MASARYK UNIVERSITY

FACULTY OF SCIENCE

HABILITATION THESIS

Filtered manifolds with distinguished transformations and transformation groups

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Abstract

This habilitation thesis discusses geometries with distinguished (local) transformations and (local) properties of the geometries following from the existence of these (local) transformations. We focus mostly on parabolic geometries, [ČaSl09], that form an important subclass of Cartan geometries, [Sha97].

Main part of the work is based on collaboration with Jan Gregorovič devoted to various generalizations of symmetric spaces. We combine several methods developed for parabolic geometries. My viewpoint is more geometric and is based on studying (local) Weyl connections, their relation to curvatures and their compatibility with (local) transformations, [Zal09, Zal10a] and [Zal10b] (Chapter 1). The viewpoint of Jan Gregorovič focuses on homogeneous geometries and their description via functorial constructions based on algebraic methods, [Gre12c, Gre12a, Gre13]. The combination of the methods allows to give complete description of (both local and global) geometric properties of generalized symmetric geometries and their classification, [GrZa16b, GrZa15b] and [GrZa17](Chapter 2). Generalized symmetries of almost CR structures, [GrZa18] (Chapter 3), serve as an example.

The next part of the work deals with submaximally symmetric parabolic geometries. Complex submaximally symmetric parabolic geometries are completely described in [KrTh14], and it follows from their discussion that real submaximally symmetric parabolic geometries shall be studied case by case. In the collaboration with Boris Kruglikov and Henrik Winther we study submaximally symmetric almost quaternionic structures, which are real parabolic geometries. We give submaximal dimensions and models in [KrWiZa18] (Chapter 4).

Finally, there are interesting applications in geometric control theory. In my collaboration with Jaroslav Hrdina, Aleš Návrat and Petr Vašík, we focus on control theory on Lie groups. For various non–holonomic mechanisms, their configuration spaces are filtered manifolds that often form parabolic geometries modeled on nilpotent Lie groups. We study controllability and optimal control of these mechanisms (with respect to a suitable sub-Riemannian metric) using properties of these geometries and their transformations, [HrZa19] (Chapter 5).

We use the CAM system Maple and the package DifferentialGeometry by Ian Anderson to realize computations, [AnTo12].

Contents

Chapter 0: Introduction

1. Motivation: Affine geometries

1.1. **Homotheties on affine geometries.** Affine spaces $\mathcal{A}_n$ and their transformations are the best known examples of geometrical objects, [KoNo63]. The affine plane $\mathcal{A}_2$ is already studied at secondary schools and affine transformations are described there as translations, scalings and homotheties, similarity transformations, reflections, rotations, shear mappings, and compositions of them in any combination and order. Affine manifolds generalize affine spaces to the curved setting. An *affine manifold* or *affine geometry* is a smooth manifold M together with a linear connection ∇. A *(local) affine transformation* f is a (local) smooth map $f : M \to M$ such that $f^*\nabla = \nabla$. There always are two basic invariants of affine manifolds preserved by all affine transformations, the *torsion* $T(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta]$ and the *curvature* $R(\xi, \eta)(\nu) = \nabla_\xi \nabla_\eta \nu - \nabla_\eta \nabla_\xi \nu - \nabla_{[\xi,\eta]}\nu$ for $\xi, \eta, \nu \in TM$. The space $\mathcal{A}_n$ is (globally) the only affine manifold equipped with a connection with vanishing curvature and torsion.

It is a general principle in geometry that the existence of non–trivial curvature or torsion implies restrictions on (the existence of) transformations. Conversely, the existence of a non–trivial transformation can restrict the curvature and the torsion of the geometry. Let us demonstrate this principle on homotheties. The affine space $\mathcal{A}_n$ carries homotheties $f_{x,k}$ of any ratio $k \neq 0$ with the center at any point x. Indeed, since $f_{x,k}(x) = x$, we can identify $\mathcal{A}_n$ with the vector space V_n with the origin x. Then the homothety is given in each basis of V_n by multiplying the coordinates by the matrix

$$\begin{pmatrix} k & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k \end{pmatrix}.$$

To define a (local) homothety of the ratio $k \neq 0$ at some point x of a general (curved) affine manifold (M, ∇), we identify T_xM with V_n. Thus we consider a (local) transformation such that $f_{x,k}(x) = x$ and $T_x f_{x,k}.\xi = k\xi$ for all $\xi \in T_xM$. Let us point out that (local) homotheties $f_{x,k}$ are (local) affine transformations with *natural tangent action* at x which means that

- the (local) morphism preserves the point x, and
- for all bases of T_xM, the coordinate descriptions of its differential (viewed as a linear endomorphism) are equal.

Then, at the center x of the (local) homothety $f_{x,k}$, the torsion T equals to $f^*_{x,k}T = kT$, because T is an invariant tensor of type $(2,1)$, and the curvature R equals to $f^*_{x,k}R = k^2R$, because R is an invariant tensor of type $(3,1)$. Thus there cannot exist (local) non–trivial homotheties at points with non–vanishing torsion. The only (local) homotheties that can exist on torsion–free but curved affine manifolds are

- the identity for $k = 1$, and
- the (local) point symmetry for $k = -1$.

Moreover, if there is a (local) point symmetry $f_{x,-1} =: f_x$, then, at the center x, ∇R equals to $f^*_x \nabla R = -\nabla R$, because ∇R is an invariant tensor of type $(4,1)$. Thus if there is a (local) point symmetry at x, then T and ∇R vanish at x.

Altogether, there can exist (local) point symmetries on affine geometries (M, ∇) only if $T = 0$ and $\nabla R = 0$, while there are no (local) homotheties for $k \neq \pm 1$. Conversely, there always exist (local) point symmetries at each point of (M, ∇) if $T = 0$ and $\nabla R = 0$, [KoNo69, Chapter XI] or [Hel01]. They are given by (local) geodesic symmetries of the geodesics of ∇, or equivalently, by symmetries in normal geodesic coordinates of ∇ (on a normal neighborhood of x). In particular, they are given uniquely and they change smoothly from point to point, i.e., form a unique smooth system of symmetries.

1.2. **Homotheties on affine geometries revisited.** Let us consider the first order frame bundle $p : \mathcal{P}^1M \to M$ over an n–dimensional manifold M, which is a principal bundle with the structure group $GL(n,\mathbb{R})$, the general linear group. The fiber over each $x \in M$ consists of all bases of T_xM and elements of $GL(n,\mathbb{R})$ are transition matrices acting as changes of bases. There is the *canonical form* $\theta \in \Omega^1(\mathcal{P}^1M,\mathbb{R}^n)$ such that $\theta(u)(\xi)$ are coordinates of the projection $Tp.\xi \in T_{p(u)}M$ at the basis u. Then we have $TM \simeq \mathcal{P}^1M \times_{GL(n,\mathbb{R})} \mathbb{R}^n$, where the pair $[\![u,X]\!] \in T_{p(u)}M$ represents the tangent vector at $p(u)$ with coordinates X in the basis u. Indeed, $[\![u,X]\!] = [\![ut,t^{-1}X]\!]$ holds for $t \in GL(n,\mathbb{R})$, which reflects the fact that the change of the basis gives the change of coordinates for the transition matrix t. Altogether, $(\mathcal{P}^1M \to M,\theta)$ is a first order G–structure with the structure group $GL(n,\mathbb{R})$, [KoMiSl93].

There is the well known 1–1 correspondence between affine connections ∇ on M and principal connections $\gamma \in \Omega^1(\mathcal{P}^1M,\mathfrak{gl}(n,\mathbb{R}))$, where $\mathfrak{gl}(n,\mathbb{R})$ is the general linear Lie algebra. The correspondence is given by inducing the connection on the associated bundle TM. Each (local) map $f : M \to M$ induces a (local) bundle map $\mathcal{P}^1f : \mathcal{P}^1M \to \mathcal{P}^1M$ preserving θ, and f is a (local) affine transformation of ∇ if and only if $\mathcal{P}^1f$ preserves γ.

It is now illustrative to study (local) affine transformations f with natural tangent action at x from this viewpoint. If $f(x) = x$, then $\mathcal{P}^1f(u) = us$ for $u \in p^{-1}(x)$ and some $s \in GL(n,\mathbb{R})$. Then the tangent map Tf satisfies at the point x

$$T_xf.[\![u,X]\!] = [\![\mathcal{P}^1f(u),X]\!] = [\![us,X]\!] = [\![u,sX]\!]. \tag{1}$$

The change of u to ut for some transition $t \in GL(n,\mathbb{R})$ gives $\mathcal{P}^1f(ut) = (ut)t^{-1}st$. Then the coordinate description of T_xf in all frames is equal if $t^{-1}stX = sX$ for all $X \in \mathbb{R}^n$ and $t \in GL(n,\mathbb{R})$. Thus maps with natural tangent actions correspond to elements s of the center $Z(GL(n,\mathbb{R}))$ of $GL(n,\mathbb{R})$. In fact, $\mathcal{P}^1f$ is constant along the fiber over x for $s \in Z(GL(n,\mathbb{R}))$. Direct computation gives

$$Z(GL(n,\mathbb{R})) = \left\{ \begin{pmatrix} k & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k \end{pmatrix} : k \in \mathbb{R}^\times \right\} = \{k \cdot \mathrm{id}_{\mathbb{R}^n} : k \in \mathbb{R}^\times\}, \tag{2}$$

where $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$. Thus the only possible (local) morphisms f with natural tangent action at x satisfy $T_xf.[\![u,X]\!] = [\![u,kX]\!]$ for $k \in \mathbb{R}^\times$ and we are back in the classical setting from Section 1.1. In particular, homotheties are the only possible (local) morphisms of affine manifolds with natural tangent actions.

Altogether, we have got a simple algebraic restriction on possible transformations with natural tangent action. Contrary to the classical viewpoint, this concept can be generalized to many types of geometric structures that can be described as Cartan and parabolic geometries, [ČaSl09].

1.3. **Symmetric and (sub)maximal affine geometries.** The above concept leads to so–called *(locally) symmetric spaces.* A manifold M with an affine connection ∇ is *locally symmetric* if $T = 0$ and $\nabla R = 0$, [KoNo69, Hel01]. Then there are point symmetries defined locally at each point of M. If the symmetries are globally defined, then the manifold M is homogeneous and the group generated by symmetries acts transitively on M. Then we call the pair (M,∇) *symmetric space*, [KoNo69, Hel01]. Let us remind that there is also algebraic definition of a symmetric space as a space M together with a map $S : M \times M \to M$ such that

- $S(x,x) = x$,
- $S(x,S(x,y)) = y$,

- $S(x, S(y, z)) = S(S(x, y), S(x, z))$, and
- x is an isolated fixed point of $S(x, -)$

for all $x, y, z \in M$, [Kow80, Loo67]. In the case M is a smooth manifold and S is a smooth system of affine transformations, we recover exactly the classical setting. In particular, (local) symmetric spaces are examples of affine manifolds carrying many (local) affine transformations. However, there are other interesting affine geometries with many (local) transformations.

It is well known, [Sha97], that affine transformations of an affine manifold form a finite–dimensional Lie group. In the case of n–dimensional affine manifold, the biggest possible dimension of this group is $n^2 + n$. This happens in the case of the affine space $\mathcal{A}_n$, where the group is exactly the group $A(n, \mathbb{R})$ of affine transformations $X \mapsto AX + b$ for the regular matrices A of order n and vectors $b \in \mathbb{R}^n$.

There is a natural question what are the possible smaller dimensions. Similarly to symmetric spaces, this question is closely related to the existence of a curvature and a torsion. It is proved in [Ego67] that if the dimension of the group of affine transformations is not maximal, than it cannot be bigger than n^2. In particular, for an affine manifold (M, ∇) with non–zero curvature, the maximal possible dimension of the transformation group is n^2 and the group acts transitively on M. Their realizations can be found in [Ego67, Section 3.A]. It is also proved there that such manifold is projectively flat. Let us note that the third possible dimension is $n^2 - 1$ and its realization can be found in [Ego67, Section 3.B].

Consider the action of $A(n, \mathbb{R})$ on $\mathcal{A}_n$. The stabilizer of a point $x \in \mathcal{A}_n$ is $GL(n, \mathbb{R}) \subset A(n, \mathbb{R})$, so the group of affine transformations of $\mathcal{A}_n^x := \mathcal{A}_n \setminus \{x\}$ has dimension n^2. However, $\mathcal{A}_n^x$ is locally equivalent to $\mathcal{A}_n$ and its transformation group is locally maximal. Moreover, local transformations do not extend generally to global ones and we are mostly interested in the local viewpoint. To avoid these problems, it is convenient to swap to infinitesimal level and to study corresponding Lie algebras instead of the transformation groups. Thus we study *infinitesimal transformations*, i.e., vector fields whose flows are affine transformations at all times, [Ego67, Section 2]. This leads to the so–called *gap phenomenon* studied for many geometric structures with finite-dimensional transformation groups, [KrTh14, Kr16, KrMaTh16, KrMa14, DoTh14]. The authors ask the following question:

- What is the maximal possible dimension of Lie algebra of infinitesimal transformations among all manifolds that are not everywhere flat?

The dimension is then called *submaximal*. In addition to [Ego67], let us mention here paper [KrMa14] where authors study submaximal metric affine structures. Let us remark that the submaximal models discovered in [Ego67] are not metric.

2. Results: Cartan geometries

2.1. **Automorphisms with natural tangent actions.** The concept of Cartan geometries provides a natural generalization of affine geometries and G-structures which allows to study wide class of geometries in a uniform way, [Sha97, ČaSl09]. Let G be a Lie group, $P \subset G$ its closed subgroup and $\mathfrak{p} \subset \mathfrak{g}$ their Lie algebras. A *Cartan geometry of type* (G, P) is a principal P–bundle $p : \mathcal{G} \to M$ together with a 1–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that

(1) $(r^p)^*\omega = \mathrm{Ad}_{p^{-1}} \circ \omega$ for all $p \in P$, where Ad denotes (the restriction of) the Adjoint action of G on $\mathfrak{g}$,
(2) $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{p}$, where ζ_X are the *fundamental vector fields*,
(3) $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

We are interested in local properties, so we can assume M connected and locally connected. A *(local) automorphism* of the Cartan geometry is a (local) principal

P–bundle morphism $\varphi : \mathcal{G} \to \mathcal{G}$ such that $\varphi^*\omega = \omega$. Each (local) automorphism φ determines a (local) underlying morphism $f : M \to M$, and we assume that the groups $P \subset G$ are chosen in such a way that there is 1–1 correspondence between (local) automorphisms φ and the (local) underlying morphisms f, [ČaSl09, Section 1.5.3]. Automorphisms of Cartan geometries always form a finite–dimensional Lie group $\mathrm{Aut}(\mathcal{G}, \omega)$, [ČaSl09, Section 1.5]. In fact, a choice of $u \in \mathcal{G}$ provides an inclusion $\mathrm{Aut}(\mathcal{G}, \omega) \hookrightarrow \mathcal{G}$ given by $\varphi \mapsto \varphi(u)$.

