrow[d, "1"] \arrow[dl, "g"'] \arrow[dl, phantom, "\overset{\Rightarrow}{\epsilon}", near end, shift left=3] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TB \arrow[r, "Tg"] \arrow[d, "1"'] \arrow[dr, phantom, "\overset{T\epsilon}{\Leftarrow}", near start] & TA \arrow[d, "a"] \arrow[dl, "Tf"] \\
TB \arrow[d, "b"'] \arrow[r, phantom, "\overset{\overline{f}}{\Nwarrow}"] & A \arrow[d, "1"] \arrow[dl, "f"'] \arrow[dl, phantom, "\overset{\eta}{\Leftarrow}", near end, shift left=3] \\
B \arrow[r, "g"'] & A
\end{tikzcd}
$$

Since lax and colax morphisms cannot be composed this relationship cannot be expressed 2-categorically or indeed $\mathcal{F}$-categorically – it can be captured using *double* *categories* as in Example 5.4 of [22]. However if we start with $(f, \overline{f})$ a pseudomorphism then it does live in the same 2-category as the resultant lax morphism $(g, \overline{g}) : (B, b) \rightsquigarrow (A, a)$. Moreover it is shown in Proposition 1.3 of [6] that the unit and counit η and ϵ then become algebra 2-cells $\eta : 1 \Rightarrow (g, \overline{g}) \circ (f, \overline{f})$ and $\epsilon : (f, \overline{f}) \circ (g, \overline{g}) \Rightarrow 1$ in T-Alg$_{\mathrm{l}}$ and so yield an adjunction $(\epsilon, (f, \overline{f}) \dashv (g, \overline{g}), \eta)$ in T-Alg$_{\mathrm{l}}$.

Dually, if $(g, \overline{g})$ were a pseudomorphism, then upon equipping f with the corresponding colax structure $(f, \overline{f})$ the adjunction $(\epsilon, f \dashv g, \eta)$ lifts to an adjunction $(\epsilon, (f, \overline{f}) \dashv (g, \overline{g}), \eta)$ in the 2-category T-Alg$_c$.

The invertibility of $\overline{f}$ does not imply the invertibility of its mate $\overline{g}$, or vice-versa. However, if both η and ϵ are invertible, then $\overline{f}$ is invertible just when $\overline{g}$ is, which is to say that if $(\epsilon, f \dashv g, \eta)$ is an adjoint equivalence and either f or g admits the structure of a pseudomorphism the adjoint equivalence lifts to an adjoint equivalence in T-Alg$_p$.

Let us now abstract these lifting properties of adjunctions and adjoint equivalences into properties of the forgetful $\mathcal{F}$-functors $U : \text{T-}\mathbb{A}\mathrm{lg}_{w_1,w_2} \to \mathcal{C}$. By an *adjunction or adjoint equivalence in an $\mathcal{F}$-category* $\mathbb{A}$ we will mean an adjunction or adjoint equivalence in its 2-category $\mathcal{A}_\lambda$ of loose morphisms. Given an $\mathcal{F}$-functor $H : \mathbb{A} \to \mathbb{B}$ and an adjunction $(\epsilon, f \dashv g, \eta)$ in $\mathbb{B}$ a *lifting of this adjunction along* H is an adjunction $(\epsilon', f' \dashv g', \eta')$ in $\mathbb{A}$ such that $Hf' = f, Hg' = g, H\epsilon' = \epsilon$ and $H\eta' = \eta$. We will likewise speak of liftings of adjoint equivalences along H. An $\mathcal{F}$-functor $H : \mathbb{A} \to \mathbb{B}$ is said to satisfy

- *weak l-doctrinal adjunction* if for each tight arrow $f : A \to B \in \mathbb{A}$ each adjunction $(\epsilon, Hf \dashv g, \eta)$ in $\mathbb{B}$ lifts along H to an adjunction in $\mathbb{A}$ with left adjoint f.
- *weak p-doctrinal adjunction* if for each tight arrow $f : A \to B \in \mathbb{A}$ each adjoint equivalence $(\epsilon, Hf \dashv g, \eta)$ in $\mathbb{B}$ lifts along H to an adjoint equivalence in $\mathbb{A}$ with left adjoint f.[2]
- *weak c-doctrinal adjunction* if for each tight arrow $f : A \to B \in \mathbb{A}$ each adjunction $(\epsilon, g \dashv Hf, \eta)$ in $\mathbb{B}$ lifts along H to an adjunction in $\mathbb{A}$ with right adjoint f.

The lifting properties for algebras described above can be rephrased as asserting exactly that *the forgetful $\mathcal{F}$-functor $U : \text{T-}\mathbb{A}\mathrm{lg}_{p,w} \to \mathcal{C}$ satisfies weak w-doctrinal adjunction for $w \in \{l, p, c\}$*. Each of these statements asserts that if we are given a pseudomorphism of algebras whose underlying arrow has some kind of adjoint, then that adjunction lifts in a certain way; of course if the starting pseudomorphism were in fact strict then the same lifting will exist so that, in particular, *the forgetful $\mathcal{F}$-functor $U : \text{T-}\mathbb{A}\mathrm{lg}_w \to \mathcal{C}$ satisfies weak w-doctrinal adjunction for $w \in \{l, p, c\}$*.

In fact such forgetful $\mathcal{F}$-functors lift these adjunctions *uniquely* – as will follow upon considering the following simple lifting properties. Recall that a 2-functor $H : \mathcal{A} \to \mathcal{B}$ *reflects identity 2-cells* or is *locally conservative* when it reflects the property of a 2-cell being an identity or an isomorphism, and is *locally faithful* if it reflects the equality of parallel 2-cells. Let us say that an $\mathcal{F}$-functor $H : \mathbb{A} \to \mathbb{B}$ has any of these three local properties when its loose part $H_\lambda : \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ has

[2]This lifting property appears biased but of course is not, since the left adjoint of an adjoint equivalence is equally its right adjoint.

them: this means that H has these properties with respect to all 2-cells and not just those between tight morphisms. The following is evident from the definition of an algebra 2-cell.

Proposition 1. *Given $w_1, w_2 \in \{s, l, p, c\}$ satisfying $w_1 \leq w_2$ the forgetful $\mathcal{F}$-functor $U : \text{T-}\mathbb{A}\mathrm{lg}_{w_1,w_2} \to \mathcal{C}$ reflects identity 2-cells, is locally conservative and locally faithful. In particular these properties are true of each $U : \text{T-}\mathbb{A}\mathrm{lg}_w \to \mathcal{C}$.*

Definition 2. Let $w \in \{l, p, c\}$. An $\mathcal{F}$-functor $H : \mathbb{A} \to \mathbb{B}$ is said to satisfy *w-doctrinal adjunction* if it satisfies the unique form of weak w-doctrinal adjunction.

For instance $H : \mathbb{A} \to \mathbb{B}$ satisfies l-doctrinal adjunction if for each tight arrow $f : A \to B \in \mathbb{A}$ each adjunction $(\epsilon, Hf \dashv g, \eta)$ in $\mathbb{B}$ lifts *uniquely* along H to an adjunction in $\mathbb{A}$ with left adjoint f. Let us note that, since the $(-)^{co}$ duality interchanges left and right adjoints in an $\mathcal{F}$-category, H satisfies c-doctrinal adjunction just when H^{co} satisfies l-doctrinal adjunction. Since adjoint equivalences are fixed H satisfies p-doctrinal adjunction just when H^{co} does.

Proposition 3. *Let $w \in \{l, p, c\}$ and consider $H : \mathbb{A} \to \mathbb{B}$.*

(1) If H satisfies weak w-doctrinal adjunction and reflects identity 2-cells it satisfies w-doctrinal adjunction.

(2) If H is locally conservative and satisfies l-doctrinal adjunction or c-doctrinal adjunction then H satisfies p-doctrinal adjunction.

Proof. (1) To prove the cases $w = l$ and $w = p$ it will suffice to show that any two adjunctions $(\epsilon, f \dashv g, \eta)$ and $(\epsilon', f \dashv g', \eta')$ in $\mathbb{A}$ with common left adjoint and common image under H necessarily coincide in $\mathbb{A}$. Since adjoints are unique up to isomorphism we have $m : g \cong g'$ given by the composite

$$\begin{tikzcd}
& A \arrow[r, "1"] \arrow[d, rightsquigarrow, "f"'] \arrow[dl, phantom, "\epsilon\Downarrow", shift right=3ex] \arrow[dr, phantom, "\eta'\Downarrow", shift left=3ex] & A \\
B \arrow[ur, rightsquigarrow, "g"] \arrow[r, "1"'] & B \arrow[ur, rightsquigarrow, "g'"'] &
\end{tikzcd}$$

Using the triangle equations for $f \dashv g$ and $f \dashv g'$ we see that $(m.f) \circ \eta = \eta'$ and that $\epsilon' \circ (f.m) = \epsilon$. Now the image of the 2-cell m under H is an identity by one of the triangle equations for $Hf \dashv Hg = Hg'$. Therefore m is an identity and $g = g'$. Since $m.f$ and $f.m$ are identities it follows that $\eta = \eta'$ and $\epsilon = \epsilon'$ too. The case $w = c$ is dual.

(2) Suppose that H satisfies l-doctrinal adjunction and is locally conservative. Then given a tight arrow $f \in \mathbb{A}$ each adjoint equivalence $(\epsilon, Hf \dashv g, \eta)$ in $\mathbb{B}$ lifts uniquely to an adjunction $(\epsilon', f \dashv g', \eta')$ in $\mathbb{A}$. Since H is locally conservative both ϵ' and η' are invertible because their images are. Therefore the lifted adjunction is an adjoint equivalence so that H satisfies p-doctrinal adjunction. The c-case is dual.

□

Corollary 4. *Let $w \in \{l, p, c\}$. Then $U : \text{T-}\mathbb{A}\mathrm{lg}_w \to \mathcal{C}$ satisfies w-doctrinal adjunction. Furthermore both $U : \text{T-}\mathbb{A}\mathrm{lg}_l \to \mathcal{C}$ and $U : \text{T-}\mathbb{A}\mathrm{lg}_c \to \mathcal{C}$ satisfy p-doctrinal adjunction.*

Proof. Since $U : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{C}$ satisfies weak w-doctrinal adjunction and reflects identity 2-cells it follows from Proposition 3.1 that U satisfies w-doctrinal adjunction. Since it is locally conservative the second part of the claim follows from Proposition 3.2. $\square$

Example 5. In the concrete setting of monoidal categories doctrinal adjunction is well known. Here we describe only those aspects relevant to our needs: namely, that the forgetful $\mathcal{F}$-functors $V : \mathbb{M}\text{onCat}_w \to \text{Cat}$ satisfy w-doctrinal adjunction for $w \in \{l, p, c\}$. Consider then a strict monoidal functor $F : \overline{A} \to \overline{B}$ and an adjunction of categories $(\epsilon, F \dashv G, \eta)$. The right adjoint G obtains the structure of a lax monoidal functor $\overline{G} = (G, g, g_0) : \overline{B} \rightsquigarrow \overline{A}$ with constraints $g_{x,y}$ and g_0 given by the composites

$$Gx \otimes Gy \xrightarrow{\eta_{Gx\otimes Gy}} GF(Gx \otimes Gy) = G(FGx \otimes FGy) \xrightarrow{G(\epsilon_x \otimes \epsilon_y)} G(x \otimes y)$$

and $\eta_{i^A} : i^A \to GFi^A = Gi^B$. It is straightforward to show that, with respect to these constraints, the natural transformations ϵ and η become monoidal transformations. Therefore we obtain the lifted adjunction $(\epsilon, F \dashv (G, g, g_0), \eta)$ in $\mathbb{M}\text{onCat}_l$ whose uniqueness follows, using Proposition 3.1, from the fact that V reflects identity 2-cells; thus $V : \mathbb{M}\text{onCat}_l \to \text{Cat}$ satisfies l-doctrinal adjunction. The cases $w = p$ and $w = c$ are entirely analogous.

Note that unless $w = l$ the forgetful $\mathcal{F}$-functor $V : \mathbb{M}\text{onCat}_w \to \text{Cat}$ does not satisfy l-doctrinal adjunction. That $V : \mathbb{M}\text{onCat}_l \to \text{Cat}$ itself does so is due to the fact that the constraints $f_{a,b} : Fa \otimes Fb \to F(a \otimes b)$ and $f_0 : i^B \to Fi^A$ point in the correct direction – into F – and are not invertible. Whilst l-doctrinal adjunction captures, to some extent, this *laxness* it does not determine in any way the *coherence axioms* for a lax monoidal functor. This is illustrated by the fact that there is an $\mathcal{F}$-category of monoidal categories, strict and *incoherent* lax monoidal functors (these have components f and f_0 oriented as above but satisfying no equations) and this too sits over Cat via a forgetful $\mathcal{F}$-functor satisfying l-doctrinal adjunction.

3.2. **Reflections and doctrinal $\mathcal{F}$-functors.** Although each forgetful $\mathcal{F}$-functor $U : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{C}$ satisfies w-doctrinal adjunction it turns out that, insofar as this property characterises such $\mathcal{F}$-functors, the only relevant adjunctions take a more specialised form. We will begin by treating the cases $w = l$ and $w = p$, before treating the case $w = c$ by duality.

Let us call an adjunction $(1, f \dashv g, \eta)$ with tight left adjoint and identity counit an *l-reflection*. If, in addition, the unit η is invertible then we call the adjunction a *p-reflection*. A p-reflection is of course an adjoint equivalence.

We remark that any adjunction $(\epsilon, f \dashv g, \eta)$ is determined by three of its four parts: in particular the unit η is the unique 2-cell $1 \Rightarrow g.f$ satisfying the triangle equation $(\epsilon.f) \circ (f.\eta) = 1$. In a w-reflection $(1, f \dashv g, \eta)$ the unit η is therefore uniquely determined by the adjoints $f \dashv g$ and the knowledge that the counit is the identity – we can thus faithfully abbreviate a w-reflection $(1, f \dashv g, \eta)$ by $f \dashv g$ if convenient. Whilst we only need to consider liftings of w-reflections we also need to consider liftings of *morphisms of w-reflections*. Consider w-reflections

$(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ and a *tight* commutative square $(r, s) : f_1 \to f_2$ as left below.

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f_1"'] & C \arrow[d, "f_2"] \\ B \arrow[r, "s"'] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} A \arrow[r, "r"] & C \\ B \arrow[r, "s"'] \arrow[u, squiggly, "g_1"] & D \arrow[u, squiggly, "g_2"'] \end{tikzcd}$$

We call $(r, s) : f_1 \to f_2$ a *morphism of w-reflections* if the above square of right adjoints also commutes and furthermore the compatibility with units $r.\eta_1 = \eta_2.r$ is met. Compatibility with the identity counits is automatic. The following lemma is sometimes useful for recognising morphisms of w-reflections.

Lemma 6. *Let $w \in \{l, p\}$. Consider w-reflections $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ and a tight commuting square $(r, s) : f_1 \to f_2$. Then (r, s) is a morphism of w-reflections just when its mate $m(r, s) : g_2.s \Rightarrow r.g_1$ is an identity 2-cell.*

Proof. It suffices to consider the case $w = l$. Given a morphism of l-reflections $(r, s) : f_1 \to f_2$

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f_1"'] & B \arrow[d, "f_2"] \\ C \arrow[r, "s"'] & D \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, "r"] \arrow[d, "f_1"] & B \arrow[r, "1"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\eta_2\Downarrow"] & B \\ C \arrow[r, "1"'] \arrow[ur, squiggly, "g_1"] & C \arrow[r, "s"'] & D \arrow[ur, squiggly, "g_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[r, "1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\eta_1\Downarrow"] & A \arrow[r, "r"] & B \\ C \arrow[r, "1"'] \arrow[ur, squiggly, "g_1"] & C \arrow[r, "s"'] \arrow[ur, squiggly, "g_1"'] & D \arrow[ur, squiggly, "g_2"'] & \end{tikzcd}$$

consider the mate of the left square: the central composite above. As (r, s) is a morphism of l-reflections we have $\eta_2.r = r.\eta_1$ so that the mate reduces to the rightmost composite: this is an identity by the triangle equation for $f_1 \dashv g_1$.
Conversely suppose that the mate of $(r, s) : f_1 \to f_2$ is an identity. Then (r, s) certainly commutes with the right adjoints so that it remains to verify compatibility with the units. This is straightforward. ☐

Note that each $\mathcal{F}$-functor preserves morphisms of w-reflections. The following lifting properties are crucial. For $w \in \{l, p\}$ we say that an $\mathcal{F}$-functor $H : \mathbb{A} \to \mathbb{B}$ satisfies:

- *w-Refl* if given a tight arrow $f : A \to B \in \mathbb{A}$ each w-reflection $(1, Hf \dashv g, \eta)$ lifts uniquely along H to a w-reflection $(1, f \dashv g', \eta')$ in $\mathbb{A}$.
- *w-Morph* if given w-reflections $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ a tight commuting square $(r, s) : f_1 \to f_2$ is a morphism of w-reflections just when its image $(Hr, Hs) : Hf_1 \to Hf_2$ is one.

We say that $H : \mathbb{A} \to \mathbb{B}$ satisfies *c-Refl* or *c-Morph* if $H^{co} : \mathbb{A}^{co} \to \mathbb{B}^{co}$ satisfies l-Refl or l-Morph respectively. We remark that these c-variants concern *c-reflections* which are adjunctions with *tight right adjoint* and *identity unit.*

Definition 7. Let $w \in \{l, p, c\}$. An $\mathcal{F}$-functor $H : \mathbb{A} \to \mathbb{B}$ is said to be *w-doctrinal* if it satisfies w-Refl, w-Morph and is locally faithful. We denote the class of w-doctrinal $\mathcal{F}$-functors by w-Doct.

The condition that W be locally faithful may seem somewhat unnatural – see the discussion after Theorem 21 for our reasons for including it.
Evidently we have that H is c-doctrinal just when H^{co} is l-doctrinal. Furthermore

H is p-doctrinal just when H^{co} is p-doctrinal. Let us now compare these lifting properties with those of Section 3.1.

Lemma 8. *Let $w \in \{l, p, c\}$ and consider $H : \mathbb{A} \to \mathbb{B}$.*

1. *If H satisfies w-doctrinal adjunction and reflects identity 2-cells then it satisfies w-Refl and w-Morph.*
2. *If H satisfies w-doctrinal adjunction, reflects identity 2-cells and is locally faithful then it is w-doctrinal.*
3. *If H is locally conservative, reflects identity 2-cells, is locally faithful and satisfies either l or c-doctrinal adjunction then it is p-doctrinal.*

Proof. We will only consider the l-case of (1) and (2), all being essentially identical, with the l and c cases dual.

1. Consider a tight morphism $f : A \to B \in \mathbb{A}$ and l-reflection $(1, Hf \dashv g, \eta)$. Because H satisfies l-doctrinal adjunction this lifts uniquely to an adjunction $(\epsilon', f \dashv g', \eta')$ in $\mathbb{A}$. Since $H\epsilon' = 1$ and H reflects identity 2-cells ϵ' is an identity. This verifies l-Refl. For l-Morph consider l-reflections $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ in $\mathbb{A}$ and a tight morphism $(r, s) : f_1 \to f_2$ such that $(Hr, Hs) : Hf_1 \to Hf_2$ is a morphism of l-reflections; we must show $(r, s) : f_1 \to f_2$ is one too. By Lemma 6 this is equally to say that the mate $m_{r,s} : r.g_1 \Rightarrow g_2.s$ of this square is an identity 2-cell, so giving a commutative square $(s, r) : g_1 \to g_2$. So it will suffice to check $Hm_{r,s}$ is an identity. But $Hm_{r,s} = m_{Hr,Hs}$ which is an identity since (Hr, Hs) is a morphism of l-reflections.
2. Since w-doctrinal just means w-Refl and w-Morph together with local faithfulness this follows immediately from Part 1.
3. Since by Proposition 3.2 such an $\mathcal{F}$-functor satisfies p-doctrinal adjunction this follows from Part 2. □

Corollary 9. *Let T be a 2-monad on $\mathcal{C}$. For each $w \in \{l, p, c\}$ the forgetful $\mathcal{F}$-functor $U : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{C}$ is w-doctrinal. Furthermore both $U : \text{T-}\mathbb{A}\text{lg}_l \to \mathcal{C}$ and $U : \text{T-}\mathbb{A}\text{lg}_c \to \mathcal{C}$ are p-doctrinal.*

Proof. Since $U : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{C}$ satisfies w-doctrinal adjunction, reflects identity 2-cells and is locally conservative the first claim follows from Lemma 8.2; since each such U is also locally conservative the second claim follows from Lemma 8.3. □

Let us conclude by mentioning a further evident example of w-doctrinal $\mathcal{F}$-functors.

Proposition 10. *Let $w \in \{l, p, c\}$. If $H : \mathbb{A} \to \mathbb{B}$ is such that $H_\lambda : \mathcal{A}_\lambda \to \mathcal{B}_\lambda$ is 2-fully faithful then H is w-doctrinal for each w. In particular each equivalence of $\mathcal{F}$-categories is w-doctrinal for each w.*

3.3. **A small orthogonality class.** Though not strictly necessary in what follows let us remark that, with the exception of weak w-doctrinal adjunction, all of the lifting/reflection properties so far considered are expressible as orthogonal lifting properties in $\mathcal{F}$-CAT. Certainly it is not hard to see that this is true of the property of reflecting identity 2-cells, or of being locally faithful or locally conservative. Less obvious is that this is true of the notion of w-doctrinal adjunction or of the conditions w-Refl and w-Morph, in particular of the condition w-Morph concerning liftings of morphisms of adjunctions. We describe the conditions l-Refl and l-Morph here – the p and c cases being similar. To this end consider the following $\mathcal{F}$-category $\mathbb{A}dj_l$ depicted in its entirety on the left below

$$
\begin{tikzcd}
0 \arrow[r, "f"] \arrow[d, "1"'] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}" very near start] & 1 \arrow[d, "1"] \arrow[dl, squiggly, "g"] \\
0 \arrow[r, "f"'] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{2} \arrow[r, "j"] \arrow[d] & \mathbb{A}dj_l \arrow[d] \arrow[dl, dotted] \\
\mathbb{A} \arrow[r, "H"'] & \mathbb{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}\text{-CAT}(\mathbb{A}dj_l, \mathbb{A}) \arrow[r, "H_*"] \arrow[d, "j^*"'] & \mathcal{F}\text{-CAT}(\mathbb{A}dj_l, \mathbb{B}) \arrow[d, "j^*"] \\
\mathcal{F}\text{-CAT}(\mathbf{2}, \mathbb{A}) \arrow[r, "H_*"'] & \mathcal{F}\text{-CAT}(\mathbf{2}, \mathbb{B})
\end{tikzcd}
$$

where $g.f$ is loose, $f.g = 1$, $f.\eta = 1$ and $\eta.g = 1$. This is the free adjunction with identity counit and tight left adjoint f. It has a single non-identity tight arrow f so that $\mathbb{A}dj_{l_\tau}$ equals the free tight arrow $\mathbf{2}$; therefore the inclusion $\mathcal{A}dj_{l_\tau} \to \mathbb{A}dj_l$ is the $\mathcal{F}$-functor $j : \mathbf{2} \to \mathbb{A}dj_l$ selecting f. Now to give a commutative square as in the middle above is to give a tight arrow $f \in \mathbb{A}$ and adjunction $(1, Hf \dashv g, \eta)$ in $\mathbb{B}$. To give a filler is to lift the adjunction along H to an adjunction $(1, f \dashv g', \eta')$ in $\mathbb{A}$; thus H is orthogonal to j when condition l-Refl is met. The conditions l-Refl and l-Morph together assert exactly that j and H are orthogonal in $\mathcal{F}$-CAT as a 2-category – this means that the right square above is a pullback in CAT. In fact by a standard argument the conditions l-Refl and l-Morph jointly amount to ordinary (1-categorical) orthogonality against the single $\mathcal{F}$-functor $j \times 1 : \mathbf{2} \times \mathbf{2} \to \mathbb{A}dj_l \times \mathbf{2}$, of which j is a retract.

Therefore the class of w-doctrinal $\mathcal{F}$-functors, w-Doct, forms an orthogonality class in the category of $\mathcal{F}$-categories and $\mathcal{F}$-functors. The following section is geared towards establishing sufficient conditions on an $\mathcal{F}$-category $\mathbb{A}$ under which the inclusion $j : \mathcal{A}_\tau \to \mathbb{A}$ belongs to ${}^{\perp}w$-Doct – is orthogonal to each w-doctrinal $\mathcal{F}$-functor.

4. Loose morphisms as spans

We now consider completeness properties of $\mathcal{F}$-categories appropriate to those of the form T-$\mathbb{A}\text{lg}_w$. In an $\mathcal{F}$-category $\mathbb{A}$ with such completeness properties we can represent loose morphisms in $\mathbb{A}$ by certain tight spans. In the lax setting, for instance, each loose morphism $f : A \rightsquigarrow B$ is represented as a tight span $C_f : A \nrightarrow B$

$$A \xleftarrow{p_f \dashv r_f} C_f \xrightarrow{q_f} B$$

by taking its *colax* limit. The tight left leg $p_f : C_f \to A$ is part of an l-reflection $(1, p_f \dashv r_f, \eta_f)$ from which f can be recovered as the composite $q_f.r_f : A \rightsquigarrow C_f \to B$. In the present section we analyse this representation of loose morphisms by tight spans in detail, beginning with a discussion of the relevant limits.

4.1. **Limits of loose morphisms.** The main reason that $\mathcal{F}$-categories were introduced in [16] was to capture the behaviour of limits in 2-categories of the form T-Alg$_w$ for $w \in \{l, p, c\}$. In such 2-categories many *2-categorical* limits have an $\mathcal{F}$-categorical aspect – namely, that the projections from the limit are *strict and jointly detect strictness*. Interpreted in the $\mathcal{F}$-category T-$\mathbb{A}\text{lg}_w$ this asserts that the *limit projections are tight and jointly detect tightness*. This latter property is exactly that which distinguishes $\mathcal{F}$-categorical limits (in the sense of $\mathcal{F}$-enriched category theory) from 2-categorical ones (see Proposition 3.6 of [16]). We will only need a few basic $\mathcal{F}$-categorical limits and our treatment, in what follows, is elementary.

There are three limits to consider here – *colax, pseudo and lax limits of loose morphisms* – these correspond, in turn, to lax, pseudo and colax morphisms. As we always lead with the case of lax morphisms we focus here primarily on colax limits. Given a loose morphism $f : A \rightsquigarrow B \in \mathbb{A}$ its (colax/pseudo/lax)-limit consists of an object $C_f/P_f/L_f$ and a (colax/pseudo/lax)-cone (p_f, λ_f, q_f) as below

$$\begin{tikzcd} & C_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\lambda_f\Downarrow"] & \\ A \arrow[rr, squiggly, "f"'] & {} & B \end{tikzcd} \qquad \begin{tikzcd} & P_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\cong_{\lambda_f}"] & \\ A \arrow[rr, squiggly, "f"'] & {} & B \end{tikzcd} \qquad \begin{tikzcd} & L_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\Uparrow\lambda_f"] & \\ A \arrow[rr, squiggly, "f"'] & {} & B \end{tikzcd}$$

with both projections p_f and q_f *tight.*

The colax limit C_f[3] is required to be the usual 2-categorical *colax limit of an arrow* [13] in $\mathcal{A}_\lambda$: this means that it has the following two universal properties.

(1) Given any *colax cone* (r, α, s) as below

$$\begin{tikzcd} & X \arrow[dl, squiggly, "r"'] \arrow[dr, squiggly, "s"] \arrow[d, phantom, "\alpha\Downarrow"] & \\ A \arrow[rr, squiggly, "f"'] & {} & B \end{tikzcd}$$

there exists a unique $t : X \rightsquigarrow C_f$ satisfying

$$p_f.t = r,\ q_f.t = s \text{ and } \lambda_f.t = \alpha$$

(2) Given a pair of colax cones (r, α, s) and (r', α', s') with common base X together with 2-cells $\theta_r : r \Rightarrow r' \in \mathcal{A}_\lambda(X, A)$ and $\theta_s : s \Rightarrow s' \in \mathcal{A}_\lambda(X, B)$ satisfying

$$\begin{tikzcd} s \arrow[r, Rightarrow, "\theta_s"] \arrow[d, Rightarrow, "\alpha"'] & s' \arrow[d, Rightarrow, "\alpha'"] \\ fr \arrow[r, Rightarrow, "f\theta_r"'] & fr' \end{tikzcd}$$

there exists a unique 2-cell $\phi : t \Rightarrow t' \in \mathcal{A}_\lambda(X, C_f)$ between the induced factorisations such that

$$p_f.\phi = \theta_r \text{ and } q_f.\phi = \theta_s\ .$$

For C_f to be the colax limit of f in the $\mathcal{F}$-categorical sense we must also have

(3) A morphism $t : X \rightsquigarrow C_f$ is tight just when $p_f.t$ and $q_f.t$ are tight – *the projections jointly detect tightness.*

In the case of the pseudolimit P_f the 2-cell λ_f is required to be invertible. If we call those colax cones with an invertible 2-cell *pseudo-cones* then the universal properties of (1) and (2) above are only changed by replacing colax cones by pseudo-cones – thus (p_f, λ_f, q_f) is the universal pseudo-cone. The $\mathcal{F}$-categorical

[3]Colax limits of arrows are usually called oplax limits of arrows. We prefer colax limits here since they are lax limits in $\mathbb{A}^{co}$ rather than $\mathbb{A}^{op}$ and, similarly, sit better with our usage of lax and colax morphisms.

aspect of (3) remains the same. The lax limit of f is simply the colax limit in $\mathbb{A}^{co}$. For $w \in \{l, p, c\}$ let us write *w-limit of a loose morphism* as an abbreviation, so that c-limit stands for colax limit, for instance. Now we mentioned above that lax morphisms correspond to colax limits and so on. To capture this let us set $\bar{l} = c$, $\bar{p} = p$ and $\bar{c} = l$ as in [16]: the correspondence is captured by the following result.

Proposition 11. *Let $w \in \{l, p, c\}$. The forgetful $\mathcal{F}$-functor $U : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{C}$ creates $\overline{w}$-limits of loose morphisms.*

In its $\mathcal{F}$-categorical formulation above this is a specialisation of Theorem 5.13 of [16] which characterises those limits created by the forgetful $\mathcal{F}$-functors $U : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{C}$. However for $\overline{w}$-limits of loose morphisms, as concern us, the result goes back to [3] and [13].
Because lax limits are colax limits in $\mathbb{A}^{co}$ we can and will avoid them entirely. In order to work with colax and pseudolimits simultaneously let us introduce a final piece of notation. For $w \in \{l, p\}$ we will use

$$\begin{tikzcd} & \bar{W}_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\lambda_f \Downarrow"] & \\ A \arrow[rr, squiggly, "f"'] & {} & B \end{tikzcd}$$

to denote the $\overline{w}$-limit of f and universal $\overline{w}$-cone: so C_f and its colax cone when $w = l$ and P_f when $w = p$. When $w = p$ the 2-cell λ_f should be interpreted as invertible.

Example 12. In Cat the colax limit of a functor $F : A \to B$ is given by the comma category B/F: this has objects $(x, \alpha : x \to Fa, a)$ and morphisms $(r, s) : (x, \alpha, a) \to (y, \beta, b)$ given by pairs of arrows $r : x \to y \in \mathcal{B}$ and $s : a \to b \in \mathcal{A}$ rendering commutative the square on the left.

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, "r"'] & Fa \arrow[d, "Fs"] \\ y \arrow[r, "\beta"'] & Fb \end{tikzcd} \qquad \begin{tikzcd} & B/F \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\lambda \Downarrow"] & \\ A \arrow[rr, "F"'] & {} & B \end{tikzcd}$$

The projections $p : B/F \to A$ and $q : B/F \to B$ of the colax cone (p, λ, q) act on a morphism (r, s) of B/F as $p(r, s) = s : a \to b$ and $q(r, s) = r : x \to y$; the value of $\lambda : q \Rightarrow pF$ at (x, α, a) is simply the morphism $\alpha : x \to Fa$ itself.
The pseudolimit of F is the full subcategory of B/F whose objects are those pairs $(\alpha : x \to Fa, a)$ with α invertible, whilst the lax limit of F is the comma category F/B.

Example 13. It is illuminating to consider the colax limit of a lax monoidal functor $F = (F, f, f_0) : \overline{A} \rightsquigarrow \overline{B}$. The forgetful $\mathcal{F}$-functor $U : \mathbb{M}\text{onCat}_l \to \text{Cat}$ creates these limits: to see how this goes first consider the colax limit of the functor F, the comma category B/F equipped with its colax cone (p, λ, q) described above. The crux of the argument is to show that this lifts uniquely to a colax cone in $\mathbb{M}\text{onCat}_l$: that *B/F admits a unique monoidal structure such that p and q become strict monoidal and λ a monoidal transformation.*

So consider two objects (x, α, a) and (y, β, b) of B/F: if p and q are to be strict monoidal the tensor product $(x, \alpha, a) \otimes (y, \beta, b)$ must certainly be of the form $(x \otimes y, \theta, a \otimes b)$; furthermore the tensor condition for λ to be a monoidal transformation interpreted at this pair asserts precisely that $(x, \alpha, a) \otimes (y, \beta, b)$ equals

$$x \otimes y \xrightarrow{\alpha \otimes \beta} Fa \otimes Fb \xrightarrow{f_{a,b}} F(a \otimes b)$$

Likewise the unit condition for a monoidal transformation forces us to define the unit of B/F to be $(f_0 : i^B \to Fi^A, i^A)$. For p and q to preserve tensor products of morphisms we must define the tensor product as $(r, s) \otimes (r', s') = (r \otimes r', s \otimes s')$ at morphisms of B/F – to say the resulting pair is a morphism of B/F is then to say that the following square is commutative.

$$\begin{tikzcd}
x \otimes y \arrow[r, "\alpha \otimes \beta"] \arrow[d, "r \otimes r'"'] & Fa \otimes Fb \arrow[r, "f_{a,b}"] \arrow[d, "Fs \otimes Fs'"] & F(a \otimes b) \arrow[d, "F(s \otimes s')"] \\
x' \otimes y' \arrow[r, "\alpha' \otimes \beta'"'] & Fa' \otimes Fb' \arrow[r, "f_{a',b'}"'] & F(a' \otimes b')
\end{tikzcd}$$

The left square trivially commutes and the right square commutes by naturality of the $f_{a,b}$. It remains to give the associator and the left and right unit constraints for the monoidal structure on B/F – this is where the coherence axioms for a lax monoidal functor finally come into play. Certainly if p and q are to be strict monoidal they must preserve the associators strictly: this means that the associator at a triple of objects $((x, \alpha, a), (y, \beta, b), (z, \gamma, c))$ of B/F must be given by $(\lambda^B_{x,y,z}, \lambda^A_{a,b,c})$. To say that this is a morphism of B/F is equally to say that the composite square

$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "(\alpha \otimes \beta) \otimes \gamma"] \arrow[d, "\lambda^B_{x,y,z}"'] & (Fa \otimes Fb) \otimes Fc \arrow[r, "f_{a,b} \otimes 1"] \arrow[d, "\lambda^B_{Fa,Fb,Fc}"] & F(a \otimes b) \otimes Fc \arrow[r, "f_{a \otimes b,c}"] & F((a \otimes b) \otimes c) \arrow[d, "F\lambda^A_{a,b,c}"] \\
x \otimes (y \otimes z) \arrow[r, "\alpha \otimes (\beta \otimes \gamma)"'] & Fa \otimes (Fb \otimes Fc) \arrow[r, "1 \otimes f_{b,c}"'] & Fa \otimes F(b \otimes c) \arrow[r, "f_{a,b \otimes c}"'] & F(a \otimes (b \otimes c))
\end{tikzcd}$$

is commutative. The left square commutes by naturality of the associators in B with the right square asserting exactly the associativity condition for a lax monoidal functor. Similarly the left and right unit constraints at (x, α, a) must be given by $(\rho^B_l x, \rho^A_l a)$ and $(\rho^B_r x, \rho^A_r a)$ – that these lift to isomorphisms of B/F likewise correspond to the left and right unit conditions for a lax monoidal functor. Having given the monoidal structure for B/F it remains to check it verifies the axioms for a monoidal category, but all of these clearly follow from the corresponding axioms for $\overline{A}$ and $\overline{B}$ because p and q preserve the structure strictly and are jointly faithful.

Finally one needs to verify that this uniquely lifted colax cone in $\mathbb{M}\mathrm{onCat}_l$ satisfies the universal property of the colax limit of (F, f, f_0) therein. That p and q jointly detect tightness follows from the fact that they jointly reflect identity arrows – from here it is straightforward to verify that B/F has the universal property of the colax limit in MonCat_l. For $w \in \{p, c\}$ one constructs the $\bar{w}$-limit of a loose morphism in $\mathbb{M}\mathrm{onCat}_w$ in an entirely similar way.

Observe that in lifting the colax cone (p, λ, q) to $\mathbb{M}\mathrm{onCat}_l$ we used all of the coherence axioms for a lax monoidal functor, and indeed these generating coherence axioms are required for the colax cone to lift. Thus while l-doctrinal adjunction is related to the *laxness* – orientation and non-invertibility – of our lax monoidal functors, colax limits of loose morphisms concern the *coherence* axioms these lax morphisms must satisfy.

4.2. **Loose morphisms as tight spans.** For $w \in \{l, p\}$ we suppose that $\mathbb{A}$ admits $\bar{w}$-limits of loose morphisms. Then given $f : A \rightsquigarrow B$ we have the commutative triangle on the left below.

$$\begin{tikzcd}
 & A \arrow[dl, "1"'] \arrow[dr, rightsquigarrow, "f"] & \\
A \arrow[rr, rightsquigarrow, "f"'] & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & A \arrow[d, rightsquigarrow, "r_f"'] \arrow[ddl, bend right, "1"'] \arrow[ddr, rightsquigarrow, bend left, "f"] & \\
 & \bar{W}_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\lambda_f \Downarrow"] & \\
A \arrow[rr, rightsquigarrow, "f"'] & {} & B
\end{tikzcd}$$

By the universal property of $\bar{W}_f$ we obtain a unique 1-cell $r_f : A \rightsquigarrow \bar{W}_f$ satisfying

$$p_f.r_f = 1,\ q_f.r_f = f \text{ and } \lambda_f.r_f = 1 \tag{4.1}$$

as expressed in the equality of pasting diagrams above. Since p_f and q_f jointly detect tightness we also have that

$$r_f \text{ is tight just when } f \text{ is.} \tag{4.2}$$

Proposition 14. *Given $f : A \rightsquigarrow B$ as above we have a w-reflection $(1, p_f \dashv r_f, \eta_f)$ where $\eta_f : 1 \Rightarrow r_f p_f$ is the unique 2-cell satisfying*

$$p_f.\eta_f = 1 \text{ and } q_f.\eta_f = \lambda_f. \tag{4.3}$$

Proof. Let us consider firstly the case $w = l$. We need to give a unit $\eta_f : 1 \Rightarrow r_f.p_f$. To give such a 2-cell is, by the 2-dimensional universal property of C_f, equally to give 2-cells $\theta_1 : p_f.(1) \Rightarrow p_f.(r_f.p_f)$ and $\theta_2 : q_f.(1) \Rightarrow q_f.(r_f.p_f)$ satisfying $\theta_1 \circ \lambda_f = (\lambda_f.r_f.p_f) \circ \theta_2$. We take θ_1 to be the identity and θ_2 to be $\lambda_f : q_f \Rightarrow f.p_f$; the required equality involving θ_1 and θ_2 is then the assertion that λ_f equals itself. We thus obtain a unique $\eta_f : 1 \Rightarrow r_f.p_f$ such $p_f.\eta_f = 1$ and $q_f.\eta_f = \lambda_f$. If the identity 2-cell $p_f.q_f = 1$ is to be the counit of the adjunction then the triangle equations become $p_f.\eta_f = 1$ and $\eta_f.r_f = 1$. So it remains to check that $\eta_f.r_f = 1$ for which it suffices, again by the 2-dimensional universal property of C_f, to show that $p_f.\eta_f.r_f = 1$ and $q_f.\eta_f.r_f = 1$. The first of these holds since $p_f.\eta_f = 1$; the second since $q_f.\eta_f = \lambda_f$ and $\lambda_f.r_f = 1$.
The case $w = p$ is essentially identical – the key point is that the 2-cells $\theta_1 = 1$ and $\theta_2 = \lambda_f$ used above to construct η_f are now both invertible. That η_f is itself invertible follows from the fact that p_f and q_f are jointly conservative – this conservativity follows from the 2-dimensional universal property of P_f. □

The above constructions have their genesis in the proof of Theorem 4.2 of [3], in which $\mathcal{F}$-categorical aspects of pseudolimits of arrows in T-Alg$_p$ were used to study establish properties of pseudomorphism classifiers. If we ignore $\mathcal{F}$-categorical aspects then the above constructions and resulting factorisations $f = q_f.r_f$ have appeared in other contexts too. In the pseudolimit case the factorisation is the (trivial cofibration, fibration)-factorisation of the natural model structure on a 2-category [14]. In Cat the factorisation $q_f.r_f : A \to C_f \to B$ of a functor f through its colax limit coincides with its factorisation $A \to B/f \to B$ through the comma category B/f – this is the factorisation (Lf, Rf) of a natural weak factorisation system on Cat described in [5].

Let us return to $f : A \rightsquigarrow B$ as in Proposition 14. By that result we have a span of tight morphisms (*a tight span*) as below:

$$A \xleftarrow{p_f \dashv r_f} \bar{W}_f \xrightarrow{q_f} B$$

whose left leg p_f is equipped with the structure of a w-reflection $(1, p_f \dashv r_f, \eta_f)$. More generally let us use the term *w-span* to refer to a tight span equipped with the structure of a w-reflection on its left leg. Now let $\bar{W}_f : A \nrightarrow B$ denote the w-span just described. Given $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ we are going to show that $\bar{W}_f : A \nrightarrow B$ and $\bar{W}_g : B \nrightarrow C$ can be composed to give a w-span $\bar{W}_g\bar{W}_f : A \nrightarrow C$ and furthermore we will study the relationship between $\bar{W}_g\bar{W}_f$ and $\bar{W}_{gf}$.
In order to consider composition of such spans we will need to consider *tight pullbacks*. Given tight morphisms $f : A \to C$ and $g : B \to C$ in $\mathbb{A}$ the tight pullback D of f and g

$$\begin{tikzcd} D \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

is the pullback in the 2-category $\mathcal{A}_\lambda$ with, moreover, both projections p and q tight and jointly detecting tightness. This is equally to say that D is a pullback in $\mathcal{A}_\tau$ which is preserved by the inclusion $j : \mathcal{A}_\tau \to \mathcal{A}_\lambda$.

Lemma 15. *Let $w \in \{l, p\}$ and $\mathbb{A}$ admit $\bar{w}$-limits of loose morphisms. At $f : A \rightsquigarrow B$ consider the induced tight projection $p_f : \bar{W}_f \to A$ from the limit. The tight pullback of p_f along any tight morphism $g : C \to A$ exists.*

Proof. In fact the tight pullback is given by $\bar{W}_{fg}$, the w-limit of the composite $fg : C \rightsquigarrow A$. For observe that by the universal property of $\bar{W}_f$ the $\overline{w}$-cone $(g.p_{fg}, \lambda_{fg}, q_{fg})$ induces a unique tight map $t : \bar{W}_{fg} \to \bar{W}_g$ such that the left square below commutes

$$\begin{tikzcd} \bar{W}_{fg} \arrow[r, "t"] \arrow[d, "p_{fg}"'] & \bar{W}_f \arrow[r, "q_f"] \arrow[d, "p_f"'] \arrow[dr, phantom, "\lambda_f \swarrow"] & B \arrow[d, "1"] \\ C \arrow[r, "g"'] & A \arrow[r, squiggly, "f"'] & B \end{tikzcd}$$

and such that $\lambda_f.t = \lambda_{gf}$. Now the universal property of $\bar{W}_{fg}$ implies that the left square is a tight pullback. $\square$

Lemma 16. *For $w \in \{l, p\}$ consider a tight pullback square*

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f_1"'] & B \arrow[d, "f_2"] \\ C \arrow[r, "s"'] & D \end{tikzcd}$$

and w-reflection $(1, f_2 \dashv g_2, \eta_2)$. There exists a unique w-reflection $(1, f_1 \dashv g_1, \eta_1)$ such that $(r, s) : f_1 \to f_2$ is a morphism of w-reflections.

Proof. First suppose that $w = l$. Then the left square below

$$\begin{tikzcd} C \arrow[r, squiggly, "g_2.s"] \arrow[d, "1"'] & B \arrow[d, "f_2"] \\ C \arrow[r, "s"'] & D \end{tikzcd} \qquad\qquad \begin{tikzcd} C \arrow[r, "s"] \arrow[d, squiggly, "g_1"'] & D \arrow[d, squiggly, "g_2"] \\ A \arrow[r, "r"'] & B \end{tikzcd}$$

commutes and so induces a unique loose morphism $g_1 : C \rightsquigarrow A$ such that $f_1.g_1 = 1$ and such that the right square commutes. These necessary commutativities will ensure the claimed uniqueness. By the 2-dimensional universal property of the pullback there exists a unique 2-cell $\eta_1 : 1 \Rightarrow g_1.f_1$ such that $r.\eta_1 = \eta_2.r$ and $f_1.\eta_1 = 1$. The other triangle equation $\eta_1.f_1 = 1$ also follows from the universal property of the pullback. Since (r, s) commutes with both adjoints and the units it is a morphism of l-reflections. Note that if η_2 is invertible then, since the pullback projections are jointly conservative, so too is η_1. This gives the case $w = p$. $\square$

Now given $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ we can form the composite span

$$\begin{tikzcd} & & \bar{W}_g\bar{W}_f \arrow[dl, "p_{g,f} \dashv r_{g,f}"'] \arrow[dr, "q_{g,f}"] & & \\ & \bar{W}_f \arrow[dl, "p_f \dashv r_f"'] \arrow[dr, "q_f"] & & \bar{W}_g \arrow[dl, "p_g \dashv r_g"'] \arrow[dr, "q_g"] & \\ A & & B & & C \end{tikzcd}$$

in which the central square is the tight pullback of p_g along q_f. (This pullback exists by Lemma 15). By Lemma 16 there exists a unique w-reflection $(1, p_{g,f} \dashv r_{g,f}, \eta_{g,f})$ such that

$$(q_{g,f}, q_f) : p_{g,f} \to p_g \text{ is a morphism of } w\text{-reflections.} \tag{4.4}$$

We can then compose the w-reflections $p_f \dashv r_f$ and $p_{g,f} \dashv r_{g,f}$ to obtain another w-reflection $p_f.p_{g,f} \dashv r_{g,f}.r_f$, so that the outer span becomes a w-span $\bar{W}_g\bar{W}_f : A \twoheadrightarrow C$.

Let us consider the relationship between $\bar{W}_g\bar{W}_f$ and $\bar{W}_{gf}$. By the universal property of $\bar{W}_{gf}$ the $\overline{w}$-cone left below

$$
\begin{tikzcd}
 & & \bar{W}_g\bar{W}_f \arrow[dl, "p_{g,f}"'] \arrow[dr, "q_{g,f}"] & & \\
 & \bar{W}_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\lambda_f \Downarrow"] & & \bar{W}_g \arrow[dl, "p_g"'] \arrow[dr, "q_g"] \arrow[d, phantom, "\lambda_g \Downarrow"] & \\
A \arrow[rr, squiggly, "f"'] & & B \arrow[rr, squiggly, "g"'] & & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & \bar{W}_g\bar{W}_f \arrow[dl, "p_{g,f}"'] \arrow[d, "k_{g,f}"] \arrow[dr, "q_{g,f}"] & & \\
 & \bar{W}_f \arrow[dl, "p_f"'] & \bar{W}_{gf} \arrow[dll, "p_{gf}"] \arrow[drr, "q_{gf}"'] \arrow[d, phantom, "\lambda_{gf} \Downarrow"] & \bar{W}_g \arrow[dr, "q_g"] & \\
A \arrow[rr, squiggly, "f"'] & & B \arrow[rr, squiggly, "g"'] & & C
\end{tikzcd}
$$

induces a unique tight arrow $k_{g,f} : \bar{W}_g\bar{W}_f \to \bar{W}_{gf}$ satisfying the equations

$$p_{gf}.k_{g,f} = p_f.p_{g,f},\ q_{gf}.k_{g,f} = q_g.q_{g,f} \text{ and } \lambda_{gf}.k_{g,f} = (g.\lambda_f.p_{g,f}) \circ (\lambda_g.q_{g,f}) \tag{4.5}$$

equally expressed in the equality of pasting diagrams above. Furthermore

Lemma 17. *Let $w \in \{l, p\}$. In the span map*

$$
\begin{tikzcd}
A \arrow[d, "1"] & \bar{W}_g\bar{W}_f \arrow[l, "p_f.p_{g,f}"'] \arrow[r, "q_g.q_{g,f}"] \arrow[d, "k_{g,f}"] & B \arrow[d, "1"] \\
A & \bar{W}_{gf} \arrow[l, "p_{gf}"'] \arrow[r, "q_{gf}"] & B
\end{tikzcd}
$$

the commuting square $(k_{g,f}, 1) : p_f.p_{g,f} \to p_{gf}$ is a morphism of w-reflections.

Proof. To show that $(k_{g,f}, 1) : p_f.p_{g,f} \to p_{gf}$ is a morphism of w-reflections it suffices, by Lemma 6, to show that the mate of this square is an identity. Because the w-reflection $p_f.p_{g,f} \dashv r_{g,f}.r_f$ has identity counit the mate of $(k_{g,f}, 1)$ is simply

$$
\begin{tikzcd}
 & \bar{W}_g\bar{W}_f \arrow[r, "k_{g,f}"] \arrow[d, "p_f.p_{g,f}"] & \bar{W}_{gf} \arrow[r, "1"] \arrow[d, "p_{gf}"', "\eta_{gf}"] \arrow[dr, phantom, "\Searrow"] & \bar{W}_{gf} \\
A \arrow[r, "1"'] \arrow[ur, squiggly, "r_{g,f}.r_f"] & A \arrow[r, "1"'] & A \arrow[ur, squiggly, "r_{gf}"'] &
\end{tikzcd}
$$

Now the 2-dimensional universal property of $\bar{W}_{gf}$ implies that the projections p_{gf} and q_{gf} jointly reflect identity 2-cells: see Lemma 3.1 of [13] in the case of the colax limit. Therefore it suffices to show that the composite of the above 2-cell with both p_{gf} and q_{gf} yields an identity. One of the triangle equations for $p_{gf} \dashv r_{gf}$ gives $p_{gf}.\eta_{gf}{=}1$; thus it remains to show $q_{gf}.\eta_{gf}.k_{g,f}.r_{g,f}.r_f = 1$. By (4.3) this equals $\lambda_{gf}.k_{g,f}.r_{g,f}.r_f$ and by definition of $k_{g,f}$ (as in (4.5)) we have $\lambda_{gf}.k_{g,f} = (g.\lambda_f.p_{g,f}) \circ (\lambda_g.q_{g,f})$. Therefore it suffices to show that the 2-cells $(g.\lambda_f.p_{g,f}).r_{g,f}.r_f$ and $(\lambda_g.q_{g,f}).r_{g,f}.r_f$ are identities separately. With regards the former we have that $\lambda_f.p_{g,f}.r_{g,f}.r_f = \lambda_f.r_f = 1$ where we first use that $p_{g,f}.r_{g,f} = 1$ and then (4.1); for the other composite we have $(\lambda_g.q_{g,f}).r_{g,f}.r_f = \lambda_g.r_g.q_f.r_f = 1$. The first equation holds by (4.4) and the second equation by (4.1). $\square$

Remark 18. Given any $\mathcal{F}$-category $\mathbb{A}$ one can define a bicategory $Span_w(\mathbb{A})$ of w-spans in $\mathbb{A}$. Their composition extends that described for w-spans of the form $\bar{W}_f : A \nrightarrow B$ above. To ensure composites exist one allows only those w-spans whose left legs admit tight pullbacks along arbitrary tight maps. Now when $\mathbb{A}$ admits w-limits of loose morphisms the assigment of $\bar{W}_f : A \nrightarrow B$ to f can be extended to an identity on objects lax functor from the underlying category of $\mathcal{A}_\lambda$ to $Span_w(\mathbb{A})$ with the span map $k_{g,f} : \bar{W}_g\bar{W}_f \to \bar{W}_{gf}$ describing one of the coherence constraints.

4.3. **Representing 2-cells via span transformations.** Lastly we consider how to represent a 2-cell $\alpha : f \Rightarrow g$ by a span map $\bar{W}_f \to \bar{W}_g$. Here the cases $w = l$ and $w = p$ diverge.

4.3.1. *The case w=l.* Suppose that $\mathbb{A}$ admits colax limits of loose morphisms. Given $\alpha : f \Rightarrow g$ the colax cone left below

$$\begin{tikzcd}
 & C_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\lambda_f\Downarrow"] & \\
A \arrow[rr, squiggly, bend left, "f"] \arrow[rr, squiggly, bend right, "g"'] \arrow[rr, phantom, "\alpha\Downarrow"] & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & C_f \arrow[d, "c_\alpha"] \arrow[ddl, bend right, "p_f"'] \arrow[ddr, bend left, "q_f"] & \\
 & C_g \arrow[dl, "p_g"'] \arrow[dr, "q_g"] \arrow[d, phantom, "\lambda_g\Downarrow"] & \\
A \arrow[rr, squiggly, "g"'] & & B
\end{tikzcd}$$

induces a unique *tight* arrow $c_\alpha : C_f \to C_g$ satisfying

$$p_f = c_\alpha.p_g, \; q_f = c_\alpha.q_g \text{ and } (\alpha.p_f) \circ \lambda_f = \lambda_g.c_\alpha \tag{4.6}$$

In particular the 2-cell α is represented by a span map $c_\alpha : C_f \to C_g$ as below.

$$\begin{tikzcd}
A \arrow[d, "1"] & C_f \arrow[l, "p_f"'] \arrow[r, "q_f"] \arrow[d, "c_\alpha"] & B \arrow[d, "1"] \\
A & C_g \arrow[l, "p_g"'] \arrow[r, "q_g"] & B
\end{tikzcd}$$

Lemma 19. *Let $c_\alpha : C_f \to C_g$ be as above and let m_α denote the mate of the square $(c_\alpha, 1) : p_f \to p_g$ through the adjunctions $p_f \dashv r_f$ and $p_g \dashv r_g$.*

$$\begin{tikzcd}
A \arrow[r, squiggly, "r_f"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow m_\alpha"] & C_f \arrow[r, "q_f"] \arrow[d, "c_\alpha"] & B \arrow[d, "1"] \\
A \arrow[r, squiggly, "r_g"'] & C_g \arrow[r, "q_g"'] & B
\end{tikzcd}$$

The composite 2-cell $q_g.m_\alpha$ equals α.

Proof. Because the counit of $p_f \dashv r_f$ is an identity the mate m_α is simply given by

$$\begin{tikzcd}
A \arrow[r, squiggly, "r_f"] \arrow[dd, "1"'] & C_f \arrow[d, "c_\alpha"] \arrow[ddl, "p_f"] \\
 & C_g \arrow[d, "1"] \arrow[dl, "p_g"'] \\
A \arrow[d, "1"'] \arrow[r, phantom, "\overset{\eta_g}{\Swarrow}"] & C_g \\
A \arrow[ur, squiggly, "r_g"'] &
\end{tikzcd}$$

Therefore $q_g.m_\alpha = q_g.\eta_g.c_\alpha.r_f = \lambda_g.c_\alpha.r_f = (\alpha.p_f.r_f) \circ (\lambda_f.r_f) = \alpha \circ 1 = \alpha$ where the second, third and fourth equalities use (4.3), (4.6) and (4.3) respectively. □

4.3.2. *The case w=p.* Suppose that $\mathbb{A}$ admits pseudolimits of loose morphisms. If $\alpha : f \Rightarrow g$ is *invertible* then we may construct a map $P_f \to P_g$ in essentially the same way as for colax limits of loose morphisms. However this approach does not work in general. The following lemma describes a representation that works for non-invertible 2-cells. It is based upon the notion of a *transformation* of *anafunctors* [19] – see also [21] and [1]. Anafunctors can be viewed as spans of categories and functors in which the left leg is a surjective on objects equivalence. If we take $\mathbb{A} = \mathrm{Cat}$ and functors $F, G : A \to B$ then the associated spans $P_F, P_G : A \nrightarrow B$ are anafunctors. In this setting the 2-cell ρ_α described below specifies precisely a transformation between the anafunctors P_F and P_G.

Lemma 20. *Given $\alpha : f \Rightarrow g$ consider the tight pullback left below (existing by Lemma 15)*

$$\begin{tikzcd}
K_{f,g} \arrow[r, "s_{f,g}"] \arrow[d, "t_{f,g}"'] \arrow[dr, "u_{f,g}"] & P_f \arrow[d, "p_f"] \\
P_g \arrow[r, "p_g"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, squiggly, "v_{f,g}"] \arrow[drr, squiggly, bend left, "r_f"] \arrow[ddr, squiggly, bend right, "r_g"'] & & \\
& K_{f,g} \arrow[r, "s_{f,g}"] \arrow[d, "t_{f,g}"'] \arrow[dr, phantom, "\Swarrow \rho_\alpha"] & P_f \arrow[d, "q_f"] \\
& P_g \arrow[r, "q_g"'] & A
\end{tikzcd}$$

with diagonal denoted $u_{f,g}$.

(1) *There is a unique p-reflection $u_{f,g} \dashv v_{f,g}$ such that $(s_{f,g}, 1) : u_{f,g} \to p_f$ and $(t_{f,g}, 1) : u_{f,g} \to p_g$ are morphisms of p-reflections. If f and g are tight so too is $v_{f,g}$.*

(2) *There exists a unique 2-cell $\rho_\alpha : q_f.s_{f,g} \Rightarrow q_g.t_{f,g}$ such that $\rho_\alpha.v_{f,g} = \alpha$.*

Proof. (1) Consider the diagram

$$\begin{tikzcd}
A \arrow[dr, squiggly, "v_{f,g}"] \arrow[drr, squiggly, bend left, "r_f"] \arrow[ddr, squiggly, bend right, "r_g"'] & & \\
& K_{f,g} \arrow[r, "s_{f,g}"'] \arrow[d, "t_{f,g}"] & P_f \arrow[d, "p_f"] \\
& P_g \arrow[r, "p_g"'] & A
\end{tikzcd}$$

in which $v_{f,g}$ is the unique loose map satisfying $s_{f,g}.v_{f,g} = r_f$ and $t_{f,g}.v_{f,g} = r_g$. It follows that $u_{f,g}.v_{f,g} = 1$. To give an invertible 2-cell $\theta_{f,g} : 1 \cong v_{f,g}.u_{f,g}$ is, by the universal property of $K_{f,g}$, equally to give invertible 2-cells $\theta_1 : s_{f,g} \cong s_{f,g}.v_{f,g}.u_{f,g}$ and $\theta_2 : t_{f,g} \cong t_{f,g}.v_{f,g}.u_{f,g}$ satisfying $p_f.\theta_1 = p_g.\theta_2$. We set $\theta_1 = \eta_f.s_{f,g}$ and $\theta_2 = \eta_g.t_{f,g}$ noting that $p_f.\eta_f.s_{f,g} = 1 = p_g.\eta_g.s_{f,g}$. The triangle equations for the p-reflection follow using the universal property of the pullback $K_{f,g}$ and that $s_{f,g} : u_{f,g} \to p_f$ and $t_{f,g} : u_{f,g} \to p_g$ are morphisms of p-reflections follows from the construction of $u_{f,g}$ and $\theta_{f,g}$.

If f and g are tight so too, by (4.2), are r_f and r_g. By the universal property of the tight pullback $K_{f,g}$ it then follows that $v_{f,g}$ is tight.

(2) From the first part we have $s_{f,g}.v_{f,g} = r_f$ and $t_{f,g}.v_{f,g} = r_g$ as in the two triangles above. Now by (4.3) we have $q_f.r_f = f$ and $q_g.r_g = g$. Therefore we can write $\alpha : (q_f.s_{f,g}).v_{f,g} \Rightarrow (q_g.t_{f,g}).v_{f,g}$. Since $v_{f,g} : A \rightsquigarrow K_{f,g}$ is an equivalence in $\mathcal{A}_\lambda$ the functor $\mathcal{A}_\lambda(v_{f,g}, A) : \mathcal{A}_\lambda(K_{f,g}, A) \to \mathcal{A}_\lambda(A, A)$ is an equivalence of categories – using its fully faithfulness we obtain $\rho_\alpha : q_f.s_{f,g} \Rightarrow q_g.t_{f,g}$. □

5. Orthogonality

The following theorem is the crucial result of the paper. The monadicity theorems of Section 6 follow easily from it. We note that both this theorem and the corollary that follows it are independent of the formalism of 2-monads.

Theorem 21. *Let $w \in \{l, p, c\}$. Consider an $\mathcal{F}$-category $\mathbb{A}$ with $\bar{w}$-limits of loose morphisms. Then the inclusion of tight morphisms $j : \mathcal{A}_\tau \to \mathbb{A}$ is orthogonal to each w-doctrinal $\mathcal{F}$-functor.*

Proof. Consider a commuting square in $\mathcal{F}$-CAT

$$\begin{tikzcd} \mathcal{A}_\tau \arrow[r, "j"] \arrow[d, "R"'] & \mathbb{A} \arrow[d, "S"] \arrow[dl, dotted, "K"] \\ \mathbb{B} \arrow[r, "H"'] & \mathbb{C} \end{tikzcd}$$

in which H is w-doctrinal. We must show there exists a unique diagonal filler K. We begin by noting that the cases $w = c$ and $w = l$ are dual since $\mathbb{A}$ satisfies the c-criteria of the theorem just when $\mathbb{A}^{co}$ satisfies the l-criteria with, equally, H c-doctrinal just when H^{co} is l-doctrinal. Therefore it will suffice to suppose $w \in \{l, p\}$.

(1) Before constructing the diagonal we fix some notation and make some observations about lifted adjunctions that will be repeatedly used in what follows. Given a w-reflection $(1, f \dashv g, \eta) \in \mathbb{A}$ we obtain a w-reflection $(1, Sf \dashv Sg, S\eta)$ in $\mathbb{C}$ with $Sf = Hf)$ since f is tight. As H is w-doctrinal this lifts uniquely along H to a w-reflection in $\mathbb{B}$ which we denote by $(1, Rf \dashv \overline{g}, \overline{\eta})$.
Next consider w-reflections $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ in $\mathbb{A}$ and a tight commuting square $(r, s) : f_1 \to f_2$ in $\mathbb{A}$ as left below

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f_1"'] & C \arrow[d, "f_2"] \\ B \arrow[r, "s"'] & D \end{tikzcd} \qquad \begin{tikzcd} RA \arrow[r, "Rr"] \arrow[d, "Rf_1"'] & RC \arrow[d, "Rf_2"] \\ RB \arrow[r, "Rs"'] & RD \end{tikzcd} \qquad \begin{tikzcd} RB \arrow[r, "Rs"] \arrow[d, squiggly, "\overline{g_1}"'] & RD \arrow[d, squiggly, "\overline{g_2}"] \\ RA \arrow[r, "Rr"'] & RC \end{tikzcd}$$

Since $\mathcal{F}$-functors preserves morphisms of w-reflections the square $(Sr, Ss) : Sf_1 \to Sf_2$ is one in $\mathbb{C}$. Now the tight commutative square $(Rr, Rs) : Rf_1 \to Rf_2$ has image under H the morphism of w-reflections $(Sr, Ss) : Sf_1 \to Sf_2$. Because H is w-doctrinal it follows that $(Rr, Rs) : Rf_1 \to Rf_2$ is a morphism between the lifted w-reflections in $\mathbb{B}$. In particular we obtain a commuting square of right adjoints $(Rs, Rr) : \overline{g_1} \to \overline{g_2}$.
Finally consider a composable pair of tight left adjoints $f_1 : A \to B$ and

$f_2 : B \to C$ with associated w-reflections $(1, f_1 \dashv g_1, \eta_1)$ and $(1, f_2 \dashv g_2, \eta_2)$ in $\mathbb{A}$. We can form the w-reflections $(1, Rf_1 \dashv \overline{g_1}, \overline{\eta_1})$ and $(1, Rf_2 \dashv \overline{g_2}, \overline{\eta_2})$ in $\mathbb{B}$ and compose these to obtain a further w-reflection $f_2.f_1 \dashv \overline{g_1}.\overline{g_2}$ in $\mathbb{B}$. It is clear that this is a lifting of the w-reflection $S(f_2.f_1) \dashv S(g_1.g_2)$ so that, by uniqueness of liftings, the w-reflections $Rf_2.Rf_1 \dashv \overline{g_1}.\overline{g_2}$ and $R(f_2.f_1) \dashv \overline{g_1.g_2}$ coincide.

(2) Now to begin constructing K observe that for the left triangle to commute we must define $KA = RA$ for each $A \in \mathbb{A}$.

(3) Given $f : A \rightsquigarrow B \in \mathbb{A}$ we recall from (4.1) its factorisation as $q_f.r_f : A \rightsquigarrow C_f \to B$ where $(1, p_f \dashv r_f, \eta_f)$. Since p_f is tight we have the lifted w-reflection $(1, Rp_f \dashv \overline{r_f}, \overline{\eta}_f)$ in $\mathbb{B}$ living over $(1, Sp_f \dashv Sr_f, S\eta_f)$. We define $Kf : RA \rightsquigarrow RB$ as the composite $Rq_f.\overline{r}_f : RA \rightsquigarrow R\bar{W}_f \to RB$.

(4) Observe that $HKf = HRq_f.H\overline{r}_f = Sq_f.Sr_f = S(q_f.r_f) = Sf$ as required. To see that K extends R observe that if $f : A \to B$ is tight then, by (4.2), r_f is tight too, so that we have a w-reflection $(1, Rp_f \dashv Rr_f, R\eta_f)$ living over $(1, HRp_f \dashv Sr_f, \eta_f)$; thus $(1, Rp_f \dashv Rr_f, R\eta_f) = (1, Rp_f \dashv \overline{r_f}, \overline{\eta_f})$ so that $Kf = Rq_f.Rr_f = Rf$. Thus K coincides with R on tight morphisms.

(5) As K agrees with R on tight morphisms we already know that it preserves identity 1-cells. To see that it preserves composition of 1-cells it will suffice to show that all of the regions of the diagram below commute.

$$\begin{tikzcd}
 & & R\bar{W}_{gf} \arrow[ddrr, "Rq_{gf}"] & & \\
 & & R\bar{W}_{g,f} \arrow[u, "Rk_{g,f}"] \arrow[dr, "Rq_{g,f}"] & & \\
 & R\bar{W}_f \arrow[ur, squiggly, "\overline{r}_{g,f}"] \arrow[dr, "Rq_f"'] & & R\bar{W}_g \arrow[dr, "Rq_g"'] & \\
RA \arrow[ur, squiggly, "\overline{r}_f"'] \arrow[uuurr, squiggly, "\overline{r}_{gf}"] & & RB \arrow[ur, squiggly, "\overline{r}_g"'] & & RC
\end{tikzcd}$$

The rightmost quadrilateral certainly commutes as it is the image of a commutative diagram in $\mathcal{A}_\tau$, from (4.5), under R. To see that the central square commutes recall from (4.4) the morphism of w-reflections $(q_{g,f}, q_f) : p_{g,f} \to p_g$ in $\mathbb{A}$. By Part 1 $(Rq_{g,f}, Rq_f) : Rp_{g,f} \to p_{g,f}$ is a morphism of w-reflections in $\mathbb{B}$: now commutativity of the central square simply asserts commutativity with the right adjoints. With regards the leftmost quadrilateral recall from Lemma 17 the morphism of w-reflections in $\mathbb{A}$ given by $(k_{g,f}, 1) : p_f.p_{g,f} \to p_{gf}$. By Part 1 we know that $(Rk_{g,f}, 1) : R(p_f.p_{g,f}) \to Rp_{gf}$ is a morphism of the lifted w-reflections so that, in particular, we have a commuting square of right adjoints $(1, Rk_{g,f}) : \overline{r_{g,f}.r_f} \to \overline{r}_{gf}$ in $\mathbb{B}$. Again by Part 1 we have $\overline{r_{g,f}.r_f} = \overline{r}_{g,f}\overline{r}_f$ and therefore the desired commutativity.

(6) We define K differently on 2-cells depending upon whether $w = l$ or $w = p$. Consider $\alpha : f \Rightarrow g$ and the case $w = l$. The commuting square $(c_\alpha, 1) : p_f \to p_g$ of Lemma 19 has image $(Rc_\alpha, 1) : Rp_f \to Rp_g$. We denote the mate of this

square through the adjunctions $Rp_f \dashv \overline{r}_f$ and $Rp_g \dashv \overline{r}_g$ by $\overline{m}_\alpha : Rc_\alpha.\overline{r}_f \Rightarrow \overline{r}_g$.

$$\begin{tikzcd}
RA \arrow[r, rightsquigarrow, "\overline{r}_f"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow\overline{m}_\alpha"] & RC_f \arrow[r, "Rq_f"] \arrow[d, "Rc_\alpha"] & B \arrow[d, "1"] \\
RA \arrow[r, rightsquigarrow, "\overline{r}_g"'] & RC_g \arrow[r, "Rq_g"'] & RB
\end{tikzcd}$$

Now we set $K\alpha = Rq_g.\overline{m}_\alpha$ as depicted above. Note that we have $\overline{m}_\alpha = \overline{\eta}_g.Rc_\alpha.\overline{r}_f$ since the counit of the l-reflection $Rp_f \dashv \overline{r}_f$ is an identity.
Let us show that $HK\alpha = S\alpha$. We have from Lemma 19 the decomposition in $\mathbb{A}$ of α as $q_g.m_\alpha = q_g.\eta_g.c_\alpha.r_f$ and also have $K\alpha = Rq_g.\overline{\eta}_g.Rc_\alpha.\overline{r}_f$ from above. Thus $HK\alpha = Sq_g.S\eta_g.Sc_\alpha.Sr_f = S(q_g.\eta_g.c_\alpha.r_f) = S\alpha$.
To see that K extends R observe that if both f and g are tight then by Part 4 we have $(1, Rp_f \dashv \overline{r}_f, \overline{\eta}_f) = (1, Rp_f \dashv Rr_f, R\eta_f)$ and likewise for g. Using the same decomposition of α we have $R\alpha = Rq_g.R\eta_g.Rc_\alpha.Rr_f = K\alpha$.
For $w = p$ consider the p-reflection $u_{f,g} \dashv v_{f,g}$ and the morphisms of p-reflections $(s_{f,g}, 1) : u_{f,g} \to p_f$ and $(t_{f,g}, 1) : u_{f,g} \to p_g$ of Lemma 20. By Part 1 their images $(Rs_{f,g}, 1) : Ru_{f,g} \to Rp_f$ and $(Rt_{f,g}, 1) : Ru_{f,g} \to Rp_g$ are morphisms of p-reflections in $\mathbb{B}$ so that the two triangles in the diagram below

$$\begin{tikzcd}
RA \arrow[dr, rightsquigarrow, "\overline{v}_{f,g}"] \arrow[drr, rightsquigarrow, bend left, "\overline{r}_f"] \arrow[ddr, rightsquigarrow, bend right, "\overline{r}_g"'] & & \\
& RK_{f,g} \arrow[r, "Rs_{f,g}"] \arrow[d, "Rt_{f,g}"'] \arrow[dr, phantom, "\swarrow R\rho_\alpha"] & RP_f \arrow[d, "Rq_f"] \\
& RP_g \arrow[r, "Rq_g"'] & A
\end{tikzcd}$$

commute. We set $K\alpha = R\rho_\alpha.\overline{v}_{f,g}$.
By Lemma 20 we have that $\alpha = \rho_\alpha.v_{f,g}$. Therefore $S\alpha = S\rho_\alpha.Sv_{f,g} = HR\rho_\alpha.H\overline{v}_{f,g} = HK\alpha$. If now both f and g are tight then so is $v_{f,g}$, again by Lemma 20. It follows that the p-reflections $Ru_{f,g} \dashv \overline{v}_{f,g}$ and $Ru_{f,g} \dashv Rv_{f,g}$ coincide, and therefore that $R\alpha = R\rho_\alpha.Rv_{f,g} = K\alpha$.

(7) Treating the cases $w = l$ and $w = p$ together again observe that because H is locally faithful the functoriality of K on 2-cells trivally follows from the functoriality of $HK = S$ on 2-cells. Thus K is an $\mathcal{F}$-functor.

(8) For uniqueness let us observe that the definition of the filler is forced upon us by the constraints. Consider any diagonal filler L. We must have $Lf = L(q_f.r_f) = Lq_f.Lr_f = Rq_f.Lr_f$. Since $HL = S$ we would need that the image of the adjunction $(1, p_f \dashv r_f, \eta_f)$ under L were a lifting of its image $(1, Sp_f \dashv Sr_f, S\eta_f)$ under HL. But by uniqueness of liftings we would then have $(1, Lp_f \dashv Lr_f, L\eta_f) = (1, Rp_f \dashv \overline{r_f}, \overline{\eta_f})$ so that $Lf = Rq_f.\overline{r}_f$. Now for $w = l$ we have, by Lemma 19, the decomposition of $\alpha : f \Rightarrow g$ as $q_g.m_\alpha = q_g.\eta_g.c_\alpha.r_f$ and so must have $L\alpha = Rq_g.\overline{\eta}_g.Rc_\alpha.\overline{r}_f$. The argument when $w = p$ is similar but uses the decomposition $\alpha = \rho_\alpha.v_{f,g}$ of Lemma 20 instead. □

Let us note that Theorem 21 remains true even if we remove the assumption that w-doctrinal $\mathcal{F}$-functors are locally faithful. The only place that we used this assumption was in establishing the functoriality of the diagonal K on 2-cells. This can alternatively be established by carefully analysing the functoriality of the assignment in Section 4.3 which represents a 2-cell by a span transformation. However the proof becomes significantly longer and more technical, very significantly when $w = p$. Since local faithfulness is true of those $\mathcal{F}$-functors $U : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{C}$ that we seek to characterise (see Theorem 25) and because the property is easily verified in practice, we have chosen to include it in our definition of w-doctrinal $\mathcal{F}$-functor.

As an immediate consequence of Theorem 21 we have:

Corollary 22. *Let $w \in \{l, p, c\}$. Consider an $\mathcal{F}$-functor $H : \mathbb{A} \to \mathcal{B}$ to a 2-category and suppose that $\mathbb{A}$ has $\bar{w}$-limits of loose morphisms and that H is w-doctrinal. Then the decomposition in* $\mathcal{F}$-CAT

$$\mathcal{A}_\tau \xrightarrow{j} \mathbb{A} \xrightarrow{H} \mathcal{B}$$

of the 2-functor $H_\tau : \mathcal{A}_\tau \to \mathcal{B}$ is an orthogonal $({}^\perp w\text{-Doct}, w\text{-Doct})$*-decomposition.*

Since orthogonal decompositions are essentially unique, this asserts that an $\mathcal{F}$-category satisfying some completeness properties and sitting over a base 2-category in a certain way – such as $\mathbb{M}\text{onCat}_w$ or $\text{T-}\mathbb{A}\text{lg}_w$ over Cat – is *uniquely determined* by how its tight part sits over the base. This is the core idea behind our monadicity results of Section 6.

5.1. **A note on alternative hypotheses.** There are other hypotheses upon $\mathbb{A}$ which ensure that $j : \mathcal{A}_\tau \to \mathbb{A}$ belongs to ${}^\perp w$-Doct. Let us focus on the lax case. Say that $\mathbb{A}$ admits *loose morphism classifiers* if the inclusion $j : \mathcal{A}_\tau \to \mathcal{A}_\lambda$ has a left 2-adjoint Q and write $p_A : A \rightsquigarrow QA$ and $q_A : QA \to A$ for the unit and counit. Each loose $f : A \rightsquigarrow B$ corresponds to a tight morphism $f' : QA \to B$ such that $f'.p_A = f$. One triangle equation gives $q_A.p_A = 1$; if for each A this happens to be the counit of an l-reflection $q_A \dashv p_A$ then each f is represented by an l-span

$$A \xleftarrow{q_A \dashv p_A} QA \xrightarrow{f'} B$$

and it can be easily shown that $j : \mathcal{A}_\tau \to \mathbb{A} \in^\perp l$-Doct. Of course these hypotheses are strong: it is not easy to check whether a given $\mathcal{F}$-category admits loose morphism classifiers.

6. Monadicity

In this section we give our monadicity theorems. We begin by extending Eilenberg-Moore comparison 2-functors to $\mathcal{F}$-functors. In Theorem 24 we show these comparisons to be natural in w, in the sense of Diagram 2 of the Introduction. Our main result on monadicity is Theorem 25.

6.1. Extending the Eilenberg-Moore comparison.

Theorem 23. *Let $H : \mathbb{A} \to \mathcal{B}$ be an $\mathcal{F}$-functor to a 2-category whose tight part $H_\tau : \mathcal{A}_\tau \to \mathcal{B}$ has a left adjoint and consider the induced Eilenberg-Moore comparison*

2-functor E left below.

$$\begin{tikzcd} \mathcal{A}_\tau \arrow[rr, "E"] \arrow[dr, "H_\tau"'] & & \text{T-Alg}_\text{s} \arrow[dl, "U_s"] \\ & \mathcal{B} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_\tau \arrow[r, "j"] \arrow[d, "E"'] & \mathbb{A} \arrow[dr, "H"] \arrow[d, dotted, "E_w"'] & \\ \text{T-Alg}_s \arrow[r, "j_w"] & \text{T-}\mathbb{A}\text{lg}_w \arrow[r, "U"'] & \mathcal{B} \end{tikzcd}$$

Let $w \in \{l, p, c\}$ and suppose that $\mathbb{A}$ admits $\overline{w}$-limits of loose morphisms. Then $E : \mathcal{A}_\tau \to \text{T-Alg}_\text{s}$ admits a unique extension to an $\mathcal{F}$-functor $E_w : \mathbb{A} \to \text{T-}\mathbb{A}\text{lg}_w$ over $\mathcal{B}$, as depicted on the right above.

Proof. The commutativity of the outside of the right diagram just says that $U_s E = H_\tau$. By Corollary 9 and Theorem 21 we know that $U : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{B}$ is w-doctrinal and that $j : \mathcal{A}_\tau \to \mathbb{A}$ is orthogonal to each w-doctrinal $\mathcal{F}$-functor, in particular U. Therefore there exists a unique $\mathcal{F}$-functor $E_w : \mathbb{A} \to \text{T-}\mathbb{A}\text{lg}_w$ satisfying the depicted equations $E_w j = j_w E$ and $UE_w = H$. These respectively assert that E_w extends E and lives over the base $\mathcal{B}$. $\square$

In order to understand the naturality in w of the above Eilenberg-Moore extensions $E_w : \mathbb{A} \to \text{T-}\mathbb{A}\text{lg}_w$ it will be convenient to briefly consider $\mathcal{F}_2$-categories. An $\mathcal{F}_2$-category consists of a 2-category equipped with three kinds of morphism: tight, loose and very loose, all satisfying the expected axioms. For instance we have the $\mathcal{F}_2$-category of monoidal categories, strict, strong and lax monoidal functors; likewise of algebras together with strict, pseudo and lax morphisms for a 2-monad. One presentation of an $\mathcal{F}_2$-category is as a triple on the left below

$$\mathcal{A}_\tau \xrightarrow{\ j\ } \mathcal{A}_\lambda \xrightarrow{\ j\ } \mathcal{A}_\phi \qquad\qquad \mathbb{A}_{\tau,\lambda} \xrightarrow{\ j\ } \mathbb{A}_{\tau,\phi}$$

in which each inclusion is the identity on objects, faithful and locally fully faithful. Thus an $\mathcal{F}_2$-category has three associated $\mathcal{F}$-categories $\mathbb{A}_{\tau,\lambda}$, $\mathbb{A}_{\lambda,\phi}$ and $\mathbb{A}_{\tau,\phi}$, and is determined by the first and third of these together with the inclusion $\mathcal{F}$-functor, right above, which views tight and loose morphisms as tight and very loose respectively.

We commonly encounter $\mathcal{F}_2$-categories sitting over a base 2-category as on the left below. See how monoidal categories, strict, strong and lax monoidal functors sit over Cat for instance.

$$\begin{tikzcd} \mathcal{A}_\tau \arrow[r, "j"] \arrow[dr, "H_\tau"'] & \mathcal{A}_\lambda \arrow[r, "j"] \arrow[d, "H_\lambda"] & \mathcal{A}_\phi \arrow[dl, "H_\phi"] \\ & \mathcal{B} & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}_{\tau,\lambda} \arrow[rr, "j"] \arrow[dr, "H_{\tau,\lambda}"'] & & \mathbb{A}_{\tau,\phi} \arrow[dl, "H_{\tau,\phi}"] \\ & \mathcal{B} & \end{tikzcd}$$

To give such a diagram is equally to give a commutative triangle in $\mathcal{F}$-CAT as on the right above – here the $\mathcal{F}$-functors $H_{\tau,\lambda}$ and $H_{\tau,\phi}$ agree as H_τ on tight parts, and have loose parts H_λ and H_ϕ respectively.

Theorem 24. *Let $w \in \{l, c\}$. Consider an $\mathcal{F}_2$-category over a 2-category as below*

$$\begin{tikzcd} \mathbb{A}_{\tau,\lambda} \arrow[rr, "j"] \arrow[dr, "H_{\tau,\lambda}"'] & & \mathbb{A}_{\tau,\phi} \arrow[dl, "H_{\tau,\phi}"] \\ & \mathcal{B} & \end{tikzcd}$$

Suppose that H_τ admits a left adjoint and that $\mathbb{A}_{\tau,\lambda}$ and $\mathbb{A}_{\tau,\phi}$ satisfy the (p/w) variants of the completeness criteria of Theorem 23 so that the Eilenberg-Moore comparison 2-functor $E : \mathcal{A}_\tau \to \text{T-Alg}_\text{s}$ extends uniquely to $\mathcal{F}$-functors $E_p : \mathbb{A}_{\tau,\lambda} \to \text{T-}\mathbb{A}\text{lg}_p$ and $E_w : \mathbb{A}_{\tau,\phi} \to \text{T-}\mathbb{A}\text{lg}_w$ over the base $\mathcal{B}$.
When all of this holds the square

$$\begin{tikzcd} \mathbb{A}_{\tau,\lambda} \arrow[r, "j"] \arrow[d, "E_p"'] & \mathbb{A}_{\tau,\phi} \arrow[d, "E_w"] \\ \text{T-}\mathbb{A}\text{lg}_p \arrow[r, "j"'] & \text{T-}\mathbb{A}\text{lg}_w \end{tikzcd}$$

commutes.

Proof. Consider the diagram left below

$$\begin{tikzcd} \mathcal{A}_\tau \arrow[r, "j"] & \mathbb{A}_{\tau,\lambda} \arrow[r, "j"] \arrow[d, "E_p"'] & \mathbb{A}_{\tau,\phi} \arrow[d, "E_w"] & \\ & \text{T-}\mathbb{A}\text{lg}_p \arrow[r, "j"'] & \text{T-}\mathbb{A}\text{lg}_w \arrow[r, "U_w"'] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_\tau \arrow[r, "j"] \arrow[d, "jE"'] & \mathbb{A}_{\tau,\lambda} \arrow[d, "H_{\tau,\lambda}"] \arrow[dl, dotted] \\ \text{T-}\mathbb{A}\text{lg}_w \arrow[r, "U_w"'] & \mathcal{B} \end{tikzcd}$$

Now E_p and E_w agree as $E : \mathcal{A}_\tau \to \text{T-Alg}_\text{s}$ on tight morphisms. Consequently both paths of the square coincide upon precomposition with $j : \mathcal{A}_\tau \to \mathbb{A}_{\tau,\lambda}$ as the composite $jE : \mathcal{A}_\tau \to \text{T-Alg}_\text{s} \to \text{T-}\mathbb{A}\text{lg}_w$. Because both Eilenberg-Moore extensions E_p and E_w lie over the base $\mathcal{B}$ we find that postcomposing both paths of the square with U_w yields the common composite $H_{\tau,\lambda} : \mathbb{A}_{\tau,\lambda} \to \mathcal{B}$. Therefore both paths of the square are diagonal fillers for the square on the right above. Now by Corollary 9 we know that $U_w : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{B}$ is p-doctrinal. But by Theorem 21 $j : \mathcal{A}_\tau \to \mathbb{A}_{\tau,\lambda}$ is orthogonal to each p-doctrinal $\mathcal{F}$-functor so that both paths coincide as the unique filler. $\square$

6.2. **2-categorical monadicity.** We now turn to monadicity. Let us say that a 2-functor with a left adjoint is *monadic* if the induced Eilenberg-Moore comparison is a 2-equivalence and *strictly monadic* if the comparison is an isomorphism.

Theorem 25. *Let $H : \mathbb{A} \to \mathcal{B}$ be an $\mathcal{F}$-functor to a 2-category $\mathcal{B}$. Let $w \in \{l, p, c\}$ and suppose that*

(1) $H_\tau : \mathcal{A}_\tau \to \mathcal{B}$ is monadic.
(2) $\mathbb{A}$ admits $\overline{w}$-limits of loose morphisms.
(3) $\mathcal{B}$ admits $\overline{w}$-limits of arrows.
(4) H is w-doctrinal (it suffices that H satisfies w-doctrinal adjunction, is locally faithful and reflects identity 2-cells).

Then the 2-equivalence $E : \mathcal{A}_\tau \to \text{T-Alg}_\text{s}$ extends uniquely to an equivalence of $\mathcal{F}$-categories $E_w : \mathbb{A} \to \text{T-}\mathbb{A}\text{lg}_w$ over $\mathcal{B}$. Moreover if H_τ is strictly monadic then $E_w : \mathbb{A} \to \text{T-}\mathbb{A}\text{lg}_w$ is an isomorphism of $\mathcal{F}$-categories.

Proof. As in Theorem 23 we have our extension $E_w : \mathbb{A} \to \text{T-}\mathbb{A}\text{lg}_w$, unique in filling the square

$$\begin{tikzcd}
\mathcal{A}_\tau \arrow[r, "j"] \arrow[d, "E"'] & \mathbb{A} \arrow[dd, "H"] \arrow[ddl, dotted, "E_w"] \\
\text{T-Alg}_\text{s} \arrow[d, "j_w"'] & \\
\text{T-}\mathbb{A}\text{lg}_w \arrow[r, "U"'] & \mathcal{B}
\end{tikzcd}$$

Recall that this filler exists because U is w-doctrinal and j orthogonal to such $\mathcal{F}$-functors.

We begin by proving the theorem in the case of strict monadicity and deduce the general case from that – so suppose that $E : \mathcal{A}_\tau \to \text{T-Alg}_\text{s}$ is an isomorphism of 2-categories. By Proposition 11 $U : \text{T-}\mathbb{A}\text{lg}_w \to \mathcal{B}$ creates $\overline{w}$-limits of loose morphisms so that $\text{T-}\mathbb{A}\text{lg}_w$ admits them. Now Theorem 21 implies that the inclusion $j_w : \text{T-Alg}_\text{s} \to \text{T-}\mathbb{A}\text{lg}_w$ is orthogonal to each w-doctrinal $\mathcal{F}$-functor; since $E : \mathcal{A}_\tau \to \text{T-Alg}_\text{s}$ is an isomorphism $j_w E : \mathcal{A} \to \text{T-}\mathbb{A}\text{lg}_w$ is also orthogonal to w-doctrinal $\mathcal{F}$-functors. Now H is w-doctrinal by assumption so that the two outer paths of the square are orthogonal decompositions of a common $\mathcal{F}$-functor. Therefore the unique filler $E_w : \mathbb{A} \to \text{T-Alg}_w$ is an isomorphism.

Suppose now that $E : \mathcal{A}_\tau \to \text{T-Alg}_\text{s}$ is only an equivalence of 2-categories. The problem now is that E will no longer be orthogonal to w-doctrinal $\mathcal{F}$-functors. We will rectify this by factoring the composite 2-functor $j_w E : \mathcal{A}_\tau \to \text{T-Alg}_\text{s} \to \text{T-Alg}_w$ as the identity on objects followed by 2-fully faithful through a 2-category $\overline{\mathcal{A}}_\lambda$ as in the commutative square below.

$$\begin{tikzcd}
\mathcal{A}_\tau \arrow[r, "\overline{j}"] \arrow[d, "E"'] & \overline{\mathcal{A}}_\lambda \arrow[d, "K"] \\
\text{T-Alg}_\text{s} \arrow[r, "j_w"] & \text{T-Alg}_w
\end{tikzcd}$$

Then K is 2-fully faithful by construction but also essentially surjective on objects since each of E, $\overline{j}$ and j_w are; as such K is a 2-equivalence. Moreover the composite $K\overline{j} = j_w E$ is both faithful and locally fully faithful since both j_w and E are, whilst K is 2-fully faithful. Therefore $\overline{j}$ is faithful and locally fully faithful too and since it is identity on objects by construction it is the inclusion of an $\mathcal{F}$-category $\overline{\mathbb{A}} : \mathcal{A}_\tau \to \overline{\mathcal{A}}_\lambda$. It now follows that the above commutative square, whose vertical legs are 2-equivalences, exhibits $L = (E, K) : \overline{\mathbb{A}} \to \text{T-}\mathbb{A}\text{lg}_w$ as an equivalence of $\mathcal{F}$-categories.

Consider the diagram defining the extension E_w again, now drawn on the left below

$$\begin{tikzcd} \mathcal{A}_\tau \arrow[r, "j"] \arrow[d, "\overline{j}"'] & \mathbb{A} \arrow[dd, "H"] \arrow[dl, dotted, "\overline{E}"'] \arrow[ddl, dotted, "E_w"] \\ \overline{\mathbb{A}} \arrow[d, "L"'] & \\ \text{T-}\mathbb{A}\mathrm{lg}_w \arrow[r, "U"'] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_\tau \arrow[rr, "j"] \arrow[d, "\overline{j}"'] & & \mathbb{A} \arrow[d, "H"] \arrow[dll, dotted, "\overline{E}"] \\ \overline{\mathbb{A}} \arrow[r, "L"'] & \text{T-}\mathbb{A}\mathrm{lg}_w \arrow[r, "U"'] & \mathcal{B} \end{tikzcd}$$

with the left leg rewritten as the composite $L \circ \overline{j}$. Since L is $\mathcal{F}$-fully faithful (*2-fully faithful on tight and loose parts*) it is orthogonal to each bijective on objects $\mathcal{F}$-functor, in particular j, so that we obtain a unique diagonal filler $\overline{E} : \mathbb{A} \to \overline{\mathbb{A}}$ making the two leftmost triangles commute.
Our goal is to show that E_w is an equivalence of $\mathcal{F}$-categories - but since L is an equivalence this is, by 2 out of 3, equivalently to show that $\overline{E}$ is an equivalence of $\mathcal{F}$-categories. Now consider the square on the right. The bottom leg is w-doctrinal as both of its components are, L by Proposition 10. Since the $\mathcal{F}$-category $\overline{\mathbb{A}}$ is equivalent to T-$\mathbb{A}\mathrm{lg}_w$ it has the same completeness properties, so that, using Theorem 21 again, the left leg is orthogonal to each w-doctrinal $\mathcal{F}$-functor. Therefore the right commutative square consists of two orthogonal decompositions of a common $\mathcal{F}$-functor and we conclude that $\overline{E}$ is an isomorphism. □

Note that although Theorem 25, in each of its variants, only asks that $\mathbb{A}$ admits certain limits it follows that H creates those limits: for U : T-$\mathbb{A}\mathrm{lg}_w \to \mathcal{B}$ does so by Proposition 11 and $E : \mathbb{A} \to$ T-$\mathbb{A}\mathrm{lg}_w$ is an equivalence of $\mathcal{F}$-categories. In applying the theorem this is often useful in that it tells us how these limits must be constructed in $\mathbb{A}$.

6.3. **$\mathcal{F}$-categorical monadicity.** Our focus has been upon 2-monads but indeed the above results extend in a routine way to cover $\mathcal{F}$-categorical monadicity too. Let us briefly explain how this goes. An $\mathcal{F}$-monad [16] is a monad in the 2-category $\mathcal{F}$-CAT and so consists of an $\mathcal{F}$-functor $T : \mathbb{A} \to \mathbb{A}$ and two $\mathcal{F}$-natural transformations satisfying the usual equations. 2-monads are just $\mathcal{F}$-monads on 2-categories viewed as $\mathcal{F}$-categories. An $\mathcal{F}$-monad T induces 2-monads T_τ and T_λ and so we have strict T_τ and T_λ-algebra morphisms as in the two leftmost diagrams below.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, rightsquigarrow, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, rightsquigarrow, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, rightsquigarrow, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & TB \arrow[d, "b"] \\ A \arrow[r, rightsquigarrow, "f"'] & B \end{tikzcd}$$

These are the tight and loose morphisms of the Eilenberg-Moore $\mathcal{F}$-category which is denoted by T-$\mathbb{A}\mathrm{lg}_s$. We also have pseudo, lax and colax T_λ-morphisms – a lax T_λ-morphism is drawn above. These are the loose morphisms of the $\mathcal{F}$-categories T-$\mathbb{A}\mathrm{lg}_w$ whose tight morphisms are the strict T_τ-algebra maps. Now for each

$w \in \{l, p, c\}$ we have the $\mathcal{F}_2$-category whose tight and loose morphisms are the strict T_τ and T_λ-morphisms and whose very loose morphisms are the w-T_λ-morphisms: this is captured by the inclusion $\mathcal{F}$-functor $j_w : \text{T-}\mathbb{A}\text{lg}_s \to \text{T-}\mathbb{A}\text{lg}_w$. We have the evident forgetful $\mathcal{F}$-functors $U_s : \text{T-}\mathbb{A}\text{lg}_s \to \mathbb{A}$ and $U_w : \text{T-}\mathbb{A}\text{lg}_w \to \mathbb{A}$ commuting with j_w over the base. The key point for our applications is that these have the same properties as in the 2-monads case: namely U_s creates all limits, U_w creates $\overline{w}$-limits of loose morphisms (this follows from Theorem 5.13 of [16]) and U_w is w-doctrinal. With these facts in place we can give our monadicity theorem for $\mathcal{F}$-monads – we leave it to the reader to formulate the naturality of the Eilenberg-Moore extensions, which can be done using $\mathcal{F}_3$-*categories*.

Theorem 26. *Consider an $\mathcal{F}_2$-category $\mathcal{A}_\tau \to \mathcal{A}_\lambda \to \mathcal{A}_\phi$ over an $\mathcal{F}$-category $\mathbb{B}$ as below*

$$\begin{tikzcd} \mathbb{A}_{\tau,\lambda} \arrow[rr, "j"] \arrow[dr, "G"'] & & \mathbb{A}_{\tau,\phi} \arrow[dl, "H"] \\ & \mathbb{B} & \end{tikzcd}$$

(this means that $G_\tau = H_\tau$). Suppose that G has a left $\mathcal{F}$-adjoint so that we have the Eilenberg-Moore comparison $\mathcal{F}$-functor $E : \mathbb{A}_{\tau,\lambda} \to \text{T-}\mathbb{A}\text{lg}_s$ over the base $\mathbb{B}$. Now let $w \in \{l, p, c\}$ and suppose that both $\mathbb{A}_{\tau,\lambda}$ and $\mathbb{A}_{\tau,\phi}$ admit $\overline{w}$-limits of loose morphisms. Then there exists a unique $\mathcal{F}$-functor $E_w : \mathbb{A}_{\tau,\phi} \to \text{T-}\mathbb{A}\text{lg}_w$ extending E and living over the base, as depicted by the following everywhere commutative diagram.

$$\begin{tikzcd} \mathbb{A}_{\tau,\lambda} \arrow[r, "j"] \arrow[d, "E"'] & \mathbb{A}_{\tau,\phi} \arrow[d, dotted, "E_w"'] \arrow[dr, "H"] & \\ \text{T-}\mathbb{A}\text{lg}_s \arrow[r, "j_w"] & \text{T-}\mathbb{A}\text{lg}_w \arrow[r, "U_w"'] & \mathbb{B} \end{tikzcd}$$

If G is $\mathcal{F}$-monadic, $\mathbb{B}$ admits $\overline{w}$-limits of loose morphisms and H is w-doctrinal then E_w is an equivalence of $\mathcal{F}$-categories, and an isomorphism of $\mathcal{F}$-categories whenever G is strictly monadic.

Proof. The outside of the diagram clearly commutes – since $Hj = G$ and $U_w j_w = U_s$ this just amounts to the fact that the Eilenberg-Moore comparison E satisfies $U_s E = G$. Now $U_w : \text{T-}\mathbb{A}\text{lg}_w \to \mathbb{B}$ is w-doctrinal so that if we can show that the inclusion $j : \mathbb{A}_{\tau,\lambda} \to \mathbb{A}_{\tau,\phi}$ belongs to ${}^\perp w$-Doct then we will obtain E_w as the unique filler. Now have a commutative triangle of inclusions

$$\begin{tikzcd} \mathcal{A}_\tau \arrow[r, "j"] \arrow[dr, "j"'] & \mathbb{A}_{\tau,\lambda} \arrow[d, "j"] \\ & \mathbb{A}_{\tau,\phi} \end{tikzcd}$$

in which the two j's moving from left to right belong to ${}^{\perp}w$-Doct by Theorem 21; thus by 2 out of 3 $j : \mathbb{A}_{\tau,\lambda} \to \mathbb{A}_{\tau,\phi} \in^{\perp} w$-Doct. As such we obtain the Eilenberg-Moore extension E_w as the unique filler. The remainder of the proof is a straightforward modification of the proof of Theorem 25. □

7. Examples and applications

We now turn to examples. We begin by completing our running example of monoidal categories. Of course it is well known that monoidal categories, and each flavour of morphism between them, can be described using 2-monads – see Section 5.5 of [15] for an argument via colimit presentations – although this has not previously been established by application of a monadicity theorem. We then turn to more complex examples. Our final example, in 7.3, is new and typical of the kind of result which cannot be established using techniques, such as colimit presentations, that require explicit knowledge of a 2-monad.

7.1. **Monoidal categories.** Let us focus, as usual, on the lax monoidal functors of $\mathbb{M}\mathrm{onCat}_l$. From Example 5 we know that $V : \mathbb{M}\mathrm{onCat}_l \to \mathrm{Cat}$ satisfies l-doctrinal adjunction. It is clearly locally faithful and reflects identity 2-cells. From Example 13 we know that $V : \mathbb{M}\mathrm{onCat}_l \to \mathrm{Cat}$ creates colax limits of loose morphisms. Therefore to apply Theorem 25 and establish monadicity it remains to verify that the 2-functor $V_s : \mathrm{MonCat}_s \to \mathrm{Cat}$ is monadic. That V_s strictly creates V_s-absolute coequalisers (in the enriched sense) is true by essentially the same argument given for groups in 6.8 of [18]; by Beck's theorem in the enriched setting [4] it follows that V_s is strictly monadic so long as it has a left 2-adjoint.
Proposition 3.1 of [3] asserts that a 2-functor preserving *cotensors with **2*** admits a left 2-adjoint just when its underlying functor admits a left adjoint. Now cotensors with **2** are, in fact, just colax limits of identity arrows. That these are created by V_s follows from the fact that $V : \mathbb{M}\mathrm{onCat}_l \to \mathrm{Cat}$ creates colax limits of loose morphisms. It therefore remains to show that the underlying functor $(V_s)_0$ admits a left adjoint. Since this functor creates all limits it suffices to show that $(V_s)_0$ satisfies the solution set condition. For this it suffices to show that given a small category A each functor $F : A \to C = U\overline{C}$ to a monoidal category factors as $ME : A \to B \to C$ with B monoidal, M strict monoidal and the cardinality of the set of morphisms $Mor(B)$ bounded by that of $Mor(A)$. Here B will be the monoidal subcategory of C generated by the image of F and M the inclusion of this monoidal subcategory.
For the induced 2-monad T on Cat we now conclude, by Theorem 25, that the isomorphism of 2-categories $E : \mathrm{MonCat}_s \to \mathrm{T\text{-}Alg_s}$ over Cat extends uniquely to an isomorphism of $\mathcal{F}$-categories $E_l : \mathbb{M}\mathrm{onCat}_l \to \mathrm{T\text{-}}\mathbb{A}\mathrm{lg}_l$ over Cat. Likewise one can verify, in an entirely similar way, that $V : \mathbb{M}\mathrm{onCat}_w \to \mathrm{Cat}$ satisfies the conditions of Theorem 25 in the cases $w \in \{p, c\}$. It follows that we have isomorphisms $E_w : \mathbb{M}\mathrm{onCat}_w \to \mathrm{T\text{-}}\mathbb{A}\mathrm{lg}_w$ over Cat for each $w \in \{l, p, c\}$. By Theorem 24 these isomorphisms are natural in $p \leq l$ and $p \leq c$ in the sense of Diagram 2 of the Introduction.

7.2. **Categories with structure and variants.** Of course there was nothing special about our taking monoidal categories in the preceding section. The same

arguments can be used to establish monadicity of categories with any kind of algebraically specified structure and their various flavours of morphisms: categories with chosen limits of some kind for instance, distributive categories and so forth. All of these cases are well known to be monadic using colimit presentations of 2-monads, although it requires substantial and laborious calculation to use that theory to establish monadicity in the detailed manner above. Such a detailed treatment using colimit presentations, one expressed in terms of isomorphisms of 2-categories or $\mathcal{F}$-categories, will not be found in the literature. Colimit presentations are one of the two standard techniques for understanding the monadicity of weaker kinds of morphisms; the other is direct calculation with a 2-monad known to exist. As a representative example of this technique consider a small 2-category $\mathcal{J}$ and the forgetful 2-functor $U : [\mathcal{J}, \mathrm{Cat}] \to [ob\mathcal{J}, \mathrm{Cat}]$ which restricts presheaves to families along the inclusion $ob\mathcal{J} \to \mathcal{J}$. U has a left 2-adjoint F given by left Kan extension and is strictly monadic by Beck's theorem; moreover the induced 2-monad $T = UF$ admits a simple pointwise description: at $X \in [ob\mathcal{J}, \mathrm{Cat}]$ we have $TX(j) = \Sigma_i \mathcal{J}(i,j) \times Xi$. Using this formula (as in [3]) one directly calculates that T-pseudomorphisms bijectively correspond to pseudonatural transformations and so on, eventually deducing an isomorphism $Ps(\mathcal{J}, \mathrm{Cat}) \to \text{T-Alg}_p$. It is in such cases, more specifically when T is not so simple, that our results have most value. An example of this kind was given in [10]. Given a complete and cocomplete symmetric monoidal closed category $\mathcal{V}$ the authors take as base the 2-category V-Cat of small $\mathcal{V}$-categories. For a small class of weights Φ they consider the 2-category Φ-Colim (we will write $\Phi\text{-Colim}_s$) of $\mathcal{V}$-categories with *chosen* Φ-weighted colimits, $\mathcal{V}$-functors preserving those colimits strictly and $\mathcal{V}$-natural transformations. This 2-category lives over V-Cat via a forgetful 2-functor $U_s : \Phi\text{-Colim}_s \to \text{V-Cat}$. One also has $\mathcal{V}$-functors preserving colimits in the usual, up to isomorphism, sense: these are the loose morphisms of the $\mathcal{F}$-category $\Phi\text{-}\mathbb{C}\mathrm{olim}_p : \Phi\text{-Colim}_s \to \Phi\text{-Colim}_p$ which sits over V-Cat via a forgetful $\mathcal{F}$-functor $U : \Phi\text{-}\mathbb{C}\mathrm{olim}_p \to \text{V-Cat}$. The authors show that $U_s : \Phi\text{-Colim}_s \to \text{V-Cat}$ is strictly monadic and then, by calculating directly with the induced 2-monad T, show that one obtains an isomorphism of 2-categories $U : \Phi\text{-}\mathbb{C}\mathrm{olim}_p \to \text{T-}\mathbb{A}\mathrm{lg}_p$. Let us show how this can be deduced from Theorem 25. Firstly observe that U satisfies p-doctrinal adjunction: this follows from the fact that any equivalence of $\mathcal{V}$-categories preserves colimits. Again since U is locally fully faithful it is certainly locally faithful and reflects identity 2-cells. It is not hard to see that $\Phi\text{-}\mathbb{C}\mathrm{olim}_p$ admits pseudo-limits of loose morphisms – indeed this is shown in Section 5 of [10]. It then follows immediately from Theorem 25 that the isomorphism of 2-categories $E : \Phi\text{-Colim}_s \to \text{T-Alg}_s$ extends uniquely to an isomorphism of $\mathcal{F}$-categories $E : \Phi\text{-}\mathbb{C}\mathrm{olim}_p \to \text{T-}\mathbb{A}\mathrm{lg}_p$ over V-Cat.
Furthermore, because left adjoints preserve colimits, $U : \Phi\text{-}\mathbb{C}\mathrm{olim}_p \to \text{V-Cat}$ has the additional property of satisfying c-doctrinal adjunction; since it is isomorphic to $\text{T-}\mathbb{A}\mathrm{lg}_p$ it also admits, by Theorem 2.6 of [3], lax limits of loose morphisms. Therefore Theorem 25 ensures that the composite $\Phi\text{-}\mathbb{C}\mathrm{olim}_p \to \text{T-}\mathbb{A}\mathrm{lg}_p \to \text{T-}\mathbb{A}\mathrm{lg}_c$ is also an isomorphism, thus explaining why the colax and pseudo T-morphisms coincide as those $\mathcal{V}$-functors which preserve Φ-colimits. Again one easily applies Theorem 25 to show that $\text{T-}\mathbb{A}\mathrm{lg}_l$ is isomorphic to the $\mathcal{F}$-category $\Phi\text{-}\mathbb{C}\mathrm{olim}_l$ whose loose morphisms are arbitary $\mathcal{V}$-functors. This final isomorphism highlights the

fact that $U_l : \text{T-Alg}_l \to \text{V-Cat}$ is 2-fully faithful: 2-monads with this property are called *lax idempotent/Kock-Zöberlein* and have been carefully studied in [9].

7.3. **In a monoidal 2-category.** In a monoidal category $\mathcal{C}$ one can consider the category of monoids $\mathrm{Mon}(\mathcal{C})$ or of commutative monoids. If the forgetful functor $U : \mathrm{Mon}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint then Beck's theorem can be applied, with no further information, to show that U is monadic. For our final example we study the analogous situation in the context of a monoidal 2-category $\mathcal{C}$, in which one can consider monoids, pseudomonoids (generalising monoidal categories), braided pseudomonoids and so on. We consider only the simplest case of monoids because, in the absence of a suitable graphical calculus, it is difficult to encode diagrams compactly. We have the 2-category of monoids, strict monoid morphisms and monoid transformations $\mathrm{Mon}(\mathcal{C})_s$ and a forgetful 2-functor $U_s : \mathrm{Mon}(\mathcal{C})_s \to \mathcal{C}$. Just as before, the enriched version of Beck's theorem [4] can be applied to show that if U_s has a left 2-adjoint then it is monadic. However we now also have (lax/pseudo/colax)-morphisms of monoids and we would like to understand that these too are monadic in the appropriate sense: this is the content, under completeness conditions on the base, of the present example.
By a monoidal 2-category $\mathcal{C}$ we will mean a monoidal $\mathcal{V}$-category where $\mathcal{V} = \mathrm{Cat}$: this satisfies the same axioms as a monoidal category with the exception that the tensor product, the associator and the other data involved are now 2-functorial and 2-natural. In working with $\mathcal{C}$ we will write as though it were strict monoidal – this is justified in the theorem that follows – and will use juxtaposition for the tensor product. A monoid in $\mathcal{C}$ is just a monoid in the usual sense. Given monoids (X, m_X, i_X) and (Y, m_Y, i_Y) a lax monoid map $(f, \overline{f}, f_0) : (X, m_X, i_X) \to (Y, m_Y, i_Y)$ consists of an arrow $f : X \to Y$ and 2-cells as below

$$\begin{tikzcd}
X^2 \arrow[r, "f^2"] \arrow[d, "m_X"'] \arrow[dr, phantom, "\overline{f}\Downarrow"] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dl, "i_X"'] \arrow[dr, "i_Y"] \arrow[d, phantom, "f_0\Downarrow"] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

such that the equation

$$\begin{tikzcd}
X^3 \arrow[r, "1ff"] \arrow[d, "1m_X"'] \arrow[dr, phantom, "1\overline{f}\Downarrow"] & XY^2 \arrow[r, "f11"] \arrow[d, "1m_Y"] & Y^3 \arrow[d, "1m_Y"] \\
X^2 \arrow[r, "1f"'] \arrow[d, "m_X"'] & XY \arrow[r, "f1"'] \arrow[d, phantom, "\overline{f}\Downarrow"] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}
=
\begin{tikzcd}
X^3 \arrow[r, "11f"] \arrow[d, "m_X1"'] & X^2Y \arrow[r, "ff1"] \arrow[d, "m_X1"'] \arrow[dr, phantom, "\overline{f}1\Downarrow"] & Y^3 \arrow[d, "m_Y1"] \\
X^2 \arrow[r, "1f"'] \arrow[d, "m_X"'] & XY \arrow[r, "f1"'] \arrow[d, phantom, "\overline{f}\Downarrow"] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

holds and such that both composite 2-cells

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "1i_X"'] & X \arrow[r, "f"] \arrow[d, "1i_Y"'] & Y \arrow[d, "1i_Y"] \\
X^2 \arrow[r, "1f"'] \arrow[d, "m_X"'] \arrow[ur, phantom, "1f_0\Downarrow"] & XY \arrow[r, "f1"'] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rr, "f"'] \arrow[urr, phantom, "1\overline{f}\Downarrow"] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_X 1"'] & Y \arrow[r, "1"] \arrow[d, "i_X 1"'] & Y \arrow[d, "i_Y 1"] \\
X^2 \arrow[r, "1f"'] \arrow[d, "m_X"'] & XY \arrow[r, "f1"'] \arrow[ur, phantom, "f_0 1\Downarrow"] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rr, "f"'] \arrow[urr, phantom, "1\overline{f}\Downarrow"] & & Y
\end{tikzcd}$$

are identities. A monoid transformation $\alpha : (f, \overline{f}, f_0) \Rightarrow (g, \overline{g}, g_0)$ is a 2-cell $\alpha : f \Rightarrow g$ satisfying the equations

$$\begin{tikzcd}
X^2 \arrow[r, bend left, "f^2"] \arrow[r, bend right, "g^2"'] \arrow[r, phantom, "\alpha^2\Downarrow"] \arrow[d, "m_X"'] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[r, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\overline{g}"] & Y
\end{tikzcd}
=
\begin{tikzcd}
X^2 \arrow[r, bend left, "f^2"] \arrow[d, "m_X"'] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] \arrow[ur, phantom, "\overline{f}\Downarrow"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dl, bend right, "i_X"'] \arrow[dr, bend left, "i_Y"] & \\
X \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\alpha\Downarrow"] & \arrow[u, phantom, "f_0\Downarrow"] & Y
\end{tikzcd}
=
\begin{tikzcd}
& I \arrow[dl, bend right, "i_X"'] \arrow[dr, bend left, "i_Y"] & \\
X \arrow[rr, bend right, "g"'] & \arrow[u, phantom, "g_0\Downarrow"] & Y
\end{tikzcd}$$

These are the 2-cells of the $\mathcal{F}$-category $\mathbb{M}\mathrm{on}(\mathcal{C})_l : \mathrm{Mon}(\mathcal{C})_s \to \mathrm{Mon}(\mathcal{C})_l$ of monoids, strict and lax monoid morphisms which sits over $\mathcal{C}$ via a forgetful $\mathcal{F}$-functor $U : \mathbb{M}\mathrm{on}(\mathcal{C})_l \to \mathcal{C}$. Likewise we have pseudo and colax monoid morphisms and forgetful $\mathcal{F}$-functors $U : \mathbb{M}\mathrm{on}(\mathcal{C})_w \to \mathcal{C}$ for each $w \in \{l, p, c\}$.
For a simple statement we assume in the following result that $\mathcal{C}$ admits *pie limits* [20]: these are a good class of limits containing w-limits of arrows for each w.

Theorem 27. *Let $\mathcal{C}$ be a monoidal 2-category admitting pie limits and suppose that $U_s : \mathrm{Mon}(\mathcal{C})_s \to \mathcal{C}$ has a left 2-adjoint. Let T be the induced 2-monad on $\mathcal{C}$. Then for each $w \in \{l, p, c\}$ we have isomorphisms of $\mathcal{F}$-categories $\mathbb{M}\mathrm{on}(\mathcal{C})_w \to \mathrm{T}\text{-}\mathbb{A}\mathrm{lg}_w$ over $\mathcal{C}$ and these are natural in w.*

Proof. Let us begin by showing that it suffices to suppose $\mathcal{C}$ to be strict monoidal. A straightforward extension of the usual argument for monoidal categories shows that $\mathcal{C}$ is equivalent to a strict monoidal 2-category $\mathcal{D}$ via a strong monoidal 2-equivalence $E : \mathcal{C} \to \mathcal{D}$.

$$\begin{tikzcd}
\mathbb{M}\mathrm{on}(\mathcal{C})_w \arrow[r, "E_*"] \arrow[d, "U_C"'] & \mathbb{M}\mathrm{on}(\mathcal{D})_w \arrow[d, "U_D"] \\
\mathcal{C} \arrow[r, "E"'] & \mathcal{D}
\end{tikzcd}$$

Such an equivalence naturally lifts to a 2-equivalence $E_* : \mathrm{Mon}(\mathcal{C})_w \to \mathrm{Mon}(\mathcal{D})_w$ for each w and so induces a commuting square of $\mathcal{F}$-categories and $\mathcal{F}$-functors as above with both horizontal legs equivalences of $\mathcal{F}$-categories. Now to apply Theorem 25 we must show that $U_C : \mathbb{M}\mathrm{on}(\mathcal{C})_w \to \mathcal{C}$ is w-doctrinal and that $\mathbb{M}\mathrm{on}(\mathcal{C})_w$ has $\overline{w}$-limits of loose morphisms. So suppose that U_D and $\mathbb{M}\mathrm{on}(\mathcal{D})_w$ have these properties and let us deduce from these the corresponding properties for $\mathcal{C}$. Certainly if U_D were w-doctrinal then U_C would be too; for both horizontal legs, being equivalences, are w-doctrinal and such $\mathcal{F}$-functors, being defined by lifting properties (as in Section 3.3), are closed under 2 out of 3. Likewise any limits existing in $\mathbb{M}\mathrm{on}(\mathcal{D})_w$ exist in the $\mathcal{F}$-equivalent $\mathbb{M}\mathrm{on}(\mathcal{C})_w$. Therefore it suffices to suppose that $\mathcal{C}$ is strict monoidal.

Now certainly $U : \mathbb{M}\mathrm{on}(\mathcal{C})_l \to \mathcal{C}$ is locally faithful and reflects identity 2-cells. Moreover given a strict monoid map $(f, \overline{f}, f_0) : X \to Y$ and adjunction $(\epsilon, f \dashv g, \eta) \in \mathcal{C}$ taking mates gives 2-cells

$$\begin{tikzcd}
Y^2 \arrow[r, "g^2"] \arrow[dd, "1"'] & X^2 \arrow[d, "m_X"] \arrow[ddl, "f^2"'] \arrow[ddl, phantom, "\Leftarrow\epsilon^2", near end, shift left=3] \\
& X \arrow[dd, "1"] \arrow[ddl, "f"'] \\
Y^2 \arrow[d, "m_Y"'] & \arrow[dl, phantom, "\Leftarrow\eta"] \\
Y \arrow[r, "g"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "i_X"] \arrow[d, "i_Y"'] & X \arrow[dl, "f"'] \arrow[d, "1"] \arrow[dl, phantom, "\Leftarrow\eta", shift left=4] \\
Y \arrow[r, "g"'] & X
\end{tikzcd}$$

It is straightforward to see, by cancelling mates, that these give g the structure of a lax monoid map $(g, \overline{g}, g_0)$, with respect to which $(\epsilon, f \dashv (g, \overline{g}, g_0), \eta)$ is an adjunction in $\mathbb{M}\mathrm{on}(\mathcal{C})_l$. Uniqueness of the lifted adjunction follows from Proposition 3.1. Therefore U satisfies l-doctrinal adjunction.

We will show that $U : \mathbb{M}\mathrm{on}(\mathcal{C})_l \to \mathcal{C}$ creates colax limits of loose morphisms. Consider a lax monoid map $(f, \overline{f}, f_0) : X \to Y$ and the colax limit C of f in $\mathcal{C}$ with colax cone as below

$$\begin{tikzcd}
& C \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\lambda\Downarrow"] & \\
X \arrow[rr, "f"'] & {} & Y
\end{tikzcd}$$

By the universal property of C the composite colax cone left below induces a unique map $m_C : CC \to C$ such that $p.m_C = m_X.p^2$, $q.m_C = m_Y.q^2$ and such that the left equation below holds. Likewise we obtain a unique $i_C : I \to C$ such that $p.i_C = i_X$, $q.i_C = i_Y$ and satisfying $\lambda.i_C = f_0$ as on the right below.

$$\begin{tikzcd}
& C^2 \arrow[dl, "p^2"'] \arrow[dr, "q^2"] \arrow[d, phantom, "\lambda^2\Downarrow"] & \\
X^2 \arrow[rr, "f^2"] \arrow[d, "m_X"'] & {} \arrow[d, phantom, "\overline{f}\Downarrow"] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rr, "f"'] & {} & Y
\end{tikzcd}
=
\begin{tikzcd}
& C^2 \arrow[dl, "p^2"'] \arrow[dr, "q^2"] \arrow[d, "m_C"] & \\
X^2 \arrow[d, "m_X"'] & C \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\lambda\Downarrow"] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rr, "f"'] & {} & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[ddl, bend right, "i_X"'] \arrow[ddr, bend left, "i_Y"] \arrow[dd, phantom, "f_0\Downarrow"] & \\
& & \\
X \arrow[rr, "f"'] & {} & Y
\end{tikzcd}
=
\begin{tikzcd}
& I \arrow[ddl, bend right, "i_X"'] \arrow[ddr, bend left, "i_Y"] \arrow[d, "i_C"] & \\
& C \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\lambda\Downarrow"] & \\
X \arrow[rr, "f"'] & {} & Y
\end{tikzcd}$$

If we can show (C, m_C, i_C) to be a monoid then these combined equations will assert exactly that p and q are strict monoid maps and $\lambda : q \Rightarrow (f, \overline{f}, f_0).p$ a monoid transformation. To show that $m_C.(m_C 1) = m_C.(1 m_C) : C^3 \to C$ amounts to showing that both paths coincide upon postcomposition with the components p, q and λ of the universal colax cone. We have that $p.m_C.(m_C 1) = m_X.p^2.(m_C 1) = m_X.(m_X 1).p^3 = m_X.(1 m_X).p^3 = m_X.p^2.(1 m_C) = p.m_C.(1 m_C)$ and similarly for q so that associativity of m_C will follow if we can show that both paths coincide upon postcomposition with λ. Consider the following series of equalities

$$
\begin{tikzcd}
 & C^3 \arrow[d, "1m_C"] & \\
 & C^2 \arrow[d, "m_C"] & \\
 & C \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\lambda\Downarrow"] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}
=
\begin{tikzcd}
 & C^3 \arrow[dl, "ppp"'] \arrow[d, "1m_C"] \arrow[dr, "qqq"] & \\
X^3 \arrow[d, "1m_X"'] & C^2 \arrow[dl, "pp"] \arrow[dr, "qq"'] \arrow[d, phantom, "\lambda\lambda\Downarrow"] & Y^3 \arrow[d, "1m_Y"] \\
X^2 \arrow[rr, "ff"'] \arrow[d, "m_X"'] & \arrow[d, phantom, "\overline{f}\Downarrow"] & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}
=
\begin{tikzcd}
 & & & C^3 \arrow[dll, "p11"'] \arrow[dd, "1m_C"] \arrow[drr, "1qq"] & & & \\
 & XC^2 \arrow[dl, "1pp"'] \arrow[dd, "1m_C"] & & & & CY^2 \arrow[dd, "1m_Y"] \arrow[dr, "q11"] & \\
X^3 \arrow[dd, "1m_X"'] & & & C^2 \arrow[dll, "p1"'] \arrow[drr, "1q"] & & & Y^3 \arrow[dd, "1m_Y"] \\
 & XC \arrow[dl, "1p"'] \arrow[drr, "1q"] \arrow[d, phantom, "1\lambda\Downarrow"] & & & & CY \arrow[dll, "p1"'] \arrow[dr, "q1"] \arrow[d, phantom, "\lambda 1\Downarrow"] & \\
X^2 \arrow[rrr, "1f"'] \arrow[d, "m_X"'] & & & XY \arrow[rrr, "f1"'] \arrow[d, phantom, "\overline{f}\Downarrow"] & & & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rrrrrr, "f"'] & & & & & & Y
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
 & & C^3 \arrow[dl, "p11"'] \arrow[dr, "1qq"] & & \\
 & XC^2 \arrow[dl, "1pp"'] \arrow[d, "1m_C"] \arrow[dr, "1qq"] & & CY^2 \arrow[dl, "p11"'] \arrow[d, "1m_Y"] \arrow[dr, "q11"] & \\
X^3 \arrow[d, "1m_X"'] & XC \arrow[dl, "1p"'] \arrow[dr, "1q"] \arrow[d, phantom, "1\lambda\Downarrow"] & XY^2 \arrow[d, "1m_Y"] & CY \arrow[dl, "p1"'] \arrow[dr, "q1"] \arrow[d, phantom, "\lambda 1\Downarrow"] & Y^3 \arrow[d, "1m_Y"] \\
X^2 \arrow[rr, "1f"'] \arrow[d, "m_X"'] & & XY \arrow[rr, "f1"'] \arrow[d, phantom, "\overline{f}\Downarrow"] & & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rrrr, "f"'] & & & & Y
\end{tikzcd}
=
\begin{tikzcd}
 & & & & C^3 \arrow[dl, "p11"'] \arrow[dr, "1qq"] & & \\
 & & & XC^2 \arrow[dlll, "1pp"'] \arrow[dr, "1qq"] \arrow[d, phantom, "1\lambda\lambda\Downarrow"] & & CY^2 \arrow[dl, "p11"'] \arrow[dr, "q11"] \arrow[d, phantom, "\lambda 11\Downarrow"] & \\
X^3 \arrow[rr, "11f"'] \arrow[d, "1m_X"'] & & X^2Y \arrow[rr, "1f1"'] \arrow[d, phantom, "1\overline{f}\Downarrow"] & & XY^2 \arrow[rr, "f11"'] \arrow[d, "1m_Y"] & & Y^3 \arrow[d, "1m_Y"] \\
X^2 \arrow[rrrr, "1f"'] \arrow[d, "m_X"'] & & & & XY \arrow[rr, "f1"'] \arrow[d, phantom, "\overline{f}\Downarrow"] & & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rrrrrr, "f"'] & & & & & & Y
\end{tikzcd}
$$

The first holds by definition of m_C, the second merely rewrites the tensor product $\lambda\lambda$ and the third rewrites the commuting central diamond. The final equality of pasting composites does two things at once: on its left side it again applies the definition of m_C postcomposed with λ; we rewrite the right hand side using $(\lambda 1).(1m_Y) = (1m_Y).(\lambda 11)$. By an entirely similar diagram chase we obtain the equality

$$
\begin{tikzcd}
 & C^3 \arrow[d, "m_C 1"] & \\
 & C^2 \arrow[d, "m_C"] & \\
 & C \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\lambda\Downarrow"] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}
=
\begin{tikzcd}
 & & & C^3 \arrow[dll, "pp1"'] \arrow[dr, "11q"] & & \\
 & X^2C \arrow[dl, "11p"'] \arrow[dr, "11q"] \arrow[d, phantom, "11\lambda\Downarrow"] & & & C^2Y \arrow[dll, "pp1"'] \arrow[dr, "qq1"] \arrow[d, phantom, "\lambda\lambda 1\Downarrow"] & \\
X^3 \arrow[rr, "11f"'] \arrow[d, "m_X 1"'] & & X^2Y \arrow[rr, "1f1"'] \arrow[d, "m_X 1"'] & & XY^2 \arrow[r, "f11"'] \arrow[d, phantom, "\overline{f}1\Downarrow"] & Y^3 \arrow[d, "m_Y 1"] \\
X^2 \arrow[rr, "1f"'] \arrow[d, "m_X"'] & & XY \arrow[rrr, "f1"'] \arrow[d, phantom, "\overline{f}\Downarrow"] & & & Y^2 \arrow[d, "m_Y"] \\
X \arrow[rrrrr, "f"'] & & & & & Y
\end{tikzcd}
$$

This final composite and the one above it each constitute λ^3 sat atop either side of the first equation for a lax monoid morphism: as such they agree and m_C is associative. Much smaller, though similar, diagram chases show that $m_C.(i_C 1) = 1$ and that $m_C.(1 i_C) = 1$; thus C is a monoid and the colax cone $(p, \lambda : q \Rightarrow pf, q)$ lifts (uniquely) to a colax cone in $\mathbb{M}\mathrm{on}(\mathcal{C})_l$.

That the lifted colax cone satisfies the universal property of the colax limit is relatively straightforward and left to the reader. That p and q detect strict monoid morphisms is a consequence of the fact that they jointly detect identity 2-cells in $\mathcal{C}$.

Now if $U_s : \mathrm{Mon}(\mathcal{C})_s \to \mathcal{C}$ has a left adjoint it is automatically strictly monadic by the enriched version of Beck's monadicity theorem [4]. Therefore, using the above, Theorem 25 asserts that the isomorphism of 2-categories $E : \mathrm{Mon}(\mathcal{C})_s \to \mathrm{T\text{-}Alg_s}$ over $\mathcal{C}$ extends uniquely to an isomorphism of $\mathcal{F}$-categories $E_l : \mathbb{M}\mathrm{on}(\mathcal{C})_l \to \mathrm{T\text{-}\mathbb{A}lg_l}$ over $\mathcal{C}$. In a similar way one verifies the conditions of Theorem 25 when $w \in \{p, c\}$ to obtain isomorphisms of $\mathcal{F}$-categories $E_w : \mathbb{M}\mathrm{on}(\mathcal{C})_w \to \mathrm{T\text{-}\mathbb{A}lg}_w$ over $\mathcal{C}$ for each w; by Theorem 24 these isomorphisms are natural in w. □

7.4. **A non-example.** All of the examples we have seen are of the strictly monadic variety and indeed this is the case whenever one studies structured objects over some same base 2-category. Now Theorem 25 is general enough to cover ordinary monadicity – up to equivalence of $\mathcal{F}$-categories – but in fact there exist situations of a weaker kind. Here is one such case. Let $\mathrm{Cat}_f \subset \mathrm{Cat}$ be a full sub 2-category of Cat whose objects form a skeleton of the finitely presentable categories (the finitely presentable objects in Cat) and let $[\mathrm{Cat}, \mathrm{Cat}]_f \subset [\mathrm{Cat}, \mathrm{Cat}]$ be the full sub 2-category consisting of those endo 2-functors preserving filtered colimits: this is the tight part of the $\mathcal{F}$-category $\mathbb{P}\mathrm{s}(\mathrm{Cat}, \mathrm{Cat})_f : [\mathrm{Cat}, \mathrm{Cat}]_f \to \mathrm{Ps}(\mathrm{Cat}, \mathrm{Cat})_f$ whose loose morphisms are pseudonatural transformations. Likewise we have an $\mathcal{F}$-category $\mathbb{P}\mathrm{s}(\mathrm{Cat}_f, \mathrm{Cat}) : [\mathrm{Cat}_f, \mathrm{Cat}] \to \mathrm{Ps}(\mathrm{Cat}_f, \mathrm{Cat})$ and now restriction along the inclusion $\mathrm{Cat}_f \to \mathrm{Cat}$ induces a forgetful $\mathcal{F}$-functor $R : \mathbb{P}\mathrm{s}(\mathrm{Cat}, \mathrm{Cat})_f \to \mathbb{P}\mathrm{s}(\mathrm{Cat}_f, \mathrm{Cat})$. Further restricting along the inclusion $ob\mathrm{Cat}_f \to \mathrm{Cat}_f$ gives a commuting triangle

$$\begin{tikzcd} \mathbb{P}\mathrm{s}(\mathrm{Cat}, \mathrm{Cat})_f \arrow[r, "R"] \arrow[dr, "SR"'] & \mathbb{P}\mathrm{s}(\mathrm{Cat}_f, \mathrm{Cat}) \arrow[d, "S"] \\ & [ob\mathrm{Cat}_f, \mathrm{Cat}] \end{tikzcd}$$

The composite $S_\tau R_\tau : [\mathrm{Cat}, \mathrm{Cat}]_f \to [ob\mathrm{Cat}_f, \mathrm{Cat}]$ is monadic though not strictly so: for the induced 2-monad T we have $\mathrm{T\text{-}\mathbb{A}lg}_p$ isomorphic to $\mathbb{P}\mathrm{s}(\mathrm{Cat}_f, \mathrm{Cat})$ with $R_\tau : [\mathrm{Cat}, \mathrm{Cat}]_f \to [\mathrm{Cat}_f, \mathrm{Cat}]$ the Eilenberg-Moore comparison 2-functor – that this is a 2-equivalence follows from Cat's being locally finitely presentable as a 2-category. Whilst R_τ is a 2-equivalence the 2-functor $R_\lambda : \mathrm{Ps}(\mathrm{Cat}, \mathrm{Cat})_f \to \mathrm{Ps}(\mathrm{Cat}_f, \mathrm{Cat})$ is not: indeed $\mathrm{Ps}(\mathrm{Cat}_f, \mathrm{Cat})$ is locally small whereas $\mathrm{Ps}(\mathrm{Cat}, \mathrm{Cat})_f$ is not. Yet R_λ turns out to be a biequivalence and R the uniquely induced $\mathcal{F}$-functor to the $\mathcal{F}$-category of algebras.

References

[1] Bartels, T. *Higher gauge theory 1:2-Bundles.* PhD thesis, University of California, Riverside, 2006.
[2] Beck, J. Triples, algebras and cohomology. *Rep. Theory Appl. Categ. 2* (2003), 1–59.
[3] Blackwell, R., Kelly, G. M., and Power, A. J. Two-dimensional monad theory. *Journal of Pure and Applied Algebra 59*, 1 (1989), 1–41.

[4] Dubuc, E. J. *Kan extensions in enriched category theory*, vol. 145 of *Lecture Notes in Mathematics*. Springer, 1970.

[5] Grandis, M., and Tholen, W. Natural weak factorization systems. *Archivum Mathematicum 42*, 4 (2006), 397–408.

[6] Kelly, G. M. Doctrinal adjunction. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 257–280.

[7] Kelly, G. M. On clubs and doctrines. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 181–256.

[8] Kelly, G. M. *Basic concepts of enriched category theory*, vol. 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982.

[9] Kelly, G. M., and Lack, S. On property-like structures. *Theory and Applications of Categories 3*, 9 (1997), 213–250.

[10] Kelly, G. M., and Lack, S. On the monadicity of categories with chosen colimits. *Theory and Applications of Categories 7*, 7 (2000), 148–170.

[11] Kelly, G. M., and Power, A. J. Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads. *Journal of Pure and Applied Algebra 89*, 1–2 (1993), 163–179.

[12] Kelly, G. M., and Street, R. Review of the elements of 2-categories. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 75–103.

[13] Lack, S. Limits for lax morphisms. *Appl. Categ. Structures 13*, 3 (2002), 189–203.

[14] Lack, S. Homotopy-theoretic aspects of 2-monads. *Journal of Homotopy and Related Structures 7*, 2 (2007), 229–260.

[15] Lack, S. A 2-categories companion. In *Towards higher categories*, vol. 152 of *IMA Vol. Math. Appl.* Springer, 2010, pp. 105–191.

[16] Lack, S., and Shulman, M. Enhanced 2-categories and limits for lax morphisms. *Advances in Mathematics 229*, 1 (2011), 294–356.

[17] Le Creurer, I., Marmolejo, F., and Vitale, E. Beck's theorem for pseudomonads. *Journal of Pure and Applied Algebra 173*, 3 (2002), 293–313.

[18] Mac Lane, S. *Categories for the Working Mathematician*, vol. 5 of *Graduate Texts in Mathematics*. Springer, 1971.

[19] Makkai, M. Avoiding the axiom of choice in general category theory. *Journal of Pure and Applied Algebra 108*, 2 (1996), 109–173.

[20] Power, J., and Robinson, E. A characterization of pie limits. *Mathematical Proceedings of the Cambridge Philosophical Society 110*, 1 (1991), 33–47.

[21] Roberts, D. Internal categories, anafunctors and localisations. *Theory and Applications of Categories 26*, 29 (2012), 788–829.

[22] Shulman, M. Comparing composites of left and right derived functors. *New York Journal of Mathematics 17* (2011), 75–125.

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 60000, Czech Republic

E-mail address: bourkej@math.muni.cz

Chapter 3

Algebraic weak factorisation systems I: Accessible AWFS

This chapter contains the article *Algebraic weak factorisation systems I: Accessible AWFS* by *John Bourke and Richard Garner*, published in the *Journal of Pure and Applied Algebra 220* (2016) 108–147. Both authors contributed an equal share to the results of this article.

ALGEBRAIC WEAK FACTORISATION SYSTEMS I: ACCESSIBLE AWFS

JOHN BOURKE AND RICHARD GARNER

Abstract. Algebraic weak factorisation systems (awfs) refine weak factorisation systems by requiring that the assignations sending a map to its first and second factors should underlie an interacting comonad–monad pair on the arrow category. We provide a comprehensive treatment of the basic theory of awfs—drawing on work of previous authors—and complete the theory with two main new results. The first provides a characterisation of awfs and their morphisms in terms of their double categories of left or right maps. The second concerns a notion of *cofibrant generation* of an awfs by a small double category; it states that, over a locally presentable base, any small double category cofibrantly generates an awfs, and that the awfs so arising are precisely those with accessible monad and comonad. Besides the general theory, numerous applications of awfs are developed, emphasising particularly those aspects which go beyond the non-algebraic situation.

1. Introduction

A *weak factorisation system* on a category $\mathcal{C}$ comprises two classes of maps $\mathcal{L}$ and $\mathcal{R}$, each closed under retracts in the arrow category, and obeying two axioms: firstly, that each map $f \in \mathcal{C}$ admit a factorisation $f = Rf \cdot Lf$ with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$; and secondly, that each $r \in \mathcal{R}$ have the right lifting property with respect to each $\ell \in \mathcal{L}$—meaning that, for every square as in the solid part of:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\ell"'] & C \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dotted] & D \end{tikzcd} \tag{1.1}$$

there should exist a commuting diagonal filler as indicated. Weak factorisation systems play a key role in *Quillen model structures* [42], which comprise two intertwined weak factorisation systems on a category; but they also arise elsewhere, for example in the categorical semantics of intensional type theory [4, 20].

Algebraic weak factorisation systems were introduced in [25]; they refine the basic notion by requiring that the factorisation process $f \mapsto (Lf, Rf)$ yield a compatible comonad L and monad R on the arrow category of $\mathcal{C}$. Given (L, R), we re-find $\mathcal{L}$ and $\mathcal{R}$ as the retract-closures of the classes of maps admitting L-coalgebra or R-algebra structure, so that one may define an algebraic weak

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factorisation system (henceforth AWFS) purely in terms of a comonad–monad pair (L, R) satisfying suitable axioms; we recall these in Section 2 below.

As shown in [22], any cofibrantly generated weak factorisation system on a well-behaved category may be realised as an AWFS, so that the algebraic notions are entirely appropriate for doing homotopy theory; this point of view has been pushed by Riehl, who in [43, 44] gives definitions of algebraic model category and algebraic monoidal model category, and in subsequent collaboration has used these notions to obtain non-trivial homotopical results [6, 12, 15].

Yet AWFS can do more than just serve as well-behaved realisations of their underlying weak factorisation systems; by making serious use of the monad R and comonad L, we may capture phenomena which are invisible in the non-algebraic setting. For example, each AWFS on $\mathcal{C}$ induces a *cofibrant replacement comonad* on $\mathcal{C}$ by factorising the unique maps out of 0; and if we choose our AWFS carefully, then the Kleisli category of this comonad Q—whose maps $A \rightsquigarrow B$ are maps $QA \to B$ in the original category—will equip $\mathcal{C}$ with a usable notion of *weak map*. For instance, there is an AWFS on the category of tricategories [24] and strict morphisms (preserving all structure on the nose) for which $\mathbf{Kl}(\mathsf{Q})$ comprises the tricategories and their trihomomorphisms (preserving all structure up to coherent equivalence); this example and others were described in [23], and will be revisited in the companion paper [13].

Probably the most important expressive advantage of AWFS is that their left and right classes can delineate kinds of map which mere weak factorisation systems cannot—the reason being that we interpret the classes of an AWFS as being composed of the L-coalgebras and R-algebras, rather than the underlying $\mathcal{L}$-maps and $\mathcal{R}$-maps. For example, here are some classes of map in **Cat** which are not the $\mathcal{R}$-maps of any weak factorisation system, but which—as shown in Examples 28 below—may be described in terms of the possession of R-algebra structure for a suitable AWFS:

- The Grothendieck fibrations;
- The Grothendieck fibrations whose fibres are groupoids;
- The Grothendieck fibrations whose fibres have finite limits, and whose reindexing functors preserve them;
- The left adjoint left inverse functors.

At a crude level, the reason that these kinds of map cannot be expressed as classes of $\mathcal{R}$-maps is that they are not retract-closed. The deeper explanation is that being an $\mathcal{R}$-map is a mere *property*, while being an R-algebra is a *structure* involving choices of basic lifting operations—and this choice allows for necessary equational axioms to be imposed between derived operations. As a further demonstration of the power this affords, we mention the result of [27] that for any monad T on a category $\mathcal{C}$ with finite products, there is an AWFS on $\mathcal{C}$ whose "algebraically fibrant objects"—R-algebras $X \to 1$—are precisely the T-algebras.

The existence of AWFS that reach beyond the scope of the non-algebraic theory opens up intriguing vistas. In a projected sequel to this paper, we will consider the theory of enrichment over monoidal AWFS [44] and use it to develop an abstract "homotopy coherent enriched category theory". In fact, we touch on this already in the current paper; Section 8 describes the *enriched small object* *argument* and sketches some of its applications to two-dimensional category theory, and to notions from the theory of quasicategories [32, 33, 40] such as *limits*, *Grothendieck fibrations* and *Kan extensions*.

The role of these examples, and others like them, will be to illuminate and justify the main contribution of this paper—that of giving a comprehensive account of the theory of unenriched AWFS. Parts of this theory can be found developed across the papers [6, 22, 25, 43, 44]; our objective is to draw the most important of these results together, and to complete them with two new theorems that clarify and simplify both the theory and the practice of AWFS.

In order to explain our two main theorems, we must first recall some *double-categorical* aspects of AWFS. A double category $\mathbb{A}$ is an internal category in **CAT**, as on the left below; we refer to objects and arrows of $\mathcal{A}_0$ as *objects* and *horizontal arrows*, and to objects and arrow of $\mathcal{A}_1$ as *vertical arrows* and *squares*. Internal functors and internal natural transformations between internal categories in **CAT** will be called *double functors* and *horizontal natural transformations*; they comprise a 2-category **DBL**.

$$\mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \xrightarrow{\ \circ\ } \mathcal{A}_1 \overset{\mathrm{dom}}{\underset{\mathrm{cod}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{\mathrm{id}}}}}} \mathcal{A}_0 \qquad\qquad \mathsf{R}\text{-}\mathbf{Alg} \times_{\mathcal{C}} \mathsf{R}\text{-}\mathbf{Alg} \xrightarrow{\ \circ\ } \mathsf{R}\text{-}\mathbf{Alg} \overset{\mathrm{dom}}{\underset{\mathrm{cod}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{\mathrm{id}}}}}} \mathcal{C}\ .$$

To each AWFS (L, R) on a category $\mathcal{C}$ we may associate a double category $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$, as on the right above, whose objects and horizontal arrows are the objects and arrows of $\mathcal{C}$, whose vertical arrows are the R-algebras, and whose squares are maps of R-algebras. The functor $\circ\colon \mathsf{R}\text{-}\mathbf{Alg} \times_{\mathcal{C}} \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{R}\text{-}\mathbf{Alg}$ encodes a canonical *composition law* on R-algebras—recalled in Section 2.8 below—which is an algebraic analogue of the fact that $\mathcal{R}$-maps in a weak factorisation system are closed under composition.

There is a forgetful double functor $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ into the double category of *squares* in $\mathcal{C}$—wherein objects are those of $\mathcal{C}$, vertical and horizontal arrows are arrows of $\mathcal{C}$, and squares are commuting squares in $\mathcal{C}$. In [43, Lemma 6.9], it was shown that the assignation sending an AWFS (L, R) on $\mathcal{C}$ to the double functor $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ constitutes the action on objects of a 2-fully faithful 2-functor

$$(–)\text{-}\mathbb{A}\mathbf{lg}\colon \mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{DBL}^{\mathbf{2}} \tag{1.2}$$

from the 2-category of AWFS, lax AWFS morphisms and AWFS 2-cells—whose definition we recall in Section 2.9 below—to the arrow 2-category $\mathbf{DBL}^{\mathbf{2}}$.

Our first main result, Theorem 6 below, gives an elementary characterisation of the essential image of (1.2) and of its dual, the fully faithful coalgebra 2-functor $(–)\text{-}\mathbb{C}\mathbf{oalg}\colon \mathbf{AWFS}_{\mathrm{oplax}} \to \mathbf{DBL}^{\mathbf{2}}$. In the theory of monads, a corresponding characterisation of the *strictly monadic* functors—those in the essential image of the 2-functor sending a monad T to the forgetful functor $U^{\mathsf{T}}\colon \mathsf{T}\text{-}\mathbf{Alg} \to \mathcal{C}$—is given by Beck's monadicity theorem; and so we term our result a *Beck theorem* for AWFS. Various aspects of this result are already in the literature—see [22, Appendix], [23, Proposition 2.8] or [43, Theorem 2.24]—and during the preparation of this paper, we became aware that Athorne had independently arrived at a similar result as [3, Theorem 2.5.3]. We nonetheless provide a complete treatment here, as this theorem is crucial to a smooth handling of AWFS, in particular allowing them to be constructed simply by giving a double category of the correct form to be one's double category of R-algebras or L-coalgebras.

The second main result of this paper deals with the appropriate notion of cofibrant generation for AWFS. Recall that a weak factorisation system $(\mathcal{L}, \mathcal{R})$ is *cofibrantly generated* if there is a mere *set* of maps J such that $\mathcal{R}$ is the precisely the class of maps with the right lifting property against each $j \in J$. Cofibrantly generated weak factorisation systems are commonplace due to Quillen's *small object argument* [42, §II.3.2], which ensures that for any set of maps J in a locally presentable category [19], the AWFS $(\mathcal{L}, \mathcal{R})$ cofibrantly generated by J exists.

In [22, Definition 3.9] is described a notion of cofibrant generation for AWFS: given a small category $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ over $\mathcal{C}^{\mathbf{2}}$, the AWFS cofibrantly generated by $\mathcal{J}$, if it exists, has as R-algebras, maps g equipped with choices of right lifting against each map Uj, naturally with respect to maps of $\mathcal{J}$. This notion is already more permissive than the usual one—as witnessed by [45, Example 13.4.5], for example—but we argue that it is still insufficient, since it excludes important examples on AWFS, such as the ones on **Cat** listed above, whose R-algebras are not retract-closed.

To rectify this, we introduce in Section 6.2 the notion of an AWFS being cofibrantly generated by a small *double category* $U\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$. This condition specifies the R-algebras as being maps equipped with liftings against Uj for each vertical map $j \in \mathbb{J}$, naturally with respect to squares of $\mathbb{J}$, but now with the extra requirement that, for each pair of composable vertical maps $j\colon x \to y$ and $k\colon y \to z$ of $\mathbb{J}$, the specified lifting against kj should be obtained by taking specified lifts first against k and then against j.

This broader notion of cofibrant generation is permissive enough to capture all our leading examples, including the ones on **Cat** listed above. Our second main theorem justifies it at a theoretical level, by characterising the cofibrantly generated AWFS on locally presentable categories as being exactly the *accessible* AWFS—those (L, R) whose comonad L and monad R preserve κ-filtered colimits for some regular cardinal κ. More precisely, we show for a locally presentable $\mathcal{C}$ that every small $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ cofibrantly generates an accessible AWFS on $\mathcal{C}$; and that every accessible AWFS on $\mathcal{C}$ is cofibrantly generated by a small $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$.

We now give a summary of the contents of the paper. We begin in Section 2 with a revision of the basic theory of AWFS: the definition, the relation with weak factorisation systems, double categories of algebras and coalgebras, morphisms of AWFS, and the fully faithful 2-functors $(-)$-$\mathbb{A}\mathbf{lg}$ and $(-)$-$\mathbb{C}\mathbf{oalg}$. In Section 3, we give our first main result, the Beck theorem for AWFS described above, together with two useful variants. Section 4 then uses the Beck theorem to give constructions of a wide range of AWFS; in particular, we discuss the AWFS for *split epis* in any category with binary coproducts; the AWFS for *lalis* (left adjoint left inverse functors) in any 2-category with oplax limits of arrows; and the construction of *injective* and *projective* liftings of AWFS.

Section 5 revisits the notion of cofibrant generation of AWFS by small categories, as introduced in [22], and uses the Beck theorem to give a simplified proof that such AWFS always exist in a locally presentable $\mathcal{C}$. This prepares the way for our second main result; in Section 6 we introduce cofibrant generation by small double categories, and in Section 33, show that over a locally presentable base $\mathcal{C}$, such AWFS always exist and are precisely the accessible AWFS on $\mathcal{C}$.

Finally in Section 8 we say a few words about *enriched cofibrant generation*. As mentioned above, a sequel to this paper will deal with this in greater detail; here, we content ourselves with giving the basic construction and a range of applications. In particular, we will see how to express notions such as *Grothendieck fibrations*, *categories with limits*, and *Kan extensions* in terms of algebras for suitable AWFS, and explain how to extend these constructions to the quasicategorical context.

2. Revision of algebraic weak factorisation systems

Algebraic weak factorisation systems were introduced in [25]—there called *natural* weak factorisation systems—and their theory developed further in [2, 6, 22, 43, 44]. They are highly structured objects, and an undisciplined approach runs the risk of foundering in a morass of calculations. The above papers, taken together, show that a smoother presentation is possible; in this introductory section, we draw together the parts of this presentation so as to give a concise account of the basic aspects of the theory.

Before beginning, let us state our foundational assumptions. κ will be a Grothendieck universe, and sets in κ will be called *small*, while general sets will be called *large*. **Set** and **SET** are the categories of small and large sets; **Cat** and **CAT** are the 2-categories of small categories (ones internal to **Set**) and of locally small categories (ones enriched in **Set**). Throughout the paper, all categories will be assumed to be locally small and all 2-categories will be assumed to be locally small (=**Cat**-enriched) except for ones whose names, like **SET** or **CAT**, are in capital letters.

2.1. **Functorial factorisations.** By a *functorial factorisation* on a category $\mathcal{C}$, we mean a functor $F\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ from the category of arrows to that of composable pairs which is a section of the composition functor $\mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$. We write $F = (L, E, R)$, to indicate that the value of F at an object f or morphism $(h, k)\colon f \to g$ of $\mathcal{C}^{\mathbf{2}}$ is given as on the left and the right of:

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y \qquad\qquad \begin{array}{ccccc} X & \xrightarrow{Lf} & Ef & \xrightarrow{Rf} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle E(h,k)} & & \downarrow{\scriptstyle k} \\ W & \xrightarrow{Lg} & Eg & \xrightarrow{Rg} & Z \end{array} .$$

Associated to (L, E, R) are the functors $L, R\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ with actions on objects $f \mapsto Lf$ and $f \mapsto Rf$, and the natural transformations $\epsilon\colon L \Rightarrow 1$ and $\eta\colon 1 \Rightarrow R$ with components at f given by the commuting squares:

$$(2.1) \qquad \begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle f} \\ Ef & \xrightarrow{Rf} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{Lf} & Ef \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} \\ B & \xrightarrow{1} & B \end{array} .$$

Note that (L, E, R) is completely determined by either (L, ϵ) or (R, η).

2.2. **Algebraic weak factorisation systems.** An *algebraic weak factorisation system* (AWFS) on $\mathcal{C}$ comprises:

(i) A functorial factorisation (L, E, R) on $\mathcal{C}$;
(ii) An extension of (L, ϵ) to a comonad $\mathsf{L} = (L, \epsilon, \Delta)$;
(iii) An extension of (R, η) to a monad $\mathsf{R} = (R, \eta, \mu)$;
(iv) A distributive law [9] of the comonad L over the monad R, whose underlying transformation $\delta \colon LR \Rightarrow RL$ satisfies $\mathrm{dom} \cdot \delta = \mathrm{cod} \cdot \Delta$ and $\mathrm{cod} \cdot \delta = \mathrm{dom} \cdot \mu$.

This definition involves less data than may be immediately apparent. The counit and unit axioms $\epsilon L \cdot \Delta = 1$ and $\mu \cdot \eta R = 1$ force the components of $\Delta \colon L \to LL$ and $\mu \colon RR \to R$ to be identities in their respective domain and codomain components, so given by commuting squares as on the left and right of:

$$
\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "Lf"'] & A \arrow[d, "LLf"] \\
Ef \arrow[r, "\Delta_f"] & ELf
\end{tikzcd}
\qquad
\begin{tikzcd}
Ef \arrow[r, "\Delta_f"] \arrow[d, "LRf"'] & ELf \arrow[d, "RLf"] \\
ERf \arrow[r, "\mu_f"] & Ef
\end{tikzcd}
\qquad
\begin{tikzcd}
ERf \arrow[r, "\mu_f"] \arrow[d, "RRf"'] & Ef \arrow[d, "Rf"] \\
B \arrow[r, "1"] & B
\end{tikzcd}
; \tag{2.2}
$$

while the conditions on δ in (iv) force its component at f to be given by the central square. Thus the only additional data beyond the underlying functorial factorisation (i) are maps $\Delta_f \colon Ef \to ELf$ and $\mu_f \colon ERf \to Ef$, satisfying suitable axioms. Note that the comonad L and monad R between them contain all of the data, so that we may unambiguously denote an AWFS simply by (L, R).

By examining (2.1) and (2.2), we see that the comonad L of an AWFS is a comonad *over* the domain functor $\mathrm{dom} \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$. By this we mean that $\mathrm{dom} \cdot L = \mathrm{dom}$ and that $\mathrm{dom} \cdot \epsilon$ and $\mathrm{dom} \cdot \Delta$ are identity natural transformations. Dually, the monad R is a monad over the codomain functor.

2.3. **Coalgebras and algebras.** To an algebraic weak factorisation system (L, R), we may associate the categories L-**Coalg** and R-**Alg** of coalgebras and algebras for L and R; these are to be thought of as constituting the respective left and right classes of the AWFS. Because L is a comonad over the domain functor, a coalgebra structure $f \to Lf$ on $f \colon A \to B$ necessarily has its domain component an identity, and so is determined by a single map $s \colon B \to Ef$; we write such a coalgebra as $\boldsymbol{f} = (f, s) \colon A \to B$. Dually, an R-algebra structure on $g \colon C \to D$ is determined by a single map $p \colon Ef \to C$, and will be denoted $\boldsymbol{g} = (g, p) \colon C \to D$. If the base category $\mathcal{C}$ has an initial object 0, then we may speak of *algebraically cofibrant objects*, meaning L-coalgebras $0 \to X$; dually, if $\mathcal{C}$ has a terminal object 1, then an *algebraically fibrant object* is an R-algebra $X \to 1$.

2.4. **Lifting of coalgebras against algebras.** Given an L-coalgebra $\boldsymbol{f} = (f, s)$, an R-algebra $\boldsymbol{g} = (g, p)$ and a commuting square as in the centre of:

$$
\begin{tikzcd}
A' \arrow[r, "a"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
B' \arrow[r, "b"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "v"'] \arrow[ur, dotted] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "g"'] & C' \arrow[d, "g'"] \\
D \arrow[r, "d"'] & D'
\end{tikzcd}
\tag{2.3}
$$

there is a canonical diagonal filler $\Phi_{\boldsymbol{f},\boldsymbol{g}}(u,v)\colon B \to C$ given by the composite $p \cdot E(u,v) \cdot s\colon B \to Ef \to Eg \to C$. These fillers are natural with respect to morphisms of L-coalgebras and R-algebras; which is to say that if we have commuting squares as on the left and right above which underlie, respectively, an L-coalgebra morphism $\boldsymbol{f'} \to \boldsymbol{f}$ and an R-algebra morphism $\boldsymbol{g} \to \boldsymbol{g'}$, then composing (2.3) with these two squares preserves the canonical filler:

$$c \cdot \Phi_{\boldsymbol{f},\boldsymbol{g}}(u,v) \cdot b = \Phi_{\boldsymbol{f'},\boldsymbol{g'}}(cua, dvb)\ .$$

Writing $U\colon \mathsf{L}\text{-}\mathbf{Coalg} \to \mathcal{C}^{\mathbf{2}}$ and $V\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ for the forgetful functors, this naturality may be expressed by saying that the canonical liftings constitute the components of a natural transformation

$$\Phi\colon \mathcal{C}^{\mathbf{2}}(U-, V?) \Rightarrow \mathcal{C}(\mathrm{cod}\ U-, \mathrm{dom}\ V?)\colon \mathsf{L}\text{-}\mathbf{Coalg}^{\mathrm{op}} \times \mathsf{R}\text{-}\mathbf{Alg} \to \mathbf{Set}\ . \tag{2.4}$$

2.5. **Factorisations with universal properties.** The two parts of the factorisation $f = Rf \cdot Lf$ of a map underlie the cofree L-coalgebra $\boldsymbol{L}f = (Lf, \Delta_f)\colon A \to Ef$ and the free R-algebra $\boldsymbol{R}f = (Rf, \mu_f)\colon Ef \to B$. The freeness of the latter says that, for any R-algebra $\boldsymbol{g} = (g,p)$ and morphism $(h,k)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$, there is a unique arrow ℓ such that the left-hand diagram in

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "Lf"] \arrow[dd, "f"'] & & C \arrow[dd, "g"] \\ & Ef \arrow[ur, dotted, "\exists!\ell"'] \arrow[d, "Rf"'] & \\ B \arrow[r, "1"] & B \arrow[r, "k"] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[dd, "f"'] & C \arrow[r, "1"] \arrow[d, "Lg"] & C \arrow[dd, "g"] \\ & Ef \arrow[dr, "Rg"] & \\ B \arrow[ur, dotted, "\exists!m"] \arrow[rr, "k"] & & D \end{tikzcd} \tag{2.5}$$

commutes and such that (ℓ, k) is an algebra morphism $\boldsymbol{R}f \to \boldsymbol{g}$. The dual universal property, as on the right above, describes the co-freeness of $\boldsymbol{L}g$ with respect to maps out of an L-coalgebra $\boldsymbol{f} = (f,s)$.

Observe also the following canonical liftings involving (co)free (co)algebras:

$$\underset{(2.6)}{\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "\boldsymbol{L}g"'] & A \arrow[d, "(g,p)"] \\ Eg \arrow[ur, dotted, "p"] \arrow[r, "Rg"'] & B \end{tikzcd}} \qquad \underset{(2.7)}{\begin{tikzcd} A \arrow[r, "Lf"] \arrow[d, "(f,s)"'] & A \arrow[d, "\boldsymbol{R}f"] \\ B \arrow[ur, dotted, "s"] \arrow[r, "1"'] & B \end{tikzcd}} \qquad \underset{(2.8)}{\begin{tikzcd} Ef \arrow[r, "\Delta_f"] \arrow[d, "\boldsymbol{L}Rf"'] & ELf \arrow[d, "\boldsymbol{R}Lf"] \\ ERf \arrow[ur, dotted, "\Delta_f \cdot \mu_f"] \arrow[r, "\mu_f"'] & Ef \end{tikzcd}}\ .$$

Here, (2.6) implies that an R-algebra is uniquely determined by its liftings against L-coalgebras, and dually in (2.7); while (2.8) expresses precisely the non-trivial axiom of the distributive law $\delta\colon \mathsf{LR} \Rightarrow \mathsf{RL}$.

2.6. **Underlying weak factorisation system.** By the preceding two sections, the classes of maps in $\mathcal{C}$ that admit L-coalgebra or R-algebra structure satisfy all of the axioms needed to be the two classes of a weak factorisation system, except maybe for closure under retracts. On taking the retract-closures of these two classes, we thus obtain a weak factorisation system $(\mathcal{L}, \mathcal{R})$, the *underlying weak factorisation system* of (L, R).

2.7. **Algebras and coalgebras from liftings.** A *coalgebra lifting operation* on a map $g\colon C \to D$ is a function φ_{-g} assigning to each L-coalgebra $\boldsymbol{f}$ and each commuting square $(u,v)\colon f \to g$ as in (2.3) a diagonal filler $\varphi_{\boldsymbol{f},g}(u,v)\colon B \to C$, naturally in maps of L-coalgebras. Maps equipped with coalgebra lifting operations are the objects of a category $\mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ over $\mathcal{C}^{\mathbf{2}}$—the nomenclature will be explained in Section 5.1 below—whose maps $(g, \varphi_{-g}) \to (g', \varphi_{-g'})$ are maps $(c,d)\colon g \to g'$ of $\mathcal{C}^{\mathbf{2}}$ for which $c \cdot \varphi_{\boldsymbol{f},g}(u,v) = \varphi_{\boldsymbol{f},g'}(cu, dv)$. Any R-algebra $\boldsymbol{g}$ induces the lifting operation $\Phi_{-\boldsymbol{g}}$ on its underlying map, and by (2.4) any algebra map respects these liftings; so we have a functor $\bar{\Phi}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ over $\mathcal{C}^{\mathbf{2}}$.

Lemma 1. $\bar{\Phi}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ *is injective on objects and fully faithful, and has in its image just those (g, φ_{-g}) such that $\varphi_{\boldsymbol{L}f,g}(u,v)\cdot\mu_f = \varphi_{\boldsymbol{L}Rf,g}(\varphi_{\boldsymbol{L}f,g}(u,v), v\cdot\mu_f)$ for all maps $(u,v)\colon Lf \to g$:*

$$
\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "\boldsymbol{L}f"'] & A \arrow[r, "u"] \arrow[dd, "\boldsymbol{L}f"] & C \arrow[dd, "g"] \\
Ef \arrow[d, "\boldsymbol{L}Rf"'] & & \\
ERf \arrow[r, "\mu_f"'] \arrow[urr, dotted, "\varphi(u,v)\cdot\mu_f"] & Ef \arrow[r, "v"'] \arrow[ur, dotted, "\varphi(u,v)"] & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\boldsymbol{L}f"'] & C \arrow[dd, "g"] \\
Ef \arrow[d, "\boldsymbol{L}Rf"'] \arrow[ur, dotted, "\varphi(u,v)"] & \\
ERf \arrow[r, "v\cdot\mu_f"'] \arrow[uur, dotted, "\varphi(\varphi(u,v),v\mu_f)"] & D
\end{tikzcd}\ . \tag{2.9}
$$

Proof. We observed in Section 2.5 that an R-algebra is determined by its liftings against coalgebras, whence $\bar{\Phi}$ is injective on objects. For full fidelity, let $\boldsymbol{g} = (g,p)$ and $\boldsymbol{h} = (h,q)$ be R-algebras and $(c,d)\colon (g, \Phi_{-\boldsymbol{g}}) \to (h, \Phi_{-\boldsymbol{h}})$ a map of underlying lifting operations; we must show (c,d) is an R-algebra map. Consider the diagram

$$
\begin{tikzcd}
C \arrow[r, "1"] \arrow[d, "\boldsymbol{L}g"'] & C \arrow[r, "c"] \arrow[d, "\boldsymbol{g}"] & D \arrow[d, "\boldsymbol{h}"] \\
Eg \arrow[r, "Rg"'] \arrow[ur, dotted, "p"] & C \arrow[r, "d"'] & C'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, "\boldsymbol{L}g"'] & C' \arrow[r, "1"] \arrow[d, "\boldsymbol{L}h"'] & C' \arrow[d, "\boldsymbol{h}"] \\
Eg \arrow[r, "E(c,d)"'] & Eh \arrow[r, "Rh"'] \arrow[ur, dotted, "q"] & D
\end{tikzcd}\ .
$$

By (2.6) $\Phi_{-\boldsymbol{g}}$ assigns the filler p to the far left square; so as (c,d) respects liftings, $\Phi_{-\boldsymbol{h}}$ assigns the filler $c \cdot p$ to the composite left rectangle. Likewise $\Phi_{-\boldsymbol{h}}$ assigns the filler q to the far right square; so as the square to its left is a (cofree) map of L-coalgebras, $\Phi_{-\boldsymbol{h}}$ assigns the filler $q \cdot E(c,d)$ to the composite right rectangle. Thus $c \cdot p = q \cdot E(c,d)$ and (c,d) is an R-algebra map as required.

We next show that each $(g, \Phi_{-\boldsymbol{g}})$ in the image of $\bar{\Phi}$ satisfies (2.9). By universality (2.5) and naturality (2.4), it suffices to take $\boldsymbol{g} = \boldsymbol{R}Lf$ and $(u,v) = (LLf, 1)$, and now

$$
\begin{aligned}
\Phi_{\boldsymbol{L}f,\boldsymbol{R}Lf}(LLf,1)\cdot\mu_f = \Delta_f \cdot \mu_f &= \Phi_{\boldsymbol{L}Rf,\boldsymbol{R}Lf}(\Delta_f, \mu_f) \\
&= \Phi_{\boldsymbol{L}Rf,\boldsymbol{R}Lf}(\Phi_{\boldsymbol{L}f,\boldsymbol{R}Lf}(LLf,1), \mu_f)
\end{aligned}
$$

by (2.6) and (2.8), as required. Finally, we show that $\bar{\Phi}$ is surjective onto those pairs (g, φ_{-g}) satisfying (2.9). Given such a pair, we define $p = \varphi_{\boldsymbol{L}g,g}(1, Rg)$; now by universality (2.5) and naturality (2.4), the pair (g, φ_{-g}) will be the image under $\bar{\Phi}$ of $\boldsymbol{g} = (g,p)$ so long as (g,p) is in fact an R-algebra. The unit axiom is already inherent in p's being a lifting; as for the multiplication axiom, we have

$$\begin{aligned} p \cdot \mu_g &= \varphi_{\boldsymbol{L}g,g}(1, Rg) \cdot \mu_g = \varphi_{\boldsymbol{L}Rg,g}(\varphi_{\boldsymbol{L}g,g}(1, Rg), Rg \cdot \mu_g) \\ &= \varphi_{\boldsymbol{L}Rg,g}(p, RRg) = \varphi_{\boldsymbol{L}Rg,g}(p, Rg \cdot E(p,1)) \\ &= \varphi_{\boldsymbol{L}g,g}(1, Rg) \cdot E(p,1) = p \cdot E(p,1) \end{aligned}$$

by definition of p, (2.9), and naturality of φ_{-g} with respect to the L-coalgebra morphism $(p, E(p,1)) \colon \boldsymbol{L}Rg \to \boldsymbol{L}g$. $\square$

It is not hard to see that the category $\mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ is in fact isomorphic to the category (R,η)-**Alg** of algebras for the mere pointed endofunctor (R,η) underlying the monad R, with the functor $\bar{\Phi}$ corresponding under this identification to the natural inclusion of R-**Alg** into (R,η)-**Alg**. Of course, all these results have a dual form characterising coalgebras in terms of their liftings against algebras.

2.8. **Double categories of algebras and coalgebras.** In a weak factorisation system the left and right classes of maps are closed under composition and contain the identities. We now describe the analogue of this for AWFS. For binary composition, given R-algebras $\boldsymbol{g} \colon A \to B$ and $\boldsymbol{h} \colon B \to C$ we may define a coalgebra lifting operation $\Phi_{-,\boldsymbol{h}\cdot\boldsymbol{g}}$ on the composite underlying map $h\cdot g$, whose liftings are obtained by first lifting against $\boldsymbol{h}$ and then against $\boldsymbol{g}$, as on the left in:

$$\Phi_{\boldsymbol{f},\boldsymbol{h}\cdot\boldsymbol{g}}(u,v) = \Phi_{\boldsymbol{f},\boldsymbol{g}}(u, \Phi_{\boldsymbol{f},\boldsymbol{h}}(gu, v)) \qquad \Phi_{\boldsymbol{f},\mathbf{1}_A}(u,v) = v \ . \tag{2.10}$$

It is easy to verify that this is a coalgebra lifting operation satisfying (2.9), so that by Lemma 1, it is the canonical lifting operation associated to a unique R-algebra $\boldsymbol{h}\cdot\boldsymbol{g} \colon A \to C$, the composite of $\boldsymbol{g}$ and $\boldsymbol{h}$. As for nullary composition, each identity map 1_A bears a *unique* coalgebra lifting operation as on the right above, which is easily seen to satisfy (2.9); so each identity map bears an R-algebra structure $\mathbf{1}_A$ which is in fact *unique*.

This composition law for algebras is associative and unital: to see associativity, we check that $\boldsymbol{k} \cdot (\boldsymbol{h} \cdot \boldsymbol{g})$ and $(\boldsymbol{k} \cdot \boldsymbol{h}) \cdot \boldsymbol{g}$ have the same lifting operations, and apply Lemma 1; unitality is similar. We may also use Lemma 1 to verify that each $(a,a) \colon \mathbf{1}_A \to \mathbf{1}_{A'}$ is a map of R-algebras, and that if $(a,b) \colon \boldsymbol{g} \to \boldsymbol{g}'$ and $(b,c) \colon \boldsymbol{h} \to \boldsymbol{h}'$ are maps of R-algebras, then so too is $(a,c) \colon \boldsymbol{h}\cdot\boldsymbol{g} \to \boldsymbol{h}'\cdot\boldsymbol{g}'$. Consequently, there is a double category $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ whose objects and horizontal arrows are those of $\mathcal{C}$, and whose vertical arrows and squares are the R-algebras and the maps thereof. There is a forgetful double functor $U^{\mathsf{R}} \colon \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ into the double category of commutative squares in $\mathcal{C}$, which, displayed as an internal functor between internal categories in **CAT**, is as on the left in:

$$\begin{tikzcd} \mathsf{R}\text{-}\mathbf{Alg} \times_{\mathcal{C}} \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, "U^{\mathsf{R}} \times_{\mathcal{C}} U^{\mathsf{R}}"] \arrow[d, shift right=3, "p"'] \arrow[d, "m"] \arrow[d, shift left=3, "q"] & \mathcal{C}^{\mathbf{3}} \arrow[d, shift right=3, "p"'] \arrow[d, "m"] \arrow[d, shift left=3, "q"] \\ \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, "U^{\mathsf{R}}"] \arrow[d, shift right=3, "d"'] \arrow[d, shift left=3, "c"] & \mathcal{C}^{\mathbf{2}} \arrow[d, shift right=3, "d"'] \arrow[d, shift left=3, "c"] \\ \mathcal{C} \arrow[u, "i"] \arrow[r, "1"'] & \mathcal{C} \arrow[u, "i"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \arrow[r, "V_1 \times_{V_0} V_1"] \arrow[d, shift right=3, "p"'] \arrow[d, "m"] \arrow[d, shift left=3, "q"] & \mathcal{C}^{\mathbf{3}} \arrow[d, shift right=3, "p"'] \arrow[d, "m"] \arrow[d, shift left=3, "q"] \\ \mathcal{A}_1 \arrow[r, "V_1"] \arrow[d, shift right=3, "d"'] \arrow[d, shift left=3, "c"] & \mathcal{C}^{\mathbf{2}} \arrow[d, shift right=3, "d"'] \arrow[d, shift left=3, "c"] \\ \mathcal{A}_0 \arrow[u, "i"] \arrow[r, "V_0 = 1"'] & \mathcal{C} \arrow[u, "i"] \end{tikzcd} .$$

Note that this double functor has object component *an identity* and arrow component a *faithful functor*. By a *concrete double category over* $\mathcal{C}$, we mean a double functor $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ (as on the right above) whose V has these two properties. For example, the L-coalgebras also constitute a concrete double category $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$ over $\mathcal{C}$.

We note before continuing that the equality (2.9) may now be re-expressed as saying that each square as on the left below is one of $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$, and each square on the right is one of $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$. These squares will be important in what follows.

$$(2.11)\qquad \begin{tikzcd} A \arrow[r, "1"] \arrow[d, "\boldsymbol{L}f"'] & A \arrow[dd, "\boldsymbol{L}f"] \\ Ef \arrow[d, "\boldsymbol{L}Rf"'] & \\ ERf \arrow[r, "\mu_f"'] & Ef \end{tikzcd} \qquad\qquad \begin{tikzcd} Ef \arrow[r, "\Delta_f"] \arrow[dd, "\boldsymbol{R}f"'] & ELf \arrow[d, "\boldsymbol{R}Lf"] \\ & Ef \arrow[d, "\boldsymbol{R}f"] \\ B \arrow[r, "1"'] & B \end{tikzcd}\ .$$

2.9. **Morphisms of AWFS.** Given AWFS (L,R) and $(\mathsf{L}',\mathsf{R}')$ on categories $\mathcal{C}$ and $\mathcal{D}$, a *lax morphism* of AWFS $(F,\alpha)\colon (\mathcal{C},\mathsf{L},\mathsf{R}) \to (\mathcal{D},\mathsf{L}',\mathsf{R}')$ comprises a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a natural family of maps α_f rendering commutative each square as on the left in:

$$(2.12)\qquad \begin{tikzcd} & FA \arrow[dl, "L'Ff"'] \arrow[dr, "FLf"] & \\ E'Ff \arrow[rr, "\alpha_f"] \arrow[dr, "R'Ff"'] & & FEf \arrow[dl, "FRf"] \\ & FB & \end{tikzcd} \qquad\qquad \begin{tikzcd} E'Ff \arrow[r, "\alpha_f"] \arrow[d, "E'(\gamma_A,\gamma_B)"'] & FEf \arrow[d, "\gamma_{Ef}"] \\ E'Gf \arrow[r, "\beta_f"'] & GEf \end{tikzcd}\ ,$$

and such that the induced $(\alpha,1)\colon R'F^{\mathbf{2}} \Rightarrow F^{\mathbf{2}}R$ and $(1,\alpha)\colon L'F^{\mathbf{2}} \Rightarrow F^{\mathbf{2}}L$ are respectively a lax monad morphism $\mathsf{R} \to \mathsf{R}'$ and a lax comonad morphism $\mathsf{L} \to \mathsf{L}'$ over $F^{\mathbf{2}}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{D}^{\mathbf{2}}$ (i.e., a *monad functor* and a *comonad opfunctor* in the terminology of [48]). A transformation $(F,\alpha) \Rightarrow (G,\beta)$ between lax morphisms is a natural transformation $\gamma\colon F \Rightarrow G$ rendering commutative the square above right for each $f\colon A \to B$ in $\mathcal{C}$. Algebraic weak factorisation systems, lax morphisms and transformations constitute a 2-category $\mathbf{AWFS}_{\mathrm{lax}}$. Dually, an *oplax morphism* of AWFS involves maps α_f with the opposing orientation to (2.12), and with the induced $(\alpha,1)$ and $(1,\alpha)$ now being *oplax* monad and comonad morphisms over $F^{\mathbf{2}}$. With a similar adaptation on 2-cells, this yields a 2-category $\mathbf{AWFS}_{\mathrm{oplax}}$.

2.10. **Adjunctions of AWFS.** Given AWFS (L,R) and $(\mathsf{L}',\mathsf{R}')$ on categories $\mathcal{C}$ and $\mathcal{D}$, and an adjunction $F \dashv G\colon \mathcal{C} \to \mathcal{D}$, there is a bijection between 2-cells α exhibiting G as a lax AWFS morphism and 2-cells β exhibiting F as oplax; this is the *doctrinal adjunction* of [35]. From α we determine the components of β by

$$\beta_f = FE'f \xrightarrow{FE'(\eta_A,\eta_B)} FE'GFf \xrightarrow{F\alpha_{Ff}} FGEFf \xrightarrow{\epsilon_{EFf}} EFf\ ,$$

so that β is the *mate* [38] of α under the adjunctions $F \dashv G$ and $F^{\mathbf{2}} \dashv G^{\mathbf{2}}$. The functoriality of this correspondence is expressed through an identity-on-objects isomorphism of 2-categories $\mathbf{AWFS}_{\mathrm{radj}}{}^{\mathrm{coop}} \cong \mathbf{AWFS}_{\mathrm{ladj}}$, where $\mathbf{AWFS}_{\mathrm{radj}}$ is defined identically to $\mathbf{AWFS}_{\mathrm{lax}}$ except that its 1-cells come equipped with chosen left adjoints, and where $\mathbf{AWFS}_{\mathrm{ladj}}$ is defined from $\mathbf{AWFS}_{\mathrm{oplax}}$ dually. By an *adjunction of* AWFS, we mean a morphism of one of these isomorphic 2-categories; thus, a pair of a lax AWFS morphism (G, α) and an oplax AWFS morphism (F, β) whose underlying functors are adjoint, and whose 2-cell data determine each other by mateship. In this situation, an easy calculation shows that

$$\Phi_{\boldsymbol{f},G\boldsymbol{g}}(u,v) = \Phi_{F\boldsymbol{f},\boldsymbol{g}}(\bar{u},\bar{v}) \tag{2.13}$$

for any L'-coalgebra $\boldsymbol{f}$, any R-algebra $\boldsymbol{g}$, and any $(u,v)\colon f \to Gg$ in $\mathcal{D}^{\mathbf{2}}$ with adjoint transpose $(\bar{u},\bar{v})\colon Ff \to g$ in $\mathcal{C}^{\mathbf{2}}$.

2.11. **Semantics 2-functors.** A lax morphism of AWFS $(F,\alpha)\colon (\mathcal{C},\mathsf{L},\mathsf{R}) \to (\mathcal{C}',\mathsf{L}',\mathsf{R}')$ has an underlying lax monad morphism, yielding as in [30, Lemma 1] a lifted functor as to the left in

$$\begin{tikzcd} \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, dotted, "\bar{F}"] \arrow[d, "U^{\mathsf{R}}"'] & \mathsf{R}'\text{-}\mathbf{Alg} \arrow[d, "U^{\mathsf{R}'}"] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"'] & \mathcal{D}^{\mathbf{2}} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \arrow[r, dotted, "\bar{F}"] \arrow[d, "U^{\mathsf{R}}"'] & \mathsf{R}'\text{-}\mathbb{A}\mathbf{lg} \arrow[d, "U^{\mathsf{R}'}"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"'] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd} \tag{2.14}$$

whose action on algebras we will abusively denote by $\boldsymbol{g} \mapsto F\boldsymbol{g}$. A short calculation from the fact that $(1,\alpha)$ is a lax comonad morphism shows that $F\Phi_{\boldsymbol{L}f,\boldsymbol{g}}(u,v)\cdot\alpha_f = \Phi_{\boldsymbol{L}'Ff,F\boldsymbol{g}}(Fu, Fv\cdot\alpha_f)$ for each R-algebra $\boldsymbol{g}$. From this and (2.10) it follows that $\Phi_{\boldsymbol{L}'Ff,F\boldsymbol{h}\cdot F\boldsymbol{g}}(Fu, Fv\cdot\alpha_{hg}) = \Phi_{\boldsymbol{L}'Ff,F(\boldsymbol{h}\cdot\boldsymbol{g})}(Fu, Fv\cdot\alpha_{hg})$ for all composable R-algebras $\boldsymbol{g}$ and $\boldsymbol{h}$; now taking $f = hg$ and $(u,v) = (1, Rhg)$ and applying (2.6), we conclude that $F\boldsymbol{h}\cdot F\boldsymbol{g} = F(\boldsymbol{h}\cdot\boldsymbol{g})$. Thus the lifted functor on the left of (2.14) preserves algebra composition; it must also preserve the (unique) algebra structures on identities, and so underlies a double functor as on the right. Moreover, each 2-cell $\gamma\colon (F,\alpha) \to (G,\beta)$ in $\mathbf{AWFS}_{\mathrm{lax}}$ has an underlying lax monad transformation, so that $\gamma\colon F \to G$ may be lifted to a transformation on categories of algebras, which—by concreteness—lifts further to a horizontal transformation on double categories.

The preceding constructions have evident duals involving $\mathbf{AWFS}_{\mathrm{oplax}}$ and coalgebras; and in this way, we obtain the left and right *semantics* 2-*functors*:

$$(\text{–})\text{-}\mathbb{C}\mathbf{oalg}\colon \mathbf{AWFS}_{\mathrm{oplax}} \to \mathbf{DBL}^{\mathbf{2}} \qquad (\text{–})\text{-}\mathbb{A}\mathbf{lg}\colon \mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{DBL}^{\mathbf{2}}\ , \tag{2.15}$$

where $\mathbf{DBL}$ denotes the 2-category of double categories, double functors and horizontal transformations. The following result is now [43, Lemma 6.9]:

Proposition 2. *The* 2*-functors* $(\text{–})\text{-}\mathbb{A}\mathbf{lg}$ *and* $(\text{–})\text{-}\mathbb{C}\mathbf{oalg}$ *are* 2*-fully faithful.*

Proof. By duality, we need only deal with the case of $(\text{–})\text{-}\mathbb{A}\mathbf{lg}$. Note that each double functor $\mathbb{S}\mathbf{q}(\mathcal{C}) \to \mathbb{S}\mathbf{q}(\mathcal{D})$ must be of the form $\mathbb{S}\mathbf{q}(F)$ for some $F\colon \mathcal{C} \to \mathcal{D}$; thus given $(\mathcal{C},\mathsf{L},\mathsf{R})$ and $(\mathcal{D},\mathsf{L}',\mathsf{R}')$ in $\mathbf{AWFS}_{\mathrm{lax}}$, a morphism between their images in $\mathbf{DBL}^{\mathbf{2}}$ amounts to a square as on the right of (2.14). Each such has its underlying action on vertical arrows and squares as on the left in (2.14), and so must be induced by a unique lax monad morphism $\gamma\colon \mathsf{R} \to \mathsf{R}'$ over $F^{\mathbf{2}}$. As R and R' are monads over the codomain functor, we must have $\gamma = (\alpha, 1)$ for natural maps α_f as in (2.12). It remains to show that $(1, \alpha)$ is a lax comonad morphism $\mathsf{L} \to \mathsf{L}'$ over $F^{\mathbf{2}}$. The compatibilities in (2.12) already show that $(1, \alpha)$ commutes with counits; we need to establish the same for the comultiplications. Consider the diagrams:

$$
\begin{tikzcd}
E'Ff \arrow[r, "\Delta'_{Ff}"] \arrow[dd, "\boldsymbol{R}'Ff"] & E'L'Ff \arrow[r, "E'(1,\alpha_f)"] \arrow[d, "\boldsymbol{R}'L'Ff"'] & E'FLf \arrow[r, "\alpha_{Lf}"] \arrow[d, "\boldsymbol{R}'FLf"'] & FELf \arrow[d, "F\boldsymbol{R}Lf"'] \\
 & E'Ff \arrow[r, "\alpha_f"'] \arrow[d, "\boldsymbol{R}'Ff"'] & FEf \arrow[r, "1"'] \arrow[d, "F\boldsymbol{R}f"'] & FEf \arrow[d, "F\boldsymbol{R}f"'] \\
FB \arrow[r, "1"'] & FB \arrow[r, "1"'] & FB \arrow[r, "1"'] & FB
\end{tikzcd}
\qquad
\begin{tikzcd}
E'Ff \arrow[r, "\alpha_f"] \arrow[dd, "\boldsymbol{R}'Ff"] & FEf \arrow[r, "F\Delta_f"] \arrow[dd, "F\boldsymbol{R}f"] & FELf \arrow[d, "F\boldsymbol{R}Lf"'] \\
 & & FEf \arrow[d, "F\boldsymbol{R}f"'] \\
FB \arrow[r, "1"'] & FB \arrow[r, "1"'] & FB\,.
\end{tikzcd}
$$

Every region here is a square of R'-$\mathbb{A}\mathbf{lg}$: the leftmost as it is of the form (2.11), the rightmost as it is the image under the lifted double functor of another such square in R-$\mathbb{A}\mathbf{lg}$, and all the rest because $(\alpha, 1)$ is a lax monad morphism. So the exterior squares are also squares in R'-$\mathbb{A}\mathbf{lg}$; but as both have the same composite $FLLf$ with the unit morphism $(L'Ff, 1)\colon Ff \to R'Ff$, they must, by freeness of $\boldsymbol{R}'Ff$, coincide; now the equality of their domain-components expresses precisely the required compatibility with comultiplications.

This completes the proof of full fidelity on 1-cells; on 2-cells, it is easy to see that in the commuting diagram of 2-functors

$$
\begin{tikzcd}
\mathbf{AWFS}_{\mathrm{lax}} \arrow[r, "(-)\text{-}\mathbb{A}\mathbf{lg}"] \arrow[d, "(\mathcal{C},\mathsf{L},\mathsf{R})\mapsto(\mathcal{C}^{\mathbf{2}},\mathsf{R})"'] & \mathbf{DBL}^{\mathbf{2}} \arrow[d, "(-)_1"] \\
\mathbf{MND}_{\mathrm{lax}} \arrow[r, "(-)\text{-}\mathbf{Alg}"] & \mathbf{CAT}^{\mathbf{2}}
\end{tikzcd}
$$

the left and bottom edges are locally fully faithful, and that the right edge is locally fully faithful on the concrete double categories in the image of $(-)$-$\mathbb{A}\mathbf{lg}$; whence the top edge is also locally fully faithful. $\square$

2.12. **Orthogonal factorisation systems.** Recall that an *orthogonal factorisation system* [18] is a weak factorisation systems $(\mathcal{L}, \mathcal{R})$ wherein liftings (1.1) of $\mathcal{L}$-maps against $\mathcal{R}$-maps are unique. The following result characterises those AWFS arising from orthogonal factorisation systems: it improves on Theorem 3.2 of [25] by requiring idempotency of only *one* of L or R. This resolves the open question posed in Remark 3.3(a) of *ibid*.

Proposition 3. *Let* (L, R) *be an* AWFS *on* $\mathcal{C}$. *The following are equivalent:*

(i) L-$\mathbf{Coalg} \to \mathcal{C}^{\mathbf{2}}$ *is fully faithful;*
(ii) L *is an idempotent comonad;*
(iii) For each L*-coalgebra* $\boldsymbol{f}\colon A \to B$*, there is a coalgebra map* $(f, 1)\colon \boldsymbol{f} \to \mathbf{1}_B$*;*
(iv) R-$\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ *is fully faithful;*

(v) R *is an idempotent monad;*
(vi) *For each* R*-algebra* $\boldsymbol{g}\colon C \to D$*, there is an algebra map* $(1, g)\colon \mathbf{1}_C \to \boldsymbol{g}$*;*
(vii) *Liftings of* L*-coalgebras against* R*-algebras are unique;*
(viii) *The underlying weak factorisation system* $(\mathcal{L}, \mathcal{R})$ *is orthogonal.*

Under these circumstances, moreover, the AWFS (L, R) *is determined up to isomorphism by the underlying* $(\mathcal{L}, \mathcal{R})$*.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (i) by standard properties of idempotent comonads, and (i) $\Rightarrow$ (iii) is trivial. We next prove (iii) $\Rightarrow$ (vii). So given $\boldsymbol{f} \in \mathsf{L}\text{-}\mathbf{Coalg}$ and $\boldsymbol{g} \in \mathsf{R}\text{-}\mathbf{Alg}$, we must show that any diagonal filler j for a square $(u, v)\colon f \to g$, as on the left below, is equal to the canonical filler $\Phi_{\boldsymbol{f},\boldsymbol{g}}(u, v)$.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dotted, "j"] & D \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[r, "j"] \arrow[d, "1"'] & C \arrow[d, "g"] \\ B \arrow[r, "1"'] & B \arrow[r, "v"'] & D \end{tikzcd} .$$

So factorise (u, v) as on the right above; by (iii), the left-hand square is a coalgebra map $\boldsymbol{f} \to \mathbf{1}_B$, whence by naturality (2.4) we have $j = \Phi_{\mathbf{1}_B,\boldsymbol{g}}(j, v) = \Phi_{\boldsymbol{f},\boldsymbol{g}}(u, v)$ as required. We next prove (vii) $\Rightarrow$ (iv); so for algebras $\boldsymbol{g}$ and $\boldsymbol{h}$, we must show that each $(c, d)\colon g \to h$ underlies an algebra map $\boldsymbol{g} \to \boldsymbol{h}$. By Lemma 1, it suffices to show that (c, d) commutes with the coalgebra lifting functions; in other words, that for each coalgebra $\boldsymbol{f}$ and each $(u, v)\colon f \to g$, we have $c \cdot \Phi_{\boldsymbol{f},\boldsymbol{g}}(u, v) = \Phi_{\boldsymbol{f},\boldsymbol{h}}(cu, dv)$. This is so by (vii) since both these maps fill the square $(cu, dv)\colon f \to h$.

Dual arguments now prove that (iv) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i); it remains to show that (i)–(vii) are equivalent to (viii). Clearly (viii) $\Rightarrow$ (vii); on the other hand, if (i) and (iv) hold, then any retract of an L-coalgebra or R-algebra is again a coalgebra or algebra—being given by the splitting of an idempotent in R-**Alg** or L-**Coalg**—so that any map in $\mathcal{L}$ or $\mathcal{R}$ is the underlying map of an L-coalgebra or R-algebra; whence by (vii) liftings of $\mathcal{L}$-maps against $\mathcal{R}$-maps are unique.

Finally, if (i)–(viii) hold, then we can reconstruct L and R up to isomorphism from $\mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ and $\mathsf{L}\text{-}\mathbf{Coalg} \to \mathcal{C}^{\mathbf{2}}$, and can reconstruct these in turn from $\mathcal{L}$ and $\mathcal{R}$ as the full subcategories of $\mathcal{C}^{\mathbf{2}}$ on the $\mathcal{L}$-maps and the $\mathcal{R}$-maps. □

3. A Beck theorem for AWFS

In this section, we give our first main result—Theorem 6 below—which provides an elementary characterisation of the concrete double categories in the essential image of the semantics 2-functors (2.15). This will allow us to construct an AWFS simply by exhibiting a double category of an appropriate form to be one's double category of algebras or coalgebras. As explained in the introduction, the essential images of the corresponding semantics 2-functors for monads and comonads are characterised by Beck's (co)monadicity theorem, and so we term our result a "Beck theorem" for algebraic weak factorisation systems.

3.1. **Reconstruction.** There are two main aspects to the Beck theorem. The first is the following reconstruction result; in the special case where R is *idempotent*, this is [29, Proposition 5.9], while the general case is given as [6, Theorem 4.15]; for the sake of a self-contained presentation, we include a proof here, which improves on that of [6] only in trivial ways.

Proposition 4. *Let $\mathsf{R}\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ be a monad over the codomain functor. The right semantics 2-functor (2.15) induces a bijection between extensions of R to an algebraic weak factorisation system (L, R) and extensions of the diagram*

$$\begin{tikzcd} \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, "U^{\mathsf{R}}"] \arrow[d, shift right, "dU^{\mathsf{R}}"'] \arrow[d, shift left, "cU^{\mathsf{R}}"] & \mathcal{C}^{\mathbf{2}} \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] \\ \mathcal{C} \arrow[r, "1"] & \mathcal{C} \end{tikzcd} \tag{3.1}$$

to a concrete double category over $\mathcal{C}$.

Proof. To give an extension of (3.1) to a concrete double category over $\mathcal{C}$ is to give a composition law $\boldsymbol{h}, \boldsymbol{g} \mapsto \boldsymbol{h} \star \boldsymbol{g}$ on R-algebras which is associative, unital, and compatible with R-algebra maps. For each such, we must exhibit a unique (L, R) whose induced R-algebra composition is $\star$. As in Section 2.1, the monad R determines the (L, ϵ) underlying L, and so we need only give the maps Δ_f satisfying appropriate axioms. Consider the diagram on the left in:

$$\begin{tikzcd} A \arrow[rr, "LLf"] \arrow[dd, "f"'] \arrow[dr, "Lf"] & & ELf \arrow[d, "RLf"] \\ & Ef \arrow[ur, dotted, "\Delta_f"] \arrow[d, "Rf"'] & Ef \arrow[d, "Rf"] \\ B \arrow[r, "1"] & B \arrow[r, "1"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dd, "f"'] \arrow[dr, "Lf"] & & B \arrow[dd, "1_B"] \\ & Ef \arrow[ur, dotted, "Rf"] \arrow[d, "Rf"'] & \\ B \arrow[r, "1"] & B \arrow[r, "1"] & B\,. \end{tikzcd}$$

The outer square commutes, and the right edge bears the R-algebra structure $\boldsymbol{R}f \star \boldsymbol{R}Lf$; now applying (2.5) yields the map Δ_f, with the induced square forming an R-algebra morphism $\boldsymbol{R}f \to \boldsymbol{R}f \star \boldsymbol{R}Lf$. If $\star$ did arise from an AWFS, this induced square would be exactly the right square of (2.11); whence these Δ_f's are the unique possible choice for L's comultiplication. By pasting together appropriate squares as in the proof of Proposition 2—involving algebra squares as on the left above, and also ones $(Rf, 1)\colon \boldsymbol{R}f \to \mathbf{1}_B$ induced by universality as on the right above—we may now check that these Δ_f's satisfy the comonad and distributivity axioms; we thus have the desired (L, R), and it remains only to show that the composition law it induces coincides with $\star$. So let $\boldsymbol{g}$ and $\boldsymbol{h}$ be composable R-algebras, write $f = hg$ for the composite of the underlying maps, and consider the left-hand diagram in:

$$
\begin{tikzcd}
A \arrow[rr, "1"] \arrow[dd, "Lf"'] \arrow[dr, "LLf"] & & A \arrow[d, "g"] \\
& ELf \arrow[ur, dotted, "\exists!\ell"] \arrow[d, "RLf"'] & B \arrow[dd, "h"] \\
Ef \arrow[r, "1"'] \arrow[d, "Rf"'] & Ef \arrow[ur, dotted, "\exists!m"'] \arrow[d, "Rf"'] & \\
C \arrow[r, "1"] & C \arrow[r, "1"] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
Ef \arrow[r, "\Delta_f"] \arrow[dd, "Rf"'] & ELf \arrow[r, "\ell"] \arrow[d, "RLf"'] & A \arrow[d, "g"] \\
& Ef \arrow[r, "m"] \arrow[d, "Rf"'] & B \arrow[d, "h"] \\
C \arrow[r, "1"] & C \arrow[r, "1"] & C\,.
\end{tikzcd}
$$

The dotted maps m and ℓ are obtained by successively applying (2.5) to the maps $(g, 1)\colon f \to h$ and $(1, m)\colon Lf \to g$; thus the two squares on the far right above are maps of R-algebras $\boldsymbol{R}Lf \to \boldsymbol{g}$ and $\boldsymbol{R}f \to \boldsymbol{h}$. The rectangle to the left of these squares is a map of R-algebras $\boldsymbol{R}f \to \boldsymbol{R}f \star \boldsymbol{R}Lf$ by definition, and also one $\boldsymbol{R}f \to \boldsymbol{R}f \cdot \boldsymbol{R}Lf$, by (2.11). Thus the composite square can be seen both as a map $\boldsymbol{R}f \to \boldsymbol{R}Lf \cdot \boldsymbol{R}f \to \boldsymbol{h} \cdot \boldsymbol{g}$ and as one $\boldsymbol{R}f \to \boldsymbol{R}Lf \star \boldsymbol{R}f \to \boldsymbol{h} \star \boldsymbol{g}$. Since precomposing further with the unit $f \to Rf$ yields the identity square on $f = hg$, we conclude by universality of $\boldsymbol{R}f$ that $(\ell\Delta_f, 1)$ is both the R-algebra structure of $\boldsymbol{h} \cdot \boldsymbol{g}$ and that of $\boldsymbol{h} \star \boldsymbol{g}$, which thus coincide. $\square$

3.2. **Monads over the codomain functor.** What is missing from the last result is a characterisation of when a monad on an arrow category is over the codomain functor. Our next result provides this; it generalises the characterisation in [29, Proposition 5.1] of *idempotent* monads over the codomain functor.

Proposition 5. *A monad* R *on* $\mathcal{C}^{\mathbf{2}}$ *is isomorphic to one over the codomain functor if and only if:*

(a) Each identity map has an R*-algebra structure* $\mathbf{1}_A$*;*
(b) For each $f\colon A \to B$*, there is an algebra map* $(f, f)\colon \mathbf{1}_A \to \mathbf{1}_B$*;*
(c) For each R*-algebra* $\boldsymbol{g}\colon A \to B$*, there is an algebra map* $(g, 1)\colon \boldsymbol{g} \to \mathbf{1}_B$*.*

Moreover, in this situation, the algebra structures on identity arrows are unique.

Proof. As the properties (a)–(c) are clearly invariant under monad isomorphism, we may assume in the "only if" direction that R is a monad over the codomain functor; now a short calculation shows that $\mathbf{1}_A = (1_A, R1_A)\colon A \to A$ is the unique R-algebra structure on 1_A, and that (b)–(c) are then satisfied. Conversely, let R satisfy (a)–(c), and consider the unit map $(r, s)\colon f \to Rf$, as on the left in:

$$
\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[d, "Rf"] \\
B \arrow[r, "s"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & C \arrow[d, "1"] \\
B \arrow[r, "1"] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "t"] \arrow[d, "Rf"] & B \arrow[d, "1"] \\
B \arrow[r, "s"] & D \arrow[r, "u"] & B\,.
\end{tikzcd}
$$

We claim that s is invertible. Since Rf underlies the free R-algebra on f, and 1_C has by (a) the algebra structure $\mathbf{1}_C$, the commuting square in the middle above induces a unique algebra map $(t, u)\colon \boldsymbol{R}f \to \mathbf{1}_C$ making the rightmost diagram commute. In particular $us = 1$, and it remains to show that $su = 1$. Using (b), we have the algebra map $(s, s) \cdot (t, u)\colon \boldsymbol{R}f \to \mathbf{1}_B \to \mathbf{1}_D$, and using (c) we have $(Rf, 1)\colon \boldsymbol{R}f \to \mathbf{1}_D$. These maps coincide on precomposition with the unit $(r, s)\colon f \to Rf$, as on the left of:

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "t"] \arrow[d, "Rf"] & B \arrow[r, "s"] \arrow[d, "1"] & D \arrow[d, "1"] \\ B \arrow[r, "s"] & D \arrow[r, "u"] & B \arrow[r, "s"] & D \end{tikzcd} = \begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "Rf"] \arrow[d, "Rf"] & D \arrow[d, "\mathbf{1}"] \\ B \arrow[r, "s"] & D \arrow[r, "1"] & D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "1"] \arrow[d, "Rf"'] & C \arrow[d, "R'f"] \\ D \arrow[r, "u"] & B \end{tikzcd}$$

So they must themselves agree; in particular, $su = 1$ and s and u are isomorphisms. So defining $R'f = uRf$, we have isomorphisms $Rf \cong R'f$ for each $f \in \mathcal{C}^{\mathbf{2}}$ as on the right above; now successively transporting the functor R and the monad structure thereon along these isomorphisms yields a monad $\mathsf{R}' \cong \mathsf{R}$ over the codomain functor, as required. □

3.3. **The Beck theorem.** Combining the preceding two results, we obtain our first main theorem, characterising the concrete double categories in the essential image of the semantics 2-functor $(-)$-$\mathbb{A}\mathbf{lg}$. Of course, we have also the dual result, which we do not trouble to state, characterising the essential image of $(-)$-$\mathbb{C}\mathbf{oalg}$.

Theorem 6. *The* 2*-functor* $(-)$-$\mathbb{A}\mathbf{lg}\colon \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL}^{\mathbf{2}}$ *has in its essential image exactly those concrete double categories* $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *such that:*

(i) The functor $V_1\colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ *on vertical arrows and squares is strictly monadic;*
(ii) For each vertical arrow $\boldsymbol{f}\colon A \to B$ *of* $\mathbb{A}$*, the following is a square of* $\mathbb{A}$*:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\boldsymbol{f}"'] & B \arrow[d, "\mathbf{1}"] \\ B \arrow[r, "1"] & B \end{tikzcd} \tag{3.2}$$

Proof. Any $U^{\mathsf{R}}\colon \mathsf{R}$-$\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ clearly has property (i) while property (ii) follows from Proposition 5. Suppose conversely that the concrete $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ satisfies (i) and (ii). As V_1 is strictly monadic, $\mathcal{A}_1$ is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to the category of algebras for the monad R induced by V_1 and its left adjoint. This R satisfies (a)–(c) of Proposition 5: (c) by virtue of (ii) above, and (a) and (b) using the vertical identities of the double category $\mathbb{A}$. Thus we have a monad $\mathsf{R}' \cong \mathsf{R}$ over the codomain functor; now transporting the double category structure of $V\colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ along the isomorphisms $\mathcal{A}_1 \cong \mathsf{R}$-$\mathbf{Alg} \cong \mathsf{R}'$-$\mathbf{Alg}$ and $\mathcal{C}^{\mathbf{2}}$ yields $V'\colon \mathbb{A}' \to \mathbb{S}\mathbf{q}(\mathcal{C})$ which, by Proposition 4, is in the image of $(-)$-$\mathbb{A}\mathbf{lg}$. □

We will call a concrete double category *right-connected* if it satisfies (ii) above, and *monadic right-connected* if it satisfies (i) and (ii). Now combining Theorem 6 with Proposition 4 and the remarks preceding it yields the following result; again, there is a dual form which we do not state characterising $\mathbf{AWFS}_{\text{oplax}}$.

Corollary 7. *The* 2*-category* $\mathbf{AWFS}_{\text{lax}}$ *is equivalent to the full sub-*2*-category of* $\mathbf{DBL}^{\mathbf{2}}$ *on the monadic right-connected concrete double categories.*

We conclude this section by describing two variations on our main result which will come in useful from time to time.

3.4. **Discrete pullback-fibrations.** Under mild conditions on the base category $\mathcal{C}$, we may rephrase Proposition 5 so as to obtain a more intuitive characterisation of the concrete double categories over $\mathcal{C}$ in the essential image of the semantics functor. Let us define a functor $p\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$ to be a *discrete pullback-fibration* if, for every $\boldsymbol{g} \in \mathcal{A}$ over $g \in \mathcal{C}^{\mathbf{2}}$ and every pullback square $(h,k)\colon f \to g$, there is a unique arrow $\varphi\colon \boldsymbol{f} \to \boldsymbol{g}$ in $\mathcal{A}$ over (h,k), and this arrow is cartesian:

$$\exists! \boldsymbol{f} \overset{\exists!\varphi}{\dashrightarrow} \boldsymbol{g} \qquad \longmapsto \qquad \begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] & D \end{tikzcd} . \tag{3.3}$$

If $\mathcal{C}$ has all pullbacks, then the codomain functor $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ is a fibration, with the pullback squares in $\mathcal{C}^{\mathbf{2}}$ as its cartesian arrows; now the fact that the lifts (3.3) are cartesian implies that the composite functor $\mathrm{cod} \cdot p\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ is also a Grothendieck fibration, with cartesian liftings preserved and reflected by p.

Proposition 8. *If $\mathcal{C}$ has pullbacks, then a monad R on $\mathcal{C}^{\mathbf{2}}$ is isomorphic to one over the codomain functor if and only if $U^{\mathsf{R}}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ is a discrete pullback-fibration.*

Under the hypotheses of this proposition, the composite $\mathrm{cod} \cdot U^{\mathsf{R}}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}$ is a Grothendieck fibration; which, loosely speaking, is the statement that "R-algebra structure is stable under pullback".