As an example, let us consider G to be the affine group $A(n, \mathbb{R}) = GL(n, \mathbb{R}) \rtimes \mathbb{R}^n$ and $P = GL(n, \mathbb{R})$. Then the frame bundle $p : \mathcal{P}^1 M \to M$ together with the form $\omega := \theta + \gamma$ for a fixed principal connection form γ forms an *affine Cartan geometry.* (Local) automorphisms of affine Cartan geometries are (local) automorphisms of the bundle $\mathcal{P}^1 M$ preserving the soldering form θ and the connection form γ. Their (local) underlying morphisms are (local) affine transformations of (M, ∇) for ∇ corresponding to the principal connection γ.

Affine Cartan geometries are typical examples of *reductive* Cartan geometries, i.e., the quotient $\mathfrak{g}/\mathfrak{p}$ may be identified with an $\mathrm{Ad}(H)$–invariant subspace of $\mathfrak{g}$. In general, Cartan geometries are not reductive. Cartan bundles can be viewed as abstract analogs of frame bundles. The points of $\mathcal{G}$ are (higher-order) frames on which P acts by transitions, while ω is a straight generalization of the affine connections, [Sha97, ČaSl09]. In general, it is not reasonable to speak about homotheties for general Cartan geometries. Nevertheless, affine Cartan geometries suggest a way to generalize the concept of automorphisms with natural tangent action to general Cartan geometries. In fact, automorphisms with natural tangent action give the only reasonable generalization of homotheties on affine spaces.

Let us consider (local) automorphisms φ with underlying (local) maps f such that $f(x) = x$ and let $u \in p^{-1}(x)$. The action of each such (local) automorphism φ is described by the transition $s \in P$ between u and $\varphi(u) = us$ at x. It holds $TM \simeq \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ via factorizing of $[u, X] \mapsto Tp.\omega^{-1}(u)(X)$ by the action of P and we get

$$(3) \qquad T_x f.[\![u, X + \mathfrak{p}]\!] = [\![\varphi(u), X + \mathfrak{p}]\!] = [\![us, X + \mathfrak{p}]\!] = [\![u, \underline{\mathrm{Ad}}_s(X + \mathfrak{p})]\!]$$

on $T_x M$, where $\underline{\mathrm{Ad}}$ is the truncated Adjoint action. In fact, if we view $u \in \mathcal{G}$ as frame, then $X + \mathfrak{p}$ plays the role of coordinates in $T_x M \simeq \mathfrak{g}/\mathfrak{p}$ in the basis u and $\underline{\mathrm{Ad}}_s$ realizes the change of coordinates for the change of the basis from u to us (viewed as an endomorphism of $\mathfrak{g}/\mathfrak{p}$). Let ut, $t \in P$, be a different frame from the fiber over x. If φ is represented by the element s at $u \in p^{-1}(x)$, then the same φ is represented by the element $t^{-1}st$ at $ut \in p^{-1}(x)$. So we get a direct analog of behavior from the affine case. Motivated by homotheties and affine transformations with natural tangent actions in the affine geometries, we are interested in the following (local) morphisms.

Definition 1. We call a (local) automorphism φ of a Cartan geometry an *automorphism with natural tangent action at x* if for the (local) underlying morphism $f : M \to M$ induced by the automorphism φ it holds

- for the point $x \in M$ we have $f(x) = x$, and
- for all frames $u \in p^{-1}(x) \subset \mathcal{G}$, the coordinate descriptions of the endomorphism $T_x f : T_x M \to T_x M$ (viewed as an endomorphism of $\mathfrak{g}/\mathfrak{p}$) are equal.

It follows from (3) that (local) automorphisms with natural tangent actions at x are exactly (local) automorphisms represented (at arbitrary frame $u \in p^{-1}(x)$) by elements $s \in P$ such that $\underline{\mathrm{Ad}}_s = \underline{\mathrm{Ad}}_{t^{-1}st}$ on $\mathfrak{g}/\mathfrak{p}$ for all $t \in P$. Thus they correspond to elements of the center $Z(\underline{\mathrm{Ad}}(P))$ of the image of P in $GL(\mathfrak{g}/\mathfrak{p})$ for $\underline{\mathrm{Ad}}$.

The *curvature* is the basic invariant of any Cartan geometry (with respect to all automorphisms). It is given by

$$\kappa : \mathcal{G} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}, \quad \kappa(u)(X,Y) = [X,Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)])(u).$$

The curvature κ vanishes if and only if the Cartan geometry is locally isomorphic with the *flat model* $(G \to G/P, \omega_G)$. Flat models are the simplest examples of *homogeneous Cartan geometries* which are geometries $(\mathcal{G} \to M, \omega)$ such that $\mathrm{Aut}(\mathcal{G}, \omega)$ is transitive on M (by means of underlying morphisms). Let us emphasize that local automorphisms of a flat model can be always globally extended, the automorphism group of the flat model coincides with G and has maximal possible dimension among all Cartan geometries of type (G, P), [ČaSl09, Section 1.4.].

In general, the existence of special (local) automorphisms like automorphisms with natural tangent actions can induce restrictions on the curvature of Cartan geometry. Let us demonstrate it on automorphisms with natural tangent actions on affine Cartan geometries. Let us write elements of $A(n,\mathbb{R})$ as $(1,n)$–block matrices $\left(\begin{smallmatrix} 1 & 0 \\ b & A \end{smallmatrix}\right)$, where $b \in \mathbb{R}^n$ and $A \in GL(n,\mathbb{R})$, and $GL(n,\mathbb{R}) \subset A(n,\mathbb{R})$ consists of elements satisfying $b = 0$. Then the Lie algebra $\mathfrak{a}(n,\mathbb{R})$ consists of $(1,n)$–block matrices $\left(\begin{smallmatrix} 0 & 0 \\ w & C \end{smallmatrix}\right)$, where $w \in \mathbb{R}^n$ and $C \in \mathfrak{gl}(n,\mathbb{R})$, and the subalgebra $\mathfrak{gl}(n,\mathbb{R})$ consists of elements with $w = 0$. We have

$$\mathrm{Ad}_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & B \end{smallmatrix}\right)} \left(\begin{smallmatrix} 0 & 0 \\ w & C \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ b & B \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ w & C \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & B^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ Bw & BCB^{-1} \end{smallmatrix}\right).$$

According to the reductivity, the Ad–action of $GL(n,\mathbb{R})$ on $\mathfrak{a}(n,\mathbb{R})$ decomposes into the action on $\mathbb{R}^n = \mathfrak{a}(n,\mathbb{R})/\mathfrak{gl}(n,\mathbb{R})$ and the action on $\mathfrak{gl}(n,\mathbb{R})$. Then $\underline{\mathrm{Ad}} = \mathrm{Ad}|_{\mathbb{R}^n}$ is the map given by $B \mapsto (X \mapsto BX)$ for all $X \in \mathbb{R}^n$. Computation gives

$$Z(\underline{\mathrm{Ad}}(P)) = \{k \cdot \mathrm{id}_{\mathbb{R}^n} : k \in \mathbb{R}^\times\} = Z(GL(n,\mathbb{R}))$$

as in (2). Thus (local) automorphisms of affine Cartan geometries with natural tangent actions are (local) automorphisms represented by elements $Z(GL(n,\mathbb{R}))$, in accordance with Section 1.2. Let us emphasize that there is 1–1 correspondence between $Z(\underline{\mathrm{Ad}}(P))$ and $Z(GL(n,\mathbb{R}))$, which can be viewed as the preimage of the center of the image of the Ad–action in P.

In the affine case, $\kappa : \mathcal{P}^1 M \to \wedge^2\mathbb{R}^{n*} \otimes (\mathbb{R}^n \oplus \mathfrak{gl}(n,\mathbb{R}))$ decomposes (with respect to the action of $GL(n,\mathbb{R})$) into the torsion valued in $\wedge^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$ and the curvature valued in $\wedge^2\mathbb{R}^{n*} \otimes \mathfrak{gl}(n,\mathbb{R})$. We get $\kappa(u) = \kappa(us) = s.\kappa(u)$ for an automorphism corresponding to $s \in Z(GL(n,\mathbb{R}))$. Assume $s = k \cdot \mathrm{id}_{\mathbb{R}^n}$. Then the restriction of $\kappa(u)$ to $\wedge^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$, which corresponds to the torsion, equals to $k \cdot \kappa(u)$, while the restriction of $\kappa(u)$ to $\wedge^2\mathbb{R}^{n*} \otimes \mathfrak{gl}(n,\mathbb{R})$, which corresponds to the curvature, equals to $k^2 \cdot \kappa(u)$. So we recover the classical setting. The flat model is the affine space $\mathcal{A}_n = \mathbb{R}^n$, $A(n,\mathbb{R})$ acts transitively on $\mathcal{A}_n$ and, in particular, it contains all homotheties.

2.2. **Automorphisms with natural tangent actions on filtered manifolds.** We focus on a special class of Cartan geometries over filtered manifolds. A *parabolic geometry* is a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G,P) for a semisimple Lie group G and its parabolic subgroup P, [ČaSl09, Section 3.1]. We assume M is connected and simply connected and G is simple[1]. We consider $\mathfrak{g}$ to be $|k|$–graded simple Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. We fix the Levi decomposition $P = G_0 \rtimes \exp(\mathfrak{p}_+)$, where $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, and G_0 is the Lie group of grading–preserving elements of P. We write $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$. There always is the P–invariant filtration of $\mathfrak{g}$ given by $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$ which induces a filtration $T^iM \simeq \mathcal{G} \times_P \mathfrak{g}^i/\mathfrak{p}$ of TM.

[1] All results can be formulated with slightly weaker assumptions, [GrZa15b].

We always consider *regular* parabolic geometries, i.e., the curvature κ only has components of positive homogeneity, [ČaSl09, Section 3.1.7.]. Each such geometry is equivalent to the *underlying structure* on the manifold which consists of

- a filtration T^iM of TM such that the *associated graded bundle* $\mathrm{gr}(TM) = \mathrm{gr}_{-k}(TM) \oplus \cdots \oplus \mathrm{gr}_{-1}(TM)$ of TM, where $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M$, satisfies $\mathrm{gr}(T_xM) \simeq \mathfrak{g}_-$ for all $x \in M$, and
- a reduction of the structure group of $\mathrm{gr}(TM)$ with respect to $\mathrm{Ad} : G_0 \to \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$.

This correspondence is 1–1 assuming that the parabolic geometry is *normal*, i.e., $\partial^* \circ \kappa = 0$ for the codifferential ∂^* in the standard complex computing Lie algebra homology of $\mathfrak{p}_+$ with coefficients in $\mathfrak{g}$, [ČaSl09, Section 3.1.10.]. Then the basic invariant of regular normal parabolic geometries is the *harmonic curvature* κ_H defined as the projection of κ to $\ker(\partial^*)/\mathrm{im}(\partial^*)$. The curvature κ can be computed explicitly from κ_H using a differential operator, [ČaSlSo01].

The existence of the canonical filtration $\{T^iM\}$ on parabolic geometries $(p : \mathcal{G} \to M, \omega)$ implies that it is not reasonable to 'fix ratios of homotheties' on the whole TM, [Zal06, GrZa13] and [Zal10b] (Chapter 1). Let us remind that (local) morphisms of Cartan geometries with natural tangent actions correspond to elements $s \in P$ such that $\underline{\mathrm{Ad}}_s = \underline{\mathrm{Ad}}_{t^{-1}st}$ on $\mathfrak{g}/\mathfrak{p}$ for all $t \in P$. For parabolic geometries, $s = g_0 \exp Z$ and $t = h_0 \exp V$, $g_0, h_0 \in G_0$ and $Z, V \in \mathfrak{p}_+$, and

$$t^{-1}st = h_0^{-1} g_0 h_0 \exp(-\mathrm{Ad}_{g_0 h_0} V) \exp(\mathrm{Ad}_{h_0} Z) \exp V.$$

In particular, we have $\mathrm{Ad}_{g_0} = \mathrm{Ad}_{h_0^{-1} g_0 h_0}$ on $\mathfrak{g}_{-1} \simeq \mathfrak{g}^{-1}/\mathfrak{p}$ and $\mathfrak{p}_+$ acts trivially on $\mathfrak{g}_{-1}$. Thus to study the action on $T^{-1}M$, we can focus on elements of the center $Z(G_0)$. Let us emphasize that viewing $\mathfrak{g}_{-1}$ as G_0–module, each element of $Z(G_0)$ acts on each irreducible component of $\mathfrak{g}_{-1}$ as multiplication by a single eigenvalue. Thus if $X, Y \in \mathfrak{g}_{-1}$ such that $\mathrm{Ad}_{g_0} X = kX$ and $\mathrm{Ad}_{g_0} Y = lY$ for $g_0 \in Z(G_0)$, then $\mathrm{Ad}_{g_0}[X, Y] = [\mathrm{Ad}_{g_0} X, \mathrm{Ad}_{g_0} Y] = kl \cdot [X, Y]$. In particular, we still have 'multiplication by eigenvalue' as for homotheties, but possible eigenvalues are different on different components.

In general, we can prescribe the tangent action only on the distinguished distribution $T^{-1}M \simeq \mathcal{G} \times_P \mathfrak{g}^{-1}/\mathfrak{p}$ in TM, because underlying morphisms for parabolic geometries respect Lie brackets. The above computation shows that on the level of the associated grading $\mathrm{gr}(TM)$, underlying morphisms with natural tangent actions are determined uniquely by their actions on $T^{-1}M$. However, it is generally difficult to study such morphisms on TM because of the action of $\exp(\mathfrak{p}_+)$.