Proof. First let R be over the codomain functor. Given an R-algebra $\boldsymbol{g} = (g,p)\colon C \to D$ and pullback square as in (3.3), form the unique map $q\colon Ef \to A$ with $fq = Rf$ and $hq = p \cdot E(h,k)$. This easily yields an R-algebra $(f,q)\colon A \to B$ for which $(h,k)\colon (f,q) \to (g,p)$ is a cartesian algebra map; moreover, any q' for which (h,k) were a map $(f,q') \to (g,p)$ would have to satisfy the defining conditions of q; whence q is unique as required. Conversely, suppose that U^{R} is a discrete pullback-fibration. Arguing as in Proposition 5, it suffices to show that each unit-component $(r,s)\colon f \to Rf$ has s invertible. Form the pullback of Rf along s and induced map t as on the left in:

$$\begin{tikzcd} A \arrow[r, dotted, "t"] \arrow[rr, bend left, "r"] \arrow[d, "f"'] & E \arrow[r, "u"] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "Rf"] \\ B \arrow[r, "1"'] & B \arrow[r, "s"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "t"] \arrow[d, "f"'] & E \arrow[d, "g"] \\ B \arrow[r, "1"'] & B \end{tikzcd} = \begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "v"] \arrow[d, "Rf"'] & E \arrow[d, "g"] \\ B \arrow[r, "s"'] & D \arrow[r, "w"'] & B \end{tikzcd} .$$

As U^{R} is a discrete pullback-fibration, there is a unique R-algebra structure $\boldsymbol{g}$ on g making $(u,s)\colon \boldsymbol{g} \to \boldsymbol{R}\boldsymbol{f}$ an algebra map; now applying freeness of $\boldsymbol{R}\boldsymbol{f}$ to the central square above yields a unique algebra map $(v,w)\colon \boldsymbol{R}\boldsymbol{f} \to \boldsymbol{g}$ making the right-hand diagram commute. In particular, $ws = 1$; on the other hand, the algebra map $(uv, sw)\colon \boldsymbol{R}\boldsymbol{f} \to \boldsymbol{g} \to \boldsymbol{R}\boldsymbol{f}$ precomposes with the unit $(r,s)\colon f \to Rf$ to yield (r,s); whence $(uv,sw) = (1,1)$ and s is invertible as required. $\square$

The evident adaptation of the proof of Theorem 6 now yields:

Theorem 9. *If $\mathcal{C}$ admits pullbacks, then a concrete double category $V \colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ is in the essential image of $(\text{–})\text{-}\mathbb{A}\mathbf{lg} \colon \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL}^{\mathbf{2}}$ just when:*

(i) $V_1 \colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic;
(ii) $V_1 \colon \mathcal{A}_1 \to \mathcal{C}^{\mathbf{2}}$ is a discrete pullback-fibration.

Remark 10. Note that for the "if" direction of this result, $\mathcal{C}$ does not need to have *all* pullbacks; a closer examination of the proof of Proposition 8 shows that only pullbacks along (the underlying maps of) R-algebras are needed. This is useful in practice; categories with pullbacks of this restricted kind arise in the study of stacks [47], or in the categorical foundations of Martin–Löf type theory [20].

3.5. **Intrinsic concreteness.** Above, we have chosen to view the semantics 2-functors (2.15) as landing in the 2-category $\mathbf{DBL}^{\mathbf{2}}$. We have done this so as to stress the analogy with the situation for monads and comonads, and also because this is the most practically useful form of our results. Yet this presentation has some redundancy, as we now explain.

In the presence of the remaining hypotheses, condition (ii) of Theorem 6 is easily seen to be equivalent to the requirement that the codomain functor $c \colon \mathcal{A}_1 \to \mathcal{A}_0$ of the double category $\mathbb{A}$ be a left adjoint left inverse for the identities functor $i \colon \mathcal{A}_0 \to \mathcal{A}_1$ (such *lalis* will appear again in Section 4.2 below). This is a property of $\mathbb{A}$, rather than extra structure; namely the property that

$$\text{each } c_{\boldsymbol{f},iB} \colon \mathcal{A}_1(\boldsymbol{f}, iB) \to \mathcal{A}_0(c\boldsymbol{f}, B) \text{ is invertible.} \tag{3.4}$$

But given only this, we may reconstruct the double functor $V \colon \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{A}_0)$. First we reconstruct the components (3.2) of the unit $\eta \colon 1 \Rightarrow ic$ using (3.4); now we reconstruct $V_1 \colon \mathcal{A}_1 \to \mathcal{A}_0^{\mathbf{2}}$ as the functor classifying the natural transformation

$$\begin{tikzcd} \mathcal{A}_1 \arrow[r, bend left, "1"] \arrow[r, bend right, "ic"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{A}_1 \arrow[r, "d"] & \mathcal{A}_0 \end{tikzcd} ; \tag{3.5}$$

finally, by pasting together unit squares and using (3.4), we verify that $V = (\mathrm{id}, V_1)$ preserves vertical composition, and so is a double functor. If $W \colon \mathbb{B} \to \mathbb{S}\mathbf{q}(\mathcal{B}_0)$ is another concrete double category whose W is determined in this manner, then *any* double functor $F \colon \mathbb{A} \to \mathbb{B}$ will by (3.4) satisfy $F_1 \cdot \eta_{\mathbb{A}} = \eta_{\mathbb{B}} \cdot F_1$, and so $\mathbb{S}\mathbf{q}(F_0) \cdot V = W \cdot F$; that is, any double functor $\mathbb{A} \to \mathbb{B}$ is concrete.

Let us, therefore, define a double category $\mathbb{A}$ to be *right-connected* if it satisfies (3.4), and *monadic right-connected* if in addition the induced functor $\mathcal{A}_1 \to (\mathcal{A}_0)^{\mathbf{2}}$ is strictly monadic. The preceding discussion now shows that:

Proposition 11. *The assignation $(\mathcal{C}, \mathsf{L}, \mathsf{R}) \mapsto \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ gives an equivalence of 2-categories between $\mathbf{AWFS}_{\text{lax}}$ and the full sub-2-category of $\mathbf{DBL}$ on the monadic right-connected double categories.*

4. Double categories at work

As a first illustration of the usefulness of the double categorical approach and our Theorem 6, we use it to construct a variety of algebraic weak factorisation systems, some well-known, and some new.

4.1. **Split epimorphisms.** Given a category $\mathcal{C}$, we write $\mathbf{SplEpi}(\mathcal{C})$ for the category of split epimorphisms therein: objects are pairs of a map $g\colon A \to B$ of $\mathcal{C}$ together with a section p of g, while morphisms $(g,p) \to (h,q)$ are serially commuting diagrams as on the left in:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, shift right, "g"'] & C \arrow[d, shift right, "h"'] \\ B \arrow[r, "s"'] \arrow[u, shift right, "p"'] & D \arrow[u, shift right, "q"'] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "m"] \arrow[dr, "1"'] & 0 \arrow[d, "e"] \arrow[dr, "me"] & \\ & 1 \arrow[r, "m"'] & 0\,. \end{tikzcd} \tag{4.1}$$

Split epimorphisms compose—by composing the sections—so that we have a double category $\mathbb{S}\mathbf{plEpi}(\mathcal{C})$ which is concrete over $\mathcal{C}$ and easily seen to be right-connected. It will thus be the double category of algebras of an AWFS on $\mathcal{C}$ whenever $U\colon \mathbf{SplEpi}(\mathcal{C}) \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic.

Now, we may identify U with $\mathcal{C}^j\colon \mathcal{C}^{\mathcal{S}} \to \mathcal{C}^{\mathbf{2}}$, where $\mathcal{S}$ is the *free split epimorphism*, drawn above right, and $j\colon \mathbf{2} \to \mathcal{S}$ is the evident inclusion. Thus U strictly creates colimits, and so will be strictly monadic whenever it has a left adjoint: which will be so whenever $\mathcal{C}$ is cocomplete enough to admit left Kan extensions along j. Using the Kan extension formula one finds that only binary coproducts are required; the free split epi Rf on $f\colon A \to B$ is $\langle f, 1\rangle\colon A+B \to B$ with section ι_B, while the unit $f \to Rf$ is given by:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "f"'] & A+B \arrow[d, shift right, "\langle f{,}1\rangle"'] \\ B \arrow[r, "1"'] & B\,. \arrow[u, shift right, "\iota_B"'] \end{tikzcd} \tag{4.2}$$

The remaining structure of the AWFS for split epis can be calculated using Proposition 4. We find that each coproduct inclusion admits a L-coalgebra structure; while if $\mathcal{C}$ is lextensive, then this coalgebra structure is *unique*, and the category of L-coalgebras is precisely the category of coproduct injections and pullback squares. This was proved in [25, Proposition 4.2], but see also Example 12(iii) below.

Finally, let us observe an important property of the split epi AWFS. Although $\mathsf{R} = (R, \eta, \mu)$ is a monad, its algebras—the split epis—are simultaneously the algebras for its underlying pointed endofunctor (R, η); which is to say that the monad R is *algebraically-free* [36, §22] on its underlying pointed endofunctor. We will make use of this property in Proposition 16 below.

4.2. **Lalis.** In the terminology of Gray [26], a *lali* (left adjoint left inverse) in a 2-category $\mathcal{C}$ is a split epi $(g,p)\colon A \to B$ with the extra property that $g \dashv p$ with identity counit. This property may be expressed either by requiring that for each $x\colon A \to X$ and $y\colon B \to X$ in $\mathcal{C}$, the function

$$(-)\cdot p\colon \mathcal{C}(A,X)(x, yg) \to \mathcal{C}(B,X)(xp, y) \tag{4.3}$$

be invertible; or by requiring the provision of a—necessarily unique—unit 2-cell $\eta\colon 1 \Rightarrow pg$ satisfying $g\eta = 1$ and $\eta p = 1$. Lalis in $\mathcal{C}$ form a category $\mathbf{Lali}(\mathcal{C})$, wherein a morphism is a commuting diagram as on the left of (4.1); it is automatic by invertibility of (4.3) that such a morphism also commutes with the unit 2-cells.

Since split epis and adjoints compose, so too do lalis; thus—writing $\mathcal{C}_0$ for the underlying category of $\mathcal{C}$—lalis in $\mathcal{C}$ form a concrete sub-double category $\mathbb{L}\mathbf{ali}(\mathcal{C})$ of $\mathbb{S}\mathbf{plEpi}(\mathcal{C}_0)$ which, since it is full on cells, inherits right-connectedness. So $\mathbb{L}\mathbf{ali}(\mathcal{C})$ will be the double category of algebras for an AWFS on $\mathcal{C}_0$ whenever $U\colon \mathbf{Lali}(\mathcal{C}) \to (\mathcal{C}_0)^{\mathbf{2}}$ is strictly monadic. Now, we may identify U with restriction

$$\mathbf{2\text{-}CAT}(j, \mathcal{C})\colon \mathbf{2\text{-}CAT}(\mathcal{L}, \mathcal{C}) \to \mathbf{2\text{-}CAT}(\mathbf{2}, \mathcal{C})$$

along the inclusion $j\colon \mathbf{2} \to \mathcal{L}$ of $\mathbf{2}$ into the *free lali* $\mathcal{L}$—which has the same underlying category as the free split epimorphism $\mathcal{S}$ and a single non-trivial 2-cell $1 \Rightarrow me$; so as before, monadicity obtains whenever $\mathcal{C}$ is cocomplete enough (as a 2-category) to admit left Kan extensions along j. In this case, the colimits needed are *oplax colimits of arrows*: given $f\colon C \to B$, its oplax colimit is the universal diagram as on the left of:

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\ell"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[dl, "p"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{1}{\Rightarrow}"] & B \arrow[dl, "1"] \\ & B\,. & \end{tikzcd}$$

Applying the one-dimensional aspect of this universality to the cocone on the right yields a retraction $g\colon A \to B$ for p; now applying the two-dimensional aspect shows that the pair (g, p) verifies invertibility in (4.3), and so is a lali, the free lali on f, with as unit $f \to Rf$ the map $(\ell, 1)$.

When $\mathcal{C} = \mathbf{Cat}$, we may characterise the corresponding category of L-coalgebras as the subcategory of $\mathbf{Cat}^{\mathbf{2}}$ whose objects are the *categorical cofibrations*—those functors arising as pullbacks of the domain inclusion $\mathbf{1} \to \mathbf{2}$—and whose morphisms are the pullback squares; see example (iv) in Section 4.4 below.[1]

On reversing or inverting the non-trivial 2-cell of $\mathcal{L}$, it becomes the free *rali* (right adjoint left inverse) $\mathcal{L}^{\mathrm{co}}$ or the free *retract equivalence* $\mathcal{L}^g$; from which we obtain AWFS whose algebras are ralis or retract equivalences on any 2-category that admits lax colimits or pseudocolimits of arrows. When $\mathcal{C} = \mathbf{Cat}$, the respective L-coalgebra structures on a morphism f are unique, and exist just when f is a pullback of the codomain inclusion $\mathbf{1} \to \mathbf{2}$, respectively, when f is injective on objects.

4.3. **Via cocategories.** The preceding examples fit into a common framework. Let $\mathcal{V}$ be a suitable base for enriched category theory, and suppose that we are given a cocategory object $\mathbb{A}$ in $\mathcal{V}$-$\mathbf{Cat}$:

$$\begin{tikzcd} \mathbb{A}_0 = \mathbf{I} \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & \mathbb{A}_1 \arrow[l, "i" description] \arrow[r, shift left=2, "p"] \arrow[r, "m" description] \arrow[r, shift right=2, "q"'] & \mathbb{A}_2 \end{tikzcd}$$

[1]This characterisation would extend to any suitably exact 2-category, except that there is no account of the appropriate two-dimensional exactness notion; it appears to be closely related to the one mentioned in the introduction to [49].

with object of co-objects the unit $\mathcal{V}$-category, with d and c jointly bijective on objects, and with $i \dashv c$ with identity unit. For any $\mathcal{V}$-category $\mathcal{C}$, the image of $\mathbb{A}$ under the limit-preserving $\mathcal{V}\text{-}\mathbf{CAT}(-, \mathcal{C})\colon \mathcal{V}\text{-}\mathbf{Cat}^{\mathrm{op}} \to \mathbf{CAT}$ is an internal category $\mathbb{A}(\mathcal{C})$ in $\mathbf{CAT}$—a double category—which is right-connected and concrete over $\mathcal{C}_0$, the underlying ordinary category of $\mathcal{C}$. The induced comparison functor $\mathbb{A}(\mathcal{C})_1 \to (\mathcal{C}_0)^{\mathbf{2}}$ is the functor

$$\mathcal{V}\text{-}\mathbf{CAT}(j, \mathcal{C})\colon \mathcal{V}\text{-}\mathbf{CAT}(\mathbb{A}_1, \mathcal{C}) \to \mathcal{V}\text{-}\mathbf{CAT}(\mathbf{2}, \mathcal{C}) \tag{4.4}$$

where $j\colon \mathbf{2} \to \mathbb{A}_1$ is the functor from the free $\mathcal{V}$-category on an arrow which classifies the d-component of the unit $\eta\colon 1 \Rightarrow ci\colon \mathbb{A}_1 \to \mathbb{A}_1$. This j is bijective on objects, and so (4.4) will be monadic whenever $\mathcal{C}$ is sufficiently cocomplete that each $\mathcal{V}$-functor $\mathbf{2} \to \mathcal{C}$ admits a left Kan extension along j; whereupon Theorem 6 induces an AWFS on $\mathcal{C}$ with $\mathbb{A}(\mathcal{C})$ as its double category of algebras. Note that the assignation $\mathcal{C} \mapsto \mathbb{A}(\mathcal{C})$ is clearly 2-functorial in $\mathcal{C}$, so that by Proposition 11, it underlies a 2-functor $\mathcal{V}\text{-}\mathbf{CAT}' \to \mathbf{AWFS}_{\mathrm{lax}}$ defined on all $\mathcal{V}$-categories with sufficient colimits and all $\mathcal{V}$-functors (not necessarily cocontinuous) between them.

Of course, the preceding examples are instances of this framework on taking $\mathcal{V} = \mathbf{Set}$ with $\mathbb{A}_1 = \mathcal{S}$ (for split epis); or taking $\mathcal{V} = \mathbf{Cat}$ with $\mathbb{A}_1 = \mathcal{L}$, $\mathcal{L}^{\mathrm{co}}$ or $\mathcal{L}^g$ (for lalis, ralis or retract equivalences).

4.4. **Stable classes of monics.** Let $\mathcal{C}$ be a category with pullbacks, and consider a class $\mathcal{M}$ of monics which contains the isomorphisms, is closed under composition, and is stable under pullback along maps of $\mathcal{C}$; we call this a *stable class of monics*. The class $\mathcal{M}$ is said to be *classified* if the category $\mathcal{M}_{\mathrm{pb}}$ of pullback squares between $\mathcal{M}$-maps has a terminal object, which we call a *generic $\mathcal{M}$-map*. A standard argument [31, p. 24] shows that the domain of a generic $\mathcal{M}$-map must be terminal in $\mathcal{C}$.

Since $\mathcal{M}$-maps and pullback squares compose, the category $\mathcal{M}_{\mathrm{pb}}$ underlies a double category $\mathbb{M}_{\mathrm{pb}}$ concrete over $\mathcal{C}$. The monicity of the $\mathcal{M}$'s ensures that for each $m \in \mathcal{M}$, the square on the left below is a pullback and so in $\mathcal{M}_{\mathrm{pb}}$. Thus $\mathbb{M}_{\mathrm{pb}}$ is left-connected, and so will comprise the L-coalgebras of an AWFS on $\mathcal{C}$ as soon as the forgetful $U\colon \mathcal{M}_{\mathrm{pb}} \to \mathcal{C}^{\mathbf{2}}$ is comonadic.

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[m]{} & B \end{array} \qquad\qquad \begin{array}{ccccc} A_1 & \xrightarrow{f_1} & B_1 & \overset{g_1}{\underset{h_1}{\rightrightarrows}} & C_1 \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ A_2 & \xrightarrow[f_2]{} & B_2 & \overset{g_2}{\underset{h_2}{\rightrightarrows}} & C_2 \end{array}$$

First we note that U always strictly creates equalisers. Indeed, if $b \rightrightarrows c$ is a parallel pair in $\mathcal{M}_{\mathrm{pb}}$ as on the far right above, then on forming the equaliser $a \to b$ in $\mathcal{C}^{\mathbf{2}}$, as to the left, the equalising square will be a pullback—since c is monic and both rows are equalisers—and so will lift uniquely to $\mathcal{M}_{\mathrm{pb}}$. Thus U will be strictly comonadic as soon as it has a right adjoint.

For this, we must assume that the generic $\mathcal{M}$-map $p\colon 1 \to \Sigma$ is exponentiable; it follows that every $\mathcal{M}$-map is exponentiable, since exponentiable maps are pullback-stable in any category with pullbacks (see [50, Corollary 2.6], for example). In particular, each map $B \times p\colon B \to B \times \Sigma$ is exponentiable, and so for any $f\colon A \to B$

of $\mathcal{C}$, we may form the object $\Pi_{B\times p}(f)$ of $\mathcal{C}/(B\times\Sigma)$; which has the universal property that, in the category $\mathcal{P}_f$ whose objects are pullback diagrams as on the left in

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\ell"'] & A \arrow[r, "f"] & B \arrow[d, "B\times p"] \\ Y \arrow[rr, "k"'] & & B\times\Sigma \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\varepsilon"] \arrow[d, "\eta"'] & A \arrow[r, "f"] & B \arrow[d, "B\times p"] \\ D \arrow[rr, "\Pi_{B\times p}(f)"'] & & B\times\Sigma \end{tikzcd}$$

and whose maps $(X,Y,h,k,\ell)\to(X',Y',h',k',\ell')$ are maps $X\to X'$ and $Y\to Y'$ satisfying the obvious equalities, there is a terminal object as on the right. Now, an object of $\mathcal{P}_f$ determines and is determined by a pair of squares

$$\begin{tikzcd} X \arrow[r, "h_1"] \arrow[d, "\ell"'] & A \arrow[r, "f"] & B \arrow[d, "1"] \\ Y \arrow[rr, "k_1"'] & & B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "!"] \arrow[d, "\ell"'] & 1 \arrow[d, "p"] \\ Y \arrow[r, "k_2"'] & \Sigma \end{tikzcd}$$

with the right-hand one a pullback. This ensures that ℓ—as a pullback of p—is an $\mathcal{M}$-map, whereupon by genericity of p, the map k_2 is uniquely determined. Thus we may identify $\mathcal{P}_f$ with the comma category $U\downarrow f$, and the terminal object therein provides the value $\eta\colon C\to D$ of the desired right adjoint at f.

For the (L,R) induced in this way, the restriction of the monad R to the slice over B is the *partial $\mathcal{M}$-map classifier monad* on $\mathcal{C}/\mathcal{B}$; see [46] and the references therein. Note also that, in the terminology of [17], the $\Pi_{B\times p}(f)$ constructed above is a *pullback complement* of f and $B\times p$.

Examples 12. (i) Let $\mathcal{E}$ be an elementary topos, and $\mathcal{M}$ the stable class of all monics. $\mathcal{M}$ is classified by $\top\colon 1\to\Omega$, which—like any map in a topos—is exponentiable; and so we have an AWFS on $\mathcal{E}$ for which L-**Coalg** is the category of monomorphisms and pullback squares. The corresponding R-algebras are discussed in [39]. More generally, we can let $\mathcal{E}$ be any quasitopos, and $\mathcal{M}$ the class of strong monomorphisms.

(ii) Let $\mathcal{E}$ be an elementary topos, j a Lawvere-Tierney topology on $\mathcal{E}$, and $\mathcal{M}$ the class of j-dense monomorphisms. This is a stable class of monics classified by $\top\colon 1\to J$, where J is the equaliser of $j,\top\colon\Omega\to\Omega$. So j-dense monomorphisms and their pullbacks form the coalgebras of an AWFS on $\mathcal{E}$. An algebraically fibrant object is a "weak sheaf": an object equipped with coherent, but not unique, choices of patchings for j-covers. An object X is a sheaf if and only if both $X\to 1$ and $X\to X\times X$ admit R-algebra structure.

(iii) Let $\mathcal{C}$ be a lextensive category and $\mathcal{M}$ the stable class of coproduct injections. This is classified by $\iota_1\colon 1\to 1+1$; which by extensivity is always exponentiable, with $\Pi_{\iota_1}=(-)+1\colon\mathcal{C}\to\mathcal{C}/(1+1)$. More generally, the right adjoint to pullback along $B\times\iota_1$ is $(-)+B\colon\mathcal{C}/B\to\mathcal{C}/(B+B)$; whence the AWFS generated on $\mathcal{C}$ is that for split epis. This re-proves the fact that, in any lextensive category, the coalgebras of the AWFS for split epis are the coproduct injections and pullback squares.

(iv) Consider, as in Section 4.2, the stable class of categorical cofibrations in **Cat**. This is classified by the domain functor $d\colon \mathbf{1} \to \mathbf{2}$, which is exponentiable in **Cat**, so that we have an AWFS wherein L-coalgebras are the categorical cofibrations and pullback squares. For any category B, the functor $\Pi_{B\times d}\colon \mathbf{Cat}/B \to \mathbf{Cat}/(B \times \mathbf{2})$ sends $f \in \mathbf{Cat}/B$ to the induced map between the oplax colimits of f and of 1_B—the latter being simply $B \times \mathbf{2}$—and it follows that this AWFS is the one for lalis. This proves the claim about its coalgebras made in Section 4.2 above.

(v) By a *left cofibration* of simplicial sets, we mean a pullback of the face inclusion $\delta_1\colon \Delta[0] \to \Delta[1]$. The left cofibrations constitute a stable class of monics, classified by δ_1; and since **SSet** is a topos, this δ_1 is exponentiable. We thus have an AWFS on simplicial sets whose L-coalgebras are left cofibrations and pullbacks. The corresponding R-coalgebras we call the *simplicial lalis*; we will have more to say about them in Section 8.2 below.

4.5. **Projective and injective liftings.** Our next example is a construction which, under suitable circumstances, allows us to lift an AWFS $(\mathcal{D}, \mathsf{L}, \mathsf{R})$ along a functor $F\colon \mathcal{C} \to \mathcal{D}$ to yield an AWFS on $\mathcal{C}$. There are two ways of doing this, well known from the model category literature: we either lift the algebras (as in [16]) or the coalgebras (as in [8]). In the former case, we form the pullback on the left in:

$$
\begin{tikzcd}
\mathbb{A} \arrow[r] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \arrow[d, "U^{\mathsf{R}}"] \\
\mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"] & \mathbb{S}\mathbf{q}(\mathcal{D})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}_1 \arrow[r] \arrow[d, "V_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{R}\text{-}\mathbf{Alg} \arrow[d, "U^{\mathsf{R}}"] \\
\mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"] & \mathcal{D}^{\mathbf{2}}
\end{tikzcd}
\tag{4.5}
$$

The double category $\mathbb{A}$ therein, as a pullback of right-connected double categories, is itself right-connected; while the component functor V_1 of V as on the right above is a pullback of the strictly monadic U^{R}, whence easily seen to strictly create coequalisers for V_1-absolute coequaliser pairs. So whenever V_1 admits a left adjoint, $\mathbb{A}$ will by Theorem 6 constitute the algebra double category of an AWFS $(\mathsf{L}', \mathsf{R}')$ on $\mathcal{C}$, the *projective lifting* of (L, R) along F. The left square of (4.5) then amounts to a lax morphism of AWFS $(\mathcal{C}, \mathsf{L}', \mathsf{R}') \to (\mathcal{D}, \mathsf{L}, \mathsf{R})$, which is easily seen to be a cartesian lifting of F along the forgetful functor $\mathbf{AWFS}_{\text{lax}} \to \mathbf{CAT}$.

Dually, we may form the pullback of double categories on the left in

$$
\begin{tikzcd}
\mathbb{B} \arrow[r] \arrow[d, "W"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \arrow[d, "U^{\mathsf{L}}"] \\
\mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"] & \mathbb{S}\mathbf{q}(\mathcal{D})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}_1 \arrow[r] \arrow[d, "W_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{L}\text{-}\mathbf{Coalg} \arrow[d, "U^{\mathsf{L}}"] \\
\mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"] & \mathcal{D}^{\mathbf{2}}
\end{tikzcd}
\tag{4.6}
$$

if on doing so, the component W_1 of W as to the right has a right adjoint, then $\mathbb{B}$ will be the coalgebra double category of an AWFS on $\mathcal{C}$, the *injective lifting* of (L, R) along F. The left square of (4.6) now corresponds to an oplax map of AWFS providing a cartesian lifting of F with respect to the forgetful $\mathbf{AWFS}_{\text{oplax}} \to \mathbf{CAT}$.

Sometimes, it may be that the required adjoints to V_1 or W_1 simply exist. For example, the projective and injective liftings of an AWFS $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ along a forgetful functor $\mathcal{C}/X \to \mathcal{C}$ always exist, and coincide; the factorisations of the resultant *slice* AWFS are given by

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "a"'] & & B \arrow[dl, "b"] \\
& X &
\end{tikzcd}
\qquad \mapsto \qquad
\begin{tikzcd}
A \arrow[r, "Lf"] \arrow[dr, "a"'] & Ef \arrow[r, "Rf"] \arrow[d, "b \cdot Rf"'] & B \arrow[dl, "b"] \\
& X &
\end{tikzcd} .
$$

More typically, we will appeal to a general result like the following, which ensures that the desired adjoint exists without necessarily giving a closed formula for it. As in the introduction, we call an AWFS on a locally presentable category *accessible* if its comonad L and monad R are so, in the sense of preserving κ-filtered colimits for some regular cardinal κ; in fact, it is easy to see that accessibility of the comonad implies that of the monad, and vice versa.

Proposition 13. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally presentable categories, and (L, R) an accessible* AWFS *on $\mathcal{D}$.*

(a) The projective lifting of (L, R) along any right adjoint $F \colon \mathcal{C} \to \mathcal{D}$ exists and is accessible;

(b) The injective lifting of (L, R) along any left adjoint $F \colon \mathcal{C} \to \mathcal{D}$ exists and is accessible.

Proof. For (a), let $F \colon \mathcal{C} \to \mathcal{D}$ be a right adjoint. To show the projective lifting exists, we must prove that the functor V_1 to the right of (4.5) has a left adjoint. Now, $\mathcal{C}^{\mathbf{2}}$ and $\mathcal{D}^{\mathbf{2}}$ are locally presentable since $\mathcal{C}$ and $\mathcal{D}$ are, by [19, §7.2(h)], while R-**Alg** is locally presentable since R is accessible, by [19, Satz 10.3]; moreover, both $F^{\mathbf{2}}$ and U^{R} are right adjoints. Since U^{R} has the isomorphism-lifting property, its pullback against $F^{\mathbf{2}}$ is also a bipullback [34]; but by [10, Theorem 2.18], the 2-category of locally presentable categories and right adjoint functors is closed under bilimits in **CAT**, whence V_1, like $F^{\mathbf{2}}$ and U^{R}, lies in this 2-category; thus it has a left adjoint K_1 and is accessible by [19, Satz 14.6]. So the projective lifting of (L, R) along F exists, and is accessible as its monad $V_1 K_1$ is so.

For (b), let $F \colon \mathcal{C} \to \mathcal{D}$ be a left adjoint; we must show that W_1 to the right of (4.6) has a right adjoint. We now argue using [10, Theorem 3.15], which states that the 2-category of locally presentable categories and left adjoint functors is closed under bilimits in **CAT**. As before, $F^{\mathbf{2}} \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{D}^{\mathbf{2}}$ lies in this 2-category, and $U^{\mathsf{L}} \colon \mathsf{L}\text{-}\mathbf{Coalg} \to \mathcal{D}^{\mathbf{2}}$ will so long as L-**Coalg** is in fact locally presentable. But L is an accessible comonad on the accessible category $\mathcal{D}^{\mathbf{2}}$; by [41, Theorem 5.1.6], the 2-category of accessible categories and accessible functors is closed in **CAT** under all bilimits, in particular under Eilenberg-Moore objects of comonads, and so L-**Coalg** is accessible; it is also cocomplete (since U^{L} creates colimits) and so is locally presentable, as required. We thus conclude that the functor W_1, like U^{L} and $F^{\mathbf{2}}$, is a left adjoint between locally presentable categories; while by [19, Satz 14.6], its right adjoint G_1 is accessible. So the projective lifting of (L, R) along F exists, and is accessible as its comonad $W_1 G_1$ is so. $\square$

In fact, we may drop the requirement of local presentability from the first part of the preceding proposition if we strengthen the hypotheses on U.

Proposition 14. *Let $\mathcal{D}$ be a cocomplete category, let T be an accessible monad on $\mathcal{D}$ and let (L, R) be an accessible* AWFS *on $\mathcal{D}$. Then the projective lifting of (L, R) along the forgetful functor $U^{\mathsf{T}} \colon \mathsf{T}\text{-}\mathbf{Alg} \to \mathcal{D}$ exists and is accessible.*

Proof. As in (4.5), we form the pullback of double categories as to the left in

$$\begin{tikzcd} \mathbb{A} \arrow[r] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \arrow[d, "U^{\mathsf{R}}"] \\ \mathbb{S}\mathbf{q}(\mathsf{T}\text{-}\mathbf{Alg}) \arrow[r, "\mathbb{S}\mathbf{q}(U^{\mathsf{T}})"] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_1 \arrow[r] \arrow[d, "V_1"'] & \mathsf{R}\text{-}\mathbf{Alg} \arrow[d] \\ (\mathsf{T}\text{-}\mathbf{Alg})^{\mathbf{2}} \arrow[r, "(U^{\mathsf{T}})^{\mathbf{2}}"] & \mathcal{D}^{\mathbf{2}} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{S} & \mathsf{R} \arrow[l, "\sigma"'] \\ \mathsf{T}^{\mathbf{2}} \arrow[u, "\nu"] & \mathrm{id} \arrow[l, "!"'] \arrow[u, "!"'] \end{tikzcd}$$

and must show that the underlying 1-component V_1, as in the centre, has a left adjoint. Now, the monad R on $\mathcal{D}^{\mathbf{2}}$ is accessible by assumption, while $\mathsf{T}^{\mathbf{2}}$ is accessible since T is so; whence by [36, Theorem 27.1] the coproduct $\mathsf{S} = \mathsf{T}^{\mathbf{2}} + \mathsf{R}$ exists in $\mathbf{MND}(\mathcal{D}^{\mathbf{2}})$ and is accessible. This coproduct is equally the pushout on the right above, and [36] guarantees that the functor $(–)\text{-}\mathbf{Alg}$ sends this square to a pullback of categories, so that we may identify V_1 with $\nu^* \colon \mathsf{S}\text{-}\mathbf{Alg} \to \mathsf{T}^{\mathbf{2}}\text{-}\mathbf{Alg} = (\mathsf{T}\text{-}\mathbf{Alg})^{\mathbf{2}}$. Now as $\mathcal{D}^{\mathbf{2}}$ is cocomplete and S is accessible, this ν^* has a left adjoint by [36, Theorem 25.4]. $\square$

A standard application of the preceding results is to the construction of AWFS on diagram categories. If the cocomplete $\mathcal{C}$ bears an accessible AWFS (L, R), and $\mathcal{I}$ is a small category, then the functor $U \colon \mathcal{C}^{\mathcal{I}} \to \mathcal{C}^{\mathrm{ob}\,\mathcal{I}}$ given by precomposition with $\mathrm{ob}\,\mathcal{I} \to \mathcal{I}$ has a left adjoint (given by left Kan extension) and is strictly monadic; so that by Proposition 14 we may projectively lift the pointwise AWFS on $\mathcal{C}^{\mathrm{ob}\,\mathcal{I}}$ along U to obtain the *projective* AWFS on $\mathcal{C}^{\mathcal{I}}$. If $\mathcal{C}$ is moreover complete, then U also has a right adjoint given by right Kan extension; if it is moreover locally presentable then we may apply Proposition 13(b) to injectively lift the pointwise AWFS along U, so obtaining the *injective* AWFS on $\mathcal{C}^{\mathcal{I}}$.

5. Cofibrant generation by a category

For our next application of Theorem 6, we use it to give a simplified treatment of *cofibrantly generated* algebraic weak factorisation systems. An AWFS is cofibrantly generated when its R-algebras are precisely morphisms equipped with a choice of liftings against some small category $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ of "generating cofibrations". This extends and tightens the familiar notion of cofibrant generation of a weak factorisation system by a set of generating cofibrations; and the main result of [22] extends and tightens Quillen's small object argument to show that, under commonly-satisfied conditions on $\mathcal{C}$, the AWFS cofibrantly generated by any small $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ exists. The proof given there is quite involved; we will give an simpler one exploiting Theorem 6. First we recall the necessary background.

5.1. **Lifting operations.** Given categories $U \colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ and $V \colon \mathcal{K} \to \mathcal{C}^{\mathbf{2}}$ over $\mathcal{C}^{\mathbf{2}}$, a $(\mathcal{J}, \mathcal{K})$-*lifting operation* is a natural family of functions $\varphi_{j,k}$ assigning to each

$j \in \mathcal{J}$ and $k \in \mathcal{K}$ and each commuting square

$$\begin{tikzcd} \mathrm{dom}\; Uj \arrow[r, "u"] \arrow[d, "Uj"'] & \mathrm{dom}\; Vk \arrow[d, "Vk"] \\ \mathrm{cod}\; Uj \arrow[r, "v"] \arrow[ur, dotted, "\varphi_{j,k}(u,v)"] & \mathrm{cod}\; Vk \end{tikzcd}$$

a diagonal filler as indicated making both triangles commute. Naturality here expresses that the $\varphi_{j,k}$'s are components of a natural transformation $\varphi\colon \mathcal{C}^{\mathbf{2}}(U\text{–}, V?) \Rightarrow \mathcal{C}(\mathrm{cod}\, U\text{–}, \mathrm{dom}\, V?)\colon \mathcal{J}^{\mathrm{op}} \times \mathcal{K} \to \mathbf{Set}$. Thus Section 2.4 describes the canonical $(\mathsf{L}\text{-}\mathbf{Coalg}, \mathsf{R}\text{-}\mathbf{Alg})$-lifting operation associated to any AWFS.

The assignation taking $(\mathcal{J}, \mathcal{K})$ to the collection of $(\mathcal{J}, \mathcal{K})$-lifting operations is easily made the object part of a functor $\mathbf{Lift}\colon (\mathbf{CAT}/\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \times (\mathbf{CAT}/\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \to \mathbf{SET}$; we now have, as in [22, Proposition 3.8]:

Proposition 15. *Each functor* $\mathbf{Lift}(\mathcal{J}, \text{–})$ *and* $\mathbf{Lift}(\text{–}, \mathcal{K})$ *is representable, so that we induce an adjunction*

$$(\mathbf{CAT}/\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \underset{(\text{–})^{\pitchfork}}{\overset{{}^{\pitchfork}(\text{–})}{\underset{\perp}{\rightleftarrows}}} \mathbf{CAT}/\mathcal{C}^{\mathbf{2}}\ . \tag{5.1}$$

Proof. The category $\mathcal{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ representing $\mathbf{Lift}(\mathcal{J}, \text{–})$, has as objects, pairs of $g \in \mathcal{C}^{\mathbf{2}}$ and $\varphi_{\text{–}g}$ a $(\mathcal{J}, g)$-lifting operation; and as maps $(g, \varphi_{\text{–}g}) \to (h, \varphi_{\text{–}h})$, those $g \to h$ of $\mathcal{C}^{\mathbf{2}}$ which commute with the lifting functions. Dually, the representing category ${}^{\pitchfork}\mathcal{K} \to \mathcal{C}^{\mathbf{2}}$ for $\mathbf{Lift}(\text{–}, \mathcal{K})$ comprises objects $f \in \mathcal{C}^{\mathbf{2}}$ equipped with $(f, \mathcal{K})$-lifting operations, together with the maps of $\mathcal{C}^{\mathbf{2}}$ between them that commute with the lifting functions. □

We have seen a particular instance of this result in Section 2.7 above; the category $\mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ described there is the representing object for $\mathbf{Lift}(\mathsf{L}\text{-}\mathbf{Coalg}, \text{–})$, and the functor $\bar{\bar{\Phi}}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ is the one induced by this representation from the canonical lifting operation of Section 2.4.

5.2. **Cofibrant generation.** If the maps $g\colon C \to D$ and $h\colon D \to E$ of $\mathcal{C}$ are equipped with lifting operations $\varphi_{\text{–}g}$ and $\varphi_{\text{–}h}$ against a category $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$, then their composite hg also bears a lifting operation $\varphi_{\text{–}hg}$, defined as in (2.10) by $\varphi_{j,hg}(u, v) = \varphi_{j,g}(u, \varphi_{j,h}(gu, v))$. This composition, together with the equipment of an identity map with its *unique* lifting operation, provides the necessary vertical structure to make $\mathcal{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ into a concrete double category $\mathcal{J}^{\pitchfork\!\!\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$; dually, we can make ${}^{\pitchfork}\mathcal{K} \to \mathcal{C}^{\mathbf{2}}$ into a concrete double category ${}^{\pitchfork\!\!\pitchfork}\mathcal{K} \to \mathbb{S}\mathbf{q}(\mathcal{C})$.

We now define an AWFS (L, R) on $\mathcal{C}$ to be *cofibrantly generated* by a small $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ if $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \cong \mathcal{J}^{\pitchfork\!\!\pitchfork}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$. If this isomorphism is verified for a $\mathcal{J}$ which is large, we say instead that $\mathcal{C}$ is *class-cofibrantly generated* by $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$. There are dual notions of fibrant or class-fibrant generation, involving an isomorphism ${}^{\pitchfork\!\!\pitchfork}\mathcal{K} \cong \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$; however, these are markedly less prevalent than their duals in categories of mathematical interest.

5.3. **Existence of cofibrantly generated awfs.** In [22, Definition 3.9] is given the notion of an AWFS being "algebraically-free" on $U \colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$: to which the notion of cofibrant generation given above, though apparently different in form, is in fact equivalent.[2] Theorem 4.4 of *ibid.* thus guarantees, among other things, that in a locally presentable category $\mathcal{C}$, the AWFS cofibrantly generated by *any* small $U \colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ exists. We now use Theorem 6 to give a shorter proof of this.

Proposition 16. *If $\mathcal{C}$ is locally presentable then the* AWFS (L, R) *cofibrantly generated by any small $U \colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ exists; its underlying monad R is algebraically-free on a pointed endofunctor and accessible.*

This result as stated is less general than [22, Theorem 4.4], which also deals with certain kinds of non-locally presentable $\mathcal{C}$. Though we have no need for this extra generality here, let us note that reincorporating it would simply be a matter of adapting the final paragraph of the following proof.

Proof. It is easy to see that $V \colon \mathcal{J}^{\pitchfork\!\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ is right-connected; so by Theorem 6, it will comprise the double category of algebras for the desired AWFS as soon as $V_1 \colon \mathcal{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic. Now an object of $\mathcal{J}^{\pitchfork}$ is equally a map g of $\mathcal{C}$ together with a section φ_{-g} of the natural transformation

$$\psi_{-g} \colon \mathcal{C}(\mathrm{cod}\ U{-}, \mathrm{dom}\ g) \Rightarrow \mathcal{C}^{\mathbf{2}}(U{-}, g) \colon \mathcal{J}^{\mathrm{op}} \to \mathbf{Set} \tag{5.2}$$

whose j-component sends $m \colon \mathrm{cod}\ Uj \to C$ to $(m \cdot Uj, gm) \colon Uj \to g$; while a map $(g, \varphi_{-g}) \to (h, \varphi_{-h})$ of $\mathcal{J}^{\pitchfork}$ is a map $g \to h$ of $\mathcal{C}^{\mathbf{2}}$ for which the induced $\psi_{-g} \to \psi_{-h}$ in $[\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}}$ commutes with the sections. We thus have a pullback as on the left in

$$\begin{tikzcd} \mathcal{J}^{\pitchfork} \arrow[r] \arrow[d, "V_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{SplEpi}([\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]) \arrow[d] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "\psi"] & {[\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}}} \end{tikzcd} \qquad \begin{tikzcd} K\psi \arrow[r, "K\eta\psi"] \arrow[d, "\varepsilon"'] & KT\psi \arrow[d] \\ 1 \arrow[r, "\rho"] \arrow[ur, phantom, very near end, "\ulcorner"] & P\,. \end{tikzcd} \tag{5.3}$$

We first show that ψ has a left adjoint. The composite $\mathrm{cod} \cdot \psi \colon \mathcal{C}^{\mathbf{2}} \to [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]$ is the singular functor $\mathcal{C}^{\mathbf{2}}(U, 1)$ which has left adjoint F_1 given by the left Kan extension of $U \colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ along the Yoneda embedding; while $\mathrm{dom} \cdot \psi$ is the functor $\mathcal{C}(\mathrm{cod} \cdot U, \mathrm{dom}) \cong \mathcal{C}^{\mathbf{2}}(\mathrm{id} \cdot \mathrm{cod} \cdot U, 1)$ (as $\mathrm{id} \dashv \mathrm{dom} \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$) and so also has a left adjoint F_2. The natural transformation $\mathrm{cod} \cdot \psi \to \mathrm{dom} \cdot \psi$ induces one $\alpha \colon F_2 \to F_1$, using which ψ admits a left adjoint K sending $f \colon X \to Y$ to the pushout of $F_2 f \colon F_2 X \to F_2 Y$ along $\alpha_X \colon F_2 X \to F_1 X$.

Now as we noted in Section 4.1 above, $\mathbf{SplEpi}([\mathcal{J}^{\mathrm{op}}, \mathbf{Set}])$ may be identified with the category of algebras for a pointed endofunctor (T, η) on $[\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}}$ with unit $\eta_f \colon f \to Tf$ given by (4.2). It follows that $\mathcal{J}^{\pitchfork}$ is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to the category of algebras for the pointed endofunctor (P, ρ) in the pushout square above right (c.f. [53, Theorem 2.1]). An easy consequence of this identification is that V_1 strictly creates coequalisers of V_1-absolute coequaliser pairs; so it will be strictly monadic whenever it has a left adjoint—that is, whenever free (P, ρ)-algebras exist. We show this existence using results of [36].

[2]This follows from Proposition 22 below.

Since $\mathcal{C}$ is locally presentable, there is a regular cardinal κ such that the domain and codomain of each Uj is κ-presentable; thus the domain and codomain of each $\psi_{j-}\colon \mathcal{C}(\mathrm{cod}\ Uj, \mathrm{dom}\ -) \Rightarrow \mathcal{C}^{\mathbf{2}}(Uj, -)$ preserves κ-filtered colimits, so that ψ does so too. Now K, being a left adjoint, is cocontinuous, while T is so by inspection of (4.2); whence the P of (5.3) is a pushout of functors preserving κ-filtered colimits, and so also preserves them. Thus by [36, Theorem 22.3], free (P, ρ)-algebras exist, which is to say that V_1 has a left adjoint F_1 as required. Since P preserves κ-filtered colimits, V_1 creates them, and so the induced monad $\mathsf{R} = V_1F_1$ preserves them; it is thus accessible. Finally, it follows from [36, Proposition 22.2] that R is the algebraically-free monad on the pointed endofunctor (P, ρ). □

This result provides a rich source of algebraic weak factorisation systems; in particular, we may make any cofibrantly generated weak factorisation system on a locally presentable category into an AWFS by applying the result to its set J of generating cofibrations, seen as a discrete category over $\mathcal{C}^{\mathbf{2}}$. For applications that make serious use of the algebraicity so obtained, see [21, 23]; for applications utilising the extra generality of cofibrant generation by a small *category*, rather than a small set of morphisms, see [5, 7].

5.4. **Inadequacy of cofibrant generation by a category.** Proposition 16 tells us that the monad of an AWFS cofibrantly generated by a small category has two specific properties: it is accessible, and it is algebraically-free on a pointed endofunctor. Our experience of monads suggests that the former property should be more common than the latter, and in fact this is the case: many AWFS of practical interest verify the accessibility, but not the freeness. For example:

Proposition 17. *The monad of the* AWFS *for lalis on* **Cat** *is accessible, but not algebraically-free on a pointed endofunctor; in particular, this* AWFS *is not cofibrantly generated by a small category.*

Proof. The monad R at issue is given by left Kan extension and restriction along the 2-functor $j\colon \mathbf{2} \to \mathcal{L}$ of Section 4.2; it is thus cocontinuous and in particular, accessible. On the other hand, if it were algebraically-free on a pointed endofunctor, then its category of algebras—**Lali**(**Cat**)—would be isomorphic to the category of algebras for a pointed endofunctor, and as such would be retract-closed: meaning that, for any lali $f \dashv u$ and any retract g of f in $\mathbf{Cat}^{\mathbf{2}}$, there would exist lali structure on g. Now, to equip the unique functor $\mathcal{A} \to 1$ with the structure of a lali is to specify a terminal object of $\mathcal{A}$; so it is enough to describe a category $\mathcal{A}$ with a terminal object, and a retract $\mathcal{B}$ of $\mathcal{A}$ without one. To this end, take $\mathcal{B}$ to be the free idempotent $t^2 = t$ as on the left in:

$$
\begin{tikzcd} 0 \arrow[loop left, "t"] \end{tikzcd} \longmapsto \begin{tikzcd} 0 \arrow[loop left, "t"] \arrow[r, "!"] & 1 \end{tikzcd} \longmapsto \begin{tikzcd} 0 \arrow[loop left, "t"] \end{tikzcd}
$$

and take $\mathcal{A}$ to be $\mathcal{B}$ with a terminal object 1 freely adjoined. The inclusion of $\mathcal{B}$ into $\mathcal{A}$ admits a retraction, which fixes t and sends $!\colon 0 \to 1$ to the identity on 0. Yet $\mathcal{B}$ does not admit a terminal object. □

In particular, this result tells us that, since lalis are not closed under retracts, they cannot be characterised as the right class of maps for a mere weak factorisation system; thus the algebraicity is, in this case, essential.

6. Cofibrant generation by a double category

In light of Proposition 17, it is natural to ask whether there is a more refined notion of cofibrant generation which encompasses such examples of AWFS as the one for lalis on **Cat**. In this section, we describe such a notion; it involves lifting properties against a small *double* category, rather than a small category, of generating cofibrations.

6.1. **Double-categorical lifting operations.** Let $U\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ and $V\colon \mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ be double categories over $\mathbb{S}\mathbf{q}(\mathcal{C})$. We define a $(\mathbb{J}, \mathbb{K})$-*lifting operation* to be a $(\mathcal{J}_1, \mathcal{K}_1)$-lifting operation in the sense of Section 5.1 which is also compatible with vertical composition in $\mathbb{J}$ and $\mathbb{K}$, in the sense that

$$
\begin{aligned}
&\varphi_{j,\ell\cdot k}(u,v) = \varphi_{j,k}(u, \varphi_{j,\ell}(Vk\cdot u, v)) \\
\text{and}\quad &\varphi_{j\cdot i,k}(u,v) = \varphi_{j,k}(\varphi_{i,k}(u, v\cdot Uj), v)
\end{aligned}
\tag{6.1}
$$

for all vertically composable maps $j \cdot i\colon A \to B \to C$ and $\ell \cdot k\colon D \to E \to F$ in $\mathbb{J}$ and in $\mathbb{K}$. For example, by virtue of (2.10) and its dual, the lifting operation (2.4) associated to an AWFS is an $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}, \mathsf{R}\text{-}\mathbb{A}\mathbf{lg})$-lifting operation. As before, the assignation sending $\mathbb{J}$ and $\mathbb{K}$ to the collection of $(\mathbb{J}, \mathbb{K})$-lifting operations underlies a functor $\mathbb{L}\mathbf{ift}\colon (\mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \times (\mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \to \mathbf{SET}$; and also as before, we have:

Proposition 18. *Each functor* $\mathbb{L}\mathbf{ift}(\mathbb{J}, –)$ *and* $\mathbb{L}\mathbf{ift}(–, \mathbb{K})$ *is representable, so that the adjunction* (5.1) *extends to one*

$$
(\mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \underset{(-)^{\pitchfork}}{\overset{{}^{\pitchfork}(-)}{\underset{\perp}{\leftrightarrows}}} \mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}) \ . \tag{6.2}
$$

Proof. Given $U\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, we have as in Section 5.2 the concrete double category ${\mathcal{J}_1}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$. The representing object $\mathbb{J}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ for $\mathbb{L}\mathbf{ift}(\mathbb{J}, –)$ is the sub-double category of ${\mathcal{J}_1}^{\pitchfork}$ with the same objects and horizontal arrows, and just those vertical arrows (g, φ_{-g}) whose lifting operations respect vertical composition in $\mathbb{J}$, together with all cells between them. The representing object ${}^{\pitchfork}\mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ for $\mathbb{L}\mathbf{ift}(–, \mathbb{K})$ is defined dually. □

Note that the 2-functor $(–)_1\colon \mathbf{DBL} \to \mathbf{CAT}$ sending a double category to its vertical category is represented by the free vertical arrow $\mathbf{2}_v$, and consequently has a left adjoint, sending $\mathcal{C}$ to the product of $\mathbf{2}_v$ with the free *horizontal* double category on $\mathcal{C}$. This lifts to a left adjoint $\mathbb{F}$ for the induced functor $(–)_1\colon \mathbf{DBL}/\mathbb{S}\mathbf{q}(\mathcal{C}) \to \mathbf{CAT}/\mathcal{C}^{\mathbf{2}}$ on slice categories; and as there are no non-trivial vertical composites in $\mathbb{F}\mathcal{J}$, we have that in fact $\mathcal{J}^{\pitchfork} = (\mathbb{F}\mathcal{J})^{\pitchfork}$; thus the double category structure on $\mathcal{J}^{\pitchfork}$, for which we offered no abstract justification in Section 5.2 above, is now explained in terms of the adjunction (6.2).

6.2. **Cofibrant generation by a double category.** An algebraic weak factorisation system (L, R) is said to be *cofibrantly generated* by a small double category $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ if $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \cong \mathbb{J}^{\pitchfork}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$; if we have this isomorphism for a large $\mathbb{J}$, then we call (L, R) *class-cofibrantly* generated by $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$. Once again, we have the dual notions of fibrant and class-fibrant generation by a double category.

If, as in the preceding section, we identify a category over $\mathcal{C}^{\mathbf{2}}$ with the free double category thereon, then this definition is a conservative extension of that given in Section 5.2 above. However, by contrast with Proposition 17, we have:

Proposition 19. *The* AWFS *for lalis on* **Cat** *is cofibrantly generated by a small double category.*

Proof. We first claim that to equip a functor $f\colon A \to B$ with lali structure is equally to give:

- A section $u\colon \operatorname{ob} B \to \operatorname{ob} A$ of the action of f on objects; and
- For each $a \in A$ and $b \in B$, a section $\gamma_a\colon B(fa, b) \to A(a, ub)$ of the action of f on morphisms; such that
- $\gamma_a(\beta \cdot f\alpha) = \gamma_{a'}(\beta) \cdot \alpha$ for all $\alpha\colon a \to a'$ in A and $\beta\colon fa' \to b$ in B; and
- $\gamma_{ub}(1_b) = 1_{ub}$ for all $b \in B$.

Indeed, if f is part of a lali $f \dashv p$ with unit η, then we obtain these data by taking u to be the action of p on objects, and taking $\gamma_a(\beta) = p(\beta) \cdot \eta_a$. Conversely, given these data, we define a section p of f on objects by $p(b) = u(b)$ and on morphisms by $p(\beta\colon b \to b') = \gamma_{ub}(\beta)\colon ub \to ub'$, and define a unit $\eta\colon 1 \Rightarrow pf$ with components $\eta_a = \gamma_a(1_{fa})$. This proves the claim.

We now define a small $V : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathbf{Cat})$ for which $\mathbb{J}^{⫛} \cong \mathbb{L}\mathbf{ali}(\mathbf{Cat})$. The objects of $\mathbb{J}$ are the ordinals $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$, and its horizontal arrows are order-preserving maps; on these, V acts as the identity. The vertical arrows of $\mathbb{J}$ are freely generated by morphisms $k\colon \mathbf{0} \to \mathbf{1}$, $\ell\colon \mathbf{1} \to \mathbf{2}$ and $m\colon \mathbf{2} \to \mathbf{3}$, which are sent by V to the appropriate initial segment inclusions between ordinals. Its squares are freely generated by the following four:

$$
\begin{tikzcd}
\mathbf{0} \arrow[r, "!"] \arrow[d, "k"'] \arrow[dr, phantom, "\text{(a)}"] & \mathbf{1} \arrow[d, "\ell"] \\
\mathbf{1} \arrow[r, "\delta_0"'] & \mathbf{2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "\delta_0"] \arrow[d, "\ell"'] \arrow[dr, phantom, "\text{(b)}"] & \mathbf{2} \arrow[d, "m"] \\
\mathbf{2} \arrow[r, "\delta_0"'] & \mathbf{3}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "\delta_1"] \arrow[d, "\ell"'] \arrow[dr, phantom, "\text{(c)}"] & \mathbf{2} \arrow[d, "m"] \\
\mathbf{2} \arrow[r, "\delta_1"'] & \mathbf{3}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{0} \arrow[r, "\mathrm{id}"] \arrow[d, "k"'] & \mathbf{0} \arrow[dd, "k"] \\
\mathbf{1} \arrow[d, "\ell"'] \arrow[r, phantom, "\text{(d)}"] & {} \\
\mathbf{2} \arrow[r, "!"'] & \mathbf{1}
\end{tikzcd},
$$

where (following standard simplicial notation) δ_0 and δ_1 denote the order-preserving injections omitting 0 and 1 respectively. We claim that $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathbf{Cat})$ cofibrantly generates the AWFS for lalis; in other words, that $\mathbb{J}^{⫛} \cong \mathbb{L}\mathbf{ali}(\mathbf{Cat})$ over $\mathbb{S}\mathbf{q}(\mathbf{Cat})$. Indeed, given a functor $f\colon A \to B$, we see that:

- To equip f with a Vk-lifting operation is to give a section u of its action on objects;
- To equip f with a $V\ell$-lifting operation is to give, for every $a \in A$ and $\beta\colon fa \to b$ in B, a map $\gamma_a(\beta)\colon a \to \bar{b}$ in A over β; to ask for the compatibility (a) with the Vk-lifting operation is to ask that, in fact, $\bar{b} = ub$.
- To equip f with a Vm-lifting operation is to give, for every $\alpha\colon a \to a'$ in A and $\beta\colon fa' \to b$, a map $\psi_\alpha(\beta)\colon a' \to \bar{b}$ in A over β. The compatibility (b) now forces $\psi_\alpha(\beta) = \gamma_{a'}(\beta)$, while that in (c) forces $\psi_\alpha(\beta) \cdot \alpha = \gamma_a(\beta \cdot f\alpha)$; so ψ is determined by γ, and we have $\gamma_a(\beta \cdot f\alpha) = \gamma_{a'}(\beta) \cdot \alpha$.

- Finally, the compatibility (d) forces $\gamma_{ub}(1_b) = 1_{ub}$.

This verifies that vertical arrows of $\mathbb{J}^{\pitchfork\!\!\!\pitchfork}$ are lalis in **Cat**; similar arguments show that squares of $\mathbb{J}^{\pitchfork\!\!\!\pitchfork}$ are ones preserving the lali structure, and that composition is composition of lalis, as required. $\square$

6.3. **Canonical class-cofibrant generation.** As further evidence for the adequacy of the notion of double-categorical cofibrant generation, we have the following result, which tells us that any AWFS is class-cofibrantly generated by its (typically large) double category of coalgebras.

Proposition 20. *Any* AWFS *is class-cofibrantly generated by* $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *and class-fibrantly generated by* $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$.

Proof. By duality, we need prove only the first statement. As above, (2.4) is an $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}, \mathsf{R}\text{-}\mathbb{A}\mathbf{lg})$-lifting operation, and so by Proposition 18 corresponds to a concrete double functor $\Lambda\colon \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork\!\!\!\pitchfork}$ over $\mathcal{C}$. Of course, Λ is the identity on objects and horizontal arrows; while its component $\Lambda_1\colon \mathsf{R}\text{-}\mathbf{Alg} \to (\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork\!\!\!\pitchfork})_1$ on vertical arrows and squares postcomposes with the inclusion $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork\!\!\!\pitchfork})_1 \hookrightarrow \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ to yield the $\bar{\Phi}$ of Lemma 1. As $\bar{\Phi}$ is injective on objects and fully faithful, so too is Λ_1; while since the left square of (2.11) is one of $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$, the condition (6.1) defining the objects in $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork\!\!\!\pitchfork})_1$ implies that in (2.9), so that Λ_1 is also surjective on objects, and thus an isomorphism. $\square$

The fact of an isomorphism $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \cong {}^{\pitchfork\!\!\!\pitchfork}\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ is often a useful tool for calculation. Many AWFS that arise in practice are cofibrantly generated by double categories in a natural way; as such, we have a concrete understanding of the R-algebras. The above isomorphism offers one technique for obtaining the corresponding L-coalgebras without explicit calculation of the values of L or R.

6.4. **Freeness of cofibrantly generated awfs.** Above, we have defined cofibrant generation of an AWFS in terms of a universal property of its double category of algebras. We conclude this section by showing that this implies another universal property: that its double category of coalgebras is *freely generated* by the given $\mathbb{J}$ with respect to left adjoint functors between concrete double categories. In the case of cofibrant generation by a mere category, this was shown in [43, Theorem 6.22]; our result generalises this, and simplifies the proof.

The key step will be to extend the adjunction (6.2) to account for change of base. To this end, we consider the 2-category $\mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})$, whose objects are double functors $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, whose 1-cells are squares as on the left of

$$(6.3)\qquad \begin{tikzcd} \mathbb{J} \arrow[r, "\bar{F}"] \arrow[d, "U"'] & \mathbb{J}' \arrow[d, "U'"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"'] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{J} \arrow[r, bend left, "\bar{F}"] \arrow[r, bend right, "\bar{F}'"'] \arrow[r, phantom, "\Downarrow\bar{\alpha}"] \arrow[d, "U"'] & \mathbb{J}' \arrow[d, "U'"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, bend left, "\mathbb{S}\mathbf{q}(F)"] \arrow[r, bend right, "\mathbb{S}\mathbf{q}(F')"'] \arrow[r, phantom, "\Downarrow\mathbb{S}\mathbf{q}(\alpha)"] & \mathbb{S}\mathbf{q}(\mathcal{D}) \end{tikzcd}$$

together with a chosen right adjoint G for F, and whose 2-cells are diagrams as on the right above. We define the 2-category $\mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})$ dually.

Proposition 21. *The adjunction* (6.2) *extends to a* 2*-adjunction*

$$
\begin{tikzcd}
(\mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}}))^{\mathrm{coop}} \arrow[r, shift right=1.5ex, "(-)^{\pitchfork}"'] \arrow[r, phantom, "\perp"] & \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}}) \arrow[l, shift right=1.5ex, "{}^{\pitchfork}(-)"']
\end{tikzcd} \;, \tag{6.4}
$$

with respect to which the canonical isomorphisms $\Lambda\colon \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork}$ *and* $\bar{\Lambda}\colon \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \to {}^{\pitchfork}\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ *of Proposition 20 are* 2*-natural.*

Proof. The 2-functor $(–)^{\pitchfork}$ is defined as before on objects, and as before on 1-cells (6.3) whose bottom edge is an identity. To complete the definition on 1-cells, it thus suffices to consider squares whose *top* edge is an identity, as on the left in

$$
\begin{tikzcd}
\mathbb{J} \arrow[r, "1"] \arrow[d, "U"'] & \mathbb{J} \arrow[d, "\mathbb{S}\mathbf{q}(F)\cdot U"] \\
\mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"'] & \mathbb{S}\mathbf{q}(\mathcal{D})
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(F\mathbb{J})^{\pitchfork} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{J}^{\pitchfork} \arrow[d] \\
\mathbb{S}\mathbf{q}(\mathcal{D}) \arrow[r, "\mathbb{S}\mathbf{q}(G)"'] & \mathbb{S}\mathbf{q}(\mathcal{C})
\end{tikzcd} \;. \tag{6.5}
$$

Let $F\mathbb{J}$ denote the double category $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{D})$ over $\mathbb{S}\mathbf{q}(\mathcal{D})$ down the right edge of this square, and let G be the chosen right adjoint for F. We claim that there is in fact a *pullback* square as displayed right above: which we will take to be the value of $(–)^{\pitchfork}$ at the left-hand square. For this, we must show that for each $g\colon C \to D$ of $\mathcal{D}$, the $(F\mathbb{J}, g)$-lifting operations $\varphi_{-,g}$ correspond functorially with $(\mathbb{J}, Gg)$-lifting operations $\bar{\varphi}_{-,Gg}$. But such lifting operations are given by natural transformations

$$\varphi_{-,g}\colon \mathcal{C}(\mathrm{cod}\ FU–, \mathrm{dom}\ g) \Rightarrow \mathcal{C}^{\mathbf{2}}(FU–, g)\colon \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$$
$$\bar{\varphi}_{-,Gg}\colon \mathcal{C}(\mathrm{cod}\ U–, \mathrm{dom}\ Gg) \Rightarrow \mathcal{C}^{\mathbf{2}}(U–, Gg)\colon \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$$

satisfying axioms; so as $F \dashv G$, these are in a clear bijective correspondence, under which the maps of lifting operations also correspond as required.

Finally, we must define $(–)^{\pitchfork}$ on 2-cells. Given a diagram as on the right of (6.3), the 2-cell $\alpha\colon F \Rightarrow F'$ therein transposes under adjunction to one $\beta\colon G' \Rightarrow G$. As the forgetful double functors $(\mathbb{J}')^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{D})$ and $\mathbb{J}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ are concrete, there is a unique possible 2-cell $\bar{\beta}\colon (\bar{F}')^{\pitchfork} \Rightarrow \bar{F}^{\pitchfork}$ lifting β; a short calculation shows that this lifting indeed exists, so that we may take $(\alpha, \bar{\alpha})^{\pitchfork} = (\beta, \bar{\beta})$. This completes the definition of $(–)^{\pitchfork}$; that of ${}^{\pitchfork}(–)$ is dual.

To verify that the extended $(–)^{\pitchfork}$ and ${}^{\pitchfork}(–)$ are 2-adjoint, we observe that for $\mathbb{J}$ and $\mathbb{K}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$ and $\mathbb{S}\mathbf{q}(\mathcal{D})$, a map $\mathbb{J} \to {}^{\pitchfork}\mathbb{K}$ in $\mathbf{DBL}/\mathbb{S}\mathbf{q}(–_{\mathrm{ladj}})$ is an adjunction $F \dashv G\colon \mathcal{D} \to \mathcal{C}$ together with a $(F\mathbb{J}, \mathbb{K})$-lifting operation, while a map $\mathbb{K} \to \mathbb{J}^{\pitchfork}$ in $\mathbf{DBL}/\mathbb{S}\mathbf{q}(–_{\mathrm{radj}})$ is an adjunction $F \dashv G$ together with a $(\mathbb{J}, G\mathbb{K})$-lifting operation; by the above argument, these data are in bijective correspondence. We argue similarly for the bijection on 2-cells. Finally, the naturality of the isomorphisms Λ and $\bar{\Lambda}$ is exactly (2.13); their 2-naturality now follows by concreteness. □

Using the result, we now give our promised generalisation of [43, Theorem 6.22].

Proposition 22. *If* (L, R) *is (class-)cofibrantly generated by* $U\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$*, then it provides a left* 2*-adjoint at* $\mathbb{J}$ *for* $(–)\text{-}\mathbb{C}\mathbf{oalg}\colon \mathbf{AWFS}_{\mathrm{ladj}} \to \mathbf{DBL}/\mathbb{S}\mathbf{q}(–_{\mathrm{ladj}})$.

Proof. We have natural isomorphisms

$$\begin{aligned}
\mathbf{AWFS}_{\mathrm{ladj}}((\mathsf{L},\mathsf{R}),\ (\mathsf{L}',\mathsf{R}')) &\cong \mathbf{AWFS}_{\mathrm{radj}}((\mathsf{L}',\mathsf{R}'),\ (\mathsf{L},\mathsf{R}))^{\mathrm{op}} \\
&\cong \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})(\mathsf{R}'\text{-}\mathbb{A}\mathbf{lg}, \mathsf{R}\text{-}\mathbb{A}\mathbf{lg})^{\mathrm{op}} \\
&\cong \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{radj}})(\mathsf{R}'\text{-}\mathbb{A}\mathbf{lg}, \mathbb{J}^{\pitchfork})^{\mathrm{op}} \\
&\cong \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})(\mathbb{J}, {}^{\pitchfork}\mathsf{R}'\text{-}\mathbb{A}\mathbf{lg}) \\
&\cong \mathbf{DBL}/\mathbb{S}\mathbf{q}(-_{\mathrm{ladj}})(\mathbb{J}, \mathsf{L}'\text{-}\mathbb{C}\mathbf{oalg})\ ,
\end{aligned}$$

by, respectively, the isomorphism $\mathbf{AWFS}_{\mathrm{ladj}} \cong \mathbf{AWFS}_{\mathrm{radj}}{}^{\mathrm{coop}}$; full fidelity of $(-)\text{-}\mathbb{A}\mathbf{lg}$; the assumed isomorphism $\mathbb{J}^{\pitchfork} \cong \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$; adjointness (6.4); and the natural isomorphisms of Proposition 20. ☐

7. A characterisation of accessible awfs

We are now ready to give our second main result, Theorem 25, which gives a characterisation of the accessible awfs on a locally presentable category. We will show that, for a locally presentable $\mathcal{C}$, any small double category over $\mathbb{S}\mathbf{q}(\mathcal{C})$ cofibrantly generates an accessible awfs, and, moreover, that every accessible awfs on $\mathcal{C}$ arises in this way.

7.1. **Existence of cofibrantly generated AWFS.** We begin by extending Proposition 16 from cofibrant generation by categories to that by double categories.

Proposition 23. *If $\mathcal{C}$ is locally presentable, then the* awfs *cofibrantly generated by any small double category $U\colon \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ exists and is accessible.*

Proof. Write $\mathcal{J}_2 = \mathcal{J}_1 \times_{\mathcal{J}_0} \mathcal{J}_1$ for the category of vertically composable pairs in $\mathbb{J}$, and $m\colon \mathcal{J}_2 \to \mathcal{J}_1$ for the vertical composition. Now the triangle on the left in

$$\begin{tikzcd}
\mathcal{J}_2 \arrow[rr, "m"] \arrow[dr, "U_1 m"'] & & \mathcal{J}_1 \arrow[dl, "U_1"] \\
& \mathcal{C}^{\mathbf{2}} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{J}_1{}^{\pitchfork} \arrow[rr, "m^{\pitchfork}"] \arrow[dr] & & \mathcal{J}_2{}^{\pitchfork} \arrow[dl] \\
& \mathbb{S}\mathbf{q}(\mathcal{C}) &
\end{tikzcd}$$

induces a morphism $m^{\pitchfork}$ of concrete double categories over $\mathcal{C}$ as on the right, which on vertical arrows sends $(g, \varphi_{-g})\colon D \to E$ to $(g, \psi_{-g})\colon D \to E$, with $(\mathcal{J}_2, g)$-lifting operation ψ_{-g} defined as on the left in

$$\psi_{(j,i),g}(u,v) = \varphi_{j\cdot i,g}(u,v) \qquad\qquad \theta_{(j,i),g}(u,v) = \varphi_{j,g}(\varphi_{i,g}(u, v\cdot Uj), v)\ .$$

We also have the $(\mathcal{J}_2, g)$ lifting operation θ_{-g} as on the right above, and the assignation $(g, \varphi_{-g}) \mapsto (g, \theta_{-g})$ in fact gives the action on vertical arrows of a further double functor $\delta_{\mathbb{J}}\colon \mathcal{J}_1{}^{\pitchfork} \to \mathcal{J}_2{}^{\pitchfork}$ concrete over $\mathcal{C}$. It is easy to see that we now have an equaliser of concrete double categories over $\mathcal{C}$ as on the left in

$$\begin{tikzcd}
\mathbb{J}^{\pitchfork} \arrow[r] \arrow[dr] & \mathcal{J}_1{}^{\pitchfork} \arrow[r, shift left, "m^{\pitchfork}"] \arrow[r, shift right, "\delta_{\mathbb{J}}"'] \arrow[d] & \mathcal{J}_2{}^{\pitchfork} \arrow[dl] \\
& \mathbb{S}\mathbf{q}(\mathcal{C}) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{J}_1{}^{\pitchfork} \arrow[r, shift left, "m^{\pitchfork}"] \arrow[r, shift right, "(\delta_{\mathbb{J}})_1"'] \arrow[d, "\cong"'] & \mathcal{J}_2{}^{\pitchfork} \arrow[d, "\cong"] \\
\mathsf{R}_1\text{-}\mathbf{Alg} \arrow[r, shift left, "s^*"] \arrow[r, shift right, "t^*"'] & \mathsf{R}_2\text{-}\mathbf{Alg}\ .
\end{tikzcd} \tag{7.1}$$

Since ${\mathcal{J}_1}^{\pitchfork\!\!\pitchfork}$ and ${\mathcal{J}_2}^{\pitchfork\!\!\pitchfork}$ are right-connected, so too will $\mathbb{J}^{\pitchfork\!\!\pitchfork}$ be; whence by Theorem 6, the AWFS cofibrantly generated by $\mathbb{J}$ will exist if and only if $(\mathbb{J}^{\pitchfork\!\!\pitchfork})_1 \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic. As $\mathcal{J}_1$ and $\mathcal{J}_2$ are small, Proposition 16 ensures that there are AWFS $(\mathsf{L}_1, \mathsf{R}_1)$ and $(\mathsf{L}_2, \mathsf{R}_2)$ on $\mathcal{C}$ with ${\mathcal{J}_1}^{\pitchfork} \cong \mathsf{R}_1\text{-}\mathbf{Alg}$ and ${\mathcal{J}_2}^{\pitchfork} \cong \mathsf{R}_2\text{-}\mathbf{Alg}$ over $\mathcal{C}^{\mathbf{2}}$; now by full fidelity of the assignment $\mathbf{Mnd}(\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \to \mathbf{CAT}/\mathcal{C}^{\mathbf{2}}$, there are unique monad maps $s, t \colon \mathsf{R}_2 \rightrightarrows \mathsf{R}_1$ rendering commutative the diagram above right. It follows that $(\mathbb{J}^{\pitchfork\!\!\pitchfork})_1$ is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to the equaliser of the lower row. Since R_1 and R_2 are accessible, the parallel pair of monad morphisms s, t admits by [36, Theorem 27.1] a coequaliser $q \colon \mathsf{R}_1 \to \mathsf{R}$ which is again accessible, and for which q^* is the equaliser of s^* and t^*. Thus $(\mathbb{J}^{\pitchfork\!\!\pitchfork})_1 \cong \mathsf{R}\text{-}\mathbf{Alg}$ over $\mathcal{C}^{\mathbf{2}}$ so that $(\mathbb{J}^{\pitchfork\!\!\pitchfork})_1 \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic for an accessible monad, as required. $\square$

We now wish to show that every accessible AWFS on a locally presentable category arises in the manner of the preceding proposition. The key idea is to show that any accessible AWFS has a small dense subcategory of L-coalgebras, and then deduce the result using Proposition 20 and the following lemma:

Lemma 24. *Given a morphism of categories over* $\mathcal{C}^{\mathbf{2}}$

$$\begin{tikzcd} \mathcal{J} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{K} \arrow[dl, "V"] \\ & \mathcal{C}^{\mathbf{2}} & \end{tikzcd},$$

the induced $F^{\pitchfork} \colon \mathcal{K}^{\pitchfork} \to \mathcal{J}^{\pitchfork}$ is an isomorphism of categories whenever the identity 2-cell exhibits V as the pointwise Kan extension $\mathrm{Lan}_F U$. In particular, this is the case whenever F is dense and V preserves the colimits exhibiting this density.

Proof. As in the proof of Proposition 16, an element of $\mathcal{K}^{\pitchfork}$ is a pair (g, φ_{-g}) where $g \colon C \to D$ in $\mathcal{C}$ and φ_{-g} is a section of the $\psi_{-g} \colon \mathcal{C}(\mathrm{cod}\, V{-}, C) \to \mathcal{C}^{\mathbf{2}}(V{-}, g)$ of (5.2); or equally (as $\mathrm{cod} \dashv \mathrm{id} \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$) a section of $(1, g) \cdot ({-}) \colon \mathcal{C}^{\mathbf{2}}(V{-}, 1_C) \to \mathcal{C}^{\mathbf{2}}(V{-}, g)$. In these terms, a map $(g, \varphi_{-g}) \to (h, \varphi_{-h})$ of $\mathcal{K}^{\pitchfork}$ is a map $g \to h$ of $\mathcal{C}^{\mathbf{2}}$ for which the induced $\mathcal{C}^{\mathbf{2}}(V{-}, g) \to \mathcal{C}^{\mathbf{2}}(V{-}, h)$ commutes with the given sections. The map $\mathcal{K}^{\pitchfork} \to \mathcal{J}^{\pitchfork}$ is induced by the restriction functor $[F^{\mathrm{op}}, 1] \colon [\mathcal{K}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]$, and sends (g, φ_{-g}) to the pair (g, φ') with second component

$$\varphi' = \varphi_{-g} F \colon \mathcal{C}^{\mathbf{2}}(U{-}, g) \to \mathcal{C}^{\mathbf{2}}(U{-}, 1_C) \ .$$

So to prove $F^{\pitchfork} \colon \mathcal{K}^{\pitchfork} \to \mathcal{J}^{\pitchfork}$ is an isomorphism, it will suffice to show that $[F^{\mathrm{op}}, 1]$ becomes fully faithful when restricted to the full subcategory of $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ on the presheaves of the form $\mathcal{C}^{\mathbf{2}}(V{-}, g)$. Now, to say that the identity 2-cell exhibits V as $\mathrm{Lan}_F U$ is equally to say that it exhibits V^{op} as $\mathrm{Ran}_{F^{\mathrm{op}}} U^{\mathrm{op}}$; the continuous $\mathcal{C}^{\mathbf{2}}({-}, g) \colon (\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \to \mathbf{Set}$ preserves this pointwise Kan extension, and so the identity 2-cell exhibits $\mathcal{C}^{\mathbf{2}}(V{-}, g)$ as the right Kan extension of $\mathcal{C}^{\mathbf{2}}(U{-}, g)$ along F^{op}. The universal property of this right Kan extension now immediately implies that the action of $[F^{\mathrm{op}}, 1]$ on morphisms

$$[\mathcal{K}^{op}, \mathbf{Set}](\mathcal{C}^{\mathbf{2}}(V{-}, f), \mathcal{C}^{\mathbf{2}}(V{-}, g)) \to [\mathcal{J}^{op}, \mathbf{Set}](\mathcal{C}^{\mathbf{2}}(U{-}, f), \mathcal{C}^{\mathbf{2}}(U{-}, g))$$

is invertible for any f and g in $\mathcal{C}^{\mathbf{2}}$, as required. Finally, note that if F is dense, then the identity 2-cell exhibits $1_{\mathcal{K}}$ as $\mathrm{Lan}_F F$, and that if V preserves the colimits exhibiting this density, then it preserves this left Kan extension, so that the identity 2-cell exhibits V as $\mathrm{Lan}_F U$. □

Theorem 25. *An* AWFS *on a locally presentable category is accessible if and only if it is cofibrantly generated by some small double category.*

Proof. One direction is Proposition 23; for the other, let (L, R) be an accessible AWFS on the locally presentable $\mathcal{C}$. By Proposition 20 we have an isomorphism $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork} \cong \mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$; it will thus suffice to find a small sub-double category $\mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg} \hookrightarrow \mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$ with the induced $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork} \to \mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg}^{\pitchfork}$ invertible. To this end, we consider the following diagram, whose left column is yet to be defined:

$$\begin{tikzcd}
\mathsf{L}_\lambda\text{-}\mathbf{Coalg}_2 \arrow[r, "j_2"] \arrow[d, shift right=2] \arrow[d] \arrow[d, shift left=2] & \mathsf{L}\text{-}\mathbf{Coalg}_2 \arrow[r, "V_2"] \arrow[d, shift right=2, "p_{\mathsf{L}}"'] \arrow[d, "m_{\mathsf{L}}" description] \arrow[d, shift left=2, "q_{\mathsf{L}}"] & \mathcal{C}^{\mathbf{3}} \arrow[d, shift right=2, "p"'] \arrow[d, "m" description] \arrow[d, shift left=2, "q"] \\
\mathsf{L}_\lambda\text{-}\mathbf{Coalg} \arrow[r, "j_1"] \arrow[d, shift right=2] \arrow[d, leftarrow] \arrow[d, shift left=2] & \mathsf{L}\text{-}\mathbf{Coalg} \arrow[r, "V_1"] \arrow[d, shift right=2, "d_{\mathsf{L}}"'] \arrow[d, leftarrow, "i_{\mathsf{L}}" description] \arrow[d, shift left=2, "c_{\mathsf{L}}"] & \mathcal{C}^{\mathbf{2}} \arrow[d, shift right=2, "d"'] \arrow[d, leftarrow, "i" description] \arrow[d, shift left=2, "c"] \\
\mathcal{C}_\lambda \arrow[r, "j_0"] & \mathcal{C} \arrow[r, "1"] & \mathcal{C}\ .
\end{tikzcd}$$

As colimits are pointwise in functor categories the right column is composed of cocontinuous functors between cocomplete categories; since V_1 creates colimits, the same is true for $\mathsf{L}\text{-}\mathbf{Coalg}$ and each of d_{L}, i_{L} and c_{L}. Moreover d_{L} has the isomorphism-lifting property since V_1 and d do, whence by [34] the pullback $\mathsf{L}\text{-}\mathbf{Coalg}_2$ of d_{L} along c_{L} is also a bipullback. Consequently, $\mathsf{L}\text{-}\mathbf{Coalg}_2$ is cocomplete and the projections q_{L} and p_{L} cocontinuous; so too is m_{L}, since d_{L} and c_{L} jointly create colimits; and similarly for V_2. Thus the right two columns are composed of cocomplete categories and cocontinuous functors.

Now $\mathcal{C}$ is locally presentable, whence also $\mathcal{C}^{\mathbf{2}}$ and $\mathcal{C}^{\mathbf{3}}$; and since L is accessible, $\mathsf{L}\text{-}\mathbf{Coalg}$ is locally presentable as in the proof of Proposition 13. Finally, $\mathsf{L}\text{-}\mathbf{Coalg}_2$ is a bipullback of cocontinuous functors between locally presentable categories, and so itself locally presentable by [10, Theorem 3.15]. So the right two columns are composed of locally presentable categories and cocontinuous functors.

Any cocontinuous functor between locally presentable categories has an accessible right adjoint [19, Satz 14.6]; thus there is a regular λ so that the right two columns are composed of locally λ-presentable categories and functors with λ-accessible right adjoints; it follows that each left adjoint preserves λ-presentable objects. We now define $\mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg}$[3] as the full sub-double category of $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}$ with as objects, isomorphism-class representatives of the λ-presentable objects of $\mathcal{C}$. This category is small, and the inclusion $j_0 \colon \mathcal{C}_\lambda \to \mathcal{C}$ is dense; we claim that j_1 and j_2 are too. For j_1, we consider the full dense subcategory $(\mathsf{L}\text{-}\mathbf{Coalg})_\lambda$ of λ-presentable objects in $\mathsf{L}\text{-}\mathbf{Coalg}$. Because d_{L} and c_{L} preserve λ-presentability it follows that each λ-presentable of $\mathsf{L}\text{-}\mathbf{Coalg}$ has λ-presentable domain and codomain; so the fully faithful dense functor $(\mathsf{L}\text{-}\mathbf{Coalg})_\lambda \to \mathsf{L}\text{-}\mathbf{Coalg}$ factors through the full inclusion $j_1 \colon \mathsf{L}_\lambda\text{-}\mathbf{Coalg} \to \mathsf{L}\text{-}\mathbf{Coalg}$, whence, by [37, Theorem 5.13] j_1 is dense. The

[3]Our notation $\mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg}$ acknowledges the fact that this is equally the double category of coalgebras associated to the restricted AWFS on $\mathcal{C}_\lambda$.

density of j_2 follows in the same manner, by considering the full dense subcategory $(\mathsf{L}\text{-}\mathbf{Coalg}_2)_\lambda$ of $\mathsf{L}\text{-}\mathbf{Coalg}_2$ and arguing that it is contained in $(\mathsf{L}_\lambda\text{-}\mathbf{Coalg})_2$.

Since both j_1 and j_2 are dense, and since all of the functors to their right are cocontinuous it follows from Lemma 24 that both $j_1{}^{\pitchfork}$ and $j_2{}^{\pitchfork}$ are isomorphisms of categories; whence $j_1{}^{\Pitchfork}$ and $j_2{}^{\Pitchfork}$ as displayed in the right two columns of

$$\begin{tikzcd}
\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}^{\Pitchfork} \arrow[r] \arrow[d, "j^{\Pitchfork}"'] & \mathsf{L}\text{-}\mathbf{Coalg}^{\Pitchfork} \arrow[r, shift left, "m^{\Pitchfork}"] \arrow[r, shift right, "\delta"'] \arrow[d, "j_1{}^{\Pitchfork}"] & (\mathsf{L}\text{-}\mathbf{Coalg}_2)^{\Pitchfork} \arrow[d, "j_2{}^{\Pitchfork}"'] \\
\mathsf{L}_\lambda\text{-}\mathbb{C}\mathbf{oalg}^{\Pitchfork} \arrow[r] & \mathsf{L}_\lambda\text{-}\mathbf{Coalg}^{\Pitchfork} \arrow[r, shift left, "m^{\Pitchfork}"] \arrow[r, shift right, "\delta"'] & (\mathsf{L}_\lambda\text{-}\mathbf{Coalg}_2)^{\Pitchfork}
\end{tikzcd}$$

are isomorphisms of double categories. But both rows are, as in (7.1), equalisers, and it follows that the induced double functor $j^{\Pitchfork}$ on the left is invertible as claimed. □

8. Fibre squares and enriched cofibrant generation

We conclude this paper with a further batch of examples based on a surprisingly powerful modification of the small object argument of Proposition 16. Given a well-behaved monoidal category $\mathcal{V}$ equipped with an AWFS (L, R), a well-behaved $\mathcal{V}$-category $\mathcal{C}$, and a small category $U \colon \mathcal{J} \to (\mathcal{C}_0)^{\mathbf{2}}$ over $(\mathcal{C}_0)^{\mathbf{2}}$, it constructs the "free (L, R)-enriched AWFS cofibrantly generated by $\mathcal{J}$". This is an AWFS on $\mathcal{C}_0$ whose algebras $\boldsymbol{g} \colon C \to D$ are maps equipped with, for every $Uj \colon A \to B$ in the image of U, a choice of R-map structure on the canonical comparison map $\mathcal{C}(B, C) \to \mathcal{C}(B, D) \times_{\mathcal{C}(A,D)} \mathcal{C}(A, C)$ in $\mathcal{V}$, naturally in j.

The "enriched small object argument" which builds AWFS of this kind is most properly understood in the context of *monoidal algebraic weak factorisation systems* [44]; and as indicated in the introduction, a comprehensive treatment along those lines must await a further paper. Here, we content ourselves with giving the construction and a range of applications.

8.1. Fibre squares. The cleanest approach to enriched cofibrant generation makes use of the following *fibre square* construction. It associates to any AWFS (L, R) on a category $\mathcal{C}$ with pullbacks an AWFS on the arrow category $\mathcal{C}^{\mathbf{2}}$ whose algebras are the R-*fibre squares*: maps $(u, v) \colon f \to g$ of $\mathcal{C}^{\mathbf{2}}$ as to the left of

$$\begin{tikzcd}
A \arrow[rrd, bend left, "u"] \arrow[rd, dotted, "k"] \arrow[rdd, bend right, "f"'] & & \\
& B \times_D C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& B \arrow[r, "v"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Fibre}_{\mathsf{R}}(\mathcal{C}) \arrow[r] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{R}\text{-}\mathbf{Alg} \arrow[d, "U^{\mathsf{R}}"] \\
(\mathcal{C}^{\mathbf{2}})^{\mathbf{2}} \arrow[r, "(u,v) \mapsto k"'] & \mathcal{C}^{\mathbf{2}}
\end{tikzcd}$$

equipped with an R-algebra structure on the comparison map $k \colon A \to B \times_D C$. The R-fibre squares are the objects of a category $\mathbf{Fibre}_{\mathsf{R}}(\mathcal{C})$, whose morphisms are determined by way of the pullback square on the right above.

We may compose R-fibre squares: given $(u, v) \colon f \to g$ with R-algebra structure $\boldsymbol{k} \colon A \to B \times_D C$ and $(s, t) \colon g \to h$ with structure $\boldsymbol{\ell} \colon C \to D \times_F E$, we can pull back $\boldsymbol{\ell}$ along $v \times_F E$ and use Proposition 8 to obtain an algebra structure $v^*\boldsymbol{\ell}\colon B \times_D C \to B \times_F E$; the composite algebra $v^*\boldsymbol{\ell} \cdot \boldsymbol{k}\colon A \to B \times_D C \to B \times_F E$ now equips $(s,t) \cdot (u,v)\colon f \to h$ with R-fibre square structure. In this way we obtain a double category $\mathbb{F}\mathbf{ibre}_{\mathsf{R}}(\mathcal{C}) \to \mathbb{S}\mathbf{q}(\mathcal{C}^{\mathbf{2}})$ which is concrete over $\mathcal{C}^{\mathbf{2}}$ and easily seen to be right-connected; so it will be the double category of algebras for an AWFS on $\mathcal{C}^{\mathbf{2}}$ as soon as $V\colon \mathbf{Fibre}_{\mathsf{R}} \to (\mathcal{C}^{\mathbf{2}})^{\mathbf{2}}$ is strictly monadic. As a pullback of the strictly monadic $U^{\mathsf{R}}\colon \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$, this V will always create V-absolute coequaliser pairs; so it suffices to show that it has a left adjoint. Given a square in $\mathcal{C}$ as on the left below, we factor the induced $k\colon A \to B \times_D C$ as $Rk \cdot Lk\colon A \to Ek \to B \times_D C$, yielding the factorisation

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "Lk"] \arrow[d, "f"'] & Ek \arrow[r, "\pi_2 \cdot Rk"] \arrow[d, "\pi_1 \cdot Rk"'] & C \arrow[d, "g"] \\ B \arrow[r, "1"'] & B \arrow[r, "v"'] & D \end{tikzcd} \tag{8.1}$$

on the right. The rightmost square, when equipped with the algebra structure $\boldsymbol{R}k\colon Ek \to B \times_D C$ on the comparison map, now comprises the free R-fibre square on (u,v). This shows that V has a left adjoint, whence by Theorem 6, $\mathbb{F}\mathbf{ibre}_{\mathsf{R}}(\mathcal{C}) \to \mathbb{S}\mathbf{q}(\mathcal{C}^{\mathbf{2}})$ is the double category of algebras for an AWFS on $\mathcal{C}^{\mathbf{2}}$; it is not hard to calculate that the corresponding coalgebras are maps $(u,v)\colon f \to g$ for which u is equipped with L-coalgebra structure and v is invertible. Note that if (L,R) is an accessible AWFS on a locally presentable category, then the fibre square AWFS on $\mathcal{C}^{\mathbf{2}}$ is also accessible, since its factorisations (8.1) are constructed using the accessible L and R and finite limits in $\mathcal{C}$.

8.2. **Enriched cofibrant generation.** Suppose now that $\mathcal{V}$ is a locally presentable symmetric monoidal closed category: thus a suitable base for enriched category theory. Let (L,R) be an accessible AWFS on $\mathcal{V}$, and let $\mathcal{C}$ be a $\mathcal{V}$-category whose underlying category $\mathcal{C}_0$ is locally presentable; finally, let $U\colon \mathcal{J} \to (\mathcal{C}_0)^{\mathbf{2}}$ be a small category over $(\mathcal{C}_0)^{\mathbf{2}}$. By applying its monad and comonad pointwise, we may lift the AWFS (L,R) on $\mathcal{V}$ to an accessible AWFS on $[\mathcal{J}^{\mathrm{op}}, \mathcal{V}]$; now applying the fibre square construction of the preceding section to this yields an accessible AWFS on $[\mathcal{J}^{\mathrm{op}}, \mathcal{V}]^{\mathbf{2}}$. There is a right adjoint functor $\tilde{U}\colon \mathcal{C}_0 \to [\mathcal{J}^{\mathrm{op}}, \mathcal{V}]^{\mathbf{2}}$ sending $x \in \mathcal{C}_0$ to the arrow $\mathcal{C}(\mathrm{cod}\, U–, x) \to \mathcal{C}(\mathrm{dom}\, U–, x)$ of $[\mathcal{J}^{\mathrm{op}}, \mathcal{V}]$; and we now define the *enriched* AWFS *cofibrantly generated by* $\mathcal{J}$ to be the projective lifting of the fibre square AWFS on $[\mathcal{J}^{\mathrm{op}}, \mathcal{V}]^{\mathbf{2}}$ along $\tilde{U}$; note that the existence of this lifting is guaranteed by Proposition 13(a). The double category of algebras of the AWFS $(\mathsf{L}_{\mathcal{J}}, \mathsf{R}_{\mathcal{J}})$ so obtained is thus defined by a pullback square

$$\begin{tikzcd} \mathsf{R}_{\mathcal{J}}\text{-}\mathbb{A}\mathbf{lg} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{F}\mathbf{ibre}_{[\mathcal{J}^{\mathrm{op}}, \mathcal{V}]}([\mathcal{J}^{\mathrm{op}}, \mathcal{V}]) \arrow[d] \\ \mathbb{S}\mathbf{q}(\mathcal{C}_0) \arrow[r, "\mathbb{S}\mathbf{q}(\tilde{U})"'] & \mathbb{S}\mathbf{q}([\mathcal{J}^{\mathrm{op}}, \mathcal{V}]^{\mathbf{2}}) \end{tikzcd} ;$$

so that in particular, an algebra structure on a map $g\colon C \to D$ is given by the choice, for each $j \in \mathcal{J}$ with image $Uj\colon A \to B$ in $\mathcal{C}^{\mathbf{2}}$, of an R-algebra structure on the comparison map $\mathcal{C}(B,C) \to \mathcal{C}(A,C) \times_{\mathcal{C}(A,D)} \mathcal{C}(B,D)$, naturally in j; we may say that g is equipped with an *enriched lifting operation* against $\mathcal{J}$.

8.3. **Examples.** The remainder of the paper will be given over to a range of examples of enriched cofibrant generation. All these examples will in fact be generated by a mere *set* of maps J, seen as a discrete subcategory $J \hookrightarrow (\mathcal{C}_0)^{\mathbf{2}}$ of a locally presentable $\mathcal{C}_0$.

Examples 26. (i) Let $\mathcal{V} = \mathbf{Set}$ and (L, R) be the AWFS for split epis thereon; then the notion of enriched cofibrant generation reduces to the unenriched one of Section 5.2. A full treatment of enriched cofibrant generation would in fact generalise each aspect of the theory developed in Sections 5 and 6; but as we have said above, this must await another paper.

(ii) More generally, let $\mathcal{V}$ be any locally presentable symmetric monoidal closed category, and let (L, R) be the split epi AWFS thereon. The enriched AWFS cofibrantly generated by $J \hookrightarrow \mathcal{C}^{\mathbf{2}}$ is precisely that obtained by the "enriched small object argument" of [45, Proposition 13.4.2].

(iii) Let $\mathcal{V} = \mathbf{Set}$ and let (L, R) be the initial AWFS thereon; its algebra category is the full subcategory of $\mathbf{Set}^{\mathbf{2}}$ on the isomorphisms. In this case, the algebra category of the enriched AWFS $(\mathsf{L}_J, \mathsf{R}_J)$ generated by $J \hookrightarrow \mathcal{C}^{\mathbf{2}}$ may be identified with the full subcategory of $\mathcal{C}^{\mathbf{2}}$ on those maps with the *unique* right lifting property against each map in J. In particular, since $\mathsf{R}_J\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$ is fully faithful, Proposition 3 applies, so that $(\mathsf{L}_J, \mathsf{R}_J)$ in fact describes the *orthogonal* factorisation system whose left class is generated by J [18, §2.2].

Examples 27. For our next examples of enriched cofibrant generation, we take $\mathcal{V} = \mathbf{Cat}$ equipped with the AWFS (L, R) for retract equivalences of Section 4.2.

(i) Take $\mathcal{C} = \mathbf{Cat}$ and let J comprise the single functor $! \colon \mathbf{0} \to \mathbf{1}$. It's easy to see that the enriched AWFS generated by J is again that for retract equivalences.

(ii) Take $\mathcal{C} = \mathbf{Cat}$ and let J comprise the single functor $\top \colon \mathbf{1} \to \mathbf{2}$ picking out the terminal object of $\mathbf{2}$. An algebra for the enriched AWFS generated by J is a functor $G \colon C \to D$ such that the induced $C^{\mathbf{2}} \to D \downarrow G$ bears retract equivalence structure, or equivalently, is fully faithful and equipped with a section on objects. Full fidelity corresponds to the requirement that every arrow of C be cartesian over D, whereupon a section on objects amounts to a choice of cartesian liftings: so an algebra is a cloven fibration whose fibres are groupoids. We find further that maps of algebras are squares strictly preserving the cleavage, and that composition of algebras is the usual composition of fibrations.

(iii) Let $(\mathcal{C}, j)$ be a small site, let $\mathcal{E} = \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})$ be the 2-category of pseudofunctors, strong transformations and modifications $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, and let $y \colon \mathcal{C} \to \mathcal{E}$ be the Yoneda embedding. If $(U_i \to U : i \in I)$ is a covering family in $\mathcal{C}$, then the associated *covering 2-sieve* $f \colon \varphi \to yU$ in $\mathcal{E}$ is the second half of the pointwise (bijective on objects, fully faithful) factorisation

$\sum_i yU_i \to \varphi \to yU$; recall that an object $X \in \mathcal{E}$ is called a *stack* if the restriction functor $\mathcal{E}(f, X)\colon \mathcal{E}(yU, X) \to \mathcal{E}(\varphi, X)$ is an equivalence of categories for each covering 2-sieve.

Consider now the 2-category $\mathcal{E}_s = \mathbf{Hom}(\mathcal{C}^{\mathrm{op}}, \mathbf{Cat})_s$ of pseudofunctors, 2-natural transformations and modifications. By the general theory of [11], there is an accessible 2-monad T on $\mathbf{Cat}^{\mathrm{ob}\,\mathcal{C}}$ whose 2-category $\mathsf{T}\text{-}\mathbf{Alg}_s$ of strict algebras and strict algebra morphisms is $\mathcal{E}_s$—so that in particular, $\mathcal{E}_s$ is locally presentable—and whose 2-category $\mathsf{T}\text{-}\mathbf{Alg}$ of strict algebras and pseudomorphisms is $\mathcal{E}$. It follows by [11, Theorem 3.13] that the inclusion 2-functor $\mathcal{E}_s \to \mathcal{E}$ has a left adjoint $Q\colon \mathcal{E}_s \to \mathcal{E}$. Thus we may identify stacks with pseudofunctors X such that the restriction functor $\mathcal{E}_s(Qf, X)\colon \mathcal{E}_s(QyU, X) \to \mathcal{E}_s(Q\varphi, X)$ is an equivalence for each covering 2-sieve f.

As $\mathcal{E}_s$ is locally presentable, we may consider the enriched AWFS thereon generated by the set $\{Qf : f \text{ a covering 2-sieve in } \mathcal{E}\}$, and by the above, we see that its algebraically fibrant objects are a slight strengthening of the notion of stack; namely, they are pseudofunctors $X\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ such that each $\mathcal{E}(f, X)\colon \mathcal{E}(yU, X) \to \mathcal{E}(\varphi, X)$ is given the structure of a *retract* equivalence.

(iv) We may make arbitrary stacks into the algebraically fibrant objects of an AWFS by way of the following observations. Given $F\colon C \to D$ in $\mathbf{Cat}$, we may form its pseudolimit $D \downarrow_{\cong} F$—the category whose objects are triples $(d \in D, c \in C, \theta\colon d \cong Fc)$—and choices of equivalence section for $D \downarrow_{\cong} F \to D$ now correspond to choices of equivalence pseudoinverse for F. It follows that if $f\colon X \to Y$ is a map in a finitely cocomplete 2-category $\mathcal{E}$, and $\bar{f}\colon X \to \bar{Y}$ is the injection of X into the pseudocolimit of f, then for each $F \in \mathcal{E}$, retract equivalence structures on $\mathcal{E}(\bar{f}, F)$ correspond with equivalence structures on $\mathcal{E}(f, F)$.

In particular, if $(\mathcal{C}, j)$ is a small site and $\mathcal{E}$ and $\mathcal{E}_s$ are as above, then we may consider the enriched AWFS on $\mathcal{E}_s$ cofibrantly generated by the set $\{\overline{Qf} : f \text{ a covering 2-sieve in } \mathcal{E}\}$. Its algebraically fibrant objects are now stacks in the usual sense; to be precise, they are pseudofunctors X such that each $\mathcal{E}(yU, X) \to \mathcal{E}(\varphi, X)$ is provided with a chosen equivalence pseudoinverse.

Examples 28. Our next collection of examples again take $\mathcal{V} = \mathbf{Cat}$, but now with (L, R) the AWFS for lalis of Section 4.2.

(i) Let $\mathcal{C} = \mathbf{Cat}$ and let J comprise the single functor $!\colon \mathbf{0} \to \mathbf{1}$; then the enriched AWFS generated by J is simply that for lalis.

(ii) Let $\mathcal{C} = \mathbf{Cat}$ and let J comprise the single functor $\top\colon \mathbf{1} \to \mathbf{2}$ picking out the terminal object of $\mathbf{2}$. To equip $G\colon C \to D$ with algebra structure for the enriched AWFS this generates is to equip the induced $C^{\mathbf{2}} \to D \downarrow G$ with a right adjoint right inverse; which by [26], is equally to equip G with the structure of a cloven fibration. As before, maps of algebras are squares strictly preserving the cleavage; and composition of algebras is the usual composition of fibrations.

(iii) Generalising the previous two examples, let Φ be some set of small categories, and let J comprise the class of functors $\{A \to A_\perp : A \in \Phi\}$ obtained by freely adjoining an initial object to each A. To equip $G\colon C \to D$ with algebra structure for the induced AWFS is to give, for each $A \in \Phi$, a right adjoint section for the comparison functor $[A_\perp, C] \to [A_\perp, D] \times_{[A,D]} [A, C]$. Since a functor out of $A_\perp$ is the same thing as a functor out of A together with a cone over it, a short calculation shows that this amounts to giving, for each $A \in \Phi$, each diagram $F\colon A \to C$, and each cone $p\colon \Delta V \to GF$ in D, a cone $q\colon \Delta W \to F$ in C with $Gq = p$, such that for any cone $q'\colon \Delta W' \to F$, each factorisation of Gq' through p via a map $k\colon GW' \to V$ lifts to a unique factorisation of q' through q.[4] In particular, an algebraically fibrant object for this AWFS is a category equipped with a choice of limits for all diagrams indexed by categories in Φ.

(iv) Generalising the preceding example, let $\mathcal{C}$ be any locally presentable 2-category. For any arrow $f\colon A \to B$ of $\mathcal{C}$, let $\bar{f}\colon A \to \bar{B}$ denote the injection into the lax colimit of the arrow f, and for any set J of arrows in $\mathcal{C}$, let $\bar{J} = \{\bar{f} : f \in J\}$. By using the universal property of the colimit, we may calculate that an algebra structure on a morphism $g\colon C \to D$ for the AWFS induced by $\bar{J}$ is given by the choice, for every $f\colon A \to B$ in J and every 2-cell θ as on the left below, of a map k and 2-cell γ as in the centre with $gk = v$ and $g\gamma = \theta$, and such that γ is *initial* over g; thus, for every diagram as on the right with $g\alpha = \theta \cdot \beta f$, there exists a unique 2-cell $\delta\colon \ell \Rightarrow k$ with $g\delta = \beta$ and $\alpha = \gamma \cdot \delta f$.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow\theta"] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[r, phantom, shift right=3, "\Uparrow\gamma"] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "k"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[r, phantom, shift right=3, "\Uparrow\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "\ell"] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] & D \end{tikzcd} .$$

In particular, the algebraically fibrant objects for this AWFS are those $C \in \mathcal{C}$ such that, for each $f\colon A \to B$ in J, the functor $\mathcal{C}(f, C)\colon \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ is equipped with a right adjoint; that is, such that each morphism $A \to C$ in $\mathcal{C}$ is equipped with a chosen right Kan extension along f.

Of course, we may construct examples dual to the preceding ones—dealing with ralis, opfibrations, final liftings of cocones, categories with colimits, and left Kan extensions—by replacing the AWFS for lalis in the above by that for ralis.

Example 29. Let $\mathcal{V}$ be the category $\mathbf{SSet} = [\Delta^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets, equipped with the AWFS for *trivial Kan fibrations*, cofibrantly generated by the set of boundary inclusions $\{\partial\Delta[n] \to \Delta[n] : n \in \mathbb{N}\}$. Given a set J of maps in the locally presentable $\mathbf{SSet}$-category $\mathcal{C}$, the algebraically fibrant objects of the enriched AWFS cofibrantly generated by J are those $X \in \mathcal{C}$ such that, for each $f\colon A \to B$ in J, the induced map $\mathcal{C}(B, X) \to \mathcal{C}(A, X)$ bears algebraic trivial fibration structure. Most typically, one would consider this in the situation where

[4] Note that when A is a discrete category, this is precisely the notion of *G-initial lifting* [1, Definition 10.57] of a discrete cone.

$\mathcal{C}$ is a simplicial model category and the J's are a class of cofibrations; then the algebraically fibrant objects above which are also fibrant for the underlying model structure are precisely the *J-local objects* [28, Definition 3.1.4]. The enriched small object argument in this particular case was described in [14, §7].

Our final examples bear on the theory of *quasicategories* [32]. Quasicategories are simplicial sets with fillers for all inner horns; they are a particular model for $(\infty, 1)$-categories—weak higher categories whose cells of dimension > 1 are weakly invertible—and have a comprehensive theory [33, 40] paralleling the classical theory of categories. In particular, there are good notions of *Grothendieck fibration*, *limit*, and *Kan extension* for quasicategories; and by analogy with Examples 27, we may hope to obtain these notions as fibrations or fibrant objects for AWFS constructed by enriched cofibrant generation, so long as we can find a good quasicategorical analogue of the AWFS for lalis.

For this we propose the AWFS of Examples 12(v) above, whose R-algebras we called the *simplicial lalis*. By using the explicit formulae of Section 4.4, we may calculate the monad R at issue, and so obtain the following concrete description of its algebras. First some notation: given a simplicial set X, we write $x\colon a \rightsquigarrow b$ to denote a simplex $x \in X_{n+1}$ whose $(n+1)$st face is a and whose final vertex is b. Now by a *simplicial lali*, we mean a simplicial map $f\colon A \to B$ together with:

- A section $u\colon B_0 \to A_0$ of the action of f on 0-simplices;
- For each $a \in A_n$ and $x\colon fa \rightsquigarrow b$ in B, a simplex $\gamma_a(x)\colon a \rightsquigarrow ub$ over x;

subject to the coherence conditions that:

- $\gamma_a(x) \cdot \delta_i = \gamma_{a\cdot\delta_i}(x \cdot \delta_i)$ and $\gamma_a(x) \cdot \sigma_i = \gamma_{a\cdot\sigma_i}(x \cdot \sigma_i)$;
- $\gamma_{ub}(b \cdot \sigma_0) = ub \cdot \sigma_0$; and
- $\gamma_{\gamma_a(x)}(x \cdot \sigma_{n+1}) = \gamma_a(x) \cdot \sigma_{n+1}$.

By comparing with Proposition 19, we see that if $A = N\mathcal{C}$ and $B = N\mathcal{D}$ are the nerves of small categories, then a simplicial lali $A \to B$ is the same thing as a lali $\mathcal{C} \to \mathcal{D}$. This, of course, does not in itself justify the notion of simplicial lali; in order to do so, we will relate it to a notion introduced by Riehl and Verity in [52, Example 4.4.7], which for the present purposes we shall refer to as a *quasicategorical lali*. A simplicial map $f\colon A \to B$ is a quasicategorical lali if equipped with a strict section $u\colon B \to A$, a simplicial homotopy $\eta\colon 1 \Rightarrow uf$ satisfying $f\eta = 1_f$ (on the nose), and a 2-homotopy

$$\begin{tikzcd} & u \arrow[dr, "1_u"] \arrow[d, phantom, "\theta"] & \\ u \arrow[ur, "\eta u"] \arrow[rr, "1_u"'] & {} & u \end{tikzcd}$$

satisfying $f\theta = 1_{1_B}$. We have established the following result; its proof is beyond the scope of this paper, but note that it is a quasicategorical analogue of the correspondence between the two views of ordinary lalis described in Proposition 19.

Proposition 30. *Let $f\colon A \to B$ be a map of simplicial sets. If f is a simplicial lali, then it is a quasicategorical lali; if it is a quasicategorical lali and an inner Kan fibration, then it is a simplicial lali.*

Since Riehl and Verity are able to use quasicategorical lalis to describe various aspects of the theory of quasicategories, the above proposition allows us to describe these same aspects using simplicial lalis, and, hence, using the theory of enriched AWFS.

Examples 31. We consider enriched cofibrant generation in the case where $\mathcal{V} =$ **SSet** and (L, R) is the AWFS for simplicial lalis.

(i) *Limits.* Let A be a quasicategory, and let $A_\perp$ denote the quasicategory $\Delta[0] \oplus A$ obtained by adjoining an initial vertex to A. Riehl and Verity show in [52, Corollary 5.2.19] that a quasicategory X admits all limits of shape A—in the sense of [32, Definition 4.5]—just when the simplicial map $X^{A_\perp} \to X^A$ bears quasicategorical lali structure. It follows that, among the quasicategories, those equipped with a choice of all limits of shape A may be realised as the algebraically fibrant objects of an AWFS on **SSet**; namely, the enriched AWFS cofibrantly generated by the single map $A \to A_\perp$. By enriched cofibrant generation with respect to the set of maps $J = \{A \to A_\perp : A \text{ a finite quasicategory}\}$ we may capture quasicategories with all finite limits as algebraically fibrant objects.

(ii) *Grothendieck fibrations.* Riehl and Verity are in the process of analysing the quasicategorical Grothendieck fibration—the “Cartesian fibrations” of [40, §2.4]—and have shown [51] that an isofibration $g \colon C \to D$ of quasicategories bears such a structure just when the comparison functor $C^{\mathbf{2}} \to D \downarrow g$ bears quasicategorical lali structure. Here, $C^{\mathbf{2}}$ denotes the exponential $C^{\Delta[1]}$, while $D \downarrow g$ is the pullback of $g \colon C \to D$ along $D^{\delta_0} \colon D^{\Delta[1]} \to D$. Thus, among the isofibrations of quasicategories, the Grothendieck fibrations can be realised as the algebras of an AWFS on **SSet**—namely, that obtained by enriched cofibrant generation with respect to the single map $\delta_0 \colon \Delta[0] \to \Delta[1]$.

(iii) *Right Kan extensions.* By [52, Example 5.0.4 and Proposition 5.1.19], a morphism of quasicategories $f \colon C \to D$ admits a right adjoint if and only if the projection $f \downarrow D \to D$ admits simplicial lali structure. So suppose that J is a set of morphisms between quasicategories; for each $f \colon A \to B$ in J, define $\bar{B}$ to be the the pushout of f along $\delta_1 \times A \colon A \to \Delta[1] \times A$, and define $\bar{f} \colon A \to \bar{B}$ to be the composite of $\delta_0 \times A \colon A \to \Delta[1] \times A$ with the pushout injection $\Delta[1] \times A \to \bar{B}$. Then a quasicategory X is an algebraically fibrant object for the enriched AWFS cofibrantly generated by $\{\bar{f} : f \in J\}$ just when each functor $X^f \colon X^B \to X^A$ has a right adjoint; that is, just when each functor $A \to X$ admits a right Kan extension along f.

Of course, by considering the AWFS for simplicial *ralis* in place of simplicial lalis, we may capture notions such as colimits, opfibrations and left Kan extensions.

References

[1] Adámek, J., Herrlich, H., and Strecker, G. E. *Abstract and concrete categories.* Pure and Applied Mathematics. John Wiley and Sons, 1990.

[2] Athorne, T. The coalgebraic structure of cell complexes. *Theory and Applications of Categories 26* (2012), 304–330.

[3] Athorne, T. *Coalgebraic cell complexes.* PhD thesis, University of Sheffield, 2013.

[4] Awodey, S., and Warren, M. Homotopy theoretic models of identity types. *Mathematical Proceedings of the Cambridge Philosophical Society 146*, 1 (2009), 45–55.

[5] Barthel, T., May, J., and Riehl, E. Six model structures for dg-modules over dgas: Model category theory in homological action. *New York Journal of Mathematics 20* (2014), 1077–1160.

[6] Barthel, T., and Riehl, E. On the construction of functorial factorizations for model categories. *Algebraic & Geometric Topology 13*, 2 (2013), 1089–1124.

[7] Batanin, M., Cisinski, D.-C., and Weber, M. Multitensor lifting and strictly unital higher category theory. *Theory and Applications of Categories 28* (2013), 804–856.

[8] Bayeh, M., Hess, K., Karpova, V., Kedziorek, M., Riehl, E., and Shipley, B. Left-induced model structures and diagram categories. Preprint, available as arXiv:1401.3651, 2014.

[9] Beck, J. Distributive laws. In *Seminar on Triples and Categorical Homology Theory (Zürich, 1966/67)*, vol. 80 of *Lecture Notes in Mathematics*. Springer, 1969, pp. 119–140.

[10] Bird, G. *Limits in 2-categories of locally-presented categories*. PhD thesis, University of Sydney, 1984.

[11] Blackwell, R., Kelly, G. M., and Power, A. J. Two-dimensional monad theory. *Journal of Pure and Applied Algebra 59*, 1 (1989), 1–41.

[12] Blumberg, A. J., and Riehl, E. Homotopical resolutions associated to deformable adjunctions. *Algebraic & Geometric Topology 14*, 5 (2014), 3021–3048.

[13] Bourke, J., and Garner, R. Algebraic weak factorisation systems II: weak maps. Preprint, available as arXiv:1412.6560, 2014.

[14] Bousfield, A. Constructions of factorization systems in categories. *Journal of Pure and Applied Algebra 9*, 2-3 (1977), 207–220.

[15] Ching, M., and Riehl, E. Coalgebraic models for combinatorial model categories. *Homology, Homotopy and Applications 16*, 2 (2014), 171–184.

[16] Crans, S. Quillen closed model structures for sheaves. *Journal of Pure and Applied Algebra 101* (1995), 35–57.

[17] Dyckhoff, R., and Tholen, W. Exponentiable morphisms, partial products and pullback complements. *Journal of Pure and Applied Algebra 49*, 1-2 (1987), 103–116.

[18] Freyd, P. J., and Kelly, G. M. Categories of continuous functors I. *Journal of Pure and Applied Algebra 2*, 3 (1972), 169–191.

[19] Gabriel, P., and Ulmer, F. *Lokal präsentierbare Kategorien*, vol. 221 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971.

[20] Gambino, N., and Garner, R. The identity type weak factorisation system. *Theoretical Computer Science 409* (2008), 94–109.

[21] Garner, R. A homotopy-theoretic universal property of Leinster's operad for weak ω-categories. *Mathematical Proceedings of the Cambridge Philosophical Society 147*, 3 (2009), 615–628.

[22] Garner, R. Understanding the small object argument. *Applied Categorical Structures 17*, 3 (2009), 247–285.

[23] Garner, R. Homomorphisms of higher categories. *Advances in Mathematics 224*, 6 (2010), 2269–2311.

[24] Gordon, R., Power, J., and Street, R. Coherence for tricategories. *Memoirs of the American Mathematical Society 117* (1995).

[25] Grandis, M., and Tholen, W. Natural weak factorization systems. *Archivum Mathematicum 42*, 4 (2006), 397–408.

[26] Gray, J. W. Fibred and cofibred categories. In *Conference on Categorical Algebra (La Jolla, 1965)*. Springer, 1966, pp. 21–83.

[27] Hébert, M. Weak reflections and weak factorization systems. *Applied Categorical Structures 19*, 1 (2011), 9–38.

[28] Hirschhorn, P. S. *Model categories and their localizations*, vol. 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2003.

[29] Im, G. B., and Kelly, G. M. On classes of morphisms closed under limits. *Journal of the Korean Mathematical Society 23*, 1 (1986), 1–18.

[30] Johnstone, P. T. Adjoint lifting theorems for categories of algebras. *Bulletin of the London Mathematical Society 7*, 3 (1975), 294–297.

[31] Johnstone, P. T. *Topos theory*, vol. 10 of *London Mathematical Society Monographs*. Academic Press, 1977.

[32] Joyal, A. Quasi-categories and Kan complexes. *Journal of Pure and Applied Algebra 175*, 1-3 (2002), 207–222.

[33] Joyal, A. The theory of quasi-categories and its applications. Quadern 45-2, CRM Barcelona, 2008.

[34] Joyal, A., and Street, R. Pullbacks equivalent to pseudopullbacks. *Cahiers de Topologie et Geométrie Différentielle Catégoriques 34*, 2 (1993), 153–156.

[35] Kelly, G. M. Doctrinal adjunction. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 257–280.

[36] Kelly, G. M. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. *Bulletin of the Australian Mathematical Society 22*, 1 (1980), 1–83.

[37] Kelly, G. M. *Basic concepts of enriched category theory*, vol. 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982. Republished as: *Reprints in Theory and Applications of Categories 10* (2005).

[38] Kelly, G. M., and Street, R. Review of the elements of 2-categories. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 75–103.

[39] Kock, A. Algebras for the partial map classifier monad. In *Category theory (Como, 1990)*, vol. 1488 of *Lecture Notes in Mathematics*. Springer, 1991, pp. 262–278.

[40] Lurie, J. *Higher Topos Theory*. Princeton University Press, 2009.

[41] Makkai, M., and Paré, R. *Accessible categories: the foundations of categorical model theory*, vol. 104 of *Contemporary Mathematics*. American Mathematical Society, 1989.

[42] Quillen, D. G. *Homotopical algebra*, vol. 43 of *Lecture Notes in Mathematics*. Springer, 1967.

[43] Riehl, E. Algebraic model structures. *New York Journal of Mathematics 17* (2011), 173–231.

[44] Riehl, E. Monoidal algebraic model structures. *Journal of Pure and Applied Algebra 217*, 6 (2013), 1069–1104.

[45] Riehl, E. *Categorical homotopy theory*, vol. 24 of *New Mathematical Monographs*. Cambridge University Press, 2014.

[46] Rosolini, G. *Continuity and effectiveness in topoi*. PhD thesis, University of Oxford, 1986.

[47] Stanculescu, A. Stacks and sheaves of categories as fibrant objects I. *Theory and Applications of Categories 29*, 24 (2014), 654–695.

[48] Street, R. The formal theory of monads. *Journal of Pure and Applied Algebra 2*, 2 (1972), 149–168.

[49] Street, R. Fibrations in bicategories. *Cahiers de Topologie et Geométrie Différentielle Catégoriques 21*, 2 (1980), 111–160.

[50] Street, R., and Verity, D. The comprehensive factorization and torsors. *Theory and Applications of Categories 23* (2010), 42–75.

[51] Verity, D. Quasi-categories, profunctors and equipments. Talk to Australian Category Seminar, August 2014.

[52] Verity, D., and Riehl, E. The 2-category theory of quasicategories. Preprint, available as http://arxiv.org/abs/1306.5144, 2013.

[53] Wolff, H. Free monads and the orthogonal subcategory problem. *Journal of Pure and Applied Algebra 13*, 3 (1978), 233–242.

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 60000, Czech Republic
E-mail address: bourkej@math.muni.cz

Department of Mathematics, Macquarie University, NSW 2109, Australia
E-mail address: richard.garner@mq.edu.au

Chapter 4

Skew structures in 2-category theory and homotopy theory

This chapter contains the article *Skew structures in 2-category theory and homotopy theory* by *John Bourke*, published in the *Journal of Homotopy and Related Structures 12* (2017) 31–81.

SKEW STRUCTURES IN 2-CATEGORY THEORY AND HOMOTOPY THEORY

JOHN BOURKE

Abstract. We study Quillen model categories equipped with a monoidal skew closed structure that descends to a genuine monoidal closed structure on the homotopy category. Our examples are 2-categorical and include permutative categories and bicategories. Using the skew framework, we adapt Eilenberg and Kelly's theorem relating monoidal and closed structure to the homotopical setting. This is applied to the construction of monoidal bicategories arising from the pseudo-commutative 2-monads of Hyland and Power.

1. Introduction

The notion of a monoidal closed category captures the behaviour of the tensor product and internal hom on classical categories such as those of sets and vector spaces. Some of the basic facts about monoidal closed categories have an intuitive meaning. For instance, the isomorphism

$$\mathcal{C}(A,B) \cong \mathcal{C}(I,[A,B]) \tag{1.1}$$

says that elements of the internal hom $[A,B]$ are the same thing as morphisms $A \to B$.

Recently some new variants have come to light. Firstly, the *skew* monoidal categories of Szlachányi [36] in which the structure maps such as $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$ have a specified orientation and are not necessarily invertible. Shortly afterwards the dual notion of a skew closed category was introduced by Street [35]. Here one has a canonical map

$$\mathcal{C}(A,B) \to \mathcal{C}(I,[A,B]) \tag{1.2}$$

but this need not be invertible. Intuitively, we might view this relaxation as saying that $[A,B]$ should contain the morphisms $A \to B$ as elements, but possibly something else too.

In the present paper a connection is drawn between skew structures and homotopy theory. We study examples of Quillen model categories $\mathcal{C}$ in which the correct internal homs $[A,B]$ have more general *weak maps* $A \rightsquigarrow B$ as elements. By the above reasoning these examples are necessarily skew. These skew closed categories form part of enveloping monoidal skew closed structures that descend to the homotopy category $Ho(\mathcal{C})$ where, in fact, they yield *genuine* monoidal closed structures. The study of skew structures on a category that induce genuine structures on the homotopy category is our main theme.

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Our examples are 2-categorical in nature – most involve tweaking better known weak 2-categorical structures to yield not strict, but skew, structures. For example, we describe a monoidal skew closed structure on the 2-category of permutative categories – symmetric strict monoidal categories – and strict maps. This contains, on restricting to the cofibrant objects, a copy of the well known monoidal bicategory of permutative categories and strong maps. More generally, we describe a skew structure for each pseudo-commutative 2-monad T on **Cat** in the sense of [13]. Other examples concern 2-categories and bicategories.

The theory developed in the present paper has a future goal, concerning Gray-categories, in mind. It was shown in [4] that there exists no homotopically well behaved monoidal biclosed structure on the category of Gray-categories. The plan is, in a future paper, to use the results developed here to understand the correct enriching structure on the category of Gray-categories.

Let us now give an overview of the paper. Section 2 is mainly background on skew monoidal, skew closed and monoidal skew closed categories. We recall Street's theorem describing the perfect correspondence between skew monoidal structures $(\mathcal{C}, \otimes, I)$ and skew closed structures $(\mathcal{C}, [-,-], I)$ in the presence of adjointness isomorphisms $\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, [B, C])$. In Theorem 2.6 we reformulate Eilenberg and Kelly's theorem [8], relating monoidal and closed structure, in the skew language. Finally, we introduce symmetric skew closed categories.

It turns out that all the examples of skew closed structures that we meet in the present paper can be seen as arising from certain multicategories in a canonical way. In Section 3 we describe the passage from such multicategories to skew closed categories.

Using the multicategory approach where convenient, Section 4 gives concrete examples of some of the skew closed structures that we are interested in. We describe the examples of categories with limits, permutative categories, 2-categories and bicategories.

Section 5 concerns the interaction between skew structures and Quillen model structures that lies at the heart of the paper. We begin by describing how a skew monoidal structure $(\mathcal{C}, \otimes, I)$ can be *left derived* to the homotopy category. This is the skew version of Hovey's construction [12]. We call $(\mathcal{C}, \otimes, I)$ *homotopy monoidal* if the left derived structure $(Ho(\mathcal{C}), \otimes_l, I)$ is genuinely monoidal. This is complemented by an analysis of how skew closed structure can be *right derived* to the homotopy category, and we obtain a corresponding notion of *homotopy closed* category. Combining these cases Theorem 5.11 describes how monoidal skew closed structure can be *derived* to the homotopy category. This is used to prove Theorem 5.12, a homotopical analogue of Eilenberg and Kelly's theorem, which allows us to recognise homotopy monoidal structure in terms of homotopy closed structure.

Section 6 returns to the examples of categories with limits and permutative categories in the more general setting of pseudo-commutative 2-monads T on **Cat**. We make minor modifications to Hyland and Power's construction [13] of a pseudo-closed structure on T-Alg to produce a skew closed structure on the 2-category T-Alg$_s$ of algebras and strict morphisms. For accessible T this forms part of an enveloping monoidal skew closed structure which, using Theorem 5.12,

we show to be homotopy monoidal. Using this, we give a complete construction of the monoidal bicategory structure on T-Alg of Hyland and Power – thus solving a problem of [13].
Section 7 consists of an in-depth analysis of the skew structure on the category of bicategories and strict homomorphisms. Though not particularly interesting in its own right, we regard this example as a preliminary to future work in higher dimensions.

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2. Skew monoidal and skew closed categories

2.1. **Skew monoidal categories.** Skew monoidal categories were introduced by Szlachányi [36] in the study of bialgebroids over rings. There are left and right versions (depending upon the orientation of the associativity and unit maps) and it is the left handed case that is of interest to us.

Definition 2.1. A (left) skew monoidal category $(\mathcal{C}, \otimes, I, \alpha, l, r)$ is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $I \in \mathcal{C}$, and natural families $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $l_A : I \otimes A \to A$ and $r_A : A \to A \otimes I$ satisfying five axioms [36].

There is no need for us to reproduce these five axioms here as we will not use them. We remark only that they are neatly labelled by the five words

$$\begin{gathered} abcd \\ iab\ aib\ abi \\ ii \end{gathered}$$

of which the first refers to MacLane's pentagon axiom.
Henceforth the term skew monoidal is taken to mean left skew monoidal. A monoidal category is precisely a skew monoidal category in which the constraints α, l and r are invertible.

2.2. **Skew closed categories.** In the modern treatment of monoidal closed categories as a basis for enrichment [16] it is the monoidal structure that is typically treated as primitive. Nonetheless, the first major treatment [8] emphasised the closed structure, presumably because internal homs are often more easily described than the corresponding tensor products. In the examples of interest to us (see Section 4) this is certainly the case. These examples will not be closed in the sense of *ibid.* but only skew closed.

Definition 2.2 (Street [35])**.** A (left) skew closed category $(\mathcal{C}, [-,-], I, L, i, j)$ consists of a category $\mathcal{C}$ equipped with a bifunctor $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ and unit object I together with

(1) components $L = L^A_{B,C} : [B, C] \to [[A, B], [A, C]]$ natural in B, C and extranatural in A,
(2) a natural transformation $i = i_A : [I, A] \to A$,
(3) components $j = j_A : I \to [A, A]$ extranatural in A,

satisfying the following five axioms.

(C1)

$$\begin{tikzcd}
 & {[[A,C],[A,D]]} \arrow[dr, "L"] & \\
{[C,D]} \arrow[ur, "L"] \arrow[d, "L"'] & & {[[[A,B],[A,C]],[[A,B],[A,D]]]} \arrow[d, "{[L,1]}"] \\
{[[B,C],[B,D]]} \arrow[rr, "{[1,L]}"'] & & {[[B,C],[[A,B],[A,D]]]}
\end{tikzcd}$$

.

$$(C2) \quad \begin{tikzcd}
{[[A,A],[A,C]]} \arrow[r, "{[j,1]}"] & {[I,[A,C]]} \arrow[d, "i"] \\
{[A,C]} \arrow[u, "L"] \arrow[r, "1"] & {[A,C]}
\end{tikzcd} \qquad (C3) \quad \begin{tikzcd}
{[B,B]} \arrow[rr, "L"] & & {[[A,B],[A,B]]} \\
 & I \arrow[ul, "j"] \arrow[ur, "j"'] &
\end{tikzcd}$$

$$(C4) \quad \begin{tikzcd}
{[B,C]} \arrow[rr, "L"] \arrow[dr, "{[i,1]}"'] & & {[[I,B],[I,C]]} \arrow[dl, "{[1,i]}"] \\
 & {[[I,B],C]} &
\end{tikzcd} \qquad (C5) \quad \begin{tikzcd}
I \arrow[rr, "j"] \arrow[dr, "1"'] & & {[I,I]} \arrow[dl, "i"] \\
 & I &
\end{tikzcd}$$

$(\mathcal{C},[-,-],I)$ is said to be *left normal* when the composite function

$$\mathcal{C}(A,B) \xrightarrow{[A,-]} \mathcal{C}([A,A],[A,B]) \xrightarrow{\mathcal{C}(j,1)} \mathcal{C}(I,[A,B]) \tag{2.1}$$

is invertible, and *right normal* if $i:[I,A]\to A$ is invertible. A *closed category* is, by definition, a skew closed category which is both left and right normal.[1]

Variants 2.3. We will regularly mention a couple of variants on the above definition and we note them here.

1. We will sometimes consider skew closed 2-categories: the **Cat**-enriched version of the above concept. The difference is that $\mathcal{C}$ is now a *2-category*, $[-,-]$ a *2-functor* and each of the three transformations *2-natural* in each variable.
2. We call a structure $(\mathcal{C},[-,-],L)$ satisfying $C1$ but without unit a *semi-closed* category.

2.3. **The correspondence between skew monoidal and skew closed categories.** A monoidal category $(\mathcal{C},\otimes,I)$ in which each functor $-\otimes A:\mathcal{C}\to\mathcal{C}$ has a right adjoint $[A,-]$ naturally gives rise to the structure $(\mathcal{C},[-,-],I)$ of a closed category. Counterexamples to the converse statement are described in Section 3 of [6]: no closed category axiom ensures the associativity of the corresponding tensor

[1] The original definition of closed category [8] involved an *underlying functor* to **Set**. We are using the modified definition of [33] (see also [26]) which eliminates the reference to **Set**.

product. An appealing feature of the skew setting is that there is a perfect correspondence between skew monoidal and skew closed structure.

Theorem 2.4 (Street [35]). *Let $\mathcal{C}$ be a category equipped with an object I and a pair of bifunctors $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ related by isomorphisms $\varphi : \mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, [B, C])$ natural in each variable. There is a bijection between extensions of $(\mathcal{C}, \otimes, I)$ to a skew monoidal structure and of $(\mathcal{C}, [-,-], I)$ to a skew closed structure.*

Our interest is primarily in the passage from the closed to the monoidal side and, breaking the symmetry slightly, we describe it now: for the full symmetric treatment see [35].

- $l : I \otimes A \to A$ is the unique map such that the diagram

(2.2)
$$\begin{tikzcd} \mathcal{C}(A,B) \arrow[r, "\mathcal{C}(l,1)"] \arrow[dr, "v"'] & \mathcal{C}(I \otimes A, B) \arrow[d, "\varphi"] \\ & \mathcal{C}(I, [A,B]) \end{tikzcd}$$

  commutes for all B. Here $v = \mathcal{C}(j, 1) \circ [A, -] : \mathcal{C}(A, B) \to \mathcal{C}(I, [A, B])$ is the morphism defining left normality. *In particular l is invertible for each A just when v is.*
- $r : A \to A \otimes I$ is the unique morphism such that the diagram

(2.3)
$$\begin{tikzcd} \mathcal{C}(A \otimes I, B) \arrow[r, "\mathcal{C}(r,1)"] \arrow[d, "\varphi"'] & \mathcal{C}(A,B) \\ \mathcal{C}(A, [I,B]) \arrow[ur, "\mathcal{C}(1,i)"'] & \end{tikzcd}$$

  commutes for all B. *In particular r is invertible for each A just when i is.*
- Transposing the identity through the isomorphism $\varphi : \mathcal{C}(A \otimes B, A \otimes B) \cong \mathcal{C}(A, [B, A \otimes B])$ yields a morphism $u : A \to [B, A \otimes B]$ natural in each variable. Write $t : [A \otimes B, C] \to [A, [B, C]]$ for the composite

(2.4)
$$[A \otimes B, C] \xrightarrow{L} [[B, A \otimes B], [B, C]] \xrightarrow{[u,1]} [A, [B, C]]$$

  which, we note, is natural in each variable. The constraint $\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is the unique morphism rendering commutative the diagram

(2.5)
$$\begin{tikzcd} \mathcal{C}(A \otimes (B \otimes C), D) \arrow[r, "\mathcal{C}(\alpha,1)"] \arrow[dd, "\varphi"'] & \mathcal{C}((A \otimes B) \otimes C, D) \arrow[d, "\varphi"] \\ & \mathcal{C}(A \otimes B, [C,D]) \arrow[d, "\varphi"] \\ \mathcal{C}(A, [B \otimes C, D]) \arrow[r, "\mathcal{C}(1,t)"'] & \mathcal{C}(A, [B, [C,D]]) \end{tikzcd}$$

for all D. *In particular α is invertible just when t is.*

Definition 2.5. A monoidal skew closed category consists of a skew monoidal category $(\mathcal{C}, \otimes, I, \alpha, l, r)$ and skew closed category $(\mathcal{C}, [-,-], I, L, i, j)$, together with natural isomorphisms $\varphi : \mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, [B, C])$ all related by the above equations.

Of course in the presence of the isomorphisms either bifunctor determines the other. Accordingly a monoidal skew closed category is determined by either the skew monoidal or closed structure together with the isomorphisms φ.
We remark that monoidal skew closed structures on the category of left R-modules over a ring R that have R as unit correspond to left bialgebroids over R. This was the reason for the introduction of skew monoidal categories in [36].
The following result – immediate from the above – is, minus the skew monoidal terminology, contained within Chapter 2 and in particular Theorem 5.3 of [8].

Theorem 2.6 (Eilenberg-Kelly)**.** *Let $(\mathcal{C}, \otimes, [-,-], I)$ be a monoidal skew closed category. Then $(\mathcal{C}, \otimes, I)$ is monoidal if and only if $(\mathcal{C}, [-,-], I)$ is closed and the transformation $t : [A \otimes B, C] \to [A, [B, C]]$ is an isomorphism for all A, B and C.*

Eilenberg and Kelly's theorem can be used to recognise monoidal structure in terms of closed structure. However it can be difficult to determine whether $t : [A \otimes B, C] \to [A, [B, C]]$ is invertible. This difficulty disappears in the presence of a suitable symmetry.

2.4. **Symmetry.** A symmetry on a skew closed category begins with a natural isomorphism $s : [A, [B, C]] \cong [B, [A, C]]$. If $\mathcal{C}$ is left normal the vertical maps

$$\begin{tikzcd}
\mathcal{C}(A, [B, C]) \arrow[r, "s_0"] \arrow[d, "v"'] & \mathcal{C}(B, [A, C]) \arrow[d, "v"] \\
\mathcal{C}(I, [A, [B, C]]) \arrow[r, "\mathcal{C}(I, s)"] & \mathcal{C}(I, [B, [A, C]])
\end{tikzcd}$$

are isomorphisms, so that we obtain an isomorphism s_0 by conjugating $\mathcal{C}(I, s)$. If $\mathcal{C}$ underlies a monoidal skew closed category this in turn gives rise to a natural isomorphism

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(B \otimes A, C)$$

and so, by Yoneda, a natural isomorphism

$$c : B \otimes A \cong A \otimes B \quad . \tag{2.6}$$

Our leading examples of skew closed categories do admit symmetries, but are *not* left normal: accordingly, the symmetries are visible on the closed side but not on the monoidal side. However they often reappear on the monoidal side upon passing to the *homotopy category* – see Theorem 5.13.

Definition 2.7. A symmetric skew closed category consists of a skew closed category $(\mathcal{C}, [-,-], I)$ together with a natural isomorphism $s : [A, [B, C]] \cong$

$[B,[A,C]]$ satisfying the following four equations.

(S1)
$$\begin{tikzcd}
{[A,[B,C]]} \arrow[rr, "1"] \arrow[dr, "s"'] & & {[A,[B,C]]} \\
& {[B,[A,C]]} \arrow[ur, "s"'] &
\end{tikzcd}$$

(S2)
$$\begin{tikzcd}
{[A,[B,[C,D]]]} \arrow[r, "s"] \arrow[d, "{[1,s]}"'] & {[B,[A,[C,D]]]} \arrow[r, "{[1,s]}"] & {[B,[C,[A,D]]]} \arrow[d, "s"] \\
{[A,[C,[B,D]]]} \arrow[r, "s"] & {[C,[A,[B,D]]]} \arrow[r, "{[1,s]}"] & {[C,[B,[A,D]]]}
\end{tikzcd}$$

(S3)
$$\begin{tikzcd}
{[A,[B,C]]} \arrow[r, "L"] \arrow[d, "s"'] & {[[D,A],[D,[B,C]]]} \arrow[r, "{[1,s]}"] & {[[D,A],[B,[D,C]]]} \arrow[d, "s"] \\
{[B,[A,C]]} \arrow[rr, "{[1,L]}"] & & {[B,[[D,A],[D,C]]]}
\end{tikzcd}$$

(S4)
$$\begin{tikzcd}
{[A,B]} \arrow[r, "L"] \arrow[rrrr, bend right, "1"'] & {[[A,A],[A,B]]} \arrow[r, "{[j,1]}"] & {[I,[A,B]]} \arrow[r, "s"] & {[A,[I,B]]} \arrow[r, "{[1,i]}"] & {[A,B]}
\end{tikzcd}$$

$\mathcal{C}$ is said to be symmetric closed if its underlying skew closed category is closed.

Variants 2.8. As in Variants 2.3 there are evident notions of symmetric skew closed 2-categories and symmetric semi-closed categories.

Remark 2.9. The notion of symmetric closed category described above coincides with that of [6], though this may not be immediately apparent. Their invertible unit map $i : X \to [I,X]$ points in the opposite direction to ours. Reversing it, their (CC4) is clearly equivalent to our (S4). Their remaining axioms are a proper subset of those above, with (C1), (C3), (C4) and (C5) omitted. But as they point out in Proposition 1.3 any symmetric closed category in their sense is a closed category and hence satisfies all four of these.

I first encountered a result close to the following one as Proposition 2.3 of [6], which shows that a symmetric closed category $\mathcal{C}$ gives rise to a symmetric promonoidal one by setting $P(A,B,C) = \mathcal{C}(A,[B,C])$. This easily implies that a symmetric closed category gives rise to a symmetric monoidal one on taking adjoints. In a discussion about that result, Ross Street pointed out that a skew monoidal category with an invertible natural isomorphism $A \otimes B \cong B \otimes A$ satisfying the braid equation $B5$ of [14] is necessarily associative. The first part of the following result essentially reformulates Street's associativity argument in terms of the closed structure. Diagram 4.8 of Chapter IV of De Schipper's book [7] – (2.7) below – proved helpful in making that reformulation.

Theorem 2.10 (Day-LaPlaza, Street). *Let $(\mathcal{C}, \otimes, [-,-], I)$ be monoidal skew closed.*

(1) The transformation $t : [A \otimes B, C] \to [A, [B, C]]$ is invertible if $(\mathcal{C}, [-,-], I)$ is left normal and admits a natural isomorphism $s : [A, [B, C]] \cong [B, [A, C]]$ satisfying S3. In particular, if $(\mathcal{C}, [-,-], I)$ is actually closed and admits such a symmetry then $(\mathcal{C}, \otimes, I)$ is monoidal.

(2) If $(\mathcal{C}, [-,-], I, s)$ is symmetric closed then $(\mathcal{C}, \otimes, I, c)$ is symmetric monoidal.

Proof. For (1) we first prove that

$$\begin{tikzcd}
{[A \otimes B, [C, D]]} \arrow[rr, "s"] \arrow[d, "t"'] & & {[C, [A \otimes B, D]]} \arrow[d, "{[1,t]}"] \\
{[A, [B, [C, D]]]} \arrow[r, "{[1,s]}"'] & {[A, [C, [B, D]]]} \arrow[r, "s"'] & {[C, [A, [B, D]]]}
\end{tikzcd} \tag{2.7}$$

commutes. From (2.4) we have $t = [u, 1] \circ L$. So the upper path equals

$$[[1, [u, 1]] \circ [1, L] \circ s = [1, [u, 1]] \circ s \circ [1, s] \circ L = s \circ [1, s] \circ [u, 1] \circ L = s \circ [1, s] \circ t$$

by S3, naturality of s twice applied, and the definition of t.
By assumption each component $v : \mathcal{C}(A, B) \to \mathcal{C}(I, [A, B])$ is invertible. Accordingly a morphism $f : [A, B] \to [C, D]$ gives rise to a further morphism $f_0 : \mathcal{C}(A, B) \to \mathcal{C}(C, D)$ by conjugating $\mathcal{C}(I, f)$. At $[A, g] : [A, B] \to [A, C]$ we obtain $[A, g]_0 = \mathcal{C}(A, g)$. At $L : [A, B] \to [[C, A], [C, B]]$ an application of $C3$ establishes that $L_0 = [C, -] : \mathcal{C}(A, B) \to \mathcal{C}([C, A], [C, B])$. Now $(-)_0$, being defined by conjugating through natural isomorphisms, preserves composition. Combining the two last cases we find that $t_0 = [u, 1]_0 \circ L_0 = \mathcal{C}(u, 1) \circ [B, -] = \varphi$, the adjointness isomorphism. Applying $(-)_0$ to the above diagram, componentwise, then gives the commutative diagram below.

$$\begin{tikzcd}
\mathcal{C}(A \otimes B, [C, D]) \arrow[rr, "s_0"] \arrow[d, "\varphi"'] & & \mathcal{C}(C, [A \otimes B, D]) \arrow[d, "{\mathcal{C}(C,t)}"] \\
\mathcal{C}(A, [B, [C, D]]) \arrow[r, "{\mathcal{C}(A,s)}"'] & \mathcal{C}(A, [C, [B, D]]) \arrow[r, "s_0"'] & \mathcal{C}(C, [A, [B, D]])
\end{tikzcd}$$

Since the left vertical path and both horizontal paths are isomorphisms, so is $\mathcal{C}(C, t)$ for each C. Therefore t is itself an isomorphism. The remainder of (1) now follows from Theorem 2.6.
As mentioned, Part 2 follows from Proposition 2.3 of [6]. We note an alternative elementary argument. Having established the commutativity of (2.7) and that t is an isomorphism, we are essentially in the presence of what De Schipper calls a *monoidal symmetric closed category*.[2] Theorem 6.2 of [7] establishes that a monoidal symmetric closed category determines a symmetric monoidal one, as required. □

[2] This is not exactly the case as De Schipper, following Eilenberg-Kelly, includes a basic functor $V : \mathcal{C} \to \mathbf{Set}$ in his definition of symmetric closed category. However this basic functor plays no role in the proof of the cited result.

3. From multicategories to skew closed categories

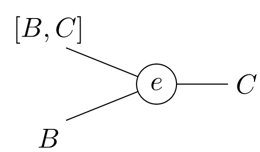
Our examples of skew closed categories in Section 4 can be seen as arising from closed multicategories equipped with further structure. In the present section we describe how to pass from such multicategories to skew closed categories.
Multicategories were introduced in [25] and have objects $A, B, C\ldots$ together with *multimaps* $(A_1, \ldots, A_n) \to B$ for each $n \in \mathbb{N}$. These multimaps can be composed and satisfy natural associativity and unit laws. We use boldface $\mathbf{C}$ for a multicategory and $\mathcal{C}$ for its underlying category of unary maps. A *symmetric* multicategory $\mathbf{C}$ comes equipped with actions of the symmetric group S_n on the sets $\mathbf{C}(A_1, \ldots A_n; B)$ of n-ary multimaps. These actions must be compatible with multimap composition. For a readable reference on the basics of multicategories we refer to [27].

3.1. **Closed multicategories.** A multicategory $\mathbf{C}$ is said to be *closed* if for all $B, C \in \mathbf{C}$ there exists an object $[B, C]$ and evaluation multimap $e : ([B, C], B) \to C$ with the universal property that the induced function

$$\mathbf{C}(A_1, \ldots, A_n; [B, C]) \to \mathbf{C}(A_1, \ldots, A_n, B; C) \tag{3.1}$$

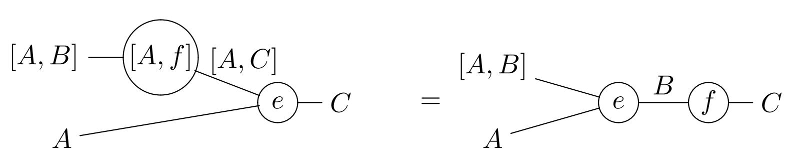
is a bijection for all $(A_1, \ldots, A_n)$ and $n \in \mathbb{N}$. We can depict the multimap $e : ([B, C], B) \to C$ as below.

(3.2)



3.1.1. *Semiclosed structure.* Using the above universal property one obtains a bifunctor $[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$. Given $f : B \to C$ the map $[A, f] : [A, B] \to [A, C]$ is the unique one such that the two multimaps

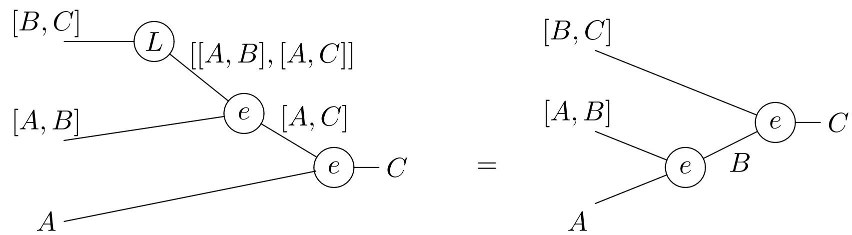
(3.3)



coincide, whilst $[f, A]$ is defined in a similar manner.
The natural bijections

$$\mathbf{C}([B, C], [A, B], A; C) \cong \mathbf{C}([B, C], [A, B]; [A, C]) \cong \mathbf{C}([B, C], [[A, B], [A, C]])$$

induce a unique morphism $L : [B, C] \to [[A, B], [A, C]]$ such that the multimaps
(3.4)



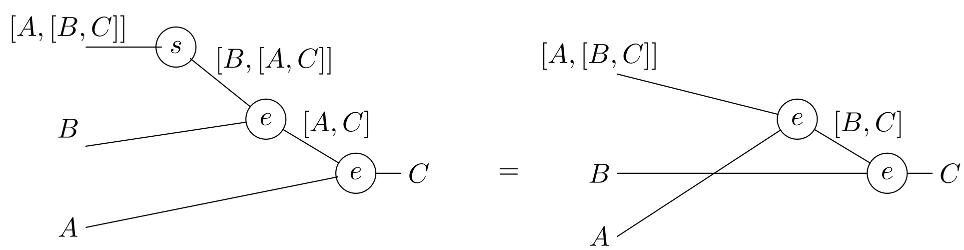
coincide.

3.1.2. *Symmetry.* If $\mathbf{C}$ is a symmetric multicategory the natural bijections

$$\mathbf{C}([A, [B, C]], B, A; C) \cong \mathbf{C}([A, [B, C]], B; [A, C]) \cong \mathbf{C}([A, [B, C]], [B, [A, C]])$$

induce a unique map $s : [A, [B, C]] \to [B, [A, C]]$ such that the multimaps

(3.5)



coincide. The multimap above right depicts the image of $e \circ (e, 1) : ([A, [B, C]], A, B) \to ([B, C], B) \to C$ under the action

$$\mathbf{C}([A, [B, C]], A, B; C) \cong \mathbf{C}([A, [B, C]], B, A; C)$$

of the symmetric group.

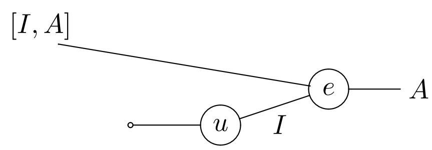
3.1.3. *Nullary map classifiers and units.*

Definition 3.1. A multicategory $\mathbf{C}$ has a nullary map classifier if there exists an object I and multimap $u : (-) \to I$ such that the induced morphism

$$\mathbf{C}(u; A) : \mathbf{C}(I, A) \to \mathbf{C}(-; A)$$

is a bijection for each A.

Equivalently, if the functor $\mathbf{C}(-; ?) : \mathcal{C} \to \mathbf{Set}$ sending an object A to the set of nullary maps $(-) \to A$ is representable. In a closed multicategory a nullary map classifier I enables the construction of morphisms $i : [I, A] \to A$ and $j : I \to [A, A]$. The former is given by



(3.6)

For j observe that the identity $1 : A \to A$ corresponds under the isomorphism $\mathbf{C}(A, A) \cong \mathbf{C}(-; [A, A])$ to a nullary map $\hat{1} : (-) \to [A, A]$. Now $j : I \to [A, A]$ is defined to be the unique map such that

$$j \circ u = \hat{1} \ . \tag{3.7}$$

In [29] the term *unit* for a closed multicategory means something stronger than a nullary map classifier: it consists of a multimap $u : (-) \to I$ for which (3.6) is invertible. By Remark 4.2 of *ibid.* a unit is a nullary map classifier.

3.2. **The results.** The following result is Proposition 4.3 of [29].

Theorem 3.2 (Manzyuk)**.** *If* $\mathbf{C}$ *is a closed multicategory with unit* I *then* $(\mathcal{C}, [-,-], I, L, i, j)$ *is a closed category.*

Since we are interested in constructing mere skew closed categories a nullary map classifier suffices.

Theorem 3.3. *If* $\mathbf{C}$ *is a closed multicategory with a nullary map classifier* I *then* $(\mathcal{C}, [-,-], I, L, i, j)$ *is a skew closed category. Furthermore if* $\mathbf{C}$ *is a symmetric multicategory then* $(\mathcal{C}, [-,-], I, L, i, j, s)$ *is symmetric skew closed.*

Proof. We only outline the proof, which involves routine multicategorical diagram chases best accomplished using string diagrams as in (3.2)–(3.6). (We note that the deductions of C1 and C3 are given in the proof of Proposition 4.3 of [29].) The axioms C1, C2 and C4 each assert the equality of two maps

$$X \rightrightarrows [Y_1, \ldots, [Y_{n-1}, [Y_n, Z]]..]$$

constructed using $[-,-]$, L, i and j. These correspond to the equality of the transposes

$$(X, Y_1, \ldots, Y_n) \rightrightarrows Z$$

obtained by postcomposition with the evaluation multimaps. Since $[-,-]$, L, i and j are defined in terms of their interaction with the evaluation multimaps only their definitions, together with the associativity and unit laws for a multicategory, are required to verify these axioms. C3 and C5 each concern the equality of two maps $I \rightrightarrows A$. Here one shows that the corresponding nullary maps $(-) \rightrightarrows A$ coincide. Again this is straightforward. The axioms S1-S4 are verified in a similar fashion. $\square$

Theorem 3.3, as stated, will not apply to the examples of interest, none of which quite has a nullary map classifier. What we need is a generalisation that deals with combinations of strict and weak maps.

Definition 3.4. Let $\mathbf{C}$ be a multicategory equipped with a subcategory $\mathcal{C}_s \subseteq \mathcal{C}$ of *strict* morphisms containing all of the identities. We say that a multimap $f : (A_1, \ldots, A_n) \to B$ is *strict in* i (or A_i abusing notation) if for all families of multimaps $\{a_j : (-) \to A_j : j \in \{1, \ldots, i-1, i+1, \ldots, n\}\}$ the unary map

$$f \circ (a_1, \ldots a_{i-1}, 1, a_{i+1}, \ldots a_n) : A_i \to B$$

is strict.

Theorem 3.5. *Let* $\mathbf{C}$ *be a closed multicategory equipped with a subcategory* $\mathcal{C}_s \to \mathcal{C}$ *of strict maps containing the identities. Suppose further that*

(1) A multimap $(A_1, \ldots, A_n, B) \to C$ *is strict in* A_i *if and only if its transpose* $(A_1, \ldots, A_n) \to [B, C]$ *is.*
(2) There is a multimap $u : (-) \to I$*, precomposition with which induces a bijection* $\mathbf{C}(u, A) : \mathcal{C}_s(I, A) \to \mathbf{C}(-; A)$ *for each* A*.*

Then $(\mathcal{C}, [-,-], L)$ *is a semi-closed category. Moreover* $[-,-]$ *and* L *restrict to* $\mathcal{C}_s$ *where they form part of a skew closed structure* $(\mathcal{C}_s, [-,-], I, L, i, j)$*.*
Furthermore if $\mathbf{C}$ *is a symmetric multicategory then* $(\mathcal{C}, [-,-], L, s)$ *is symmetric semi-closed and* $(\mathcal{C}_s, [-,-], I, L, i, j, s)$ *is symmetric skew closed.*

Proof. We must show that these assumptions ensure that the bifunctor $[-,-] : \mathcal{C}^{op} \times \mathcal{C}$ restricts to $\mathcal{C}_s$ and that the transformations L, i, j and s have strict components. Beyond this point the proof is identical to that of Theorem 3.3.
A consequence of Definition 3.4 is that multimaps strict in a variable are *closed under composition*: that is, given $f : (A_1, A_2 \ldots A_n) \to B_k$ strict in A_i and $g : (B_1, B_2 \ldots B_m) \to C$ strict in B_k the composite multimap

$$(B_1 \ldots B_{k-1}, A_1 \ldots A_i \ldots A_n, B_{k+1} \ldots B_m) \to C$$

is strict in A_i. We use this fact freely in what follows.
Observe that since $1 : [A, B] \to [A, B]$ is strict its transpose, the evaluation multimap $e : ([A, B], A) \to B$, is strict in $[A, B]$. It follows that if $f : B \to C$ is strict then the composite multimap $f \circ e : ([A, B], A) \to B \to C$ of (3.3) is strict in $[A, B]$. Accordingly its transpose $[A, f] : [A, B] \to [A, C]$ is strict. Likewise $[f, A]$ is strict if f is. Hence $[-,-]$ restricts to $\mathcal{C}_s$.
Since evaluation multimaps are strict in the first variable the composite $e \circ (1, e) : ([B, C], [A, B], A) \to C$ of (3.4) is strict in $[B, C]$. Since transposing this twice yields $L : [B, C] \to [[A, B], [A, C]]$ we conclude that L is strict. The composite $i = e \circ (1, u) : [I, A] \to A$ of (3.6) is strict as $e : ([I, A], I) \to A$ is strict in the first variable. Clearly j is strict.
In a symmetric multicategory the actions of the symmetric group commute with composition. It follows that if $F : (A_1, \ldots A_n) \to B$ is strict in A_i and $\varphi \in Sym(n)$ then $\varphi(F) : (A_{\varphi(1)}, \ldots, A_{\varphi(n)}) \to B$ is strict in $A_{\varphi(i)}$. Therefore the composite $([A, [B, C]], B, A) \to C$ on the right hand side of (3.5) is strict in $[A, [B, C]]$. Since $s : [A, [B, C]] \cong [B, [A, C]]$ is obtained by transposing this twice, it follows that s is strict. □

In the next section we will encounter several examples of **Cat**-enriched multicategories, hence 2-multicategories [13]. A 2-multicategory $\mathbf{C}$ has *categories*

$\mathbf{C}(A_1, \ldots, A_n; B)$ of multilinear maps and transformations between, and an extension of multicategorical composition dealing with these transformations. There is an evident notion of *closed 2-multicategory*, in which the bijection (3.1) is replaced by an isomorphism, and of symmetric 2-multicategory. Theorem 3.5 generalises straightforwardly to 2-multicategories as we now record.

Theorem 3.6. *Let* $\mathbf{C}$ *be a closed 2-multicategory equipped with a locally full sub 2-category* $\mathcal{C}_s \to \mathcal{C}$ *of strict maps containing the identities. Suppose further that*

(1) A multimap $(A_1, \ldots, A_n, B) \to C$ *is strict in* A_i *if and only if its transpose* $(A_1, \ldots, A_n) \to [B, C]$ *is.*
(2) There is a multimap $u : (-) \to I$*, precomposition with which induces an isomorphism* $\mathbf{C}(u, A) : \mathcal{C}_s(I, A) \to \mathbf{C}(-; A)$ *for each* A*.*

Then $(\mathcal{C}, [-,-], L)$ *is a semi-closed 2-category. Moreover* $[-,-]$ *and* L *restrict to* $\mathcal{C}_s$ *where they form part of a skew closed 2-category* $(\mathcal{C}_s, [-,-], I, L, i, j)$*.*
Furthermore if $\mathbf{C}$ *is a symmetric 2-multicategory then* $(\mathcal{C}, [-,-], L, s)$ *is symmetric semi-closed and* $(\mathcal{C}_s, [-,-], I, L, i, j, s)$ *is a symmetric skew closed 2-category.*

Remark 3.7. Theorem 5.1 of [29] shows that the notions of closed multicategory with unit and closed category are, in a precise sense, equivalent. We do not know whether skew closed categories are equivalent to some kind of multicategorical structure.

4. Examples of skew closed structures

The goal of this section is to describe a few concrete examples of the kind of skew closed structures that we are interested in. All can be seen to arise from multicategories although sometimes it will be easier to describe the skew closed structure directly.
In each case we meet a category, or 2-category, $\mathcal{C}$ of weak maps equipped with a subcategory $\mathcal{C}_s$ of strict maps. The subcategory of strict maps is well behaved – locally presentable, for instance – whereas $\mathcal{C}$ is not. The objects of the internal hom $[A, B]$ are the *weak maps* but these only form part of a skew closed structure on the subcategory $\mathcal{C}_s$ of *strict maps.*

4.1. **Categories with structure.** The following examples can be understood as arising from pseudo-commutative 2-monads in the sense of [13] – this more abstract approach is described in Section 6.

4.1.1. *Categories with specified limits.* Let $\mathbf{D}$ be a set of small categories, thought of as diagram types. There is a symmetric 2-multicategory **D-Lim** whose objects $\mathbf{A}$ are categories A equipped with a choice of $\mathbf{D}$-limits. The objects of the category $\mathbf{D\text{-}Lim}(\mathbf{A}_1, \ldots, \mathbf{A}_n; \mathbf{B})$ are functors $F : A_1 \times \ldots \times A_n \to B$ preserving $\mathbf{D}$-limits in each variable, and the morphisms are just natural transformations. For the case $n = 0$ we have $\mathbf{D\text{-}Lim}(-; \mathbf{B}) = B$.
The morphisms in the 2-category of unary maps $\mathbf{D}$-Lim are the $\mathbf{D}$-limit preserving functors, amongst which we have the 2-category $\mathbf{D}$-Lim_s of strict $\mathbf{D}$-limit preserving functors and the inclusion $j : \mathbf{D}\text{-}\mathrm{Lim}_s \to \mathbf{D}\text{-}\mathrm{Lim}$. Accordingly a multimap $\mathbf{F} : (\mathbf{A}_1, \ldots, \mathbf{A}_n) \to \mathbf{B}$ is strict in $\mathbf{A}_i$ just when each functor

$F(a_1, \ldots, a_{i-1}, -, a_{i+1}, \ldots, a_n) : A_i \to B$ preserves **D**-limits strictly.
The functor category $[A, B]$ has a canonical choice of **D**-limits inherited pointwise from $\mathbf{B}$. Since **D**-*limits commute with* **D**-*limits* the full subcategory $j : \mathbf{D}\text{-Lim}(\mathbf{A}, \mathbf{B}) \to [A, B]$ is closed under their formation (although $\mathbf{D}\text{-Lim}_s(\mathbf{A}, \mathbf{B})$ is not!) and we write $[\mathbf{A}, \mathbf{B}]$ for $\mathbf{D}\text{-Lim}(A, B)$ equipped with this choice of **D**-limits. It is routine to verify that the objects $[\mathbf{A}, \mathbf{B}]$ exhibit **D**-Lim as a closed 2-multicategory and moreover that a multimap $\mathbf{F} : (\mathbf{A}_1, \ldots, \mathbf{A}_n, \mathbf{B}) \to \mathbf{C}$ is strict in $\mathbf{A}_i$ just when its transpose $(\mathbf{A}_1, \ldots, \mathbf{A}_n) \to [\mathbf{B}, \mathbf{C}]$ is. With regards units, the key point is that the forgetful 2-functor $U : \mathbf{D}\text{-Lim}_s \to \mathbf{Cat}$ has a left 2-adjoint F. This follows from [3] but see also Section 6.2. Accordingly we have a natural isomorphism $\mathbf{D\text{-}Lim}(-; \mathbf{A}) \cong \mathbf{Cat}(1, A) \cong \mathbf{D}\text{-Lim}_s(F1, \mathbf{A})$. By Theorem 3.6 we obtain the structure of a symmetric semi-closed 2-category $(\mathbf{D}\text{-Lim}, [-,-], L, s)$ restricting to a symmetric skew closed 2-category $(\mathbf{D}\text{-Lim}, [-,-], F1, L, i, j, s)$.
The skew closed structure on $\mathbf{D}\text{-Lim}_s$ fails to extend to **D**-Lim because the unit map $j : F1 \to [\mathbf{A}, \mathbf{A}]$ is only *pseudo-natural* in morphisms of **D**-Lim. (It does, however, extend to a *pseudo-closed* structure on **D**-Lim in the sense of [13]). The skew closed $\mathbf{D}\text{-Lim}_s$ is neither left nor right normal: for example, the canonical functor $\mathbf{D}\text{-Lim}_s(\mathbf{A}, \mathbf{B}) \to \mathbf{D}\text{-Lim}_s(F1, [\mathbf{A}, \mathbf{B}])$ is isomorphic to the inclusion $\mathbf{D}\text{-Lim}_s(\mathbf{A}, \mathbf{B}) \to \mathbf{D}\text{-Lim}(\mathbf{A}, \mathbf{B})$ and this is not in general invertible.

4.1.2. *Permutative categories and so on.* An example amenable to calculation concerns symmetric strict monoidal – or permutative – categories. The symmetric skew closed structure can be seen as arising from a symmetric 2-multicategory, described in [9]. Because the relevant definition of multilinear map is rather long, we treat the skew closed structure directly.
Let Perm_s and Perm denote the 2-categories of permutative categories with the strict symmetric monoidal and strong symmetric monoidal functors between. Using the symmetry of $\mathbf{B}$ the category $\text{Perm}(\mathbf{A}, \mathbf{B})$ inherits a pointwise structure $[\mathbf{A}, \mathbf{B}] \in \text{Perm}$. Namely we set $(F \otimes G)- = F - \otimes G-$. The structure isomorphism $(F \otimes G)(a \otimes b) \cong (F \otimes G)a \otimes (F \otimes G)b$ combines the structure isomorphisms $F(a \otimes b) \cong Fa \otimes Fb$ and $G(a \otimes b) \cong Ga \otimes Gb$ with the symmetry as below:

$$F(a \otimes b) \otimes G(a \otimes b) \cong Fa \otimes Fb \otimes Ga \otimes Gb \cong Fa \otimes Ga \otimes Fb \otimes Gb \quad .$$

The structural isomorphism concerning monoidal units is obvious. The hom objects $[A, B]$ extend in the obvious way to a 2-functor $[-,-] : \text{Perm}^{op} \times \text{Perm} \to \text{Perm}$. Moreover, the functor $[\mathbf{C}, -] : \text{Perm}(\mathbf{A}, \mathbf{B}) \to \text{Perm}([\mathbf{C}, \mathbf{A}], [\mathbf{C}, \mathbf{B}])$ lifts to a strict map $L = [\mathbf{C}, -] : [\mathbf{A}, \mathbf{B}] \to [[\mathbf{C}, \mathbf{A}], [\mathbf{C}, \mathbf{B}]]$ because both domain and codomain have structure inherited *pointwise from* $\mathbf{B}$. We omits details of the symmetry isomorphism $s : [\mathbf{A}, [\mathbf{B}, \mathbf{C}]] \cong [\mathbf{B}, [\mathbf{A}, \mathbf{C}]]$.
The unit $F1$ is the free permutative category on 1: the category of finite ordinals and bijections. The unit map $i : [F1, \mathbf{A}] \to \mathbf{A}$ is given by evaluation at 1 whilst $j : F1 \to [\mathbf{A}, \mathbf{A}]$ is the unique symmetric strict monoidal functor with $j(1) = 1_{\mathbf{A}}$. Again the skew closed structure is neither left nor right normal.
This example can be generalised to deal with general symmetric monoidal categories. A careful analysis of both tensor products and internal homs on the 2-category SMon of symmetric monoidal categories and strong symmetric monoidal functors was given by Schmitt [31].

4.2. **2-categories and bicategories.** Examples of skew closed structures not arising from pseudo-commutative 2-monads, even in the extended sense of [28], include 2-categories and bicategories. We focus upon the more complex case of bicategories. Let Bicat denote the category of bicategories and homomorphisms (also called pseudofunctors) and Bicat_s the subcategory of bicategories and strict homomorphisms. We describe a symmetric skew closed structure on Bicat_s with internal hom $Hom(A, B)$ the bicategory of pseudofunctors, pseudonatural transformations and modifications from A to B.

This skew closed structure arises from a closed symmetric multicategory. We begin by briefly recalling the multicategory structure, which was introduced and studied in depth in Section 1.3 of [37] by Verity, and to which we refer for further details. The multicategory **Bicat** – denoted by $\underline{Hom_s}$ in *ibid.* – has bicategories as objects. The multimaps are a variant of the cubical functors of [11]. More precisely, a multimap $F : (A_1, \ldots, A_n) \to B$ consists of

- for each n-tuple $(a_1, \ldots, a_n)$ an object $F(a_1, \ldots, a_n)$ of B;
- for each $1 \le i \le n$ a homomorphism $F(a_1, \ldots, a_{i-1}, -, a_{i+1} \ldots, a_n)$ extending the above function on objects;
- for each pair $1 \le i < j \le n$, n-tuple of objects $(a_1, \ldots, a_n)$ and morphisms $f : a_i \to a_i' \in A_i$ and $f_j : a_j \to a_j' \in A_j$, an invertible 2-cell:

$$\begin{tikzcd} F(a_i, a_j) \arrow[r, "F(f_i, a_j)"] \arrow[d, "F(a_i, f_j)"'] \arrow[dr, phantom, "\swarrow F(f_i, f_j)"] & F(a_i', a_j) \arrow[d, "F(a_i', f_j)"] \\ F(a_i, a_j') \arrow[r, "F(f_i, a_j')"'] & F(a_i', a_j') \end{tikzcd}$$

  where we have omitted to label the inactive parts of the n-tuple $(a_1, \ldots, a_n)$ under the action of F. These invertible 2-cells are required to form the components of pseudonatural transformations both vertically – $F(-_i, f_j) : F(-_i, a_j) \to F(-_i, a_j')$ – and horizontally – $F(f_i, -_j) : F(a_i, -_j) \to F(a_i', -_j)$ – and satisfy a further cubical identity involving trios of morphisms.

A nullary morphism $(-) \to B$ is simply defined to be an object of B. Observe that the category of unary maps of **Bicat** is simply Bicat. It is established in Section 1.3 of *ibid.* – see Lemma 1.3.4 and the discussion that follows – that the symmetric multicategory **Bicat** is closed, with hom-object given by the bicategory $Hom(A, B)$ of homomorphisms, pseudonatural transformations and modifications from A to B.

A multimap $F : (A_1, \ldots, A_n) \to B$ is strict in A_i just when each homomorphism $F(a_1, \ldots, a_{i-1}, -, a_{i+1} \ldots, a_n)$ is strict. An inspection of the bijection of Lemma 1.3.4 of *ibid.* makes it clear that the natural bijection

$$\mathbf{Bicat}(A_1, \ldots, A_n, B; C) \cong \mathbf{Bicat}(A_1, \ldots, A_n; Hom(B, C))$$

respects strictness in A_i.

Turning to the unit, recall that $\mathbf{Bicat}(-; A) = A_0$. The forgetful functor $(-)_0 : \mathrm{Bicat}_s \to \mathbf{Set}$ has a left adjoint F for general reasons – see Section 7 for more on this. It follows that we have a bijection $\mathrm{Bicat}_s(F1, A) \cong \mathbf{Bicat}(-; A)$

where $F1$ is the free bicategory on 1. Concretely, $F1$ has a single object $\bullet$ and a single generating 1-cell $e : \bullet \to \bullet$. General morphisms are (non-empty) bracketed copies of e such as $((ee)e)$, and two such morphisms are connected by a unique 2-cell, necessarily invertible.

By Theorem 3.5 we obtain a symmetric semi-closed category $(\mathrm{Bicat}_s, Hom, L)$ which restricts to a symmetric skew closed structure $(\mathrm{Bicat}_s, Hom, F1, L, i, j)$. As in the preceding examples the skew closed structure on Bicat_s is not closed and fails to extend to Bicat.

In Section 7 we further analyse this symmetric skew closed structure. Accordingly we describe a few aspects of it in more detail. Firstly, let us describe the action of the functor $Hom(-,-) : \mathrm{Bicat}^{op} \times \mathrm{Bicat} \to \mathrm{Bicat}$. From a homomorphism $f : A \rightsquigarrow B$ the homomorphism $Hom(f,1) : Hom(B,C) \to Hom(A,C)$ obtained by precomposition is always strict, and straightforward to describe. The post-composition map $Hom(1,f) = Hom(B,C) \to Hom(B,D)$ induced by a strict homomorphism $f : C \to D$ is equally straightforward.

Though not strictly required in what follows, for completness we mention the slightly more complex case where f is non-strict. At $\eta : g \to h \in Hom(B,C)$ the pseudonatural transformation $f\eta : fg \to fh$ has components $f\eta_a : fga \to fha$ at $a \in B$; at $\alpha : a \to b$ the invertible 2-cell $(f\eta)_\alpha$:

$$fh\alpha \circ f\eta_a \xRightarrow{\lambda_f} f(h\alpha \circ \eta_a) \xRightarrow{f\eta_\alpha} f(\eta_b \circ g\alpha) \xRightarrow{{\lambda_f}^{-1}} f\eta_b \circ fg\alpha$$

conjugates $f\eta_\alpha$ by the coherence constraints for f. The action of f_* on 2-cells is straightforward. The coherence constraints $f(\eta \circ \mu) \cong f(\eta) \circ f(\mu)$ and $f(id_g) \cong id(fg)$ for f^* are pointwise those for f.

The only knowledge required of $L : Hom(B,C) \to Hom(Hom(A,B), Hom(A,C))$ is that it has underlying function

$$Hom(A,-) : \mathrm{Bicat}(B,C) \to \mathrm{Bicat}(Hom(A,B), Hom(A,C)) \quad .$$

The unit map $i : Hom(F1, A) \to A$ evaluates at the single object $\bullet$ of $F1$ whilst $j : F1 \to Hom(A,A)$ is the unique strict homomorphism sending $\bullet$ to the identity on A.

This example can be modified to deal with 2-categories. Let **2-Cat** $\subset$ **Bicat** and $2\text{-}\mathrm{Cat}_s \subset \mathrm{Bicat}_s$ be the symmetric multicategory and category obtained by restricting the objects from bicategories to 2-categories. Since $Hom(A,B)$ is a 2-category if B is, we obtain a closed multicategory **2-Cat** by restriction. In this case we have a natural bijection $2\text{-}\mathrm{Cat}_s(1,A) \cong A_0 = \mathbf{2\text{-}Cat}(-;A)$. It follows that we obtain a symmetric skew closed structure $(2\text{-}\mathrm{Cat}_s, Hom, L, 1, i, j)$ with the same semi-closed structure as before, but with the simpler unit 1.

4.3. **Lax morphisms.** Each of the above examples describes a symmetric skew closed structure arising from a symmetric closed multicategory. In each case there are non-symmetric variants dealing with lax structures, of which we mention a few now. These have the same units but different internal homs. In $\mathbf{D}\text{-}\mathrm{Lim}_s$ the hom $[\mathbf{A}, \mathbf{B}]$ is the functor category $[A, B]$ equipped with $\mathbf{D}$-limits pointwise in B. In Perm_s the internal hom $[\mathbf{A}, \mathbf{B}]$ consists of lax monoidal functors and monoidal transformations. For Bicat_s one can take $[A, B]$ to be the bicategory of homomorphisms and lax natural transformations from A to B.

5. Skew structures descending to the homotopy category

In the present section we consider categories $\mathcal{C}$ equipped with a Quillen model structure as well as a skew monoidal or skew closed structure. We describe conditions under which the skew structures descend to the homotopy category $Ho(\mathcal{C})$ and call the skew monoidal/closed structures on $\mathcal{C}$ *homotopy monoidal/closed* if the induced structures on $Ho(\mathcal{C})$ are *genuinely monoidal/closed*. Theorem 5.11 gives a complete description of how monoidal skew closed structure descends to the homotopy category. Our analogue of Eilenberg and Kelly's theorem is Theorem 5.12: it allows us to recognise homotopy monoidal structure in terms of homotopy closed structure.

We assume some familiarity with the basics of Quillen model categories, as introduced in [30], and covered in Chapter 1 of [12]. Let us fix some terminology and starting assumptions. We assume that all model categories $\mathcal{C}$ have functorial factorisations. It follows that $\mathcal{C}$ is equipped with cofibrant and fibrant replacement functors Q and R together with natural transformations $p : Q \to 1$ and $q : 1 \to R$ whose components are respectively trivial fibrations and trivial cofibrations. Let $j : \mathcal{C}_c \to \mathcal{C}$ and $j : \mathcal{C}_f \to \mathcal{C}$ denote the full subcategories of cofibrant and fibrant objects, through which Q and R respectively factor. The four functors preserve weak equivalences and hence extend to the homotopy category. At that level we obtain adjoint equivalences

$$Ho(\mathcal{C}_c) \underset{Ho(Q)}{\overset{Ho(j)}{\rightleftarrows}} Ho(\mathcal{C}) \qquad\qquad Ho(\mathcal{C}_f) \underset{Ho(R)}{\overset{Ho(j)}{\rightleftarrows}} Ho(\mathcal{C})$$

with counit and unit given by $Ho(p)$ and its inverse, and $Ho(q)$ and its inverse respectively. If a functor between model categories $F : \mathcal{C} \to \mathcal{D}$ preserves weak equivalences between cofibrant objects we can form its *left derived functor* $F_l = Ho(FQ) : Ho(\mathcal{C}) \to Ho(\mathcal{D})$, equally $Ho(Fj)Ho(Q) : Ho(\mathcal{C}) \to Ho(\mathcal{C}_c) \to Ho(\mathcal{D})$. If G preserves weak equivalences between fibrant objects then $G_r = Ho(GR) = Ho(Gj)Ho(R)$ is its *right derived functor.*

5.1. **Skew monoidal structure on the homotopy category.** Let $\mathcal{C}$ be a model category equipped with a skew monoidal structure $(\mathcal{C}, \otimes, I, \alpha, l, r)$. Our interest is in left deriving this to a skew monoidal structure on $Ho(\mathcal{C})$. In the monoidal setting this was done in [12] and the construction in the skew setting, described below, is essentially identical.

Axiom M. $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves cofibrant objects and weak equivalences between them and the unit I is cofibrant.

The above assumption ensures that the skew monoidal structure on $\mathcal{C}$ restricts to one on $\mathcal{C}_c$ and that the restricted functor $\otimes : \mathcal{C}_c \times \mathcal{C}_c \to \mathcal{C}_c$ preserves weak equivalences. Accordingly we obtain a skew monoidal structure $(Ho(\mathcal{C}_c), Ho(\otimes), I)$ with the same components as before. Transporting this along the adjoint equivalence $Ho(j) : Ho(\mathcal{C}_c) \leftrightarrows Ho(\mathcal{C}) : Ho(Q)$ yields a skew monoidal structure

$$(Ho(\mathcal{C}), \otimes_l, I, \alpha_l, l_l, r_l)$$

on $Ho(\mathcal{C})$. We will often refer to $(Ho(\mathcal{C}), \otimes_l, I)$ as the *left-derived* skew monoidal structure since $\otimes_l$ is the left derived functor of $\otimes$. On objects we have $A \otimes_l B = QA \otimes QB$ and an easy calculation shows that the constraints for the skew monoidal structure are given by the following maps in $Ho(\mathcal{C})$.

$$\begin{tikzcd} Q(QA \otimes QB) \otimes QC \arrow[r, "p\otimes 1"] & (QA \otimes QB) \otimes QC \arrow[d, "\alpha"] & \\ & QA \otimes (QB \otimes QC) \arrow[r, "(1\otimes p)^{-1}"] & QA \otimes Q(QB \otimes QC) \end{tikzcd} \tag{5.1}$$

$$QI \otimes QA \xrightarrow{p\otimes 1} I \otimes QA \xrightarrow{l} QA \xrightarrow{p} A \tag{5.2}$$

$$A \xrightarrow{p^{-1}} QA \xrightarrow{r} QA \otimes I \xrightarrow{(1\otimes p)^{-1}} QA \otimes QI \tag{5.3}$$

Definition 5.1. Let $(\mathcal{C}, \otimes, I)$ be a skew monoidal structure on a model category $\mathcal{C}$ satisfying Axiom M. We say that $\mathcal{C}$ is *homotopy monoidal* if $(Ho(\mathcal{C}), \otimes_l, I)$ is genuinely monoidal.

Proposition 5.2. *Let $(\mathcal{C}, \otimes, I)$ be a skew monoidal category with a model structure satisfying Axiom M. The following are equivalent.*

(1) $(\mathcal{C}, \otimes, I)$ is homotopy monoidal.
(2) For all cofibrant X, Y, Z the map $\alpha : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is a weak equivalence, and for all cofibrant X both maps $r : X \to X \otimes I$ and $l : I \otimes X \to X$ are weak equivalences.

Proof. Observe that the constraints (5.1),(5.2) and (5.3) are $\alpha_{QA,QB,QC}$, l_{QA} and r_{QA} conjugated by isomorphisms in $Ho(\mathcal{C})$. It follows that $(Ho(\mathcal{C}), \otimes_l, I)$ is genuine monoidal just when for $\alpha_{QA,QB,QC}$, l_{QA} and r_{QA} are isomorphisms in $Ho(\mathcal{C})$ for all A, B and C. Axiom M ensures for that cofibrant A, B, C that we have isomorphisms $\alpha_{A,B,C} \cong \alpha_{QA,QB,QC}$, $l_A \cong l_{QA}$ and $r_A \cong r_{QA}$ in $Ho(\mathcal{C})^{\mathbf{2}}$ so that the former maps are isomorphisms just when the latter ones are. This proves the claim. $\square$

Notation 5.3. We call $(\mathcal{C}, \otimes, I)$ *homotopy symmetric monoidal* if $(Ho(\mathcal{C}), \otimes_l, I)$ admits the further structure of a symmetric monoidal category, but emphasise that this refers to a symmetry on $Ho(\mathcal{C})$ not necessarily arising from a symmetry on $\mathcal{C}$ itself.

5.2. **Skew closed structure on the homotopy category.** Let $\mathcal{C}$ be a model category equipped with a skew closed structure $(\mathcal{C}, [-,-], I, L, i, j)$. Our intention is to *right derive* the skew closed structure to the homotopy category. This construction, more complex than its monoidal counterpart, is closely related to the construction of a skew closed category $(\mathcal{C}, [Q-,-], I)$ from a closed comonad Q [35].

Axiom C. For cofibrant X the functor $[X,-]$ preserves fibrant objects and trivial fibrations. For fibrant Y the functor $[-,Y]$ preserves weak equivalences between cofibrant objects. The unit I is cofibrant.[3]

It follows from Axiom C that if A is cofibrant then $[1,p_B]:[A,QB]\to[A,B]$ is a trival fibration. Accordingly we obtain a lifting $k_{A,B}$ as below.

(5.4)
$$\begin{tikzcd} Q[A,B] \arrow[rr, "k_{A,B}"] \arrow[dr, "p_{[A,B]}"'] & & {[A,QB]} \arrow[dl, "{[1,p_B]}"] \\ & {[A,B]} & \end{tikzcd}$$

Because I is cofibrant we also have a lifting e as below.

(5.5)
$$\begin{tikzcd} I \arrow[rr, "e"] \arrow[dr, "1"'] & & QI \arrow[dl, "p_I"] \\ & I & \end{tikzcd}$$

Lemma 5.4. *Let $(\mathcal{C},[-,-],I)$ satisfy Axiom C. Then each of the following four diagrams*

(5.6)
$$\begin{tikzcd} Q[QB,C] \arrow[r, "QL"] \arrow[d, "k"'] & Q[[QA,QB],[QA,C]] \arrow[r, "{Q[k,1]}"] & Q[Q[QA,B],[QA,C]] \arrow[r, "k"] & {[Q[QA,B],Q[QA,C]]} \arrow[d, "{[1,k]}"] \\ {[QB,QC]} \arrow[r, "L"'] & {[[QA,QB],[QA,QC]]} \arrow[rr, "{[k,1]}"'] & & {[Q[QA,B],[QA,QC]]} \end{tikzcd}$$

(5.7)
$$\begin{tikzcd} Q[I,B] \arrow[rr, "k"] \arrow[dr, "Qi"'] & & {[I,QB]} \arrow[dl, "i"] \\ & QB & \end{tikzcd}$$

(5.8)
$$\begin{tikzcd} I \arrow[r, "e"] \arrow[rrrr, bend right, "j"'] & QI \arrow[r, "Qj"] & Q[A,A] \arrow[r, "{Q[p,1]}"] & Q[QA,A] \arrow[r, "k"] & {[QA,QA]} \end{tikzcd}$$

(5.9)
$$\begin{tikzcd} Q[B,C] \arrow[r, "k_{B,C}"] \arrow[d, "{Q[f,g]}"'] & {[B,QC]} \arrow[d, "{[f,Qg]}"] \\ Q[A,D] \arrow[r, "k_{A.D}"'] & {[A,QD]} \end{tikzcd}$$

[3] We could weaken Axiom C by requiring that $[X,-]$ preserves only weak equivalences between fibrant objects, rather than all trivial fibrations. This is still enough to construct the skew closed structure of Theorem 5.5 though the proof becomes slightly longer. Because we need the stronger Axiom MC in the crucial monoidal skew closed case anyway, we emphasise the convenient Axiom C.

commutes up to left homotopy. Moreover, if X is fibrant then the image of each diagram under $[-,X]$ commutes in $Ho(\mathcal{C})$.

Note that in (5.9) A and B are cofibrant and the morphisms $f : B \to A$ and $g : C \to D$ are arbitrary.

Proof. In each case we are presented with a pair of maps $f, g : U \rightrightarrows V$ with U cofibrant. To prove that f and g are left homotopic it suffices, by Proposition 1.2.5(iv) of [12], to show that there exists a trivial fibration $h : V \to W$ with $h \circ f = h \circ g$. We take the trivial fibrations $[1, p_{[QA,QC]}]$, p_B, $[1, p_A]$ and $[1, p_D]$ respectively. Each diagram, postcomposed with the relevant trivial fibration, is easily seen to commute.
For the second point observe that any functor $\mathcal{C} \to \mathcal{D}$ sending weak equivalences between cofibrant objects to isomorphisms identifies left homotopic maps - this follows the proof of Corollary 1.2.9 of *ibid.* Applying this to the composite of $[-,X] : \mathcal{C} \to \mathcal{C}^{op}$ and $\mathcal{C}^{op} \to Ho(\mathcal{C})^{op}$ gives the result. $\square$

Axiom C ensures that the right derived functor

$$[-,-]_r : Ho(\mathcal{C})^{op} \times Ho(\mathcal{C}) \to Ho(\mathcal{C})$$

exists with value $[A,B]_r = [QA, RB]$. The unit for the skew closed structure will be I. Using Axiom C we form transformations L_r, i_r and j_r on $Ho(\mathcal{C})$ as below.
(5.10)

$$\begin{tikzcd}
{[QA,RB]} \arrow[d, "{[Qq,1]^{-1}}"'] & & {[Q[QC,RA],R[QC,RB]]} \\
{[QRA,RB]} \arrow[r, "L"] & {[[QC,QRA],[QC,RB]]} \arrow[r, "{[k,1]}"] & {[Q[QC,RA],[QC,RB]]} \arrow[u, "{[1,q]}"']
\end{tikzcd}$$

(5.11)
$$[QI, RA] \xrightarrow{[e,1]} [I, RA] \xrightarrow{i} RA \xrightarrow{q^{-1}} A$$

(5.12)
$$I \xrightarrow{j} [A,A] \xrightarrow{[p,1]} [QA,A] \xrightarrow{[1,q]} [QA,RA]$$

Theorem 5.5. *Let $\mathcal{C}$ be a model category equipped with a skew closed structure $(\mathcal{C}, [-,-], I)$ satisfying Axiom C. Then $Ho(\mathcal{C})$ admits a skew closed structure $(Ho(\mathcal{C}), [-,-]_r, I)$ with constraints as above.*

Proof. In order to keep the calculations relatively short we will first describe a slightly simpler skew closed structure on $Ho(\mathcal{C}_f)$. We then obtain the skew closed structure on $Ho(\mathcal{C})$ by transport of structure.
So our main task is to construct a suitable skew closed structure on $Ho(\mathcal{C}_f)$. Now Axiom C ensures that $[QA,B]$ is fibrant whenever B is. The restricted bifunctor $[Q-,-] : \mathcal{C}_f^{op} \times \mathcal{C}_f \to \mathcal{C}_f$ then preserves weak equivalences in each variable and so extends to a bifunctor $Ho([Q-,-])$ on $Ho(\mathcal{C}_f)$. For the unit on $Ho(\mathcal{C}_f)$ we take RI.
The constraints are given by the following three maps.

(5.13)
$$[QA,B] \xrightarrow{L} [[QC,QA],[QC,B]] \xrightarrow{[k,1]} [Q[QC,A],[QC,B]]$$

$$[QRI, A] \xrightarrow{[Qq,1]} [QI, A] \xrightarrow{[e,1]} [I, A] \xrightarrow{\ i\ } A \tag{5.14}$$

$$RI \xrightarrow{q^{-1}} I \xrightarrow{\ j\ } [A, A] \xrightarrow{[p,1]} [QA, A] \tag{5.15}$$

We should explain why the above components are natural on $Ho(\mathcal{C}_f)$ – in the appropriate variance – since consideration of extraordinary naturality is perhaps non-standard.

Given $F, G : \mathcal{A} \rightrightarrows \mathcal{B}$ and a family of maps $\{\eta_A : FA \to GA : A \in \mathcal{A}\}$ we can consider the class of morphisms $Nat(\eta) \subseteq Mor(\mathcal{A})$ with respect to which η is natural. $Nat(\eta)$ is closed under composition and inverses in $\mathcal{A}$. If $(\mathcal{A}, \mathcal{W})$ is a category equipped with a collection of weak equivalences $\mathcal{W}$ then each arrow of $Ho(\mathcal{A})$ is composed of morphisms in $\mathcal{A}$ together with formal inverses w^{-1} where $w \in \mathcal{W}$. It follows that the family $\{\eta_A : FA \to GA : A \in \mathcal{A}\}$ is natural in $Ho(\mathcal{A})$ just when it is natural where restricted to $\mathcal{A}$. Similarly given $S : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{B}$ and a family of morphisms $\{\theta_A : X \to S(A, A) : A \in \mathcal{A}\}$ we can consider the class $Ex(\theta) \subseteq Mor(\mathcal{A})$ with respect to which θ is extranatural. This has the same closure properties as before. It follows that the family $\theta_A : X \to S(A, A)$ is extranatural in $Ho(\mathcal{A})$ just when it is extranatural when restricted to maps in $\mathcal{A}$. Using this reasoning we deduce that (5.14) and (5.15) are natural. We likewise obtain the naturality of the L-component of (5.13) in each variable. So it suffices to show that $(k_{C,A}, 1) : [[QC, QA], [QC, B]] \to [Q[QC, A], [QC, B]]$ is natural in each variable. This follows from Diagram (5.9) of Lemma 5.4.

We verify the diagrams (C1-C5) below. Each involves an instance of the corresponding diagram (C1-C5) for the skew closed structure on $\mathcal{C}$ itself, an application of Lemma 5.4 and straightforward applications of naturality.

(C2)

$$\begin{tikzcd}
{[Q[QA,A],[QA,C]]} \arrow[r, "{[Q[p,1],1]}"] & {[Q[A,A],[QA,C]]} \arrow[r, "{[Qj,1]}"] & {[QI,[QA,C]]} \arrow[r, "{[Qq,1]^{-1}}"] \arrow[dr, "1"'] & {[QRI,[QA,C]]} \arrow[d, "{[Qq,1]}"] \\
 & & & {[QI,[QA,C]]} \arrow[d, "{[e,1]}"] \\
{[[QA,QA],[QA,C]]} \arrow[uu, "{[k,1]}"] \arrow[rrr, "{[j,1]}"'] & & & {[I,[QA,C]]} \arrow[d, "i"] \\
{[QA,C]} \arrow[u, "L"] \arrow[rrr, "1"'] & & & {[QA,C]}
\end{tikzcd}$$

(C3)

$$\begin{tikzcd}
RI \arrow[r, "q^{-1}"] & I \arrow[r, "j"] \arrow[dr, "j"'] & {[B,B]} \arrow[r, "{[p,1]}"] \arrow[d, "L"] & {[QB,B]} \arrow[d, "L"] \\
 & & {[[QA,B],[QA,B]]} \arrow[r, "{[[1,p],1]}"] \arrow[dr, "{[p,1]}"'] & {[[QA,QB],[QA,B]]} \arrow[d, "{[k,1]}"] \\
 & & & {[Q[QA,B],[QA,B]]}
\end{tikzcd}$$

(C4)

$$\begin{tikzcd}
 & & {[[QRI,QB],[QRI,C]]} \arrow[r, "{[k,1]}"] \arrow[dr, "{[1,[Qq,1]]}"'] & {Q[QRI,B],[QRI,C]]} \arrow[dr, "{[1,[Qq,1]]}"] & \\
 & & & {[[QRI,QB],[QI,C]]} \arrow[r, "{[k,1]}"] & {[Q[QRI,B],[QI,C]]} \arrow[dd, "{[1,[e,1]]}"] \\
{[QB,C]} \arrow[uurr, "L"] \arrow[rr, "L"] \arrow[dr, "L"] \arrow[ddr, bend right, "{[i,1]}"] \arrow[ddd, "{[Qi,1]}"'] & & {[[QI,QB],[QI,C]]} \arrow[ur, "{[[Qq,1],1]}"] \arrow[r, "{[k,1]}"] \arrow[d, "{[1,[e,1]]}"] & {[Q[QI,B],[QI,C]]} \arrow[ur, "{[Q[Qq,1],1]}"'] \arrow[d, "{[1,[e,1]]}"] & \\
 & {[[I,QB],[I,C]]} \arrow[r, "{[[e,1],1]}"] \arrow[d, "{[1,i]}"] & {[[QI,QB],[I,C]]} \arrow[r, "{[k,1]}"] \arrow[d, "{[1,i]}"] & {[Q[QI,B],[I,C]]} \arrow[r, "{[Q[Qq,1],1]}"] \arrow[d, "{[1,i]}"] & {[Q[QRI,B],[I,C]]} \arrow[d, "{[1,i]}"] \\
 & {[[I,QB],C]} \arrow[r, "{[[e,1],1]}"] \arrow[dl, "{[k,1]}"'] & {[[QI,QB],C]} \arrow[r, "{[k,1]}"] & {[Q[QI,B],C]} \arrow[r, "{[Q[Qq,1],1]}"] & {[Q[QRI,B],C]} \\
{[Q[I,B],C]} \arrow[urrr, bend right, "{[Q[e,1],1]}"'] & & & &
\end{tikzcd}$$

(C5)

$$\begin{tikzcd}
RI \arrow[r, "q^{-1}"] & I \arrow[r, "j"] \arrow[d, "j"'] \arrow[dd, bend right=60, "1"'] & {[RI,RI]} \arrow[r, "{[p,1]}"] \arrow[d, "{[q,1]}"] & {[QRI,RI]} \arrow[d, "{[Qq,1]}"] \\
 & {[I,I]} \arrow[r, "{[1,q]}"] \arrow[d, "i"'] & {[I,RI]} \arrow[r, "{[p,1]}"] \arrow[d, "i"'] \arrow[dr, "1"'] & {[QI,RI]} \arrow[d, "{[e,1]}"] \\
 & I \arrow[r, "q"'] & RI & {[I,RI]} \arrow[l, "i"]
\end{tikzcd}$$

(C1)

$$
\begin{tikzcd}
 & & {[Q[QA,C],[QA,D]]} \arrow[r, "L"] & {[[Q[QA,B],Q[QA,C]],[Q[QA,B],[QA,D]]]} \arrow[d, "{[k,1]}"] \\
{[QC,D]} \arrow[r, "L"] \arrow[dd, "L"] & {[[QA,QC],[QA,D]]} \arrow[r, "L"] \arrow[ur, "{[k,1]}"] \arrow[d, "L"] & {[[[Q[QA,B],[QA,QC]],[Q[QA,B],[QA,D]]]} \arrow[ur, "{[[1,k],1]}"] \arrow[d, "{[[k,1],1]}"] & {[Q[Q[QA,B],[QA,C]],[Q[QA,B],[QA,D]]]} \arrow[dd, "{[Q[k,1],1]}"] \\
 & {[[[QA,QB],[QA,QC]],[[QA,QB],[QA,D]]]} \arrow[r, "{[1,[k,1]]}"] \arrow[d, "{[L,1]}"] & {[[[QA,QB],[QA,QC]],[Q[QA,B],[QA,D]]]} \arrow[d, "{[L,1]}"] & \\
{[[QB,QC],[QB,D]]} \arrow[r, "{[1,L]}"] \arrow[d, "{[k,1]}"] & {[[QB,QC],[[QA,QB],[QA,D]]]} \arrow[r, "{[1,[k,1]]}"] \arrow[dr, "{[k,1]}"] & {[[QB,QC],[Q[QA,B],[QA,D]]]} \arrow[dr, "{[k,1]}"] & {[Q[[QA,QB],[QA,C]],[Q[QA,B],[QA,D]]]} \arrow[d, "{[QL,1]}"] \\
{[Q[QB,C],[QB,D]]} \arrow[rr, "{[1,L]}"] & & {[Q[QB,C],[[QA,QB],[QA,D]]]} \arrow[r, "{[1,[k,1]]}"] & {[Q[QB,C],[Q[QA,B],[QA,D]]]}
\end{tikzcd}
$$

We now transport the skew closed structure along the adjoint equivalence $Ho(j) : Ho(\mathcal{C}_f) \leftrightarrows Ho(\mathcal{C}) : Ho(R)$. The skew closed structure obtained in this way has bifunctor

$$Ho(\mathcal{C})^{op} \times Ho(\mathcal{C}) \xrightarrow{Ho(R)^{op} \times Ho(R)} Ho(\mathcal{C}_f)^{op} \times Ho(\mathcal{C}_f) \xrightarrow{Ho([Q-,-])} Ho(\mathcal{C}_f) \xrightarrow{Ho(j)} Ho(\mathcal{C})$$

and unit given by

$$I \xrightarrow{RI} Ho(\mathcal{C}_f) \xrightarrow{Ho(j)} Ho(\mathcal{C}) \quad .$$

So $Ho([QR-,R-])$ and RI respectively. Neither is quite as claimed. We finally obtain the skew closed structure stated in the theorem by transferring this last skew closed structure along the isomorphisms of bifunctors $[Qq,1] : Ho([QR-,R-]) \to Ho([Q-,R-])$ and of units $q^{-1} : RI \to I$. $\square$

We often refer to $(Ho(\mathcal{C}), [-,-]_r, I)$ as the *right derived* skew closed structure since $[-,-]_r$ is the right derived functor of $[-,-]$.

Definition 5.6. Let $\mathcal{C}$ be a model category with a skew closed structure $(\mathcal{C}, [-,-], I)$ satisfying Axiom C. We say that $(\mathcal{C}, [-,-], I)$ is *homotopy closed* if the right derived skew closed structure $(Ho(\mathcal{C}), [-,-]_r, I)$ is genuinely closed.

Proposition 5.7. *Let $(\mathcal{C}, [-,-], I)$ be a skew closed category satisfying Axiom C. Then $(\mathcal{C}, [-,-], I)$ is homotopy closed if and only if the following two conditions are met.*

(1) For all cofibrant A and fibrant B the map

$$v = \mathcal{C}(j,1) \circ [A,-] : \mathcal{C}(A,B) \to \mathcal{C}(I,[A,B])$$

is a bijection on homotopy classes of maps.

(2) For all fibrant A the map $i : [I,A] \to A$ is a weak equivalence.

Proof. We will show that (1) and (2) amount to left and right normality of $(Ho(\mathcal{C}), [-,-]_r, I)$ respectively. Now $Ho(\mathcal{C})$ is left normal just when

$$Ho(\mathcal{C})(A,B) \xrightarrow{Ho(\mathcal{C})(j_r,1) \circ [A,-]_r} Ho(\mathcal{C})(I,[A,B]_r)$$

is a bijection for all A and B. As in any skew closed category this map is natural in both variables. Since we have isomorphisms $QA \to A$ and $B \to RB$ in $Ho(\mathcal{C})$ the above map will be an isomorphism for all A, B just when it is so for all cofibrant A and fibrant B. For such A and B we consider the diagram

$$\begin{tikzcd}
\mathcal{C}(A,B) \arrow[r, "\mathcal{C}(j,1)\circ\mathcal{C}({[p,q]},1)\circ{[QA,R-]}"] \arrow[d] & \mathcal{C}(I,[QA,RB]) \arrow[d] \\
Ho(\mathcal{C})(A,B) \arrow[r, "{[Ho(\mathcal{C})(j_r,1)\circ[A,-]_r}"] & Ho(\mathcal{C})(I,[QA,RB])
\end{tikzcd}$$

which is commutative by definition of $[A,-]_r$ and j_r. The left and right vertical morphisms are surjective and identify precisely the homotopic maps. It follows that the bottom row is invertible just when the top row induces a bijection on homotopy classes. By naturality of p and q we can rewrite the top row as

$$\mathcal{C}(A,B) \xrightarrow{\mathcal{C}(j,1)\circ[A,-]} \mathcal{C}(I,[A,B]) \xrightarrow{\mathcal{C}(1,[p,q])} \mathcal{C}(I,[QA,RB]) \quad .$$

Axiom C ensures that $[p,q] : [A,B] \to [QA,RB]$ is a weak equivalence between fibrant objects. Since I is cofibrant it follows that $\mathcal{C}(1,[p,q])$ is a bijection on homotopy classes. Therefore the above composite is a bijection on homotopy classes just when its left component is.

Now $Ho(\mathcal{C})$ is right normal just when $i_r = q^{-1} \circ i_{RA} \circ [e,1] : [QI,RA] \to [I,RA] \to RA \to A$ is invertible; equally just when $i_{RA} : [I,RA] \to RA$ is invertible for each A. For fibrant A we have $i_A \cong i_{RA}$ in $Ho(\mathcal{C})^{\mathbf{2}}$ and the result follows. ☐

5.2.1. *The symmetric case.* Consider a skew closed structure $(\mathcal{C},[-,-],I)$ satisfying Axiom C. By Theorem 5.5 we may form the right derived skew closed structure $(Ho(\mathcal{C}),[-,-]_r,I)$. Then from a symmetry isomorphism $s : [A,[B,C]] \cong [B,[A,C]]$ we can define a symmetry isomorphism $s_r : [A,[B,C]_r]_r \cong [B,[A,C]_r]_r$ on the right derived internal hom as below.

$$[QA,R[QB,RC]] \xrightarrow{[1,q]^{-1}} [QA,[QB,RC]] \xrightarrow{s} [QB,[QA,RC]] \xrightarrow{[1,q]} [QB,R[QA,RC]] \tag{5.16}$$

Proposition 5.8. *Consider $(\mathcal{C},[-,-],I)$ satisfying Axiom C, so we have the skew closed structure $(Ho(\mathcal{C}),[-,-]_r,I)$ of Theorem 5.5.*
If a natural isomorphism $s : [A,[B,C]] \cong [B,[A,C]]$ satisfies any of $S1 - S4$ then so does $s_r : [A,[B,C]_r]_r \cong [B,[A,C]_r]_r$. In particular, if $(\mathcal{C},[-,-],I,s)$ is symmetric skew closed then so is $(Ho(\mathcal{C}),[-,-]_r,I,s_r)$.

Proof. As in the proof of Theorem 5.5 we transport the structure from $Ho(\mathcal{C}_f)$. We can extend the skew closed structure $(Ho(\mathcal{C}_f),Ho[Q-,-],RI)$ described therein by a symmetry transformation s_r whose component at (A,B,C) is

$$[QA,[QB,C]] \xrightarrow{s_{QA,QB,C}} [QB,[QA,C]]$$

Since the components of s_r are just those of s it follows that $S1$ and $S2$ hold in $Ho(\mathcal{C}_f)$ if they do so in $\mathcal{C}$. The diagrams for $S3$ and $S4$ are below.

(S3)

$$\begin{tikzcd}
[QA,[QB,C]] \arrow[r, "L"] \arrow[dd, "s"'] & {[[QD,QA],[QD,[QB,C]]]} \arrow[r, "{[k,1]}"] \arrow[d, "{[1,s]}"] & {[Q[QD,A],[QD,[QB,C]]]} \arrow[d, "{[1,s]}"] \\
 & {[[QD,QA],[QB,[QD,C]]]} \arrow[r, "{[k,1]}"] \arrow[d, "s"] & {[Q[QD,A],[QB,[QD,C]]]} \arrow[d, "s"] \\
{[QB,[QA,C]]} \arrow[r, "{[1,L]}"] & {[QB,[[QD,QA],[QD,C]]]} \arrow[r, "{[1,[k,1]]}"] & {[QB,[Q[QD,A],[QD,C]]]}
\end{tikzcd}$$

(S4)

$$\begin{tikzcd}
{[QA,B]} \arrow[r, "L"] \arrow[ddd, "1"'] & {[[QA,QA],[QA,B]]} \arrow[r, "{[k,1]}"] \arrow[dd, "{[j,1]}"] & {[Q[QA,A],[QA,B]]} \arrow[r, "{[Q[p,1],1]}"] & {[Q[A,A],[QA,B]]} \arrow[d, "{[Qj,1]}"] \\
& & & {[QI,[QA,B]]} \arrow[d, "{[Qq,1]^{-1}}"] \arrow[dl, "1"'] \\
& {[I,[QA,B]]} \arrow[d, "s"] & {[QI,[QA,B]]} \arrow[l, "{[e,1]}"] \arrow[d, "s"] & {[QRI,[QA,B]]} \arrow[l, "{[Qq,1]}"] \arrow[d, "s"] \\
{[QA,B]} & {[QA,[I,B]]} \arrow[l, "{[1,i]}"] & {[QA,[QI,B]]} \arrow[l, "{[1,[e,1]]}"] & {[QA,[QRI,B]]} \arrow[l, "{[1,[Qq,1]]}"]
\end{tikzcd}$$

Therefore $(Ho(\mathcal{C}_f), Ho[Q-,-], RI, s_r)$ satisfies any of $S1-S4$ when $(\mathcal{C}, [-,-], I, s)$ does so. The desired structure on $Ho(\mathcal{C})$ is obtained by transporting from the equivalent $Ho(\mathcal{C}_f)$ as in Theorem 5.5. ☐

5.3. **Monoidal skew closed structure on the homotopy category and the homotopical version of Eilenberg and Kelly's theorem.** Let $\mathcal{C}$ be a model category equipped with a monoidal skew closed structure $(\mathcal{C}, \otimes, [-,-], I)$. In order to derive the tensor-hom adjunctions $- \otimes QA \dashv [QA, -]$ to the homotopy category, we will make use of the concept of a Quillen adjunction.