The above observations motivate to study (local) automorphisms of parabolic geometries with natural tangent actions on $T^{-1}M$, [GrZa15b, GrZa17, GrZa16b].

Definition 2. A (local) automorphism φ of a parabolic geometry has a *natural tangent action at x on $T^{-1}M$* if

- the (local) underlying morphism f of φ satisfies $f(x) = x$, and
- the restricted tangent action $T_xf : T_x^{-1}M \to T_x^{-1}M$ of f is such that for all frames $u \in p^{-1}(x)$, the coordinate descriptions of the endomorphism $T_xf : T_x^{-1}M \to T_x^{-1}M$ (viewed as an endomorphism of $\mathfrak{g}^{-1}/\mathfrak{p} \simeq \mathfrak{g}_{-1}$) are equal.

Thus we get from above that the (local) automorphisms φ with natural tangent actions at x on $T^{-1}M$ enjoy the following two equivalent descriptions:

(1) they are represented by elements s such that $\underline{\mathrm{Ad}}_s = \underline{\mathrm{Ad}}_{t^{-1}st}$ on $\mathfrak{g}^{-1}/\mathfrak{p}$ for all $t \in P$,
(2) they correspond to elements of the center $Z(\underline{\mathrm{Ad}}|_{\mathfrak{g}^{-1}/\mathfrak{p}}(P))$ of the image of $\underline{\mathrm{Ad}} : P \to GL(\mathfrak{g}^{-1}/\mathfrak{p})$.

The preimage of $Z(\underline{\mathrm{Ad}}|_{\mathfrak{g}^{-1}/\mathfrak{p}}(P))$ in P has the form $Z(G_0)\rtimes\exp(\mathfrak{p}_+)$, where elements of $Z(G_0)$ determine the action on $\mathfrak{g}_{-1} \simeq \mathfrak{g}^{-1}/\mathfrak{p}$ while $\exp(\mathfrak{p}_+)$ acts trivially on $\mathfrak{g}^{-1}/\mathfrak{p}$. The existence of $\exp(\mathfrak{p}_+)$ implies that elements having the same action on $\mathfrak{g}^{-1}/\mathfrak{p}$ are not unique. Let us remind that there is 1–1 correspondence between elements of the center of the Ad–action and their preimages in P in the case of reductive Cartan geometries, while $\exp(\mathfrak{p}_+)$ makes the situation more complicated for parabolic geometries.

We can also describe (local) automorphisms φ with natural tangent actions at x on $T^{-1}M$ corresponding to $s\exp(Z) \in Z(G_0) \rtimes \exp(\mathfrak{p}_+)$ locally on a neighborhood of x. Let us consider the *normal coordinate system* at $u \in p^{-1}(x)$ given by flows of constant fields $\omega^{-1}(X)$ for X in some neighborhood of 0 in $\mathfrak{g}_-$. Then we have

$$\varphi : \mathrm{Fl}_1^{\omega^{-1}(X)}(u) \mapsto \mathrm{Fl}_1^{\omega^{-1}(X)}(us\exp(Z))$$

over the normal neighborhood of x. Then s provides a linear change of coordinates of X in $\mathfrak{g}_-$ while $\exp(\mathfrak{p}_+)$ changes the normal coordinate system itself and its action is non–linear. (Local) automorphisms represented by elements $s \in Z(G_0)$ at some u then form a distinguished class of automorphisms which are given by a geodesic transformation in suitable normal coordinates.

Let us emphasize that there is just one normal coordinate system in the case of affine Cartan geometries which reflects the non–existence of $\exp(\mathfrak{p}_+)$ in the affine case.

Definition 3. We call a (local) automorphism φ with natural tangent action at x on $T^{-1}M$ *(local) s–symmetry* for $s \in Z(G_0)$ or *(local) generalized symmetry* at x, if there is a frame $u \in p^{-1}(x)$ such that $\varphi(u) = us$. Thus (local) s–symmetries are morphisms of the form

$$\varphi(\mathrm{Fl}_1^{\omega^{-1}(X)}(u)) = \mathrm{Fl}_1^{\omega^{-1}(X)}(us) = \mathrm{Fl}_1^{\omega^{-1}(\mathrm{Ad}_s X)}(u)$$

for X in a maximal possible neighborhood of 0 in $\mathfrak{g}_-$ preserved by Ad_s, [GrZa15b, Definition 2], [GrZa17, Definition 1.1](Chapter 2).

The term 'generalized' justifies the fact that manifolds carrying generalized symmetries cover various generalizations of symmetric spaces, [Kow80, KaZai00, Loo67]. We study relations of automorphisms with natural tangent actions on $T^{-1}M$ and generalized symmetries on homogeneous geometries in [GrZa15b]. The main result of [GrZa15b] given in Theorem 4.1 states that on homogeneous parabolic geometries, there are automorphisms with natural tangent actions at x on $T^{-1}M$ corresponding to some $s\exp(Z) \in Z(G_0)\exp(\mathfrak{p}_+)$ if and only there are s–symmetries. This result does not apply for general (non–homogeneous) parabolic geometries. However, it turns out that the existence of (local) generalized symmetries implies (local) homogeneity for many interesting types of parabolic geometries and it is reasonable to focus only on (local) generalized symmetries in general.

2.3. **Existence and uniqueness of generalized symmetries.** The only non–identical (local) generalized symmetries that can exist on affine geometries are (local) point reflexions and there can be at most one (local) symmetry at each point depending on the curvature and torsion of the affine geometry, [KoNo69, Hel01]. They are given by geodesic symmetries in the only normal coordinate system.

This is not necessarily true for parabolic geometries. For example, if there is an s–symmetry for some $s \in Z(G_0)$ at the frame $e \in P \subset G$ on the flat model $(G \to G/P, \omega)$, then there are infinitely many symmetries at eP represented by s at arbitrary frame $g \in P$, and transitivity gives symmetries at each point of G/P. In particular, for G and $s \in Z(G_0)$, the flat model G/P is s–symmetric and there is infinite number of s–symmetries at each point, [Zal06]. We proved in our work [GrZa17, GrZa16b, Zal09, Zal10b, GrZa13] that uniqueness of (local) s–symmetries also often occurs for non–flat parabolic geometries of various types (G, P) for various elements $s \in Z(G_0)$.

An important feature of (regular, normal) parabolic geometries is that we can study the actions of generalized symmetries on components of the harmonic curvature κ_H instead of the whole curvature κ. Let us stress that the harmonic curvature $\kappa_H : \mathcal{G} \to \ker(\partial^*)/\mathrm{im}(\partial^*)$ is P–equivariant and $\exp(\mathfrak{p}_+)$ acts trivially on $\ker(\partial^*)/\mathrm{im}(\partial^*)$. Thus κ_H descends to a G_0–equivariant function valued in $\ker(\partial^*)/\mathrm{im}(\partial^*) \simeq H^2(\mathfrak{g}_-, \mathfrak{g})$, which is completely reducible G_0–module, [ČaSl09, Section 3.1.12]. Thus κ_H decomposes according to the decomposition of $H^2(\mathfrak{g}_-, \mathfrak{g})$ into irreducible components μ and there is an algorithmic way to describe its components using the Kostant's theorem, [ČaSl09, Section 3.3.].

Since elements of $Z(G_0)$ act on each irreducible component of $\mathfrak{g}$ by a multiple, the same holds for components of $H^2(\mathfrak{g}_-, \mathfrak{g})$ and we can firstly mimic naively ideas from the affine case. Assume there is a (local) s–symmetry φ for $s \in Z(G_0)$ at x and a component $\kappa_\mu : \mathcal{G} \to \mu \subset \mathfrak{g}^*_{a_1} \wedge \mathfrak{g}^*_{a_2} \otimes \mathfrak{g}_{a_3}$ of κ_H (where μ is irreducible as a G_0–submodule of $H^2(\mathfrak{g}_-, \mathfrak{g})$). The element s acts (in the frame $u \in p^{-1}(x)$) by multiples k_i on corresponding components of $\mathfrak{g}_{a_i}$. At the point x then

$$\varphi^*(\kappa_\mu)(u) = \kappa_\mu(\varphi(u)) = \kappa_\mu(us) = s.\kappa_\mu(u) = \frac{k_3}{k_1 k_2} \cdot \kappa_\mu(u), \tag{4}$$

where . denotes the action on corresponding tensor product. This gives the rough restriction $k_3(k_1k_2)^{-1} = 1$ on the possible multiples k_i. Thus the curvature κ_H can have values only in the components of $H^2(\mathfrak{g}_-, \mathfrak{g})$ satisfying these restrictions for given s. In this way, we get the following trivial but the crucial result, which is the first step in restricting possible generalized symmetries and components of harmonic curvature, [Zal09],[Zal10b](Chapter 1) and [GrZa17](Chapter 2).

Theorem 1. *If $s \in Z(G_0)$ is such that μ is not contained in an 1–eigenspace of the action of s, then there is no s–symmetry at the points with non–zero κ_H valued in μ.*

However, it is often not sufficient to study only the G_0–action because of the possible action of $\exp(\mathfrak{p}_+)$. At this point, it is reasonable to swap to the setting of Lie algebras. For $\phi \in \mu \subset H^2(\mathfrak{g}_-, \mathfrak{g})$, let us denote by

$$ann(\phi) := \{A \in \mathfrak{g}_0 : A.\phi = 0\} \tag{5}$$

the *annihilator* of ϕ in $\mathfrak{g}_0$, which contains the infinitesimal version of elements from (4). In the next step, we define the *ith prolongation of the annihilator* of ϕ by

$$pr(\phi)_i = \{Z \in \mathfrak{g}_i : \mathrm{ad}(X_1)\dots\mathrm{ad}(X_i)(Z) \in ann(\phi) \text{ for all } X_1, \dots X_i \in \mathfrak{g}_{-1}\}. \tag{6}$$

Then we define the crucial property of the geometry generalizing classical prolongation rigidity, [KrTh14, Yam93].

Definition 4. For arbitrary fixed $s \in Z(G_0)$ and irreducible component $\mu \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ (viewed as a component of κ_H), we say that the triple $(\mathfrak{g}, \mathfrak{p}, \mu)$ is *prolongation rigid outside of the 1–eigenspace of s*, if for all $\phi \in \mu$, all prolongations $pr(\phi)_i$ are contained in the 1–eigenspace of s, [GrZa17, Definition 1.5.].

Assume there are two (local) s–symmetries φ_1, φ_2 at x for $s \in Z(G_0)$. If $u \in p^{-1}(x)$ is such that $\varphi_1(u) = us$, then $\varphi_2(u) = us\exp(Z)$. We prove in [GrZa17, Proposition 3.1] that if $Z = Z_i + \cdots + Z_k$ for $Z_j \in \mathfrak{g}_j$, where i is the smallest index such that Z_i is non–zero, then s acts non–trivially on each component of $Z_i \in \mathfrak{g}_i$. Moreover, we prove in [GrZa17, Proposition 3.2] that $Z_i \in pr(\kappa_H(u))_i$, where $Z = Z_i + \cdots + Z_k$ as above. These facts are summarized in the following statement, [GrZa17, Theorem 1.6] (Chapter 2).

Theorem 2. *If $s \in Z(G_0)$ is such that $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation–rigid outside of the 1–eigenspace of s, then there is at most one (local) s–symmetry of a parabolic geometry of type (G, P) at each point x with a non–zero component of harmonic curvature in μ.*

Thus prolongation rigidity outside of the 1–eigenspace of s is a crucial algebraic condition for studying (local) generalized symmetries.

Let us emphasize that although proofs of [GrZa17, Proposition 3.1 and 3.2] are technical, the main idea of the proofs is based on inductive derivation of κ_H with respect to suitable admissible connections and studying actions of both (local) symmetries φ_1, φ_2 on them, so it generalizes the observation on parallel curvature for affine geometries with (local) point symmetries. In particular, as a consequence of the proof of [GrZa17, Proposition 3.2] we get that

$$\nabla_\xi \kappa_H(x) = \nabla_{\xi_{\mathrm{fix}}} \kappa_H(x)$$

for a distinguished admissible connection, where $\xi_{\mathrm{fix}} \in T_xM$ is the component of $\xi \in T_xM$ on which s acts trivially, [GrZa17, Corollary 3.3].

We give the complete classification of triples that are prolongation rigid outside of the 1–eigenspace of s for all possible $s \in Z(G_0)$ in [GrZa17, Section 5]. We also provide some geometric properties of corresponding (local) s–symmetric geometries. These geometries display very diverse properties and therefore the classification list is split according to them.

Remark 1. *The result also follows from* [GrZa16b, Theorem 1.3] *in the case of homogeneous geometries. However, we discuss non–homogeneous s–symmetric examples in* [GrZa16b, Section 6]. *For AHS and parabolic contact geometries in the case of s such that* $\mathrm{Ad}_s = -\mathrm{id}$ *on $\mathfrak{g}_{-1}$, the result was already proved using more direct but less generally applicable methods in* [Zal09] *and* [Zal10b]*(Chapter 1).*

In general, there need not exist (local) s–symmetries at each point of M. If there is a (local) s–symmetry φ_x at each $x \in M$, the geometry is *(locally) s–symmetric*. Then we consider the *system of (local) symmetries* $S : M \to \mathrm{Aut}(\mathcal{G}, \omega)$ given by $x \mapsto \varphi_x$ and the corresponding system $f : M \to \mathrm{Diff}(M)$ of (local) underlying morphisms f_x on M. Contrary to the affine symmetric spaces, such systems are not necessarily smooth in general. One can easily find non–smooth systems on flat models. We constructed examples of non–homogeneous locally–flat s–symmetric geometries that do not carry any smooth system of s–symmetries in [Zal14, GrZa15a], motivated by [Pod89], and examples of such non–flat geometries can be found in [GrZa16a].

However, we give the following crucial characterization of smooth systems on non–flat geometries, [GrZa17, Theorem 1.8] (Chapter 2).

Theorem 3. *Suppose $s \in Z(G_0)$ is such that $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s. Suppose that the parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) has got an everywhere non–zero component of the harmonic curvature in μ.*

(1) *The geometry is (locally) s–symmetric if and only if there is a smooth system S of (local) s–symmetries.*
(2) *In such case, the system S is unique.*

Remark 2. *For AHS and parabolic contact geometries in the case of s such that* $\mathrm{Ad}_s = -\mathrm{id}$ *on $\mathfrak{g}_{-1}$, the result and some further consequences were already shown in* [Zal10a, GrZa13].