An adjunction $F : \mathcal{C} \leftrightarrows D : U$ of model categories is said to be a *Quillen adjunction* if the right adjoint $U : \mathcal{D} \to \mathcal{C}$ preserves fibrations and trivial fibrations. One says that U is *right Quillen*. This is equivalent to asking that F preserves cofibrations and trivial cofibrations, in which case F is said to be *left Quillen*. The derived functors F_l and U_r then exist and form an adjoint pair $F_l : Ho(\mathcal{C}) \leftrightarrows Ho(\mathcal{D}) : U_r$. This works as follows. At cofibrant A and fibrant B the isomorphisms $\varphi_{A,B} : \mathcal{D}(FA, B) \cong \mathcal{C}(A, UB)$ are well defined on homotopy classes and so give natural bijections

$$\varphi_{A,B} : Ho(\mathcal{D})(FA, B) \cong Ho(\mathcal{C})(A, UB)$$

– we use the same labelling. At $A, B \in Ho(\mathcal{C})$ the hom-set bijections

$$\varphi^d_{A,B} : Ho(\mathcal{D})(F_l A, B) \cong Ho(\mathcal{C})(A, U_r B)$$

are then given by conjugating the $\varphi_{A,B}$ as below:

$$Ho(\mathcal{D})(FQA,B) \xrightarrow{(1,q)} Ho(\mathcal{D})(FQA,RB) \xrightarrow{\varphi_{A,B}} Ho(\mathcal{C})(QA,URB) \xrightarrow{(p,1)^{-1}} Ho(\mathcal{C})(A,URB)$$

It follows that the unit of the derived adjunction – the *derived unit* – is given by

$$A \xrightarrow{p_A^{-1}} QA \xrightarrow{\eta_{QA}} UFQA \xrightarrow{Uq_{FQA}} URFQA$$

whilst the derived counit admits a dual description.

Proposition 5.9. *For $\mathcal{C}$ a model category and $(\mathcal{C}, \otimes, [-,-], I)$ monoidal skew closed the following are equivalent.*

(1) For cofibrant X the functor $- \otimes X$ is left Quillen and for each cofibrant Y the functor $Y \otimes -$ preserves weak equivalences between cofibrant objects.

(2) For cofibrant X the functor $[X,-]$ is right Quillen and for fibrant Y the functor $[-,Y]$ preserves weak equivalences between cofibrant objects.

Proof. Certainly $-\otimes X$ is left Quillen just when $[X,-]$ is right Quillen. Let $f : A \to B$ be a weak equivalence between cofibrant objects. Then the natural transformation of left Quillen functors $-\otimes f : -\otimes A \to -\otimes B$ and the natural transformation of right Quillen functors $[f,-] : [B,-] \to [A,-]$ are *mates*. By Corollary 1.4.4(b) of [12] it follows that $Y \otimes f$ is a weak equivalence for all cofibrant Y if and only if $[f,Y]$ is a weak equivalence for all fibrant Y. □

Axiom MC. $(\mathcal{C},\otimes,[-,-],I)$ satisfies either of the equivalent conditions of Proposition 5.9 and the unit I is cofibrant.

Proposition 5.10. *Axiom MC implies Axioms M and C.*

Proof. When Axiom MC holds Ken Brown's lemma (1.1.12 of [12]) ensures that for cofibrant X the functor $X\otimes -$ preserves weak equivalences between cofibrant objects. So Axiom MC is a strengthening of Axiom M. That it implies Axiom C is clear: if the right adjoint $[X,-]$ preserves fibrations it preserves fibrant objects. □

For $(\mathcal{C},\otimes,[-,-],I,\varphi)$ satisfying Axiom MC it follows that we can form the left derived skew monoidal and right derived skew closed structures on $Ho(\mathcal{C})$. The following result establishes, as expected, that these form part of a monoidal skew closed structure on $Ho(\mathcal{C})$.

Theorem 5.11. *Let $(\mathcal{C},\otimes,[-,-],I,\varphi)$ be a monoidal skew closed category satisfying Axiom MC. Then the left derived skew monoidal structure $(Ho(\mathcal{C}),\otimes_l,I)$ and the right derived skew closed structure $(\mathcal{C},[-,-]_r,I)$ together with the isomorphisms*

$$\varphi^d : Ho(\mathcal{C})(QA\otimes QB,C) \cong Ho(\mathcal{C})(A,[QB,RC])$$

form a monoidal skew closed structure on $Ho(\mathcal{C})$.

The straightforward but long proof is deferred until the appendix. The following result is the homotopical version of Eilenberg and Kelly's theorem. Note that by Proposition 5.7 conditions (1) and (2) amount to $(\mathcal{C},[-,-],I)$ being homotopy closed.

Theorem 5.12. *Let $(\mathcal{C},\otimes,[-,-],I)$ be a monoidal skew closed category satisfying Axiom MC. Then $(\mathcal{C},\otimes,I)$ is homotopy monoidal if and only if the following three conditions are satisfied.*

(1) For all cofibrant A and fibrant B the function $v : \mathcal{C}(A,B) \to \mathcal{C}(I,[A,B])$ is a bijection on homotopy classes of maps,
(2) For all fibrant A the map $i : [I,A] \to A$ is a weak equivalence,
(3) The transformation $t : [A\otimes B,C] \to [A,[B,C]]$ is a weak equivalence whenever A and B are cofibrant and C is fibrant.

Proof. Combining Theorems' 5.11 and 2.6 we have that $(Ho(\mathcal{C}),\otimes_l,I)$ is monoidal just when $(Ho(\mathcal{C}),[-,-]_r,I)$ is closed and the induced transformation $t^d : [QA\otimes QB,RC] \to [QA,R[QB,RC]]$ is an isomorphism in $Ho(\mathcal{C})$. By Proposition 5.7

closedness amounts to (1) and (2) above. From the proof of Theorem 5.11 the transformation t^d is given by the composite

$$[Q(QA{\otimes}QB),RC] \xrightarrow{[p,1]^{-1}} [QA{\otimes}QB,RC] \xrightarrow{t} [QA,[QB,RC]] \xrightarrow{[1,q]} [QA,R[QB,RC]]$$

and therefore is invertible just when the central component

$$t_{QA,QB,RC} : [QA \otimes QB, RC] \to [QA, [QB, RC]]$$

is so for all A, B and C. For A, B cofibrant and C fibrant $t_{QA,QB,RC}$ is isomorphic to $t_{A,B,C} : [A \otimes B, C] \to [A, [B, C]]$. Since QA and QB are cofibrant and RC is fibrant it follows that $t_{QA,QB,RC}$ is invertible for all A, B, C just when $t_{A,B,C}$ is invertible for all cofibrant A,B and fibrant C. □

Again we have a symmetric variant.

Theorem 5.13. *Let $(\mathcal{C}, \otimes, [-,-], I)$ be a monoidal skew closed category satisfying Axiom MC.*

(1) If $(\mathcal{C}, [-,-], I)$ is homotopy closed and admits a natural symmetry isomorphism $s : [A, [B, C]] \cong [B, [A, C]]$ satisfying S3 *then $(\mathcal{C}, \otimes, I)$ is homotopy monoidal.*

(2) If, in addition to (1), $(\mathcal{C}, [-,-], I, s)$ is symmetric skew closed then $(\mathcal{C}, \otimes, I)$ is homotopy symmetric monoidal.

Proof. By Proposition 5.8 if s satisfies S3 then so does s_r with respect to $(Ho(\mathcal{C}), [-,-]_r, I)$. From Theorem 2.10 it follows that $(Ho(\mathcal{C}), \otimes_l, I)$ is monoidal. The second part follows again by application of Proposition 5.8 and 2.10. □

6. Pseudo-commutative 2-monads and monoidal bicategories

In the category $CMon$ of commutative monoids the set $CMon(A, B)$ forms a commutative monoid $[A, B]$ with respect to the pointwise structure of B. This is the internal hom of a symmetric monoidal closed structure on $CMon$ whose tensor product represents functions $A \times B \to C$ that are homomorphisms in each variable. From the monad-theoretic viewpoint the enabling property is that the commutative monoid monad on **Set** is a *commutative monad*.

Extending this to dimension 2, Hyland and Power [13] introduced the notion of a *pseudo-commutative 2-monad* T on **Cat**. Examples include the 2-monads for categories with a class of limits, permutative categories, symmetric monoidal categories and so on. For such T they showed that the 2-category of strict algebras and pseudomorphisms admits the structure of a *pseudo-closed 2-category* – a slight weakening of the notion of a closed category with a 2-categorical element. Theorem 2 of *ibid.* described a bicategorical version of Eilenberg and Kelly's theorem, designed to produce a monoidal bicategory structure on T-Alg. However they did not give the details of the proof, which involved lengthy calculations of a bicategorical nature, and expressed their dissatisfaction with the argument.[4]

[4] From [13]:"Naturally, we are unhappy with the proof we have just outlined. Since the data we start from is in no way symmetric we expect some messy difficulties: but the calculations we do not give are very tiresome, and it would be only too easy to have made a slip. Hence we would like a more conceptual proof."

In this section we take a slightly different route to the monoidal bicategory structure on T-Alg. We begin by making minor modifications to Hyland and Power's construction to produce a skew closed structure on the 2-category T-Alg$_s$ of algebras and *strict morphisms.* This is simply the restriction of the pseudo-closed 2-category structure on T-Alg. We then obtain a monoidal skew closed structure on T-Alg$_s$ and, using Theorem 5.12, establish that it is homotopy monoidal. The monoidal bicategory structure on T-Alg is obtained by transport of structure from the full sub 2-category of T-Alg$_s$ containing the cofibrant objects.

6.1. **Background on commutative monads.** If V is a symmetric monoidal closed category and T an endofunctor of V then enrichments of T to a V-functor correspond to giving a *strength*: that is, a natural transformation $t_{A,B} : A \otimes TB \to T(A \otimes B)$ subject to associativity and identity conditions. One obtains a *costrength* $t^*_{A,B} : TA \otimes B \to T(A \otimes B)$ related to the strength by means of the symmetry isomorphism $c_{A,B} : A \otimes B \to B \otimes A$.
If (T, η, μ) is a V-enriched monad then we can consider the following diagram

$$
\begin{tikzcd}
TA \otimes TB \arrow[r, "t"] \arrow[d, "t^\star"'] & T(TA \otimes B) \arrow[r, "Tt^\star"] & T^2(A \otimes B) \arrow[d, "\mu"] \\
T(A \otimes TB) \arrow[r, "Tt"] & T^2(A \otimes B) \arrow[r, "\mu"] & T(A \otimes B)
\end{tikzcd} \tag{6.1}
$$

and if this commutes for all A and B then T is said to be a *commutative monad* [17].
Now if T is commutative and V sufficiently complete and cocomplete then the category of algebras V^T is itself symmetric monoidal closed [18, 15]. Both tensor product and internal hom represent T*-bilinear maps* – this perspective was explored in [19] and more recently in [32]. More generally, a T-multilinear map consists of a morphism $f : A_1 \otimes \ldots \otimes A_n \to B$ which is a T*-algebra map in each variable.* This means that the diagram

$$
\begin{tikzcd}
A_1 \otimes \ldots \otimes TA_i \otimes \ldots \otimes A_n \arrow[r, "t"] \arrow[d, "1 \otimes \ldots \otimes a_i \otimes \ldots \otimes 1"'] & T(A_1 \otimes \ldots \otimes A_n) \arrow[r, "Tf"] & TB \arrow[d, "b"] \\
A_1 \otimes \ldots \otimes A_i \otimes \ldots \otimes A_n \arrow[rr, "f"] & & B
\end{tikzcd}
$$

is commutative for each i where the top row $t : A_1 \otimes \ldots \otimes TA_i \otimes \ldots \otimes A_n \to T(A_1 \otimes \ldots \otimes A_n)$ is the unique map constructible from the strengths and costrengths. T-multilinear maps form the morphisms of a *multicategory* of T-algebras. Surprisingly, the multicategory perspective appears to have first been explored in the more general 2-categorical setting of [13].

6.2. **Background on 2-monads.** The category of small categories **Cat** is cartesian closed and hence provides a basis suitable for enriched category theory. In particular one has the notions of **Cat**-enriched category – hence 2-category – and of **Cat**-enriched monads – hence 2-monads. The appendage "2-" will always refer to strict **Cat**-enriched concepts.

Given a 2-monad $T = (T, \eta, \mu)$ on a 2-category $\mathcal{C}$ one has the Eilenberg-Moore 2-category T-Alg$_s$ of algebras. In T-Alg$_s$ everything is completely strict. There are the usual *strict* algebras $\mathbf{A} = (A, a)$ satisfying $a \circ Ta = a \circ \mu_A$ and $a \circ \eta_A = 1$. The *strict* morphisms $f : \mathbf{A} \to \mathbf{B}$ of T-Alg$_s$ satisfy the usual equation $b \circ Tf = f \circ a$ on the nose, whilst the 2-cells $\alpha : f \Rightarrow g \in$ T-Alg$_s(\mathbf{A}, \mathbf{B})$ satisfy $b \circ T\alpha = \alpha \circ a$. T-Alg$_s$ is just as well behaved as its **Set**-enriched counterpart. Important facts for us are the following ones.

- The usual (free, forgetful)-adjunction lifts to a 2-adjunction $F \dashv U$ where $U :$ T-Alg$_s \to \mathcal{C}$ is the evident forgetful 2-functor.
- Suppose that $\mathcal{C}$ is a locally presentable 2-category: one, like **Cat**, that is cocomplete in the sense of enriched category theory [16] and whose underlying category is locally presentable [1]. If T is accessible – preserves λ-filtered colimits for some regular cardinal λ – then T-Alg$_s$ is also locally presentable.

There are accessible 2-monads T on **Cat** whose strict algebras are categories with **D**-limits, permutative categories, symmetric monoidal categories and so on. In particular the examples of skew closed 2-categories from Section 4.1 reside on 2-categories of the form T-Alg$_s$ for T an accessible 2-monad on **Cat**.

So far we have discussed strict aspects of two-dimensional monad theory. Though there are several possibilities, the only weak structures of interest here are *pseudomorphisms* of *strict* T-algebras. A pseudomorphism $\mathbf{f} : \mathbf{A} \rightsquigarrow \mathbf{B}$ consists of a morphism $f : A \to B$ and invertible 2-cell $\overline{f} : b \circ Tf \cong f \circ a$ satisfying two coherence conditions [3]. These are the morphisms of the 2-category T-Alg into which T-Alg$_s$ includes via an identity on objects 2-functor $\iota :$ T-Alg$_s \to$ T-Alg. The inclusion commutes with the forgetful 2-functors

$$
\begin{tikzcd}
\text{T-Alg}_s \arrow[rr, "\iota"] \arrow[dr, "U"'] & & \text{T-Alg} \arrow[dl, "U"] \\
& \mathcal{C} &
\end{tikzcd}
\tag{6.2}
$$

to the base. Pseudomorphisms of T-algebras capture functors preserving categorical structure up to isomorphism. For example, in the case that T is the 2-monad for categories with **D**-limits or permutative categories we obtain the 2-categories **D**-Lim and Perm as T-Alg.

An important tool in the study of pseudomorphisms are *pseudomorphism classifiers*. If T is a reasonable 2-monad – for instance, an accessible 2-monad on **Cat** – then by Theorem 3.3 of [3] the inclusion $\iota :$ T-Alg$_s \to$ T-Alg has a left 2-adjoint Q. We call $Q\mathbf{A}$ the pseudomorphism classifier of $\mathbf{A}$ since each pseudomorphism $\mathbf{f} : \mathbf{A} \rightsquigarrow \mathbf{B}$ factors uniquely through the unit $\mathbf{q_A} : \mathbf{A} \rightsquigarrow Q\mathbf{A}$ as a strict morphism $Q\mathbf{A} \to \mathbf{B}$. The counit $p_{\mathbf{A}} : Q\mathbf{A} \to \mathbf{A}$ is a strict map with homotopy theoretic content – see Section 6.4.1 below.

6.3. **From pseudo-commutative 2-monads to monoidal skew closed 2-categories.** Given a 2-monad T on **Cat** we have, in particular, the corresponding strengths $t : T(A \times B) \to A \times TB$ and costrengths $T(A \times B) \to TA \times B$ and can enquire as to whether T is commutative. For those structures – such as categories with finite products or symmetric monoidal categories – that involve an aspect of weakness in their definitions the relevant diagram (6.1) rarely commutes on the nose, but often commutes up to natural isomorphism. This leads to the notion of a *pseudo-commutative 2-monad* T which is a 2-monad T equipped with invertible 2-cells

$$\begin{tikzcd} TA \times TB \arrow[r, "t"] \arrow[d, "t^\star"'] & T(TA \times B) \arrow[r, "Tt^\star"] \arrow[d, phantom, "\Downarrow \alpha_{A,B}"] & T^2(A \times B) \arrow[d, "\mu"] \\ T(A \times TB) \arrow[r, "Tt"'] & T^2(A \times B) \arrow[r, "\mu"'] & T(A \times B) \end{tikzcd}$$

subject to axioms (see Definition 5 of [13]) asserting the equality of composite 2-cells built from the above ones. If α commutes with the symmetry isomorphism – in the sense that $\alpha_{B,A} = Tc_{A,B} \circ \alpha_{A,B} \circ c_{TB,TA}$ – then T is said to be a *symmetric* pseudo-commutative 2-monad.

The 2-monad for categories with **D**-limits is symmetric pseudo-commutative [28] as are the 2-monads for permutative and symmetric monoidal categories [13]. An example of a pseudo-commutative 2-monad which is not symmetric is the 2-monad for braided strict monoidal categories [5].

6.3.1. *The 2-multicategory of algebras.* For T pseudo-commutative one can define T-multilinear maps. A T-multilinear map $\mathbf{f} : (\mathbf{A}_1, \ldots, \mathbf{A}_n) \to \mathbf{B}$ consists of a functor $f : A_1 \times \ldots A_n \to B$ together with a family of invertible 2-cells f_i:

$$\begin{tikzcd} A_1 \times \ldots \times TA_i \times \ldots \times A_n \arrow[r, "t"] \arrow[d, "1\times\ldots\times a_i\times\ldots\times 1"'] & T(A_1 \times \ldots \times A_n) \arrow[r, "Tf"] \arrow[d, phantom, "\Downarrow f_i"] & TB \arrow[d, "c"] \\ A_1 \times \ldots \times A_i \times \ldots \times A_n \arrow[rr, "f"] & & B \end{tikzcd}$$

satisfying indexed versions of the pseudomorphism equations, and a compatibility condition involving the pseudo-commutativity. A nullary map $(-) \to \mathbf{B}$ is defined to be an object of the category B.

There are transformations of multilinear maps and these are the morphisms of a category $\mathbf{T\text{-}Alg}(\mathbf{A}_1, \mathbf{A}_2 \ldots \mathbf{A}_n; \mathbf{B})$. Proposition 18 of *ibid.* shows that these are the hom-categories of a 2-multicategory of T-algebras **T-Alg** and that, moreover, if T is symmetric pseudo-commutative then **T-Alg** is a symmetric 2-multicategory. T-Alg is itself recovered as the 2-category of unary maps.

Of course we can speak of multimaps $(\mathbf{A}_1, \ldots, \mathbf{A}_n) \to \mathbf{B}$ which are strict in $\mathbf{A}_i$: those for which the natural transformation f_i depicted above is an identity. Note that this agrees with the formulation given in Definition 3.4.

Theorem 6.1 (Hyland-Power [13])**.** *The 2-multicategory* **T-Alg** *is closed. Moreover a multimap* $(\mathbf{A}_1, \mathbf{A}_2 \ldots \mathbf{A}_n, \mathbf{B}) \to \mathbf{C}$ *is strict in* $\mathbf{A}_i$ *just when the corresponding map* $(\mathbf{A}_1, \mathbf{A}_2 \ldots \mathbf{A}_n) \to [\mathbf{B}, \mathbf{C}]$ *is so.*

6.3.2. *The skew closed structure.* By definition $\mathbf{T\text{-}Alg}(-;\mathbf{A}) = A$. Since we have a natural isomorphism $\text{T-Alg}_s(F1,\mathbf{A}) \cong \mathbf{Cat}(1,A) \cong A$ and a suitable closed 2-multicategory **T-Alg** we can apply Theorem 3.6 to obtain a skew closed structure on T-Alg_s.

Theorem 6.2. *Let T be a pseudo-commutative 2-monad on* **Cat**.

(1) Then $(\text{T-Alg}, [-,-], L)$ is a semi-closed 2-category. Moreover $[-,-]$ and L restrict to T-Alg_s where they form part of a skew closed 2-category $(\text{T-Alg}_s, [-,-], F1, L, i, j)$.
(2) If T is symmetric then $(\text{T-Alg}, [-,-], L)$ is symmetric semi-closed and $(\text{T-Alg}_s, [-,-], F1, L, i, j)$ is symmetric skew closed.

The skew closed 2-category $(\text{T-Alg}_s, [-,-], F1, L, i, j)$ has components as constructed in Section 3. Let us record, for later use, some further information about these components.

(1) The underlying category of $[\mathbf{A},\mathbf{B}]$ is just $\text{T-Alg}(\mathbf{A},\mathbf{B})$. More generally $U \circ [-,-] = \text{T-Alg}(-,-) : \text{T-Alg}^{op} \times \text{T-Alg} \to \mathbf{Cat}$.
(2) The underlying functor of $L : [\mathbf{A},\mathbf{B}] \to [[\mathbf{C},\mathbf{A}],[\mathbf{C},\mathbf{B}]]$ is given by $[\mathbf{C},-]_{\mathbf{A},\mathbf{B}} : \text{T-Alg}(\mathbf{A},\mathbf{B}) \to \text{T-Alg}([\mathbf{C},\mathbf{A}],[\mathbf{C},\mathbf{B}])$.
(3) The underlying functor of $i : [F1,\mathbf{A}] \to \mathbf{A}$ is the composite

$$\text{T-Alg}(\mathbf{F}1,\mathbf{A}) \xrightarrow{U_{\mathbf{F1},\mathbf{A}}} \mathbf{Cat}(T1,A) \xrightarrow{\mathbf{Cat}(\eta_1,A)} \mathbf{Cat}(1,A) \xrightarrow{ev_\bullet} A$$

whose last component is the evaluation isomorphism.
(4) $j : F1 \to [\mathbf{A},\mathbf{A}]$ is the transpose of the functor $\hat{1} : 1 \to \text{T-Alg}(\mathbf{A},\mathbf{A})$ selecting the identity on $\mathbf{A}$.

(1) follows from the construction of the hom algebra $[\mathbf{A},\mathbf{B}]$ in [13] as a 2-categorical limit in T-Alg created by $U : \text{T-Alg} \to \mathbf{Cat}$.[5] Theorem 11 of *ibid.* gives a full description of the isomorphisms $\mathbf{T\text{-}Alg}(\mathbf{A}_1,\ldots,\mathbf{A}_n,\mathbf{B};\mathbf{C}) \cong \mathbf{T\text{-}Alg}(\mathbf{A}_1\ldots\mathbf{A}_n;[\mathbf{B},\mathbf{C}])$. From this, it follows that the evaluation multimap $ev : ([\mathbf{A},\mathbf{B}],\mathbf{A}) \to \mathbf{B}$ has underlying functor $\text{T-Alg}(\mathbf{A},\mathbf{B}) \times \mathbf{A} \to \mathbf{B}$ acting by application, which is what is required for (3). (2) follows from the analysis, given in Proposition 21 of *ibid*, of how the same adjointness isomorphisms behave with respect to underlying maps. (4) is by definition.

6.3.3. *The monoidal skew closed structure on* T-Alg_s. We now describe left 2-adjoints to the 2-functors $[\mathbf{A},-] : \text{T-Alg}_s \to \text{T-Alg}_s$. For this let us further suppose that T is an accessible 2-monad. We must show that each $\mathbf{B}$ admits a reflection $\mathbf{B} \otimes \mathbf{A}$ along $[\mathbf{A},-]$. Since T-Alg_s is cocomplete the class of algebras admitting such a reflection is closed under colimits; because each algebra is a coequaliser of frees it therefore suffices to show that each free algebra admits a

[5] This construction is accomplished in three stages by firstly forming an iso-inserter and then a pair of equifiers and amounts to the construction of a descent object.

reflection. With this is mind observe that the triangle

$$\begin{tikzcd} \text{T-Alg}_s \arrow[rr, "{[\mathbf{A},-]}"] \arrow[dr, "{\text{T-Alg}(\mathbf{A},\iota-)}"'] & & \text{T-Alg}_s \arrow[dl, "U"] \\ & \mathbf{Cat} & \end{tikzcd} \tag{6.3}$$

commutes. Because T is accessible we have the 2-adjunction $Q \dashv \iota$ and corresponding isomorphism $\text{T-Alg}_s(Q\mathbf{A}, -) \cong \text{T-Alg}(\mathbf{A}, \iota-)$. Now the representable $\text{T-Alg}_s(Q\mathbf{A}, -)$ has a left adjoint $-.Q\mathbf{A}$ given by taking copowers. It follows that at $C \in \mathbf{Cat}$ the reflection $FC \otimes \mathbf{A}$ is given by $C.Q\mathbf{A}$. We conclude:

Proposition 6.3. *If T is an accessible pseudo-commutative 2-monad on* **Cat** *then each $[\mathbf{A}, -] : \text{T-Alg}_s \to \text{T-Alg}_s$ has a left 2-adjoint $- \otimes \mathbf{A}$. In particular $(\text{T-Alg}_s, \otimes, [-,-], F1)$ is a monoidal skew closed 2-category.*

6.4. **From** T-Alg$_s$ **as a monoidal skew closed 2-category to** T-Alg **as a monoidal bicategory.** Our next goal is to show that $(\text{T-Alg}_s, \otimes, [-,-], F1)$ is homotopy monoidal. In order to do so requires understanding the Quillen model structure on T-Alg$_s$ and its relationship with pseudomorphisms. We summarise the key points below and refer to the original source [22] for further details.

6.4.1. *Homotopy theoretic aspects of 2-monads.* Thought of as a mere category, **Cat** admits a Quillen model structure in which the weak equivalences are the equivalences of categories. The cofibrations are the injective on objects functors and the fibrations are the isofibrations: functors with the isomorphism lifting property. It follows that all objects are cofibrant and fibrant.
Equipped with the cartesian closed structure, **Cat** is a monoidal model category [12]. Therefore one can speak of model 2-categories, of which **Cat** is the leading example. It was shown in Theorem 4.5 of [22] that for an accessible 2-monad T on **Cat** the model structure lifts along $U : \text{T-Alg}_s \to \mathbf{Cat}$ to a model 2-category structure on T-Alg$_s$: a morphism of T-Alg$_s$ is a weak equivalence or fibration just when its image under U is one. It follows immediately that the adjunction $F \dashv U : \text{T-Alg}_s \leftrightarrows \mathbf{Cat}$ is a Quillen adjunction.
Since F preserves cofibrations each free algebra is cofibrant. In fact, the cofibrant objects are the *flexible algebras* of [3] and were studied long before the connection with model categories was made in [22]. Another source of cofibrant algebras comes from pseudomorphism classifiers: each $Q\mathbf{A}$ is cofibrant. In fact the counit $p_{\mathbf{A}} : Q\mathbf{A} \to \mathbf{A}$ of the adjunction $Q \dashv \iota : \text{T-Alg}_s \leftrightarrows \text{T-Alg}$ is a trivial fibration in T-Alg$_s$; thus $Q\mathbf{A}$ is a cofibrant replacement of $\mathbf{A}$.
Theorem 4.7 of [3] ensures that if $\mathbf{A}$ is flexible then, for all $\mathbf{B}$, the fully faithful inclusion

$$\iota_{\mathbf{A},\mathbf{B}} : \text{T-Alg}_s(\mathbf{A}, \mathbf{B}) \to \text{T-Alg}(\mathbf{A}, \mathbf{B})$$

is essentially surjective on objects: that is, an *equivalence of categories*. This important fact can also be deduced from the model 2-category structure: the inclusion $\iota_{\mathbf{A},\mathbf{B}}$ is isomorphic to $\text{T-Alg}_s(p_{\mathbf{A}}, \mathbf{B}) : \text{T-Alg}_s(\mathbf{A}, \mathbf{B}) \to \text{T-Alg}_s(Q\mathbf{A}, \mathbf{B})$ which is an equivalence since $p_{\mathbf{A}} : Q\mathbf{A} \to \mathbf{A}$ is a weak equivalence of cofibrant objects.

Finally we note that a parallel pair of algebra morphisms $f, g : \mathbf{A} \rightrightarrows \mathbf{B}$ are right homotopic just when they are isomorphic in $\text{T-Alg}_s(\mathbf{A}, \mathbf{B})$. This follows from the fact that for each algebra $\mathbf{B}$ the power algebra $[I, \mathbf{B}]$ is a path object where I is the walking isomorphism. In particular, if A is cofibrant then f and g are homotopic just when they are isomorphic.

6.4.2. *Homotopical behaviour of the skew structure.*

Theorem 6.4. *Let T be an accessible pseudo-commutative 2-monad on* **Cat**.

(1) Then the monoidal skew closed 2-category $(\text{T-Alg}_s, \otimes, [-,-], F1)$ *satisfies Axiom MC and* $(\text{T-Alg}_s, \otimes, F1)$ *is homotopy monoidal.*
(2) If T is symmetric then $(\text{T-Alg}_s, \otimes, F1)$ *is homotopy symmetric monoidal.*

Proof. We observed above that each free algebra is cofibrant. Therefore the unit $F1$ is cofibrant. We now verify Axiom MC in its closed form: in fact we establish the stronger result that for all $\mathbf{A}$ the 2-functor $[\mathbf{A}, -]$ is right Quillen and that $[-, \mathbf{A}]$ preserves all weak equivalences. For the first part consider the equality $U \circ [\mathbf{A}, -] = \text{T-Alg}_s(\mathbf{A}, \iota -)$ of (6.3). Since U reflects weak equivalences and fibrations it suffices to show that $\text{T-Alg}(\mathbf{A}, \iota -) : \text{T-Alg}_s \to \mathbf{Cat}$ is right Quillen. Now $\text{T-Alg}(\mathbf{A}, \iota -) \cong \text{T-Alg}_s(Q\mathbf{A}, -) : \text{T-Alg}_s \to \mathbf{Cat}$ and this last 2-functor is right Quillen since $Q\mathbf{A}$ is cofibrant and T-Alg_s a model 2-category.
For the second part we use the commutativity $U \circ [-, \mathbf{A}] \cong \text{T-Alg}(\iota -, \mathbf{A})$. Arguing as before it suffices to show that $\text{T-Alg}(\iota -, \mathbf{A}) : \text{T-Alg}_s \to \mathbf{Cat}$ preserves all weak equivalences or, equally, that the isomorphic $\text{T-Alg}_s(Q\iota -, \mathbf{A})$ does so. Now if $f : \mathbf{B} \to \mathbf{C}$ is a weak equivalence then $Q\iota f$ is a weak equivalence of cofibrant objects. As $\mathbf{A}$, like all objects, is fibrant and T-Alg_s a model 2-category the functor $\text{T-Alg}_s(Q\iota f, \mathbf{A})$ is an equivalence.
We now apply Theorem 5.12 to establish that T-Alg_s is homotopy monoidal. To verify the three conditions requires only the information on the underlying functors of $[-,-]$, L and i given in Section 6.3.2. Firstly we must show that the underlying function of

$$v_{\mathbf{A},\mathbf{B}} : \text{T-Alg}_s(\mathbf{A}, \mathbf{B}) \to \text{T-Alg}_s(F1, [\mathbf{A}, \mathbf{B}])$$

induces a bijection on homotopy classes of maps for cofibrant $\mathbf{A}$. Since morphisms with cofibrant domain are homotopic just when isomorphic it will suffice to show that $v_{\mathbf{A},\mathbf{B}}$ is an equivalence of categories. To this end consider the composite:

$$\text{T-Alg}_s(\mathbf{A}, \mathbf{B}) \xrightarrow{v_{\mathbf{A},\mathbf{B}}} \text{T-Alg}_s(F1, [\mathbf{A}, \mathbf{B}]) \xrightarrow{\varphi} \mathbf{Cat}(1, \text{T-Alg}(\mathbf{A}, \mathbf{B})) \xrightarrow{ev_\bullet} \text{T-Alg}(\mathbf{A}, \mathbf{B})$$

in which φ is the adjointness isomorphism – recall that $U \circ [-,-] = \text{T-Alg}(-,-)$ – and in which $ev_\bullet$ is the evaluation isomorphism. It suffices to show that the composite is an equivalence. $v_{\mathbf{A},\mathbf{B}}$ sends $f : \mathbf{A} \to \mathbf{B}$ to $[\mathbf{A}, \mathbf{f}] \circ j : F1 \to [\mathbf{A}, \mathbf{A}] \to [\mathbf{A}, \mathbf{B}]$, whose image under φ is the functor $\text{T-Alg}(A, f) \circ \hat{1}_\mathbf{A} : 1 \to \text{T-Alg}(\mathbf{A}, \mathbf{A}) \to \text{T-Alg}(\mathbf{A}, \mathbf{B})$. Evaluating at $\bullet$ thus returns f viewed as a pseudomap. The action on 2-cells is similar and we conclude that the composite is the inclusion $\iota_{\mathbf{A},\mathbf{B}} : \text{T-Alg}_s(\mathbf{A}, \mathbf{B}) \to \text{T-Alg}(\mathbf{A}, \mathbf{B})$. As per Section 6.4.1 this is an equivalence since $\mathbf{A}$ is cofibrant.

Secondly we show that

$$i_{\mathbf{A}} : [F1, \mathbf{A}] \to \mathbf{A}$$

is a weak equivalence for all $\mathbf{A}$: that its underlying functor $i_{\mathbf{A}} : \text{T-Alg}(F1, \mathbf{A}) \to A$ is an equivalence of categories. Since $F1$ is cofibrant this is equally to show that the composite

$$i_{\mathbf{A}} \circ \iota_{F1,\mathbf{A}} : \text{T-Alg}_s(F1, \mathbf{A}) \to \text{T-Alg}(F1, \mathbf{A}) \to A$$

is an equivalence. An easy calculation shows that this is equally the composite $ev_{\bullet} \circ \varphi : \text{T-Alg}_s(F1, \mathbf{A}) \to \mathbf{Cat}(1, A) \to A$ of the canonical adjunction and evaluation isomorphisms. Hence $i_{\mathbf{A}}$ is an equivalence for all $\mathbf{A}$.
Let $u : \mathbf{A} \to [\mathbf{B}, \mathbf{A} \otimes \mathbf{B}]$ denote the unit of the adjunction $- \otimes \mathbf{B} \dashv [\mathbf{B}, -]$. We are to show that the morphism $t_{\mathbf{A},\mathbf{B},\mathbf{C}}$ given by the composite

$$[u, 1] \circ L : [\mathbf{A} \otimes \mathbf{B}, \mathbf{C}] \to [[\mathbf{B}, \mathbf{A} \otimes \mathbf{B}], [\mathbf{B}, \mathbf{C}]] \to [\mathbf{A}, [\mathbf{B}, \mathbf{C}]]$$

is a weak equivalence for cofibrant $\mathbf{A}$ and $\mathbf{B}$. Now the underlying functor of this composite is just the top row below.

$$\begin{tikzcd}
\text{T-Alg}(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}) \arrow[r, "{[\mathbf{B},-]}"] & \text{T-Alg}([\mathbf{B}, \mathbf{A} \otimes \mathbf{B}], [\mathbf{B}, \mathbf{C}]) \arrow[r, "{\text{T-Alg}(u,1)}"] & \text{T-Alg}(\mathbf{A}, [\mathbf{B}, \mathbf{C}]) \\
\text{T-Alg}_s(\mathbf{A} \otimes \mathbf{B}, \mathbf{C}) \arrow[r, "{[\mathbf{B},-]}"] \arrow[u, "\iota"] & \text{T-Alg}_s([\mathbf{B}, \mathbf{A} \otimes \mathbf{B}], [\mathbf{B}, \mathbf{C}]) \arrow[r, "{\text{T-Alg}_s(u,1)}"] \arrow[u, "\iota"] & \text{T-Alg}_s(\mathbf{A}, [\mathbf{B}, \mathbf{C}]) \arrow[u, "\iota"]
\end{tikzcd}$$

In this diagram the left square commutes since $[\mathbf{B}, -]$ restricts from T-Alg to T-Alg_s and the right square since u is a strict algebra map. The outer vertical arrows are equivalences since both $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{A}$ are cofibrant: the former using Axiom MC and the latter by assumption. The bottom row is the adjointness isomorphism so that the top row is an equivalence by two from three.
Finally if T is symmetric then, by Theorem 6.2, the skew closed 2-category $(\text{T-Alg}_s, [-,-], F1)$ is symmetric skew closed. It now follows from Theorem 5.13 that $(\text{T-Alg}_s, \otimes, F1)$ is homotopy symmetric monoidal. ☐

6.4.3. *The monoidal bicategory* T-Alg. A monoidal bicategory is a bicategory $\mathcal{C}$ equipped with a tensor product $\mathcal{C} \times \mathcal{C} \rightsquigarrow \mathcal{C}$ and unit I together with equivalences $\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $l : I \otimes A \to A$ and $r : A \to A \otimes I$ pseudonatural in each variable, and satisfying higher dimensional variants of the axioms for a monoidal category [10]. Note that here we mean equivalences in the *2-categorical or bicategorical sense*, as opposed to weak equivalences.
In particular, each skew monoidal 2-category in which the components α, l and r are equivalences provides an example of a monoidal bicategory. The skew monoidal 2-category $(\text{T-Alg}_s, \otimes, F1)$ is *not* itself a monoidal bicategory.[6] However $(\text{T-Alg}_s, \otimes, F1)$ satisfies Axiom MC and hence, by Proposition 5.10, it satisfies Axiom M. Therefore the skew monoidal structure restricts to the full sub 2-category $(\text{T-Alg}_s)_c$ of cofibrant objects. Since $(\text{T-Alg}_s, \otimes, F1)$ is homotopy monoidal each component α, l and r is a weak equivalence of cofibrant objects.

[6] In fact $l : F1 \otimes \mathbf{A} \to \mathbf{A}$ is an equivalence just when $\mathbf{A}$ is equivalent to a flexible algebra. Such algebras are called semiflexible [3].

Since all objects are fibrant such weak equivalences are homotopy equivalences: thus equivalences in the 2-categorical sense. We conclude:

Proposition 6.5. *The skew monoidal structure* $(\text{T-Alg}_s, \otimes, F1)$ *restricts to a skew monoidal 2-category* $((\text{T-Alg}_s)_c, \otimes, F1)$ *which is a monoidal bicategory.*

In fact $(\text{T-Alg}_s)_c$ is biequivalent to the 2-category T-Alg of algebras and pseudomorphisms.

Lemma 6.6. *The 2-adjunction* $Q \dashv \iota : \text{T-Alg}_s \leftrightarrows \text{T-Alg}$ *restricts to a 2-adjunction* $Q \dashv \iota : (\text{T-Alg}_s)_c \leftrightarrows \text{T-Alg}$ *whose unit and counit are pointwise equivalences. In particular, the composite inclusion* $\iota : (\text{T-Alg}_s)_c \to \text{T-Alg}$ *is a biequivalence.*

Proof. Because each $Q\mathbf{A}$ is cofibrant/flexible the adjunction restricts. The unit $q : \mathbf{A} \rightsquigarrow Q\mathbf{A}$ is an equivalence by Theorem 4.2 of [3]. Since $\mathbf{A}$ is flexible the counit $p_{\mathbf{A}} : Q\mathbf{A} \to \mathbf{A}$ is an equivalence in T-Alg_s by Theorem 4.4 of *ibid.* □

Just as monoidal structure can be transported along an adjoint equivalence of categories, so the structure of a monoidal bicategory may be transported along an adjoint biequivalence. And we obtain the following result: see Theorem 14 of [13]. The present argument has the advantage of dealing solely with the strict concepts of **Cat**-enriched category theory until the last possible moment.

Theorem 6.7. *For T an accessible pseudo-commutative 2-monad on* **Cat** *the 2-category* T-Alg *admits the structure of a monoidal bicategory.*

7. Bicategories

We now return to the skew closed category $(\text{Bicat}_s, Hom, F1)$ of Section 4.2 and show that it forms part of a monoidal skew closed category that is homotopy symmetric monoidal. A similar, but simpler, analysis yields the corresponding result for the skew structure on 2-Cat_s discussed in Section 4.2 – this is omitted.

7.1. **Preliminaries on** Bicat_s**.** To begin with, it will be helpful to discuss some generalities concerning homomorphism classifiers and the algebraic nature of Bicat_s.

To this end, let us recall that the category **Cat**-Gph of **Cat**-enriched graphs is naturally a *2-category* – called **CG** in [23]. **CG** is locally presentable as a 2-category: that is, cocomplete as a 2-category and its underlying category **Cat**-Gph is locally presentable. Section 4 of *ibid.* describes a filtered colimit preserving 2-monad T on **CG** whose strict algebras are the bicategories, and whose strict morphisms and pseudomorphisms are the strict homomorphisms and homomorphisms respectively. The algebra 2-cells are called *icons* [24]. We write Icon_s and Icon_p for the corresponding extensions of Bicat_s and Bicat to 2-categories with icons as 2-cells. It follows from [3] that the inclusion $\iota : \text{Icon}_s \to \text{Icon}_p$ has a left 2-adjoint Q: this assigns to a bicategory A its *homomorphism classifier* QA.

As mentioned **Cat**-Gph is locally presentable. Since T preserves filtered colimits it follows that the category of algebras Bicat_s is locally presentable too, and that the forgetful right adjoint $U : \text{Bicat}_s \to$ **Cat**-Gph preserve limits and filtered colimits. Now the three functors from **Cat**-Gph to **Set** sending a **Cat**-graph to its set of (0/1/2)-cells respectively are represented by finitely presentable **Cat**-graphs. It follows that the composite of each of these with U – the functors

$$(-)_0, (-)_1, (-)_2 : \mathrm{Bicat}_s \to \mathbf{Set}$$

sending a bicategory to its set of (0/1/2)-cells – preserves limits and filtered colimits. Now a functor between locally presentable categories has a left adjoint just when it preserves limits and is accessible: preserves λ-filtered colimits for some regular cardinal λ. See, for instance, Theorem 1.66 of [1]. It follows that each of the above three functors has a left adjoint – we used the adjoint F to $(-)_0$ to construct the unit $F1$ in Section 4.2.

7.2. **The monoidal skew closed structure on** Bicat_s**.** Our goal now is to show that $Hom(A, -) : \mathrm{Bicat}_s \to \mathrm{Bicat}_s$ has a left adjoint for each A. We will establish this by showing that $Hom(A, -)$ preserves limits and is accessible. As pointed out above, the functors $(-)_0, (-)_1, (-)_2 : \mathrm{Bicat}_s \to \mathbf{Set}$ preserve limits and filtered colimits. Since they jointly reflect isomorphisms they also jointly reflect limits and filtered colimits. Accordingly it will be enough to show that the three functors

$$Hom(A, -)_0, Hom(A, -)_1, Hom(A, -)_2 : \mathrm{Bicat}_s \to \mathbf{Set}$$

preserve limits and are accessible. We argue case by case.

(1) $Hom(A, B)_0$ is the set of homomorphisms from A to B. Hence $Hom(A, -)_0$ is naturally isomorphic to $\mathrm{Bicat}_s(QA, -)$ where Q is the homomorphism classifier. Like any representable functor $Hom(A, -)_0$ preserves limits and is accessible.

(2) $Hom(A, B)_1$ is the set of pseudonatural transformations between homomorphisms. Let $Cyl(B)$ denote the following bicategory – first constructed, in the lax case, in [2] . The objects of $Cyl(B)$ are the morphisms of B whilst morphisms $(r, s, \theta) : f \to g$ are diagrams as below left

(7.1)
$$\begin{tikzcd} a \arrow[r, "r"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & b \arrow[d, "g"] \\ c \arrow[r, "s"'] & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "r"] \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\theta'}{\Rightarrow}"] & b \arrow[d, bend left=30, "g'"] \\ c \arrow[r, "s"'] & d \end{tikzcd} = \begin{tikzcd} a \arrow[r, "r"] \arrow[d, bend right=30, "f"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & b \arrow[d, bend right=50, "g"'] \arrow[d, bend left=50, "g'"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] \\ c \arrow[r, "s"'] & d \end{tikzcd}$$

in which θ is invertible. 2-cells of $Cyl(B)$ consist of pairs of 2-cells (α, β) satisfying the equality displayed above right. Note that here are strict projection homomorphisms $d, c : Cyl(B) \rightrightarrows B$ which, on objects, respectively select the domain and codomain of an arrow.

It is straightforward to see that we have a natural isomorphism of functors $Hom(A, -)_1 \cong \mathrm{Bicat}(A, Cyl(-))$. Combining this with $\mathrm{Bicat}(A, -) \cong \mathrm{Bicat}_s(QA, -)$ gives an isomorphism $Hom(A, -)_1 \cong \mathrm{Bicat}_s(QA, Cyl(-))$. Since this is the composite $\mathrm{Bicat}_s(QA, -) \circ Cyl(-) : \mathrm{Bicat}_s \to \mathrm{Bicat}_s \to \mathbf{Set}$ whose second component is representable, it will suffice to show that $Cyl(-)$ preserves limits and is accessible.

For this, arguing as before, it is enough to show that each of $Cyl(-)_0$, $Cyl(-)_1$ and $Cyl(-)_2$ preserves limits and is accessible. Certainly we have

$Cyl(B)_0 \cong B_1$ naturally in B and $(-)_1$ preserves limits and filtered colimits. We construct $Cyl(-)_1$ as a finite limit in four stages. These stages correspond to the sets constructed below:

$$\begin{aligned}
&Opp(B_2) = \{(x,y) : x,y \in B_2, sx = ty, tx = sy\}\\
&Comp(B_1) = \{(a,b) : a,b \in B_1, ta = sb\}\\
&Iso(B_2) = \{(x,y) \in Opp(B_2) : y \circ x = i_{ty}, x \circ y = i_{tx}\}\\
&Cyl(B)_1 = \{(a,b,c,d,x,y) : (x,y) \in Iso(B_2),\\
&\qquad\qquad (a,b),(c,d) \in Comp(B_1), sx = b \circ a, tx = d \circ c\}
\end{aligned}$$

the first three of which, in turn, define the sets of pairs of 2-cells pointing in the opposite direction, of composable pairs of 1-cells, and of invertible 2-cells. Each stage corresponds to the finite limit in $\mathbf{CAT}(\mathrm{Bicat}_s, \mathbf{Set})$ below.

$$\begin{tikzcd}
Opp(B_2) \arrow[r] & (B_2)^2 \arrow[r, shift left, "(sx,tx)"] \arrow[r, shift right, "(ty,sy)"'] & (B_1)^2
\end{tikzcd}$$

$$\begin{tikzcd}
Comp(B_1) \arrow[r] \arrow[d] & B_1 \arrow[d, "s"'] \\
B_1 \arrow[r, "t"] & B_0
\end{tikzcd}
\quad
\begin{tikzcd}
Iso(B_2) \arrow[r] \arrow[d] & Opp(B_2) \arrow[d, "(y\circ x, x\circ y)"'] \\
(B_1)^2 \arrow[r, "i^2"] & (B_2)^2
\end{tikzcd}
\quad
\begin{tikzcd}
Cyl(B)_1 \arrow[r] \arrow[d] & Iso(B_2) \arrow[d, "(sx,tx)"'] \\
Comp(B_1)^2 \arrow[r, "(b\circ a, d\circ c)"] & (B_1)^2
\end{tikzcd}$$

Now each of the functors $(-)_0$,$(-)_1$ and $(-)_2$ preserve finite limits and filtered colimits. Since finite limits commute with limits and filtered colimits in **Set** it follows that each constructed functor, and in particular, $Cyl(-)_1$ preserves limits and filtered colimits. Another pullback followed by an equaliser constructs $Cyl(-)_2$ and shows it to have the same preservation properties: we leave this case to the reader.

(3) Finally observe that we can express $Hom(A,B)_2$ as the equaliser of the two functions

$$Hom(A, Cyl(B))_1 \rightrightarrows \mathrm{Bicat}(A,B)^2$$

sending an element $\alpha : f \Rightarrow g$ of $Hom(A, Cyl(B))_1$ to the pair (df, cf) and (dg, cg) respectively. This is natural in B. Therefore $Hom(A,-)_2$ is a finite limit of functors, each of which has already been shown to preserve limits and be accessible. Since finite limits in **Set** commute with limits and with λ-filtered colimits for each regular cardinal λ it follows that $Hom(A,-)_2$ preserves limits and is itself accessible.

We conclude:

Proposition 7.1. *For each bicategory A the functor $Hom(A,-) : \mathrm{Bicat}_s \to \mathrm{Bicat}_s$ has a left adjoint $- \otimes A$. In particular we obtain a monoidal skew closed category* $(\mathrm{Bicat}_s, \otimes, Hom, F1)$.

7.3. **Homotopical behaviour of the skew structure.** We turn to the homotopical aspects of the skew structure.

7.3.1. *The model structure on* Bicat_s. A 1-cell $f : X \to Y$ in a bicategory A is said to be an *equivalence* if there exists $g : Y \to X$ and isomorphisms $1_X \cong gf$ and $1_Y \cong fg$. Now a homomorphism of bicategories $F : A \rightsquigarrow B$ is said to be a biequivalence if it is essentially surjective up to equivalence (given $Y \in B$ there exists $X \in A$ and an equivalence $Y \to FX$) and locally an equivalence: each functor $F_{X,Y} : A(X,Y) \to B(FX,FY)$ is an equivalence of categories.
The relevant model structure on Bicat_s was constructed in [21]. The weak equivalences are those strict homomorphisms that are biequivalences. A strict homomorphism $F : A \to B$ is said to be a fibration if it has the following two properties (1) if $f : Y \to FX$ is an equivalence then there exists an equivalence $f^\star : Y^\star \to X$ with $Ff^\star = f$ and (2) each $F_{X,Y} : A(X,Y) \to B(FX,FY)$ is an isofibration of categories. We note that all objects are fibrant.
The only knowledge that we require of the cofibrant objects is that each homomorphism classifier QA is cofibrant. To see this observe that if $f : A \to B$ is a trivial fibration then there exists a homomorphism $g : B \rightsquigarrow A$ with $f \circ g = 1$. Since the inclusion $\iota : \mathrm{Bicat}_s \to \mathrm{Bicat}$ sends each trivial fibration to a split epimorphism, and since split epis can be lifted through any object, an adjointness argument applied to $Q \dashv \iota$ shows that each QA is cofibrant. By Theorem 4.2 of [3] the counit $p_A : QA \to A$ is a surjective equivalence – equivalence plus split epi – in the 2-category Icon_p. Therefore p_A is a trivial fibration and so exhibits QA as a *cofibrant replacement* of A.
The right homotopy relation on $\mathrm{Bicat}_s(A,B)$ is *equivalence* in the bicategory $Hom(A,B)$. Where needed, we will use the term *pseudonatural equivalence* for clarity. We note that a morphism $\eta : F \to G \in Hom(A,B)$ is an equivalence just when each component $\eta_X : FX \to GX$ is an equivalence in B. That pseudonatural equivalence coincides with right homotopy follows from the fact, used in *ibid.*, that the full sub-bicategory PB of $Cyl(B)$, with objects the equivalences, is a path object for B. In particular, if A is a cofibrant bicategory then $F, G : A \rightrightarrows B$ are homotopic just when they are equivalent in $Hom(A,B)$.

7.3.2. *Homotopy monoidal structure.* Finally, we are in a position to prove the main theorem of this section.

Theorem 7.2. *The monoidal skew closed structure* $(\mathrm{Bicat}_s, \otimes, Hom, F1)$ *satisfies Axiom MC and is homotopy symmetric monoidal.*

Proof. Firstly we show that the unit $F1$ is cofibrant. Recall that F is left adjoint to $(-)_0 : \mathrm{Bicat}_s \to \mathbf{Set}$. Since $(-)_0$ sends trivial fibrations to surjective functions, and since surjective functions can be lifted through 1, it follows by adjointness that $F1$ is cofibrant.
In order to verify the remainder of Axiom MC we use the well known fact, see for example [34], that a homomorphism $F : A \rightsquigarrow B$ is a biequivalence if and only if there exists $G : B \rightsquigarrow A$ and equivalences $1_A \to GF$ and $1_B \to FG$. A consequence is that if $F : A \to B$ is a biequivalence then so is $Hom(C,F)$ and $Hom(F,D)$ for all C and D.
To verify Axiom MC, it remains to show that if C is cofibrant and F a fibration, then $Hom(C,F)$ is a fibration: in fact, we will show that this is true for all C. To see that $Hom(C,F) : Hom(C,A) \to Hom(C,B)$ is locally an isofibration,

consider $\alpha : G \to H \in Hom(C, A)$ and $\theta : \beta \cong F\alpha$. Then each component θ_X is invertible in B and so lifts along F as depicted below.

$$\begin{tikzcd} GX \arrow[r, bend left, "\beta^\star_X"] \arrow[r, bend right, "\alpha_X"'] \arrow[r, phantom, "\Downarrow\theta^\star_X"] & HX \arrow[r, mapsto, "F"] & FGX \arrow[r, bend left, "\beta_X"] \arrow[r, bend right, "F\alpha_X"'] \arrow[r, phantom, "\Downarrow\theta_X"] & FHX \end{tikzcd}$$

The components $\beta^\star_X : GX \to HX$ admit a unique extension to a pseudonatural transformation $\beta^\star$ such that $\theta^\star : \beta^\star \to \alpha$ is a modification: at $f : X \to Y$ the 2-cell $\beta^\star{}_f$ is given by:

$$Hf \circ \beta^\star_X \xRightarrow{Hf \circ \theta^\star_X} Hf \circ \alpha_X \xRightarrow{\alpha_f} \alpha_Y \circ Gf \xRightarrow{(\theta^\star_Y)^{-1} \circ Gf} \beta^*_Y \circ Gf$$

Then $F\beta^\star = \beta$ and we conclude that $Hom(C, F)$ is locally an isofibration.
It remains to show that $Hom(C, F)$ has the equivalence lifting property. So consider $G : C \to A$ and an equivalence $\alpha : H \to FG \in Hom(C, B)$: a pseudonatural transformation with each component $\alpha_X : HX \to FGX$ an equivalence in B. Since F is a fibration there exists an equivalence $\beta_X : H^\star X \to GX \in A$ with $F\beta_X = \alpha_X$. Each such equivalence forms part of an adjoint equivalence $(\eta_x, \beta_X \dashv \rho_x, \epsilon_x)$ and at $f : X \to Y$ we define $H^\star(f) : H^\star X \to H^\star Y$ as the conjugate

$$H^\star X \xrightarrow{\beta_X} GX \xrightarrow{Gf} GY \xrightarrow{\rho_Y} H^\star Y$$

in which we take, as a matter of convention, this to mean $(\rho_Y \circ Gf) \circ \beta_X$. With the evident extension to 2-cells $H^\star$ becomes a homomorphism. Moreover the morphisms β_X naturally extend to an equivalence $\beta : H^\star \to G \in Hom(C, A)$.
Although $FH^\star X = HX$ for all X it is not necessarily the case that $Hf = FH^\star f$. Rather, we only have invertible 2-cells $\varphi_f : Hf \cong FH^\star f$ corresponding to the pasting diagram below.

$$\begin{tikzcd} & HY \arrow[dr, "\alpha_Y"] \arrow[drr, bend left, "1_Y"] \arrow[drr, phantom, "\Downarrow F\eta_Y"] & & \\ HX \arrow[ur, "Hf"] \arrow[dr, "\alpha_X"'] \arrow[rr, phantom, "\Downarrow(\alpha_f)^{-1}"] & & FGY \arrow[r, "F\rho_Y"'] & HY \\ & FGX \arrow[ur, "FGf"'] & & \end{tikzcd}$$

Indeed $\varphi : H \cong FH^\star$ is an invertible icon in the sense of [24]. Since F is locally an isofibration these lift to invertible 2-cells $\varphi^\star(f) : H^{\star\star}(f) \cong H^\star f$. Moreover $H^{\star\star}$ becomes a homomorphism, unique such that the above 2-cells yield an invertible icon $\varphi^\star : H^{\star\star} \cong H^\star$. Composing $\varphi^\star : H^{\star\star} \cong H^\star$ and $\beta : H^\star \to G$ gives the sought after lifted equivalence. This completes the verification of Axiom MC.
From Section 4.2 we know that $(\mathrm{Bicat}_s, Hom, F1)$ forms a symmetric skew closed category. According to Theorem 5.13 the skew monoidal $(\mathrm{Bicat}_s, \otimes, F1)$ will form part of a homotopy symmetric monoidal category so long as $(\mathrm{Bicat}_s, Hom, F1)$ is homotopy closed.
Firstly we show that $i = ev_\bullet : Hom(F1, A) \to A$ is a biequivalence for each A. As

pointed out in Section 4.2 $F1$ has a single object and each parallel pair of 1-cells is connected by a unique invertible 2-cell. Therefore the map $! : F1 \to 1$ is a biequivalence. Hence $Hom(!, A) : Hom(1, A) \to Hom(F1, A)$ is a biequivalence whereby it suffices to show that the composite $ev_\bullet \circ Hom(!, A)$ is a biequivalence. This is just $ev_\bullet : Hom(1, A) \to A$. It is straightforward, albeit tedious, to verify that this last map is a biequivalence directly. For a quick proof we can use the fact that for each bicategory A there is a strict 2-category $st(A)$ and biequivalence $p : A \rightsquigarrow st(A)$. Since evaluation is natural in all homomorphisms the square below left commutes

$$\begin{tikzcd}
Hom(1,A) \arrow[r, squiggly, "Hom(1,p)"] \arrow[d, "ev_\bullet"'] & Hom(1,st(A)) \arrow[d, "ev_\bullet"'] & Ps(1,st(A)) \arrow[l, "\iota"'] \arrow[dl, "ev_\bullet"] \\
A \arrow[r, squiggly, "p"'] & st(A) &
\end{tikzcd}$$

and since both horizontal arrows are biequivalences it suffices to show that $ev_\bullet : Hom(1, st(A)) \to st(A)$ is a biequivalence. Now let $Ps(1, st(A)) \to Hom(1, st(A))$ be the full sub 2-category containing the 2-functors. It is easy to see that ι is essentially surjective up to equivalence – 1 *is* a cofibrant 2-category! – and hence a biequivalence. Therefore we need only show that the composite $ev_\bullet : Ps(1, st(A)) \to st(A)$ is a biequivalence. It is an isomorphism.

Finally we show that the function $v : \mathrm{Bicat}_s(A, B) \to \mathrm{Bicat}_s(F1, Hom(A, B))$ given by the composite

$$\mathrm{Bicat}_s(A,B) \xrightarrow{Hom(A,-)} \mathrm{Bicat}_s(Hom(A,A), Hom(A,B)) \xrightarrow{\mathrm{Bicat}_s(j,1)} \mathrm{Bicat}_s(F1, Hom(A,B))$$

is a bijection on homotopy classes of maps for each cofibrant A. Firstly consider the strict homomorphism

$$Hom(A,B) \xrightarrow{L} Hom(Hom(A,A), Hom(A,B)) \xrightarrow{Hom(j,1)} Hom(F1, Hom(A,B))$$

of bicategories. By (C2) it composes with $i : Hom(F1, Hom(A, B)) \to Hom(A, B)$ to give the identity. Since this last map is a biequivalence so too is $Hom(j, 1) \circ L$ by two from three. It follows that its underlying function $\mathrm{Bicat}(j, 1) \circ Hom(A, -) : \mathrm{Bicat}(A, B) \to \mathrm{Bicat}(F1, Hom(A, B))$ induces a bijection on equivalence classes of objects: pseudonatural equivalence classes of homomorphisms.

Now we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Bicat}_s(A,B) \arrow[r, "\mathrm{Bicat}_s(j,1)\circ Hom(A,-)"] \arrow[d, "\iota"'] & \mathrm{Bicat}_s(F1, Hom(A,B)) \arrow[d, "\iota"] \\
\mathrm{Bicat}(A,B) \arrow[r, "\mathrm{Bicat}(j,1)\circ Hom(A,-)"] & \mathrm{Bicat}(F1, Hom(A,B))
\end{tikzcd}$$

in which the vertical functions are the inclusions. Each of the four functions is well defined on pseudonatural equivalence classes: it follows, by two from three, that the top function will determine a bijection on pseudonatural equivalence classes if the two vertical inclusions do so. More generally, if X is a cofibrant bicategory the inclusion $\iota_{X,Y} : \mathrm{Bicat}_s(X, Y) \to \mathrm{Bicat}(X, Y)$ induces a bijection on pseudonatural equivalence classes. For we can identify this inclusion, up to isomorphism, with the function

$$\mathrm{Bicat}_s(p_X, 1) : \mathrm{Bicat}_s(X, Y) \to \mathrm{Bicat}_s(QX, Y)$$

where $p_X : QX \to X$ is the counit of the adjunction $Q \dashv \iota$. Since $p_X : QX \to X$ exhibits QX as a cofibrant replacement of X, and so is a weak equivalence between cofibrant objects, it follows – see, for instance, Proposition 1.2.5 of [12] – that $\mathrm{Bicat}_s(p_X, 1)$ induces a bijection on homotopy classes, that is, pseudonatural equivalence classes, of morphisms.
From Section 4.2 we know that $(\mathrm{Bicat}_s, Hom, F1)$ forms a symmetric skew closed category. Since it is homotopy closed we conclude from Theorem 5.13 that the skew monoidal $(\mathrm{Bicat}_s, \otimes, F1)$ is homotopy symmetric monoidal. $\square$

8. Appendix

8.1. **Proof of Theorem 5.11.** The isomorphism

$$\varphi^d : Ho(\mathcal{C})(A \otimes_l B, C) \cong Ho(\mathcal{C})(A, [B, C]_r)$$

given by the composite

$$Ho(\mathcal{C})(QA \otimes QB, C) \xrightarrow{(1,q)} Ho(\mathcal{C})(QA \otimes QB, RC) \xrightarrow{\varphi} Ho(\mathcal{C})(QA, [QB, RC]) \xrightarrow{(p,1)^{-1}} Ho(\mathcal{C})(A, [QB, RC])$$

is natural in each variable in $Ho(\mathcal{C})$ since each component is natural in $\mathcal{C}$.
Now the left and right derived structures have components $(Ho(\mathcal{C}), \otimes_l, I, \alpha_l, l_l, r_l)$ and $(Ho(\mathcal{C}), [-,-]_r, I, L_r, i_r, j_r)$ respectively. We must prove that these components are related by the equations (2.2), (2.3) and (2.5) of Section 2.3.
For (2.2) we must show that the diagram

$$\begin{tikzcd} Ho(\mathcal{C})(A, B) \arrow[r, "(l_l,1)"] \arrow[dr, "v_r"'] & Ho(\mathcal{C})(QI \otimes QA, B) \arrow[d, "\varphi^d"] \\ & Ho(\mathcal{C})(I, [QA, RB]) \end{tikzcd} \tag{8.1}$$

commutes for all A and B. By naturality it suffices to verify commutativity in the case that A is cofibrant and B is fibrant. By definition v_r is the composite

$$Ho(\mathcal{C})(A, B) \xrightarrow{Ho([QA, R-])} Ho(\mathcal{C})([QA, RA], [QA, RB]) \xrightarrow{([p,q],1)} Ho(\mathcal{C})([A, A], [QA, RB]) \xrightarrow{(j,1)} Ho(\mathcal{C})(I, [QA, RB])$$

Since A is cofibrant and B fibrant we can identify $Ho(\mathcal{C})(A, B)$ with the set of homotopy classes $[f] : A \to B$ of morphisms $f : A \to B$; then $v_r([f])$ is the homotopy class of

$$I \xrightarrow{j} [A, A] \xrightarrow{[p,q]} [QA, RA] \xrightarrow{[1,Rf]} [QA, RB]$$

which, by naturality of p and q, coincides with the homotopy class of

$$I \xrightarrow{j} [A, A] \xrightarrow{[1,f]} [A, B] \xrightarrow{[p,q]} [QA, RB] \qquad .$$

Therefore the shorter path in the diagram below is v_r. The longer path below is, by definition, the longer path of the triangle (8.1). Accordingly we must show that the following diagram commutes.

(8.2)
$$\begin{tikzcd}
Ho(\mathcal{C})(A,B) \arrow[r, "{(p,1)}"] \arrow[dr, "{(p,q)}"'] \arrow[dd, "v"'] & Ho(\mathcal{C})(QA,B) \arrow[r, "{(l,1)}"] \arrow[d, "{(1,q)}"] & Ho(\mathcal{C})(I\otimes QA,B) \arrow[r, "{(p\otimes 1,1)}"] \arrow[d, "{(1,q)}"'] & Ho(\mathcal{C})(QI\otimes QA,B) \arrow[d, "{(1,q)}"] \\
 & Ho(\mathcal{C})(QA,RB) \arrow[r, "{(l,1)}"] \arrow[dr, "v"'] & Ho(\mathcal{C})(I\otimes QA,RB) \arrow[r, "{(p\otimes 1,1)}"] \arrow[d, "\varphi"] & Ho(\mathcal{C})(QI\otimes QA,RB) \arrow[d, "\varphi"] \\
Ho(\mathcal{C})(I,[A,B]) \arrow[rr, "{(1,[p,q])}"] & & Ho(\mathcal{C})(I,[QA,RB]) \arrow[r, "{(p,1)}"] \arrow[dr, "1"'] & Ho(\mathcal{C})(QI,[QA,RB]) \arrow[d, "{(p^{-1},1)}"] \\
 & & & Ho(\mathcal{C})(I,[QA,RB]
\end{tikzcd}$$

Each object above is of the form $Ho(\mathcal{C})(X,Y)$ for X cofibrant and Y fibrant and we view each $Ho(\mathcal{C})(X,Y)$ as the set of homotopy classes of maps from X to Y. The morphisms are of two kinds. Firstly there are those of the form $Ho(\mathcal{C})(f,1)$ or $Ho(\mathcal{C})(1,f)$ for f a morphism of $\mathcal{C}$. Such morphisms respect the homotopy relation and we view them as acting on homotopy classes. The other morphisms are of the form v or φ and, because A is cofibrant and B fibrant, each occurence is well defined on homotopy classes. Accordingly, to verify that the above diagram commutes it suffices to verify that each sub-diagram commutes. Now apart from the commutative triangle on the bottom right, each sub-diagram of (8.2) consists of a diagram involving the hom-sets of $\mathcal{C}$, but with components viewed as acting on homotopy classes. Since in $\mathcal{C}$ itself these sub-diagrams commute, by naturality or (2.2), they certainly commute when viewed as acting on homotopy classes. Therefore (8.2) commutes.

According to (2.3) we must show that

(8.3)
$$\begin{tikzcd}
Ho(\mathcal{C})(QA\otimes QI,B) \arrow[d, "\varphi^d"'] \arrow[dr, "{(r_l,1)}"] & \\
Ho(\mathcal{C})(A,[QI,RB]) \arrow[r, "{(1,i_r)}"'] & Ho(\mathcal{C})(A,B)
\end{tikzcd}$$

commutes for each A and B. Note that in $Ho(\mathcal{C})$ the morphism $1\otimes e : QA\otimes I \to QA\otimes QI$ is inverse to $1\otimes p : QA\otimes QI \to QA\otimes I$. Accordingly we can rewrite r_l as

$$A \xrightarrow{p^{-1}} QA \xrightarrow{r} QA\otimes I \xrightarrow{1\otimes e} QA\otimes QI$$

by substituting $1 \otimes e$ for $(1 \otimes p)^{-1}$. The following diagram then establishes the commutativity of (8.3).

$$\begin{tikzcd}
Ho(\mathcal{C})(QA{\otimes}QI,B) \arrow[r, "(1\otimes e{,}1)"] \arrow[d, "(1{,}q)"'] & Ho(\mathcal{C})(QA{\otimes}I,B) \arrow[d, "(1{,}q)"'] \arrow[dr, "(r{,}1)"] & & & \\
Ho(\mathcal{C})(QA{\otimes}QI,RB) \arrow[r, "(1\otimes e{,}1)"] \arrow[d, "\varphi"'] & Ho(\mathcal{C})(QA{\otimes}I,RB) \arrow[d, "\varphi"'] \arrow[dr, "(r{,}1)"] & Ho(\mathcal{C})(QA,B) \arrow[d, "(1{,}q)"'] \arrow[dr, "(p^{-1}{,}1)"] & & \\
Ho(\mathcal{C})(QA,[QI,RB]) \arrow[r, "(1{,}[e{,}1])"] \arrow[d, "(p^{-1}{,}1)"'] & Ho(\mathcal{C})(QA,[I,RB]) \arrow[r, "(1{,}i)"'] \arrow[d, "(p^{-1}{,}1)"'] & Ho(\mathcal{C})(QA,RB) \arrow[dr, "(p^{-1}{,}1)"'] & Ho(\mathcal{C})(A,B) \arrow[d, "(1{,}q)"'] \arrow[dr, "1"] & \\
Ho(\mathcal{C})(A,[QI,RB]) \arrow[r, "(1{,}[e{,}1])"'] & Ho(\mathcal{C})(A,[I,RB]) \arrow[rr, "(1{,}i)"'] & & Ho(\mathcal{C})(A,RB) \arrow[r, "(1{,}q^{-1})"'] & Ho(\mathcal{C})(A,B)
\end{tikzcd}$$

Next we calculate that $t^d : [Q(QA{\otimes}QB), RC] \to [QA, R[QB, RC]]$ as constructed in (2.4) has value:

$$[Q(QA{\otimes}QB),RC] \xrightarrow{[p,1]^{-1}} [QA{\otimes}QB,RC] \xrightarrow{t} [QA,[QB,RC]] \xrightarrow{[1,q]} [QA,R[QB,RC]].$$

This calculation is given overleaf by the commutative diagram (8.4). All subdiagrams of (8.4) commute in a routine manner. Apart from basic naturalities we use the defining equation $t = [u, 1] \circ L$ of (2.4), the equation $[1, p] \circ k = p$ of (5.4) and naturality of k as in (5.9). Furthermore, on the bottom right corner, we use that the morphisms $[Qp_A, 1], [p_{QA}, 1] : [QA, R[QB, RC]] \rightrightarrows [[QQA, R[QB, RC]]$ coincide in $Ho(\mathcal{C})$. To see that this is so we argue as in the proof of Lemma 5.4. Namely, p_{QA} and Qp_A are left homotopic because they are coequalised by the trivial fibration p_A in $\mathcal{C}$, and, since $R[QB, RC]$ is fibrant, the desired equality follows.

Finally, we use the calculation of t^d to prove that the diagram

$$\begin{tikzcd}
Ho(\mathcal{C})(QA \otimes Q(QB \otimes QC), D) \arrow[r, "Ho(\mathcal{C})(\alpha_l{,}1)"] \arrow[dd, "\varphi^d"'] & Ho(\mathcal{C})(Q(QA \otimes QB) \otimes QC, D) \arrow[d, "\varphi^d"] \\
 & Ho(\mathcal{C})(QA \otimes QB, [QC, RD]) \arrow[d, "\varphi^d"] \\
Ho(\mathcal{C})(A, [QB \otimes QC, RD]) \arrow[r, "Ho(\mathcal{C})(1{,}t^d)"'] & Ho(\mathcal{C})(A, [QB, R[QC, RD]])
\end{tikzcd}$$

instantiating (2.5) commutes for all A, B, C and D. This is established overleaf in the large, but straightforward, commutative diagram (8.5) whose only non-trivial step is an application of (2.5) in $\mathcal{C}$ itself.

$$
\begin{tikzcd}
{[Q(QA{\otimes}QB),RC]} \arrow[r, "{[Qq,1]^{-1}}"] \arrow[ddd, "{[p,1]^{-1}}"'] \arrow[dr, "1"'] & {[QR(QA{\otimes}QB),RC]} \arrow[r, "L"] \arrow[d, "{[Qq,1]}"] & {[[QB,QR(QA{\otimes}QB)],[QB,RC]]} \arrow[r, "{[k,1]}"] \arrow[d, "{[[1,Qq],1]}"] & {[Q[QB,R(QA{\otimes}QB)],[QB,RC]]} \arrow[r, "{[1,q]}"] \arrow[d, "{[Q[1,q],1]}"] & {[Q[QB,R(QA{\otimes}QB)],R[QB,RC]]} \arrow[d, "{[Q[1,q],1]}"] \\
 & {[Q(QA{\otimes}QB),RC]} \arrow[r, "L"] & {[[QB,Q(QA{\otimes}QB)],[QB,RC]]} \arrow[r, "{[k,1]}"] & {[Q[QB,QA{\otimes}QB],[QB,RC]]} \arrow[r, "{[1,q]}"] \arrow[d, "{[Qu,1]}"] & {[Q[QB,QA{\otimes}QB],R[QB,RC]]} \arrow[d, "{[Qu,1]}"] \\
 & & {[[QB,QA{\otimes}QB],[QB,RC]]} \arrow[u, "{[[1,p],1]}"] \arrow[ur, "{[p,1]}"] \arrow[d, "{[u,1]}"] & {[QQA,[QB,RC]]} \arrow[r, "{[1,q]}"] & {[QQA,R[QB,RC]]} \\
{[QA{\otimes}QB,RC]} \arrow[uur, "{[p,1]}"] \arrow[urr, "L"] \arrow[rr, "t"'] & & {[QA,[QB,RC]]} \arrow[ur, "{[p,1]}"] \arrow[rr, "{[1,q]}"'] & & {[QA,R[QB,RC]]} \arrow[u, "{[p_{QA},1]=[Qp_A,1]}"']
\end{tikzcd}
\tag{8.4}
$$

$$
\begin{tikzcd}[column sep=small]
Ho(\mathcal{C})(QA\otimes Q(QB\otimes QC),D) \arrow[r, "((1\otimes p)^{-1}{,}1)"] \arrow[dd, "(1{,}q)"'] & Ho(\mathcal{C})(QA\otimes(QB\otimes QC),D) \arrow[r, "(\alpha{,}1)"] \arrow[dd, "(1{,}q)"'] & Ho(\mathcal{C})((QA\otimes QB)\otimes QC,D) \arrow[r, "(p\otimes 1{,}1)"] \arrow[d, "(1{,}q)"] & Ho(\mathcal{C})(Q(QA\otimes QB)\otimes QC,D) \arrow[d, "(1{,}q)"] \\
 & & Ho(\mathcal{C})((QA\otimes QB)\otimes QC,RD) \arrow[r, "(p\otimes 1{,}1)"] \arrow[dd, "\varphi"'] & Ho(\mathcal{C})(Q(QA\otimes QB)\otimes QC,RD) \arrow[d, "\varphi"] \\
Ho(\mathcal{C})(QA\otimes Q(QB\otimes QC),RD) \arrow[r, "((1\otimes p)^{-1}{,}1)"] \arrow[dd, "\varphi"'] & Ho(\mathcal{C})(QA\otimes(QB\otimes QC),RD) \arrow[ur, "(\alpha{,}1)"] \arrow[dd, "\varphi"'] & & Ho(\mathcal{C})(Q(QA\otimes QB),[QC,RD]) \arrow[d, "(p^{-1}{,}1)"] \\
 & & Ho(\mathcal{C})(QA\otimes QB,[QC,RD]) \arrow[ur, "(p{,}1)"] \arrow[r, "1"] \arrow[dr, "(1{,}q)"'] \arrow[dd, "\varphi"'] & Ho(\mathcal{C})(QA\otimes QB,[QC,RD]) \arrow[d, "(1{,}q)"] \\
Ho(\mathcal{C})(QA,[Q(QB\otimes QC),RD]) \arrow[r, "(1{,}[p{,}1]^{-1})"] \arrow[dd, "(p^{-1}{,}1)"'] & Ho(\mathcal{C})(QA,[QB\otimes QC,RD]) \arrow[dr, "(1{,}t)"'] \arrow[dd, "(p^{-1}{,}1)"'] & & Ho(\mathcal{C})(QA\otimes QB,R[QC,RD]) \arrow[d, "\varphi"] \\
 & & Ho(\mathcal{C})(QA,[QB,[QC,RD]]) \arrow[r, "(1{,}[1{,}q])"] \arrow[d, "(p^{-1}{,}1)"] & Ho(\mathcal{C})(QA,[QB,R[QC,RD]]) \arrow[d, "(p^{-1}{,}1)"] \\
Ho(\mathcal{C})(A,[Q(QB\otimes QC),RD]) \arrow[r, "(1{,}[p{,}1]^{-1})"'] & Ho(\mathcal{C})(A,[QB\otimes QC,RD]) \arrow[r, "(1{,}t)"'] & Ho(\mathcal{C})(A,[QB,[QC,RD]]) \arrow[r, "(1{,}[1{,}q])"'] & Ho(\mathcal{C})(A,[QB,R[QC,RD]])
\end{tikzcd}
\tag{8.5}
$$

References

[1] Adámek, J., and Rosický, J. *Locally Presentable and Accessible Categories*, vol. 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1994.

[2] Bénabou, J. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, vol. 47 of *Lecture Notes in Mathematics*. Springer, 1967, pp. 1–77.

[3] Blackwell, R., Kelly, G. M., and Power, A. J. Two-dimensional monad theory. *Journal of Pure and Applied Algebra 59*, 1 (1989), 1–41.

[4] Bourke, J., and Gurski, N. A cocategorical obstruction to tensor products of Gray-categories. *Theory and Applications of Categories*, 11 (2015), 387–409.

[5] Corner, A., and Gurski, N. Operads with general groups of equivariance and some 2-categorical aspects of operads in Cat (2013). Preprint available at http://arxiv.org/abs/1312.5910.

[6] Day, B., and Laplaza, M. On embedding closed categories. *Bulletin of the Australian Mathematical Society 18*, 3 (1978), 357–371.

[7] de Schipper, W. *Symmetric closed categories*. Mathematisch Centrum, Amsterdam, 1975.

[8] Eilenberg, S., and Kelly, G. M. Closed categories. In *Conference on Categorical Algebra (La Jolla, 1965)*. Springer, New York, 1966, pp. 421–562.

[9] Elmendorf, A. D., and Mandell, M. A. Rings, modules and algebras in infinite loop space theory. *Advances in Mathematics*, 205 (2006), 163–228.

[10] Gordon, R., Power, J., and Street, R. Coherence for Tricategories. *Memoirs of the American Mathematical Society 117* (1995).

[11] Gray, J. W. *Formal category theory: adjointness for 2-categories*, vol. 391 of *Lecture Notes in Mathematics*, 1974.

[12] Hovey, M. *Model categories*, vol. 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1999.

[13] Hyland, M., and Power, J. Pseudo-commutative monads and pseudo-closed 2-categories. *Journal of Pure and Applied Algebra 175* (2002), 141–185.

[14] Joyal, A., and Street, R. Braided monoidal categories. Macquarie Math Reports 860081 (1986), 1–54.

[15] Keigher, W. Symmetric monoidal closed categories generated by commutative adjoint monads. *Cahiers de Topologie et Geométrie Différentielle 19* no. 3, (1978), 269–293.

[16] Kelly, G. M. *Basic concepts of enriched category theory*, vol. 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982.

[17] Kock, A. Monads on symmetric monoidal closed categories. *Archiv der Mathematik 21*, 1 (1970), 1–10.

[18] Kock, A. Closed categories generated by commutative monads. Journal of the Australian Mathematical Society, 12 (1971), 405–424.

[19] Kock, A. Bilinearity and cartesian closed monads. *Math. Scand. 29* (1971), 161–174.

[20] Lack, S. A Quillen model structure for 2-categories. *K-Theory*, 26 (2002), 171–205.

[21] Lack, S. A Quillen model structure for bicategories. *K-Theory*, 33 (2004), 185–197.

[22] Lack, S. Homotopy-theoretic aspects of 2-monads. *Journal of Homotopy and Related Structures 7*, 2 (2007), 229–260.

[23] Lack, S., and Paoli, S. 2-nerves for bicategories. *K-Theory 38*, 2 (2008), 153–175.

[24] Lack, S. Icons. *Applied Categorical Structures 18*, 3 (2010), 289–307.

[25] Lambek, J. Deductive systems and categories. II. Standard constructions and closed categories In *Category theory, Homology theory and their Applications, I.* Springer, Berlin. 1969, pp. 76–122.

[26] Laplaza, M. Embedding of closed categories into monoidal closed categories. *Transactions of the American Mathematical Society 233* (1977), 85–91.

[27] Leinster, T. Higher operads, higher categories. *London Mathematical Society Lecture Notes 298, Cambridge University Press* (2004) 1–433.

[28] López-Franco, I. Pseudo-commutativity of KZ 2-monads. *Advances in Mathematics 228*, 5 (2011), 2557–2605.

[29] Manzyuk, O. Closed categories vs. closed multicategories. *Theory and Applications of Categories*, 5 (2012), 132–175.

[30] Quillen, D. *Homotopical Algebra*, vol. 43 of *Lecture Notes in Mathematics*. Springer, 1967.

[31] Schmitt, V. Tensor product for symmetric monoidal categories (2007). Preprint available at http://arxiv.org/abs/0711.0324.

[32] Seal, G. J. Tensors, monads and actions. *Theory and Applications of Categories*, 15 (2013), 403–433.

[33] Street, R. Elementary Cosmoi I. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 135–180.

[34] Street, R. Fibrations in bicategories. Cahiers de Topologie et Geométrie Différentielle 21 no. 2, (1980), 111-160.

[35] Street, R. Skew-closed categories. *Journal of Pure and Applied Algebra 217*, 6 (2013), 973–988.

[36] Szlachányi, K. Skew-monoidal categories and bialgebroids. *Advances in Mathematics 231*, 3-4 (2012), 1694–1730.

[37] Verity, D. *Enriched categories, internal categories and change of base*. PhD thesis, Cambridge University, 1992. Republished in *Reprints in Theory and Applications of Categories*, No. 20 (2011), 1–266.

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 60000, Czech Republic

E-mail address: `bourkej@math.muni.cz`

Chapter 5

Monads and theories

This chapter contains the article *Monads and theories* by *John Bourke and Richard Garner*, published in *Advances in Mathematics 351* (2019) 1024–1071. Both authors contributed an equal share to the results of this article.

MONADS AND THEORIES

JOHN BOURKE AND RICHARD GARNER

Abstract. Given a locally presentable enriched category $\mathcal{E}$ together with a small dense full subcategory $\mathcal{A}$ of *arities*, we study the relationship between monads on $\mathcal{E}$ and identity-on-objects functors out of $\mathcal{A}$, which we call $\mathcal{A}$-pretheories. We show that the natural constructions relating these two kinds of structure form an adjoint pair. The fixpoints of the adjunction are characterised on the one side as the $\mathcal{A}$-*nervous monads*—those for which the conclusions of Weber's *nerve theorem* hold—and on the other, as the $\mathcal{A}$-*theories* which we introduce here.

The resulting equivalence between $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories is best possible in a precise sense, and extends almost all previously known monad–theory correspondences. It also establishes some completely new correspondences, including one which captures the globular theories defining Grothendieck weak ω-groupoids.

Besides establishing our general correspondence and illustrating its reach, we study good properties of $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories that allow us to recognise and construct them with ease. We also compare them with the monads with arities and theories with arities introduced and studied by Berger, Melliès and Weber.

1. Introduction

Category theory provides two approaches to classical universal algebra. On the one hand, we have finitary monads on **Set** and on the other hand, we have Lawvere theories. Relating the two approaches we have Linton's result [26], which shows that the category of finitary monads on **Set** is equivalent to the category of Lawvere theories. An essential feature of this equivalence is that it respects semantics, in the sense that the algebras for a finitary monad coincide up to equivalence over **Set** with the models of the associated theory, and vice versa.

There have been a host of generalisations of the above story, each dealing with algebraic structure borne by objects more general than sets. In many of these [32, 31, 22, 23], one starts on one side with the monads on a given category that preserve a specified class of colimits. This class specifies, albeit indirectly, the arities of operations that may arise in the algebraic structures encoded by such monads, and from this one may define, on the other side, corresponding notions of theory and model. These are subtler than in the classical setting, but

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once the correct definitions have been found, the equivalence with the given class of monads, and the compatibility with semantics, follows much as before.

The most general framework for a monad–theory correspondence to date involves the notions of *monad with arities* and *theory with arities*. In this setting, the permissible arities of operations are part of the basic data, given as a small, dense, full subcategory of the base category. The monads with arities were introduced first, in [35], as a setting for an abstract nerve theorem. Particular cases of this theorem include the classical nerve theorem, identifying categories with simplicial sets satisfying the Segal condition of [33], and also Berger's nerve theorem [8] for the globular higher categories of [7]. More saliently, when Weber's nerve theorem is specialised to the settings appropriate to the monad–theory correspondences listed above, it becomes exactly the fact that the functor sending the algebras for a monad to the models of the associated theory is an equivalence. This observation led [29] and [9] to introduce *theories with arities*, and to prove, by using Weber's nerve theorem, their equivalence with the monads with arities. The monad–theory correspondence obtained in this way is general enough to encompass all of the instances from [32, 31, 22, 23].

Our own work in this paper has two motivations: one abstract and one concrete. Our abstract motivation is a desire to explain the apparently *ad hoc* design choices involved in the monad–theory correspondences outlined above. For indeed, while these choices must be carefully balanced in order to obtain an equivalence, there is no reason to believe that different careful choices might not yield more general or more expressive results.

Our concrete motivation comes from the study of the Grothendieck weak ω-groupoids introduced by Maltsiniotis [27], which, by definition, are models of a globular theory in the sense of Berger [8]. Globular theories describe algebraic structure on globular sets with arities drawn from the dense subcategory of *globular cardinals*; see Example 8(v) below. However, globular theories are not necessarily theories with arities, and in particular, those capturing higher groupoidal structures are not. As such, they do not appear to one side of any of the monad–theory correspondences described above.

The first goal of this paper is to describe a new schema for monad–theory correspondences which addresses the gaps in our understanding noted above. In this schema, once we have fixed the process by which a theory is associated to a monad, everything else is forced. This addresses our first, abstract motivation. The correspondence obtained in this way is in fact *best possible*, in the sense that any other monad–theory correspondence for the same kind of algebraic structure must be a restriction of this particular one. In many cases, this best possible correspondence coincides with one in the literature, but in others, our correspondence goes beyond what already exists. In particular, an instance of our schema will identify the globular theories of [8] with a suitable class of monads on the category of globular sets. This addresses our second, concrete motivation.

The further goal of this paper is to study the classes of monads and theories that arise from our correspondence-schema. We do so both at a general level, where we will see that both the monads and the theories are closed under essentially all the constructions one could hope for; and also at a practical level, where we will see how these general constructions allow us to give expressive and intuitive *presentations* for the structure captured by a monad or theory.

To give a fuller account of our results, we must first describe how a typical monad–theory correspondence arises. As in [35], the basic setting for such a correspondence can be encapsulated by a pair consisting of a category $\mathcal{E}$ and a small, full, dense subcategory $K\colon \mathcal{A} \hookrightarrow \mathcal{E}$. For example, the Lawvere theory–finitary monad correspondence for finitary algebraic structure on sets is associated to the choice of $\mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$ the full subcategory of finite cardinals.

Given $\mathcal{E}$ and $K\colon \mathcal{A} \hookrightarrow \mathcal{E}$, the goal is to establish an equivalence between a suitable category of $\mathcal{A}$-*monads* and a suitable category of $\mathcal{A}$-*theories*. The $\mathcal{A}$-monads will be a certain class of monads on $\mathcal{E}$; while the $\mathcal{A}$-theories will be a certain class of identity-on-objects functors out of $\mathcal{A}$. We are being deliberately vague about the conditions on each side, as they are among the seemingly *ad hoc* design choices we spoke of earlier. But regardless of this, the monad–theory correspondence itself always arises through application of the following two constructions.

Construction A. For an $\mathcal{A}$-monad T on $\mathcal{E}$, the associated $\mathcal{A}$-theory $\Phi(\mathsf{T})$ is the identity-on-objects functor $J_\mathsf{T}\colon \mathcal{A} \to \mathcal{A}_\mathsf{T}$ arising from the (identity-on-objects, fully faithful) factorisation

$$\mathcal{A} \xrightarrow{J_\mathsf{T}} \mathcal{A}_\mathsf{T} \xrightarrow{V_\mathsf{T}} \mathcal{E}_\mathsf{T} \tag{1.1}$$

of the composite $F_\mathsf{T} K\colon \mathcal{A} \to \mathcal{E} \to \mathcal{E}_\mathsf{T}$. Here F_T is the free functor into the Kleisli category $\mathcal{E}_\mathsf{T}$, so $\mathcal{A}_\mathsf{T}$ is equally the full subcategory of $\mathcal{E}_\mathsf{T}$ with objects those of $\mathcal{A}$.

Construction B. For an $\mathcal{A}$-theory $J\colon \mathcal{A} \to \mathcal{T}$, the associated $\mathcal{A}$-monad $\Psi(\mathcal{T})$ is obtained from the category of *concrete* $\mathcal{T}$*-models*, which is by definition the pullback

$$\begin{tikzcd}
\mathbf{Mod}_c(\mathcal{T}) \arrow[r] \arrow[d, "U^{\mathcal{T}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{T}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\
\mathcal{E} \arrow[r, "N_K = \mathcal{E}(K-,1)"] & {[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]} .
\end{tikzcd} \tag{1.2}$$

Since $U^{\mathcal{T}}$ is a pullback of the strictly monadic $[J^{\mathrm{op}}, 1]$, it will be strictly monadic so long as it has a left adjoint. The assumption that $\mathcal{E}$ is locally presentable ensures that this is the case, and so we can take $\Psi(\mathcal{T})$ to be the monad whose algebras are the concrete $\mathcal{T}$-models.

There remains the problem of choosing the appropriate conditions on a monad or theory for it to be an $\mathcal{A}$-monad or $\mathcal{A}$-theory. Of course, these must be carefully balanced so as to obtain an equivalence, but this still seems to leave too many degrees of freedom; one might hope that everything could be determined from $\mathcal{E}$ and $\mathcal{A}$ alone. The main result of this paper shows that this is so: there are notions of $\mathcal{A}$-monad and $\mathcal{A}$-theory which require no further choices to be made, and which rather than being plucked from the air, may be derived in a principled manner.

The key observation is that Constructions A and B make sense when given as input *any* monad on $\mathcal{E}$, or *any* "$\mathcal{A}$-pretheory"—by which we mean simply an identity-on-objects functor out of $\mathcal{A}$. When viewed in this greater generality, these constructions yield an adjunction

$$\mathbf{Mnd}(\mathcal{E}) \underset{\Phi}{\overset{\Psi}{\underset{\perp}{\leftrightarrows}}} \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \tag{1.3}$$

between the category of monads on $\mathcal{E}$ and the category of $\mathcal{A}$-pretheories. Like any adjunction, this restricts to an equivalence between the objects at which the counit is invertible, and the objects at which the unit is invertible. Thus, if we *define* the $\mathcal{A}$-monads and $\mathcal{A}$-theories to be the objects so arising, then we obtain a monad–theory equivalence. By construction, it will be the *largest* possible equivalence whose two directions are given by Constructions A and B.

Having defined the $\mathcal{A}$-monads and $\mathcal{A}$-theories abstractly, it behooves us to give tractable concrete characterisations. In fact, we give a number of these, allowing us to relate our correspondence to existing ones in the literature. We also investigate further aspects of the general theory, and provide a wide range of examples illustrating the practical utility of our results.

Before getting started, we conclude this introduction with a more detailed outline of the paper's contents. In Section 2, we begin by introducing our basic setting and notions. We then construct, in Theorem 6, the adjunction (1.3) between monads and pretheories. In Section 3, with this abstract result in place, we introduce a host of running examples of our basic setting. To convince the reader of the expressive power of our notions, we construct, via colimit presentations, specific pretheories for a variety of mathematical structures.

In Section 4 we obtain our main result by characterising the fixpoints of the monad–theory adjunction: the $\mathcal{A}$-monads and $\mathcal{A}$-theories described above. The $\mathcal{A}$-monads are characterised as what we term the *$\mathcal{A}$-nervous monads*, since they are precisely those monads for which Weber's nerve theorem holds. The $\mathcal{A}$-theories turn out to be precisely those $\mathcal{A}$-pretheories for which each representable is a model; in the motivating case where $\mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$, they are exactly the Lawvere theories. With these characterisations in place, we obtain our main Theorem 19, which describes the "best possible" equivalence between $\mathcal{A}$-theories and $\mathcal{A}$-nervous monads.

Section 5 develops some of the general results associated to our correspondence-schema. We begin by showing that our monad–theory correspondence commutes, to within isomorphism, with the taking of semantics on each side. We also prove that the functors taking semantics are valued in monadic right adjoint functors between locally presentable categories. The final important result of this section states that colimits of $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories are algebraic, meaning that the semantics functors send them to limits.

Section 6 is devoted to exploring what the $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories amount to in our running examples. In order to understand the $\mathcal{A}$-nervous monads, we prove the important result that they are equally the colimits, amongst all monads, of free monads on $\mathcal{A}$-signatures. We also introduce the notion of a *saturated class of arities* as a setting in which, like in [32, 31, 22, 23], the $\mathcal{A}$-nervous monads can be characterised in terms of a colimit-preservation property. With these results in place, we are able to exhibit many of these existing monad–theory correspondences as instances of our general framework.

In Section 7, we examine the relationship between the monads and theories of our correspondence, and the *monads with arities* and *theories with arities* of [35, 29, 9]. In particular, we see that every monad with arities $\mathcal{A}$ is an $\mathcal{A}$-nervous monad but that the converse implication need not be true: so $\mathcal{A}$-nervous monads are *strictly* more general. Of course, the same is also true on the theory side. We also exhibit a further important point of difference: colimits of monads with arities, unlike those of nervous monads, need not be algebraic. This means that there is no good notion of *presentation* for monads or theories with arities.

Finally, in Section 8, we give a number of proofs deferred from Section 6.

2. Monads and pretheories

2.1. **The setting.** In this section we construct the monad–pretheory adjunction

$$\begin{tikzcd} \mathbf{Mnd}(\mathcal{E}) \arrow[r, shift right, "\Phi"'] \arrow[r, phantom, "\perp"] & \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \arrow[l, shift right, "\Psi"'] \end{tikzcd} . \tag{2.1}$$

The setting for this, and the rest of the paper, involves two basic pieces of data:

(i) A locally presentable $\mathcal{V}$-category $\mathcal{E}$ with respect to which we will describe the monad– pretheory adjunction; and

(ii) A notion of *arities* given by a small, full, dense sub-$\mathcal{V}$-category $K\colon \mathcal{A} \hookrightarrow \mathcal{E}$.

We will discuss examples in Section 2.1 below, but for now let us clarify some of the terms appearing above. While in the introduction, we focused on the unenriched context, we now work in the context of category theory enriched over a symmetric monoidal closed category $\mathcal{V}$ which is *locally presentable* as in [13]. In this context, a *locally presentable $\mathcal{V}$-category* [18] is one which is cocomplete as a $\mathcal{V}$-category, and whose underlying ordinary category is locally presentable.

We recall also some notions pertaining to density. Given a $\mathcal{V}$-functor $K\colon \mathcal{A} \to \mathcal{E}$ with small domain, the *nerve functor* $N_K\colon \mathcal{E} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is defined by $N_K(X) = \mathcal{E}(K\text{–}, X)$. We call a presheaf in the essential image of N_K a *K-nerve*, and we write K-$\mathbf{Ner}(\mathcal{V})$ for the full sub-$\mathcal{V}$-category of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ determined by these.

We say that K is *dense* if N_K is fully faithful; whereupon N_K induces an equivalence of categories $\mathcal{E} \simeq K$-$\mathbf{Ner}(\mathcal{V})$. Finally, we call a small sub-$\mathcal{V}$-category $\mathcal{A}$ of a $\mathcal{V}$-category $\mathcal{E}$ *dense* if its inclusion functor $K\colon \mathcal{A} \hookrightarrow \mathcal{E}$ is so.

2.2. **Monads.** We write $\mathbf{Mnd}(\mathcal{E})$ for the (ordinary) category whose objects are $\mathcal{V}$-monads on $\mathcal{E}$, and whose maps $\mathsf{S} \to \mathsf{T}$ are $\mathcal{V}$-transformations $\alpha\colon S \Rightarrow T$ compatible with unit and multiplication. For each $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ we have the $\mathcal{V}$-category of algebras $U^{\mathsf{T}}\colon \mathcal{E}^{\mathsf{T}} \to \mathcal{E}$ over $\mathcal{E}$, but also the *Kleisli $\mathcal{V}$-category* $F_{\mathsf{T}}\colon \mathcal{E} \to \mathcal{E}_{\mathsf{T}}$ under $\mathcal{E}$, arising from an (identity-on-objects, fully faithful) factorisation

$$\begin{tikzcd} & \mathcal{E} \arrow[dl, "F_{\mathsf{T}}"'] \arrow[dr, "F^{\mathsf{T}}"] & \\ \mathcal{E}_{\mathsf{T}} \arrow[rr, "W_{\mathsf{T}}"] & & \mathcal{E}^{\mathsf{T}} \end{tikzcd} \tag{2.2}$$

of the free $\mathcal{V}$-functor $F^{\mathsf{T}}\colon \mathcal{E} \to \mathcal{E}^{\mathsf{T}}$; concretely, we may take $\mathcal{E}_{\mathsf{T}}$ to have objects those of $\mathcal{E}$, hom-objects $\mathcal{E}_{\mathsf{T}}(A, B) = \mathcal{E}(A, TB)$, and composition and identities derived using the monad structure of T. Each monad morphism $\alpha\colon \mathsf{S} \to \mathsf{T}$ induces, functorially in α, $\mathcal{V}$-functors α^* and $\alpha_!$ fitting into diagrams

$$\begin{tikzcd} \mathcal{E}^{\mathsf{T}} \arrow[rr, "\alpha^*"] \arrow[dr, "U^{\mathsf{T}}"'] & & \mathcal{E}^{\mathsf{S}} \arrow[dl, "U^{\mathsf{S}}"] \\ & \mathcal{E} & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{E} \arrow[dl, "F_{\mathsf{S}}"'] \arrow[dr, "F_{\mathsf{T}}"] & \\ \mathcal{E}_{\mathsf{S}} \arrow[rr, "\alpha_!"] & & \mathcal{E}_{\mathsf{T}} \end{tikzcd} ; \tag{2.3}$$

here α^* sends an algebra $a\colon TA \to A$ to $a \circ \alpha_A\colon SA \to A$ and is the identity on homs, while $\alpha_!$ is the identity on objects and has action on homs given by the postcomposition maps $\alpha_B \circ (-)\colon \mathcal{E}_{\mathsf{S}}(A, B) \to \mathcal{E}_{\mathsf{T}}(A, B)$. In fact, every $\mathcal{V}$-functor $\mathcal{E}^{\mathsf{T}} \to \mathcal{E}^{\mathsf{S}}$ over $\mathcal{E}$ or $\mathcal{V}$-functor $\mathcal{E}_{\mathsf{S}} \to \mathcal{E}_{\mathsf{T}}$ under $\mathcal{E}$ is of the form α^* or $\alpha_!$ for a unique map of monads α—see, for example, [30]—and in this way, we obtain *fully faithful* functors

$$\mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \qquad \text{and} \qquad \mathbf{Mnd}(\mathcal{E}) \xrightarrow{\mathrm{Kl}} \mathcal{E}/\mathcal{V}\text{-}\mathbf{CAT} . \tag{2.4}$$

2.3. **Pretheories.** An $\mathcal{A}$-*pretheory* is an identity-on-objects $\mathcal{V}$-functor $J\colon \mathcal{A} \to \mathcal{T}$ with domain $\mathcal{A}$. We write $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ for the ordinary category whose objects are $\mathcal{A}$-pretheories and whose morphisms are $\mathcal{V}$-functors commuting with the maps from $\mathcal{A}$. While the $\mathcal{A}$-pretheory is only fully specified by both pieces of data $\mathcal{T}$ and J, we will often, by abuse of notation, leave J implicit and refer to such a pretheory simply as $\mathcal{T}$.

Just as any $\mathcal{V}$-monad has a $\mathcal{V}$-category of algebras, so any $\mathcal{A}$-pretheory has a $\mathcal{V}$-category of models. Generalising (1.2), we define the $\mathcal{V}$-category of *concrete $\mathcal{T}$-models* $\mathbf{Mod}_c(\mathcal{T})$ by a pullback of $\mathcal{V}$-categories as below left; so a concrete $\mathcal{T}$-model is an object $X \in \mathcal{E}$ together with a chosen extension of $\mathcal{E}(K-, X)\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ along $J^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{T}^{\mathrm{op}}$. The reason for the qualifier "concrete" will be made clear in Section 5.2 below, where we will identify a more general notion of model.

$$\begin{tikzcd} \mathbf{Mod}_c(\mathcal{T}) \arrow[r, "P_{\mathcal{T}}"] \arrow[d, "U_{\mathcal{T}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mod}_c(\mathcal{S}) \arrow[r, "P_{\mathcal{S}}"] \arrow[d, "U_{\mathcal{S}}"'] & {[\mathcal{S}^{\mathrm{op}}, \mathcal{V}]} \arrow[r, "{[H^{\mathrm{op}},1]}"] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[rr, "N_K"] & & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} \tag{2.5}$$

Remark 1. Avery considers a notion very similar to our $\mathcal{A}$-pretheories under the name *prototheories* [4, Definition 4.1.1]. The differences are that Avery's prototheories $\mathcal{A} \to \mathcal{T}$ are not enriched, and the hom-sets of $\mathcal{T}$ need not be small. He also defines a category of (concrete) models for a prototheory, relative to a given functor $\mathcal{E} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$ called an *aritation.* When this functor is the nerve $N_K\colon \mathcal{E} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, his category of models agrees with our $\mathbf{Mod}_c(\mathcal{T})$.

Any $\mathcal{A}$-pretheory map $H\colon \mathcal{T} \to \mathcal{S}$ gives a functor $H^*\colon \mathbf{Mod}_c(\mathcal{S}) \to \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$ by applying the universal property of the pullback left above to the commuting square on the right. In this way, we obtain a semantics functor:

$$\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Mod}_c} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} . \tag{2.6}$$

However, unlike (2.4), this is *not* always fully faithful. Indeed, in Example 10 below, we will see that non-isomorphic pretheories can have isomorphic categories of concrete models over $\mathcal{E}$.

2.4. **Monads to pretheories.** We now define the functor $\Phi\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ in (2.1). As in Construction A of the introduction, this will take the $\mathcal{V}$-monad T to the $\mathcal{A}$-pretheory $J_{\mathsf{T}}\colon \mathcal{A} \to \mathcal{A}_{\mathsf{T}}$ arising as the first part of an (identity-on-objects, fully faithful) factorisation of $F_{\mathsf{T}}K\colon \mathcal{A} \to \mathcal{E}_{\mathsf{T}}$, as to the left in:

(2.7)
$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "J_{\mathsf{T}}"] \arrow[d, "K"'] & \mathcal{A}_{\mathsf{T}} \arrow[d, "V_{\mathsf{T}}"] \\
\mathcal{E} \arrow[r, "F_{\mathsf{T}}"] & \mathcal{E}_{\mathsf{T}}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{A} \arrow[r, "J_{\mathsf{T}}"] \arrow[d, "K"'] & \mathcal{A}_{\mathsf{T}} \arrow[d, "K_{\mathsf{T}}"] \\
\mathcal{E} \arrow[r, "F^{\mathsf{T}}"] & \mathcal{E}^{\mathsf{T}}
\end{tikzcd} .
$$

Since the comparison $W_{\mathsf{T}}\colon \mathcal{E}_{\mathsf{T}} \to \mathcal{E}^{\mathsf{T}}$ is fully faithful, we can also view J_{T} as arising from an (identity-on-objects, fully faithful) factorisation as above right; the relationship between the two is that $K_{\mathsf{T}} = W_{\mathsf{T}} \circ V_{\mathsf{T}}$. Both perspectives will be used in what follows, with the functor $K_{\mathsf{T}}\colon \mathcal{A}_{\mathsf{T}} \to \mathcal{E}^{\mathsf{T}}$ of particular importance.

To define Φ on morphisms, we make use of the *orthogonality* of identity-on-objects $\mathcal{V}$-functors to fully faithful ones; this asserts that any commuting square of $\mathcal{V}$-functors as below, with F identity-on-objects and G fully faithful, admits a unique diagonal filler J making both triangles commute.

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C} \arrow[d, "G"] \\
\mathcal{B} \arrow[r, "K"'] \arrow[ur, dashed, "J"] & \mathcal{D}
\end{tikzcd}
$$

Explicitly, J is given on objects by $Ja = Ha$, and on homs by

$$
\mathcal{B}(a,b) \xrightarrow{K_{a,b}} \mathcal{D}(Ka,Kb) = \mathcal{D}(GHa,GHb) \xrightarrow{(G_{Ha,Hb})^{-1}} \mathcal{C}(Ha,Hb) .
$$

In particular, given a map $\alpha\colon \mathsf{S} \to \mathsf{T}$ of $\mathbf{Mnd}(\mathcal{E})$, this orthogonality guarantees the existence of a diagonal filler in the diagram below, whose upper triangle we take to be the map $\Phi(\alpha)\colon \Phi(\mathsf{S}) \to \Phi(\mathsf{T})$ in $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$:

$$
\begin{tikzcd}
\mathcal{A} \arrow[rr, "J_{\mathsf{T}}"] \arrow[d, "J_{\mathsf{S}}"'] & & \mathcal{A}_{\mathsf{T}} \arrow[d, "V_{\mathsf{T}}"] \\
\mathcal{A}_{\mathsf{S}} \arrow[r, "V_{\mathsf{S}}"'] \arrow[urr, dashed] & \mathcal{E}_{\mathsf{S}} \arrow[r, "\alpha_!"'] & \mathcal{E}_{\mathsf{T}}
\end{tikzcd} .
$$

2.5. **Pretheories to monads.** Thus far we have not exploited the local presentability of $\mathcal{E}$. It *will* be used in the next step, that of constructing the left adjoint to $\Phi\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$. We first state a general result which, independent of local presentability, gives a sufficient condition for an individual pretheory to have a reflection along Φ. Here, by a *reflection* of an object $c \in \mathcal{C}$ along a functor $U\colon \mathcal{B} \to \mathcal{C}$, we mean a representation for the functor $\mathcal{C}(c, U{-})\colon \mathcal{B} \to \mathbf{Set}$.

Theorem 2. *A pretheory $J\colon \mathcal{A} \to \mathcal{T}$ admits a reflection along Φ whenever the forgetful functor $U_{\mathcal{T}}\colon \mathbf{Mod}_c(\mathcal{T}) \to \mathcal{E}$ from the category of concrete models has a left adjoint $F_{\mathcal{T}}$. In this case, the reflection $\Psi\mathcal{T}$ is characterised by an isomorphism $\mathcal{E}^{\Psi\mathcal{T}} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$, or equally, by a pullback square*

$$
\begin{tikzcd}
\mathcal{E}^{\Psi\mathcal{T}} \arrow[r] \arrow[d, "U^{\Psi\mathcal{T}}"'] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\
\mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]}\ .
\end{tikzcd} \tag{2.8}
$$

To prove this result, we will need a preparatory lemma, relating to the notion of *discrete isofibration*: this is a $\mathcal{V}$-functor $U\colon \mathcal{D} \to \mathcal{C}$ such that, for each $f\colon c \cong Ud$ in $\mathcal{C}$, there is a unique $f'\colon c' \cong d$ in $\mathcal{D}$ with $U(f') = f$.

Example 3. For any $\mathcal{V}$-monad T on $\mathcal{C}$, the forgetful $\mathcal{V}$-functor $U^{\mathsf{T}}\colon \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is a discrete isofibration. Indeed, if $x\colon Ta \to a$ is a T-algebra and $f\colon b \cong a$ in $\mathcal{C}$, then $y = f^{-1} \circ x \circ Tf\colon Tb \to b$ is the unique algebra structure on b for which $f\colon (b, y) \to (a, x)$ belongs to $\mathcal{C}^{\mathsf{T}}$. In particular, for any identity-on-objects $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ between small $\mathcal{V}$-categories, the functor $[F, 1]\colon [\mathcal{B}, \mathcal{V}] \to [\mathcal{A}, \mathcal{V}]$ has a left adjoint and strictly creates colimits, whence is strictly monadic. It is therefore a discrete isofibration by the above argument.

Lemma 4. *Let $U\colon \mathcal{A} \to \mathcal{B}$ be a discrete isofibration and $\alpha\colon F \Rightarrow G\colon \mathcal{X} \to \mathcal{B}$ an invertible $\mathcal{V}$-transformation. The displayed projections give isomorphisms between liftings of F through U, liftings of α through U, and liftings of G through U:*

$$
\begin{tikzcd}
& \mathcal{A} \arrow[d, "U"] & & & \mathcal{A} \arrow[d, "U"] & & & \mathcal{A} \arrow[d, "U"] \\
\mathcal{X} \arrow[r, "F"'] \arrow[ur, dashed, "\bar F"] & \mathcal{B} & \arrow[l, "\mathrm{dom}"'] & \mathcal{X} \arrow[ur, dashed, bend left, "\bar F"] \arrow[ur, dashed, bend right, "\bar G"'] \arrow[ur, phantom, "\Downarrow\bar\alpha"] \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} & \arrow[r, "\mathrm{cod}"] & \mathcal{X} \arrow[r, "G"'] \arrow[ur, dashed, "\bar G"] & \mathcal{B}\ .
\end{tikzcd}
$$

Proof. Given $\bar G\colon \mathcal{X} \to \mathcal{A}$ as to the right, there is for each $x \in \mathcal{X}$ a unique lifting of the isomorphism $\alpha_x\colon Fx \cong U\bar G x$ to one $\bar\alpha_x\colon \bar F x \cong \bar G x$. There is now a unique way of extending $x \mapsto \bar F x$ to a $\mathcal{V}$-functor $\bar F\colon \mathcal{X} \to \mathcal{A}$ so that $\bar\alpha\colon \bar F \cong \bar G$; namely, by taking the action on homs to be $\bar F_{x,y} = \mathcal{A}(\bar\alpha_x, \bar\alpha_y{}^{-1}) \circ \bar G_{x,y}\colon X(x,y) \to \mathcal{A}(\bar F x, \bar F y)$. In this way, we have found a unique lifting of α through U whose codomain is the given lifting of G through U. So the right-hand projection is invertible; the argument for the left-hand one is the same on replacing α by α^{-1}. $\square$

We can now give:

Proof of Theorem 2. $U_{\mathcal{T}}$ has a left adjoint by assumption, and—as a pullback of the strictly monadic $[J^{\mathrm{op}}, 1]\colon [\mathcal{T}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$—strictly creates coequalisers for $U_{\mathcal{T}}$-absolute pairs. It is therefore strictly monadic. Taking $\Psi\mathcal{T} = U_{\mathcal{T}}F_{\mathcal{T}}$ to be the induced monad, we thus have an isomorphism $\mathcal{E}^{\Psi\mathcal{T}} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$.

It remains to exhibit isomorphisms $\mathbf{Mnd}(\mathcal{E})(\Psi\mathcal{T}, \mathsf{S}) \cong \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})(\mathcal{T}, \Phi\mathsf{S})$ natural in S. We do so by chaining together the following sequence of natural bijections. Firstly, by full fidelity in (2.4), monad maps $\alpha_0 \colon \Psi\mathcal{T} \to \mathsf{S}$ correspond naturally to functors $\alpha_1 \colon \mathcal{E}^{\mathsf{S}} \to \mathcal{E}^{\Psi\mathcal{T}}$ rendering commutative the left triangle in

$$\text{(2.9)} \qquad \begin{tikzcd} \mathcal{E}^{\mathsf{S}} \arrow[rr, "\alpha_1"] \arrow[dr, "U^{\mathsf{S}}"'] & & \mathcal{E}^{\Psi\mathcal{T}} \arrow[dl, "U^{\Psi\mathcal{T}}"] \\ & \mathcal{E} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^{\mathsf{S}} \arrow[r, "\alpha_2"] \arrow[d, "U^{\mathsf{S}}"'] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} .$$

Since $\mathcal{E}^{\Psi\mathcal{T}}$ is defined by the pullback (2.14), such functors α_1 correspond naturally to functors α_2 rendering commutative the square above right. Next, we observe that there is a natural isomorphism in the triangle below left

$$\text{(2.10)} \qquad \begin{tikzcd} \mathcal{E}^{\mathsf{S}} \arrow[rr, "U^{\mathsf{S}}"] \arrow[dr, "N_{F^{\mathsf{S}}K}"'] & \cong & \mathcal{E} \arrow[dl, "N_K"] \\ & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^{\mathsf{S}} \arrow[rr, "\alpha_3"] \arrow[dr, "N_{F^{\mathsf{S}}K}"'] & & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[dl, "{[J^{\mathrm{op}},1]}"] \\ & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} & \end{tikzcd}$$

with components the adjointness isomorphisms $\mathcal{E}(Ka, U^{\mathsf{S}}b) \cong \mathcal{E}^{\mathsf{S}}(F^{\mathsf{S}}Ka, b)$. Since J^{op} is identity-on-objects, $[J^{\mathrm{op}}, 1]$ is a discrete isofibration by Example 3, whence by Lemma 4 there is a natural bijection between functors α_2 as in (2.9) and ones α_3 as in (2.10). We should now like to transpose this last triangle through the following natural isomorphisms (taking $\mathcal{X} = \mathcal{A}, \mathcal{T}$):

$$\text{(2.11)} \qquad \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}^{\mathsf{S}}, [\mathcal{X}^{\mathrm{op}}, \mathcal{V}]) \cong \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{X}^{\mathrm{op}}, [\mathcal{E}^{\mathsf{S}}, \mathcal{V}]) .$$

However, since $\mathcal{E}^{\mathsf{S}}$ is large, the functor category $[\mathcal{E}^{\mathsf{S}}, \mathcal{V}]$ will not always exist as a $\mathcal{V}$-category, and so (2.11) is ill-defined. To resolve this, note that $N_{F^{\mathsf{S}}K}$ is, by its definition, *pointwise representable*; whence so too is α_3, since J is identity-on-objects. We may thus transpose the right triangle of (2.10) through the legitimate isomorphisms

$$\text{(2.12)} \qquad \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}^{\mathsf{S}}, [\mathcal{X}^{\mathrm{op}}, \mathcal{V}])_{\mathrm{pwr}} \cong \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{X}^{\mathrm{op}}, [\mathcal{E}^{\mathsf{S}}, \mathcal{V}]_{\mathrm{rep}})$$

where on the left we have the category of pointwise representable $\mathcal{V}$-functors, and on the right, the *legitimate* $\mathcal{V}$-category of representable $\mathcal{V}$-functors $\mathcal{E}^{\mathsf{S}} \to \mathcal{V}$. In this way, we establish a natural bijection between functors α_3 and functors α_4 rendering commutative the left square in:

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}} \arrow[r, "J^{\mathrm{op}}"] \arrow[d, "(F^{\mathsf{S}}K)^{\mathrm{op}}"'] & \mathcal{T}^{\mathrm{op}} \arrow[d, "\alpha_4"] \arrow[dl, dotted, "(\alpha_5)^{\mathrm{op}}"'] \\ (\mathcal{E}^{\mathsf{S}})^{\mathrm{op}} \arrow[r, "Y"'] & {[\mathcal{E}^{\mathsf{S}}, \mathcal{V}]_{\mathrm{rep}}} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "J"] \arrow[d, "J_{\mathsf{S}}"'] & \mathcal{T} \arrow[d, "\alpha_5"] \arrow[dl, dotted, "\alpha_6"'] \\ \mathcal{A}_{\mathsf{S}} \arrow[r, "K_{\mathsf{S}}"'] & \mathcal{E}^{\mathsf{S}} \end{tikzcd} .$$

Now orthogonality of the identity-on-objects J^{op} and the fully faithful Y draws the correspondence between functors α_4 and functors α_5 satisfying $\alpha_5 \circ J = F^{\mathsf{S}}K$ as left above. Finally, since $\mathcal{A}_{\mathsf{S}}$ fits in to an (identity-on-objects, fully faithful) factorisation of $F^{\mathsf{S}}K$, orthogonality also gives the correspondence, as right above, between functors α_5 and functors α_6 satisfying $\alpha_6 \circ J = J_{\mathsf{S}}$, as required. $\square$

We now show that the assumed local presentability of $\mathcal{E}$ ensures that every pretheory has a reflection along $\Phi \colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$, which consequently has a left adjoint. The key result about locally presentable categories enabling this is the following lemma.

Lemma 5. *Consider a pullback square of $\mathcal{V}$-categories*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "U"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "V"] \\ \mathcal{C} \arrow[r, "G"] & \mathcal{D} \end{tikzcd} \tag{2.13}$$

in which G and V are right adjoints between locally presentable $\mathcal{V}$-categories and V is strictly monadic. Then U and F are right adjoints between locally presentable $\mathcal{V}$-categories and U is strictly monadic.

Proof. Since V is strictly monadic, it is a discrete isofibration, and so its pullback against G is, by [14, Corollary 1], also a bipullback. By [10, Theorem 6.11] the 2-category of locally presentable $\mathcal{V}$-categories and right adjoint functors is closed under bilimits in $\mathcal{V}$-$\mathbf{CAT}$, so that both U and F are right adjoints between locally presentable categories. Finally, since U is a pullback of the strictly monadic V, it strictly creates coequalisers for U-absolute pairs. Since it is already known to be a right adjoint, it is therefore also strictly monadic. □

With this in place, we can now prove:

Theorem 6. *Let $\mathcal{E}$ be locally presentable. Then $\Phi \colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint $\Psi \colon \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Mnd}(\mathcal{E})$, whose value at the pretheory $J \colon \mathcal{A} \to \mathcal{T}$ is characterised by an isomorphism $\mathcal{E}^{\Psi(\mathcal{T})} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$, or equally, by a pullback square*

$$\begin{tikzcd} \mathcal{E}^{\Psi(\mathcal{T})} \arrow[r] \arrow[d, "U^{\Psi(\mathcal{T})}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \ . \end{tikzcd} \tag{2.14}$$

Proof. Let $J \colon \mathcal{A} \to \mathcal{T}$ be a pretheory. The pullback square (2.5) defining $\mathbf{Mod}_c(\mathcal{T})$ is a pullback of a right adjoint functor between locally presentable categories along a strictly monadic one: so it follows from Lemma 5 that $U_{\mathcal{T}} \colon \mathbf{Mod}_c(\mathcal{T}) \to \mathcal{E}$ is a right adjoint, whence the result follows from Theorem 2. □

Remark 7. In Avery's study of prototheories, he establishes a *structure–semantics* adjunction [4, Theorem 4.4.8] of the form $\mathbf{Proto}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \leftrightarrows \mathbf{CAT}/\mathcal{E}$, where here $\mathbf{CAT}$ is the category of *large* categories. By restricting to the locally small prototheories to the left and to the strictly monadic functors to the right of this adjunction, one can recover, via (2.4), the unenriched case of our adjunction (2.1).

3. Pretheories as presentations

In the next section, we will describe how the monad–pretheory adjunction (2.1) restricts to an equivalence between suitable subcategories of $\mathcal{A}$-*theories* and of

$\mathcal{A}$-*nervous monads*. However, the results we have so far are already practically useful. The notion of $\mathcal{A}$-pretheory provides a tool for presenting certain kinds of algebraic structure, by exhibiting them as categories of concrete $\mathcal{T}$-models for a suitable pretheory in a manner reminiscent of the theory of *sketches* [6]. Equivalently, via the functor Ψ, we can see $\mathcal{A}$-pretheories as a way of presenting certain monads on $\mathcal{E}$.

3.1. **Examples of the basic setting.** Before giving examples of algebraic structures presented by pretheories, we first describe a range of examples of the basic setting of Section 2.1 above.

Examples 8. We begin by considering the unenriched case where $\mathcal{V} = \mathbf{Set}$.

(i) Taking $\mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$ the full subcategory of finite cardinals captures the classical case of *finitary* algebraic structure borne by *sets*; so examples like groups, rings, lattices, Lie algebras, and so on.

(ii) Taking $\mathcal{E}$ a locally finitely presentable category and $\mathcal{A} = \mathcal{E}_f$ a skeleton of the full subcategory of finitely presentable objects, we capture *finitary* algebraic structure borne by $\mathcal{E}$-*objects*. Examples when $\mathcal{E} = \mathbf{Cat}$ include *finite product*, *finite colimit*, and *monoidal closed* structure; for $\mathcal{E} = \mathbf{CRng}$, we have *commutative* k-*algebra*, *differential ring* and *reduced ring* structure.

(iii) We can replace "finitary" above by "λ-ary" for any regular cardinal λ. For example, when $\lambda = \aleph_1$, this allows for the structure of poset with joins of ω-chains [28] when $\mathcal{E} = \mathbf{Set}$, and for *countable product* structure when $\mathcal{E} = \mathbf{Cat}$. When $\mathcal{E} = [\mathcal{O}(X)^{\mathrm{op}}, \mathbf{Set}]$ for some space X, and λ is suitably chosen, it also permits *sheaf* or *sheaf of rings* structure.

(iv) Let $\mathbb{G}_1$ be the category freely generated by the graph $0 \rightrightarrows 1$, so that $\mathcal{E} = [\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ is the category of directed multigraphs, and let $\mathcal{A} = \Delta_0$ be the full subcategory of $[\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ on graphs of the form

$$[n] := 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n \qquad\qquad \text{for } n > 0.$$

Δ_0 is dense in $[\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ because it contains the representables $[0]$ and $[1]$. This example captures structure borne by graphs in which the operations build vertices and arrows from *paths* of arrows: for example, the structures of *categories*, *involutive categories*, and *groupoids*.

(v) The *globe category* $\mathbb{G}$ is freely generated by the graph

$$0 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} 1 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} 2 \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \cdots$$

subject to the coglobular relations $\sigma\sigma = \sigma\tau$ and $\tau\sigma = \tau\tau$. This means that for each $m > n$, there are precisely two maps $\sigma^{m-n}, \tau^{m-n} \colon n \rightrightarrows m$, which by abuse of notation we will write simply as σ and τ.

The category $\mathcal{E} = [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ is the category of *globular sets*; it has a dense subcategory $\mathcal{A} = \Theta_0$, first described by Berger [8], whose objects have been termed *globular cardinals* by Street [34]. The globular cardinals include the representables—the n-globes Yn for each n—but also shapes such as the globular set with distinct cells as depicted below.

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r, bend left] \arrow[r] \arrow[r, bend right] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & \bullet \end{tikzcd} \tag{3.1}$$

The globular cardinals can be parametrised in various ways, for instance using trees [7, 8]; following [27], we will use *tables of dimensions*—sequences $\vec{n} = (n_1, \dots, n_k)$ of natural numbers of odd length with $n_{2i-1} > n_{2i} < n_{2i+1}$. Given such a table $\vec{n}$ and a functor $D\colon \mathbb{G} \to \mathcal{C}$, we obtain a diagram

$$\begin{tikzcd}[column sep=small] & Dn_2 \arrow[dl, "D\tau"'] \arrow[dr, "D\sigma"] & & Dn_4 \arrow[dl, "D\tau"'] \arrow[dr, "D\sigma"] & & \cdots & & Dn_{k-1} \arrow[dl, "D\tau"'] \arrow[dr, "D\sigma"] & \\ Dn_1 & & Dn_3 & & Dn_5 & & Dn_{k-2} & & Dn_k \end{tikzcd}$$

whose colimit in $\mathcal{C}$, when it exists, will be written as $D(\vec{n})$, and called the *D-globular sum indexed by $\vec{n}$*. Taking $D = Y : \mathbb{G} \to [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$, the category Θ_0 of globular cardinals is now defined as the full subcategory of $[\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ spanned by the Y-globular sums. For example, the globular cardinal in (3.1) corresponds to the Y-globular sum $Y(1, 0, 2, 1, 2)$.

This example captures algebraic structures on globular sets in which the operations build globes out of diagrams with shapes like (3.1); these include *strict ω-categories* and *strict ω-groupoids*, but also the (globular) *weak ω-categories* and *weak ω-groupoids* studied in [7, 25, 3].

We now turn to examples over enriched bases.

(vi) Let $\mathcal{V}$ be a locally finitely presentable symmetric monoidal category whose finitely presentable objects are closed under the tensor product (cf. [18]). By taking $\mathcal{E} = \mathcal{V}$ and $\mathcal{A} = \mathcal{V}_f$ a skeleton of the full sub-$\mathcal{V}$-category of finitely presentable objects, we capture *$\mathcal{V}$-enriched* finitary algebraic structure on $\mathcal{V}$-objects as studied in [32]. When $\mathcal{V} = \mathbf{Cat}$ this means structure on categories $\mathcal{C}$ built from functors and natural transformations $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ for finitely presentable $\mathcal{I}$: which includes *symmetric monoidal* or *finite limit* structure, but not *symmetric monoidal closed* or *factorization system* structure. Similarly, when $\mathcal{V} = \mathbf{Ab}$, it includes *$A$-module* structure but not *commutative ring* structure.

(vii) Taking $\mathcal{V}$ as before, taking $\mathcal{E}$ to be any locally finitely presentable $\mathcal{V}$-category [18] and taking $\mathcal{A} = \mathcal{E}_f$ a skeleton of the full subcategory of finitely presentable objects in $\mathcal{E}$, we capture $\mathcal{V}$-enriched finitary algebraic structure on $\mathcal{E}$-objects as studied in [31]. As before, there is the obvious generalization from finitary to λ-ary structure.

(viii) This example builds on [23]. Let $\mathcal{V}$ be a locally presentable symmetric monoidal closed category, and consider a class of $\mathcal{V}$-enriched limit-types Φ with the property that the free Φ-completion of a small $\mathcal{V}$-category is again small. A $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{V}$ with small domain is called *Φ-flat* if its cocontinuous extension $\mathrm{Lan}_y F\colon [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{V}$ preserves Φ-limits, and $a \in \mathcal{V}$ is *Φ-presentable* if $\mathcal{V}(a, –)\colon \mathcal{V} \to \mathcal{V}$ preserves colimits by Φ-flat weights.

Suppose that *if $\mathcal{C}$ is small and Φ-complete, then every Φ-continuous $F\colon \mathcal{C} \to \mathcal{V}$ is Φ-flat*; this is Axiom A of [23]. Then by Proposition 3.4 and §7.1 of *ibid.*, we obtain an instance of our setting on taking $\mathcal{E} = \mathcal{V}$ and $\mathcal{A} = \mathcal{V}_\Phi$ a skeleton of the full sub-$\mathcal{V}$-category of Φ-presentable objects.

A key example takes $\mathcal{V} = \mathcal{E} = \mathbf{Cat}$ and Φ the class of finite products; whereupon $\mathcal{V}_\Phi$ is the subcategory $\mathbb{F}$ of finite cardinals, seen as discrete categories. This example captures *strongly finitary* [19] structure on categories involving functors and transformations $\mathcal{C}^n \to \mathcal{C}$; this includes *monoidal* or *finite product* structure, but not *finite limit* structure.

(ix) More generally, we can take $\mathcal{E} = \Phi\text{-}\mathbf{Cts}(\mathcal{C}, \mathcal{V})$, the $\mathcal{V}$-category of Φ-continuous functors $\mathcal{C} \to \mathcal{V}$ for some small Φ-complete $\mathcal{C}$, and take $\mathcal{A}$ to be the full image of the Yoneda embedding $Y : \mathcal{C}^{\mathrm{op}} \to \Phi\text{-}\mathbf{Cts}(\mathcal{C}, \mathcal{V})$. This example is appropriate to the study of "Φ-ary algebraic structure on $\mathcal{E}$-objects"—subsuming most of the preceding examples.

3.2. **Pretheories as presentations.** We will now describe examples of pretheories and their models in various contexts; in doing so, it will be useful to avail ourselves of the following constructions. Given a pretheory $\mathcal{A} \to \mathcal{T}$ and objects $a, b \in \mathcal{T}$, to *adjoin a morphism* $f\colon a \to b$ is to form the $\mathcal{V}$-category $\mathcal{T}[f]$ in the pushout square to the left of:

$$\begin{tikzcd} 2 \arrow[r, "\langle a,b\rangle"] \arrow[d, "\iota"'] & \mathcal{T} \arrow[d, "\bar\iota"] \\ \mathbf{2} \arrow[r, "f"] & \mathcal{T}[f] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbf{2} +_2 \mathbf{2} \arrow[r, "\langle f,g\rangle"] \arrow[d, "\langle \mathrm{id},\mathrm{id}\rangle"'] & \mathcal{T} \arrow[d, "\bar\iota"] \\ \mathbf{2} \arrow[r, "f=g"] & \mathcal{T}[f=g] \end{tikzcd} . \tag{3.2}$$

Here, $\iota\colon 2 \to \mathbf{2}$ is the inclusion of the free $\mathcal{V}$-category on the set $\{0, 1\}$ into the free $\mathcal{V}$-category $\mathbf{2} = \{0 \to 1\}$ on an arrow. Since ι is identity-on-objects, its pushout $\bar\iota$ may also be chosen thus, so that we may speak of adjoining an arrow to a pretheory $J\colon \mathcal{A} \to \mathcal{T}$ to obtain the pretheory $J[f] = \bar\iota \circ J\colon \mathcal{A} \to \mathcal{T}[f]$.

Recall from (2.5) that a concrete $\mathcal{T}$-model comprises $X \in \mathcal{E}$ and $F \in [\mathcal{T}^{\mathrm{op}}, \mathcal{V}]$ for which $F \circ J^{\mathrm{op}} = \mathcal{E}(K\text{–}, X)\colon \mathcal{A} \to \mathcal{V}$. Thus, by the universal property of the pushout (3.2), a concrete $\mathcal{T}[f]$-model is the same as a concrete $\mathcal{T}$-model (X, F) together with a map $[f]\colon \mathcal{E}(Kb, X) \to \mathcal{E}(Ka, X)$ in $\mathcal{V}$.

Similarly given parallel morphisms $f, g\colon a \rightrightarrows b$ in the underlying category of $\mathcal{T}$ we can form the pushout above right. In this way we may speak of adjoining an equation $f = g$ to a pretheory $J\colon \mathcal{A} \to \mathcal{T}$ to obtain the pretheory $J[f=g] = \bar\iota \circ J\colon \mathcal{A} \to \mathcal{T}[f=g]$. In this case, we see that a concrete $\mathcal{T}[f=g]$-model is a concrete $\mathcal{T}$-model (X, F) such that $Ff = Fg\colon \mathcal{E}(Kb, X) \to \mathcal{E}(Ka, X)$.

Example 9. In the context of Examples 8(i) appropriate to classical finitary algebraic theories— so $\mathcal{V} = \mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$—we will construct a pretheory $J\colon \mathbb{F} \to \mathcal{M}$ whose category of concrete models is the category of monoids.

We start from the initial pretheory $\mathrm{id}\colon \mathbb{F} \to \mathbb{F}$ whose concrete models are simply sets, and construct from it a pretheory $J_1\colon \mathbb{F} \to \mathcal{M}_1$ by adjoining morphisms

$$m\colon 1 \to 2 \qquad \text{and} \qquad i\colon 1 \to 0 \tag{3.3}$$

representing the monoid multiplication and unit operations, and also morphisms

$$m1, 1m \colon 2 \rightrightarrows 3 \qquad \text{and} \qquad i1, 1i \colon 2 \rightrightarrows 1 \tag{3.4}$$

which will be necessary later to express the monoid equations. Note that our directional conventions mean that the input arity of these operations is in the *codomain* rather than the domain. It follows from the preceding remarks that a concrete $\mathcal{M}_1$-model is a set X equipped with functions

$$[m] \colon X^2 \to X \ , \quad [i] \colon 1 \to X \ , \quad [m1], [1m] \colon X^3 \rightrightarrows X^2 \ , \quad [i1], [1i] \colon 1 \rightrightarrows X$$

interpreting the morphisms adjoined above. We now adjoin to $\mathcal{M}_1$ the eight equations necessary to render commutative the following squares in $\mathcal{M}_1$:

$$\begin{tikzcd} 1 \arrow[r, "m"] \arrow[d, "\iota_1"'] & 2 \arrow[d, "\iota_1"] \\ 1+1 \arrow[r, "m1"] & 2+1 \\ 1 \arrow[u, "\iota_2"] \arrow[r, "\mathrm{id}"] & 1 \arrow[u, "\iota_2"'] \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r, "\mathrm{id}"] \arrow[d, "\iota_1"'] & 1 \arrow[d, "\iota_1"] \\ 1+1 \arrow[r, "1m"] & 1+2 \\ 1 \arrow[u, "\iota_2"] \arrow[r, "m"] & 2 \arrow[u, "\iota_2"'] \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r, "i"] \arrow[d, "\iota_1"'] & 0 \arrow[d, "!"] \\ 1+1 \arrow[r, "i1"] & 1 \\ 1 \arrow[u, "\iota_2"] \arrow[r, "\mathrm{id}"] & 1 \arrow[u, "\mathrm{id}"'] \end{tikzcd} \quad \begin{tikzcd} 1 \arrow[r, "\mathrm{id}"] \arrow[d, "\iota_1"'] & 1 \arrow[d, "\iota_1"] \\ 1+1 \arrow[r, "1i"] & 1 \\ 1 \arrow[u, "\iota_2"] \arrow[r, "i"] & 0 \arrow[u, "!"'] \end{tikzcd} \tag{3.5}$$

where ι_1, ι_2 and ! are the images under J_1 of the relevant coproduct injections or maps from 0 in $\mathbb{F}$; together with three equations which render commutative:

$$\begin{tikzcd} 1 \arrow[r, "m"] \arrow[d, "m"'] & 2 \arrow[d, "1m"] \\ 2 \arrow[r, "m1"] & 3 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "m"] \arrow[dr, "1"'] & 2 \arrow[d, "i1"] \\ & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "m"] \arrow[dr, "1"'] & 2 \arrow[d, "1i"] \\ & 1 \end{tikzcd} . \tag{3.6}$$

A concrete model for the resulting theory $J \colon \mathbb{F} \to \mathcal{M}$ is a concrete $\mathcal{M}_1$-model (X, F) for which $F^{\mathrm{op}} \colon \mathcal{M}_1 \to \mathbf{Set}^{\mathrm{op}}$ sends each diagram in (3.5) and (3.6) to a commuting one. Commutativity in (3.5) forces $[m1] = [m] \times \mathrm{id} \colon X^3 \to X^2$ and so on; whereupon commutativity of (3.6) expresses precisely the monoid axioms, so that concrete $\mathcal{M}$-models are monoids, as desired. Extending this analysis to morphisms we see that $\mathbf{Mod}_c(\mathcal{M})$ is isomorphic to the category of monoids and monoid homomorphisms.

Example 10. In the same way we can describe $\mathbb{F}$-pretheories modelling any of the categories of classical universal algebra—groups, rings and so on. Note that the same structure can be presented by distinct pretheories: for instance, we could extend the pretheory $\mathcal{M}$ of the preceding example by adjoining a further morphism $m11 \colon 3 \to 4$ and two equations forcing it to become $[m] \times 1 \times 1 \colon X^4 \to X^3$ in any model; on doing so, we would not change the category of concrete models. However, in $\mathcal{M}$, all of the maps $3 \to 4$ belong to $\mathbb{F}$ while in the new pretheory, $m11$ does not. This non-canonicity will be rectified by the *theories* introduced in Section 4 below; in particular, Corollary 24 implies that, to within isomorphism, there is at most one $\mathbb{F}$-theory which captures a given type of structure.

Example 11. In the situation of Examples 8(iv), where $\mathcal{E} = [\mathbb{G}_0^{\mathrm{op}}, \mathbf{Set}]$ is the category of directed graphs and $\mathcal{A} = \Delta_0$, we will describe a pretheory $\Delta_0 \to \mathcal{C}$ whose concrete models are categories. The construction is largely identical to the example of monoids above. Starting from the initial Δ_0-pretheory, we adjoin composition and unit maps $m\colon [1] \to [2]$ and $i\colon [1] \to [0]$ as well as the morphisms $1m, m1\colon [2] \rightrightarrows [3]$ and $i1, 1i\colon [2] \rightrightarrows [1]$ required to describe the category axioms.

We now adjoin the necessary equations. First, we have four equations ensuring that composition and identities interact appropriately with source and target:

$$\begin{tikzcd} {[0]} \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & {[1]} \arrow[d, "m"] \\ {[1]} \arrow[r, "\iota_1"] & {[2]} \end{tikzcd} \qquad \begin{tikzcd} {[0]} \arrow[r, "\tau"] \arrow[d, "\tau"'] & {[1]} \arrow[d, "m"] \\ {[1]} \arrow[r, "\iota_2"] & {[2]} \end{tikzcd} \qquad \begin{tikzcd} {[0]} \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}"'] & {[1]} \arrow[d, "i"] \\ & {[0]} \end{tikzcd} \qquad \begin{tikzcd} {[0]} \arrow[r, "\tau"] \arrow[dr, "\mathrm{id}"'] & {[1]} \arrow[d, "i"] \\ & {[0]} \end{tikzcd}$$

where here we write $\sigma, \tau\colon [0] \rightrightarrows [1]$ for the two endpoint inclusions, and ι_1, ι_2 for the two colimit injections into $[1]\,{}_{\tau}{+}_{\sigma}\,[1] = [2]$. We also require analogues of the eight equations of (3.5) and three equations of (3.6). The modifications are minor: replace n by $[n]$, the coproduct inclusions $\iota_1\colon n \to n + m \leftarrow m\colon \iota_2$ by the pushout inclusions $\iota_1\colon [n] \to [n]\,{}_{\tau}{+}_{\sigma}\,[m] \leftarrow [m]\colon \iota_2$, the first appearance of $!\colon 0 \to 1$ by $\sigma\colon [0] \to [1]$ and its second appearance by $\tau\colon [0] \to [1]$. After adjoining these six morphisms and fifteen equations, we find that the concrete models of the resulting pretheory $\Delta_0 \to \mathcal{C}$ are precisely *small categories*.

We can extend this pretheory to one for groupoids. To do so, we adjoin a morphism $c\colon [1] \to [1]$ modelling the inversion plus the further maps $1c\colon [2] \to [2]$ and $c1\colon [2] \to [2]$ required for the axioms. Now four equations must be adjoined to force the correct interpretation of $1c$ and $c1$, plus the two equations for left and right inverses. On doing so, the resulting pretheory $\Delta_0 \to \mathcal{G}$ has as its concrete models the *small groupoids*.

Example 12. In the situation of Examples 8(v), where $\mathcal{E}$ is the category of globular sets and $\mathcal{A} = \Theta_0$ is the full subcategory of globular cardinals, one can similarly construct pretheories whose concrete models are strict ω-categories or strict ω-groupoids. For instance, one encodes binary composition of n-cells along a k-cell boundary (for $k < n$) by adjoining morphisms $m_{n,k}\colon Y(n) \to Y(n, k, n)$ to Θ_0. In fact, all of the standard flavours of globular *weak* ω-category and *weak* ω-groupoid can also be encoded using Θ_0-pretheories; see Examples 44(v) below.

Example 13. Consider the case of Examples 8(viii) where $\mathcal{V} = \mathcal{E} = \mathbf{Cat}$ and $\mathcal{A} = \mathbb{F}$, the full subcategory of finite cardinals (seen as discrete categories). We will describe an $\mathbb{F}$-pretheory capturing the structure of a monoidal category. In doing so, we exploit the fact that our pretheories are no longer mere categories, but 2-categories; so we may speak not only of adjoining morphisms and equations between such, but also of *adjoining an (invertible) 2-cell*—by taking a pushout of the inclusion $\mathbf{2} +_2 \mathbf{2} \to D_2$ of the parallel pair 2-category into the free 2-category on an (invertible) 2-cell—and similarly of *adjoining an equation between 2-cells*.

To construct a pretheory for monoidal categories, we start essentially as for monoids: freely adjoining the usual maps $m, i, m1, 1m, i1, 1i$ to the initial pretheory, but now also morphisms $m11, 1m1, 11m\colon 3 \to 4$ and $1i1\colon 3 \to 2$ needed for the monoidal category coherence axioms; thus, ten morphisms in all.

We now add the $8 \times 2 = 16$ equations asserting that each of the morphisms beyond m and i has the expected interpretation in a model, plus[1] the equation $1m \circ m11 = m1 \circ 11m \colon 2 \to 4$. This being done, we next adjoin invertible 2-cells

$$
\begin{tikzcd}
1 \arrow[r, "m"] \arrow[d, "m"'] \arrow[dr, phantom, "\Downarrow\alpha"] & 2 \arrow[d, "m1"] \\
2 \arrow[r, "1m"'] & 3
\end{tikzcd}
\qquad
\begin{tikzcd}
& 2 \arrow[d, "i1"] \\
1 \arrow[ur, bend left, "m"] \arrow[r, "1"'] \arrow[ur, phantom, "\Downarrow\lambda"] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
& 2 \arrow[d, "1i"] \\
1 \arrow[ur, bend left, "m"] \arrow[r, "1"'] \arrow[ur, phantom, "\Uparrow\rho"] & 1
\end{tikzcd}
$$

expressing the associativity and unit coherences, as well as the invertible 2-cells

$$
\begin{tikzcd}
2 \arrow[r, "m1"] \arrow[d, "m1"'] \arrow[dr, phantom, "\Downarrow\alpha1"] & 3 \arrow[d, "m11"] \\
3 \arrow[r, "1m1"'] & 4
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[r, "1m"] \arrow[d, "1m"'] \arrow[dr, phantom, "\Downarrow1\alpha"] & 3 \arrow[d, "1m1"] \\
3 \arrow[r, "11m"'] & 4
\end{tikzcd}
\qquad
\begin{tikzcd}
& 3 \arrow[d, "1i1"] \\
2 \arrow[ur, bend left, "1m"] \arrow[r, "1"'] \arrow[ur, phantom, "\Downarrow1\lambda"] & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
& 3 \arrow[d, "1i1"] \\
2 \arrow[ur, bend left, "m1"] \arrow[r, "1"'] \arrow[ur, phantom, "\Uparrow\rho1"] & 2
\end{tikzcd}
$$

which will be needed to express the coherence axioms. Finally, we must adjoin equations between 2-cells: the $2 \times 4 = 8$ equations ensuring that $\alpha1$, 1α, 1λ and $\rho1$ have the intended interpretation in any model, plus two equations expressing the coherence axioms:

$$
\begin{tikzcd}
& 2 \arrow[r, "m1"] \arrow[d, phantom, "\Downarrow\alpha"] & 3 \arrow[dr, "m11"] & \\
1 \arrow[ur, "m"] \arrow[r, "m"] \arrow[dr, "m"'] & 2 \arrow[ur, "1m"'] \arrow[dr, "m1"] \arrow[d, phantom, "\Downarrow\alpha"] & & 4 \\
& 2 \arrow[r, "1m"'] & 3 \arrow[ur, "11m"'] &
\end{tikzcd}
=
\begin{tikzcd}
& 2 \arrow[r, "m1"] \arrow[dr, "m1"'] \arrow[d, phantom, "\Downarrow\alpha"] & 3 \arrow[dr, "m11"] \arrow[d, phantom, "\Downarrow\alpha1"] & \\
1 \arrow[ur, "m"] \arrow[dr, "m"'] & {} & 3 \arrow[r, "1m1"] \arrow[d, phantom, "\Downarrow1\alpha"] & 4 \\
& 2 \arrow[ur, "1m"] \arrow[r, "1m"'] & 3 \arrow[ur, "11m"'] &
\end{tikzcd}
\tag{3.7}
$$

$$
\begin{tikzcd}
& 2 \arrow[dr, "m1"'] \arrow[drr, bend left, "\mathrm{id}"] \arrow[dr, phantom, "\Downarrow\rho1", shift left=3] & & \\
1 \arrow[ur, "m"] \arrow[dr, "m"'] \arrow[r, phantom, "\Downarrow\alpha"] & {} & 3 \arrow[r, "1i1"] & 2 \\
& 2 \arrow[ur, "1m"] \arrow[urr, bend right, "\mathrm{id}"'] \arrow[ur, phantom, "\Downarrow1\lambda", shift right=3] & &
\end{tikzcd}
=
\begin{tikzcd}
& 2 \arrow[drr, bend left, "\mathrm{id}"] & & \\
1 \arrow[ur, "m"] \arrow[dr, "m"'] & & \Downarrow\mathrm{id} & 2 \\
& 2 \arrow[urr, bend right, "\mathrm{id}"'] & &
\end{tikzcd}
. \tag{3.8}
$$

All told, we have adjoined ten morphisms, seventeen equations between morphisms, seven invertible 2-cells, and nine equations between 2-cells to obtain a pretheory $J \colon \mathbb{F} \to \mathcal{MC}$ whose concrete models are precisely monoidal categories.

4. The monad–theory correspondence

In this section, we return to the general theory and establish our "best possible" monad–theory correspondence. This will be obtained by restricting the adjunction (2.1) to its *fixpoints*: the objects on the left and right at which the counit and the unit are invertible. The categories of fixpoints are the *largest*

[1]It may be prima facie unclear why this is necessary; after all, if $1m, m11, m1$ and $11m$ have the intended interpretations in a model, then it is certainly the case that they will verify this equality. Yet this equality is not forced to hold *in the pretheory*, and we need it to do so in order for (3.7) to type-check.

subcategories on which the adjunction becomes an adjoint equivalence, and it is in this sense that our monad–theory correspondence is the best possible.

4.1. **A pullback lemma.** The following lemma will be crucial in characterising the fixpoints of (2.1) on each side. Note that the force of (2) below is in the "if" direction; the "only if" is always true.

Lemma 14. *A commuting square in* $\mathcal{V}$-**CAT**

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "H"'] & \mathcal{B} \arrow[d, "K"] \\ \mathcal{C} \arrow[r, "G"] & \mathcal{D} \end{tikzcd}$$

with G fully faithful and H, K discrete isofibrations is a pullback just when:

(1) F is fully faithful; and
(2) An object $b \in \mathcal{B}$ is in the essential image of F if and only if Kb is in the essential image of G.

Proof. If the square is a pullback, then F is fully faithful as a pullback of G. As for (2), if $Kb \cong Gc$ in $\mathcal{D}$ then since K is an isofibration we can find $b \cong b'$ in $\mathcal{B}$ with $Kb' = Gc$; now by the pullback property we induce $a \in \mathcal{A}$ with $Fa = b'$ so that $b \cong Fa$ as required. Suppose conversely that (1) and (2) hold. We form the pullback $\mathcal{P}$ of K along G and the induced map L as below.

$$(4.1) \qquad \begin{tikzcd} \mathcal{A} \arrow[dr, dashed, "L"] \arrow[drr, bend left, "F"] \arrow[ddr, bend right, "H"'] & & \\ & \mathcal{P} \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "K"] \\ & \mathcal{C} \arrow[r, "G"] & \mathcal{D} \end{tikzcd}$$

P is fully faithful as a pullback of G, and F is so by assumption; whence by standard cancellativity properties of fully faithful functors, L is also fully faithful.

In fact, discrete isofibrations are also stable under pullback, and also have the same cancellativity property; this follows from the fact that they are the exactly the maps with the unique right lifting property against the inclusion of the free $\mathcal{V}$-category on an object into the free $\mathcal{V}$-category on an isomorphism. Consequently, in (4.1), Q is a discrete isofibration as a pullback of K, and H is so by assumption; whence by cancellativity, L is also a discrete isofibration.

If we can now show L is also essentially surjective, we will be done: for then L is a discrete isofibration and an equivalence, whence invertible. So let $(b, c) \in \mathcal{P}$. Since $Kb = Gc$, by (2) we have that b is in the essential image of F. So there is $a \in \mathcal{A}$ and an isomorphism $\beta \colon b \cong Fa$. Now $K\beta \colon Gc = Kb \cong KFa = GHa$ so by full fidelity of G there is $\gamma \colon c \cong Ha$ with $G\gamma = K\beta$; and so we have $(\beta, \gamma) \colon (b, c) \cong La$ exhibiting (b, c) as in the essential image of L, as required. $\square$

4.2. **$\mathcal{A}$-theories.** We first use the pullback lemma to describe the fixpoints of (2.1) on the pretheory side.

Definition 15. An $\mathcal{A}$-pretheory $J\colon \mathcal{A} \to \mathcal{T}$ is said to be an $\mathcal{A}$-*theory* if each $\mathcal{T}(J{-}, a) \in [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is a K-nerve. We write $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$ for the full subcategory of $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ on the $\mathcal{A}$-theories.

In the language of Section 5.2 below, a pretheory $\mathcal{T}$ is an $\mathcal{A}$-theory just when each representable $\mathcal{T}(-, a)\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ is a (*non-concrete*) $\mathcal{T}$-model. When $\mathcal{V} = \mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$, an $\mathcal{A}$-pretheory is an $\mathcal{A}$-theory precisely when it is a Lawvere theory; see Examples 44(i) below.

Theorem 16. *An $\mathcal{A}$-pretheory $J\colon \mathcal{A} \to \mathcal{T}$ is an $\mathcal{A}$-theory if and only if the unit component $\eta_{\mathcal{T}}\colon \mathcal{T} \to \Phi\Psi\mathcal{T}$ of* (2.1) *is invertible.*

Proof. The unit $\eta_{\mathcal{T}}\colon \mathcal{T} \to \Phi\Psi\mathcal{T}$ is obtained by starting with $\alpha_0 = 1\colon \Psi\mathcal{T} \to \Psi\mathcal{T}$ and chasing through the bijections of Theorem 6 to obtain $\alpha_6 = \eta_{\mathcal{T}}$. Doing this, we quickly arrive at α_2 equal to P, the projection in the depicted pullback square

$$\begin{tikzcd}
\mathcal{E}^{\Psi\mathcal{T}} \arrow[r, "P"] \arrow[d, "U^{\Psi\mathcal{T}}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] & \mathcal{A}^{\mathrm{op}} \arrow[rr, "J^{\mathrm{op}}"] \arrow[d, "(J_{\Psi\mathcal{T}})^{\mathrm{op}}"'] & & \mathcal{T}^{\mathrm{op}} \arrow[d, "\alpha_4"] \arrow[dll, dotted, "\alpha_6^{\mathrm{op}}"'] \arrow[dl, dotted, "\alpha_5{}^{\mathrm{op}}"] \\
\mathcal{E} \arrow[r, "N_K"] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} & (\mathcal{A}_{\Psi\mathcal{T}})^{\mathrm{op}} \arrow[r, "(K_{\Psi\mathcal{T}})^{\mathrm{op}}"'] & (\mathcal{E}^{\Psi\mathcal{T}})^{\mathrm{op}} \arrow[r, "Y"'] & {[\mathcal{E}^{\Psi\mathcal{T}}, \mathcal{V}]_{\mathrm{rep}}}
\end{tikzcd} \tag{4.2}$$

defining $\mathcal{E}^{\Psi\mathcal{T}}$. Now $\alpha_3\colon \mathcal{E}^{\Psi\mathcal{T}} \to [\mathcal{T}^{\mathrm{op}}, \mathcal{V}]$ is obtained by lifting an isomorphism through $[J^{\mathrm{op}}, 1]$ and so we have $\alpha_3 \cong P$. We obtain α_4 by transposing α_3 through the isomorphism $(-)^t\colon \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}^{\Psi\mathcal{T}}, [\mathcal{T}^{\mathrm{op}}, \mathcal{V}])_{\mathrm{pwr}} \cong \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{T}^{\mathrm{op}}, [\mathcal{E}^{\Psi\mathcal{T}}, \mathcal{V}]_{\mathrm{rep}})$ displayed in (2.12). The relationships between α_4, α_5 and the unit component $\eta_{\mathcal{T}} = \alpha_6$ are depicted in the commutative diagram above right.

The identity-on-objects unit $\eta_{\mathcal{T}} = \alpha_6$ will be invertible just when it is fully faithful which, since $K_{\Psi\mathcal{T}}$ is fully faithful, will be so just when α_5 is fully faithful. Now, since $P \cong \alpha_3 = (\alpha_4)^t = (Y \circ \alpha_5^{\mathrm{op}})^t = N_{\alpha_5}$, and P is fully faithful, as the pullback of the fully faithful N_K, it follows that $N_{\alpha_5}\colon \mathcal{E}^{\Psi\mathcal{T}} \to [\mathcal{T}^{\mathrm{op}}, \mathcal{V}]$ is also fully faithful. As a consequence, α_5 is fully faithful just when there exists a factorisation to within isomorphism:

$$Y \cong N_{\alpha_5} \circ G\colon \mathcal{T} \to \mathcal{E}^{\Psi\mathcal{T}} \to [\mathcal{T}^{\mathrm{op}}, \mathcal{V}]\ . \tag{4.3}$$

Indeed, in one direction, if α_5 is fully faithful then the canonical natural transformation $Y \Rightarrow N_{\alpha_5} \circ \alpha_5$ is invertible. In the other, given a factorisation as displayed, G is fully faithful since N_{α_5} and Y are. Moreover we have isomorphisms

$$\mathcal{E}^{\Psi\mathcal{T}}(\alpha_5 b, -) \cong [\mathcal{T}^{\mathrm{op}}, \mathcal{V}](Yb, N_{\alpha_5}-) \cong [\mathcal{T}^{\mathrm{op}}, \mathcal{V}](N_{\alpha_5}Gb, N_{\alpha_5}-) \cong \mathcal{E}^{\Psi\mathcal{T}}(Gb, -)$$

natural in b. So by Yoneda, $\alpha_5 \cong G$ and so α_5 is fully faithful since G is so.

This shows that $\eta_{\mathcal{T}}$ is invertible just when there is a factorisation (4.3). Since N_{α_5} is fully faithful this in turn is equivalent to asking that each $Yb = \mathcal{T}(-, b)$ lies in the essential image of α_5, or equally in the essential image of the isomorphic P. As the left square of (4.2) is a pullback, Lemma 14 asserts that this is, in turn, equivalent to each $[J^{\mathrm{op}}, 1](Yb) = \mathcal{T}(J{-}, b)$ being in the essential image of N_K; which is precisely the condition that J is an $\mathcal{A}$-theory. $\square$

4.3. **$\mathcal{A}$-nervous monads.** We now characterise the fixpoints on the monad side. In the following definition, $\mathcal{A}_\mathsf{T}$, J_T and K_T are as in (2.7).

Definition 17. A $\mathcal{V}$-monad T on $\mathcal{E}$ is called $\mathcal{A}$-*nervous* if

(i) The fully faithful $K_\mathsf{T} \colon \mathcal{A}_\mathsf{T} \to \mathcal{E}^\mathsf{T}$ is dense;
(ii) A presheaf $X \in [\mathcal{A}_\mathsf{T}{}^{\mathrm{op}}, \mathcal{V}]$ is a K_T-nerve if and only if $X \circ J_\mathsf{T}{}^{\mathrm{op}}$ is a K-nerve.

We write $\mathbf{Mnd}_\mathcal{A}(\mathcal{E})$ for the full subcategory of $\mathbf{Mnd}(\mathcal{E})$ on the $\mathcal{A}$-nervous monads.

Note that the adjointness isomorphisms $\mathcal{E}^\mathsf{T}(K_\mathsf{T} J_\mathsf{T} X, Y) = \mathcal{E}^\mathsf{T}(F^\mathsf{T} K X, Y) \cong \mathcal{E}(KX, U^\mathsf{T} Y)$ for the adjunction $F^\mathsf{T} \dashv U^\mathsf{T}$ give a pseudo-commutative square

$$\begin{tikzcd} \mathcal{E}^\mathsf{T} \arrow[r, "N_{K_\mathsf{T}}"] \arrow[d, "U^\mathsf{T}"'] \arrow[dr, phantom, "\cong"] & {[\mathcal{A}_\mathsf{T}{}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J_\mathsf{T}{}^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[r, "N_K"'] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} \; ; \tag{4.4}$$

as a result of which, $[J_\mathsf{T}^{\mathrm{op}}, 1]$ maps K_T-nerves to K-nerves. Thus the force of clause (ii) of the preceding definition lies in the *if* direction.

Theorem 18. *The counit component $\varepsilon_\mathsf{T} \colon \Psi\Phi\mathsf{T} \to \mathsf{T}$ of (2.1) at a monad T on $\mathcal{E}$ is invertible if and only if T is $\mathcal{A}$-nervous.*

Proof. ε_T is obtained by taking $\alpha_6 = 1 \colon J_\mathsf{T} \to J_\mathsf{T}$ and proceeding in reverse order through the series of six natural isomorphisms in the proof of Theorem 6. Doing this, we quickly reach $\alpha_3 = N_{K_\mathsf{T}}$. Then $\alpha_2 \colon \mathcal{E}^\mathsf{T} \to [(\mathcal{A}_\mathsf{T})^{\mathrm{op}}, \mathcal{V}]$ is obtained by lifting the natural isomorphism φ of (4.4) through the discrete isofibration $[J_\mathsf{T}^{\mathrm{op}}, 1]$, yielding a commutative square as left below.

$$\begin{tikzcd} \mathcal{E}^\mathsf{T} \arrow[r, "\alpha_2"] \arrow[d, "U^\mathsf{T}"'] & {[(\mathcal{A}_\mathsf{T})^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J_\mathsf{T}^{\mathrm{op}},1]}"] \\ \mathcal{E} \arrow[r, "N_K"'] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^\mathsf{T} \arrow[r, "\alpha_1"] \arrow[dr, "U^\mathsf{T}"'] & \mathcal{E}^{\Psi\Phi\mathsf{T}} \arrow[d, "U^{\Psi\Phi\mathsf{T}}"] \\ & \mathcal{E}. \end{tikzcd} \tag{4.5}$$

The map $\alpha_1 \colon \mathcal{E}^\mathsf{T} \to \mathcal{E}^{\Psi\Phi\mathsf{T}}$ is the unique map to the pullback, and $\alpha_0 = \varepsilon_\mathsf{T}$ the corresponding morphism of monads. It follows that ε_T is invertible if and only the square to the left of (4.5) is a pullback. Both vertical legs are discrete isofibrations and N_K is fully faithful, so by Lemma 14 this happens just when, firstly, α_2 is fully faithful, and, secondly, $X \in [\mathcal{A}_\mathsf{T}^{\mathrm{op}}, \mathcal{V}]$ is in the essential image of α_2 if and only if XJ_T is a K-nerve. But as $\alpha_2 \cong N_{K_\mathsf{T}}$, and natural isomorphism does not change either full fidelity or essential images, this happens just when T is $\mathcal{A}$-nervous. ☐

4.4. **The monad–theory equivalence.** Putting together the preceding results now yields the main result of this paper.

Theorem 19. *The adjunction (2.1) restricts to an adjoint equivalence*

$$\begin{tikzcd} \mathbf{Mnd}_\mathcal{A}(\mathcal{E}) \arrow[r, shift right, "\Phi"'] \arrow[r, phantom, "\perp"] & \mathbf{Th}_\mathcal{A}(\mathcal{E}) \arrow[l, shift right, "\Psi"'] \end{tikzcd} \tag{4.6}$$

between the category of $\mathcal{A}$-nervous monads and the category of $\mathcal{A}$-theories.

Proof. Any adjunction restricts to an adjoint equivalence between the objects with invertible unit and counit components respectively, and Theorems 16 and 18 identify these objects as the $\mathcal{A}$-theories and the $\mathcal{A}$-nervous monads. □

Note that there is an asymmetry between the conditions found on each side. On the one hand, the condition characterising the $\mathcal{A}$-theories among the $\mathcal{A}$-pretheories is intrinsic, and easy to check in practice. On the other hand, the condition defining an $\mathcal{A}$-nervous monad refers to the associated pretheory, and is non-trivial to check in practice. Indeed, one of the main points of [35, 9] is to provide a general set of *sufficient* conditions under which a monad can be shown to be $\mathcal{A}$-nervous.

In the sections which follow, we will provide a number of more tractable characterisations of the $\mathcal{A}$-theories and $\mathcal{A}$-nervous monads; the crucial fact which drives all of these is that the adjunction (2.1) is in fact idempotent. Recall that an adjunction $L \dashv R \colon \mathcal{D} \to \mathcal{C}$ is *idempotent* if the monad RL on $\mathcal{C}$ is idempotent, and that this is equivalent to asking that the comonad LR is idempotent, or that any one of the natural transformations $R\varepsilon$, εL, ηR and $L\eta$ is invertible.

Theorem 20. *The adjunction* (2.1) *is idempotent.*

Proof. We show for each $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ that the unit $\eta_{\Phi\mathsf{T}} \colon \Phi\mathsf{T} \to \Phi\Psi\Phi\mathsf{T}$ is invertible. By Theorem 16, this is equally to show that $J_\mathsf{T} \colon \mathcal{A} \to \mathcal{A}_\mathsf{T}$ is an $\mathcal{A}$-theory, i.e., that each $\mathcal{A}_\mathsf{T}(J_\mathsf{T}–, J_\mathsf{T}a) \in [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is a K-nerve. But $\mathcal{A}_\mathsf{T}(J_\mathsf{T}–, J_\mathsf{T}a) \cong \mathcal{E}^\mathsf{T}(F^\mathsf{T}K–, F^\mathsf{T}Ka) \cong \mathcal{E}(K–, U^\mathsf{T}F^\mathsf{T}Ka) = \mathcal{E}(K–, TKa)$ as required. □

Exploiting the alternative characterisations of idempotent adjunctions listed above, we immediately obtain the following result, which tells us in particular that a monad T is $\mathcal{A}$-nervous if and only if it can be presented by some $\mathcal{A}$-pretheory.

Corollary 21. *A monad T on $\mathcal{E}$ is $\mathcal{A}$-nervous if and only if $\mathsf{T} \cong \Psi\mathcal{T}$ for some $\mathcal{A}$-pretheory $J \colon \mathcal{A} \to \mathcal{T}$; while an $\mathcal{A}$-pretheory $J \colon \mathcal{A} \to \mathcal{T}$ is an $\mathcal{A}$-theory if and only if $\mathcal{T} \cong \Phi\mathsf{T}$ for some monad T on $\mathcal{E}$.*

The next result also follows directly from the definition of idempotent adjunction.

Corollary 22. *The full subcategory $\mathbf{Mnd}_\mathcal{A}(\mathcal{E}) \subseteq \mathbf{Mnd}(\mathcal{E})$ is coreflective via $\Psi\Phi$, while the full subcategory $\mathbf{Th}_A(\mathcal{E}) \subseteq \mathbf{Preth}_\mathcal{A}(\mathcal{E})$ is reflective via $\Phi\Psi$.*

5. Semantics

In the next section, we will explicitly identify the $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories for the examples listed in Section 2.1. Before doing this, we study further aspects of the general theory, namely those related to the taking of semantics.

5.1. **Interaction with the semantics functors.** We begin by examining the interaction of our monad–theory correspondence with the semantics functors of Section 2. In fact, we begin at the level of the monad–pretheory adjunction (2.1).

Proposition 23. *There is a natural isomorphism θ as on the left in:*

$$\begin{tikzcd} \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[r, "\Psi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Mod}_c"'] & \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \arrow[d, phantom, "\overset{\theta}{\Leftarrow}"] \arrow[dl, "\mathrm{Alg}"] \\ & \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \arrow[r, "\Phi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Alg}"'] & \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Mod}_c"] \\ & \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}\ . \end{tikzcd}$$

Let $\bar{\theta}$ be its mate under the adjunction $\Phi^{\mathrm{op}} \dashv \Psi^{\mathrm{op}}$, as right above. The component of $\bar{\theta}$ at $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is invertible if and only if T is $\mathcal{A}$-nervous.

Proof. For the first claim, Theorem 6 provides the necessary natural isomorphisms $\theta_{\mathcal{T}} \colon \mathcal{E}^{\Psi\mathcal{T}} \to \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$. For the second, if we write as before $\varepsilon_{\mathsf{T}} \colon \Psi\Phi\mathsf{T} \to \mathsf{T}$ for the counit component of (2.1) at $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$, then the T-component of $\bar{\theta}$ is the composite $\theta_{\Psi\mathsf{T}} \circ (\varepsilon_{\mathsf{T}})^* \colon \mathcal{E}^{\mathsf{T}} \to \mathcal{E}^{\Psi\Phi\mathsf{T}} \to \mathbf{Mod}_c(\Phi\mathcal{T})$ over $\mathcal{E}$. Since $\theta_{\Psi\mathcal{T}}$ is invertible and since Alg is fully faithful, $\bar{\theta}_{\mathsf{T}}$ will be invertible just when ε_{T} is so; that is, by Theorem 18, just when T is $\mathcal{A}$-nervous. ☐

From this and the fact that each monad $\Psi\mathcal{T}$ is $\mathcal{A}$-nervous, it follows that *an $\mathcal{A}$-pretheory $\mathcal{T}$ and its associated theory $\Phi\Psi\mathcal{T}$ have isomorphic categories of concrete models*. By contrast, the passage from a monad T to its $\mathcal{A}$-nervous coreflection $\Psi\Phi\mathsf{T}$ may well change the category of algebras. For example, the power-set monad on **Set**, whose algebras are complete lattices, has its $\mathbb{F}$-nervous coreflection given by the finite-power-set monad, whose algebras are $\vee$-semilattices. However, if we restrict to $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories, then the situation is much better behaved.

Theorem 24. *The monad–theory equivalence* (4.6) *commutes with the semantics functors; that is, we have natural isomorphisms:*

$$\begin{tikzcd} \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[r, "\Psi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Mod}_c"'] & \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Alg}"] \\ & \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[r, "\Phi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Alg}"'] & \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Mod}_c"] \\ & \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}\ . \end{tikzcd} \tag{5.1}$$

Moreover, both $\mathrm{Mod}_c \colon \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *and* $\mathrm{Alg} \colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *are fully faithful functors.*

Proof. The first statement follows from Proposition 23. For the second, note that $\mathrm{Alg} \colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ is obtained by restricting the fully faithful $\mathrm{Alg} \colon \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ along a full embedding, and so is itself fully faithful. It follows that $\mathrm{Mod}_c \cong \mathrm{Alg} \circ \Psi^{\mathrm{op}} \colon \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ is also fully faithful. ☐

Full fidelity of $\mathrm{Mod}_c \colon \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ means that an $\mathcal{A}$-theory is determined to within isomorphism by its category of concrete models over $\mathcal{E}$. This rectifies the non-uniqueness of pretheories noted in Example 10 above.

5.2. **Non-concrete models.** In Section 2.3 we defined a *concrete model* of an $\mathcal{A}$-pretheory $\mathcal{T}$ to be an object $X \in \mathcal{E}$ endowed with an extension of $\mathcal{E}(K-, X) \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ to a functor $\mathcal{T}^{\mathrm{op}} \to \mathcal{V}$. In the literature, one often encounters a looser notion of model for a theory, in which an underlying object in $\mathcal{E}$ is not provided. In our setting, this notion becomes the following one: by an (unqualified) $\mathcal{T}$*-model*, we mean a functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ whose restriction $FJ^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a K-nerve.

The $\mathcal{T}$-models span a full sub-$\mathcal{V}$-category $\mathbf{Mod}(\mathcal{T})$ of $[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]$. Recalling from Section 2.1 that K-$\mathbf{Ner}(\mathcal{V})$ denotes the full sub-$\mathcal{V}$-category of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ on the K-nerves, we may also express $\mathbf{Mod}(\mathcal{T})$ as a pullback as to the right in:

$$\begin{tikzcd}
\mathbf{Mod}_c(\mathcal{T}) \arrow[r, dashed] \arrow[d, "U_{\mathcal{T}}"'] \arrow[rr, bend left, "P_{\mathcal{T}}"] & \mathbf{Mod}(\mathcal{T}) \arrow[r, hook] \arrow[d, "W_{\mathcal{T}}"'] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[J^{\mathrm{op}},1]}"] \\
\mathcal{E} \arrow[r, "N_K"] & K\text{-}\mathbf{Ner}(\mathcal{V}) \arrow[r, hook] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]}\ .
\end{tikzcd} \tag{5.2}$$

On the other hand, $\mathbf{Mod}_c(\mathcal{T})$ is the pullback around the outside, and so there is a canonical induced map $\mathbf{Mod}_c(\mathcal{T}) \to \mathbf{Mod}(\mathcal{T})$ as displayed. By the usual cancellativity properties, the left square above is now also a pullback. Moreover, $W_{\mathcal{T}}$ is an isofibration, as a pullback of the discrete isofibration $[J^{\mathrm{op}}, 1]$, and $N_K\colon \mathcal{E} \to K$-$\mathbf{Ner}(\mathcal{V})$ is an equivalence. Since equivalences are stable under pullback along isofibrations, we conclude that:

Proposition 25. *The comparison* $\mathbf{Mod}_c(\mathcal{T}) \to \mathbf{Mod}(\mathcal{T})$ *in* (5.2) *is an equivalence.*

Taking non-concrete models gives rise to a semantics functor landing in $\mathcal{V}$-$\mathbf{CAT}/K$-$\mathbf{Ner}(\mathcal{V})$ which, like before, is *not* fully faithful on $\mathcal{A}$-pretheories, but is so on the subcategory of $\mathcal{A}$-theories. Note that the "underlying K-nerve" of a $\mathcal{T}$-model is more natural than it might seem, being the special case of the functor $\mathbf{Mod}(\mathcal{T}) \to \mathbf{Mod}(\mathcal{S})$ induced by a morphism of $\mathcal{A}$-pretheories for which $\mathcal{S}$ is the initial pretheory. However, in the following result, for simplicity, we view the semantics functors for $\mathcal{T}$-models as landing simply in $\mathcal{V}$-$\mathbf{CAT}$.

Theorem 26. *The monad–theory equivalence* (4.6) *commutes with the non-concrete semantics functors in the sense that we have natural transformations*

$$\begin{tikzcd}
\mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[rr, "\Psi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Mod}"'] & \arrow[d, phantom, "\overset{\theta}{\Leftarrow}"] & \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Alg}"] \\
& \mathcal{V}\text{-}\mathbf{CAT} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[rr, "\Phi^{\mathrm{op}}"] \arrow[dr, "\mathrm{Alg}"'] & \arrow[d, phantom, "\overset{\bar\theta}{\Rightarrow}"] & \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \arrow[dl, "\mathrm{Mod}"] \\
& \mathcal{V}\text{-}\mathbf{CAT} &
\end{tikzcd}$$

whose components are equivalences in $\mathcal{V}$-$\mathbf{CAT}$.

Proof. First postcompose the natural isomorphisms (5.1) with the forgetful functor $\mathcal{V}$-$\mathbf{CAT}/\mathcal{E} \to \mathcal{V}$-$\mathbf{CAT}$. Then paste with the natural transformation $\mathrm{Mod}_c \Rightarrow \mathrm{Mod}\colon \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}$-$\mathbf{CAT}$ coming from the previous proposition. □

5.3. **Local presentability and algebraic left adjoints.** Next in this section, we consider the categorical properties of the $\mathcal{V}$-categories and $\mathcal{V}$-functors in the image of the semantics functors. We begin with the case of pretheories.

Proposition 27. *(i) If* $J\colon \mathcal{A} \to \mathcal{T}$ *is an* $\mathcal{A}$*-pretheory then* $\mathbf{Mod}_c(\mathcal{T})$ *is locally presentable and* $U_{\mathcal{T}}\colon \mathbf{Mod}_c(\mathcal{T}) \to \mathcal{E}$ *is a strictly monadic right adjoint.*

(ii) If $H\colon \mathcal{T} \to \mathcal{S}$ *is a map of* $\mathcal{A}$*-pretheories, then* $H^*\colon \mathbf{Mod}_c(\mathcal{S}) \to \mathbf{Mod}_c(\mathcal{T})$ *is a strictly monadic right adjoint.*

Proof. (i) follows from Lemma 5 and the description in (2.5) of $\mathbf{Mod}_c(\mathcal{T}) \to \mathcal{E}$ as a pullback. For (ii), applying the standard cancellativity properties to the pullbacks defining $\mathbf{Mod}_c(\mathcal{S})$ and $\mathbf{Mod}_c(\mathcal{T})$ yields a pullback square

$$\begin{tikzcd} \mathbf{Mod}_c(\mathcal{S}) \arrow[r, "P_{\mathcal{S}}"] \arrow[d, "H^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{S}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "{[H^{\mathrm{op}},1]}"] \\ \mathbf{Mod}_c(\mathcal{T}) \arrow[r, "P_{\mathcal{T}}"] & {[\mathcal{T}^{\mathrm{op}}, \mathcal{V}]} \end{tikzcd} .$$

Since $[H^{\mathrm{op}}, 1]$ is strictly monadic and $P_{\mathcal{T}}$ is a right adjoint between locally presentable categories, the result follows again from Lemma 5. $\square$

Composing with the equivalence $\mathbf{Mod}_c(\mathcal{T}) \simeq \mathbf{Mod}(\mathcal{T})$ of Proposition 25, this result immediately implies the local presentability of the category $\mathbf{Mod}(\mathcal{T})$ of non-concrete models. Likewise, in the non-concrete setting, the analogue of Proposition 27 remains true on replacing “strict monadicity” by “monadicity” throughout. On the other hand, taken together with Proposition 23, it immediately implies the corresponding result for nervous monads. We state this here as:

Proposition 28. *(i) If T is an $\mathcal{A}$-nervous monad then $\mathcal{E}^{\mathsf{T}}$ is locally presentable, and $U^{\mathsf{T}} \colon \mathcal{E}^{\mathsf{T}} \to \mathcal{E}$ is a strictly monadic right adjoint.*

(ii) If $\alpha \colon \mathsf{T} \to \mathsf{S}$ is a map of $\mathcal{A}$-nervous monads, then $\alpha^* \colon \mathcal{E}^{\mathsf{S}} \to \mathcal{E}^{\mathsf{T}}$ is a strictly monadic right adjoint.

5.4. **Algebraic colimits of monads and theories.** To conclude this section, we examine the interaction of the semantics functors with colimits. We begin with the more-or-less classical case of the semantics functor for monads $\mathrm{Alg} \colon \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$.

In general, $\mathbf{Mnd}(\mathcal{E})$ need not be cocomplete. Indeed, when $\mathcal{V} = \mathcal{E} = \mathbf{Set}$, it does not even have all binary coproducts; see [5, Proposition 6.10]. However many colimits of monads do exist, and an important point about these is that, in the terminology of [16], they are *algebraic*. That is, they are sent to limits by the semantics functor $\mathrm{Alg} \colon \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$.

To prove this, we use the following lemma, which is a mild variant of the standard result that right adjoints preserve limits.

Lemma 29. *Let $\mathcal{C}$ be a complete (ordinary) category with a strongly generating class of objects X and consider a functor $U \colon \mathcal{A} \to \mathcal{C}$. If each $x \in X$ admits a reflection along U then U preserves any limits that exist in $\mathcal{A}$.*

Proof. As X is a strong generator, the functors $\mathcal{C}(x, –)$ with $x \in X$ jointly reflect isomorphisms, and so jointly reflect limits. Accordingly U preserves any limits that are preserved by $\mathcal{C}(x, U–)$ for each $x \in X$. But each $\mathcal{C}(x, U–)$ is representable and so preserves all limits; whence U preserves any limits that exist. $\square$

In the setting of **Set**-enriched categories the following result, expressing the algebraicity of colimits of monads, is a special case of Proposition 26.3 of [16].

Proposition 30. $\mathrm{Alg} \colon \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *preserves limits.*

Proof. The $\mathcal{V}$-functors $F\colon \mathcal{X} \to \mathcal{E}$ with small domain form a strong generator for $\mathcal{V}$-$\mathbf{CAT}/\mathcal{E}$. Moreover, it is shown in [12, Theorem II.1.1] that each such F has a reflection along $\mathrm{Alg}\colon \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ given by its codensity monad $\mathrm{Ran}_F(F)\colon \mathcal{E} \to \mathcal{E}$. The result thus follows from Lemma 29. □

We now adapt the above results concerning $\mathbf{Mnd}(\mathcal{E})$ to the cases of $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$, $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ and $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$. In Theorem 38 below, we will see that these categories are locally presentable; in particular, and by contrast with $\mathbf{Mnd}(\mathcal{E})$, they are cocomplete. It is also not difficult to prove the cocompleteness directly.

Proposition 31. *Each of the semantics functors* $\mathrm{Alg}\colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$*,* $\mathrm{Mod}_c\colon \mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *and* $\mathrm{Mod}_c\colon \mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}$ *preserves limits.*

Proof. These three functors are isomorphic to the respective composites:

$$\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{incl}^{\mathrm{op}}} \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \tag{5.3}$$

$$\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\Psi^{\mathrm{op}}} \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E} \tag{5.4}$$

$$\mathbf{Th}_{\mathcal{A}}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\Psi^{\mathrm{op}}} \mathbf{Mnd}(\mathcal{E})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} \mathcal{V}\text{-}\mathbf{CAT}/\mathcal{E}\ ; \tag{5.5}$$

for (5.3) this is clear, while for (5.4) and (5.5) it follows from Proposition 23. The common second functor in each composite is limit-preserving by Proposition 30, while the first functor is limit-preserving in each case since it is the opposite of a left adjoint functor—by Corollary 22, Theorem 6 and Theorem 19 (taken together with Corollary 22) respectively. □

We leave it to the reader to formulate this result also for non-concrete models.

6. The monad–theory correspondence in practice

In this section, we return to the examples of our general setting described in Section 2.1, with the goal of describing as explicitly as possible what the $\mathcal{A}$-nervous monads, the $\mathcal{A}$-theories, and the corresponding models look like in each case. By way of these descriptions, we will re-find many of the monad–theory correspondences existing in the literature as instances of our main Theorem 19.

To obtain our explicit descriptions, we will require some further results which characterise $\mathcal{A}$-theories and $\mathcal{A}$-nervous monads in particular situations. We begin this section by describing these results.

6.1. **Theories in the presheaf context.** A number of the examples of our basic setting described in Section 3.1 arise in the following manner. We take $\mathcal{E} = [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ a presheaf category, and take $\mathcal{A}$ to be any full subcategory of $\mathcal{E}$ containing the representables. In this situation, we then have a factorisation

$$\mathcal{C} \xrightarrow{I} \mathcal{A} \xrightarrow{K} [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] = \mathcal{E} \tag{6.1}$$

of the Yoneda embedding. The Yoneda lemma implies that $Y\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is dense, whereupon by Theorem 5.13 of [17], both I and K are too. In particular, K provides an instance of our basic setting; we will call this the *presheaf context*. Each of Examples 8(i), (iv), (v), (vi), and (viii) arise in this way.

Lemma 32. *In the presheaf context, we have $N_I \cong K$ and $N_K \cong \mathrm{Ran}_{I^{\mathrm{op}}}$. Moreover, a functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a K-nerve just when it is the right Kan extension of its restriction along $I^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$.*

Proof. For the first isomorphism we calculate that

$$N_I(x) = \mathcal{A}(I-, x) \cong [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](KI-, Kx) = [\mathcal{C}^{\mathrm{op}}, \mathcal{V}](Y-, Kx) \cong Kx \tag{6.2}$$

by full fidelity of K and the Yoneda lemma. For the second, since $\mathrm{Lan}_Y K \dashv N_K$ and $[I^{\mathrm{op}}, 1] \dashv \mathrm{Ran}_{I^{\mathrm{op}}}$ it suffices to show $\mathrm{Lan}_Y K \cong [I^{\mathrm{op}}, 1]\colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. Since both are cocontinuous, it suffices to show $(\mathrm{Lan}_Y K)Y \cong [I^{\mathrm{op}}, 1]Y$, which follows since $(\mathrm{Lan}_Y K)Y \cong K \cong N_I = [I^{\mathrm{op}}, 1]Y$ using full fidelity of Y and (6.2). Finally, since I^{op} is fully faithful, $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a right Kan extension along I^{op} just when it is the right Kan extension of its own restriction. Thus the final claim follows using the isomorphism $N_K \cong \mathrm{Ran}_{I^{\mathrm{op}}}$. ☐

In this setting, we have practically useful characterisations of the $\mathcal{A}$-theories and their (non-concrete) models.

Proposition 33. *Let $J\colon \mathcal{A} \to \mathcal{T}$ be an $\mathcal{A}$-pretheory in the presheaf context (6.1).*

(i) A functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ is a $\mathcal{T}$-model just when $FJ^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is the right Kan extension of its restriction along $I^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$;

(ii) $J\colon \mathcal{A} \to \mathcal{T}$ is itself an $\mathcal{A}$-theory just when it is the pointwise left Kan extension of its restriction along $I\colon \mathcal{C} \to \mathcal{A}$.

Proof. (i) follows immediately from Lemma 32 since, by definition, F is a $\mathcal{T}$-model just when FJ^{op} is a K-nerve. For (ii), note that by Proposition 4.46 of [17], $J\colon \mathcal{A} \to \mathcal{T}$ is the pointwise left Kan extension of its restriction along I just when, for each $x \in \mathcal{T}$, the functor $\mathcal{T}(J-, x)\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is the right Kan extension of its restriction along I^{op}. By Lemma 32, this happens just when each $\mathcal{T}(J-, x)$ is a K-nerve—that is, just when J is a $\mathcal{A}$-theory. ☐

We can sharpen these results using Day's notion of *density presentation* [11]. The density of an ordinary functor $K\colon \mathcal{C} \to \mathcal{D}$ is often introduced as the assertion that each object of $\mathcal{D}$ is the colimit of a certain diagram in the image of K. It is this perspective that the notion of density presentation generalises.

A *family of colimits* Φ in the ordinary category $\mathcal{D}$ is a class of diagrams $(D_i\colon \mathcal{J}_i \to \mathcal{D})_{i \in I}$ each of which has a colimit in $\mathcal{D}$. In the enriched case, a *family of colimits* Φ in the $\mathcal{V}$-category $\mathcal{D}$ is a class of pairs $(W_i \in [\mathcal{J}_i^{\mathrm{op}}, \mathcal{V}], D_i\colon \mathcal{J}_i \to \mathcal{D})_{i \in I}$ such that each weighted colimit $W_i \star D_i$ exists in $\mathcal{D}$. In either case, a full replete subcategory $\mathcal{B}$ of $\mathcal{D}$ is *closed in $\mathcal{D}$ under Φ-colimits* if it contains the (weighted) colimit of any D_i in Φ whenever it contains each vertex of D_i. We say that $\mathcal{D}$ is the *closure of $\mathcal{B}$ under Φ-colimits* if the only replete full subcategory of $\mathcal{D}$ containing $\mathcal{B}$ and closed under Φ-colimits is $\mathcal{D}$ itself.

Now given a fully faithful $K\colon \mathcal{C} \to \mathcal{D}$, we say that a colimit in $\mathcal{D}$ is *K-absolute* if it is preserved by N_K, or equivalently, by each representable $\mathcal{D}(Kx, -)\colon \mathcal{D} \to \mathcal{V}$. If $\mathcal{D}$ is the closure of $\mathcal{C}$ under a family Φ of K-absolute colimits then Φ is said to be a *density presentation* for K. The nomenclature is justified by Theorem 5.19 of [17], which, among other things, says that the fully faithful K has a density presentation just when it is dense.

We will make use of density presentations in the presheaf context (6.1) with respect not to the dense K, but to the dense I. By Lemma 32 we have $N_I \cong K$, and so the I-absolute colimits are in this case those preserved by $K\colon \mathcal{A} \to \mathcal{E}$. We will see numerous instances of this situation in Section 6.3 below; we give a couple of examples now to clarify the ideas.

Examples 34.

(i) Example 8(i) corresponds to the presheaf context

$$1 \xrightarrow{I} \mathbb{F} \xrightarrow{K} \mathbf{Set} \ ,$$

and here I has a density presentation given by all *finite copowers of* $1 \in \mathbb{F}$; these are I-absolute since K preserves them. In fact, $\mathbb{F}$ has all finite coproducts and these are preserved by K, so that there is a larger density presentation given by *all finite coproducts* in $\mathbb{F}$.

(ii) Example 8(iv) yields the presheaf context below, wherein I has a density presentation given by the wide pushouts $[n] \cong [1] +_{[0]} [1] +_{[0]} \dots +_{[0]} [1]$:

$$\mathbb{G}_1 \xrightarrow{I} \Delta_0 \xrightarrow{K} [{\mathbb{G}_1}^{\mathrm{op}}, \mathbf{Set}] \ .$$

The reason we care about density presentations is the following result, which comprises various parts of Theorem 5.29 of [17].

Proposition 35. *Let $K\colon \mathcal{C} \to \mathcal{D}$ be fully faithful and dense. The following are equivalent:*

(i) $F\colon \mathcal{D} \to \mathcal{E}$ is the pointwise left Kan extension of its restriction along K;
(ii) F sends Φ-colimits to colimits for any density presentation Φ of K;
(iii) F sends K-absolute colimits to colimits.

Combined with Proposition 33, this yields the desired sharper characterisation of the $\mathcal{A}$-theories and their models.

Theorem 36. *Let $J\colon \mathcal{A} \to \mathcal{T}$ be an $\mathcal{A}$-pretheory in the presheaf context* (6.1)*, and let Φ be a density presentation for I.*

(i) A functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ is a $\mathcal{T}$-model just when $FJ^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ sends Φ-colimits in $\mathcal{A}$ to limits in $\mathcal{V}$;
(ii) $J\colon \mathcal{A} \to \mathcal{T}$ is an $\mathcal{A}$-theory just when it sends Φ-colimits to colimits.

6.2. **Nervous monads, signatures and saturated classes.** We now turn from characterisations for $\mathcal{A}$-theories to characterisations for $\mathcal{A}$-nervous monads. We know from Corollary 21 that a monad is $\mathcal{A}$-nervous just when it is isomorphic to $\Psi\mathcal{T}$ for some $\mathcal{A}$-pretheory $J\colon \mathcal{A} \to \mathcal{T}$, and the examples in Section 3 make it an intuitively reasonable idea that these are the monads which can be "presented by operations and equations with arities from $\mathcal{A}$".

Our first characterisation result makes this idea precise by exhibiting the category of $\mathcal{A}$-nervous monads as monadic over a category of *signatures*. We defer the proof of this result to Section 8.

Definition 37. The category $\mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ of *signatures* is the category $\mathcal{V}\text{-}\mathbf{CAT}(\mathrm{ob}\,\mathcal{A}, \mathcal{E})$. We write $V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ for the functor sending T to $(Ta)_{a\in\mathcal{A}}$.

Theorem 38. *$V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint $F\colon \mathbf{Sig}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Mnd}(\mathcal{E})$ taking values in $\mathcal{A}$-nervous monads. Moreover:*

(i) The restricted functor $V\colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is monadic;
(ii) A monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is $\mathcal{A}$-nervous if and only if it is a colimit in $\mathbf{Mnd}(\mathcal{E})$ of monads in the image of F;
(iii) Each of $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$, $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ and $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$ is locally presentable.

The idea behind this result originates in [20]. A signature $\Sigma \in \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ specifies for each $a \in \mathcal{A}$ an $\mathcal{E}$-object Σa of "operations of input arity a". The free monad $F\Sigma$ on this signature has as its algebras the Σ-*structures*: objects $X \in \mathcal{E}$ endowed with a function $\mathcal{E}(a, X) \to \mathcal{E}(\Sigma a, X)$ for each $a \in \mathcal{A}$. The above result implies that a monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is $\mathcal{A}$-nervous just when it admits a presentation as a coequaliser $F\Gamma \rightrightarrows F\Sigma \twoheadrightarrow \mathsf{T}$—that is, a presentation by a signature Σ of basic operations together with a family Γ of equations between derived operations.

We now turn to our second characterisation result for $\mathcal{A}$-nervous monads. This is motivated by the fact, noted in the introduction, that in many monad–theory correspondences the class of monads can be characterised by a colimit-preservation property. To reproduce this result in our setting, we require a closure property of the arities in the subcategory $\mathcal{A}$ which, roughly speaking, says that substituting $\mathcal{A}$-ary operations into $\mathcal{A}$-ary operations again yields $\mathcal{A}$-ary operations.

Definition 39. An endo-$\mathcal{V}$-functor $F\colon \mathcal{E} \to \mathcal{E}$ is called $\mathcal{A}$-*induced* if it is the pointwise left Kan extension of its restriction along K. We call $\mathcal{A}$ a *saturated class of arities* if $\mathcal{A}$-induced endofunctors of $\mathcal{E}$ are closed under composition.

Example 40. In the case of $K\colon \mathbb{F} \hookrightarrow \mathbf{Set}$, there is a density presentation for K given by *all filtered colimits in* $\mathbf{Set}$, so that by Proposition 35, an endofunctor $\mathbf{Set} \to \mathbf{Set}$ is $\mathbb{F}$-induced just when it preserves filtered colimits. Thus $\mathbb{F} \hookrightarrow \mathbf{Set}$ is a saturated class of arities.

Example 41. More generally, if Φ is a class of enriched colimit-types and $K\colon \mathcal{A} \to \mathcal{E}$ exhibits $\mathcal{E}$ as the free cocompletion of $\mathcal{A}$ under Φ-colimits, then there is a density presentation of K given by all Φ-colimits, and an endofunctor of $\mathcal{E}$ is K-induced just when it preserves Φ-colimits. Thus $\mathcal{A}$ is a saturated class of arities.

Example 42. Let $K\colon \mathcal{A} \hookrightarrow \mathbf{Set}$ be the inclusion of the one-object full subcategory $\mathcal{A}$ on the two-element set $2 = \{0, 1\}$. Since the dense generator 1 of $\mathbf{Set}$ is a retract of 2, and taking retracts does not change categories of presheaves, $\mathcal{A}$ is dense in $\mathbf{Set}$. We claim it does *not* give a saturated class of arities.

To see this, note first that $(-)^2\colon \mathbf{Set} \to \mathbf{Set}$ is $\mathcal{A}$-induced, being a left Kan extension along K of the representable $\mathcal{A}(2, -)\colon \mathcal{A} \to \mathbf{Set}$. We claim that $(-)^2 \circ (-)^2$ is *not* $\mathcal{A}$-induced. For indeed, by the Yoneda lemma, any $X \in [\mathcal{A}, \mathbf{Set}]$ has an epimorphic cover by copies of the unique representable $\mathcal{A}(2, -)$. Since left Kan extension preserves epimorphisms, each $\mathrm{Lan}_K(X)$ admits an epimorphic cover by copies of $(-)^2$. But $(-)^2 \circ (-)^2 \cong (-)^4$ can admit no such cover, since the identity map on 4 does not factor through 2, and so cannot be $\mathcal{A}$-induced.

The proof of the following result will again be deferred to Section 8 below.

Theorem 43. *Let $\mathcal{A}$ be a saturated class of arities in $\mathcal{E}$. The following are equivalent properties of a monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$:*

(i) T *is $\mathcal{A}$-nervous;*
(ii) $T \colon \mathcal{E} \to \mathcal{E}$ *is $\mathcal{A}$-induced;*
(iii) $T \colon \mathcal{E} \to \mathcal{E}$ *preserves Φ-colimits for any density presentation Φ of K.*

6.3. **The monad–theory equivalence in practice.** We now apply our characterisation results to the examples of Section 2.1. In many cases, the explicit descriptions we obtain of the $\mathcal{A}$-nervous monads, the $\mathcal{A}$-theories, and their models will allow us to reconstruct a familiar monad–theory correspondence from the literature.

Examples 44. As before, we begin with the unenriched examples where $\mathcal{V} = \mathbf{Set}$.

(i) The case $\mathcal{E} = \mathbf{Set}$ and $\mathcal{A} = \mathbb{F}$ corresponds to the instance of the presheaf context described in Examples 34(i). Applied to the density presentations for I given there, Theorem 36 tells us that an $\mathbb{F}$-pretheory $J \colon \mathbb{F} \to \mathcal{T}$ is an $\mathbb{F}$-theory just when it preserves finite copowers of 1, or equally (using the larger density presentation) all finite coproducts. It thus follows that the $\mathbb{F}$-theories are the *Lawvere theories* of [24]. Moreover a functor $F \colon \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ is a $\mathcal{T}$-model if and only if $FJ^{\mathrm{op}} \colon \mathbb{F}^{\mathrm{op}} \to \mathbf{Set}$ preserves finite products. Since, in this case, J also *reflects* finite coproducts, this happens just when $F \colon \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ is itself finite-product-preserving, that is, just when F is a model of the Lawvere theory $\mathcal{T}$.

On the other hand, by Example 40, $\mathbb{F}$ is a saturated class of arities, and the $\mathbb{F}$-induced endofunctors are the finitary ones; so by Theorem 43, a monad on $\mathbf{Set}$ is $\mathbb{F}$-nervous just when it is finitary. Theorem 19 thus specialises to the classical *finitary monad–Lawvere theory* correspondence, while Theorem 26 recaptures its compatibility with semantics.

(ii) When $\mathcal{E}$ is locally finitely presentable and $\mathcal{A} = \mathcal{E}_f$, the category of K-nerves is, by [13, Kollar 7.9], the full subcategory of $[\mathcal{E}_f^{\mathrm{op}}, \mathbf{Set}]$ on the *finite-limit-preserving* functors. So an $\mathcal{E}_f$-pretheory $J \colon \mathcal{E}_f \to \mathcal{T}$ is an $\mathcal{E}_f$-theory just when each $\mathcal{T}(J–, a) \colon {\mathcal{E}_f}^{\mathrm{op}} \to \mathbf{Set}$ preserves finite limits. By the Yoneda lemma, this happens just when J preserves finite colimits, so that the $\mathcal{E}_f$-theories are precisely [31]'s *Lawvere $\mathcal{E}$-theories*.

The concrete $\mathcal{T}$-models in this setting are exactly the models of [31, Definition 2.2]. The general $\mathcal{T}$-models are those functors $F \colon \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ for which $FJ^{\mathrm{op}} \colon \mathcal{E}_f^{\mathrm{op}} \to \mathbf{Set}$ is a K-nerve, i.e., finite-limit-preserving; these are the more general models of [22, Definition 12], and the correspondence between the two notions in Proposition 25 recaptures Proposition 15 of *ibid*.

On the monad side, since $K \colon \mathcal{E}_f \to \mathcal{E}$ exhibits $\mathcal{E}$ as the free filtered-colimit completion of $\mathcal{E}_f$, Example 41 and Theorem 43 imply that $\mathcal{E}_f$ is a saturated class, and that the $\mathcal{E}_f$-nervous monads are the finitary ones. So in this case, Theorem 19 and Corollary 24 reconstruct (the unenriched version of) the monad–theory correspondence given in [31, Theorem 5.2].

(iii) More generally, when $\mathcal{E}$ is locally λ-presentable and $\mathcal{A} = \mathcal{E}_\lambda$ is a skeleton of the full subcategory of λ-presentable objects, the $\mathcal{E}_\lambda$-theories are those pretheories $J \colon \mathcal{E}_\lambda \to \mathcal{T}$ which preserve λ-small colimits; the $\mathcal{T}$-models are functors $F\colon \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ for which FJ^{op} preserves λ-small limits; and the $\mathcal{E}_\lambda$-nervous monads are those whose endofunctor preserves λ-filtered colimits.

(iv) When $\mathcal{E} = [\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ and $\mathcal{A} = \Delta_0$, we are in the presheaf context of Examples 34(ii). For the density presentation for I given there, Theorem 36 tells us that a pretheory $J\colon \Delta_0 \to \mathcal{T}$ is a Δ_0-theory just when it preserves the wide pushouts $[n] \cong [1] +_{[0]} [1] +_{[0]} \dots +_{[0]} [1]$. Moreover, a functor $X\colon \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ is a $\mathcal{T}$-model just when it sends each of these wide pushouts to a limit in $\mathbf{Set}$. This is precisely the *Segal condition* of [33]; in elementary terms, it requires the invertibility of each canonical map

$$Xn \longrightarrow X1 \times_{X0} X1 \times_{X0} \cdots \times_{X0} X1\ . \tag{6.3}$$

In Corollary 49 below we will see that Δ_0 is *not* a saturated class of arities, and so we have no more direct characterisations of the Δ_0-nervous monads than is given by Corollary 21 or Theorem 38. However, Example 11 provides us with natural examples of Δ_0-nervous monads: namely, the monads T and T_g for *categories* and for *groupoids* on $[\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$. As was already noted in [35], the nervosity of T recaptures the classical nerve theorem relating categories and simplicial sets. Indeed, the Δ_0-theory associated to T is the first part of the (bijective-on-objects, fully faithful) factorisation

$$\Delta_0 \xrightarrow{J_{\mathsf{T}}} \Delta \xrightarrow{K_{\mathsf{T}}} \mathbf{Cat}$$

of the composite $F_{\mathsf{T}}K\colon \Delta_0 \to \mathbf{Cat}$. The interposing object here is the topologist's simplex category Δ, with K_T the standard inclusion into $\mathbf{Cat}$. Thus, to say that T is Δ_0-nervous is to say that:

(a) The classical nerve functor $N_{K_{\mathsf{T}}}\colon \mathbf{Cat} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful;
(b) The essential image of $N_{K_{\mathsf{T}}}$ comprises those $X \in [\Delta^{\mathrm{op}}, \mathbf{Set}]$ for which XJ_{T} is a K-nerve.

This much is already done in [35], but our use of density presentations allows for a small improvement. To say that XJ_{T} is a K-nerve in (b) is equally to say that X is a $\mathcal{T}$-model, or equally that X satisfies the Segal condition expressed by the invertibility of each (6.3). This is a mild sharpening of [35], where the "Segal condition" is left in the abstract form given in (b) above.

In a similar way, the nervosity of the monad T_g for small groupoids captures the "symmetric nerve theorem". This states that the functor $\mathbf{Gpd} \to [\mathbb{F}_+^{\mathrm{op}}, \mathbf{Set}]$ sending a groupoid to its *symmetric nerve*—indexed by the category of non-empty finite sets—is fully faithful, and characterises the essential image once again as the functors satisfying the Segal condition (6.3).

(v) With $\mathcal{E} = [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]$ and $\mathcal{A} = \Theta_0$, we are now in the presheaf context

$$\mathbb{G} \xrightarrow{I} \Theta_0 \xrightarrow{K} [\mathbb{G}^{\mathrm{op}}, \mathbf{Set}]\ .$$

I has a density presentation given by the *I-globular sums* $(n_1, \dots, n_k) \cong (n_1) +_{(n_2)} +(n_3) + \dots +_{(n_{k-1})} (n_k)$ in Θ_0; whence by Theorem 36, a pretheory $J\colon \Theta_0 \to \mathcal{T}$ is a Θ_0-theory when it preserves these I-globular sums—that is, when it is a *globular theory* in the sense of [8][2]. A functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathbf{Set}$ is a $\mathcal{T}$-model when it sends I-globular sums to limits, thus when each map

$$X\vec{n} \longrightarrow Xn_1 \times_{Xn_2} Xn_3 \times_{Xn_4} \ldots \times_{Xn_{k-1}} Xn_k$$

is invertible. Once again, Θ_0 is *not* a saturated class of arities, and so there is no direct characterisation of the Θ_0-nervous monads; however, their interaction with Θ_0-theories is important in the literature on globular approaches to higher category theory, as we now outline.

Globular theories can describe structures on globular sets such as strict or weak ω-categories and ω-groupoids. For the strict variants, we pointed out in Section 3.2 that these may be modelled by Θ_0-pretheories; and since reflecting a pretheory $\mathcal{T}$ into a theory $\Phi\Psi\mathcal{T}$ does not change the models, it is immediate that there are Θ_0-theories modelling these structures too.

The original definition of globular weak ω-category was given by Batanin in [7]; he defines them be globular sets equipped with algebraic structure controlled by a *globular operad*. Globular operads can be understood as certain cartesian monads on globular sets. Berger [8] introduced globular theories and described the passage from a globular operad T to a globular theory Θ_T just as in Section 2.4 above. In our language, his Theorem 1.17 states exactly that each globular operad T is Θ_0-nervous, so that algebras for the globular operad are the same as models of the associated theory Θ_T. In particular, Batanin's weak ω-categories are the models of a globular theory[3]. On the other hand, Grothendieck weak ω-groupoids [27] are, *by definition*, models for certain globular theories called *coherators*.