The result can be strengthen in the case of homogeneous parabolic geometries, because the existence of an s–symmetry at one point implies the existence of a system of s–symmetries, [GrZa16b]. The general result for homogeneous geometries can be found in [GrZa16b, Proposition 7.1]. As an example, let us present here the following result for homogeneous AHS geometries proved also by more direct methods in [Zal09, Zal10a].

Corollary 1. *Let us consider an AHS structure of type (G, P). Assume $\kappa \neq 0$ and assume there is a non–trivial s–symmetry at a point of a homogeneous parabolic geometry of type (G, P). Then M is either (locally) symmetric or (locally) $\mathbb{Z}_3$–symmetric space (i.e., s^3 equals the identity).*

2.4. **Invariant admissible connections.** In the case of affine symmetric spaces, there is a distinguished admissible affine connection which is invariant with respect to all symmetries, [KoNo69, Hel01]. There is a class of *Weyl connections* on each parabolic geometry playing a significant role in the theory of parabolic geometries, [ČaSl09, Section 5.1.] and [ČaSl03]. These are admissible affine connections satisfying a certain normalization condition on their curvature and torsion. They form an affine space modeled over the space of one–forms on M and there are explicit formulas for transformations of Weyl connection in [ČaSl03, Proposition 3.9]. Formulas are generally complicated and one shortly writes $\hat{\nabla} = \nabla + \Upsilon$ for $\Upsilon \in \Omega^1(M)$.

Analogously to (local) point symmetries in the affine case, there are *(local) geodesic transformations* l_x^{∇} of geodesics of Weyl connections ∇ with a fixed point x, which are simply given in the normal coordinate system for the Weyl connection ∇ as (local) diffeomorphisms with coordinates $\mathrm{Ad}_s \in GL(\mathfrak{g}_-)$ for $s \in Z(G_0)$ at x. Clearly, these (local) transformations can be different for different Weyl connection. It is natural to question the invariance of Weyl connections with respect to (local) generalized symmetries. Thus, at a point x we ask whether there is a Weyl connection allowing (local) s–symmetries as its (local) geodesic transformation. If there is a (local) s–symmetry at each point, then one seeks for a Weyl connection (or at least a class of Weyl connections) such that the above holds for the (local) s–symmetries at all points.

Assume there is a (local) s–symmetry φ at some point x. The simplest situation appears if there is an invariant Weyl connection ∇, i.e., $f^*\nabla = \nabla$ holds on a neighborhood of x for the underlying morphism f of φ. The following result, motivated by the affine case, was achieved in [GrZa17, Theorem 1.8] (Chapter 2).

Theorem 4. *Suppose $s \in Z(G_0)$ is such that $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s. Suppose that a (locally) s–symmetric parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) has everywhere non–zero component of the harmonic curvature valued in μ. If $\mathrm{Ad}(s) \in GL(\mathfrak{g}_-)$ has no eigenvalue 1, then there is exactly one Weyl connection invariant with respect to the (local) s–symmetries at all points.*

Remark 3. *Special cases for AHS and parabolic contact geometries were studied earlier in* [Zal09, Zal10a, Zal10b, GrZa15a, GrZa13]*, based on different direct methods.*

In general, there need not exist invariant Weyl connections. However, there can still exist an invariant class of Weyl connections. It turns out that the correct concept is the following, [GrZa17, Definition 1.7] (Chapter 2).

Definition 5. Let $[\hat{\nabla}]$ be the maximal subclass of the class of Weyl connections such that

- for all connections $\nabla \in [\hat{\nabla}]$, the corresponding geodesic transformations l_x^{∇} share the same tangent action $T_x l_x^{\nabla}$ at each $x \in M$, and
- all connections from $[\hat{\nabla}]$ restrict to the same partial connection on all smooth subbundles of TM for all eigenvalues of $T_x l_x^{\nabla}$ different from 1.

Let f be a smooth map assigning to each $x \in M$ the (local) diffeomorphism l_x^{∇} for some Weyl connection ∇ (depending on x) in $[\hat{\nabla}]$.

(1) The class $[\hat{\nabla}]$ is *f–invariant* if $f(x)^*\nabla \in [\hat{\nabla}]$ and $f(x)^*\nabla(x) = \nabla(x)$ for each Weyl connection $\nabla \in [\hat{\nabla}]$.
(2) The Weyl connections ∇ in the f–invariant class $[\hat{\nabla}]$ are called *almost f–invariant* Weyl connections.
(3) An almost f–invariant Weyl connection ∇ is *invariant at x* if $f(x)^*\nabla = \nabla$.
(4) An almost f–invariant Weyl connection ∇ is *invariant* if $f(x)^*\nabla = \nabla$ for all $x \in M$.

Remark 4. *Details about invariant and almost invariant Weyl connections can be also found in* [GrZa16b, Definition 5] *for homogeneous case, and* [Zal09, Zal10a, GrZa13] *for AHS and parabolic contact geometries.*

We summarize the results of [GrZa17](Chapter 2) concerning almost invariant Weyl connections in the following statement.

Theorem 5. *Suppose $s \in Z(G_0)$ is such that $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s. Suppose that the parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) has everywhere non–zero component of the harmonic curvature in μ. The geometry is (locally) s–symmetric if and only if there is a non–trivial f–invariant class of Weyl connections.*

This statement clearly generalizes the Theorem 4 in which the class consists of exactly one connection which is invariant.

The complete description of (locally) s–symmetric geometries with invariant (classes of) Weyl connections and the description of their geometric properties was given in [GrZa17, Section 5]. These properties depend on the specific type of the triple which is prolongation rigid outside of the 1–eigenspace of s and on the properties of (almost) invariant Weyl connections. In particular, they cannot be summarized easily.

In the case of homogeneous s–symmetric geometries, we can describe geometries with invariant (classes of) Weyl connections and their properties nicely in the language of holonomy reductions for parabolic geometries, [ČaGoHa14]. For homogeneous geometries, we can describe explicitly the group generated by all (global) generalized symmetries and we consider holonomy reductions with respect to this group, [GrZa16b]. We give two methods of construction of examples of s–symmetric homogeneous parabolic geometries in [GrZa16b, GrZa15b]. Many examples and partial classifications can be also found in [Gre12a, Gre13, Gre12b].

2.5. **Example: Symmetries of almost CR structures.** Non–degenerate partially integrable almost CR structures of hypersurface type are the best known examples of parabolic contact structures, [ČaSl09, Section 4.2.]. We study s–symmetries for s such that $\mathrm{Ad}_s = -\mathrm{id}$ on $\mathfrak{g}_{-1}$, i.e., we consider s–symmetries inducing $-\mathrm{id}$ on the contact distribution. These symmetries on almost CR geometries were firstly studied in [KaZai00, AltMeNa10], where authors however involve a compatible metric on the distribution. We get the existence of a compatible metric as a new result.

An *almost CR structure (of hypersurface type)* is a smooth manifold M of dimension $2n+1$ together with a distribution $\mathcal{H} \subset TM$ of dimension $2n$ and an *almost complex structure* J on $\mathcal{H}$, i.e., $J : \mathcal{H} \to \mathcal{H}$ is an endomorphism with the property $J^2 = -\mathrm{id}$. We assume $n > 1$. The almost CR structure is *non–degenerate* if $\mathcal{H}$ is completely non–integrable and thus defines a contact structure on M. The endomorphism J extends by complex linearity to an endomorphism of the complexification $\mathbb{C}\mathcal{H}$ of $\mathcal{H}$. It decomposes as $\mathbb{C}\mathcal{H} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}$ into *holomorphic* and *anti–holomorphic* bundles which are exactly eigenbundles for eigenvalues i and $-i$ of J. An almost CR structure is *partially integrable* if $[\mathcal{H}^{1,0}, \mathcal{H}^{1,0}] \subset \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}$, or equivalently, if $[\xi,\eta]-[J(\xi),J(\eta)]\in\Gamma(\mathcal{H})$ for all $\xi,\eta\in\Gamma(\mathcal{H})$. The component of $[\mathcal{H}^{1,0},\mathcal{H}^{1,0}]$ in $\mathcal{H}^{0,1}$ corresponds (up to a multiple) exactly to the complexification of the Nijenhuis tensor

$$N(\xi,\eta)=J([\xi,\eta]-[J(\xi),J(\eta)])-[J(\xi),\eta]-[\xi,J(\eta)]$$

for all $\xi,\eta\in\Gamma(\mathcal{H})$. An almost CR structure is *integrable* if the Nijenhuis tensor vanishes. The *signature* of an almost CR structure is the signature of its Levi form.

We consider oriented non–degenerate partially integrable almost CR structures $(M,\mathcal{H},J)$ of signature (p,q). Such almost CR structures can be equivalently described as regular normal parabolic geometries of type $(PSU(p+1,q+1),P)$, where the group $PSU(p+1,q+1)$ is the projectivization of the group of matrices preserving the pseudo–Hermitian form

$$m((u_0,\dots,u_{n+1}),(v_0,\dots,v_{n+1}))=u_0\overline{v_{n+1}}+u_{n+1}\overline{v_0}+\sum_{k=1}^{p}u_k\overline{v_k}-\sum_{k=p+1}^{n}u_k\overline{v_k}$$

on $\mathbb{C}^{n+2}$ and P is the stabilizer of the complex line generated by the first basis vector in the standard basis of $\mathbb{C}^{n+2}$. In particular, the *flat model* $PSU(p+1,q+1)/P$ is a smooth real hypersurface in $\mathbb{C}P^{n+1}$ that can be also viewed as the projectivization of the null cone of m in $\mathbb{C}^{n+2}$. The Lie algebra $\mathfrak{su}(p+1,q+1)$ of $PSU(p+1,q+1)$ consists of elements represented by $(1,n,1)$–block matrices

$$\begin{pmatrix} a & Z & iz \\ X & A & -IZ^* \\ ix & -X^*I & -\bar{a} \end{pmatrix},$$

where $\mathfrak{g}_0=\mathfrak{csu}(p,q)=\{(a,A):a\in\mathbb{C},\ A\in\mathfrak{u}(n),\ a+tr(A)-\bar{a}=0\}$, $X\in\mathfrak{g}_{-1}=\mathbb{C}^n$, $Z\in\mathfrak{g}_1=\mathbb{C}^{n*}$, $x\in\mathfrak{g}_{-2}=\mathbb{R}$ and $z\in\mathfrak{g}_2=\mathbb{R}^*$ is a contact $|2|$–grading. Here I denotes the diagonal matrix with the first p entries equal to 1 and the remaining q entries equal to -1. The Lie algebra $\mathfrak{p}$ consists of the $(1,n,1)$–block upper triangular matrices and decomposes as $\mathfrak{p}=\mathfrak{csu}(p,q)\oplus\mathbb{C}^{n*}\oplus\mathbb{R}^*$. We have $P\cong CSU(p,q)\exp(\mathbb{C}^{n*}\oplus\mathbb{R}^*)$. Then $\mathcal{H}=T^{-1}M$ carries natural complex structure induced from $\mathfrak{g}_{-1}=\mathbb{C}^n$, [ČaSl09, Section 4.2.4].

A *(local) CR transformation* of an almost CR structure $(M,\mathcal{H},J)$ is a (local) diffeomorphism of M such that the tangent map preserves $\mathcal{H}$ and its restriction to $\mathcal{H}$ is complex linear. (Local) CR transformations are precisely (local) diffeomorphisms covered by (local) automorphisms of parabolic geometries of type $(PSU(p+1,q+1),P)$. Direct computation gives that the diagonal matrix $s=diag(-1,1,\dots,1,-1)$ represents an element of $Z(CSU(p,q))$ satisfying $\mathrm{Ad}_s=-\mathrm{id}$ on $\mathfrak{g}_{-1}$.

Definition 6. *(Local) symmetries* s_x on CR geometries are the underlying morphisms of s–symmetries with $\mathrm{Ad}_s=-\mathrm{id}$ on $\mathfrak{g}_{-1}$.

Thus each (local) symmetry s_x at x on CR geometry satisfies

- $s_x(x)=x$,
- $T_xs_x=-\mathrm{id}$ on $\mathcal{H}$, and
- s_x is a (local) CR transformation.

We can describe all symmetries and compatible metrics on the flat model, [Zal10b] (Chapter 1) and [GrZa18](Chapter 3).

Theorem 6. *There exists an infinite number of symmetries at each point kP of $PSU(p+1,q+1)/P$ given by matrices of the form $ks_{Z,z}k^{-1}$ for all $Z\in\mathbb{C}^{n*}$ and $z\in\mathbb{R}^*$, where*

$$s_{Z,z}=\begin{pmatrix} -1 & -Z & iz+\frac{1}{2}ZIZ^* \\ 0 & E & -IZ^* \\ 0 & 0 & -1 \end{pmatrix}.$$

Here E denotes identity matrix. In particular:

(1) *There exists an infinite number of involutive symmetries at each point characterized by the condition $z = 0$. For each such symmetry, there is a different metric preserved by this symmetry compatible with the CR geometry.*
(2) *There exists an infinite number of non–involutive symmetries at each point characterized by the condition $z \neq 0$. They do not preserve any metric compatible with the CR geometry.*

General (curved) CR geometries have two components of harmonic curvature which are the *harmonic torsion* valued in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ and the *harmonic curvature* valued in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$. The harmonic torsion coincides (up to a multiple) with the Nijenhuis tensor N and the harmonic curvature is known as the *Chern-Moser tensor* or *Weyl tensor* W. The existence of (local) symmetries gives the following restriction on curvature, [Zal10b, GrZa18] (Chapters 1 and 3).

Theorem 7. *Following facts hold for an almost CR structure $(M, \mathcal{H}, J)$.*
(1) *If there is a (local) symmetry at $x \in M$, then $N(x) = 0$.*
(2) *If there is a non–involutive (local) symmetry at x, then $W(x) = 0$.*
(3) *There is at most one (local) symmetry at each x with $W(x) \neq 0$.*
(4) *The (local) symmetry at x is involutive if and only if there is an invariant Weyl connection ∇ at x defined locally on a neighborhood of x.*
(5) *It holds $\nabla_\xi W(x) = 0$ for each invariant Weyl connection at x and $\xi \in \mathcal{H}$.*

In particular, $T_x s_x$ is involutive and $T_x M$ decomposes into ± 1–eigenspaces, where $\mathcal{H}_x$ is the -1–eigenspace and there is the one–dimensional 1–eigenspace $T_x^+ M$ complementary to $\mathcal{H}$.