We now proceed to our examples over a more general base for enrichment $\mathcal{V}$.

(vi) With $\mathcal{V} = \mathcal{E}$ a locally finitely presentable symmetric monoidal category and with $\mathcal{A} = \mathcal{V}_f$, we are in the presheaf context

$$\mathcal{I} \xrightarrow{I} \mathcal{V}_f \xrightarrow{K} \mathcal{V} \ ,$$

wherein I has a density presentation given by the class of all *finite tensors*—tensors by finitely presentable objects of $\mathcal{V}$. Thus by Theorem 36, the $\mathcal{V}_f$-theories are the pretheories $J\colon \mathcal{V}_f \to \mathcal{T}$ which preserve finite tensors, which are precisely the *Lawvere $\mathcal{V}$-theories* of [32, Definition 3.1]. Furthermore, like in (i), a functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ is a $\mathcal{T}$-model just when it preserves finite cotensors, just as in Definition 3.2 of *ibid.* On the other hand, $\mathcal{V}_f \to \mathcal{V}$ exhibits $\mathcal{V}$ as the free filtered-colimit completion of $\mathcal{V}_f$; whence by Example 41 it is a saturated class of arities, and by Theorem 43 the $\mathcal{V}_f$-nervous monads are again the finitary ones. So Theorems 19 and 26 specialise to Theorems 4.3, 3.4 and 4.2 of [32].

[2]The definition of globular theory in [8] has the extra condition, satisfied in most cases, that J be a faithful functor.

[3]As an aside, we note that a complete understanding of those globular theories corresponding to globular operads was obtained in Theorem 6.6.8 of [2]. See also Section 3.12 of [9].

(vii) Now taking $\mathcal{E}$ to be any locally finitely presentable presentable $\mathcal{V}$-category and $\mathcal{A} = \mathcal{E}_f$, we may argue as in (ii) to recapture the fully general enriched monad–theory correspondence of [31], and its interaction with semantics.

(viii) Now suppose we are in the situation of Examples 8(viii), provided with a class Φ of enriched colimit-types satisfying Axiom A of [23]. With $\mathcal{E} = \mathcal{V}$ and $\mathcal{A} = \mathcal{V}_\Phi$, we are now in the presheaf context

$$\mathcal{I} \xrightarrow{I} \mathcal{V}_\Phi \xrightarrow{K} \mathcal{V} \ .$$

By [17, Theorem 5.35], I has a density presentation given by Φ-tensors (i.e., tensors by objects in Φ) while by [23, Theorem 3.1], K exhibits $\mathcal{V}$ as the free Φ-flat cocompletion of $\mathcal{V}_\Phi$. Arguing as in the preceding parts, we see that $\mathcal{V}_\Phi$-theories are pretheories $J \colon \mathcal{V}_\Phi^{\mathrm{op}} \to \mathcal{T}$ which preserve Φ-tensors, that $\mathcal{T}$-models are Φ-tensor-preserving functors $F \colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$, and that a monad is $\mathcal{V}_\Phi$-nervous if its underlying endofunctor preserves Φ-flat colimits. This sharpens slightly the results obtained in [23] in the special case $\mathcal{E} = \mathcal{V}$.

(ix) Finally, in the situation of Examples 8(ix), we find that the $\mathcal{A}$-theories are the Φ-colimit preserving pretheories $J \colon \mathcal{A} \to \mathcal{T}$; that the $\mathcal{T}$-models are functors $F \colon \mathcal{T}^{\mathrm{op}} \to \mathcal{V}$ such that FJ^{op} preserves Φ-limits; and that a monad is $\mathcal{A}$-nervous just when it preserves Φ-flat colimits. In this way, our Theorems 19 and 26 reconstruct Theorems 7.6 and 7.7 of [23].

7. Monads with arities and theories with arities

In the introduction, we mentioned the general framework for monad–theory correspondences obtained in [35, 9]. Similar to this paper, the basic setting involves a category $\mathcal{E}$ and a small, dense subcategory $K \colon \mathcal{A} \hookrightarrow \mathcal{E}$; given these data, one defines notions of *monad with arities* $\mathcal{A}$ and *theory with arities* $\mathcal{A}$, and proves an equivalence between the two that is compatible with semantics.

In this section, we compare this framework with ours by comparing the classes of monads and of theories. We will see that our setting yields *strictly* larger classes of monads and theories which are better-behaved in practically useful ways. On the other hand, in the more restrictive setting of [35, 9], checking that a monad or theory is in the required class may give greater combinatorial insight into the structure which it describes.

7.1. **Monads with arities versus nervous monads.** In [35, 9] the authors work in the *unenriched* setting; the introduction to [9] states that the results "should be applicable" also in the enriched one. To ease the comparison to our results, we take it for granted that this is true, and transcribe their framework into the enriched context without further comment.

Another difference is that we assume local presentability of $\mathcal{E}$ while [35] assumes only *cocompleteness*, and [9] not even that. Given a small dense subcategory, there is no readily discernible difference between cocompleteness and local presentability[4]; however, cocompleteness is substantively different from nothing, so that

[4] Indeed, if there were, then it would negate the large cardinal axiom known as *Vopěnka's principle* [1, Chapter 6].

in this respect [9]'s results are more general than ours. However all known applications are in the context of a locally presentable $\mathcal{E}$, and so we do not lose much in restricting to this context. In conclusion, when we make our comparison we will work in exactly the same general setting as in Section 2.1, and now have:

Definition 45. [35, Definition 4.1] An endofunctor $T\colon \mathcal{E} \to \mathcal{E}$ is said to *have arities* $\mathcal{A}$ if the composite $\mathcal{V}$-functor $N_K T\colon \mathcal{E} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is the left Kan extension of its own restriction along K. A monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is a *monad with arities* $\mathcal{A}$ if its underlying endofunctor has arities $\mathcal{A}$.

We consider the following way of restating this to be illuminating.

Proposition 46. *An endofunctor $T\colon \mathcal{E} \to \mathcal{E}$ has arities $\mathcal{A}$ if and only if it sends K-absolute colimits to K-absolute colimits. In particular, each endofunctor with arities $\mathcal{A}$ is $\mathcal{A}$-induced.*

Proof. By Proposition 35, T has arities $\mathcal{A}$ just when $N_K T$ sends K-absolute colimits to colimits. Since N_K is fully faithful, it reflects colimits, and so T has arities $\mathcal{A}$ just when T sends K-absolute colimits to colimits which are preserved by N_K—that is, to K-absolute colimits.

For the second claim, recall from Definition 39 that an endofunctor $T\colon \mathcal{E} \to \mathcal{E}$ is $\mathcal{A}$-induced if it is the left Kan extension of its own restriction to $\mathcal{A}$, or equivalently, by Proposition 35, when it sends K-absolute colimits to colimits. $\square$

Recall also that we call a class of arities $\mathcal{A}$ *saturated* when $\mathcal{A}$-induced endofunctors are closed under composition. Example 42 shows that this condition is not always satisfied. In light of the preceding result, the endofunctors with arities $\mathcal{A}$ can be seen as a natural subclass of the $\mathcal{A}$-induced endofunctors for which composition-closure is *always* verified.

The reason that Weber introduced monads with arities was in order to prove his *nerve theorem* [35, Theorem 4.10], which in our language may be restated as:

Theorem 47. *Monads with arities $\mathcal{A}$ are $\mathcal{A}$-nervous.*

One may reasonably ask whether the classes of monads with arities and $\mathcal{A}$-nervous monads in fact coincide. In many cases, this is true; in particular, in the situation of Example 41, where $K\colon \mathcal{A} \to \mathcal{E}$ exhibits $\mathcal{E}$ as the free Φ-cocompletion of $\mathcal{A}$ for some class of colimit-types Φ. Indeed, this condition implies that a monad T is $\mathcal{A}$-nervous precisely when T sends Φ-colimits to Φ-colimits; since Φ-colimits are K-absolute, this in turn implies that $N_K T$ sends Φ-colimits to colimits, and so is the left Kan extension of its own restriction along K. So in this case, every $\mathcal{A}$-nervous monad has arities $\mathcal{A}$; so in particular, the two notions coincide in each of Examples 8(i), (ii), (iii), (vi), (vii), (viii) and (ix).

However, they do *not* coincide in general. That is, in some instances of our basic setting, there exist monads which are $\mathcal{A}$-nervous but do not have arities $\mathcal{A}$. We give three examples of this. The first two arise in the setting of Example 8(iv), and concern the monads for groupoids and involutive graphs respectively.

Proposition 48. *The monad T on $\mathbf{Grph} := [\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ whose algebras are groupoids is Δ_0-nervous but does not have Δ_0-induced underlying endofunctor. It follows that T does not have arities Δ_0.*

Proof. From Example 11 we know that T is Δ_0-nervous. To see that T is not Δ_0-induced, consider the graph X with vertices and arrows as to the left in:

$$
a \xrightarrow{r} b \xleftarrow{s} c \qquad\qquad
\begin{tikzcd}
{[0]} \arrow[r, "\tau"] \arrow[d, "\tau"'] & {[1]} \arrow[d, "s"] \\
{[1]} \arrow[r, "r"] & X
\end{tikzcd} . \tag{7.1}
$$

This X is equally the K-absolute pushout right above; so if T were Δ_0-induced then it would preserve this pushout. But $T[1] +_{T[0]} T[1]$ is the graph

$$
\begin{tikzcd}
a \arrow[loop left, "1_a"] \arrow[r, shift right, "r"'] & b \arrow[l, shift right, "r^{-1}"'] \arrow[loop above, "1_b"] \arrow[r, shift left, "s^{-1}"] & c \arrow[l, shift left, "s"] \arrow[loop right, "1_c"]
\end{tikzcd}
$$

wherein, in particular, there is no edge $a \to c$; while in TX we have $s^{-1} \circ r \colon a \to c$. So the pushout is not preserved. This shows that T is not Δ_0-induced and so, by Proposition 46, that T does not have arities Δ_0. $\square$

Since the above result exhibits a Δ_0-nervous monad whose underlying endofunctor is not Δ_0-induced, we can apply Theorem 43 to deduce:

Corollary 49. $K \colon \Delta_0 \hookrightarrow \mathbf{Grph}$ *is not a saturated class of arities.*

Our second example, originally due to Melliès [29, Appendix III], shows that even monads with Δ_0-induced endofunctor need not have arities Δ_0. In this example, we call a graph $s, t \colon X_1 \rightrightarrows X_0$ *involutive* if it comes endowed with an order-2 automorphism $i \colon X_1 \to X_1$ reversing source and target, i.e., with $si = t$ (and hence also $ti = s$).

Proposition 50. *The monad* T *on* $\mathbf{Grph} := [\mathbb{G}_1^{\mathrm{op}}, \mathbf{Set}]$ *whose algebras are involutive graphs is* Δ_0*-nervous and has* Δ_0*-induced underlying endofunctor, but does not have arities* Δ_0*.*

Proof. The value of T at $s, t \colon X_1 \rightrightarrows X_0$ is given by $\langle s, t\rangle, \langle t, s\rangle \colon X_1 + X_1 \rightrightarrows X_0$. It follows that T is *cocontinuous* and so certainly Δ_0-induced. To see it does not have arities Δ_0, consider again the graph (7.1) and its K-absolute pushout presentation. If this were preserved by $N_K T \colon \mathbf{Grph} \to [\Delta_0^{\mathrm{op}}, \mathbf{Set}]$ then, on evaluating at $[2]$, the maps $\mathbf{Grph}([2], T[1]) \rightrightarrows \mathbf{Grph}([2], TX)$ given by postcomposition with Tr and Ts would be jointly surjective. To show this is not so, consider the map $f \colon [2] \to TX$ picking out the composable pair $(r \colon a \to b, i(s) \colon b \to c)$. Since neither Tr nor Ts are surjective on objects, the bijective-on-objects f cannot factor through either of them. This shows that T does not have arities Δ_0. $\square$

Our final example shows that not even free monads on $\mathcal{A}$-signatures—which are $\mathcal{A}$-nervous by Theorem 38 above—need necessarily have arities $\mathcal{A}$.

Proposition 51. *Let* $\mathcal{V} = \mathcal{E} = \mathbf{Set}$ *and let* $\mathcal{A}$ *be the one-object full subcategory on a two-element set. The free monad on the terminal* $\mathcal{A}$*-signature does not have* $\mathcal{A}$*-induced underlying endofunctor and therefore does not have arities* $\mathcal{A}$*.*

Proof. The algebras for the free monad T on the terminal signature are sets equipped with a binary operation. Elements of the free T-algebra on X are binary trees with leaves labelled by elements of X, yielding the formula

$$TX = \textstyle\sum_{n\in\mathbb{N}} C_n \times X^{n+1}$$

where C_n is the nth Catalan number. In particular, T contains at least one coproduct summand $(–)^4$ and so, as in Example 42, is not $\mathcal{A}$-induced; in particular, by Proposition 46, it does not have arities $\mathcal{A}$. $\square$

7.2. **Theories with arities $\mathcal{A}$ versus $\mathcal{A}$-theories.** The paper [9] introduced *theories with arities* $\mathcal{A}$. These are $\mathcal{A}$-pretheories $J\colon \mathcal{A} \to \mathcal{T}$ for which the composite

$$[\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \xrightarrow{\mathrm{Lan}_J} [\mathcal{T}^{\mathrm{op}}, \mathcal{V}] \xrightarrow{[J^{\mathrm{op}},1]} [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \tag{7.2}$$

takes K-nerves to K-nerves. This functor takes the representable $\mathcal{A}(–, x)$ to $\mathcal{T}(J–, x)$, so that in this language, we may describe the $\mathcal{A}$-theories as the pretheories for which (7.2) takes each representable to a K-nerve. It follows that:

Proposition 52. *Theories with arities $\mathcal{A}$ are $\mathcal{A}$-theories.*

Proof. It suffices to observe that each representable $\mathcal{A}(–, x)$ is a K-nerve since $\mathcal{A}(–, x) \cong \mathcal{E}(K–, Kx) = N_K(Kx)$. $\square$

Theorem 3.4 of [9] establishes an equivalence between the categories of monads with arities $\mathcal{A}$ and of theories with arities $\mathcal{A}$. The functor taking a monad with arities to the corresponding theory with arities is defined in the same way as the Φ of Section 2.4, and so it follows that:

Proposition 53. *The equivalence of monads with arities $\mathcal{A}$ and theories with arities $\mathcal{A}$ is a restriction of the equivalence between $\mathcal{A}$-nervous monads and $\mathcal{A}$-theories.*

In particular, there exist $\mathcal{A}$-theories which are not theories with arities $\mathcal{A}$; it is this statement which was verified in in [29, Appendix III].

7.3. **Colimits of monads with arities.** In Theorem 38 we saw that the $\mathcal{A}$-nervous monads are the closure of the free monads on $\mathcal{A}$-signatures under colimits in $\mathbf{Mnd}(\mathcal{E})$. Since colimits of monads are algebraic, this allows us to give intuitive presentations for $\mathcal{A}$-nervous monads as suitable colimits of frees. The pretheory presentations of Section 3 can be understood as particularly direct descriptions of such colimits.

Since not every $\mathcal{A}$-nervous monad has arities $\mathcal{A}$, the monads with arities are *not* the colimit-closure of the frees on signatures. We already saw one explanation for this in Proposition 51: the free monads on signatures need not have arities. However, this leaves open the possibility that the monads with arities $\mathcal{A}$ are the colimit-closure of some smaller class of basic monads—which would allow for the same kind of intuitive presentation as we have for $\mathcal{A}$-nervous monads. The following result shows that even this is not the case.

Theorem 54. *Monads with arities $\mathcal{A}$ need not be closed in* $\mathbf{Mnd}(\mathcal{E})$ *under colimits.*

Proof. We saw in Proposition 50 that, when $\mathcal{E} = \mathbf{Grph}$ and $\mathcal{A} = \Delta_0$, the monad T for involutive graphs does not have arities Δ_0. To prove the result it will therefore suffice to exhibit T as a colimit in $\mathbf{Mnd}(\mathbf{Grph})$ of a diagram of monads with arities Δ_0. This diagram will be a coequaliser involving a pair of monads P and Q, whose respective algebras are:

- For P: graphs X endowed with a function $u\colon X_1 \to X_0$;
- For Q: graphs X endowed with an order-2 automorphism $i\colon X_1 \to X_1$.

We construct this coequaliser of monads in terms of the categories of algebras. The category $\mathbf{Grph}^{\mathsf{T}}$ of involutive graphs is an equaliser in $\mathbf{CAT}$ as to the left in:

$$\mathbf{Grph}^{\mathsf{T}} \overset{E}{\rightarrowtail} \mathbf{Grph}^{\mathsf{Q}} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{Grph}^{\mathsf{P}} \qquad\qquad \mathsf{P} \underset{\gamma}{\overset{\varphi}{\rightrightarrows}} \mathsf{Q} \overset{\varepsilon}{\twoheadrightarrow} \mathsf{T}$$

where the functors F and G send a Q-algebra (X, i) to the respective P-algebras (X, si) and (X, t). Since each of these functors commutes with the the forgetful functors to $\mathbf{Grph}$, we have an equaliser of forgetful functors in $\mathbf{CAT}/\mathbf{Grph}$. Since the functor $\mathrm{Alg}\colon \mathbf{Mnd}(\mathbf{Grph})^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Grph}$ is fully faithful, this equaliser must be the image of a coequaliser diagram in $\mathbf{Mnd}(\mathbf{Grph})$ as right above.

It remains to show that in this coequaliser presentation both P and Q have arities Δ_0. By Proposition 35, this means showing that $N_K P$ and $N_K Q$ send K-absolute colimits to colimits, or equally, that each $\mathbf{Grph}([n], P\text{–})$ and $\mathbf{Grph}([n], Q\text{–})$ sends K-absolute colimits to colimits. To see this, we calculate P and Q explicitly. On the one hand, the free P-algebra on a graph X is obtained by freely adjoining an element $u(f)$ to X_0 for each $f \in X_1$. On the other hand, the free Q-algebra on X is obtained by freely adjoining an element $i(f) \in X_1$ for each $f \in X_1$. Thus we have

$$PX = X + X_1 \cdot [0] \qquad \text{and} \qquad QX = X + X_1 \cdot [1]$$

where we use $\cdot$ to denote copower. Since each $[n] \in \mathbf{Grph}$ is connected, and since each hom-set $\mathbf{Gph}([n], [m])$ has cardinality $\max(0, m - n + 1)$, we conclude that

$$\mathbf{Grph}([n], PX) = \begin{cases} \mathbf{Grph}([0], X) + \mathbf{Grph}([1], X) & \text{if } n = 0; \\ \mathbf{Grph}([n], X) & \text{if } n > 0. \end{cases}$$
$$\mathbf{Grph}([n], QX) = \begin{cases} \mathbf{Grph}([0], X) + 2 \cdot \mathbf{Grph}([1], X) & \text{if } n = 0; \\ \mathbf{Grph}([1], X) + \mathbf{Grph}([1], X) & \text{if } n = 1; \\ \mathbf{Grph}([n], X) & \text{if } n > 1. \end{cases} \tag{7.3}$$

Now by definition, N_K sends K-absolute colimits to colimits, whence also each $\mathbf{Grph}([n], \text{–})\colon \mathbf{Grph} \to \mathbf{Set}$. The functors with this property are closed under colimits in $[\mathbf{Grph}, \mathbf{Set}]$, and so (7.3) ensures that each $\mathbf{Grph}([n], P\text{–})$ and $\mathbf{Grph}([n], Q\text{–})$ sends K-absolute colimits to colimits as desired. □

It is not even clear to us if the category of monads with arities $\mathcal{A}$ is always cocomplete. The argument for local presentability of $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ in Theorem 38 does not seem to adapt to the case of monads with arities, and no other obvious argument presents itself. In any case, the preceding result shows that, even if the category of monads with arities does have colimits, they do not always coincide with the usual colimits of monads, and, in particular, are not always *algebraic*. This dashes any hope we might have had of giving a sensible notion of presentation for monads with arities.

8. Deferred proofs

8.1. **Identifying the monads.** In this section, we complete the proofs of the results deferred from Section 6 above, beginning with Theorem 38. Recall that the category $\mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ of *signatures* is the (ordinary) category $\mathcal{V}\text{-}\mathbf{CAT}(\mathrm{ob}\,\mathcal{A}, \mathcal{E})$, and that $V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is the functor sending T to $(Ta)_{a\in\mathcal{A}}$.

Proposition 55. *$V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint F which takes values in $\mathcal{A}$-nervous monads.*

Proof. We can decompose V as the composite

$$\mathbf{Mnd}(\mathcal{E}) \xrightarrow{V_1} \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}, \mathcal{E}) \xrightarrow{V_2} \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$$

where V_1 takes the underlying endofunctor, and V_2 is given by evaluation at each $a \in \mathrm{ob}\,\mathcal{A}$. Since V_2 is equally given by restriction along $\mathrm{ob}\,\mathcal{A} \to \mathcal{A} \to \mathcal{E}$, it has a left adjoint F_2 given by pointwise left Kan extension, with the explicit formula:

$$F_2(\Sigma) = \textstyle\sum_{a\in\mathcal{A}} \mathcal{E}(Ka, \text{–}) \cdot \Sigma a\colon \mathcal{E} \to \mathcal{E}\ ,$$

where $\cdot$ denotes $\mathcal{V}$-enriched copower. So it suffices to show that the free monad on each endofunctor $F_2(\Sigma)$ exists and is $\mathcal{A}$-nervous. By [16, Theorem 23.2], such a free monad T is characterised by the property that $\mathcal{E}^{F_2(\Sigma)} \cong \mathcal{E}^{\mathsf{T}}$ over $\mathcal{E}$, where on the left we have the $\mathcal{V}$-category of algebras for the mere endofunctor $F_2(\Sigma)$. Thus, to complete the proof, it suffices by Theorem 6 to exhibit $\mathcal{E}^{F_2(\Sigma)}$ as isomorphic to the $\mathcal{V}$-category of concrete models of some $\mathcal{A}$-pretheory.

To this end, we let $\mathcal{B}$ be the *collage* of the $\mathcal{V}$-functor $N_K\Sigma\colon \mathrm{ob}\,\mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$. Thus $\mathcal{B}$ is the $\mathcal{V}$-category with object set $\mathrm{ob}\,\mathcal{A} + \mathrm{ob}\,\mathcal{A}$ and the following hom-objects, where we write $\ell, r\colon \mathrm{ob}\,\mathcal{A} \to \mathrm{ob}\,\mathcal{B}$ for the two injections:

$$\begin{aligned} \mathcal{B}(\ell a', \ell a) &= \mathcal{A}(a', a) & \qquad \mathcal{B}(ra', ra) &= (\mathrm{ob}\,\mathcal{A})(a', a) \\ \mathcal{B}(\ell a', ra) &= \mathcal{E}(Ka', \Sigma a) & \qquad \mathcal{B}(ra', \ell a) &= 0\ . \end{aligned}$$

Let $\ell\colon \mathcal{A} \to \mathcal{B}$ and $r\colon \mathrm{ob}\,\mathcal{A} \to \mathcal{B}$ be the two injections into the collage, and now form the pushout $J\colon \mathcal{A} \to \mathcal{T}$ of $\langle \ell, r\rangle\colon \mathcal{A} + \mathrm{ob}\,\mathcal{A} \to \mathcal{B}$ along $\langle 1, \iota\rangle\colon \mathcal{A} + \mathrm{ob}\,\mathcal{A} \to \mathcal{A}$. Since $\langle \ell, r\rangle$ is identity-on-objects, so is $J\colon \mathcal{A} \to \mathcal{T}$, and so we have an $\mathcal{A}$-pretheory. To conclude the proof, it now suffices to show that $\mathcal{E}^{F_2(\Sigma)} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$.

By the universal property of the collage and the pushout, to give a functor $H\colon \mathcal{T} \to \mathcal{X}$ is equally to give a functor $F = HJ\colon \mathcal{A} \to \mathcal{X}$ together with $\mathcal{V}$-natural transformations $\alpha_a\colon \mathcal{E}(K\text{–}, \Sigma a) \Rightarrow \mathcal{X}(F\text{–}, Fa)$ for each $a \in \mathrm{ob}\,\mathcal{A}$. In particular, taking $\mathcal{X} = \mathcal{V}^{\mathrm{op}}$ and $F = \mathcal{E}(K\text{–}, X)$, we see that a concrete $\mathcal{T}$-model structure on $X \in \mathcal{E}$ is given by an $\mathrm{ob}\,\mathcal{A}$-indexed family of $\mathcal{V}$-natural transformations

$$\alpha_a\colon \mathcal{E}(K\text{–}, \Sigma a) \Rightarrow [\mathcal{E}(Ka, X), \mathcal{E}(K\text{–}, X)]$$

or equally under transpose, by a family of maps

$$\mathcal{E}(Ka, X) \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](\mathcal{E}(K\text{–}, \Sigma a), \mathcal{E}(K\text{–}, X))\ .$$

By full fidelity of N_K, the right-hand side above is isomorphic to $\mathcal{E}(\Sigma a, X)$, and so concrete $\mathcal{T}$-model structure on X is equally given by a family of maps $\mathcal{E}(Ka, X) \to \mathcal{E}(\Sigma a, X)$. Finally, using the universal properties of copowers and coproducts, this is equivalent to giving a single map

$$\bar{\alpha} \colon \textstyle\sum_{a \in \mathcal{A}} \mathcal{E}(Ka, X) \cdot \Sigma a \to X$$

exhibiting X as an $F_2(\Sigma)$-algebra. We thus have a bijection over $\mathcal{E}$ between objects of $\mathcal{E}^{F_2(\Sigma)}$ and objects of $\mathbf{Mod}_c(\mathcal{T})$.

A similar analysis shows that a morphism $A \to \mathcal{E}(X, Y)$ in $\mathcal{V}$ lifts through the monomorphism $\mathbf{Mod}_c(\mathcal{T})((X, \alpha), (Y, \beta)) \to \mathcal{E}(X, Y)$ if and only if it lifts through the monomorphism $\mathcal{E}^{F_2(\Sigma)}((X, \bar{\alpha}), (Y, \bar{\beta})) \to \mathcal{E}(X, Y)$. It follows that we have an isomorphism of $\mathcal{V}$-categories $\mathcal{E}^{F_2(\Sigma)} \cong \mathbf{Mod}_c(\mathcal{T})$ over $\mathcal{E}$ as desired. ☐

In proving the rest of Theorem 38, the following lemma will be useful.

Lemma 56. *Let $\mathcal{C}_1 \subseteq \mathcal{C}_2$ be replete, full, colimit-closed sub-$\mathcal{V}$-categories of $\mathcal{C}$; for example, they could be coreflective. If $V \colon \mathcal{C} \to \mathcal{D}$ has a left adjoint F taking values in $\mathcal{C}_1$, and the restriction $V|_{\mathcal{C}_2} \colon \mathcal{C}_2 \to \mathcal{D}$ is monadic, then $\mathcal{C}_1 = \mathcal{C}_2$.*

Proof. Since F takes values in $\mathcal{C}_1 \subseteq \mathcal{C}_2$, the left adjoint to $V|_{\mathcal{C}_2} \colon \mathcal{C}_1 \to \mathcal{D}$ is still given by F. So monadicity of $V|_{\mathcal{C}_2}$ means that each $X \in \mathcal{C}_2$ can be written as a coequaliser in $\mathcal{C}_2$, and hence also in $\mathcal{C}$, of objects in the image of F. Since $\operatorname{im} F \subseteq \mathcal{C}_1$ and since $\mathcal{C}_1$ is closed in $\mathcal{C}$ under colimits, it follows that $X \in \mathcal{C}_1$. ☐

Theorem 38. *$V \colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint $F \colon \mathbf{Sig}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Mnd}(\mathcal{E})$ taking values in $\mathcal{A}$-nervous monads. Moreover:*

(i) The restricted functor $V \colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is monadic;
(ii) A monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$ is $\mathcal{A}$-nervous if and only if it is a colimit in $\mathbf{Mnd}(\mathcal{E})$ of monads in the image of F;
(iii) Each of $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$, $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ and $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$ is locally presentable.

Proof. We begin with (i). Let $H \colon \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \to \mathcal{V}\text{-}\mathbf{CAT}(\operatorname{ob} \mathcal{A}, [\mathcal{A}^{\mathrm{op}}, \mathcal{V}])$ be the functor sending a pretheory $J \colon \mathcal{A} \to \mathcal{T}$ to the family of presheaves $(\mathcal{T}(J-, Ja))_{a \in \mathcal{A}}$. Since an $\mathcal{A}$-pretheory is a theory just when each of these presheaves is a K-nerve, we have a pullback square as to the right in:

(8.1)
$$\begin{tikzcd}
\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \arrow[r, "\Phi"] \arrow[d, "V"'] \arrow[dr, phantom, "\cong"] & \mathbf{Th}_{\mathcal{A}}(\mathcal{E}) \arrow[r, hook] \arrow[d, "P"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \arrow[d, "H"] \\
\mathcal{V}\text{-}\mathbf{CAT}(\operatorname{ob} \mathcal{A}, \mathcal{E}) \arrow[r, "N_K \circ (-)"'] & \mathcal{V}\text{-}\mathbf{CAT}(\operatorname{ob} \mathcal{A}, K\text{-}\mathbf{Ner}) \arrow[r, hook] & \mathcal{V}\text{-}\mathbf{CAT}(\operatorname{ob} \mathcal{A}, [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]) \, .
\end{tikzcd}$$

Since $K\text{-}\mathbf{Ner} \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is replete, this square is a pullback along a discrete isofibration, and so by [14, Corollary 1] also a bipullback. On the other hand, to the left, we have a pseudocommuting square as witnessed by the isomorphisms:

$$(PJ_{\mathsf{T}})(A) = \mathcal{A}_{\mathsf{T}}(J_{\mathsf{T}}-, J_{\mathsf{T}}A) = \mathcal{E}^{\mathsf{T}}(F^{\mathsf{T}}K-, F^{\mathsf{T}}KA) \cong \mathcal{E}(K-, TKA) = N_K(TKA) \, .$$

Since both horizontal edges of this square are equivalences, it is also a bipullback.

To show the required monadicity, we must prove that V creates V-absolute coequalisers. Since the large rectangle is a bipullback—as the pasting of two bipullbacks—it suffices to show that H creates H-absolute coequalizers. As the definition of H depends only on $\mathcal{A}$ and not $\mathcal{E}$, we lose no generality in proving this if we assume that $\mathcal{E} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ and $K = Y$. In this case, *every* presheaf on $\mathcal{A}$ is a K-nerve, and so the horizontal composites in (8.1) are equivalences; and so, finally, it suffices to prove that V is monadic when $\mathcal{E} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ and $K = Y$.

Note that, in this case, $\mathcal{A}$ is a saturated class of arities: for indeed, by the universal property of free cocompletion, a functor $F \colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is $\mathcal{A}$-induced if and only if it is *cocontinuous*. It thus follows from Proposition 58 below that the restriction $V_c \colon \mathbf{Mnd}_c(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ of V to cocontinuous monads is monadic; so we will be done if $\mathbf{Mnd}_c(\mathcal{E}) = \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$. In this case, $\Psi \colon \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ sends $J \colon \mathcal{A} \to \mathcal{T}$ to a monad which is isomorphic to that induced by the adjunction $\mathrm{Lan}_J \colon [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \leftrightarrows [\mathcal{T}^{\mathrm{op}}, \mathcal{V}] \colon [J^{\mathrm{op}}, 1]$, and so $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \subseteq \mathbf{Mnd}_c(\mathcal{E})$. To obtain equality, we apply Lemma 56. We have that:

- $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ and $\mathbf{Mnd}_c(\mathcal{E})$ are coreflective in $\mathbf{Mnd}(\mathcal{E})$ by Corollary 22 and Lemma 57 respectively;
- $V \colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ has a left adjoint taking values in $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$;
- The restriction $V_c \colon \mathbf{Mnd}_c(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is monadic;

and so $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) = \mathbf{Mnd}_c(\mathcal{E})$. This proves monadicity of V in the special case $\mathcal{E} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, whence also, by the preceding argument, in the general case.

In order to prove (ii), we let $\mathcal{C}_1$ be the colimit-closure in $\mathbf{Mnd}(\mathcal{E})$ of the image of F. Since $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ contains this image and is colimit-closed, we have $\mathcal{C}_1 \subseteq \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \subseteq \mathbf{Mnd}(\mathcal{E})$. Thus, applying Lemma 56 to this triple and $V \colon \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ gives $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) = \mathcal{C}_1$ as desired.

Finally we prove (iii). The monadicity of V above implies that of P and hence also of H (by taking $\mathcal{E} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$). Since filtered colimits of $\mathcal{A}$-pretheories can be computed at the level of underlying *graphs*, the forgetful H preserves them; which is to say that $\mathbf{Preth}_{\mathcal{A}}(\mathcal{E})$ is *finitarily* monadic over the locally presentable $\mathcal{V}\text{-}\mathbf{CAT}(\mathrm{ob}\,\mathcal{A}, [\mathcal{A}^{\mathrm{op}}, \mathcal{V}])$, whence locally presentable by [13, Satz 10.3]. So in the right-hand and the large bipullback squares in (8.1), the bottom and right sides are right adjoints between locally presentable categories. Since by [10, Theorem 2.18], the 2-category of locally presentable categories and right adjoint functors is closed under bilimits in $\mathbf{CAT}$, we conclude that each $\mathbf{Th}_{\mathcal{A}}(\mathcal{E})$ and each $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ is also locally presentable. □

8.2. **Saturated classes.** We now turn to the deferred proof of Theorem 43. Recall the context: an endo-$\mathcal{V}$-functor $F \colon \mathcal{E} \to \mathcal{E}$ is called $\mathcal{A}$-*induced* when the pointwise left Kan extension of its restriction along K, and $\mathcal{A}$ is a *saturated class of arities* if $\mathcal{A}$-induced endofunctors of $\mathcal{E}$ are composition-closed.

We begin by recording the basic properties of this situation. We write $\mathcal{A}\text{-}\mathbf{End}(\mathcal{E})$ and $\mathcal{A}\text{-}\mathbf{Mnd}(\mathcal{E})$ for the full subcategories of $\mathbf{End}(\mathcal{E}) = \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{E}, \mathcal{E})$ and $\mathbf{Mnd}(\mathcal{E})$ on, respectively, the $\mathcal{A}$-induced endofunctors, and the monads with $\mathcal{A}$-induced underlying endofunctor.

Lemma 57. *$\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is coreflective in $\mathbf{End}(\mathcal{E}) = \mathcal{V}$-$\mathbf{CAT}(\mathcal{E},\mathcal{E})$ via the coreflector $R(F) = \mathrm{Lan}_K(FK)$, as on the left in:*

$$(8.2) \qquad \begin{tikzcd} \mathcal{A}\text{-}\mathbf{End}(\mathcal{E}) \arrow[r, shift right, "I"'] \arrow[r, phantom, "\top"] & \mathbf{End}(\mathcal{E}) \arrow[l, shift right, "R"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{A}\text{-}\mathbf{Mnd}(\mathcal{E}) \arrow[r, shift right, "I"'] \arrow[r, phantom, "\top"] & \mathbf{Mnd}(\mathcal{E}) \arrow[l, shift right, "R"'] \end{tikzcd} .$$

If $\mathcal{A}$ is a saturated class, then $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is right-closed monoidal, and the coreflection left above lifts to the corresponding categories of monads as on the right.

Proof. Restriction and left Kan extension along the fully faithful K exhibits $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ as equivalent to $\mathcal{V}$-$\mathbf{CAT}(\mathcal{A},\mathcal{E})$, whence locally presentable. Since restriction along K is a coreflector of $\mathbf{End}(\mathcal{E})$ into $\mathcal{V}$-$\mathbf{CAT}(\mathcal{A},\mathcal{E})$, it follows that $R(F) = \mathrm{Lan}_K(FK)$ is a coreflector of $\mathbf{End}(\mathcal{E})$ into $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$.

If $\mathcal{A}$ is saturated then $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is monoidal under composition. Since each endofunctor $(\text{–}) \circ F$ of $\mathbf{End}(\mathcal{E})$ is cocontinuous, and $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is closed in $\mathbf{End}(\mathcal{E})$ under colimits, each endofunctor $(\text{–}) \circ F$ of $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is cocontinuous, and so has a right adjoint by local presentability. Thus $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is right-closed monoidal.

Furthermore, the inclusion of $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ into $\mathbf{End}(\mathcal{E})$ is strict monoidal, whence by [15, Theorem 1.5] the coreflection to the left of (8.2) lifts to a coreflection in the 2-category $\mathbf{MONCAT}$ of monoidal categories, lax monoidal functors and monoidal transformations. Applying the 2-functor $\mathbf{MONCAT}(1, \text{–})\colon \mathbf{MONCAT} \to \mathbf{CAT}$ yields the coreflection to the right of (8.2). □

The key step towards establishing Theorem 43 above is now:

Proposition 58. *The left adjoint F of $V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ takes values in $\mathcal{A}$-induced monads; furthermore, the restriction of V to $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ is monadic.*

Proof. For any $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$, its $\mathcal{A}$-induced coreflection $\varepsilon_{\mathsf{T}}\colon IR(\mathsf{T}) \to \mathsf{T}$ has as underlying map in $\mathbf{End}(\mathcal{E})$ the component $\mathrm{Lan}_K(TK) \to T$ of the counit of the adjunction given by restriction and left Kan extension along K. Since K is fully faithful, the restriction of this map along K is invertible, whence in particular, $V\varepsilon\colon VIR \Rightarrow V\colon \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ is invertible. So $\eta\colon \mathrm{id} \Rightarrow VF$ factors through $V\varepsilon_F\colon VIRF \Rightarrow VF$ whence, by adjointness, $\mathrm{id}\colon F \Rightarrow F$ factors through ε_F. Therefore each $F(\Sigma)$ is a retract of $IRF(\Sigma)$; since $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ is closed under colimits in $\mathbf{Mnd}(\mathcal{E})$, it is retract-closed and so each $F(\Sigma)$ belongs to $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$.

It remains to prove that the restriction of V to $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ is monadic. To do so, we decompose this restriction as

$$\mathcal{A}\text{-}\mathbf{Mnd}(\mathcal{E}) \xrightarrow{V_1} \mathcal{A}\text{-}\mathbf{End}(\mathcal{E}) \xrightarrow{V_2} \mathbf{Sig}_{\mathcal{A}}(\mathcal{E}) ,$$

where V_1 forgets the monad structure and V_2 is given by precomposition with $\mathrm{ob}\,\mathcal{A} \to \mathcal{A} \to \mathcal{E}$, and apply the following result, which is [21, Theorem 2]:

Theorem. *Let $\mathcal{M}$ be a right-closed monoidal category, and $V_2\colon \mathcal{M} \to \mathcal{N}$ a monadic functor for which there exists a functor $\diamond\colon \mathcal{M} \times \mathcal{N} \to \mathcal{N}$ with natural isomorphisms $X \diamond VY \cong V(X \otimes Y)$. If the forgetful functor $V_1\colon \mathbf{Mon}(\mathcal{M}) \to \mathcal{M}$ has a left adjoint, then the composite $V_2V_1\colon \mathbf{Mon}(\mathcal{M}) \to \mathcal{N}$ is monadic.*

Indeed, by Lemma 57, $\mathcal{A}$-$\mathbf{End}(\mathcal{E})$ is a right-closed monoidal category, and $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ the category of monoids therein. Under the equivalence $\mathcal{A}$-$\mathbf{End}(\mathcal{E}) \simeq \mathcal{V}$-$\mathbf{CAT}(\mathcal{A}, \mathcal{E})$, we may identify V_2 with precomposition along $\mathrm{ob}\,\mathcal{A} \to \mathcal{A}$. It is thus cocontinuous, and has a left adjoint given by left Kan extension; whence is monadic. Now since V_2V_1 has a left adjoint and V_2 is monadic, it follows that V_1 also has a left adjoint. Finally, we have a functor

$$\diamond \colon \mathcal{A}\text{-}\mathbf{End}(\mathcal{E}) \times \mathbf{Sig}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$$

defined by $(F, G) \mapsto FG$, and this clearly has the property that $M(FG) = F \diamond M(G)$. So applying the above theorem yields the desired monadicity. ☐

We are now ready to prove:

Theorem 43. *Let $\mathcal{A}$ be a saturated class of arities in $\mathcal{E}$. The following are equivalent properties of a monad $\mathsf{T} \in \mathbf{Mnd}(\mathcal{E})$:*

(i) T is $\mathcal{A}$-nervous;
(ii) $T \colon \mathcal{E} \to \mathcal{E}$ is $\mathcal{A}$-induced;
(iii) $T \colon \mathcal{E} \to \mathcal{E}$ preserves Φ-colimits for any density presentation Φ of K.

Proof. For (i) $\Leftrightarrow$ (ii), the monadicity of $V \colon \mathcal{A}$-$\mathbf{Mnd}(\mathcal{E}) \to \mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ verified in the previous proposition implies, as in the proof of Theorem 38(iii), that $\mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ is the colimit-closure in $\mathbf{Mnd}(\mathcal{E})$ of the free monads on signatures. Since $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ is also this closure, we have $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E}) = \mathcal{A}$-$\mathbf{Mnd}(\mathcal{E})$ as desired. For (ii) $\Leftrightarrow$ (iii), we apply Proposition 35. ☐

References

[1] Adámek, J., and Rosický, J. *Locally presentable and accessible categories*, vol. 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1994.

[2] Ara, D. *Sur les ∞-groupoïdes de Grothendieck et une variante ∞-catégorique*. PhD thesis, Université Paris VII, 2010.

[3] Ara, D. On the homotopy theory of Grothendieck ∞-groupoids. *Journal of Pure and Applied Algebra 217* (2013), 1237–1278.

[4] Avery, T. *Structure and semantics*. PhD thesis, University of Edinburgh, 2017.

[5] Barr, M. Coequalizers and free triples. *Mathematische Zeitschrift 116* (1970), 307–322.

[6] Barr, M., and Wells, C. *Toposes, triples and theories*, vol. 278 of *Grundlehren der Mathematischen Wissenschaften*. Springer, 1985.

[7] Batanin, M. Monoidal globular categories as a natural environment for the theory of weak n-categories. *Advances in Mathematics 136* (1998), 39–103.

[8] Berger, C. A cellular nerve for higher categories. *Advances in Mathematics 169* (2002), 118–175.

[9] Berger, C., Melliès, P.-A., and Weber, M. Monads with arities and their associated theories. *Journal of Pure and Applied Algebra 216* (2012), 2029–2048.

[10] Bird, G. *Limits in 2-categories of locally-presented categories*. PhD thesis, University of Sydney, 1984.

[11] Day, B. On closed categories of functors II. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 20–54.

[12] Dubuc, E. J. *Kan extensions in enriched category theory*, vol. 145 of *Lecture Notes in Mathematics*. Springer, 1970.

[13] Gabriel, P., and Ulmer, F. *Lokal präsentierbare Kategorien*, vol. 221 of *Lecture Notes in Mathematics*. Springer, 1971.

[14] Joyal, A., and Street, R. Pullbacks equivalent to pseudopullbacks. *Cahiers de Topologie et Geométrie Différentielle Catégoriques 34* (1993), 153–156.

[15] Kelly, G. M. Doctrinal adjunction. In *Category Seminar (Sydney, 1972/1973)*, vol. 420 of *Lecture Notes in Mathematics*. Springer, 1974, pp. 257–280.

[16] Kelly, G. M. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. *Bulletin of the Australian Mathematical Society 22* (1980), 1–83.

[17] Kelly, G. M. *Basic concepts of enriched category theory*, vol. 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982. Republished as: *Reprints in Theory and Applications of Categories 10* (2005).

[18] Kelly, G. M. Structures defined by finite limits in the enriched context I. *Cahiers de Topologie et Geométrie Différentielle Catégoriques 23* (1982), 3–42.

[19] Kelly, G. M., and Lack, S. Finite-product-preserving functors, Kan extensions and strongly-finitary 2-monads. *Applied Categorical Structures 1* (1993), 85–94.

[20] Kelly, G. M., and Power, A. J. Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads. *Journal of Pure and Applied Algebra 89* (1993), 163–179.

[21] Lack, S. On the monadicity of finitary monads. *Journal of Pure and Applied Algebra 140* (1999), 65–73.

[22] Lack, S., and Power, J. Gabriel-Ulmer duality and Lawvere theories enriched over a general base. *Journal of Functional Programming 19* (2009), 265–286.

[23] Lack, S., and Rosický, J. Notions of Lawvere theory. *Applied Categorical Structures 19* (2011), 363–391.

[24] Lawvere, F. W. *Functorial semantics of algebraic theories*. PhD thesis, Columbia University, 1963. Also *Proc. Nat. Acad. Sci. U.S.A. 50* (1963), 869–872. Republished as: *Reprints in Theory and Applications of Categories 5* (2004).

[25] Leinster, T. Operads in higher-dimensional category theory. *Theory and Applications of Categories 12* (2004), 73–194.

[26] Linton, F. E. J. Some aspects of equational categories. In *Conference on Categorical Algebra (La Jolla, 1965)*. Springer, 1966, pp. 84–94.

[27] Maltsiniotis, G. Grothendieck ∞-groupoids, and still another definition of ∞-categories. Unpublished, available as https://arxiv.org/abs/1009.2331, 2010.

[28] Markowsky, G. Chain-complete posets and directed sets with applications. *Algebra Universalis 6*, 1 (1976), 53–68.

[29] Melliès, P.-A. Segal condition meets computational effects. In *25th Annual IEEE Symposium on Logic in Computer Science LICS 2010*. IEEE Computer Society Press, 2010, pp. 150–159.

[30] Meyer, J.-P. Induced functors on categories of algebras. *Mathematische Zeitschrift 142* (1975), 1–14.

[31] Nishizawa, K., and Power, J. Lawvere theories enriched over a general base. *Journal of Pure and Applied Algebra 213* (2009), 377–386.

[32] Power, J. Enriched Lawvere theories. *Theory and Applications of Categories 6* (1999), 83–93.

[33] Segal, G. Classifying spaces and spectral sequences. *Institut des Hautes Études Scientifiques. Publications Mathématiques 34* (1968), 105–112.

[34] Street, R. The petit topos of globular sets. *Journal of Pure and Applied Algebra 154* (2000), 299–315.

[35] Weber, M. Familial 2-functors and parametric right adjoints. *Theory and Applications of Categories 18* (2007), 665–732.

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 61137, Czech Republic

E-mail address: bourkej@math.muni.cz

Department of Mathematics, Macquarie University, NSW 2109, Australia

E-mail address: richard.garner@mq.edu.au

Chapter 6

Adjoint functor theorems for homotopically enriched categories

This chapter contains the article *Adjoint functor theorems for homotopically enriched categories* by *John Bourke, Steve Lack and Lukáš Vokřínek*. This 2020 article is available on the Arxiv at `https://arxiv.org/abs/2006.07843` and has been submitted for publication. All three authors contributed an equal share to the results of this article.

ADJOINT FUNCTOR THEOREMS FOR HOMOTOPICALLY ENRICHED CATEGORIES

JOHN BOURKE, STEPHEN LACK, AND LUKÁŠ VOKŘÍNEK

Abstract. We prove a weak adjoint functor theorem in the setting of categories enriched in a monoidal model category $\mathcal{V}$ admitting certain limits. When $\mathcal{V}$ is equipped with the trivial model structure this recaptures the enriched version of Freyd's adjoint functor theorem. For non-trivial model structures, we obtain new weak adjoint functor theorems of a homotopical flavour — in particular, when $\mathcal{V}$ is the category of simplicial sets we obtain a homotopical adjoint functor theorem appropriate to the ∞-cosmoi of Riehl and Verity. We also investigate accessibility in the enriched setting, in particular obtaining homotopical cocompleteness results for accessible ∞-cosmoi.

1. Introduction

The general adjoint functor theorem of Freyd describes conditions under which a functor $U : \mathcal{B} \to \mathcal{A}$ has a left adjoint: namely,

- U satisfies the solution set condition and
- $\mathcal{B}$ is complete and $U : \mathcal{B} \to \mathcal{A}$ preserves limits.

The first applications of this result that one typically learns about are the construction of left adjoints to forgetful functors between algebraic categories, and the cocompleteness of algebraic categories. In the present paper we shall describe a strict generalisation of Freyd's result, applicable to categories of a homotopical nature, and shall demonstrate similar applications to homotopical algebra.

The starting point of our generalisation is to pass from ordinary **Set**-enriched categories to categories enriched in a *monoidal model category* $\mathcal{V}$. For instance, we could take $\mathcal{B}$ to be

- the 2-category of monoidal categories and strong monoidal functors (here $\mathcal{V} = \mathbf{Cat}$);

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- the ∞-cosmos of quasicategories admitting a class of limits and morphisms preserving those limits (here $\mathcal{V} = \mathbf{SSet}$);
- the category of fibrant objects in a model category (here $\mathcal{V} = \mathbf{Set}$ but with what we call the split model structure)

and $U : \mathcal{B} \to \mathcal{A}$ the appropriate forgetful functor to the base.

Now in none of these cases is the $\mathcal{V}$-category $\mathcal{B}$ complete in the sense of enriched category theory, but each does admit certain limits captured by the model structure in which we are enriching — in particular, *cofibrantly-weighted limits*.

If we keep the usual solution set condition, which holds in many examples including those described above, the question turns to what kind of left adjoint we can hope for? Our answer here will fit into the framework of *enriched weakness* developed by the second-named author and Rosický in [25]. Given a class $\mathcal{E}$ of morphisms in $\mathcal{V}$, one says that $U : \mathcal{B} \to \mathcal{A}$ admits an $\mathcal{E}$-weak left adjoint if for each object $A \in \mathcal{A}$ there exists an object $A' \in \mathcal{B}$ and a morphism $\eta_A : A \to UA'$ for which the induced map

$$\mathcal{B}(A', B) \xrightarrow{U} \mathcal{A}(UA', UB) \xrightarrow{\mathcal{A}(\eta_A, UB)} \mathcal{A}(A, UB) \tag{1.1}$$

belongs to $\mathcal{E}$ for all $B \in \mathcal{B}$. Here one recovers classical adjointness on taking $\mathcal{E}$ to be the isomorphisms. In this paper, we take $\mathcal{E}$ to be the class of morphisms in the model category $\mathcal{V}$ with the *dual strong deformation retract property* (see Section 3 below). We follow [9] in calling such morphisms *shrinkable*.[1]

Our first main result, Theorem 7.8, is our generalisation of Freyd's adjoint functor theorem, and gives a sufficient condition for a $\mathcal{V}$-functor to have such an $\mathcal{E}$-weak left adjoint. It *is* a generalisation, since when $\mathcal{V} = \mathbf{Set}$ with the trivial model structure, all colimits are cofibrantly-weighted, the shrinkable morphisms are the isomorphisms, and $\mathcal{E}$-weak left adjoints are just ordinary left adjoints.

Our second main result, Theorem 8.9, applies our adjoint functor theorem to the construction of *$\mathcal{E}$-weak colimits*. Namely, we prove that if $\mathcal{C}$ is an *accessible* $\mathcal{V}$-category admitting certain limits, then $\mathcal{C}$ admits $\mathcal{E}$-weak colimits.

The remainder of the paper is devoted to giving applications of our two main results in various settings. In particular, we interpret them for 2-categories and ∞-cosmoi, where we obtain new results concerning adjoints of a homotopical flavour and homotopy colimits. We also provide numerous concrete examples to which these results apply.

[1]And we thank Karol Szumilo for suggesting the name.

Let us now give a more detailed overview of the paper. In Section 2 we introduce our basic setting, including four running examples to which we return throughout the paper. These four examples capture *classical category theory*, *weak category theory*, *2-category theory* and ∞-*cosmoi*. Furthermore, we assign to each $\mathcal{V}$-category $\mathcal{C}$ its "homotopy **ER**-category" $\mathrm{h}_*\mathcal{C}$: a certain category enriched in equivalence relations. Since this is a simple kind of 2-category, it admits notions of equivalence, bi-initial objects, and so on — accordingly, we can transport these notions to define equivalences and bi-initial objects in the $\mathcal{V}$-category $\mathcal{C}$.

The next five sections walk through generalisations of the various steps in the proof of the adjoint functor theorem given in [27], which we summarise below.

- The technical core of the general adjoint functor theorem is the result that a complete category $\mathcal{C}$ with a weakly initial set of objects has an initial object. *In Section 5 we generalise this result, establishing conditions under which $\mathcal{C}$ has a bi-initial object.*
- In the classical setting the next step is to apply the preceding result to obtain an initial object in each comma category A/U. *Accordingly Section 4 is devoted to the subtle world of enriched comma categories.*
- The next step, trivial in the classical case, is the observation that if $\eta\colon A \to UA'$ is initial in A/U then (1.1) is invertible. Another way of saying this is that (1.1) is *terminal* in the slice $\mathbf{Set}/\mathcal{B}(A, UB)$. *In Section 3 we consider the class $\mathcal{E}$ of shrinkable morphisms, which turn out to be bi-terminal objects in enriched slice categories.*
- Putting all this together in Section 6, we prove our generalised adjoint functor theorem in the case where the unit $I \in \mathcal{V}$ is terminal. In Section 7 we adapt this to cover the general case by passing from $\mathcal{V}$-enrichment to $\mathcal{V}/I$-enrichment.

In Section 8 we consider accessible $\mathcal{V}$-categories and prove that each accessible $\mathcal{V}$-category with powers and enough cofibrantly-weighted limits admits $\mathcal{E}$-weak colimits.

In Section 9 we interpret our two main theorems in each of our four running examples. In the first two examples we recover, and indeed extend, the classical theory. Our focus, however, is primarily on 2-categories and ∞-cosmoi, where we describe many examples to which our two main results apply. In particular, we show that many natural examples of ∞-cosmoi are accessible, which allows us to conclude that they admit *flexibly-weighted homotopy colimits* in the sense of [33].

2. The setting

We start with a monoidal model category $\mathcal{V}$, as in [12] for example, in which the unit object I is cofibrant. We may sometimes allow ourselves to write as if the monoidal structure were strict.

We shall now give four examples, to which we shall return throughout the paper. Several further examples may be found in Section 3 below.

Example 2.1. Any complete and cocomplete symmetric monoidal closed category $(\mathcal{V}, \otimes, I)$ may be equipped with the *trivial model structure*, in which the weak equivalences are the isomorphisms, and all maps are cofibrations and fibrations. The case of classical category theory corresponds the cartesian closed category $(\mathbf{Set}, \times, 1)$.

Example 2.2. For our second example, involving weak category theory, we take our motivation from the case of **Set**, in which case the relevant model structure has cofibrations the injections, fibrations the surjections, and all maps as weak equivalences.

For general $\mathcal{V}$ we refer to the corresponding model structure as the *split model structure*, which we now define. By Proposition 2.6 of [36], in any category with binary coproducts there is a weak factorisation whose left class consists of the retracts of coproduct injections $inj\colon X \to X + Y$, while the right class consists of the split epimorphisms. Here the canonical factorisation of an arrow $f\colon X \to Y$ is given by the cograph factorisation

$$X \xrightarrow{inj} X + Y \xrightarrow{\langle f,1\rangle} Y.$$

As with any weak factorisation system (on a category with, say, finite limits and colimits), we can extend it to a model structure whose cofibrations and fibrations are the two given classes, and for which all maps are weak equivalences. Doing this in the present case produces what we shall refer to as the split model structure. It is a routine exercise to show that if $(\mathcal{V}, \otimes, I)$ is a complete and cocomplete symmetric monoidal closed category, then the split model structure on $\mathcal{V}$ is monoidal.

Example 2.3. The cartesian closed category $(\mathbf{Cat}, \times, 1)$ becomes a monoidal model category when it is equipped with the *canonical model structure*: here the weak equivalences are the equivalences of categories, the cofibrations are the functors which are injective on objects, and the fibrations are the isofibrations. All objects are cofibrant and fibrant. The trivial fibrations are the equivalences which are surjective on objects; these are often known as surjective equivalences or retract equivalences.

Example 2.4. Our final main example is the cartesian closed category $(\mathbf{SSet}, \times, 1)$ of simplicial sets equipped with the Joyal model structure. (Using the Kan model structure would give another example, but it turns out that this leads to a strictly weaker adjoint functor theorem.)

We follow the standard notation in enriched category theory of writing $\mathcal{A}_0$ for the underlying ordinary category of a $\mathcal{V}$-category $\mathcal{A}$. This has the same objects as $\mathcal{A}$, and the hom-set $\mathcal{A}_0(A, B)$ is given by the set of morphisms $I \to \mathcal{A}(A, B)$ in $\mathcal{V}$. Similarly, we write $F_0 \colon \mathcal{A}_0 \to \mathcal{B}_0$ for the underlying ordinary functor of a $\mathcal{V}$-functor $F \colon \mathcal{A} \to \mathcal{B}$.

We shall begin by showing how to associate to any $\mathcal{V}$-category a 2-category whose hom-categories are both groupoids and preorders.

2.1. **Intervals.** An *interval* in $\mathcal{V}$ will be an object J equipped with a factorisation

$$I + I \xrightarrow{(d\ c)} J \xrightarrow{e} I$$

of the codiagonal for which $(d\ c)$ is a cofibration and e a weak equivalence. We shall then say that "(J, d, c, e) is an interval". Observe that if $(d\ c)$ is a cofibration, then since I is cofibrant, d and c are also cofibrations, and the following are equivalent:

- e is a weak equivalence
- d is a trivial cofibration
- c is a trivial cofibration.

We may always obtain an interval by factorising the codiagonal $I + I \to I$ as a cofibration followed by a trivial fibration; an interval of this type will be called a *standard interval.* (J, d, c, e) is an interval if and only if (J, c, d, e) is one. There are also two constructions on intervals that will be needed.

Proposition 2.5. *If (J_1, d_1, c_1, e_1) and (J_2, d_2, c_2, e_2) are intervals, so is $(J, d_2'd_1, c_1'c_2, e)$ where J is constructed as the pushout*

$$\begin{tikzcd}
 & J_1 \arrow[dr, "d_2'"] \arrow[drr, bend left, "e_1"] & & \\
I \arrow[ur, "c_1"] \arrow[dr, "d_2"'] & & J \arrow[r, "e"] & I. \\
 & J_2 \arrow[ur, "c_1'"'] \arrow[urr, bend right, "e_2"'] & &
\end{tikzcd}$$

Proof. Since d_2' and c_1' are pushouts of trivial cofibrations they are trivial cofibrations. Thus both $d_2'd_1$ and $c_1'c_2$ are trivial cofibrations.

On the other hand we can construct J as the pushout

$$\begin{tikzcd}
& I+I+I+I \arrow[r, "(d_1\ c_1)+(d_2\ c_2)"] \arrow[d, "1+\nabla+1"'] & J_1+J_2 \arrow[d, "d_2'+c_1'"] \\
I+I \arrow[r, "i_{1,3}"'] & I+I+I \arrow[r] & J
\end{tikzcd}$$

and so the lower horizontal is also a cofibration. □

Proposition 2.6. *If (J_1,d_1,c_1) and (J_2,d_2,c_2) are intervals then $J_1\otimes J_2$ becomes an interval when equipped with the maps*

$$I \xrightarrow{\cong} I\otimes I \xrightarrow{d_1\otimes d_2} J_1\otimes J_2$$

$$I \xrightarrow{\cong} I\otimes I \xrightarrow{c_1\otimes c_2} J_1\otimes J_2$$

$$J_1\otimes J_2 \xrightarrow{e_1\otimes e_2} I\otimes I \xrightarrow{\cong} I.$$

2.2. **ER-categories.** Let **ER** be the full subcategory of **Cat** consisting of the *equivalence relations*: that is, the groupoids which are also preorders. This is closed in **Cat** under both products and internal homs; it follows that **ER** becomes a cartesian closed category in its own right, and that we can consider categories enriched over **ER**.

An **ER**-category is just a 2-category for which each hom-category is an equivalence relation. Equivalently, it is an ordinary category in which each hom-set is equipped with an equivalence relation, and this relation is respected by composition on either side (whiskering, if you will).

Since an **ER**-category is a special sort of 2-category, we can consider standard 2-categorical notions. Thus morphisms $f,g\colon A\to B$ are isomorphic in the 2-category just when they are related under the given equivalence relation. We may as well therefore use $\cong$ to denote this relation. An *equivalence* in an **ER**-category consists of morphisms $f\colon A\to B$ and $g\colon B\to A$ for which both $gf\cong 1_A$ and $fg\cong 1_B$.

Definition 2.7. An object T of an **ER**-category $\mathcal{C}$ is *bi-terminal* if for each object C, the induced map $\mathcal{C}(C,T)\to 1$ is an equivalence.

In elementary terms, this means that there exists a morphism $C\to T$ and it is unique up to isomorphism. Dually there are *bi-initial* objects.

2.3. **The homotopy ER-category of a $\mathcal{V}$-category.** Any monoidal category has a monoidal functor to **Set** given by homming out of the unit object; in our case, this has the form $\mathcal{V}(I,-)\colon \mathcal{V} \to \mathbf{Set}$. The monoidal structure is defined by

$$\begin{aligned} \mathcal{V}(I,X)\times\mathcal{V}(I,Y) &\longrightarrow \mathcal{V}(I,X\otimes Y) \\ (x,y) &\longmapsto x\otimes y \end{aligned} \tag{2.1}$$

with unit defined by the map $1 \to \mathcal{V}(I,I)$ picking out the identity.

Given $X \in \mathcal{V}$ and $x,y\colon I \to X$, write $x \sim y$ if there is an interval (J,d,c,e) and a morphism $h\colon J \to X$ with $hd = x$ and $hc = y$. This clearly defines a symmetric reflexive relation on $\mathcal{V}(I,X)$; and it is transitive by Proposition 2.5. Given $f\colon X \to Y$, the function $\mathcal{V}(I,f)$ is a morphism of equivalence relations, since if $x = hd \sim hc = y$ then also $fx = fhd \sim fhc = fy$ for any f. Thus we have a functor $\mathcal{V} \to \mathbf{ER}$. Thanks to Proposition 2.6, the functions (2.1) lift to morphisms of equivalence relations. Since the unit map $1 \to \mathcal{V}(I,I)$ is trivially a morphism of equivalence relations, the monoidal functor $\mathcal{V}(I,-)\colon \mathcal{V} \to \mathbf{Set}$ lifts to a monoidal functor $\mathrm{h}\colon \mathcal{V} \to \mathbf{ER}$.

This h induces a 2-functor $\mathrm{h}_*\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{ER}\text{-}\mathbf{Cat}$. Explicitly, for any $\mathcal{V}$-category $\mathcal{C}$, the induced $\mathbf{ER}$-category $\mathrm{h}_*(\mathcal{C})$ is given by the underlying ordinary category of $\mathcal{C}$, equipped with the equivalence relation $\cong$ on each hom-set $\mathcal{C}_0(A,B)$, where $f \cong g$ just when, seen as maps $I \to \mathcal{C}(A,B)$, they are $\sim$-related; in other words, if there is a factorisation

$$\begin{tikzcd} I+I \arrow[r, "(f\ g)"] \arrow[d, "(d\ c)"'] & \mathcal{C}(A,B) \\ J \arrow[ur, "h"'] & \end{tikzcd}$$

for some interval (J,d,c,e). We call this relation $\cong$ the $\mathcal{V}$-*homotopy relation*.

Remark 2.8. For f and g as above, let us refer to a given extension h as above as a $(\mathcal{V},J)$-*homotopy*, and denote it as $h\colon f \cong_J g$. We say that such an $h\colon J \to \mathcal{C}(A,B)$ is *trivial* if it factorises through $e\colon J \to I$, in which case of course $f = g$.

Given $a\colon X \to A$ and $b\colon B \to Y$ we naturally obtain $(\mathcal{V},J)$-homotopies $h\circ a\colon f\circ a \cong_J g\circ a$ and $b\circ h\colon b\circ f \cong_J b\circ g$. It is only when considering transitivity of the relation $\cong$ that we need to change the interval J.

Remark 2.9. A priori, we now have two "homotopy relations" for morphisms $x,y\colon I \to X$ in $\mathcal{V}$: on the one hand $x \sim y$ and on the other $x \cong y$. But these are clearly equivalent.

More generally, morphisms $f, g\colon X \to Y$ in $\mathcal{V}$ are $\mathcal{V}$-homotopic if any (and thus all) of the following diagrams has a filler:

$$\begin{tikzcd} X+X \arrow[r, "(f\ g)"] \arrow[d, "(d\cdot X\ c\cdot X)"'] & Y \\ J\cdot X \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} I+I \arrow[r, "(f\ g)"] \arrow[d, "(d\ c)"'] & {[X,Y]} \\ J \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, dotted] \arrow[dr, "\binom{f}{g}"'] & Y^J \arrow[d, "\binom{Y^d}{Y^c}"] \\ & Y\times Y. \end{tikzcd}$$

In the case that X is cofibrant then $e \cdot X\colon J \cdot X \to X$ is a weak equivalence so that $J \cdot X$ is a *cylinder object* for X; in this case the left diagram above shows $\mathcal{V}$-homotopy implies *left homotopy* in the model categorical sense of [31]. Similarly if Y is fibrant the third diagram shows that $\mathcal{V}$-homotopy implies *right homotopy*, but in general there need be no relation between $\mathcal{V}$-homotopy and left or right homotopy.

Since a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ induces an **ER**-functor $\mathrm{h}_*(F)\colon \mathrm{h}_*(\mathcal{A}) \to \mathrm{h}_*(\mathcal{B})$, if $f \cong g$ in $\mathcal{A}$ then $Ff \cong Fg$ in $\mathcal{B}$. This is one useful feature of $\mathcal{V}$-homotopy not true of the usual left and right homotopy relations in the model category $\mathcal{V}$.

Definition 2.10. An object of a $\mathcal{V}$-category $\mathcal{A}$ is *bi-terminal* or *bi-initial* if it is so in $\mathrm{h}_*(\mathcal{A})$.

In elementary terms, T is bi-terminal in $\mathcal{A}$ if for each $A \in \mathcal{A}$ there is a morphism $A \to T$, which is unique up to $\mathcal{V}$-homotopy.

If $\mathcal{A}$ has an actual terminal object 1, then T is bi-terminal if and only if the unique map $T \to 1$ is an equivalence. Explicitly, this means that there is a morphism $t\colon 1 \to T$ with $t\circ ! \cong 1_T$. The dual remarks apply to bi-initial objects.

Proposition 2.11. *Right adjoint $\mathcal{V}$-functors preserve bi-terminal objects.*

Proof. Let $U\colon \mathcal{B} \to \mathcal{A}$ be a $\mathcal{V}$-functor with $F \dashv U$ a left adjoint, and let $T \in \mathcal{B}$ be bi-terminal. For any $A \in \mathcal{A}$, there is a morphism $FA \to T$ in $\mathcal{B}$ and so a morphism $A \to UT$ in $\mathcal{A}$. If $f, g\colon A \to UT$ are two such morphisms in $\mathcal{A}$, then their adjoint transposes $f', g'\colon FA \to T$ are $\mathcal{V}$-homotopic in $\mathcal{B}$, hence $f = Uf' \circ \eta_A \cong Ug' \circ \eta_A = g$ in $\mathcal{A}$. $\square$

3. Shrinkable morphisms and $\mathcal{E}$-weak adjoints

Definition 3.1. A morphism $f\colon A \to B$ in a $\mathcal{V}$-category $\mathcal{C}$ is said to be *shrinkable* if there exists a morphism $s\colon B \to A$ with $f \circ s = 1_B$, an interval J and a $(\mathcal{V}, J)$-homotopy $h\colon sf \cong_J 1_A$ such that $fh\colon f = fsf \cong_J f$ is trivial.

In elementary terms, such an h is a morphism as below

$$\begin{tikzcd} I+I \arrow[r, "(1_A\ s\circ f)"] \arrow[d, "(d\ c)"'] & & \mathcal{C}(A,A) \arrow[d, "\mathcal{C}(A,f)"] \\ J \arrow[urr, "h"] \arrow[r, "e"'] & I \arrow[r, "f"'] & \mathcal{C}(A,B) \end{tikzcd} \tag{3.1}$$

rendering the diagram commutative.

Let us record a few easy properties.

Proposition 3.2. *Shrinkable morphisms are closed under composition, contain the isomorphisms, and are preserved by any $\mathcal{V}$-functor.*

Proof. Closure under composition follows from the fact — see Proposition 2.5 — that intervals can be composed. Verification of the remaining facts is routine. □

Proposition 3.3. *Let $\mathcal{C}$ be a model $\mathcal{V}$-category. Any trivial fibration $f\colon A \to B$ in $\mathcal{C}$ with cofibrant domain and codomain is shrinkable.*

Proof. Since B is cofibrant, f has a section s. Since A is cofibrant, the induced $A + A \to J \cdot A$ is a cofibration, and so we have an induced $h\colon J \cdot A \to A$, corresponding to the desired $J \to \mathcal{C}(A, A)$. □

Proposition 3.4. *Let $\mathcal{C}$ be a model $\mathcal{V}$-category. If $f\colon A \to B$ is shrinkable in $\mathcal{C}$ and either $d \cdot A\colon A \to J \cdot A$ or $d \pitchfork A\colon J \pitchfork A \to A$ is a weak equivalence then so is f. In particular, any shrinkable morphism whose domain is either cofibrant or fibrant is a weak equivalence.*

Proof. Suppose first that $d \cdot A$ is a weak equivalence. We have commutative diagrams

$$\begin{tikzcd} A \arrow[rr, bend left, "1"] \arrow[rr, bend right, "sf"'] \arrow[r, shift left, "d\cdot A"] \arrow[r, shift right, "c\cdot A"'] & J\cdot A \arrow[r, "h"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, bend left, "1"] \arrow[rr, bend right, "1"'] \arrow[r, shift left, "d\cdot A"] \arrow[r, shift right, "c\cdot A"'] & J\cdot A \arrow[r, "e\cdot A"] & A \end{tikzcd}$$

and since $d \cdot A$ is a weak equivalence it follows successively that $e \cdot A$, $c \cdot A$, h, and finally sf are so. But since $fs = 1$ it follows that f (and s) are also weak equivalences.

Since d is a trivial cofibration, if A is cofibrant then $d \cdot A$ will be a trivial cofibration, and in particular a weak equivalence.

The cases of $d \pitchfork A$ a weak equivalence, and of A fibrant, are similar. □

Using the preceding two propositions, we have the following result.

Corollary 3.5. *If all objects of $\mathcal{C}$ are cofibrant, then each trivial fibration is shrinkable and each shrinkable morphism is a weak equivalence with a section.*

Remark 3.6. If all objects of $\mathcal{V}$ are fibrant and $I = 1$, then in any interval (J, d, c, e), the morphism $e\colon J \to I = 1$ is not just a weak equivalence but a trivial fibration. Thus in this case any two intervals give the same notion of $\mathcal{V}$-homotopy, and the shrinkable morphisms can be described using any one fixed interval.

Definition 3.7. We write $\mathcal{E}$ for the collection of all shrinkable morphisms in $\mathcal{V}$ itself.

Examples 3.8. In each of our four main examples, all objects are cofibrant, and so by Corollary 3.5 every trivial fibration is shrinkable, and every shrinkable morphism is a weak equivalence with a section. In the first three of them, every weak equivalence with a section is a trivial fibration, and so the shrinkable morphisms are precisely the trivial fibrations.

(1) For $\mathcal{V}$ equipped with the trivial model structure, the trivial fibrations and the shrinkable morphisms are both just the isomorphisms.
(2) For $\mathcal{V}$ with the split model structure, the trivial fibrations and the shrinkable morphisms are both just the split epimorphisms.
(3) For **Cat** equipped with the canonical model structure, the trivial fibrations and the shrinkable morphisms are both just the surjective equivalences.
(4) Now consider **SSet** equipped with the Joyal model structure. By Corollary 3.5 the shrinkable morphisms lie between the weak equivalences and the trivial fibrations.

Not every shrinkable morphism is a trivial fibration. For example, let J be the nerve of the free isomorphism, with its unique interval structure, then form the pushout of the two morphisms $1 \to J$ to obtain a new interval J', as in Proposition 2.5. The induced map $J' \to J$ is a shrinkable morphism but is not a trivial fibration: in particular, J is fibrant while J' is not, so $J' \to J$ cannot be a fibration.

In general, all that we can say is that the shrinkable morphisms are weak equivalences with sections; however for a shrinkable morphism $f\colon X \to Y$ between quasicategories, the fibrant objects, we can say a little more. First recall that an *equivalence* of quasicategories is a map $f\colon X \to Y$ for which there exists a $g\colon Y \to X$ with $fg \cong_J 1_Y$ and $gf \cong_J 1_X$ where J is the nerve of the free isomorphism as discussed above. Let us call f a *surjective/retract equivalence* if in fact $fg = 1_Y$. Now since all objects are cofibrant in **SSet** each weak equivalence between quasicategories is a homotopy equivalence relative to any choice of interval; in particular, each shrinkable morphism $f\colon X \to Y$ between quasicategories is a surjective equivalence. (Note, however, that we do not assert that the $(\mathcal{V}, J)$-homotopy $gf \cong_J 1_X$ is f-trivial — indeed, while there is an f-trivial $\mathcal{V}$-homotopy relative to some interval, it does not seem that f-triviality is independent of the choice of interval.)

Example 3.9. The category **CGTop** of compactly generated topological spaces is cartesian closed, and the standard model structure for topological spaces restricts to **CGTop** (see [12, Section 2.4] for example). Here the notion of shrinkable morphism is dual to that of strong deformation retract, and it is in this case that the name "shrinkable" originated, as explained in the introduction.

Example 3.10. Another example is given by the monoidal model category **Gray**: this is the category of (strict) 2-categories and 2-functors with the model structure of [22], as corrected in [23]: the weak equivalences are the biequivalences, the trivial fibrations are the 2-functors which are surjective on objects, and retract equivalences on homs. The monoidal structure is the Gray tensor product, as in [11]. We shall see that in this case, the shrinkable morphisms are exactly the trivial fibrations which have a section.

Not all objects are cofibrant, and not all trivial fibrations have sections, so not all trivial fibrations are shrinkable. On the other hand, all objects are fibrant, so we can use a standard interval for all $\mathcal{V}$-homotopies; also all shrinkable morphisms are weak equivalences. Taking as standard interval the free adjoint equivalence, a $\mathcal{V}$-homotopy between two 2-functors is then a pseudonatural equivalence. If $p\colon E \to B$ is shrinkable, via $s\colon B \to E$ and a pseudonatural equivalence $sp \simeq 1$, then clearly p is surjective on objects and an equivalence on hom-categories. Thus it will be a trivial fibration provided that it is full on 1-cells. For this, given $x, y \in E$ and a morphism $\beta\colon px \to py$ in B, we may compose with suitable components of the pseudonatural equivalence to get a morphism

$$x \xrightarrow{\sigma} spx \xrightarrow{s\beta} spy \xrightarrow{\sigma'} y$$

and by p-triviality of σ, both $p\sigma$ and $p\sigma'$ are identities, and so p is indeed full on 1-cells.

Suppose conversely that $p\colon E \to B$ is a trivial fibration, and that $s\colon B \to E$ is a section of p. For each object $x \in E$, we have $pspx = px$ and so there is a morphism $\sigma_x\colon spx \to x$ with $p\sigma_x = 1$ and similarly a morphism $\sigma'_x\colon x \to spx$, also over the identity. These form part of an equivalence, also over the identity. For any $\alpha\colon x \to y$ the square

$$\begin{tikzcd} spx \arrow[r, "\sigma_x"] \arrow[d, "sp\alpha"'] & x \arrow[d, "\alpha"] \\ spy \arrow[r, "\sigma_y"'] & y \end{tikzcd}$$

need not commute, but both paths lie over the same morphism $p\alpha$ in B, and so there is a unique invertible 2-cell in the diagram, which is sent by p to an identity. It now follows that σ determines an equivalence $sp \simeq 1$ over B, and so that p is shrinkable.

Example 3.11. Let R be a finite-dimensional cocommutative Hopf algebra, which is Frobenius as a ring. Then the category of modules over R is a monoidal model category [12, Section 2.2 and Proposition 4.2.15]. All objects are cofibrant. The trivial fibrations are the surjections with projective kernel. In particular, the codiagonal $R \oplus R \to R$ is a trivial fibration, and so $R \oplus R$ itself is an interval. Thus the shrinkable morphisms are just the split epimorphisms. Every trivial fibration is shrinkable, while the converse is true (if and) only if every module is projective.

Example 3.12. Let R be a commutative ring, and $\mathrm{Ch}(R)$ the symmetric monoidal closed category of unbounded chain complexes of R-modules. This has a model structure [12, Section 2.3] for which the fibrations are the pointwise surjections and the weak equivalences the quasi-isomorphisms. A morphism $f\colon X \to Y$ is shrinkable when it has a section s and a chain homotopy between sf and 1 which is f-trivial. This is of course rather stronger than being a trivial fibration.

The paper [25] introduced the notion of $\mathcal{E}$-weak left adjoint for any class of morphisms $\mathcal{E}$. Here we consider the instance of this concept obtained by taking the class $\mathcal{E}$ of shrinkable morphisms.

Definition 3.13. Let $U\colon \mathcal{B} \to \mathcal{A}$ be a $\mathcal{V}$-functor. An $\mathcal{E}$-weak reflection of $A \in \mathcal{A}$ is a morphism $\eta_A\colon A \to UA'$ with the property that for all $B \in \mathcal{B}$ the induced morphism

$$\mathcal{B}(A', B) \xrightarrow{U} \mathcal{A}(UA', UB) \xrightarrow{\mathcal{A}(\eta, UB)} \mathcal{A}(A, UB)$$

belongs to $\mathcal{E}$. If each object $A \in \mathcal{A}$ admits an $\mathcal{E}$-weak reflection then we say that U admits an $\mathcal{E}$-weak left adjoint.

Examples 3.14. (1) For $\mathcal{V}$ with the trivial model structure, a $\mathcal{V}$-functor $U\colon \mathcal{B} \to \mathcal{A}$ as above admits an $\mathcal{E}$-weak left adjoint just when it admits a genuine left adjoint.

(2) For $\mathcal{V}$ with the split model structure, $U\colon \mathcal{B} \to \mathcal{A}$ admits an $\mathcal{E}$-weak left adjoint if we have morphisms $\eta\colon A \to UA'$ for each A such that the induced $\mathcal{B}(A', B) \to \mathcal{A}(A, UB)$ is a split epimorphism. In the case that $\mathcal{V} = \mathbf{Set}$, this recovers ordinary weakness — that is, for each $f\colon A \to UB$ there exists $f'\colon A' \to B$ with $Uf' \circ \eta_A = f$. In this case, setting $FA = A'$ for each $A \in \mathcal{B}$ and $Ff = (\eta_B \circ f)'\colon FA \to FB$ for $f\colon A \to B$ equips F with the structure of a graph morphism, but there is no reason why it should preserve composition.

(3) In the **Cat** case each $\mathcal{B}(A', B) \to \mathcal{A}(A, UB)$ is a retract equivalence. In this setting we can form F as in the previous example, and with a similar definition $\alpha \mapsto F\alpha$ on 2-cells. Then given $f\colon A_1 \to A_2$ and $g\colon A_2 \to A_3$, since $\mathcal{B}(A_1', A_3') \to \mathcal{A}(A_1, UA_3')$ is a retract equivalence, we obtain a unique invertible 2-cell such that $F_{f,g}\colon Fg \circ Ff \cong F(g \circ f)$. Continuing in this way, F becomes a pseudofunctor and we obtain a biadjunction in the sense of [20].

(4) For **SSet** equipped with the Joyal model structure, if $\mathcal{B}$ and $\mathcal{A}$ are *locally fibrant* — that is, if they are enriched in quasicategories — then each $\mathcal{B}(A', B) \to \mathcal{A}(A, UB)$ is a retract equivalence of quasicategories. As before, we can define F as a graph morphism as in the preceding two examples. Then given $f\colon A_1 \to A_2$ and $g\colon A_2 \to A_3$ let us write $k\colon \mathcal{B}(A_1', A_3') \to \mathcal{A}(A_1, UA_3')$ for the retract equivalence, with section s, and write $a = F(g \circ f)$ and $b = Fg \circ Ff$. Then the $(\mathbf{SSet}, J)$-homotopies $1 \cong sk$ and $sk \cong 1$ give a composite homotopy

$$F(g \circ f) = a \cong ska = skb \cong b = Fg \circ Ff$$

in the hom-quasicategory $\mathcal{B}(A_1', A_3')$, suggesting that F could be turned into a pseudomorphism of simplicially enriched categories given a suitable formalisation of that concept.

4. Comma categories and slice categories

The proof of Freyd's general adjoint functor theorem given in [27] uses the fact that in order to construct a left adjoint to $U\colon \mathcal{B} \to \mathcal{A}$, it suffices to construct an initial object in each comma category A/U, for $A \in \mathcal{A}$. Indeed, the universal property of the initial object $\eta_A\colon A \to$

UA' implies that the induced map

$$\mathcal{B}(A',B) \xrightarrow{U} \mathcal{A}(UA',UB) \xrightarrow{\mathcal{A}(\eta_A,UB)} \mathcal{A}(A,UB)$$

is a *bijection*, so that the maps $\eta_A \colon A \to UA'$ give the components of the unit for a left adjoint.

In order to frame this in a matter suitable for generalisation, observe that to say that the above function is a *bijection* is equally to say that it is a *terminal object* in the slice category $\mathbf{Set}/\mathcal{A}(A,UB)$ — thus one passes from an initial object in the comma category to a terminal object in the slice.

In the present section we develop the necessary results about comma categories in the enriched context. In our setting, we shall only be able to construct *bi-initial objects* in the comma category A/U and so are led to consider *bi-terminal objects* in enriched slice categories, which turn out to be the shrinkable morphisms of the previous section.

4.1. **Comma categories and slice categories.** Given $\mathcal{V}$-functors $G \colon \mathcal{C} \to \mathcal{A}$ and $U \colon \mathcal{B} \to \mathcal{A}$ the enriched comma-category G/U is the *comma object*, below left

$$\begin{tikzcd} G/U \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\swArrow\theta"] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{B} \arrow[r, "U"'] & \mathcal{A} \end{tikzcd} \qquad \begin{tikzcd} G/U \arrow[r] \arrow[d, "\binom{P}{Q}"'] & \mathcal{A}^{\mathbf{2}} \arrow[d, "\binom{\mathrm{dom}}{\mathrm{cod}}"] \\ \mathcal{C}\times\mathcal{B} \arrow[r, "G\times U"'] & \mathcal{A}\times\mathcal{A} \end{tikzcd}$$

in the 2-category $\mathcal{V}$-**Cat** of $\mathcal{V}$-categories. This has the universal property that, given $\mathcal{V}$-functors $R \colon \mathcal{D} \to \mathcal{C}$ and $S \colon \mathcal{D} \to \mathcal{B}$, and a $\mathcal{V}$-natural transformation $\lambda \colon G \circ R \Rightarrow U \circ S$, there exists a unique $\mathcal{V}$-functor $K \colon \mathcal{D} \to G/U$ such that $P \circ K = R$, $Q \circ K = S$ and $\theta \circ K = \lambda$, as well as a 2-dimensional aspect characterising $\mathcal{V}$-natural transformations out of such a K.

For the reader unfamiliar with 2-dimensional limits, we point out that it is equally the pullback on the right above in which $\mathcal{A}^{\mathbf{2}}$ is the (enriched) arrow category, and $\binom{\mathrm{dom}}{\mathrm{cod}}$ the projection which sends a morphism to its domain and codomain.

The objects of G/U consist of triples $(C, \alpha \colon GC \to UB, B)$ where $C \in \mathcal{C}$, $B \in \mathcal{B}$, and $\alpha \colon GC \to UB$ is in $\mathcal{A}_0$; and with hom-objects as in the pullback below.

$$\begin{tikzcd}
G/U((C,\alpha,B),(C',\alpha',B')) \arrow[rr] \arrow[dd] & & \mathcal{C}(C,C') \arrow[d, "G"] \\
& & \mathcal{A}(GC,GC') \arrow[d, "\mathcal{A}(1,\alpha')"] \\
\mathcal{B}(B,B') \arrow[r, "U"'] & \mathcal{A}(UB,UB') \arrow[r, "\mathcal{A}(\alpha,1)"'] & \mathcal{A}(GC,UB')
\end{tikzcd}$$

A standard result of importance to us is the following one.

Proposition 4.1. *Suppose that $\mathcal{B}$ and $\mathcal{C}$ admit W-limits for some weight $W\colon \mathcal{D} \to \mathcal{V}$ and that $U\colon \mathcal{B} \to \mathcal{A}$ preserves them. Then G/U admits W-weighted limits for any $G\colon \mathcal{C} \to \mathcal{A}$, and they are preserved by the projections $P\colon G/U \to \mathcal{C}$ and $Q\colon G/U \to \mathcal{B}$.*

Proof. Consider a diagram $T = (R, \lambda, S)\colon \mathcal{D} \to G/U$ and let us write

$$W\text{-cone}(X,T) := [\mathcal{D},\mathcal{V}](W, G/U(X, T-))$$

for $X = (C,\alpha,B) \in G/U$. We must prove that $W\text{-cone}(-,T)\colon G/U^{op} \to \mathcal{V}$ is representable. First observe that the definition of the hom-objects in G/U gives us the components of a pullback

$$\begin{tikzcd}
G/U(X,T-) \arrow[rr, "P"] \arrow[dd, "Q"'] & & \mathcal{C}(C,S-) \arrow[d, "G"] \\
& & \mathcal{A}(GC,GS-) \arrow[d, "\lambda_*"] \\
\mathcal{B}(B,R-) \arrow[r, "U"'] & \mathcal{A}(UB,UR-) \arrow[r, "\alpha^*"'] & \mathcal{A}(GC,UR-)
\end{tikzcd}$$

in $[\mathcal{D},\mathcal{V}]$. Applying $[\mathcal{D},\mathcal{V}](W,-)$, we obtain a pullback

$$\begin{tikzcd}
W\text{-cone}(X,T) \arrow[rr, "P_*"] \arrow[dd, "Q_*"'] & & W\text{-cone}(C,S) \arrow[d, "G_*"] \\
& & W\text{-cone}(GC,GS) \arrow[d, "\lambda_*"] \\
W\text{-cone}(B,R) \arrow[r, "U_*"'] & W\text{-cone}(UB,UR) \arrow[r, "\alpha^*"'] & W\text{-cone}(GC,UR)
\end{tikzcd} \tag{4.1}$$

in $\mathcal{V}$. Since the limits $\{W,S\}$ and $\{W,R\}$ exist and since U preserves the latter, this pullback square is isomorphic to

$$\begin{tikzcd}
W\text{-cone}(X,T) \arrow[rr] \arrow[dd] & & \mathcal{C}(C,\{W,S\}) \arrow[d, "G"] \\
& & \mathcal{A}(GC,G\{W,S\}) \arrow[d, "\varphi_*"] \\
\mathcal{B}(B,\{W,R\}) \arrow[r, "U"'] & \mathcal{A}(UB,U\{W,R\}) \arrow[r, "\alpha^*"'] & \mathcal{A}(GC,U\{W,R\})
\end{tikzcd}$$

for $\varphi\colon G\{W,S\}\to U\{W,R\}$ the morphism induced by

$$W \longrightarrow \mathcal{C}(\{W,S\},S-) \xrightarrow{G} \mathcal{A}(G\{W,S\},GS-) \xrightarrow{\lambda_*} \mathcal{A}(G\{W,S\},UR-)$$

and the universal property of $U\{W,R\}$. By definition, the pullback in this square is $G/U((C,\alpha,B),(\{W,S\},\varphi,\{W,R\}))$. Hence the object $(\{W,S\},\varphi,\{W,R\})$ represents W-cone$(-,T)$ and is therefore the limit $\{W,T\}$. $\square$

Let $\mathcal{I}$ denote the *unit $\mathcal{V}$-category*, which has a single object with object of endomorphisms the monoidal unit I of $\mathcal{V}$. An object $A\in\mathcal{A}$ then corresponds to a $\mathcal{V}$-functor $A\colon\mathcal{I}\to\mathcal{A}$. We denote by A/U the comma object depicted below left

$$\begin{tikzcd}
A/U \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\swarrow\theta"] & \mathcal{I} \arrow[d, "A"] \\
\mathcal{B} \arrow[r, "U"'] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A}/A \arrow[r, "P"] \arrow[d, "Q"'] \arrow[dr, phantom, "\swarrow\theta"] & \mathcal{A} \arrow[d, "1"] \\
\mathcal{I} \arrow[r, "A"'] & \mathcal{A}
\end{tikzcd}$$

in which we shall consider bi-initial objects, and the *enriched slice category* $\mathcal{A}/A$ as above right in which we shall consider bi-terminal ones. We record for later use that if $(B,b\colon B\to A)$ and $(C,c\colon C\to A)$ are objects of $\mathcal{A}/A$, the corresponding hom is given by the pullback

$$\begin{tikzcd}
(\mathcal{A}/A)((B,b),(C,c)) \arrow[r] \arrow[d] & \mathcal{A}(B,C) \arrow[d, "\mathcal{A}(B,c)"] \\
I \arrow[r, "b"'] & \mathcal{A}(B,A).
\end{tikzcd} \tag{4.2}$$

Remark 4.2. If the unit object of $\mathcal{V}$ is also terminal, then $\mathcal{I}$ is the terminal $\mathcal{V}$-category, and is complete and cocomplete. We may then use Proposition 4.1 to deduce that, if $\mathcal{B}$ has W-limits and $U\colon\mathcal{B}\to\mathcal{A}$ preserves them, then A/U has W-limits for any $A\in\mathcal{A}$. But if $I\neq 1$, then $\mathcal{I}$ need not be complete; indeed, almost by definition, $\mathcal{I}$ has a terminal object if and only if $I=1$.

Proposition 4.3. *Let $\mathcal{A}$ be a $\mathcal{V}$-category and A an object of $\mathcal{A}$. The identity morphism on A is a terminal object in the slice $\mathcal{V}$-category $\mathcal{A}/A$ if and only if $I = 1$.*

Proof. For any $f\colon B \to A$ we have the pullback

$$\begin{tikzcd}
(\mathcal{A}/A)((B,f),(A,1_A)) \arrow[r] \arrow[d] & \mathcal{B}(B,A) \arrow[d, "\mathcal{B}(B,1_A)"] \\
I \arrow[r, "f"'] & \mathcal{B}(B,A)
\end{tikzcd}$$

and since the pullback of an isomorphism is an isomorphism, we conclude that $(\mathcal{A}/A)((B,f),(A,1_A)) \cong I$, which is terminal if and only if I is so. $\square$

Example 4.4. By the preceding result, $1\colon A \to A$ is never terminal in the $\mathcal{V}$-category $\mathcal{A}/A$ unless $I = 1$. For a concrete example, let $\mathcal{V} = \mathbf{Ab}$, and consider $\mathbf{Ab}/A$ for a given abelian group A. An object consists of a homomorphism $f\colon B \to A$, and a morphism from (B,f) to $(A,1_A)$ consists of a pair (h,n) where $h\colon B \to A$ is a homomorphism, $n \in \mathbb{Z}$, and $h = n.f$. There is such a morphism $(n.f, n)$ for any $n \in \mathbb{Z}$, and these are all distinct, thus $(A,1_A)$ is not terminal.

Proposition 4.5. *Suppose that $I = 1$, and let $U\colon \mathcal{B} \to \mathcal{A}$ and $A \in \mathcal{A}$.*

(1) Then A/U has underlying category A/U_0. Furthermore, given parallel morphisms in A/U

$$\begin{tikzcd}
& A \arrow[dl, "b"'] \arrow[dr, "c"] & \\
UB \arrow[rr, shift left, "Uf"] \arrow[rr, shift right, "Ug"'] & & UC
\end{tikzcd}$$

a $\mathcal{V}$-homotopy $f \cong g$ in A/U is a $\mathcal{V}$-homotopy $h\colon f \cong g$ in $\mathcal{B}$ such that $Uh \circ b$ is trivial, which we will refer to as a $\mathcal{V}$-homotopy under A.

(2) Then $\mathcal{A}/A$ has underlying category $\mathcal{A}_0/A$. Furthermore, given parallel morphisms in $\mathcal{A}/A$

$$\begin{tikzcd}
B \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "b"'] & & C \arrow[dl, "c"] \\
& A &
\end{tikzcd}$$

a $\mathcal{V}$-homotopy $f \cong g$ in $\mathcal{A}/A$ is a $\mathcal{V}$-homotopy $h\colon f \cong g$ in $\mathcal{B}$ such that $c \circ h$ is trivial, which we will refer to as a $\mathcal{V}$-homotopy over A.

Proof. A morphism in $(A/U)_0$ is given by a map into the pullback as in

$$
\begin{tikzcd}
I \arrow[dr, dotted] \arrow[drrr, bend left, "f'=1"] \arrow[ddr, bend right, "f"'] & & & \\
& (A/U)((B,b),(C,c)) \arrow[rr] \arrow[d] & & I \arrow[d, "c"] \\
& \mathcal{B}(B,C) \arrow[r, "U"'] & \mathcal{A}(UB,UC) \arrow[r, "b^*"'] & \mathcal{A}(A,UC)
\end{tikzcd}
$$

where the map $f'\colon I \to I$ is necessarily the identity by virtue of the assumption that $I = 1$. Therefore it amounts to a morphism $f\colon B \to C \in \mathcal{B}_0$ such that $Uf \circ b = c$: that is, a morphism of A/U_0. Similarly, for J an interval, a $(\mathcal{V}, J)$-homotopy $f \cong g$ in A/U is specified by a map from J into the pullback whose component in I is fixed as the structure map for the interval $e\colon J \to I$, again by the assumption that $I = 1$. The component in $\mathcal{B}(B,C)$ then specifies a $(\mathcal{V}, J)$-homotopy $h\colon f \cong g$ for which $Uh \circ b$ is trivial.

This analysis applies to the special case $A/\mathcal{A}$ and so, by duality, establishes the corresponding claim for $\mathcal{A}/A$. □

Proposition 4.6. *If $I = 1$ then $f\colon A \to B$ is shrinkable if and only if it is bi-terminal in $\mathcal{C}/B$.*

Proof. By Proposition 4.3 the identity 1_B is terminal in $\mathcal{C}/B$. Therefore $f\colon A \to B$ is bi-terminal if and only if the unique map $f\colon (A,f) \to (B,1_B)$ is an equivalence. Now a morphism $s\colon (B,1_B) \to (A,f)$ is specified by a section $s\colon B \to A$ of f so that we have $f \circ s = 1_B\colon (B,1_B) \to (B,1_B)$. By Proposition 4.5 a $\mathcal{V}$-homotopy $s \circ f \cong 1_A$ in $\mathcal{C}/B$ is specified by a $\mathcal{V}$-homotopy $h\colon s \circ f \cong 1_A$ in $\mathcal{C}$ such that $f \circ h$ is trivial, as required. □

In the case I is not terminal, the above proposition need not hold. We deal with this, and other issues related to a non-terminal I, by passing from $\mathcal{V}$ to the slice category $\mathcal{V}/I$, which has an induced monoidal structure for which the unit is terminal. Moreover, the comma $\mathcal{V}$-categories A/U and $\mathcal{A}/A$ naturally give rise to $\mathcal{V}/I$-categories. See Section 7 below.

Theorem 4.7. *Suppose that $I = 1$. Let $\mathcal{B}$ be a $\mathcal{V}$-category with powers, and $U\colon \mathcal{B} \to \mathcal{A}$ a $\mathcal{V}$-functor which preserves powers; if $\eta\colon A \to UA'$ is bi-initial in A/U then*

$$\mathcal{B}(A', B) \xrightarrow{U} \mathcal{A}(UA', UB) \xrightarrow{\mathcal{A}(\eta, UB)} \mathcal{A}(A, UB) \tag{4.3}$$

is bi-terminal in $\mathcal{V}/\mathcal{A}(A, UB)$.

Proof. By Proposition 4.5 we need to find, for any object α of $\mathcal{V}/\mathcal{A}(A, UB)$ a morphism

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "\alpha"'] & & \mathcal{B}(A', B) \arrow[dl] \\ & \mathcal{A}(A, UB) & \end{tikzcd}$$

and for any two such morphisms a $\mathcal{V}$-homotopy between them over $\mathcal{A}(A, UB)$.

Using the universal property of the power $X \pitchfork B$ and the fact that $U(X \pitchfork B) \cong X \pitchfork UB$, the above problem is translated to A/U as below.

$$\begin{tikzcd} & A \arrow[dl, "\eta"'] \arrow[dr, "\alpha^\sharp"] & \\ UA' \arrow[rr, dashed, "Uf^\sharp"'] & & U(X \pitchfork B) \end{tikzcd}$$

$$\begin{tikzcd} A' \arrow[r, dashed, "f^\sharp"'] & X \pitchfork B \end{tikzcd}$$

Since η is bi-initial, the required morphism $f^\sharp$ and so f exists. If f and g are two such morphisms then we can, by bi-initiality of η, find a $\mathcal{V}$-homotopy between $f^\sharp$ and $g^\sharp$ under A; writing it as

$$h^\sharp\colon A' \to J \pitchfork (X \pitchfork B) = X \pitchfork (J \pitchfork B),$$

we obtain $h\colon X \to \mathcal{B}(A', J \pitchfork B) \cong J \pitchfork \mathcal{B}(A', B)$ and this is the required $\mathcal{V}$-homotopy between f and g over $\mathcal{A}(A, UB)$ $\square$

Remark 4.8. A more conceptual explanation for the preceding result is as follows. Given $U\colon \mathcal{B} \to \mathcal{A}$ and objects $B \in \mathcal{B}$ and $A \in \mathcal{A}$, we obtain a $\mathcal{V}$-functor $K\colon (A/U)^{op} \to \mathcal{V}/\mathcal{A}(A, UB)$ sending $\eta\colon A \to UA'$ to the morphism (4.3). If $\mathcal{B}$ has powers and U preserves them, then K has a left $\mathcal{V}$-adjoint. The preceding result then follows immediately.

5. Enough cofibrantly-weighted limits

In this section we describe the completeness and continuity conditions which will arise in our weak adjoint functor theorem. We start by recalling the notion of cofibrantly-weighted limit.

5.1. **Cofibrantly-weighted limits.** Let $\mathcal{D}$ be a small $\mathcal{V}$-category, and consider the $\mathcal{V}$-category $[\mathcal{D},\mathcal{V}]$ of $\mathcal{V}$-functors from $\mathcal{D}$ to $\mathcal{V}$. The morphisms in (the underlying ordinary category of) $[\mathcal{D},\mathcal{V}]$ are the $\mathcal{V}$-natural transformations. We shall say that such a $\mathcal{V}$-natural transformation $p\colon F \to G$ is a *trivial fibration* if it is so in the pointwise/levelwise sense: that is, if each component $pD\colon FD \to GD$ is a trivial fibration in $\mathcal{V}$.

We say that an object $Q \in [\mathcal{D},\mathcal{V}]$ is *cofibrant*, if for each trivial fibration $p\colon F \to G$ in $[\mathcal{D},\mathcal{V}]$ the induced morphism

$$[\mathcal{D},\mathcal{V}](Q,p)\colon [\mathcal{D},\mathcal{V}](Q,F) \to [\mathcal{D},\mathcal{V}](Q,G)$$

is a trivial fibration in $\mathcal{V}$. In many cases, the projective model structure on $[\mathcal{D},\mathcal{V}]$ will exist and make $[\mathcal{D},\mathcal{V}]$ into a model $\mathcal{V}$-category, and then these notions of cofibrant object and trivial fibration will be the ones that apply there.

Example 5.1. Each representable $\mathcal{D}(D,-)$ is cofibrant: this follows immediately from the Yoneda lemma, since $[\mathcal{D},\mathcal{V}](\mathcal{D}(D,-),p)$ is just $pD\colon FD \to GD$.