Remark 5. *The results were proved in* [Zal10b] *using methods analogous to the affine case. We also confirmed the results using prolongation rigidity outside of the* 1*–eigenspace of s of the triple $(\mathfrak{su}(p+1, q+1), \mathfrak{p}, W)$ in* [GrZa17, Section 5.4]*. Finally, we improved the results and their proofs in* [GrZa18].

Altogether, (locally) symmetric almost CR structures are integrable. Moreover, they admit only systems of involutive (local) symmetries if W is not vanishing.

Let us now swap to global symmetries and globally symmetric CR geometries. The local version of most of the following result is also available and can be found in [GrZa17, Section 5.4] (Chapter 2). The results are more difficult to formulate because of technicalities but they are analogous to global results.

We show in [GrZa17, Section 5.4] that the triple $(\mathfrak{su}(p+1, q+1), \mathfrak{p}, W)$ is prolongation rigid outside of the 1–eigenspace of s. The results summarized in previous two sections then imply that if the geometry is symmetric and $W \neq 0$, then the uniquely given symmetries form a smooth system. Let us remark that we have many smooth as well as non–smooth systems of symmetries on flat geometries (which particularly satisfy $N = 0$, $W = 0$). We describe the smooth system by a smooth map $S : M \times M \to M$ given by $S(x, y) = s_x(y)$ for the symmetry s_x at $x \in M$.

In [GrZa18, Proposition 2] (Chapter 3), we prove an important characterization of symmetric CR structures. The following object plays an important role there. A *reflexion space* (in the sense of [Loo67]) is a space M together with a map $S : M \times M \to M$ satisfying the following conditions

- $S(x, x) = x$,
- $S(x, S(x, y)) = y$, and
- $S(x, S(y, z)) = S(S(x, y), S(x, z))$

for all $x, y, z \in M$. Let us emphasize that contrary to the definition of symmetric spaces, reflexion spaces do not assume that x is an isolated fixed point of $S(x, -)$.

Theorem 8. *Suppose that $(M, \mathcal{H}, J)$ is a symmetric CR geometry. Then either*

(1) $W = 0$ *and the CR geometry is locally equivalent to the flat model* $PSU(p+1, q+1)/P$*, or*
(2) $W \neq 0$ *and the group generated by symmetries is a Lie group that acts transitively on* M*, i.e., the CR geometry is homogeneous and* (M, S) *is a homogeneous reflexion space, where* S *is the smooth system of uniquely given symmetries.*

Remark 6. *The result was firstly proved under stronger assumptions in* [GrZa17] *and partially follows from results given in Section 2.3. In* [GrZa18] *we give a complete proof of this statement.*

We study properties of parabolic contact structures with smooth systems of symmetries in detail in [GrZa13]. In particular, we get the following more detailed description of symmetric CR geometries.

Theorem 9. *Let* S *be a smooth system of involutive symmetries on* $(M, \mathcal{H}, J)$*. The following statements are equivalent:*

(1) *The pair* (M, S) *is a reflexion space.*
(2) *There exists an almost* S*–invariant Weyl connection.*
(3) *All symmetries from the system* S *preserve the subbundle* T^+M *formed by the subspaces* $T^+_x M$*.*
(4) *The smooth system* S *induces a structure of a symmetric space on the leaf space* N *of the foliation for the involutive distribution* T^+M*.*

In particular, if $(M, \mathcal{H}, J)$ *is a CR geometry with a smooth system of symmetries* S *such that* (M, S) *is a reflexion space, then*

- $M = K/H$*, where* K *is the Lie group generated by symmetries from* S *and* H *is the stabilizer of a point,*
- *the stabilizer* L *of the leaf* F *going through* eH *is a closed subgroup of* K*,*
- $N = K/L$ *and* $F = L/H$*.*

We discuss in [GrZa13, Section 3] conditions under which the underlying symmetric space is a pseudo–Hermitean symmetric space.

Let us finally focus on pseudo–Riemannian metrics compatible with symmetries on non–flat CR geometries $(M, \mathcal{H}, J)$. The following crucial result studies distinguished compatible Weyl connections and pseudo–Riemannian submetrics and metrics, [GrZa18, Theorem 16] (Chapter 3).

Theorem 10. *Let* K *be the Lie group generated by all symmetries of a non–flat symmetric CR geometry* $(M, \mathcal{H}, J)$*. Suppose that* $\mathrm{Ad}(H^0)|_{\mathfrak{q}/\mathfrak{h}} = \mathrm{Ad}(H)|_{\mathfrak{q}/\mathfrak{h}}$*, where* H^0 *denotes connected component of identity of the stabilizer* $H \subset K$ *of a point and* $\mathfrak{q}$ *is the* 1*–eigenspace of* s *in* $\mathfrak{k}$*. There exist*

- *a distinguished Weyl connection* ∇ *preserving the corresponding Reeb field,*
- *a* K*–invariant pseudo–Riemannian metric* $\bar{g}$ *on* $\mathcal{H}$*, and*
- *a* K*–invariant Webster metric* g *on* TM*,*

such that

(1) $\nabla \bar{g} = 0, \nabla g = 0$*,*
(2) $g|_{\mathcal{H}} = \bar{g}$ *and the Reeb field of* ∇ *is orthogonal to* $\mathcal{H}$ *and has length* 1*,*
(3) *choosing the Reeb field of* ∇ *as a trivialization of* $TM/\mathcal{H} \otimes \mathbb{C}$*, the pseudo–Riemannian metric* $\bar{g}$ *on* $\mathcal{H}$ *coincides with the real part of the Levi form up to a constant multiple,*
(4) *the symmetry at* x *is linear in geodesic coordinates of* ∇ *at* x*, reverses the directions of* $\mathcal{H}_x$ *and preserves the direction of the Reeb field of* ∇ *at* x*.*

2.6. **Submaximal models for almost quaternionic structures.** An *infinitesimal transformation* of a parabolic geometry $(\mathcal{G} \to M, \omega)$ is a P–equivariant vector field ξ on $\mathcal{G}$ such that $\mathcal{L}_\xi \omega = 0$. The crucial fact is that flows of ξ at all times then are automorphisms of the parabolic geometry. For all parabolic geometries of type (G, P), the maximal possible dimension of the Lie algebra of infinitesimal transformations is $\dim(\mathfrak{g})$. This is realized for geometries that are locally isomorphic to the flat model $(G \to G/P, \omega_G)$. Let us remark that the maximal possible dimension of transformation group is $\dim(\mathfrak{g})$, too, and this is realized for the flat model. However, it can be smaller for geometries that are only locally flat. One can e.g. remove a point from the flat model to get a manifold with the dimension of the Lie algebra of transformations being $\dim(\mathfrak{g})$ and the dimension of the Lie group of transformations being $\dim(\mathfrak{p})$, see Section 1.3.

There is a natural question on submaximal dimension of the Lie algebra of infinitesimal transformations and realizations of corresponding geometries, because submaximal geometries are other interesting examples of geometries with many symmetries. Thus we ask the following question:

- What is the maximal possible dimension of the Lie algebra of infinitesimal transformations among all (regular normal) parabolic geometries of type (G, P) that are not everywhere flat?

Let us emphasize that regularity and normality are technical assumptions from this viewpoint, because one is always interested in (infinitesimal) transformations of underlying structures and among all parabolic geometries that induce the same underlying structure, there always is a regular normal one, [ČaSl09, Section 3.1.].

In [KrTh14] authors study submaximal dimensions and corresponding models for (regular normal) complex parabolic geometries. They use the theory of Tanaka prolongations, [Tan70, Tan79], prolongation rigidity, [Yam93], and properties of minimal orbits, [OnVi90], to give a complete classification of submaximal dimensions for complex parabolic geometries, [KrTh14, Appendix C]. They also show that their restriction is *sharp* which means that they can always realize explicit examples of submaximal geometries. Indeed, they give a method of construction of these examples in [KrTh14, Section 4]. The method of construction of submaximal models is based on deformations of flat models. In fact, the method is closely related to one of methods of the construction of symmetric parabolic geometries, [GrZa15b]. Indeed, submaximal models are homogeneous and we show in [GrZa15b, Zal09] that they are always symmetric for suitable $s \in Z(G_0)$.

It turns out that the theory of [KrTh14] cannot be applied in the case of real parabolic geometries of non–split type, although main ideas still work, and they must be treated case by case. There are known results e.g. for CR geometries and c–projective geometries, [Kr16, KrMaTh16]. There also are results in [DoTh14] for conformal parabolic geometries developed by slightly different method. We give results for almost quaternionic structures in [KrWiZa18] (Chapter 4).

Classically, an *almost quaternionic structure* on a manifold M is a rank three subbundle $Q \subset \mathrm{End}(TM)$ such that locally (in a neighborhood of each point) we can find a basis I, J, K of Q with $I^2 = J^2 = K^2 = -id$ and $IJ = K$. A manifold M with a fixed almost quaternionic structure Q is an *almost quaternionic manifold*. A (local) automorphism of (M, Q) is a (local) diffeomorphism of M that preserves Q, [AleMar96]. An almost quaternionic manifold (M, Q) can be described as a normal parabolic geometry $(\mathcal{G} \to M, \omega)$ of type $(PGL(n{+}1, \mathbb{H}), P)$, where P is the stabilizer of a quaternionic line in $\mathbb{H}^{n+1}$, [ČaSl09, Section 4.1.8.]. The flat model then is the quaternionic projective space $\mathbb{H}P^n$ which is the set of quaternionic lines in $\mathbb{H}^{n+1}$. The group $PGL(n{+}1, \mathbb{H})$ acts transitively on $\mathbb{H}P^n$ as automorphisms of the natural quaternionic structure. In particular, maximal possible dimension of a Lie algebra of infinitesimal transformations for quaternionic manifolds of quaternionic dimension n is $\dim(\mathfrak{sl}(n+1), \mathbb{H}) = 4(n+1)^2 - 1$.

In the case of general (curved) almost quaternionic structures, the harmonic curvature κ_H has two components:

- the *structure torsion* κ_1 of homogeneity 1 valued in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$, and
- the *quaternionic Weyl curvature* κ_2 of homogeneity 2 valued in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$.

Let us sketch the idea of finding the submaximal dimension: Each choice of $u \in \mathcal{G}$ provides an inclusion $\mathrm{Aut}(\mathcal{G}, \omega) \hookrightarrow \mathcal{G}$. On the level of corresponding algebra of infinitesimal transformations $\mathfrak{inf}(\mathcal{G}, \omega)$ we get the inclusion $\mathfrak{inf}(\mathcal{G}, \omega) \hookrightarrow \mathfrak{g}$, $\xi \mapsto \omega(\xi(u))$. The image $\mathfrak{f}(u)$ together with the new bracket $[X, Y]_{\mathfrak{f}(u)} := [X, Y] - \kappa(u)(X, Y)$ for $X, Y \in \mathfrak{f}(u)$ forms a filtered Lie algebra. We consider its associated grading $\mathfrak{s}(u) = \mathrm{gr}(\mathfrak{f}(u)) = \mathfrak{s}_{-1}(u) \oplus \mathfrak{s}_0(u) \oplus \mathfrak{s}_1(u)$. It holds

$$dim(\mathfrak{inf}(\mathcal{G}, \omega)) = dim(\mathfrak{s}(u)).$$

In particular, $\mathfrak{s}_0(u) \subset ann(\kappa_H(u)) \subset \mathfrak{g}_0$. Moreover, the annihilator is of maximal dimension if and only if the G_0–orbit through the component of κ_H has minimal dimension in the projectivization of $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$. Since almost quaternionic structures are prolongation rigid, [KrTh14, Corollary 3.4.8], the component $\mathfrak{s}_1(u)$ does not appear. Thus

$$\mathfrak{s}(u) \subset \mathfrak{g}_{-1} \oplus ann(\kappa_H(u)) \subset \mathfrak{g}_{-1} \oplus \mathfrak{g}_0.$$

Finally, there are two possibilities for $\kappa = \kappa_H = \kappa_1$ and $\kappa = \kappa_H = \kappa_2$ giving the possible maximal bound. In [KrWiZa18] (Chapter 4), we prove the following statement.

Theorem 11. *The maximal dimension of the Lie algebra of infinitesimal transformations for almost quaternionic structures (M, Q) with $dim(M) = 4n > 4$ and $\kappa_H = \kappa_1 + \kappa_2 \neq 0$ is $4n^2 - 4n + 9$. This is realized in both cases, when $\kappa = \kappa_H = \kappa_1$ and $\kappa = \kappa_H = \kappa_2$.*

We exclude the case $n = 1$ because it is equivalent to a conformal structure and it is studied in [DoTh14]. In this case, the submaximal dimension is 8.

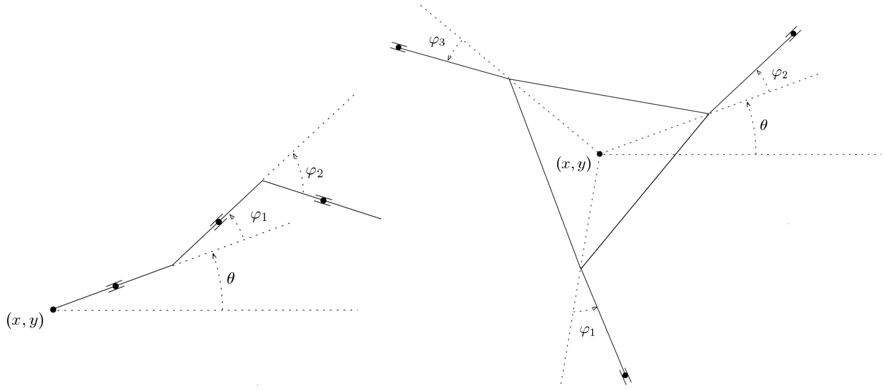
We give the proof of the submaximal dimension in [KrWiZa18, Section 4] (Chapter 4). In [KrWiZa18, Section 5], we show that the dimension is realizable in both cases and we give a complete description of the models. It turns out that the model for $\kappa_2 \neq 0$ is locally an affine symmetric space which is not Riemannian. Among all Weyl connections, there is the distinguished Ricci–flat connection with vanishing torsion and parallel curvature, so this is the connection compatible with the symmetries. The model for the case $\kappa_1 \neq 0$ is locally representable as a Lie group.

3. Applications: Geometric control theory

Transformations of Cartan geometries find interesting applications in geometric control theory and robotics. We focus on (local) control theory of various mechanical systems, [HrZa19] (Chapter 5). Instead of reminding the whole theory in detail, we just review the general concepts and we demonstrate them on the example of vertical rolling disc, which is one of the simplest mechanical systems, [Blo15, Bro82].

It is a natural question in robotics to discuss the controllability and find local extremal trajectories of various mechanisms that move in the plane. Examples of such mechanisms are snake robots, [HrNáVa16], or trident snake robots, [Ish04, PiTho14, HrNáMaVa17]. For example, the 3-link snake consists of thee rigid links of constant length interconnected by rotating joints and to each link, there is a wheel attached at the center of the link. The classical trident snake consists of a body in the shape of an equilateral triangle and three rigid links of constant length connected to the vertices of the body by rotating joints.

Configuration spaces of such mechanisms locally coincide with $\mathbb{R}^n$ for suitable n. For example, the configuration space of the 3–link snake is locally $\mathbb{R}^5$ and local coordinates can be chosen as $[x, y, \theta, \varphi_1, \varphi_2]$ according to Figure 1. The configuration space of classical trident snake coincides locally with $\mathbb{R}^6$, where coordinates can be chosen as $[x, y, \theta, \varphi_1, \varphi_2, \varphi_3]$ according to Figure 1.



(a) 3–link snake (b) classical trident snake

FIGURE 1. Snake robots

We can also generalize these classical mechanisms by changing their configuration spaces. For example, we consider a generalized trident snake which consists of a body in the shape of an equilateral triangle and three prismatic links connected to the vertices of the body, where two of them are attached by fixed joints and one of them by rotating joint according to Figure 2. Its configuration space locally coincides with $\mathbb{R}^7$ and coordinates can be chosen as $[x, y, \theta, \varphi_2, \ell_1, \ell_2, \ell_3]$.

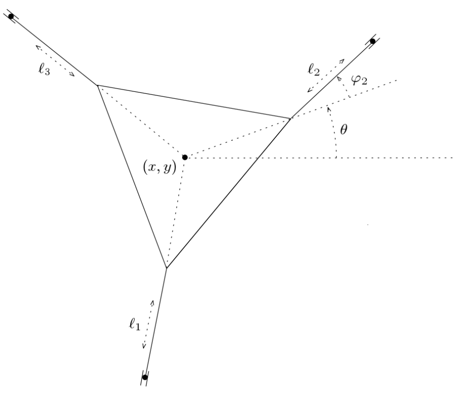


FIGURE 2. Generalized trident snake

Moreover, there are natural non–holonomic constraints coming from natural restrictions on the movement of mechanisms given by the fact that wheels cannot move in directions perpendicular to their velocities. These restrictions give additional (local) structure on the configuration space. Indeed, such condition on each wheel determines one pfaffian form and altogether, we get a pfaffian system which determines (locally, at regular points) a distribution on $\mathbb{R}^n$. If this distribution is *bracket generating* at some point of the configuration space, then the mechanical system is controllable in a neighborhood of this point, i.e., we can reach each point in this neighborhood, [AgBaBo19]. In such case, the distribution determines (locally, at these points) a filtration of the configuration space and often carries additional geometric structure that can be described as a Cartan or even parabolic geometry. For example, the 3-link snake corresponds to the famous $(2, 3, 5)$-*geometry* of Cartan, [Car1894, Car10], and the classical trident snake corresponds to the famous $(3, 6)$-*geometry* of Bryant, [Bry16]. The generalized trident snake carries the so-called *generalized path geometry* in dimension 7, [ČaSl09, Section 4.4.3].

Example 1. Let us consider a vertical disc rolling in the plane according to Figure 3. Its configuration space is (locally but not globally) $\mathbb{R}^3$ with coordinates $[x, y, \theta]$

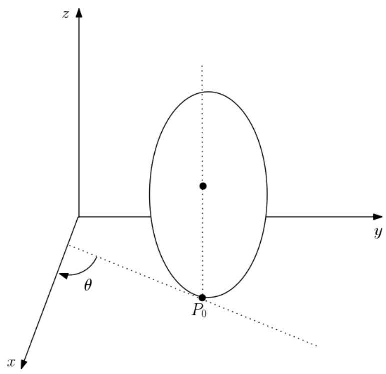


FIGURE 3. Vertical rolling disc

given by the touch point $P_0 = [x, y] \in \mathbb{R}^2$ and the velocity orientation $\theta \in [0, 2\pi]$. The constraint given by the fact that the disc cannot move in the direction perpendicular to its velocity gives the pfaffian $-\sin(\theta)dx + \cos(\theta)dy$, and its kernel is (locally) the bracket generating 2–distribution on $\mathbb{R}^3$ generated by vector fields $v_1 := \frac{\partial}{\partial\theta}$ and $v_2 := \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y}$. In fact, these two fields correspond to two natural movements of the disc which are

- the rotation around an axis parallel with z at P_0, and
- the move in the plane in the direction $(\cos(\theta), \sin(\theta))$ from P_0.

Their Lie bracket $[v_1, v_2] = v_3 := -\sin(\theta)\frac{\partial}{\partial x} + \cos(\theta)\frac{\partial}{\partial y}$ is not contained in the distribution. Thus we have (locally) a bracket-generating distribution giving a filtration $(2, 3)$ and the system is everywhere controllable. Let us remark that together with $[v_1, v_3] = -v_2$ these two brackets are the only non-trivial brackets.

Let us emphasize that the above Cartan and parabolic geometries are defined locally and they can generally have non–trivial invariants (i.e., the harmonic curvature in the case of parabolic geometries). The usual way to avoid this problem in robotics is to swap to *nilpotent approximation* at a suitable point, [AgSar87, Her86]. Geometrically, the nilpotent approximation is a suitable representative of the associated grading and it can be viewed as the maximally symmetric model of the geometry (i.e., flat model for Cartan and parabolic geometries).

Example 2. The nilpotent approximation at the origin $[0, 0, 0]$ of configuration space of the vertical rolling disc is given by a distribution generated by fields $n_1 := \frac{\partial}{\partial\theta}$, $n_2 := \frac{\partial}{\partial x} + \theta\frac{\partial}{\partial y}$. Their bracket equals to $[n_1, n_2] = n_3 := \frac{\partial}{\partial y}$. This is the only non-trivial bracket and all remaining brackets are trivial. We get a contact distribution giving a filtration $(2, 3)$ and the vector fields model the three-dimensional Heisenberg algebra.

Methods of control theory can be used efficiently to study extremal trajectories and local minimizers for optimal control problems that are related to plane mechanisms, [AgBaBo19, AgSac04].

Example 3. Let us start with the original mechanical system. Configuration space of the vertical rolling disc can be (locally) viewed as a Lie group S with solvable Lie algebra generated by the elements e_1, e_2, e_3 corresponding to the vector fields v_1, v_2, v_3. The Lie algebra is given by $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$. So we can use the concepts of control theory on Lie groups.

To study optimal movements, we consider the *canonical control metric* g_S on the distribution $D_S = \langle e_1, e_2 \rangle$, i.e., e_1, e_2 are orthogonal and both have the length 1. Let us emphasize that D_S and g_S are invariant with respect to the action given by multiplication in S and we have an invariant control system (S, D_S, g_S). We would like to study the following optimal control problem

$$\dot{c}(t) = u_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + u_2 \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}$$

for $c \subset S$ and the control $u = (u_1, u_2) \in \mathbb{R}^2$ with the boundary condition $c(0) = s_1$, $c(t_0) = s_2$ for arbitrary fixed $s_1, s_2 \in S$, where we minimize $\frac{1}{2}\int_0^{t_0}(u_1^2 + u_2^2)dt$.

We use the Hamiltonian viewpoint and the Pontryagin's maximum principle to find a system whose solutions are extremal trajectories, [AgBaBo19, Section 7]. Thus on T^*S we get the Pontryagin's system

$$\begin{aligned} h_1'(t) &= -h_3(t)h_2(t), & h_2'(t) &= h_3(t)h_1(t), & h_3'(t) &= -h_1(t)h_2(t), \\ x'(t) &= h_2(t)\cos(\theta(t)), & y'(t) &= h_2(t)\sin(\theta(t)), & \theta'(t) &= h_1(t), \end{aligned}$$

where (x, y, θ) are base coordinates and $h_i = \langle \lambda, v_i \rangle$, $\lambda \in T^*S$ are (fiber) coordinate functions. The system can be solved explicitly but the solution is given using Jacobi functions, [SaMo10], and it is not possible to use it in practical applications.

Let us thus swap to the nilpotent approximation. It corresponds to the Lie group N with Heisenberg algebra, i.e., the nilpotent algebra given by elements f_1, f_2, f_3 corresponding to n_1, n_2, n_3 such that $[f_1, f_2] = f_3$. We again consider the *canonical control metric* g_N on the distribution $D_N = \langle f_1, f_2 \rangle$, i.e., f_1, f_2 are orthogonal and both have the length 1. Thus we have an invariant control system (N, D_N, g_N) on the nilpotent Lie group N. Thus we study the following optimal control problem

$$\dot{c}(t) = u_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + u_2 \begin{pmatrix} 1 \\ \theta \\ 0 \end{pmatrix}$$

for $c \in N$ and the control $u = (u_1, u_2)$ with the boundary condition $c(0) = p_1$, $c(t_0) = p_2$ for arbitrary fixed $p_1, p_2 \in N$, where we minimize $\frac{1}{2}\int_0^{t_0}(u_1^2 + u_2^2)dt$. In this case, we get on T^*N the Pontryagin's system

$$\begin{aligned} h_1'(t) &= -h_3(t)h_2(t), & h_2'(t) &= h_3(t)h_1(t), & h_3'(t) &= 0, \\ x'(t) &= h_2(t), & y_2'(t) &= h_2(t)\theta(t), & \theta'(t) &= h_1(t), \end{aligned}$$

where (x, y, θ) are base coordinates and $h_i = \langle \lambda, n_i \rangle$, $\lambda \in T^*N$ are (fiber) coordinate functions. This system can be solved explicitly. Its solutions are extremal trajectories and these project to local minimizers $(x(t), y(t), \theta(t))$. The fiber (vertical) system has (in the generic case $h_3 \neq 0$) the solutions of the form

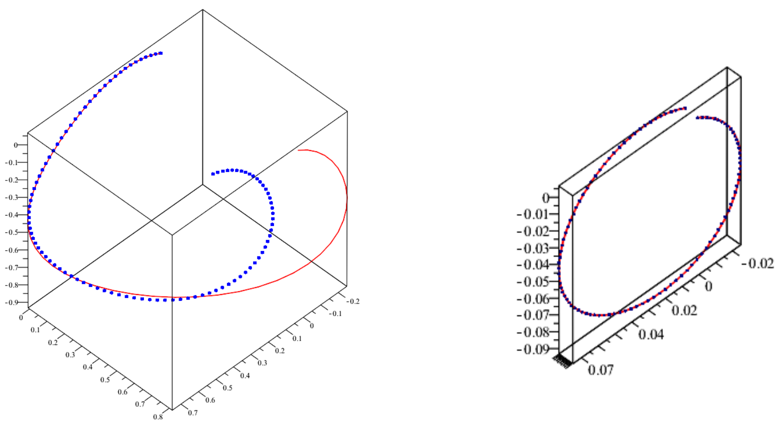
$$\begin{aligned} h_1 &= C_2 \sin(C_1 t) + C_3 \cos(C_1 t) \\ h_2 &= C_3 \sin(C_1 t) - C_2 \cos(C_1 t) \\ h_3 &= C_1 \end{aligned}$$

for constants C_2, C_3 and $C_1 \neq 0$. Then the base (horizontal) system has solutions satisfying the initial condition $x(0) = y(0) = \theta(0) = 0$ of the form

$$
\begin{aligned}
x(t) &= \frac{1}{C_1}(C_3 - C_2 \sin(C_1 t) - C_3 \cos(C_1 t)) \\
y(t) &= \frac{1}{4C_1^2}(2C_1(C_2^2 + C_3^2)t - 4C_2C_3 \cos(C_1 t) + 2C_2C_3 \cos(2C_1 t) \\
&\quad - 4C_2^2 \sin(C_1 t) + (C_2^2 - C_3^2) \sin(2C_1 t) + 2C_2C_3) \\
\theta(t) &= \frac{1}{C_1}(C_2 - C_2 \cos(C_1 t) + C_3 \sin(C_1 t))
\end{aligned}
\tag{7}
$$

for constants C_1, C_2, C_3 as above. These curves are arc-length parametrized if $C_2^2 + C_3^2 = 1$. Let us point out that it is sufficient to consider solutions with $x(0) = y(0) = \theta(0) = 0$ because solutions with different starting point can be found using multiplication in N.

Finally, let us look at explicit local minimizers in order to see how the solutions of the nilpotent system approximate the solutions of the original system, in a neighborhood of the origin. We can solve the original system numerically in Maple. Then we can compare the numeric solutions with solutions of the nilpotent system with the same initial condition. On the figures, we provide two examples of local minimizers $c(t), t \in [0, 2\pi]$ with the initial conditions $x(0) = y(0) = \theta(0) = 0$, $h_1(0) = \frac{1}{2}$, $h_2(0) = \frac{\sqrt{3}}{2}$ and for two choices of $h_3(0)$, namely $h_3(0) = 2$ and $h_3(0) = 20$. Here the dot line denotes the numeric solution of the original system and the solid line denotes the analytic solution of the nilpotent system.



(a) Choice $h_3(0) = 2$.

(b) Choice $h_3(0) = 20$.