Proposition 5.2. *The cofibrant objects in $[\mathcal{D},\mathcal{V}]$ are closed under coproducts and under copowers by cofibrant objects of $\mathcal{V}$.*

Proof. The case of coproducts is well-known. As for copowers, let $Q\colon \mathcal{D} \to \mathcal{V}$ and $X \in \mathcal{V}$ be cofibrant. If $p\colon F \to G$ is a trivial fibration in $[\mathcal{D},\mathcal{V}]$ then $[\mathcal{D},\mathcal{V}](X\cdot Q,p)$ is (up to isomorphism) given by $[X,[\mathcal{D},\mathcal{V}](Q,p)]$. Since Q is cofibrant and p is a trivial fibration, it follows that $[\mathcal{D},\mathcal{V}](Q,p)$ is a trivial fibration; and now since also X is cofibrant it follows that $[X,[\mathcal{D},\mathcal{V}](Q,p)]$ is also a trivial fibration. $\square$

Many of the $\mathcal{V}$-categories to which we shall apply our weak adjoint functor theorem have all cofibrantly-weighted limits, but the hypotheses of the theorem will be weaker than this. There are several reasons for this, and in particular for not requiring idempotents to split. For instance, the weak adjoint functor theorem of Kainen [16] which we wish to generalise does not assume them. Furthermore, in 2-category theory there exist many natural examples of 2-categories admitting enough (but not all) cofibrantly-weighted limits. An example is the

2-category of *strict* monoidal categories and *strong* monoidal functors: see [8, Section 6.2]. We start with the case where $I = 1$.

5.2. **Enough cofibrantly-weighted limits (when $I = 1$).** We now define our key completeness and continuity conditions in the case $I = 1$; these will be modified later to deal with the case where $I \neq 1$.

Definition 5.3. Suppose that the unit object I of $\mathcal{V}$ is terminal. We say that a $\mathcal{V}$-category $\mathcal{B}$ has *enough cofibrantly-weighted limits* if for any small $\mathcal{V}$-category $\mathcal{D}$, there is a chosen cofibrant weight $Q\colon \mathcal{D} \to \mathcal{V}$ for which the unique map $Q \to 1$ is a trivial fibration, and for which $\mathcal{B}$ has Q-weighted limits. Similarly, a $\mathcal{V}$-functor $U\colon \mathcal{B} \to \mathcal{A}$ preserves enough cofibrantly-weighted limits if it preserves these Q-weighted limits for each small $\mathcal{D}$.

In particular, $\mathcal{B}$ will have enough cofibrantly-weighted limits if it has all cofibrantly-weighted limits.

Remark 5.4. In our definition of "enough cofibrantly-weighted limits", we have asked that there be a chosen fixed weight $Q\colon \mathcal{D} \to \mathcal{V}$ for each small $\mathcal{D}$. But in fact it is possible to allow Q to depend upon the particular diagram $\mathcal{D} \to \mathcal{B}$ of which one wishes to form the (weighted) limit. The cost of doing this is that it becomes more complicated to express what it means to preserve enough cofibrantly-weighted limits, but our main results can still be proved with essentially unchanged proofs.

Example 5.5. If $\mathcal{V}$ has the trivial model structure, then the trivial fibrations in $[\mathcal{D}, \mathcal{V}]$ are just the isomorphisms, and all objects are cofibrant. Thus in this case a $\mathcal{V}$-category has enough cofibrantly-weighted limits if and only if it has all (unweighted) limits of $\mathcal{V}$-functors (with small domain).

Example 5.6. If $\mathcal{V}$ has the split model structure, then a $\mathcal{V}$-category will have enough cofibrantly-weighted limits provided it has products. To see this, let $\mathcal{D}$ be a small $\mathcal{V}$-category and consider the terminal weight $1\colon \mathcal{D} \to \mathcal{V}$. Then $\sum_{D\in\mathcal{D}} \mathcal{D}(D, -)$ is a coproduct of representables, and so is cofibrant. The unique map $q\colon \sum_{D\in\mathcal{D}} \mathcal{D}(D, -) \to 1$ has component at $C \in \mathcal{D}$ given by $\sum_{D\in\mathcal{D}} \mathcal{D}(D, C) \to 1$, which has a section picking out the identity $1 \to \mathcal{D}(C, C)$. Thus q is a pointwise split epimorphism, and so a trivial fibration in $[\mathcal{D}, \mathcal{V}]$. And the $\sum_D \mathcal{D}(D, -)$-weighted limit of $S\colon \mathcal{D} \to \mathcal{B}$ is just $\prod_{D\in\mathcal{D}} SD$.

Example 5.7. In the case where $\mathcal{V} = \mathbf{Cat}$, the enriched projective model structure on $[\mathcal{D}, \mathbf{Cat}]$ exists and the cofibrant weights are precisely the flexible ones in the sense of [4]: see Theorem 5.5 and Section 6 of [24]. Thus a 2-category will have all cofibrantly-weighted limits just when it has all flexible limits. But it will have enough cofibrantly-weighted limits provided that it has all weighted pseudolimits, and so in particular if it has products, inserters, and equifiers; in other words, if it has PIE limits in the sense of [30]. Once again, the 2-category of strict monoidal categories and strong monoidal functors is an example which has PIE limits but not flexible ones: see [8, Section 6.2].

Example 5.8. In the case of **SSet**, the enriched projective model structure on $[\mathcal{D}, \mathbf{SSet}]$ exists (see Proposition A.3.3.2 and Remark A.3.3.4 of [26]) and has generating cofibrations

$$\mathcal{I} = \{\partial\Delta^n \cdot \mathcal{D}(X,-) \to \Delta^n \cdot \mathcal{D}(X,-) \colon n \in \mathbb{N}, X \in \mathcal{D}\}$$

obtained by copowering the boundary inclusions of simplices by representables. The $\mathcal{I}$-cellular weights are, in this context, what Riehl and Verity call *flexible weights* — in particular, these are particular cofibrant weights. Since Quillen's small object argument applied to the set $\mathcal{I}$ produces a cellular cofibrant replacement of each weight, a simplicially enriched category will have enough cofibrantly-weighted limits provided that it has all flexible limits — that is, those weighted by flexible weights. In particular, each ∞-cosmos in the sense of [35] admits enough cofibrantly-weighted limits.

Lemma 5.9. *Let $Q \colon \mathcal{D} \to \mathcal{V}$ be a cofibrant weight for which the unique map $Q \to 1$ is a trivial fibration. Then Q is bi-terminal in the full subcategory of $[\mathcal{D}, \mathcal{V}]$ consisting of the cofibrant weights.*

Proof. Let G be cofibrant. Then the unique map from $[\mathcal{D}, \mathcal{V}](G, Q)$ to $[\mathcal{D}, \mathcal{V}](G, 1)$ is a trivial fibration. Since the left vertical in each of the following diagrams is a cofibration in $\mathcal{V}$

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & {[\mathcal{D},\mathcal{V}](G,Q)} \arrow[d] \\
I \arrow[r] \arrow[ur, dotted] & 1 = {[\mathcal{D},\mathcal{V}](G,1)}
\end{tikzcd}
\qquad
\begin{tikzcd}
I + I \arrow[r] \arrow[d] & {[\mathcal{D},\mathcal{V}](G,Q)} \arrow[d] \\
J \arrow[r] \arrow[ur, dotted] & 1 = {[\mathcal{D},\mathcal{V}](G,1)}
\end{tikzcd}$$

it follows that each diagram has a filler. This implies the existence and essential uniqueness of maps $G \to Q$. ☐

Consider a diagram $J \colon \mathcal{D} \to \mathcal{B}$, and let $L = \{Q, J\}$. Now Q is bi-terminal with respect to cofibrant objects by Lemma 5.9, and each representable is cofibrant by Example 5.1, so that there exists a morphism $s_D \colon \mathcal{D}(D,-) \to Q$. Since weighted limits are (contravariantly) functorial in their weights, we obtain a morphism

$$p_D \colon L = \{Q, J\} \to \{\mathcal{D}(D,-), J\} = JD.$$

By bi-terminality of Q once again, for any $f\colon C \to D$ in $\mathcal{D}$ there exists a $\mathcal{V}$-homotopy in the triangle below left

$$\begin{tikzcd} & \mathcal{D}(C,-) \arrow[dr, "s_C"] \arrow[d, phantom, "\cong"] & \\ \mathcal{D}(D,-) \arrow[ur, "\mathcal{D}(f,-)"] \arrow[rr, "s_D"'] & & Q \end{tikzcd} \qquad \begin{tikzcd} L \arrow[rr, "p_C"] \arrow[dr, "p_D"'] & \cong & JC \arrow[dl, "Jf"] \\ & JD & \end{tikzcd}$$

and so, on taking limits of J by the given weights, a $\mathcal{V}$-homotopy in the triangle on the right.

Lemma 5.10. *If $J\colon \mathcal{D} \to \mathcal{B}$ is fully faithful, and $L = \{Q, J\}$ as above, then the projection $p_D\colon L \to JD$ satisfies $f \circ p_D \cong 1_L$ for any morphism $f\colon JD \to L$.*

Proof. Using fully faithfulness of J we define $k\colon Q \to \mathcal{D}(D,-)$ as the unique morphism rendering commutative the upper square below

$$\begin{tikzcd} Q \arrow[r, "\eta"] \arrow[d, "k"'] & \mathcal{B}(L, J-) \arrow[d, "\mathcal{B}(f,J-)"] \\ \mathcal{D}(D,-) \arrow[r, "J", "\cong"'] \arrow[d, "s_D"'] & \mathcal{B}(JD, J-) \arrow[d, "\mathcal{B}(p_D,J-)"] \\ Q \arrow[r, "\eta"'] & \mathcal{B}(L, J-) \end{tikzcd}$$

in which η is the unit of the limit $L = \{Q, J\}$, and the lower commutes essentially by definition of p_D. Bi-terminality of Q among cofibrant objects in $[\mathcal{D}, \mathcal{V}]$ implies that the composite $s_D \circ k$ is $\mathcal{V}$-homotopic to the identity.

Consider the weighted limit $\mathcal{V}$-functor $\{-, J\}\colon [\mathcal{D}, \mathcal{V}]^{\mathrm{op}}_{\mathcal{B}} \to \mathcal{B}$, where the domain is restricted to the full subcategory containing those weights W for which $\mathcal{B}$ admits all W-weighted limits. Since $[\mathcal{D}, \mathcal{V}]_{\mathcal{B}}$ contains Q and $\mathcal{D}(D,-)$ we use the fact that $\mathcal{V}$-functors preserve $\mathcal{V}$-homotopies to deduce that $\{k, J\} \circ \{s_D, J\} \cong 1$; in other words that $f \circ p_D \cong 1$. □

Example 5.11. Let **Eq** be the free parallel pair, involving maps $f, g\colon A \to B$, made into a free $\mathcal{V}$-category. Consider the constant functor $\Delta I\colon \mathbf{Eq} \to \mathcal{V}$, which is equally the terminal weight. We first show how a (standard) interval gives a cofibrant replacement of ΔI and interpret the corresponding limit. Then we relate this special type of cofibrant replacement to a general one.

Given a standard interval — that is, an interval

$$\begin{tikzcd} I \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & J \arrow[r, "e"] & I \end{tikzcd}$$

in which e is a trivial fibration, one can show that the pair d, c constitutes a cofibrant weight $Q\colon \mathbf{Eq} \to \mathcal{V}$ and admits a pointwise trivial fibration

$$\begin{tikzcd} I \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "1"'] & J \arrow[d, "e"] \\ I \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] & I \end{tikzcd}$$

to the constant functor ΔI. A Q-cone from an object X to a diagram f, $g\colon A \rightrightarrows B$ amounts to a natural transformation

$$\begin{tikzcd} I \arrow[r, "k"] \arrow[d, shift right, "d"'] \arrow[d, shift left, "c"] & \mathcal{C}(X,A) \arrow[d, shift right, "f_*"'] \arrow[d, shift left, "g_*"] \\ J \arrow[r, "h"'] & \mathcal{C}(X,B) \end{tikzcd}$$

or equivalently a morphism $k\colon X \to A$ equipped with a homotopy $h\colon fk \cong gk$. Thus, a limit $\{Q, F\}$ possesses a universal such pair and is thus what one might call a Q-*isoinserter*, or *isoinserter* if Q is understood.

Next, suppose that $Q'\colon \mathbf{Eq} \to \mathcal{V}$ is cofibrant and $q'\colon Q' \to \Delta I$ is a pointwise trivial fibration, as in

$$\begin{tikzcd} I' \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] \arrow[d, "a"'] & J' \arrow[d, "b"] \\ I \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] & I \end{tikzcd}$$

and let u be a section of the trivial fibration a. Factorise the induced map $(d'u\ c'u)$ as a cofibration followed by a trivial fibration as in

$$\begin{tikzcd} I+I \arrow[r, "u+u"] \arrow[d, "(d\ c)"'] & I'+I' \arrow[d, "(d'\ c')"] \\ J \arrow[r, "v"] & J' \end{tikzcd}$$

to give an interval

$$\begin{tikzcd} I \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & J \arrow[r, "bv"] & I. \end{tikzcd}$$

together with a natural transformation

$$\begin{tikzcd} I \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "u"'] & J \arrow[d, "v"] \\ I' \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] & J'. \end{tikzcd}$$

Thus, any Q'-cone gives a Q-cone and, in particular, a morphism $k\colon X \to A$ and a homotopy $h\colon fk \cong gk$. We call the limit $\{Q', F\}$, the *Q'-isoinserter* of f and g, or just the *isoinserter* if Q' is understood.

5.3. **Enough cofibrantly-weighted limits ($I \neq 1$).** For $\mathcal{I}$ the unit $\mathcal{V}$-category we write $I\colon \mathcal{I} \to \mathcal{V}$ for the $\mathcal{V}$-functor selecting the object I.

Definition 5.12. We say that a $\mathcal{V}$-category $\mathcal{B}$ has *enough cofibrantly-weighted limits* if for each small $\mathcal{V}$-category $\mathcal{D}$ and each $\mathcal{V}$-functor $P\colon \mathcal{D} \to \mathcal{I}$ there is a cofibrant $\mathcal{V}$-weight $Q\colon \mathcal{D} \to \mathcal{V}$ and pointwise trivial fibration $Q \to IP$ for which $\mathcal{B}$ has Q-weighted limits. Similarly, $U\colon \mathcal{B} \to \mathcal{A}$ preserves enough cofibrantly-weighted limits if it preserves such Q-limits.

In the case where $I = 1$, the $\mathcal{V}$-category $\mathcal{I}$ is terminal, and so any $\mathcal{V}$-category $\mathcal{D}$ has a unique such P; furthermore, the composite IP is then the terminal object of $[\mathcal{D}, \mathcal{V}]$, and so this does agree with our earlier definition.

Example 5.13. For the trivial model structure on $\mathcal{V}$, no longer assuming that I is terminal, a $\mathcal{V}$-category $\mathcal{K}$ will have enough cofibrantly-weighted limits if and only if it has all limits weighted by functors of the form $IP\colon \mathcal{D} \to \mathcal{V}$ for a small $\mathcal{V}$-category $\mathcal{D}$, a $\mathcal{V}$-functor $P\colon \mathcal{D} \to \mathcal{I}$, and $I\colon \mathcal{I} \to \mathcal{V}$ the weight picking out the object $I \in \mathcal{V}$.

Example 5.14. For the split model structure on $\mathcal{V}$, no longer assuming that I is terminal, each weight $W \in [\mathcal{D}, \mathcal{V}]$ admits a canonical cofibrant replacement $q\colon W' \to W$ given by the evaluation map

$$\textstyle\sum_{D \in \mathcal{D}} WD \cdot \mathcal{D}(D, -) \xrightarrow{q} W$$

To see this, observe that the left hand side is a coproduct of copowers of representables, and so cofibrant by Proposition 5.2. Furthermore, each component of q is a split epimorphism with section

$$WA \cong WA \cdot I \to WA \cdot \mathcal{D}(A, A) \to \Sigma_{D \in \mathcal{D}} WA \cdot \mathcal{D}(D, A)$$

obtained by first copowering the map $I \to \mathcal{D}(A, A)$ defining the identity morphism on A, and then composing this with the coproduct inclusion.

Thus q is a pointwise split epimorphism, and so a trivial fibration in $[\mathcal{D}, \mathcal{V}]$. In the case that $W = IP$, as in Definition 5.12, each WA is just I. Thus the right hand side above reduces to $\sum_{D\in\mathcal{D}} \mathcal{D}(D, -)$, whence a $\mathcal{V}$-category has enough cofibrantly-weighted limits provided it has products.

6. The weak adjoint functor theorem in the case $I = 1$

The key technical step in the proof of Freyd's general adjoint functor theorem is to prove that a complete category with a weakly initial set of objects has an initial object. We shall now show that, on replacing completeness by homotopical completeness, we can still construct a bi-initial object. It is then straightforward to deduce the weak adjoint functor theorem.

Theorem 6.1. *Suppose that the unit object I of $\mathcal{V}$ is terminal. If $\mathcal{B}$ is a $\mathcal{V}$-category with enough cofibrantly-weighted limits, and $\mathcal{B}_0$ has a small weakly initial family of objects, then $\mathcal{B}$ has a bi-initial object.*

Proof. Let $\mathcal{G}$ be the full subcategory of $\mathcal{B}$ consisting of the objects appearing in the small weakly initial family. We write $J : \mathcal{G} \to \mathcal{B}$ for the inclusion of this small full subcategory. By assumption, $\mathcal{B}$ admits Q-weighted limits for some cofibrant weight $Q : \mathcal{G} \to \mathcal{V}$ for which the unique map $Q \to 1$ is a trivial fibration. We shall show that the limit $L = \{Q, J\}$ is bi-initial in $\mathcal{B}$.

Following the notation of Section 5.2, we obtain morphisms $p_C : L \to JC$ for each $C \in \mathcal{G}$, such that for $f : C \to D$ the triangle

$$\begin{tikzcd} L \arrow[rr, "p_C"] \arrow[dr, "p_D"'] & \arrow[d, phantom, "\cong"] & JC \arrow[dl, "Jf"] \\ & JD & \end{tikzcd}$$

commutes up to $\mathcal{V}$-homotopy.

Let $B \in \mathcal{B}$. Since $\mathcal{G}$ is weakly initial, there exists a $C \in \mathcal{G}$ and a morphism $f : JC \to B$, and now $f \circ p_C : L \to B$ gives the existence part of the required property of L.

Suppose now that $B \in \mathcal{B}$ and that $f, g : L \to B$ are two morphisms. As in Example 5.11, we may form the isoinserter $k : K \to L$ of f and g, and $fk \cong gk$. By weak initiality of $\mathcal{G}$ there is a morphism $u : JC \to K$ for some $C \in \mathcal{G}$. Now

$$f \circ k \circ u \circ p_C \cong g \circ k \circ u \circ p_C$$

and so it will suffice to show that $k \circ u \circ p_C \cong 1$. Lemma 5.10 gives $v \circ p_C \cong 1$ for any $v : JC \to L$ and, in particular, for $v = k \circ u$. $\square$

Theorem 6.2. *Suppose that the unit object I of $\mathcal{V}$ is terminal. Let $\mathcal{B}$ be a $\mathcal{V}$-category with powers and enough cofibrantly-weighted limits, and let $U\colon \mathcal{B} \to \mathcal{A}$ be a $\mathcal{V}$-functor that preserves them. Then U has an $\mathcal{E}$-weak left adjoint if and only if U_0 satisfies the solution set condition.*

Proof. Suppose firstly that U has an $\mathcal{E}$-weak left adjoint, and $\eta_A\colon A \to UA'$ is an $\mathcal{E}$-weak reflection. Then the singleton family consisting of (A', η) is a solution set for U_0. This proves the "only if" direction.

Since I is terminal the unit $\mathcal{V}$-category $\mathcal{I}$ is terminal in $\mathcal{V}$-**Cat** and hence complete as a $\mathcal{V}$-category. Then for any $A \in \mathcal{A}$, it follows from Proposition 4.1 that the comma category A/U has any type of limit which $\mathcal{B}$ has and $U\colon \mathcal{B} \to \mathcal{A}$ preserves, and so has enough cofibrantly-weighted limits. By Proposition 4.5 we know that $(A/U)_0 \cong A/U_0$, which has a weakly initial set of objects since U_0 satisfies the solution set condition. Hence, by Theorem 6.1, A/U has a bi-initial object $\eta\colon A \to UA'$. Since $\mathcal{B}$ has powers and U preserves them, it follows from Theorem 4.7 that

$$\mathcal{B}(A', B) \xrightarrow{U} \mathcal{A}(UA', UB) \xrightarrow{\mathcal{A}(\eta, UB)} \mathcal{A}(A, UB)$$

is bi-terminal in $\mathcal{V}/\mathcal{A}(A, UB)$ — in other words, by Proposition 4.6, it is shrinkable. $\square$

7. $\mathcal{V}/I$-categories and the weak adjoint functor theorem in the case $I \neq 1$

The proof given in the previous section relied on the fact that the enriched comma categories A/U had enough cofibrantly-weighted limits, which we were able to prove only in the case where $I = 1$, since only then could we be sure that the unit $\mathcal{V}$-category $\mathcal{I}$ had enough cofibrantly-weighted limits. We shall overcome this problem by viewing A/U as a category enriched in the monoidal category $\mathcal{V}/I$, whose unit *is* terminal. To this end, observe that, since the unit I is (trivially) a commutative monoid in $\mathcal{V}$, the slice category $\mathcal{V}/I$ becomes a symmetric monoidal category, with unit $1\colon I \to I$ and with the tensor product of $(X, x\colon X \to I)$ and $(Y, y\colon Y \to I)$ given by $(X \otimes Y, x \otimes y\colon X \otimes Y \to I)$. In particular, the unit is indeed terminal in $\mathcal{V}/I$. The resulting symmetric monoidal category $\mathcal{V}/I$ is also closed, with the internal hom of (Y, y) and (Z, z) given by the left vertical in the pullback

$$\begin{tikzcd}
\langle (Y,y),(Z,z) \rangle \arrow[rr] \arrow[d] & & {[Y,Z]} \arrow[d, "{[Y,z]}"] \\
I \arrow[r, "j"'] & {[I,I]} \arrow[r, "{[y,I]}"'] & {[Y,I]}
\end{tikzcd} \tag{7.1}$$

The category $\mathcal{V}/I$ is once again complete and cocomplete. It becomes a monoidal model category if we define a morphism to be a cofibration, weak equivalence, or fibration just when its underlying morphism in $\mathcal{V}$ is one.

A category enriched in $\mathcal{V}/I$ involves a collection of objects together with, for each pair X, Y of objects, an object

$$C_{X,Y} : \mathcal{C}(X,Y) \to I$$

of $\mathcal{V}/\mathcal{I}$. Translating through the remaining structure, this is equally to specify a $\mathcal{V}$-category $\mathcal{C}$ together with a functor $C : \mathcal{C} \to \mathcal{I}$ to the unit $\mathcal{V}$-category. We sometimes write $\mathbb{C} = (\mathcal{C}, C)$ for such a $\mathcal{V}/I$-enriched category, and we refer to the functor C as the *augmentation*.

Proposition 7.1. *This correspondence extends to an isomorphism of categories $\mathcal{V}/I$-**Cat** $\cong$ $\mathcal{V}$-**Cat**$/\mathcal{I}$.*

Proof. A $\mathcal{V}/I$-functor $F : \mathcal{C} \to \mathcal{D}$ consists of a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ for which the action $F_{C,D} : \mathcal{C}(C,D) \to \mathcal{D}(FC,FD)$ on homs commutes with the maps into I; equivalently, such that F commutes with the augmentations. □

Example 7.2. The augmentations $P : A/U \to \mathcal{I}$ and $Q : \mathcal{A}/A \to \mathcal{I}$ equip the $\mathcal{V}$-categories A/U and $\mathcal{A}/A$ with the structure of $\mathcal{V}/I$-categories.

The next four results extend Propositions 4.3, 4.5, 4.6, and Theorem 4.7 respectively, with essentially the same proof as before in each case.

Proposition 7.3. *The identity morphism on A is a terminal object in the slice $\mathcal{V}/I$-category $\mathcal{A}/A$.*

Proof. From (4.2), for any $f : B \to A$ we have the pullback

$$\begin{tikzcd} (\mathcal{A}/A)((B,f),(A,1_A)) \arrow[r] \arrow[d] & \mathcal{B}(B,A) \arrow[d, "\mathcal{B}(B,1_A)"] \\ I \arrow[r, "f"'] & \mathcal{B}(B,A) \end{tikzcd}$$

and since the pullback of an isomorphism is an isomorphism, the left vertical map is an isomorphism over I. As such, it is terminal as an object of $\mathcal{V}/I$. □

Proposition 7.4. *Let $U : \mathcal{B} \to \mathcal{A}$ and $A \in \mathcal{A}$.*

(1) *Then the $\mathcal{V}/I$-category A/U has underlying category A/U_0. Furthermore, parallel morphisms $f, g : (B,b) \rightrightarrows (C,c)$ are $\mathcal{V}/I$-homotopic if and only if f and g are $\mathcal{V}$-homotopic under A.*

(2) *Then $\mathcal{A}/A$ has underlying category $\mathcal{A}_0/A$. Furthermore, parallel morphisms $f, g\colon (B,b) \rightrightarrows (C,c)$ are $\mathcal{V}/I$-homotopic if and only if f and g are $\mathcal{V}$-homotopic over A.*

Proof. Since the unit in $\mathcal{V}/I$ is $1\colon I \to I$, a morphism in $(A/U)_0$ is given by a map into the pullback

$$\begin{tikzcd}
I \arrow[r] & (A/U)((B,b),(C,c)) \arrow[rr] \arrow[d] & & I \arrow[d, "c"] \\
& \mathcal{B}(B,C) \arrow[r, "U"'] & \mathcal{A}(UB,UC) \arrow[r, "b^*"'] & \mathcal{A}(A,UC)
\end{tikzcd}$$

whose component in I is the identity — we no longer require the assumption $I = 1$ for this to be true. As in the proof of Proposition 4.5 this amounts to a morphism of A/U_0. Now a $\mathcal{V}/I$-interval is an interval in $\mathcal{V}$ viewed as a diagram

$$(I+I,\nabla) \xrightarrow{(d\ \ c)} (J,e) \xrightarrow{e} (I,1)$$

in $\mathcal{V}/I$. Therefore a (J,e)-homotopy $f \cong g$ in A/U is specified by a map from J into the pullback whose component in I is fixed as $e\colon J \to I$, and so as in the proof of Proposition 4.5 amounts to a $(\mathcal{V}, J)$-homotopy $h\colon f \cong g$ for which $Uh \circ b$ is trivial. The case of $\mathcal{A}/A$ follows by duality, as before. □

Proposition 7.5. *A morphism $f\colon A \to B \in \mathcal{C}$ is shrinkable if and only if it is biterminal in the slice $\mathcal{V}/\mathcal{I}$-category $\mathcal{C}/B$.*

Proof. The proof of Proposition 4.6 carries over unchanged on replacing the use of Propositions 4.3 and 4.5 by their respective generalisations, Propositions 7.3 and 7.4. □

Theorem 7.6. *Let $\mathcal{B}$ be a $\mathcal{V}$-category with powers, and $U\colon \mathcal{B} \to \mathcal{A}$ a $\mathcal{V}$-functor which preserves them; if $\eta\colon A \to UA'$ is bi-initial in the $\mathcal{V}/I$-category A/U then*

$$\mathcal{B}(A',B) \xrightarrow{U} \mathcal{A}(UA',UB) \xrightarrow{\mathcal{A}(\eta,UB)} \mathcal{A}(A,UB)$$

is bi-terminal in the $\mathcal{V}/I$-category $\mathcal{V}/\mathcal{A}(A,UB)$.

Proof. The proof of Theorem 4.7 carries over unchanged on replacing the use of Proposition 4.5 by the more general Proposition 7.4. □

The proof of the next proposition relies on two lemmas which we defer until after the proof of our main result, Theorem 7.8 below.

Proposition 7.7. *If $\mathcal{B}$ has enough cofibrantly-weighted limits and U preserves them, in the $\mathcal{V}$-sense, then A/U has enough cofibrantly-weighted limits in the $\mathcal{V}/I$-sense.*

Proof. By Lemma 7.11 below $R(\mathcal{B})$ has enough cofibrantly-weighted limits and $R(U)\colon R(\mathcal{A}) \to R(\mathcal{B})$ preserves them. Therefore, by Proposition 4.1, the comma $\mathcal{V}/I$-category $A/R(U)$ has enough cofibrantly-weighted limits. By Lemma 7.9 below, $A/R(U)$ is isomorphic to the $\mathcal{V}/I$-category $P\colon A/U \to \mathcal{I}$, which consequently has enough cofibrantly-weighted limits too. □

Theorem 7.8. *Let $\mathcal{B}$ be a $\mathcal{V}$-category with powers and enough cofibrantly-weighted limits, and let $U\colon \mathcal{B} \to \mathcal{A}$ be a $\mathcal{V}$-functor that preserves them. Then U has an $\mathcal{E}$-weak left adjoint if and only if U_0 satisfies the solution set condition.*

Proof. If U has an $\mathcal{E}$-weak left adjoint, and $\eta_A\colon A \to UA'$ is an $\mathcal{E}$-weak reflection, then the singleton family consisting of (A', η) is a solution set for U_0. This proves the "only if" direction.

Conversely, by Proposition 7.7 it follows that the $\mathcal{V}/I$-category A/U has enough cofibrantly-weighted limits. By Proposition 7.4 we know that the underlying category $(A/U)_0$ of the $\mathcal{V}/I$-category A/U is isomorphic to A/U_0. Since U_0 satisfies the solution set condition A/U_0, and so also $(A/U)_0$, have weakly initial sets of objects. Thus, by Theorem 6.1, A/U has a bi-initial object $\eta\colon A \to UA'$, and now by Theorem 7.6 the morphism

$$\mathcal{B}(A', B) \xrightarrow{U} \mathcal{A}(UA', UB) \xrightarrow{\mathcal{A}(\eta, UB)} \mathcal{A}(A, UB)\ .$$

is bi-terminal in the $\mathcal{V}/I$-category $\mathcal{V}/\mathcal{A}(A, UB)$, and so is shrinkable by Proposition 7.5. □

In order to complete the proof of Proposition 7.7, we need to consider weighted limits in $\mathcal{V}/I$-categories of the form $P\colon A/U \to \mathcal{I}$. First, we show that these are comma categories in the $\mathcal{V}/I$-enriched sense.

To this end, first observe that the forgetful functor $\mathcal{V}\text{-}\mathbf{Cat}/\mathcal{I} \to \mathcal{V}\text{-}\mathbf{Cat}$ has a right adjoint R, sending $\mathcal{A}$ to the $\mathcal{V}/I$-category $\pi_2\colon \mathcal{A} \times \mathcal{I} \to \mathcal{I}$.

Lemma 7.9. *Given a $\mathcal{V}$-functor $U\colon \mathcal{B} \to \mathcal{A}$ and an object $A \in \mathcal{A}$, the $\mathcal{V}/I$-category $P\colon A/U \to \mathcal{I}$ is isomorphic to the comma $\mathcal{V}/I$-category $A/R(U)$.*

Proof. By Proposition 7.4, an object of the $\mathcal{V}/I$-category A/U is given by a pair $(B, f\colon I \to \mathcal{A}(A, UB) \in \mathcal{V})$, which bijectively corresponds to a pair $(B, (f, 1)\colon (I, 1) \to (\mathcal{A}(A, UB) \times I, \pi_2) \in \mathcal{V}/I)$; that is, to an object of $A/R(U)$.

Given objects $(B, (f, 1))$ and $(C, (g, 1))$ of $A/R(U)$, consider the following diagram, in which each region is a pullback.

$$\begin{tikzcd}
(A/R(U))((B,(f,1)),(C,(g,1))) \arrow[r] \arrow[dd] & \mathcal{B}(B,C) \times I \arrow[r, "\pi_1"] \arrow[d, "U\times 1"] & \mathcal{B}(B,C) \arrow[d, "U"] \\
& \mathcal{A}(UB,UC) \times I \arrow[r, "\pi_1"] \arrow[d, "\mathcal{A}(f,UB)\times 1"] & \mathcal{A}(UB,UC) \arrow[d, "\mathcal{A}(f,UB)"] \\
I \arrow[r, "{(g,1)}"] \arrow[rr, bend right, "g"'] & \mathcal{A}(A,UC) \times I \arrow[r, "\pi_1"] & \mathcal{A}(A,UC)
\end{tikzcd}$$

We may deduce an isomorphism

$$(A/R(U))((B,(f,1)),(C,(g,1))) \cong A/U((B,f),(C,g))$$

in $\mathcal{V}/I$, since the left hand side is defined by the pullback on the left, and the right hand side is defined by the composite pullback. The remaining verifications of functoriality are straightforward. $\square$

To complete the argument, we need to consider weighted limits in the $\mathcal{V}/I$-enriched sense. To get started, observe, on comparing (4.2) and (7.1), that $\mathcal{V}/I$, as a $\mathcal{V}/I$-enriched category, is given by the slice $\mathcal{V}$-category $Q\colon \mathcal{V}/I \to \mathcal{I}$. We denote it by $\mathbb{V}/\mathbb{I} = (\mathcal{V}/I, Q)$.

Let $\mathbb{D} = (\mathcal{D}, D)$ be a $\mathcal{V}/I$-category. Then a $\mathcal{V}/I$-weight $\mathbb{D} \to \mathbb{V}/\mathbb{I}$ is specified by a commutative triangle as on the left below.

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[dr, "D"'] & \mathcal{V}/I \arrow[r] \arrow[d, "Q"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{V} \arrow[d, equal] \\
& \mathcal{I} \arrow[r, "I"'] & \mathcal{V}
\end{tikzcd}
\quad \equiv \quad
\begin{tikzcd}
\mathcal{D} \arrow[rr, "W"] \arrow[dr, "D"'] & \arrow[d, phantom, "\Downarrow w"] & \mathcal{V} \\
& \mathcal{I} \arrow[ur, "I"'] &
\end{tikzcd}$$

By the universal property of the slice $\mathcal{V}$-category $\mathcal{V}/\mathcal{I}$ this amounts to a $\mathcal{V}$-weight $W\colon \mathcal{D} \to \mathcal{V}$ together with a $\mathcal{V}$-natural transformation $w\colon W \Rightarrow ID$. We write $\mathbb{W} = (W, w)$ for the $\mathcal{V}/I$-weight.

Building on the above description, it is straightforward to see that the presheaf $\mathcal{V}/I$-category $[\mathbb{D}, \mathbb{V}/\mathbb{I}]$ is given by the slice $\mathcal{V}$-category $[\mathcal{D}, \mathcal{V}]/ID$ equipped with its natural projection to $\mathcal{I}$. In particular, its homs are defined by the following pullbacks.

$$\begin{tikzcd}
{[\mathbb{D},\mathbb{V}/\mathbb{I}](\mathbb{F},\mathbb{G})} \arrow[r] \arrow[d, "U"'] & I \arrow[d, "f"] \\
{[\mathcal{D},\mathcal{V}](F,G)} \arrow[r, "g_*"'] & {[\mathcal{D},\mathcal{V}](F,ID)}
\end{tikzcd} \tag{7.2}$$

Finally, observe that given a diagram $S\colon \mathbb{D} \to \mathbb{A}$ and an object $X \in \mathbb{A}$ the induced presheaf

$$\mathbb{A}(X, S-)\colon \mathbb{D} \to \mathbb{V}/\mathbb{I}$$

consists of the $\mathcal{V}$-weight $\mathcal{A}(X, S-)\colon \mathcal{D} \to \mathcal{V}$ together with the augmentation $\mathcal{A}(X, S-) \to ID$ having components $A_{X,SY}\colon \mathcal{A}(X, SY) \to I$.

Lemma 7.10. *Let $\mathbb{W} = (W, w)$ be a $\mathcal{V}/I$-weight. If $\mathcal{B}$ has W-weighted limits then $R(\mathcal{B})$ has $\mathbb{W}$-weighted limits; if $U\colon \mathcal{B} \to \mathcal{A}$ preserves W-weighted limits then $R(U)$ preserves $\mathbb{W}$-weighted limits.*

Proof. Consider a diagram $\mathbb{D} \to R(\mathcal{B}) = (\mathcal{B} \times \mathcal{I}, \pi_2)$, which is necessarily of the form $\overline{S} = (S, D)$ for $S\colon \mathcal{D} \to \mathcal{B}$. We claim that the limit $\{W, S\}$ also has the universal property of $\{\mathbb{W}, \overline{S}\}$.

Using the definition of homs in $[\mathbb{D}, \mathbb{V}/\mathbb{I}]$ we have a pullback

$$\begin{tikzcd}
{[\mathbb{D},\mathbb{V}/\mathbb{I}](\mathbb{W},R(\mathcal{B})(B,\overline{S}-))} \arrow[r] \arrow[d, "U"'] & I \arrow[d, "w"] \\
{[\mathcal{D},\mathcal{V}](W,\mathcal{B}(B,S-)\times ID)} \arrow[r, "(\pi_2)_*"'] & {[\mathcal{D},\mathcal{V}](W,ID)}
\end{tikzcd}$$

and so, using the universal property of the product $\mathcal{B}(B, S-) \times ID$, an isomorphism

$$[\mathbb{D}, \mathbb{V}/\mathbb{I}](\mathbb{W}, R(\mathcal{B})(B, \overline{S}-)) \cong [\mathcal{D}, \mathcal{V}](W, \mathcal{B}(B, S-)) \times I$$

over I, and now the right hand side is naturally isomorphic to

$$\mathcal{B}(B, \{W, S\}) \times I = R(\mathcal{B})(B, \{W, S\}),$$

as required. This proves that $\{W, S\}$ also has the universal property of $\{\mathbb{W}, \overline{S}\}$. Given this, the corresponding statement about preservation is straightforward. $\square$

Lemma 7.11. *If $\mathcal{B}$ has enough cofibrantly-weighted limits and $U\colon \mathcal{B} \to \mathcal{A}$ preserves them, then $R(\mathcal{B})$ has enough cofibrantly-weighted limits (in the $\mathcal{V}/I$-enriched sense) and $R(U)\colon R(\mathcal{B}) \to R(\mathcal{A})$ preserves them.*

Proof. Let $\mathbb{D}$ be a $\mathcal{V}/I$-category. Re-expressing the definition of enough cofibrantly-weighted limits in these terms, we see that there exists a weight $\mathbb{W} = (W, w)$ with W cofibrant and $w\colon W \to ID$ a trivial fibration, and such that $\mathcal{B}$ has W-weighted limits and U preserves them. By Lemma 7.10, this implies that $R(\mathcal{B})$ has $\mathbb{W}$-weighted limits and $R(U)$ preserves them.

Now $(ID, 1)$ is the terminal $\mathcal{V}/I$-weight so that the trivial fibration $w\colon W \to ID$ equally specifies a trivial fibration $w\colon \mathbb{W} \to 1$. Therefore, if we can show that W cofibrant implies $\mathbb{W}$ cofibrant, we will have shown that $R(\mathcal{B})$ has enough cofibrantly-weighted limits as a $\mathcal{V}/I$-category and that $R(U)$ preserves them.

To this end, suppose that $\alpha\colon \mathbb{F} \to \mathbb{G}$ is a trivial fibration and consider the commutative square on the left below.

$$\begin{tikzcd}
{[\mathbb{D}, \mathbb{V}/\mathbb{I}](\mathbb{W}, \mathbb{F})} \arrow[r, "\alpha_*"] \arrow[d, "U_{W,F}"'] & {[\mathbb{D}, \mathbb{V}/\mathbb{I}](\mathbb{W}, \mathbb{G})} \arrow[r] \arrow[d, "U_{W,G}"] & I \arrow[d, "w"] \\
{[\mathcal{D}, \mathcal{V}](W, F)} \arrow[r, "\alpha_*"'] & {[\mathcal{D}, \mathcal{V}](W, G)} \arrow[r, "g_*"'] & {[\mathcal{D}, \mathcal{V}](W, ID)}
\end{tikzcd}$$

Pasting this with the pullback square (7.2) defining $[\mathbb{D}, \mathbb{V}/\mathbb{I}](\mathbb{W}, \mathbb{G})$ as above right yields the pullback square defining $[\mathbb{D}, \mathbb{V}/\mathbb{I}](\mathbb{W}, \mathbb{F})$. Therefore the left square above is a pullback in $\mathcal{V}$. Now since W is cofibrant, the lower α_* in the left square is a trivial fibration whence so is its pullback on the top row, as required. $\square$

In the case of the ordinary adjoint functor theorem, there is of course a converse: if a functor has a left adjoint then it satisfies the solution set condition and preserves any existing limits. In our setting, if there is a weak left adjoint, then the solution set condition does hold (as stated in the theorem), but the preservation of enough cofibrantly-weighted limits need not, as the following example shows.

Example 7.12. Let $\mathcal{V} = \mathbf{Set}$ with the split model structure, so that the shrinkable morphisms are just the surjections, and we are dealing with classical weak category theory. Let G be a non-trivial group, and $\mathbf{Set}^G$ the category of G-sets. The forgetful functor $U\colon \mathbf{Set}^G \to \mathbf{Set}$ has a left adjoint F. Let $P\colon \mathbf{Set}^G \to \mathbf{Set}$ be the functor sending a G-set to its set of orbits. There is a natural transformation $\pi\colon U \to P$ whose components are surjective. For each set X, the composite of the unit $X \to UFX$ and $\pi\colon UFX \to PFX$ exhibits F as a weak left adjoint to P. But P does not preserve powers (or products): in particular, it does not preserve the power G^2.

In good cases, however, there is a sort of converse involving preservation of homotopy limits, as we now describe.

Theorem 7.13. *Let $\mathcal{B}$ and $\mathcal{A}$ be locally fibrant $\mathcal{V}$-categories, and let $U: \mathcal{B} \to \mathcal{A}$ have a $\mathcal{E}$-weak left adjoint. Let $Q: \mathcal{D} \to \mathcal{V}$ be a cofibrant weight for which $[\mathcal{D}, \mathcal{V}](Q, -): [\mathcal{D}, \mathcal{V}] \to \mathcal{V}$ sends levelwise shrinkable morphisms between levelwise fibrant objects to weak equivalences. Then U preserves homotopy Q-weighted limits.*

Proof. The induced maps $\mathcal{B}(A', B) \to \mathcal{A}(A, UB)$ are shrinkable morphisms between fibrant objects and so are weak equivalences (between fibrant objects). Suppose that $\varphi: Q \to \mathcal{B}(L, S)$ exhibits L as the homotopy weighted limit $\{Q, S\}_h$, meaning that for each $B \in \mathcal{B}$ the induced morphism $\mathcal{B}(B, L) \to [\mathcal{D}, \mathcal{V}](Q, \mathcal{B}(B, S))$ is a weak equivalence in $\mathcal{V}$. Then given $A \in \mathcal{A}$ there is a commutative diagram

$$\begin{tikzcd}
\mathcal{B}(A', L) \arrow[r] \arrow[d] & {[\mathcal{D}, \mathcal{V}](Q, \mathcal{B}(A', S))} \arrow[d] \\
\mathcal{A}(A, UL) \arrow[r] & {[\mathcal{D}, \mathcal{V}](Q, \mathcal{A}(A, US))}
\end{tikzcd}$$

in which the upper horizontal is the weak equivalence expressing the universal property of $L = \{Q, S\}$, the left vertical is a shrinkable morphism (and so a weak equivalence) expressing the universal property of A', and the right vertical is given by $[\mathcal{D}, \mathcal{V}](Q, -)$ applied to a levelwise shrinkable morphism between levelwise fibrant objects, and so is a weak equivalence. Then the bottom map is also a weak equivalence, and so U preserves the homotopy limit. $\square$

Example 7.14. In [25], Q-weighted limits were said to be *$\mathcal{E}$-stable* if $\mathcal{E}$, seen as a full subcategory of $\mathcal{V}^{\mathbf{2}}$, is closed under Q-weighted limits; in other words, if $[\mathcal{D}, \mathcal{V}](Q, -): [\mathcal{D}, \mathcal{V}] \to \mathcal{V}$ sends a $\mathcal{V}$-natural transformation whose components are in $\mathcal{E}$ to a morphism in $\mathcal{E}$. If moreover the shrinkable morphisms are weak equivalences then the hypothesis in the theorem holds, and so U will preserve homotopy Q-weighted limits. This is the case for all cofibrant Q in the case of Examples 2.1, 2.2, and 2.3, but not in Example 2.4.

Example 7.15. If the enriched projective model structure on $[\mathcal{D}, \mathcal{V}]$ exists, as is the case for Example 2.4, then once again the hypothesis holds for all cofibrant Q. For a levelwise shrinkable morphism between levelwise fibrant objects is a weak equivalence between fibrant objects in the projective model structure, and so sent by $[\mathcal{D}, \mathcal{V}](Q, -)$ to a weak equivalence (between fibrant objects).

8. Accessibility and $\mathcal{E}$-weak colimits

In ordinary category theory an accessible category which is complete is also cocomplete — this provides one simple way to see that algebraic categories, which are obviously complete, are also cocomplete.

The present section is devoted to generalising this to our setting, which we do in Theorem 8.9. In Section 9 we shall use Theorem 8.9 to deduce homotopical cocompleteness of various enriched categories of (higher) categorical structures.

8.1. **$\mathcal{E}$-weak colimits.** The paper [25] introduced the notion of $\mathcal{E}$-weak colimit for any class of morphisms $\mathcal{E}$. We shall use this in our current setting where $\mathcal{E}$ consists of the shrinkable morphisms.

Definition 8.1. Let $W\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$ be a weight, and consider a diagram $S\colon \mathcal{D} \to \mathcal{A}$. A $\mathcal{V}$-natural transformation

$$W \xrightarrow{\eta} \mathcal{A}(S-, C)$$

in $[\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$ exhibits C as the $\mathcal{E}$-weak colimit of S weighted by W if the induced map

$$\mathcal{A}(C, A) \xrightarrow{\eta^* \circ \mathcal{A}(S-,1)} [\mathcal{D}^{\mathrm{op}}, \mathcal{V}](W, \mathcal{A}(S-, A)) \tag{8.1}$$

belongs to $\mathcal{E}$ for all $A \in \mathcal{A}$.

This is equally the assertion that η exhibits C as an $\mathcal{E}$-weak reflection of W along the $\mathcal{V}$-functor $\mathcal{A}(S-, 1)\colon \mathcal{A} \to [\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$ sending A to $\mathcal{A}(S-, A)$.

Example 8.2. For $\mathcal{V}$ with the trivial model structure, $\mathcal{E}$-weak colimits are weighted colimits in the usual sense.

Example 8.3. For $\mathcal{V}$ with the split model structure, $\eta\colon W \to \mathcal{A}(S-, C)$ exhibits C as the $\mathcal{E}$-weak colimit just when the induced map $\mathcal{A}(C, A) \to [\mathcal{D}^{\mathrm{op}}, \mathcal{V}](W, \mathcal{A}(S-, A))$ is a split epimorphism in $\mathcal{V}$. This implies in particular that given $f\colon W \to \mathcal{A}(S-, A)$ there exists $f'\colon C \to A$ such that the triangle

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "\eta"'] & \mathcal{A}(S-, A) \\ \mathcal{A}(S-, C) \arrow[ur, "\mathcal{A}(-, f')"'] & \end{tikzcd} \tag{8.2}$$

commutes. Indeed, when $\mathcal{V} = \mathbf{Set}$, it amounts to precisely this condition, and then if W is the terminal weight, the $\mathcal{E}$-weak colimit reduces to the ordinary weak (conical) colimit of S.

Example 8.4. In the **Cat** case, the map

$$\mathcal{A}(C,A) \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](W, \mathcal{A}(S-,A))$$

is asked to be a surjective equivalence. So given $f\colon W \to \mathcal{A}(S-,A)$ we have a factorisation as in (8.2) above; and furthermore, given $\alpha\colon f \Rightarrow g \in [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](W, \mathcal{A}(S-,A))$ there exists a unique $\alpha'\colon f' \Rightarrow g'$ such that $\mathcal{A}(S-,\alpha') \circ \eta = \alpha$. For instance, the $\mathcal{E}$-weak coequaliser

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "e"] & C \end{tikzcd}$$

satisfies $e \circ f = e \circ g$ and has the following properties:

(1) if $h\colon Y \to D$ satisfies $h \circ f = h \circ g$, there *exists* $h'\colon C \to D$ such that $h' \circ e = h$;
(2) if moreover $k\colon Y \to D$ satisfies $k \circ f = k \circ g$, and $\theta\colon h \Rightarrow k$ satisfies $\theta \circ f = \theta \circ g$, there exists a unique $\theta'\colon h' \Rightarrow k'$ such that $\theta' \circ e = \theta$.

Let us compare $\mathcal{E}$-weak colimits with the better known notion of bicolimits — given W and S as before, the W-weighted bicolimit $C = W *_b S$ [20] is defined by a *pseudonatural transformation* $\eta\colon W \to \mathcal{A}(S-,C)$ such that the induced map

$$\mathcal{A}(C,A) \to \mathrm{Ps}(\mathcal{D}^{\mathrm{op}}, \mathbf{Cat})(W, \mathcal{A}(S-,A))$$

is an equivalence of categories, where $\mathrm{Ps}(\mathcal{D}^{\mathrm{op}}, \mathbf{Cat})$ denotes the 2-category of 2-functors, pseudonatural transformations and modifications from $\mathcal{D}^{\mathrm{op}}$ to **Cat**. To make the comparison, recall from [5, Remark 3.15] that the identity on objects inclusion of $[\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}]$ in $\mathrm{Ps}(\mathcal{D}^{\mathrm{op}}, \mathbf{Cat})$ has a left adjoint $(-)'$ called the *pseudomorphism classifier*, with counit $q_W\colon W' \to W$. The morphism $q_W\colon W' \to W$ is in fact a cofibrant replacement of the weight W in the projective model structure on $[\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}]$: see [24, Sections 5.9 and 6]. Given the isomorphism

$$[\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](W', \mathcal{A}(S-,A)) \cong \mathrm{Ps}(\mathcal{D}^{\mathrm{op}}, \mathbf{Cat})(W, \mathcal{A}(S-,A)),$$

the bicolimit of S weighted by W equally amounts to a 2-natural transformation $\eta\colon W' \to \mathcal{A}(S-,C)$ for which the induced map

$$\mathcal{A}(C,A) \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Cat}](W', \mathcal{A}(S-,A))$$

is an *equivalence of categories* for all A. In particular, the $\mathcal{E}$-weak colimit of S weighted by W' is a W-weighted bicolimit of S, but satisfies the stronger condition that genuine factorisations, as in (8.2), exist. Thus any 2-category admitting $\mathcal{E}$-weak colimits also admits bicolimits (with a stronger universal property), but also admits some $\mathcal{E}$-weak colimits, such as $\mathcal{E}$-weak coequalisers, whose defining weights are not cofibrant, and so do not correspond to any bicolimit.

Example 8.5. In the **SSet**-case, the map

$$\mathcal{A}(C, A) \to [\mathcal{D}^{\mathrm{op}}, \mathbf{SSet}](W, \mathcal{A}(S-, A))$$

is a shrinkable morphism and so a weak equivalence (with respect to the Joyal model structure) with a section.

In the case that W is flexible and $\mathcal{A}$ is locally fibrant — as, for instance, if $\mathcal{A}$ is an ∞-cosmos — we can say rather more. For then, since $\mathcal{A}(S-, A)$ is pointwise fibrant in $[\mathcal{D}^{\mathrm{op}}, \mathbf{SSet}]$ and W is cofibrant, the hom $[\mathcal{D}^{\mathrm{op}}, \mathbf{SSet}](W, \mathcal{A}(S-, A))$ is also fibrant; that is, a quasicategory. It follows, by Example 3.8, that the shrinkable morphism $\mathcal{A}(C, A) \to [\mathcal{D}^{\mathrm{op}}, \mathbf{SSet}](W, \mathcal{A}(S-, A))$ is a *surjective equivalence of quasicategories.*

For $\mathcal{A}$ an ∞-cosmos and W a flexible weight, Riehl and Verity [33] define the *flexibly-weighted homotopy colimit* of S weighted by W as an object C together with a morphism $W \to \mathcal{A}(S-, C)$ for which the induced map $\mathcal{A}(C, A) \to [\mathcal{D}^{\mathrm{op}}, \mathbf{SSet}](W, \mathcal{A}(S-, A))$ is an equivalence of quasicategories for all A. In particular, if $\mathcal{A}$ admits $\mathcal{E}$-weak colimits it admits flexibly-weighted homotopy colimits with the stronger property that the comparison equivalence of quasicategories is in fact a surjective equivalence.

8.2. **Accessible $\mathcal{V}$-categories with enough cofibrantly-weighted limits have $\mathcal{E}$-weak colimits.** In the present section we suppose that (the unenriched category) $\mathcal{V}_0$ is locally presentable. It follows by [21, Proposition 2.4] that there is a regular cardinal λ_0 such that $\mathcal{V}_0$ is locally λ_0-presentable, the unit object I is λ_0-presentable, and the tensor product of two λ_0-presentable objects is λ_0-presentable. Moreover, the corresponding statements will remain true for any regular $\lambda \geq \lambda_0$.

Notation 8.6. We let λ_0 denote a fixed regular cardinal as in the previous paragraph. Whenever we consider λ-presentability or λ-accessibility in the $\mathcal{V}$-enriched context, we shall always suppose that λ is a regular cardinal and $\lambda \geq \lambda_0$.

If $\mathcal{H}$ is a small category we can speak of conical $\mathcal{H}$-shaped colimits in $\mathcal{C}$: these are $\mathcal{H}$-shaped colimits in $\mathcal{C}_0$ which are preserved by $\mathcal{C}(-, C)_0 \colon \mathcal{C}_0 \to \mathcal{V}_0^{\mathrm{op}}$ for each $C \in \mathcal{C}$. In particular we can speak of λ-filtered colimits in $\mathcal{C}$, corresponding to $\mathcal{H}$-shaped colimits for all λ-filtered categories $\mathcal{H}$.

An object A in $\mathcal{C}$ is said to be λ-*presentable* if $\mathcal{C}(A,-)\colon \mathcal{C} \to \mathcal{V}$ preserves λ-filtered colimits. We say that $\mathcal{C}$ is λ-*accessible*[2] if it admits λ-filtered colimits and a set $\mathcal{G}$ of λ-presentable objects such that each object of $\mathcal{C}$ is a λ-filtered colimit of objects in $\mathcal{G}$. It follows, arguing as usual, that the full subcategory of λ-presentable objects in $\mathcal{C}$ is essentially small and we denote by $J\colon \mathcal{C}_\lambda \to \mathcal{C}$ a small skeletal full subcategory of λ-presentables. It follows from [18, Theorem 5.19] that J is *dense*, and from [18, Theorem 5.29] that J then exhibits $\mathcal{C}$ as the free completion of $\mathcal{C}_\lambda$ under λ-filtered colimits. (Indeed, the λ-accessible $\mathcal{V}$-categories are equally the free completions of small $\mathcal{V}$-categories under λ-filtered colimits.)

Notation 8.7. Let $D\colon \mathcal{A} \to \mathcal{C}$ be a $\mathcal{V}$-functor with small domain. We write N_D for the induced $\mathcal{V}$-functor $\mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ sending an object $C \in \mathcal{C}$ to the presheaf $\mathcal{C}(D-,C)$ which in turn sends $A \in \mathcal{A}$ to $\mathcal{C}(DA,C)$. Other authors have written $\mathcal{C}(D,1)$ or $\tilde{D}$ for N_D; our notation is designed to remind that this is a generalised (N)erve.

Let $\mathcal{C}$ be a λ-accessible $\mathcal{V}$-category, and $J\colon \mathcal{C}_\lambda \to \mathcal{C}$ the inclusion of the full sub-$\mathcal{V}$-category of λ-presentable objects. Then J is dense, so that the associated functor $N_J\colon \mathcal{C} \to [\mathcal{C}_\lambda^{\mathrm{op}}, \mathcal{V}]$ is fully faithful.

Proposition 8.8. *If $\mathcal{C}$ is λ-accessible, with powers and enough cofibrantly-weighted limits, then $N_J\colon \mathcal{C} \to [\mathcal{C}_\lambda^{\mathrm{op}}, \mathcal{V}]$ admits an $\mathcal{E}$-weak left adjoint.*

Proof. The composite of N_J with the evaluation functor ev_X at a λ-presentable object X is the representable $\mathcal{C}(JX,-)$. Each such $\mathcal{V}$-functor preserves λ-filtered colimits; so, since the evaluation functors are jointly conservative, N_J also preserves λ-filtered colimits. Now $[\mathcal{C}_\lambda^{\mathrm{op}}, \mathcal{V}]$ is locally λ-presentable as a $\mathcal{V}$-category by [19, Examples 3.4]. In particular, since $(N_J)_0\colon \mathcal{C}_0 \to [\mathcal{C}_\lambda^{\mathrm{op}}, \mathcal{V}]_0$ is a λ-filtered colimit preserving functor between λ-accessible categories, it satisfies the solution set condition. The result now follows from our adjoint functor theorem, Theorem 7.8. $\square$

Theorem 8.9. *Let $\mathcal{C}$ be an accessible $\mathcal{V}$-category with powers and enough cofibrantly-weighted limits. Then $\mathcal{C}$ admits all $\mathcal{E}$-weak colimits.*

Proof. Given Proposition 8.8, this follows directly from Proposition 4.3 of [25], the argument of which we repeat here for convenience. Let $D\colon \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small. We must show that $N_D\colon \mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ has an $\mathcal{E}$-weak left adjoint; then the value of this adjoint at $W \in [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$

[2] A different notion of enriched accessibility was given in [6].

will give the $\mathcal{E}$-weak colimit of D weighted by W. First suppose that $\mathcal{C}$ is λ-accessible and consider the dense inclusion $J\colon \mathcal{C}_\lambda \to \mathcal{C}$. We then have the following diagram.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "D"] & \mathcal{C} \arrow[r, "N_J"] \arrow[dr, "N_D"'] & {[\mathcal{C}_\lambda^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "N_{(N_J D)} = [\mathcal{C}_\lambda^{\mathrm{op}}, \mathcal{V}](N_J D-, 1)"] \\
 & & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]}
\end{tikzcd}$$

Since N_J is fully faithful, there are natural isomorphisms

$$N_{N_J D} N_J C \cong [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](N_J D-, N_J C) \cong \mathcal{C}(D-, C) \cong N_D C$$

and so the triangle commutes up to isomorphism. Thus it will suffice to show that N_J and $N_{N_J D}$ have $\mathcal{E}$-weak left adjoints. The first of these does so by Proposition 8.8, while the second has a genuine left adjoint, namely the weighted colimit functor $N_J D \star -$. $\square$

We follow [25] in saying that W-weighted limits are *$\mathcal{E}$-stable* if the enriched full subcategory $\mathcal{E} \hookrightarrow \mathcal{V}^{\mathbf{2}}$ is closed under W-weighted limits. Similarly, we say that *enough cofibrantly-weighted limits are $\mathcal{E}$-stable* if $\mathcal{E}$ has, and the inclusion $\mathcal{E} \to \mathcal{V}^{\mathbf{2}}$ preserves, enough cofibrantly-weighted limits. In that case we can use results from [25] to prove a converse to the previous theorem.

Theorem 8.10. *Suppose that the shrinkable morphisms in $\mathcal{V}$ are closed in $\mathcal{V}^{\mathbf{2}}$ under enough cofibrantly-weighted limits, and let $\mathcal{C}$ be an accessible $\mathcal{V}$-category. Then the following are equivalent.*

(i) $\mathcal{C}$ admits $\mathcal{E}$-stable limits.
(ii) $\mathcal{C}$ admits powers and enough cofibrantly-weighted limits.
(iii) $\mathcal{C}$ admits $\mathcal{E}$-weak colimits.

Proof. Since shrinkable morphisms are always closed under powers, powers are $\mathcal{E}$-stable. Since also enough cofibrantly-weighted limits are $\mathcal{E}$-stable by assumption, ($1 \implies 2$). The implication ($2 \implies 3$) holds by Theorem 8.9. For ($3 \implies 1$), suppose that $\mathcal{C}$ is λ-accessible and consider the dense inclusion $J\colon \mathcal{C}_\lambda \to \mathcal{C}$. Then $N_J\colon \mathcal{C} \hookrightarrow [\mathcal{C}_\lambda^{\mathrm{op}}, \mathcal{V}]$ is fully faithful and has an $\mathcal{E}$-weak left adjoint sending W to the $\mathcal{E}$-weak colimit of J weighted by W. Furthermore, since each morphism of $\mathcal{E}$ is a split epimorphism, the representable $\mathcal{V}(I, -)\colon \mathcal{V}_0 \to \mathbf{Set}$ sends morphisms of $\mathcal{E}$ to surjections: in the language of [25] this says that I is *$\mathcal{E}$-projective*. Using Propositions 6.1 and 2.6 of [25] we deduce that $\mathcal{C}$ is closed in $[\mathcal{C}_\lambda^{\mathrm{op}}, \mathcal{V}]$ under $\mathcal{E}$-stable limits, as required. $\square$

8.3. **Recognising accessible enriched categories.** Examples of accessible $\mathcal{V}$-categories $\mathcal{C}$ are much easier to identify in the case that $\mathcal{C}$ admits powers by a suitable strong generator $\mathcal{G}$ of $\mathcal{V}_0$. As observed in [18, Section 3.8], if the $\mathcal{V}$-category $\mathcal{C}$ has powers, the difference between conical colimits in $\mathcal{C}$ and conical colimits in the underlying category $\mathcal{C}_0$ disappears. But since the map $\mathcal{C}(\operatorname{colim}_i D_i, C) \to \lim_i \mathcal{C}(D_i, C)$ expressing the universal property of a conical colimit will be invertible if and only if the induced $\mathcal{V}_0(G, \mathcal{C}(\operatorname{colim}_i D_i, C)) \to \mathcal{V}_0(G, \lim_i \mathcal{C}(D_i, C))$ is so for each $G \in \mathcal{G}$, it suffices for $\mathcal{C}$ to have powers by objects in $\mathcal{G}$. Building on this well-known argument, the following proposition enables us to recognise accessible $\mathcal{V}$-categories by looking at their underlying categories.

Recall that λ_0 satisfies the standing assumptions of Notation 8.6.

Proposition 8.11. *Let $\mathcal{V}_0$ be locally presentable, let $\mathcal{G}$ be a strong generator of $\mathcal{V}_0$, and let $\lambda \geq \lambda_0$ be a regular cardinal such that each $G \in \mathcal{G}$ is λ-presentable in $\mathcal{V}_0$. Then for any $\mathcal{V}$-category $\mathcal{C}$ with powers by objects in $\mathcal{G}$, the following are equivalent:*

(1) $\mathcal{C}$ is λ-accessible as a $\mathcal{V}$-category;
(2) $\mathcal{C}_0$ is λ-accessible and $\mathcal{C}(C, -)_0 \colon \mathcal{C}_0 \to \mathcal{V}_0$ is a λ-accessible functor, for each C in some strong generator of $\mathcal{C}_0$;
(3) $\mathcal{C}_0$ is λ-accessible and $(G \pitchfork -)_0 \colon \mathcal{C}_0 \to \mathcal{C}_0$ is a λ-accessible functor, for each $G \in \mathcal{G}$.

Moreover, for such λ and $\mathcal{C}$, an object $C \in \mathcal{C}$ is λ-presentable if and only if it is λ-presentable in $\mathcal{C}_0$.

Proof. Because of the powers, $\mathcal{C}$ has λ-filtered colimits if and only if $\mathcal{C}_0$ does so, and a class $\mathcal{H}$ of objects of $\mathcal{C}$ generates $\mathcal{C}$ under λ-filtered colimits if and only if it generates $\mathcal{C}_0$ under λ-filtered colimits. So the only possible difference between λ-accessibility of $\mathcal{C}$ and λ-accessibility of $\mathcal{C}_0$ lies in the possible difference between λ-presentability in $\mathcal{C}$ and λ-presentability in $\mathcal{C}_0$.

Since I is λ-presentable in $\mathcal{V}_0$, if C is λ-presentable in $\mathcal{C}$ then the composite

$$\mathcal{C}_0 \xrightarrow{\mathcal{C}(C,-)_0} \mathcal{V}_0 \xrightarrow{\mathcal{V}_0(I,-)} \mathbf{Set}$$

preserves λ-filtered colimits, but this composite is $\mathcal{C}_0(C, -)$, and so C is λ-presentable in $\mathcal{C}_0$. Thus $(\mathcal{C}_\lambda)_0 \subseteq (\mathcal{C}_0)_\lambda$; while if $\mathcal{C}$ is λ-accessible as a $\mathcal{V}$-category, then $\mathcal{C}_0$ is an accessible ordinary category, and (1) implies (2).

For an arbitrary $\mathcal{G}$-powered $\mathcal{V}$-category $\mathcal{C}$, however, a λ-presentable object of $\mathcal{C}_0$ might fail to be λ-presentable in $\mathcal{C}$.

For the remaining implications, consider the diagram

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "(G\pitchfork-)_0"] \arrow[d, "\mathcal{C}(C,-)_0"'] & \mathcal{C}_0 \arrow[d, "\mathcal{C}_0(C,-)"] \\ \mathcal{V}_0 \arrow[r, "\mathcal{V}_0(G,-)"'] & \mathbf{Set} \end{tikzcd} \tag{8.3}$$

in which $C \in \mathcal{C}$, $G \in \mathcal{G}$, and so the lower horizontal preserves λ-filtered colimits.

Suppose that (2) holds, so that there is a strong generator $\mathcal{H}$ for $\mathcal{C}_0$ consisting of objects which are λ-presentable in $\mathcal{C}$ (and so also in $\mathcal{C}_0$). As C ranges through $\mathcal{H}$, the lower composite $\mathcal{V}_0(G, \mathcal{C}(C,-)_0)$ in (8.3) preserves λ-filtered colimits, while the $\mathcal{C}_0(C,-)$ preserve and jointly reflect them. Thus $(G \pitchfork -)_0$ preserves them and (3) holds.

Now suppose that (3) holds. If C is λ-presentable in $\mathcal{C}_0$, then the upper composite $\mathcal{C}_0(C, G \pitchfork -)$ in (8.3) preserves λ-filtered colimits, while the $\mathcal{V}_0(G,-)$ preserve and jointly reflect them, thus $\mathcal{C}(C,-)_0$ also preserves them, and C is λ-presentable in $\mathcal{C}$. Thus $(\mathcal{C}_0)_\lambda \subseteq (\mathcal{C}_\lambda)_0$ and (1) follows. □

In practice, it is often convenient to lift enriched accessibility from one (typically locally presentable) $\mathcal{V}$-category to another using the following.

Corollary 8.12. *Let $\mathcal{A}$ and $\mathcal{C}$ be $\mathcal{V}$-categories with powers by objects in $\mathcal{G}$ and let $U\colon \mathcal{A} \to \mathcal{C}$ be a* conservative *$\mathcal{V}$-functor preserving them. Suppose that $\mathcal{C}$ is an accessible $\mathcal{V}$-category, $\mathcal{A}_0$ is an accessible category, and $U_0\colon \mathcal{A}_0 \to \mathcal{C}_0$ is an accessible functor. Then in fact $\mathcal{A}$ is accessible as a $\mathcal{V}$-category.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} \mathcal{A}_0 \arrow[r, "(G\pitchfork-)_0"] \arrow[d, "U_0"'] & \mathcal{A}_0 \arrow[d, "U_0"] \\ \mathcal{C}_0 \arrow[r, "(G\pitchfork-)_0"'] & \mathcal{C}_0 \end{tikzcd}$$

where $G \in \mathcal{G}$.

The categories $\mathcal{A}_0$ and $\mathcal{C}_0$ are both accessible, U_0 is accessible by assumption, and the lower horizontal by Proposition 8.11 and the fact that $\mathcal{C}$ is an accessible $\mathcal{V}$-category. By the uniformization theorem [1, Theorem 2.19] for accessible categories, we can choose $\lambda \geq \lambda_0$

such that each of the aforementioned categories and functors is λ-accessible. Since U_0 is conservative, the upper horizontal is also λ-accessible. Therefore $\mathcal{A}$ is a λ-accessible $\mathcal{V}$-category by Proposition 8.11 once again. $\square$

9. Examples and applications

To summarise, we have two main results. First, our weak adjoint functor theorem, and second, our result on the $\mathcal{E}$-weak cocompleteness of accessible $\mathcal{V}$-categories with sufficient limits. In this final section we illustrate the scope of these results by describing what they capture in our various settings, with a particular emphasis on 2-categories and ∞-cosmoi.

9.1. **The classical case.** In the case of **Set** with the trivial model structure, a category $\mathcal{A}$ has powers and enough cofibrantly-weighted limits just when it is complete. The shrinkable morphisms in **Set** are the bijections, and these are of course closed in $\mathbf{Set}^{\mathbf{2}}$ under (all) limits. Therefore our weak adjoint functor theorem specialises exactly to the general adjoint functor theorem of Freyd [27]. Theorem 8.10 becomes the well-known result that an accessible category is complete if and only if it is cocomplete: see for example [1, Corollary 2.47]. Since preservation of homotopy limits is just preservation of limits in this case, Theorem 7.13 is just the elementary fact that right adjoints preserve limits.

For general $\mathcal{V}$ equipped with the trivial model structure, a $\mathcal{V}$-category $\mathcal{A}$ has powers and enough cofibrantly-weighted limits just when it is complete, now in the sense of weighted limits, and the shrinkable morphisms are again the isomorphisms. Therefore our weak adjoint functor theorem specialises exactly to the adjoint functor theorem for enriched categories. Theorem 7.13 says that right adjoints preserve limits, and Theorem 8.10 yields the well-known result that an accessible $\mathcal{V}$-category is complete if and only if it is cocomplete.

9.2. **Ordinary weakness.** In the case of **Set** with the split model structure we know from Example 5.6 that for a category $\mathcal{A}$ to have enough cofibrantly-weighted limits it suffices that it admit products. Furthermore the shrinkable morphisms are the surjections, and these are closed under products. Since powers are products in the **Set**-enriched setting our weak adjoint functor theorem thus yields the weak adjoint functor theorem of Kainen [16]. Theorem 8.10 yields the well known result that an accessible category has products if and only if it has weak colimits: see for example [1, Theorem 4.11].

For general $\mathcal{V}$ equipped with the split model structure, our results appear to be completely new. In this setting a $\mathcal{V}$-category $\mathcal{A}$ has powers and enough cofibrantly-weighted limits provided that it has powers and products, and the shrinkable morphisms are the split epimorphisms, which are of course closed under powers and products. Using this, our main results (in slightly weakened form) are:

- Let $\mathcal{B}$ be a $\mathcal{V}$-category with products and powers, and let $U\colon \mathcal{B} \to \mathcal{A}$ be a $\mathcal{V}$-functor that preserves them. Then U has a (split epi)-weak left adjoint if and only if it satisfies the solution set condition.
- An accessible $\mathcal{V}$-category has products and powers if and only if it has (split epi)-weak colimits.

9.3. **2-category theory.** One of our guiding topics is that of 2-category theory, understood as **Cat**-enriched category theory, and where **Cat** is equipped with the canonical model structure.

As recalled in Example 5.7, in this case the cofibrantly-weighted limits are precisely the flexible limits of [4], and any 2-category with flexible limits has powers. The shrinkable morphisms are the retract equivalences, and these are closed in $\mathbf{Cat}^\mathbf{2}$ under cofibrantly-weighted limits as observed, for example, in [25, Section 9].

Our primary applications will be in the accessible setting which we describe now. Since each accessible functor between accessible categories satisfies the solution set condition, our weak adjoint functor in this setting gives the first part of the following, the second part of which is the instantiation of (part of) Theorem 8.10.

- Let $U\colon \mathcal{B} \to \mathcal{A}$ be an accessible 2-functor between accessible 2-categories. If $\mathcal{B}$ has flexible limits and U preserves them then it has an $\mathcal{E}$-weak left adjoint.
- An accessible 2-category has flexible limits if and only if it has $\mathcal{E}$-weak colimits.

The first result is new; the second is part of Theorem 9.4 of [25]. The utility of these results lies in the fact that many, though not all, 2-categories of pseudomorphisms are in fact accessible with flexible limits. For instance, the 2-category of monoidal categories and strong monoidal functors is accessible (although, as recalled in Section 5, the 2-category of *strict* monoidal categories and strong monoidal functors is not [8, Section 6.2]). The difference between the two cases is that the definition of strict monoidal category involves equations between objects whereas that of monoidal category does not — in terms of the associated 2-monads this corresponds to the fact that the 2-monad for strict monoidal categories is not cofibrant (=flexible) whereas that for monoidal categories is. One result generalising this is Corollary 7.3 of [8], which asserts that if T is a finitary flexible 2-monad on **Cat** then the 2-category T-Alg_p of strict algebras and pseudomorphisms is accessible with flexible limits and filtered colimits, and these are preserved by the forgetful functor to **Cat**. It now follows from our theorem that any such 2-category admits $\mathcal{E}$-weak colimits, and in particular bicolimits, and also that if $f\colon S \to T$ is a morphism of such 2-monads, then the induced map T-$\mathrm{Alg}_p \to$ S-Alg_p has an $\mathcal{E}$-weak left adjoint, and so a left biadjoint. These results concerning bicolimits and biadjoints are not new, being special cases of the results of Section 5 of [5].

However, as shown in [8, Section 6.5], many structures beyond the scope of two-dimensional monad theory are also accessible. For instance, the 2-category of small regular categories and regular functors is accessible with flexible limits and filtered colimits preserved by the forgetful 2-functor to **Cat**. Similar results hold for Barr-exact categories, coherent categories, distributive categories, and so on. It follows from the above theorems that the 2-categories of such structures admit $\mathcal{E}$-weak colimits, and so bicolimits, and this is the simplest proof of bicocompleteness in these examples that we know. Again, for such structures the weak adjoint functor theorem provides an easy technique for constructing $\mathcal{E}$-weak left adjoints, and so biadjoints, to forgetful 2-functors between them.

Theorem 7.13 asserts that a 2-functor with an $\mathcal{E}$-weak left adjoint preserves bilimits — since such 2-functors are right biadjoints, this is a special case of the well known fact that right biadjoints preserve bilimits.

9.4. **Simplicial enrichment and accessible ∞-cosmoi.** Our second main motivation concerns the ∞-cosmoi of Riehl and Verity [35]. Here the base for enrichment is **SSet** equipped with the Joyal model structure. In this setting, for a simplicially enriched category $\mathcal{B}$ to admit powers and enough cofibrantly-weighted limits it suffices, as observed in Example 5.8, that it admit flexible limits in the sense of [35]. As observed in Example 3.8, every shrinkable morphism is a weak equivalence which has a section.

In this example it can be shown that the shrinkable morphisms are *not* closed in $\mathbf{SSet^2}$ under cofibrantly-weighted limits, so we have to content ourselves with Theorem 8.9 rather than Theorem 8.10. On the other hand, enriched projective model structures do exist, so Theorem 7.13 does imply that $\mathcal{E}$-weak right adjoints preserve cofibrantly-weighted homotopy limits.

Specialising our main results, we then obtain:

- Let $\mathcal{B}$ be a **SSet**-category with flexible limits, and let $U : \mathcal{B} \to \mathcal{A}$ be a $\mathcal{V}$-functor which preserves them. Then U has an $\mathcal{E}$-weak left adjoint provided that it satisfies the solution set condition.
- If $\mathcal{A}$ is an accessible **SSet**-category with flexible limits then it has $\mathcal{E}$-weak colimits.

We wish to examine these results further in the setting of ∞-cosmoi. By definition, an ∞-cosmos is a simplicially enriched category $\mathcal{A}$ equipped with a class of morphisms called isofibrations satisfying a number of axioms — see Chapter 1 of [35]. For the purposes of the present paper, the reader need only know that the homs of an ∞-cosmos are quasicategories and that it admits flexible limits, in the sense of Examples 5.8.

The idea is that the objects of an ∞-cosmos are ∞-categories, broadly interpreted, and that the axioms for an ∞-cosmos provide what is needed to define and work with key structures — such as limits and adjoints — that arise in the ∞-categorical world. For instance one can define what it means for an object of an ∞-cosmos to have limits. This is model-independent, in the sense that one recovers the usual notion of quasicategory with limits or complete Segal space with limits on choosing the appropriate ∞-cosmos.

The natural structure preserving morphisms between ∞-cosmoi are called *cosmological functors*, and cosmological functors preserve flexible limits. By an *accessible ∞-cosmos*, we mean an ∞-cosmos which is accessible as a simplicial category[3], whilst a cosmological functor is accessible if it is so as a simplicial functor.

Our main insight here is that many of the ∞-cosmoi that arise in practice are accessible, as are the cosmological functors between them. Thus the following specialisations of our main results are broadly applicable. We remind the reader that in this context the relevant morphisms of $\mathcal{E}$ are certain surjective equivalences of quasicategories — see Examples 3.8.

- Let $U : \mathcal{B} \to \mathcal{A}$ be an accessible cosmological functor between accessible ∞-cosmoi. Then U has an $\mathcal{E}$-weak left adjoint.
- If $\mathcal{A}$ is an accessible ∞-cosmos then it has $\mathcal{E}$-weak colimits, and in particular flexibly-weighted homotopy colimits (see Example 8.5.)

The first examples of accessible ∞-cosmoi come from the following result, the first part of which is due to Riehl and Verity [35].

[3]There are further conditions that one could ask for, such as the accessibility of the class of isofibrations, but we shall not consider these in the present paper.

Proposition 9.1. *(1) Let $\mathcal{A}$ be a model category that is enriched over the Joyal model structure on simplicial sets, and in which every fibrant object is cofibrant. Then the simplicial subcategory $\mathcal{A}_{\mathrm{fib}}$ spanned by its fibrant objects is an ∞-cosmos, and is closed in $\mathcal{A}$ under flexible limits.*

(2) If $\mathcal{A}$ is furthermore a combinatorial *model category then $\mathcal{A}_{\mathrm{fib}}$ is an* accessible *∞-cosmos and the inclusion $\mathcal{A}_{\mathrm{fib}} \hookrightarrow \mathcal{A}$ is an accessible embedding.*

Proof. Part (1) holds by Proposition E.1.1 of [35]. In Part (2), $\mathcal{A}_0$ is locally presentable and $\mathcal{A}$ is cocomplete as a $\mathcal{V}$-category by assumption. Since for each $X \in \mathbf{SSet}$ the functor $(X \pitchfork -)_0 \colon \mathcal{A}_0 \to \mathcal{A}_0$ has left adjoint $(X.-)_0$ given by taking copowers (or tensors) it is accessible. Therefore $\mathcal{A}$ is accessible as a $\mathbf{SSet}$-category by Proposition 8.11. Since $\mathcal{A}_{\mathrm{fib}}$ is closed in $\mathcal{A}$ under flexible limits, it is closed under powers. Since $\mathcal{A}$ is combinatorial, $(\mathcal{A}_{\mathrm{fib}})_0 = \mathrm{Inj}(J) \hookrightarrow \mathcal{A}_0$ where J is the set of generating trivial cofibrations in $\mathcal{A}$. Now by Theorem 4.8 of [1] $\mathrm{Inj}(J)$ is accessible and accessibly embedded, so by Corollary 8.12 we conclude that $\mathcal{A}_{\mathrm{fib}}$ is accessible and therefore an accessible ∞-cosmos. $\square$

As explained in Appendix E of [35], one obtains as instances of the above the ∞-cosmoi of quasicategories, complete Segal spaces, Segal categories and Θ_n-spaces. Since all of the defining model structures are combinatorial, it follows furthermore that each of these ∞-cosmoi is accessible.

The above ∞-cosmoi can be regarded as basic. In future work, we aim to vastly extend the scope of these examples by establishing that accessible ∞-cosmoi (satisfying natural additional properties) are stable under a variety of constructions, such as the passage from an ∞-cosmos $\mathcal{A}$ to the ∞-cosmos $\mathbf{Cart}(\mathcal{A})$ of cartesian fibrations therein. For (not necessarily accessible) ∞-cosmoi this was done in Chapter 6 of [35], and it remains to add accessibility to the mix. (The corresponding stability results for the appropriate class of accessible 2-categories have been established in [8].)

In the present paper we content ourselves with describing two examples of accessible ∞-cosmoi built from the quasicategories example: the ∞-cosmoi of quasicategories with a class of limits, and the ∞-cosmos of cartesian fibrations of quasicategories. In order to construct these ∞-cosmoi systematically, we shall make use of the generalised sketch categories $C|\mathcal{C}$ of Makkai [29], which we now recall.

Given a category $\mathcal{C}$ and object $C \in \mathcal{C}$ we form the category $C|\mathcal{C}$, an object of which consists of an object $A \in \mathcal{C}$ together with a subset $A_C \subseteq \mathcal{C}(C, A)$, which we often refer to as a *marking*. A morphism

$f\colon (A, A_C) \to (B, B_C)$ consists of a morphism $f\colon A \to B$ in $\mathcal{C}$ such that if $x \in A_C$ then $f \circ x \in B_C$. The forgetful functor $C|\mathcal{C} \to \mathcal{C}$ functor has a left adjoint equipping A with the empty marking $(A, \varnothing)$.

The following straightforward result is established in Item (9) of Section (1) of [29], except that Makkai works with locally *finitely* presentable categories. In its present form, it is a special case of Proposition 2.2 of [13].

Proposition 9.2. *Let $\mathcal{C}$ be locally presentable and $C \in \mathcal{C}$. Then $C|\mathcal{C}$ is also locally presentable and the forgetful functor $C|\mathcal{C} \to \mathcal{C}$ preserves filtered colimits.*

If $\mathcal{J}$ is a set of morphisms in $C|\mathcal{C}$ then we may form the category $\mathrm{Inj}(\mathcal{J})$ of $\mathcal{J}$-injectives in $C|\mathcal{C}$. We follow Makkai's terminology in calling such a category $\mathrm{Inj}(\mathcal{J})$ a *doctrine* in $\mathcal{C}$.

Corollary 9.3. *Let $\mathcal{C}$ be locally presentable, with $C \in \mathcal{C}$ and $\mathcal{J}$ a set of morphisms in $C|\mathcal{C}$. Then $\mathrm{Inj}(\mathcal{J})$ is accessible, and the composite forgetful functor $U\colon \mathrm{Inj}(\mathcal{J}) \to \mathcal{C}$ is accessible.*

Proof. By the preceding proposition $C|\mathcal{C}$ is locally presentable whence, by Theorem 4.8 of [1], the full subcategory $\mathrm{Inj}(\mathcal{J}) \hookrightarrow C|\mathcal{C}$ is accessible and accessibly embedded. The composite $\mathrm{Inj}(\mathcal{J}) \to C|\mathcal{C} \to \mathcal{C}$ is accessible since its two components are. $\square$

We now combine this with simplicial enrichment.

Corollary 9.4. *Let $\mathcal{A}$ be an ∞-cosmos and $U\colon \mathcal{A} \to \mathcal{C}$ a conservative simplicially enriched functor preserving flexible limits to a locally presentable simplicially enriched category. Suppose that $U_0\colon \mathcal{A}_0 \to \mathcal{C}_0$ has the form $\mathrm{Inj}(\mathcal{J}) \to C|\mathcal{C}_0 \to \mathcal{C}_0$ for some $C \in \mathcal{C}$ and set $\mathcal{J}$ of morphisms in $C|\mathcal{C}_0$. Then $\mathcal{A}$ is accessible as a simplicial category, and so is an accessible ∞-cosmos.*

Proof. Since flexible limits include powers, this follows immediately from Corollaries 8.12 and 9.3. $\square$

In the following two examples, we consider the ∞-cosmoi $\mathbf{QCat}_D$ of quasicategories with D-limits, and $\mathbf{Cart}(\mathbf{QCat})$ of cartesian fibrations respectively.

These are shown to be ∞-cosmoi in Propositions 6.4.11 and 6.4.12, respectively, of [35]; moreover the forgetful simplicial functors $\mathbf{QCat}_D \to \mathbf{SSet}$ and $\mathbf{Cart}(\mathbf{QCat}) \to \mathbf{SSet}^{\mathbf{2}}$ preserve flexible limits and are conservative. To prove that they are accessible ∞-cosmoi, it suffices by Corollary 9.4 to describe the categories $(\mathbf{QCat}_D)_0$ and $\mathbf{Cart}(\mathbf{QCat})_0$ as doctrines, respectively, in $\mathbf{SSet}_0$ and in $(\mathbf{SSet}^{\mathbf{2}})_0$.

9.4.1. *Quasicategories with limits of a given class.* To begin with, we treat the simpler case of quasicategories with a terminal object. To this end, consider $\Delta^0|\mathbf{SSet}$, whose objects are pairs (X, U) where X is a simplicial set and $U \subseteq X_0$ a distinguished subset of marked 0-simplices, and whose morphisms are simplicial maps preserving marked 0-simplices. We shall describe the category $\mathbf{QCat}_T$ of quasicategories admitting a terminal object and morphisms preserving them as the small injectivity class in $\Delta^0|\mathbf{SSet}$ consisting of those (X, U) for which X is a quasicategory and $U \subseteq X_0$ is the set of terminal objects in X. To begin with, we view the inner horn inclusions

$$\{\Lambda_k[n] \colon \Lambda^n_k \to \Delta^n \mid 0 < k < n\} \tag{9.1}$$

as morphisms of $\Delta^0|\mathbf{SSet}$ in which no 0-simplex is marked in their source or target — then (X, U) is injective with respect to the inner horns just when X is a quasicategory. Now recall (see Definition 4.1 of [14] for the dual case of an initial object), that a 0-simplex a of a quasicategory X is *terminal* if the solid part of each diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d, "j_n"'] & X \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd} \tag{9.2}$$

in which $f \colon \partial\Delta^n \to X$ has value a at the final vertex $n \in \partial\Delta^n$ admits a filler. Accordingly, if (X, U) is a quasicategory with marked 0-simplices, then *each marked* 0*-simplex is terminal* precisely if the solid part of each diagram

$$\begin{tikzcd} (\partial\Delta^n, \{n\}) \arrow[r] \arrow[d, "j_n"'] & (X, U) \\ (\Delta^n, \{n\}) \arrow[ur, dotted] & \end{tikzcd} \tag{9.3}$$

in $\Delta^0|\mathbf{SSet}$ admits a filler. The existence of a terminal object can then be expressed via injectivity of (X, U) with respect to the morphism

$$(\varnothing, \varnothing) \to (\Delta^0, \{0\})$$

where the unique 0-simplex 0 of Δ^0 is marked. Combining this morphism with the inner horn inclusions (9.1) and the marked boundary inclusions (9.3) an injective object (X, U) consists of a quasicategory X together with a non-empty subset of terminal objects in X. If we stopped here, the evident full inclusion $\mathbf{QCat}_T \to \Delta^0|\mathbf{SSet}$ would not however be essentially surjective on objects; for this we require our injectives (X, U) to have the property that U consists of *all* terminal objects in X. Since each terminal object is equivalent to any other, it therefore suffices to add the repleteness condition that each object equivalent to one in U belongs to U. To this end, we consider the nerve J of the free isomorphism which has two 0-simplices 0 and 1. The repleteness condition is captured by the morphism

$$(J, \{0\}) \xrightarrow{\mathrm{id}} (J, \{0,1\})$$

whose underlying simplicial map is the identity.

It is straightforward to extend this example to quasicategories with limits of shape D for D a general simplicial set. To see this, recall the join $\star\colon \mathbf{SSet} \times \mathbf{SSet} \to \mathbf{SSet}$ of simplicial sets. As explained in [14], there is a natural inclusion $D \hookrightarrow A \star D$, whereby for fixed D, we obtain a functor $- \star D\colon \mathbf{SSet} \to D/\mathbf{SSet}$ with this value at A. This has a right adjoint $D/\mathbf{SSet} \to \mathbf{SSet}$ which sends $t\colon D \to X$ to a simplicial set X/t. Now the limit of t is defined to be a terminal object $\Delta^0 \to X/t$. By adjointness, this amounts to a morphism

$$\Delta^0 \star D \to X$$

extending t along the inclusion $j\colon D \to \Delta^0 \star D$, which should be thought of as cones over t, satisfying lifting properties obtained, by adjointness, from those of (9.2).

We work this time with the locally finitely presentable category $(\Delta^0 \star D)|\mathbf{SSet}$, whose objects (X, U) are simplicial sets X equipped with a set U of marked cones $\Delta^0 \star D \to X$, and whose morphisms are simplicial maps preserving marked cones. There is a full embedding $(\mathbf{QCat}_D) \hookrightarrow (\Delta^0 \star D)|\mathbf{SSet}$ sending a quasicategory with limits of shape D to its underlying simplicial set with all limit cones marked. Now, given $(X, U) \in (\Delta^0 \star D)|\mathbf{SSet}$, the condition that a cone of U is a limit cone amounts to asking that each

$$\begin{tikzcd} (\partial\Delta^n \star D, \{n \star D\}) \arrow[r] \arrow[d, "j_n \star D"'] & (X, U) \\ (\Delta^n \star D, \{n \star D\}) \arrow[ur, dotted] & \end{tikzcd} \tag{9.4}$$

has a filler. Similarly, injectivity with respect to the inclusion

$$j\colon (D, \varnothing) \to (\Delta^0 \star D, \{\mathrm{id}\})$$

asserts that each diagram $t\colon D \to X$ admits a limit, with limit cone in U. The repleteness condition capturing the fact that each limit cone belongs to U is expressed by injectivity with respect to

$$(J \star D, \{0 \star D\}) \xrightarrow{\mathrm{id}} (J \star D, \{0 \star D, 1 \star D\}) \ ,$$

while finally we equip the inner horns

$$\{\Lambda_k[n] \colon \Lambda_k^n \to \Delta^n \colon 0 < k < n\}$$

with the empty markings to encode the fact that the X in (X, U) is a quasicategory.

9.4.2. *Cartesian fibrations.* An inner fibration $p \colon X \to Y$ of simplicial sets is a morphism having the right lifting property with respect to the inner horn inclusions. Such an inner fibration is said to be a *cartesian fibration* if each 1-simplex $f \colon x \to py$ in Y admits a lifting along p to a *p-cartesian* 1-simplex $f' \colon x' \to y \in X$, where a morphism $g \colon a \to b \in X$ is said to be p-cartesian if each diagram

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[rr, bend left, "g"] \arrow[dr, "i"'] & \Lambda_n^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where i includes the vertices of Δ^1 as the vertices $n-1, n \in \Delta^n$, admits a filler as depicted. A commutative square is said to be a morphism of cartesian fibrations if it preserves cartesian 1-simplices. As such, we obtain a category $\mathbf{Cart}(\mathbf{SSet})$ of cartesian fibrations.

Let $\mathbf{SSet}^\mathbf{2}$ denote the arrow category. Then an object of $(\mathrm{id} \colon \Delta^1 \to \Delta^1)|\mathbf{SSet}^\mathbf{2}$ is a morphism $p \colon X \to Y$ of simplicial sets together with a subset $U \subseteq X_1$ of marked 1-simplices, whilst a morphism is a commutative square whose domain component preserves marked 1-simplices; we write $p \colon (X, U) \to Y$ for such an object. We wish to describe $\mathbf{Cart}(\mathbf{SSet})$ as a small injectivity class in $\mathrm{id}_{\Delta^1}|\mathbf{SSet}^\mathbf{2}$; that is, we shall describe a set of morphisms $\mathcal{J}$ such that $p \colon (X, U) \to Y$ is $\mathcal{J}$-injective if and only if p is a cartesian fibration with U the set of all p-cartesian 1-simplices. Note that p is an inner fibration just when it is injective in $\mathbf{SSet}^\mathbf{2}$ with respect to each square

$$\begin{tikzcd} \Lambda_k^n \arrow[r] \arrow[d] & \Delta^n \arrow[d, "1"] \\ \Delta^n \arrow[r, "1"'] & \Delta^n \end{tikzcd}$$

with $0 < k < n$. (Here we are employing the trick (see, for instance, Lemma 1 of [7]) that g has the right lifting property with respect to f if and only if g is injective in the arrow category with respect to the map $f \to 1_{\mathrm{cod}(f)}$ determined by f and $1_{\mathrm{cod}(f)}$). Thus equipping the above squares with the empty markings captures as injectives those

$p\colon (X,U) \to Y$ with p an inner fibration. To express the requirement that the elements of U be cartesian 1-simplices we use injectivity with respect to the squares

$$\begin{tikzcd}
(\Lambda^n_n, n-1 \to n) \arrow[r] \arrow[d] & (\Delta^n, n-1 \to n) \arrow[d, "1"] \\
\Delta^n \arrow[r, "1"'] & \Delta^n
\end{tikzcd}$$

whose sources have the unique 1-simplex $n-1 \to n$ marked. The existence of cartesian liftings is expressed by injectivity against

$$\begin{tikzcd}
(\Delta^0, \varnothing) \arrow[r, "d_0"] \arrow[d, "d_0"] & (\Delta^1, 0 \to 1) \arrow[d, "1"] \\
\Delta^1 \arrow[r, "1"'] & \Delta^1.
\end{tikzcd}$$

In order to ensure that each cartesian 1-simplex belong to U we add a repleteness condition based on the following lemma.

Lemma 9.5. *Let $p\colon X \to Y$ be a cartesian fibration. Let $f : a \to b$ and $g\colon c \to b$ be 1-simplices in X, with g a cartesian lifting of $(pf : pa \to pb, b)$ so that we obtain a 2-simplex*

$$\begin{tikzcd}
a \arrow[r, "\alpha"] \arrow[dr, "f"'] & c \arrow[d, "g"] \\
& b
\end{tikzcd}$$

where $p\alpha\colon pa \to pa$ is degenerate. Then $f : a \to b$ is cartesian if and only if α is an equivalence.

Proof. The degeneracy $p\alpha$ is an equivalence. Thus, by Lemma 2.4.1.5 of [26], α is an equivalence if and only it is cartesian. And now since g is cartesian, by Proposition 2.4.1.7 of [26], α is cartesian if and only if f is. □

Given this, consider the pushout below left

$$\begin{tikzcd}
\Delta^1 \arrow[r, "d_2"] \arrow[d] & \Delta^2 \arrow[d] \\
J \arrow[r] & (\Delta^2)_{0\cong 1}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
0 \arrow[r, "\simeq"] \arrow[dr] & 1 \arrow[d] \\
& 2
\end{tikzcd}$$

which is the generic 2-simplex with $0 \to 1$ an equivalence. There is a unique morphism $(\Delta^2)_{0\cong 1} \to \Delta^1$ sending the equivalence $0 \to 1$ to the degeneracy on 0, and 2 to 1. Then injectivity of $p\colon (X,U) \to Y$ against the square below left

$$\begin{tikzcd}
((\Delta^2)_{0\cong 1}, 1 \to 2) \arrow[r] \arrow[d] & ((\Delta^2)_{0\cong 1}, \{1 \to 2, 0 \to 2\}) \arrow[d, "1"] \\
\Delta^1 \arrow[r, "1"'] & \Delta^1
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\alpha"] \arrow[dr, "f"'] & c \arrow[d, "g"] \\
& b
\end{tikzcd}$$

asserts: for any 2-simplex in X as on the right above in which $g \in U$ and α is an equivalence sent by p to a degeneracy, the 1-simplex f is in U. Since each morphism of U is cartesian this implies, by Lemma 9.5, that all cartesian morphisms belong to U. In this way, we obtain the category of cartesian fibrations $\mathbf{Cart}(\mathbf{SSet}) \hookrightarrow (\mathrm{id}\colon \Delta^1 \to \Delta^1)|\mathbf{SSet^2}$ as the full subcategory of injectives.

Finally, to encode the full subcategory $\mathbf{Cart}(\mathbf{QCat}) \hookrightarrow \mathbf{Cart}(\mathbf{SSet})$ of cartesian fibrations between *quasicategories* we need to further encode that the target Y of $p\colon (X,U) \to Y$ is a quasicategory — of course it then follows that X is also a quasicategory, since p is an inner fibration. To this end, we add the injectivity condition

$$\begin{tikzcd}
(\varnothing, \varnothing) \arrow[r, "1"] \arrow[d] & (\varnothing, \varnothing) \arrow[d, "p"] \\
\Lambda^k_n \arrow[r] & \Delta^n
\end{tikzcd}$$

for each inner horn.

References

[1] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[2] J. Adámek and J. Rosický. Weakly locally presentable categories. *Cahiers Topologie Géom. Différentielle Catég.*, 35(3):179–186, 1994.

[3] Gregory Bird. *Limits in 2-categories of locally-presented categories.* PhD thesis, University of Sydney, 1984.

[4] G. J. Bird, G. M. Kelly, A. J. Power, and R. H. Street. Flexible limits for 2-categories. *J. Pure Appl. Algebra*, 61(1):1–27, 1989.

[5] R. Blackwell, G. M. Kelly, and A. J. Power. Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59(1):1–41, 1989.

[6] F. Borceux and C. Quinteriro. Enriched accessible categories. *Bull. Austral. Math. Soc.*, 54(3):489–501, 1996.

[7] John Bourke. Equipping weak equivalences with algebraic structure. *Math. Z.*, 294(3-4):995–1019, 2020.

[8] John Bourke. Accessible aspects of 2-category theory, *arXiv e-prints*, page arXiv:2003.06375, Mar 2020.

[9] Albrecht Dold. Partitions of unity in the theory of fibrations. *Ann. of Math. (2)*, 78:223–255, 1963.

[10] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Springer-Verlag, Berlin, 1971.

[11] R. Gordon, A. J. Power, and R. Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995.

[12] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[13] Valery Isaev. Model category of marked objects. *arXiv e-prints*, page arXiv:1610.08459, Oct 2016.

[14] André Joyal. Quasi-categories and Kan complexes. *J. Pure Appl. Algebra*, 175(1-3):207–222, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[15] André Joyal. The theory of quasi-categories and its applications. Quadern 45-2, CRM Barcelona, 2008.

[16] Paul C. Kainen. Weak adjoint functors. *Math. Z.*, 122:1–9, 1971.

[17] G. M. Kelly. Doctrinal adjunction. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 257–280. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[18] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137 pp. (electronic), 2005. Originally published as LMS Lecture Notes 64, 1982.

[19] G. M. Kelly. Structures defined by finite limits in the enriched context. I. *Cahiers Topologie Géom. Différentielle*, 23(1):3–42, 1982.

[20] G. M. Kelly. Elementary observations on 2-categorical limits. *Bull. Austral. Math. Soc.*, 39(2):301–317, 1989.

[21] G. M. Kelly and Stephen Lack. $\mathcal{V}$-Cat is locally presentable or locally bounded if $\mathcal{V}$ is so. *Theory Appl. Categ.*, 8:555–575, 2001.

[22] Stephen Lack. A Quillen model structure for 2-categories. *K-Theory*, 26(2):171–205, 2002.

[23] Stephen Lack. A Quillen model structure for bicategories. *K-Theory*, 33(3):185–197, 2004.

[24] Stephen Lack. Homotopy-theoretic aspects of 2-monads. *J. Homotopy Relat. Struct.*, 2(2):229–260, 2007.

[25] Stephen Lack and Jiří Rosický. Enriched weakness. *J. Pure Appl. Algebra*, 216(8-9):1807–1822, 2012.

[26] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[27] Saunders Mac Lane. *Categories for the working mathematician*. Springer-Verlag, New York, 1971.

[28] M. Makkai and R. Paré. *Accessible categories: the foundations of categorical model theory*, volume 104 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.

[29] Michael Makkai. Generalized sketches as a framework for completeness theorems. I. *J. Pure Appl. Algebra*, 115(1):49–79, 1997.

[30] J. Power and E. Robinson. A characterization of pie limits. *Math. Proc. Cambridge Philos. Soc.*, 110(1):33–47, 1991.

[31] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.
[32] E. Riehl and D. Verity. The 2-category theory of quasi-categories. *Adv. Math.*, 280:549–642, 2015.
[33] E. Riehl and D. Verity. On the construction of limits and colimits in ∞-categories. *arXiv e-prints*, page arXiv:1808.09835, Aug 2018.
[34] E. Riehl and D. Verity. Fibrations and Yoneda's lemma in an ∞-cosmos. *J. Pure Appl. Algebra*, 221(3):499–564, 2017.
[35] E. Riehl and D. Verity. Elements of ∞-category theory. Book draft downloaded 10 June 2020 from `http://www.math.jhu.edu/~eriehl/elements.pdf`.
[36] J. Rosický and W. Tholen. Factorization, fibration and torsion. *J. Homotopy Relat. Struct.*, 2(2):295–314, 2007.

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 61137, Czech Republic
E-mail address: `bourkej@math.muni.cz`

Department of Mathematics and Statistics, Macquarie University NSW 2109, Australia
E-mail address: `steve.lack@mq.edu.au`

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 61137, Czech Republic
E-mail address: `koren@math.muni.cz`