FIGURE 4. Comparison of solutions

By a *transformation of a control system* we consider a (local) diffeomorphism of the configuration space preserving the control distribution and the control metric. As usual, the *infinitesimal transformations* are vector fields whose flows are transformations of a control system at each time. In many cases, they simply are (local) infinitesimal transformations of the corresponding parabolic geometry that in addition preserve the control metric. In the case of nilpotent control systems (e.g. nilpotent approximations determined by mechanical systems), infinitesimal transformations form a Lie algebra which is a subalgebra of $\mathfrak{g}$ due to the flatness, see Section 2.1. In particular, there can be transformations preserving the origin, i.e., the isotropy subalgebra is often non–trivial. For example, in the case of the trident snake, infinitesimal transformations of the corresponding nilpotent control system form a 9–dimensional subalgebra of $\mathfrak{so}(3,4)$, where 6 fields reflect the brackets generating $(3,6)$-structure and the isotropy subalgebra is isomorphic to $\mathfrak{so}(3)$, [Mya02]. Infinitesimal transformations of corresponding nilpotent control system of generalized trident snake with filtration $(4,7)$ form a 10–dimensional subalgebra of $\mathfrak{sl}(5,\mathbb{R})$, where 7 fields reflect the brackets generating $(4,7)$-structure and the isotropy subalgebra is isomorphic to $\mathfrak{so}(3)$, [HrZa19] (Chapter 5). Let us emphasize that in the $(3,6)$-case, the isotropy subalgebra is simply the orthogonal algebra of the metric on the distribution at the origin while in the $(4,7)$-case, it is the restriction of the orthogonal algebra to a 3–dimensional subspace of the distribution at the origin given canonically by the generalized path geometry, [ČaSl09].

Infinitesimal transformations of nilpotent control systems are usually easy to find and they give a tool to study points at which local minimizers $c(t)$ stop to be optimal. Consider a trajectory $c(t)$ starting at the origin o of the system (N, D_N, g_N) to some $p \in N$ and assume there is a transformation f of (N, D_N, g_N) such that $f(o) = o$ and $f(p) = p$. If the path $c(t)$ is not contained in the fixed point set of f, then $\hat{c}(t) := f \circ c(t)$ gives another local minimizer of the same length such that $\hat{c}(o) = o$ and $\hat{c}(p) = p$. In this way, we construct two different paths from o to p of the same length, so the local minimizer cannot be optimal after the point p. This is done by [Mya02, MonMor17] for the $(3,6)$-case. We study infinitesimal transformations of nilpotent system for the mechanism with the growth vector $(4,7)$ and their possible action on local minimizers in [HrZa19].

Example 4. In the case of the nilpotent control system (N, D_N, g_N) for the vertical rolling disc form, the Lie algebra of infinitesimal transformations is generated by fields $t_1 = \frac{\partial}{\partial x}$, $t_2 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}$, $t_3 = \frac{\partial}{\partial y}$ and $t_4 = \theta\frac{\partial}{\partial x} - x\frac{\partial}{\partial \theta} + \frac{\theta^2 - x^2}{2}\frac{\partial}{\partial y}$. The transformations t_1, t_2, t_3 satisfy $[t_1, t_2] = t_3$ and generate the Heisenberg algebra. The only isotropy transformation is t_4. The fixed point set of t_4 is of the form $B = \{[0, b, 0] : b \in \mathbb{R}\}$. One can verify that each local minimizer (7) intersects with B at the point $[0, \frac{(C_2^2 + C_3^2)\pi}{C_1^2}, 0]$ at the time $t = \frac{2\pi}{C_1}$. This is the first point with this property and there is a one-parametric family of such minimizers. In particular, each arc-length parametrized local minimizer (7) intersects with B at the point $[0, \frac{\pi}{C_1^2}, 0]$ at the time $t = \frac{2\pi}{C_1}$.

Let us finally remark that the analytic curves (solid-line) in Figure 4 are pictured exactly to the points where they stop to be optimal.

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Chapter 1: Symmetries of parabolic contact structures

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SYMMETRIES OF PARABOLIC CONTACT STRUCTURES

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Abstract. We generalize the concept of locally symmetric spaces to parabolic contact structures. We show that symmetric normal parabolic contact structures are torsion-free and some types of them have to be locally flat. We prove that each symmetry given at a point with non-zero harmonic curvature is involutive. Finally we give restrictions on number of different symmetries which can exist at such a point.

Affine locally symmetric spaces are well known and studied objects in differential geometry. The classical definition says that a local symmetry at x on a manifold M with an affine connection ∇ is a locally defined affine transformation such that $s_x(x) = x$ and $T_x s_x = -\mathrm{id}$ on T_xM. We can understand ∇ as a geometric structure on M such that the symmetry s_x preserves this structure. There is a natural generalization of this concept: For a manifold with an arbitrary geometric structure, one can define a local symmetry as an automorphism of this geometric structure, which satisfies the two above conditions. Best known examples of this concept are Riemannian symmetric spaces, which play an important role in theoretical physics, see [1, 2], and projective symmetric spaces, see [3]. This concept also generalizes nicely to geometric structures which can be described as $|1|$-graded parabolic geometries, see [4, 5].

In this article, we are interested in symmetries of contact manifolds endowed with some additional structures which can be described as parabolic geometries, the so-called parabolic contact structures, see [6]. Discussion of the Levi bracket implies that we cannot define a symmetry in the classical sense, see [4]. Motivated by the definition of a symmetry for Cauchy-Riemann structures from [7], we define a symmetry at x as a morphism of the contact geometry such that $s_x(x) = x$ and $T_x s_x = -\mathrm{id}$ on the contact distribution at x. This definition works nicely for all parabolic contact structures. Then, to study symmetries on parabolic contact structures, we can use general techniques known from the theory of parabolic geometries, see [8, 6].

In this article, we discuss the curvature of symmetric parabolic contact geometries in detail. The theory of harmonic curvature for parabolic geometries allows us to prove that symmetric normal parabolic contact geometries must be torsion-free. Moreover, some types of them have to be locally flat, if they are symmetric.

Contrary to the classical case, symmetries of parabolic contact structures do not satisfy $s_x \circ s_x = \mathrm{id}$ in general, i.e. they are not necessarily involutive. We use the theory of Weyl structures to study this question. More precisely, we show that only locally flat geometries can carry non-involutive symmetries at each point. Moreover, for each involutive symmetry on a parabolic contact geometry, there exists an admissible affine connection which is invariant with respect to the symmetry. Finally we show that in many cases, there can exist at most one symmetry at points with non-zero curvature.

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1. Parabolic contact structures

We recall here basic definitions and summarize the properties of parabolic geometries. We discuss parabolic contact structures in detail. We also introduce Weyl structures, which will be our main tool to deal with parabolic contact geometries. We follow the concepts and notation of [6, 8] and the reader can find all the details and proofs therein.

1.1. **Contact structures and parabolic geometries.** Consider a manifold M endowed with a distribution H of TM of corank one. Then $H \subset TM$ forms a filtration and, on the graded bundle $\mathrm{gr}(TM) = H \oplus TM/H$, there is the *Levi bracket* $\mathcal{L} : H \times H \to TM/H$ which is a bilinear bundle map induced by the Lie bracket of vector fields. The well known definition says that $H \subset TM$ forms a *contact structure* on M if the Levi bracket is non-degenerate at each point. The subbundle H is then called *contact distribution*.

We will discuss here contact manifolds endowed with some additional structures which can be described as parabolic geometries. Let us recall that, for a semisimple Lie group G and its parabolic subgroup P, a *parabolic geometry* of type (G, P) is a pair $(p : \mathcal{G} \to M, \omega)$ consisting of a principal P-bundle $\mathcal{G} \to M$ and a 1-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, called the *Cartan connection*, which is P-equivariant, reproduces generators of fundamental vector fields and induces a linear isomorphism $T_u\mathcal{G} \cong \mathfrak{g}$ for each $u \in \mathcal{G}$. The Lie algebra $\mathfrak{g}$ of G is then equipped (up to the choice of Levi factor $\mathfrak{g}_0$ in $\mathfrak{p}$) with a grading of the form $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ such that the Lie algebra of P is exactly $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. There is the usual notation such that $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$, $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and $P_+ \subset P$ is the subgroup corresponding to $\mathfrak{p}_+$. By G_0 we denote the subgroup in P, with Lie algebra $\mathfrak{g}_0$, consisting of all elements whose Ad-action preserves the grading of $\mathfrak{g}$. Each element g of P can be uniquely written as $g_0 \exp Z_1 \cdots \exp Z_k$ for suitable $g_0 \in G_0$ and $Z_i \in \mathfrak{g}_i$, thus $\exp Z_1 \cdots \exp Z_k \in P_+$. Let us recall that for each parabolic geometry, there is an element $E \in \mathfrak{g}_0$ with the property $[E, X] = iX$ for each $X \in \mathfrak{g}_i$, the so-called *grading element*. To study contact structures, we have to focus on a special case of $|2|$-grading: A *contact grading* of a simple Lie algebra $\mathfrak{g}$ is a grading $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $\mathfrak{g}_{-2}$ is one-dimensional and the Lie bracket $[-,-] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non-degenerate. Let us remark that for each contact grading, the subspace $[\mathfrak{g}_{-2}, \mathfrak{g}_2]$ coincides with the subspace generated by E.

It is well known that the Cartan connection ω provides an identification $TM \simeq \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$. Suppose we have a parabolic geometry corresponding to a contact grading. Because each contact grading of $\mathfrak{g}$ induces P-invariant filtration of the form $\mathfrak{g} = \mathfrak{g}^{-2} \supset \mathfrak{g}^{-1} \supset \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 = \mathfrak{g}_2$, the subspace $\mathfrak{g}^{-1}/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$ defines a subbundle $T^{-1}M := \mathcal{G} \times_P \mathfrak{g}^{-1}/\mathfrak{p}$ of corank one in TM. There is the Levi bracket on $\mathrm{gr}(TM) = T^{-1}M \oplus TM/T^{-1}M$ and the geometry is called *regular* if the Levi bracket corresponds to the Lie bracket $[-,-] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ under the above identification. Then, for regular parabolic geometries corresponding to contact gradings, the underlying filtration $T^{-1}M \subset TM$ defines a contact structure on M, and each such geometry is called *parabolic contact structure* or *parabolic contact geometry*. Moreover, define $\mathcal{G}_0 := \mathcal{G}/P_+$, which is a principal G_0-bundle over M. This is the reduction of the natural frame bundle of $\mathrm{gr}(TM)$ with respect to $\mathrm{Ad} : G_0 \to Gl(\mathfrak{g}_{-1})$ and, in this way, we get an additional geometric structure on $T^{-1}M$.

Let us recall some more facts on parabolic contact structures that will be needed: The P-bundle $G \to G/P$ together with the (left) Maurer-Cartan form $\omega_G \in \Omega^1(G, \mathfrak{g})$ forms a geometry that is called *homogeneous model*. A *morphism* between geometries of type (G, P) from $(\mathcal{G} \to M, \omega)$ to $(\mathcal{G}' \to M', \omega')$ is a P-bundle morphism $\varphi : \mathcal{G} \to \mathcal{G}'$ such that $\varphi^*\omega' = \omega$. We will suppose that the maximal normal subgroup of G which is contained in P is trivial. With this assumption, there is one-to-one correspondence between morphisms of parabolic geometries and their base morphisms. Let us recall that such geometries are called *effective.*

The curvature is described by P-equivariant mapping $\kappa : \mathcal{G} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, the so-called *curvature function.* The Maurer-Cartan equation implies that the curvature of the homogeneous model vanishes. Conversely, it can be proved that if the curvature of a geometry vanishes, then it is locally isomorphic to the homogeneous model of the same type. If κ has its values in a subbundle $\wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$, we call the geometry *torsion-free.* The regular geometry is called *normal* if the curvature satisfies $\partial^* \circ \kappa = 0$, where ∂^* is the differential in the standard complex computing Lie algebra homology of $\mathfrak{p}_+$ with coefficients in $\mathfrak{g}$. Then we can define the *harmonic curvature* κ_H which is the composition of the curvature function with the projection $\ker(\partial^*) \to \ker(\partial^*)/\mathrm{im}(\partial^*)$. There is the following general statement, see [6]:

Theorem. *On a regular normal parabolic geometry, the curvature κ vanishes if and only if the harmonic curvature κ_H vanishes.*

It can be proved that $\ker(\partial^*)/\mathrm{im}(\partial^*)$ is a G_0-submodule of $\wedge^2\mathfrak{g}_-^* \otimes \mathfrak{g}$ and decomposes into the direct sum of components each of which is contained in some $\mathfrak{g}_{-i}^* \wedge \mathfrak{g}_{-j}^* \otimes \mathfrak{g}_k$. According to this decomposition, κ_H decomposes into the sum of components of homogeneity $\ell = i + j + k$ that we denote by $\kappa^{(\ell)}$. One can use the Kostant's version of the Bott-Borel-Weil theorem to find all the components of κ_H. For parabolic contact geometries, it turns out that there can exist only the following three types of components:

- $\kappa^{(1)}$ valued in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$,
- $\kappa^{(2)}$ valued in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$ and
- $\kappa^{(4)}$ valued in $\mathfrak{g}_{-2}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1$.

See the Appendix for summary of all contact gradings and corresponding geometries with their components of harmonic curvature. Detailed description and computation of components of harmonic curvature for parabolic contact structures can be found in Section 4.2. of [6].

1.2. **Adjoint tractor bundles and Weyl structures.** Here, let us briefly introduce the concept of adjoint tractor bundles, which allows us to write formulas and make computations in a more convenient form. The *adjoint tractor bundle* is the natural bundle $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$ corresponding to the restriction of Ad-action of G on $\mathfrak{g}$. For each parabolic contact geometry, the filtration of $\mathfrak{g}$ induces a filtration $\mathcal{A}M = \mathcal{A}^{-2}M \supset \mathcal{A}^{-1}M \supset \mathcal{A}^0M \supset \mathcal{A}^1M \supset \mathcal{A}^2M$ such that $\mathcal{A}^iM = \mathcal{G} \times_P \mathfrak{g}^i$, and there is the associated graded bundle $\mathrm{gr}(\mathcal{A}M) = \mathcal{A}_{-2}M \oplus \mathcal{A}_{-1}M \oplus \mathcal{A}_0M \oplus \mathcal{A}_1M \oplus \mathcal{A}_2M$, where $\mathcal{A}_iM = \mathcal{A}^iM/\mathcal{A}^{i+1}M$ equals to $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_i$. Clearly, $TM \simeq \mathcal{A}M/\mathcal{A}^0M$ and $T^*M \simeq \mathcal{A}^1M$. On the graded bundle $\mathrm{gr}(\mathcal{A}M)$, there is the *algebraic bracket* $\{-,-\} : \mathcal{A}_iM \times \mathcal{A}_jM \to \mathcal{A}_{i+j}M$ defined by means of the Lie bracket on $\mathfrak{g}$. Clearly, its part $\mathcal{A}_{-1}M \times \mathcal{A}_{-1}M \to \mathcal{A}_{-2}M$ on $\mathrm{gr}(TM) = \mathcal{A}_{-2}M \oplus \mathcal{A}_{-1}M$ coincides with the Levi bracket thanks to the regularity. Since each fiber of $\mathrm{gr}(\mathcal{A}M)$ is isomorphic to $\mathfrak{g}$, the grading element defines a unique element $E(x) \in \mathrm{gr}_0(\mathcal{A}_xM)$ such that $\{E(x), -\}$ is a multiplication by i on $\mathrm{gr}_i(\mathcal{A}_xM)$. In fact, these elements form a section E of $\mathrm{gr}_0(\mathcal{A}M)$ which is called the *grading section.* Let us also remark that we simultaneously get an action $\bullet$ of $\mathrm{gr}(\mathcal{A}M)$ on arbitrary tensor products of $\mathrm{gr}(\mathcal{A}M)$ which is given using the tensoriality of the algebraic bracket. In particular, the grading section acts on each homogeneous component of the tensor as multiplication by its homogeneity.

Now we should recall basic facts on Weyl structures. For any parabolic contact geometry $(\mathcal{G} \to M, \omega)$ with the underlying G_0-bundle $p_0 : \mathcal{G}_0 \to M$, a *Weyl structure* is a global smooth G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the canonical projection $\pi : \mathcal{G} \to \mathcal{G}_0$. Weyl structures always exist, and for any two Weyl structures σ and $\hat{\sigma}$, there are G_0-equivariant functions $\Upsilon_1 : \mathcal{G}_0 \to \mathfrak{g}_1$ and $\Upsilon_2 : \mathcal{G}_0 \to \mathfrak{g}_2$ such that

$$\hat{\sigma}(u_0) = \sigma(u_0) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0)$$

for all $u_0 \in \mathcal{G}_0$. Clearly, $\Upsilon_i \in \Gamma(\mathcal{A}_i M)$ and $\Upsilon := (\Upsilon_1, \Upsilon_2)$ is a smooth section of $\mathrm{gr}(T^*M)$. Moreover, the Campbell-Baker-Hausdorff formula implies $\exp \Upsilon = \exp \Upsilon_1 \exp \Upsilon_2 = \exp(\Upsilon_1 + \Upsilon_2) = \exp \Upsilon_2 \exp \Upsilon_1$, see [9].

For each Weyl structure σ, we can form the pullback $\sigma^*\omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$. This decomposes as $\sigma^*\omega = \sigma^*\omega_- + \sigma^*\omega_0 + \sigma^*\omega_+$, of which the part $\sigma^*\omega_- \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_-)$ is called the *soldering form.* Each Weyl structure σ induces, by means of its soldering form, an isomorphism $TM \simeq \mathrm{gr}(TM)$ which we write as $\xi \mapsto (\xi_{-2}, \xi_{-1})$. If $\sigma \exp \Upsilon_1 \exp \Upsilon_2$ is another Weyl structure, the isomorphism changes as $\xi \mapsto (\xi_{-2}, \xi_{-1} - \{\Upsilon_1, \xi_{-2}\})$. In particular, σ and $\sigma \exp \Upsilon_2$ induce the same isomorphism for an arbitrary $\Upsilon_2 : \mathcal{G}_0 \to \mathfrak{g}_2$.

The part $\sigma^*\omega_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ defines a principal connection on $p_0 : \mathcal{G}_0 \to M$ which we call the *Weyl connection.* This connection induces connections on all associated bundles. In particular, for each σ we get a preferred connection on $\mathrm{gr}(TM) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$ and, via the above isomorphism, we get a preferred connection on the tangent bundle, cotangent bundle and their tensor products. We call each such connection Weyl connection, too. For a Weyl structure σ, we denote the corresponding connection by ∇^σ. For σ and $\hat{\sigma} = \sigma \exp \Upsilon_1 \exp \Upsilon_2$ we have

$$\nabla^{\hat{\sigma}}_\xi s = \nabla^\sigma_\xi s + \big(\frac{1}{2}\{\Upsilon_1, \{\Upsilon_1, \xi_{-2}\}\} - \{\Upsilon_2, \xi_{-2}\} - \{\Upsilon_1, \xi_{-1}\}\big) \bullet s, \tag{1}$$

where $\xi \in \mathfrak{X}(M)$ and s is a section of an appropriate bundle. The positive part $\sigma^*\omega_+ \in \Omega^1(\mathcal{G}_0, \mathfrak{p}_+)$ is called *Rho-tensor* and is denoted by P^σ. We will not need it explicitly, see [6, 8] for details.

Let us finally recall the so-called normal Weyl structures. A *normal Weyl structure* at u is the only G_0-equivariant section $\sigma_u : \mathcal{G}_0 \to \mathcal{G}$ satisfying $\sigma_u \circ \pi \circ \mathrm{Fl}_1^{\omega^{-1}(X)}(u) = \mathrm{Fl}_1^{\omega^{-1}(X)}(u)$, where by $\mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ we denote flows of constant vector fields $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{G})$. Each normal Weyl structure σ_u is defined locally over some neighborhood of $p(u)$ and depends only on the G_0-orbit of $u \in \mathcal{G}$, see [6].

2. Basic facts on symmetries

We formulate here the definition of a symmetry on a parabolic contact geometry and describe its basic properties. We study the action of symmetries on Weyl structures and describe some interesting subclasses of them. We focus here on the question of involutivity of our symmetries.

2.1. **Definitions.** Let $(\mathcal{G} \to M, \omega)$ be a parabolic contact structure. A (*local*) *symmetry with the center at* $x \in M$ is a (locally defined) diffeomorphism s_x on M such that:

(1) $s_x(x) = x$,
(2) $T_x s_x = -\mathrm{id}$ on $T_x^{-1}M$,
(3) s_x is a base morphism of some (locally defined) automorphism φ of the parabolic contact geometry.

The geometry is called (*locally*) *symmetric* if there is a (local) symmetry at each point $x \in M$.

Clearly, each symmetry is a local symmetry. In this article, we discuss local symmetries and local properties of locally symmetric geometries and for conciseness will say ‘symmetry at x’ and ‘symmetric’ instead of ‘local symmetry at x’ and ‘locally symmetric’, respectively. Global symmetries and their systems we will discuss elsewhere. Moreover, we will also call the automorphism φ of $\mathcal{G}$ and its underlying automorphism φ_0 of $\mathcal{G}_0$ a 'symmetry at x'.

2.2. **Basic properties of symmetries.** Let s_x be a symmetry on a parabolic contact geometry and let φ be as above. Since each symmetry s_x preserves x, the (uniquely given) automorphism φ has to preserve the fiber over x. Then for each frame $u \in p^{-1}(x)$ we have $\varphi(u) = ug_0 \exp Z = ug_0 \exp Z_1 \exp Z_2$ for suitable $g_0 \in G_0$, $Z_1 \in \mathfrak{g}_1$ and $Z_2 \in \mathfrak{g}_2$, where $Z = Z_1 + Z_2$. Let us describe the element $g_0 \exp Z_1 \exp Z_2$ in detail:

For each $\xi(x) = [\![u, X]\!] \in T_x^{-1}M$, i.e. for each $X \in \mathfrak{g}^{-1}/\mathfrak{p}$, we have

$$\begin{aligned} T_x s_x.\xi(x) &= [\![\varphi(u), X]\!] = [\![ug_0 \exp Z_1 \exp Z_2, X]\!] = \\ &= [\![u, \underline{\mathrm{Ad}}_{\exp(-Z_2)} \underline{\mathrm{Ad}}_{\exp(-Z_1)} \mathrm{Ad}_{g_0^{-1}} X]\!] \end{aligned} \tag{2}$$

and simultaneously $T_x s_x.\xi(x) = -\xi(x) = [\![u, -X]\!]$. All together, the element $g_0 \exp Z_1 \exp Z_2$ has to induce $-\mathrm{id}$ on $\mathfrak{g}^{-1}/\mathfrak{p}$ by the $\underline{\mathrm{Ad}}$-action. Moreover, $\exp Z_1 \exp Z_2$ acts trivially on $\mathfrak{g}^{-1}/\mathfrak{p}$. Indeed, there is the formula

$$\mathrm{Ad}_{\exp Z} X = \sum_{j=0}^{\infty} \frac{1}{j!} \mathrm{ad}_Z^j X = X + [Z, X] + \frac{1}{2}[Z, [Z, X]] + \cdots \tag{3}$$

for all $X \in \mathfrak{g}_-$ and $Z \in \mathfrak{p}_+$ and if $X \in \mathfrak{g}_{-1}$, all brackets on the right hand side belong to $\mathfrak{p}$. In fact, along the fiber over x, the P_+-parts of the above elements are determined by φ and can be arbitrary in general, one only has to impose the compatibility of φ with the right action of P. Then the element g_0 has to cause the sign change on $\mathfrak{g}^{-1}/\mathfrak{p} \simeq \mathfrak{g}_{-1}$. Since our geometries are effective, there can exist at most one element $g_0 \in G_0$ which gives $-\mathrm{id}$ on $\mathfrak{g}_{-1}$ and it has to be the same element along the fiber. In particular, the underlying morphism φ_0 is of the form $\varphi_0(u_0) = u_0 g_0$ for each $u_0 \in p_0^{-1}(x)$. Clearly, the element g_0 has to induce identity on $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$.

One of basic properties of classical symmetries is their involutivity and there is a natural question on involutivity of our symmetries. Thus let us focus on $s_x^2 := s_x \circ s_x$. Let us first point out that $\varphi \circ \varphi = \mathrm{id}_{\mathcal{G}}$ if and only if $s_x \circ s_x = \mathrm{id}_M$, which follows directly from effectivity. Clearly, φ_0 is then involutive, too. Thus it suffice to study the morphism φ^2. In the above notation, $\varphi^2(u) = ug_0 \exp Z_1 \exp Z_2 g_0 \exp Z_1 \exp Z_2$ holds, and using the known fact $\exp X g_0 = g_0 \exp(\mathrm{Ad}_{g_0} X)$, we can rewrite this as $ug_0^2 \exp(-Z_1) \exp Z_2 \exp Z_1 \exp Z_2$. Moreover, g_0^2 acts as id on $\mathfrak{g}_{-1}$ and thus on $\mathfrak{g}$, because $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$, $\mathfrak{p} = \mathfrak{g}_-^*$ and $\mathfrak{g}_0 \subset \mathfrak{g}_-^* \otimes \mathfrak{g}_-$. Thus it lies in the kernel of the Ad-action which coincides with the maximal normal subgroup of G which is contained in P. Effectivity then gives $g_0^2 = e$. All together, we have got $\varphi^2(u) = u \exp 2Z_2$. In another frame uh for $h \in P$ we then have $\varphi^2(uh) = uh \exp 2\mathrm{Ad}_h Z_2$. Thus we can simply view Z_2 as P-equivariant function $Z_2 : p^{-1}(x) \to \mathfrak{g}_2$. In fact, we have proved the following statement:

Lemma. *On a parabolic contact geometry, each symmetry s_x at x defines uniquely a covector $Z_2 \in T_x^*M$ through the equation*

$$\varphi^2(u) = u \exp 2Z_2(u) \tag{4}$$

along the fiber over x. The symmetry s_x is involutive if and only if the covector Z_2 equals to zero.

Thanks to the above observations, it is easy to describe the differential of each symmetry at its center.

Proposition. *For each symmetry s_x at x on a parabolic contact geometry, the mapping $T_x s_x : T_x M \to T_x M$ is involutive, thus $T_x M$ decomposes into two eigenspaces* *with eigenvalues* -1 *and* 1. *The eigenspace corresponding to the eigenvalue* -1 *has to coincide with* $T_x^{-1}M$, *the contact distribution, and there exists a one-dimensional eigenspace corresponding to the eigenvalue* 1.

Proof. For each $\xi(x) = [\![u, X]\!]$ from T_xM we have

$$\begin{aligned} T_x s_x^2.\xi(x) &= [\![\varphi^2(u), X]\!] = [\![u \exp 2Z_2(u), X]\!] = [\![u, \underline{\mathrm{Ad}}_{\exp(-2Z_2(u))}X]\!] \\ &= [\![u, X]\!] = \xi(x), \end{aligned}$$

which follows directly from formulas (4) and (3). The rest follows immediately from the definition of the symmetry. □

2.3. **Action of symmetries on Weyl structures.** Let us now discuss relations of various Weyl structures to the symmetry s_x. For each Weyl structure σ we can write

$$\varphi(\sigma(u_0)) = \sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0) \tag{5}$$

for each $u_0 \in \mathcal{G}_0$ and for suitable functions $\Upsilon_1 : \mathcal{G}_0 \to \mathfrak{g}_1$ and $\Upsilon_2 : \mathcal{G}_0 \to \mathfrak{g}_2$ which are generally determined by φ and σ. With the notation from the last section, we have $\Upsilon_2(u_0) = Z_2(\sigma(u_0))$ in the fiber over x. The Lemma 2.2 then shows that Υ_2 does not depend on the choice of a Weyl structure σ and coincides for all Weyl structures at x. Clearly, Υ_2 vanishes at x if and only if s_x is involutive. The function Υ_1 depends on the choice of a Weyl structure σ at x and with the above notation, $\Upsilon_1(u_0) = Z_1$ for $\sigma(u_0) = u$.

Let us now focus on the role of Υ_1 for the isomorphism $TM \simeq \mathrm{gr}(TM)$ given by the Weyl structure σ: For a tangent vector $\xi(x) = [\![\sigma(u_0), X_{-2} + X_{-1}]\!]$ where $X_i \in \mathfrak{g}_i$ we have

$$\begin{aligned} T_x s_x.[\![\sigma(u_0), X_{-2} + X_{-1}]\!] &= [\![\sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0), X_{-2} + X_{-1}]\!] \\ &= [\![\sigma(u_0), X_{-2} - X_{-1} - [\Upsilon_1(u_0), X_{-2}]]\!], \end{aligned}$$

which follows from the fact that $\varphi_0(u_0) = u_0 g_0$ for g_0 giving $-\mathrm{id}$ on $\mathfrak{g}_{-1}$ and from formulas (5) and (3). In particular, the isomorphism $TM \simeq \mathrm{gr}(TM)$ given by a Weyl structure σ reflects the decomposition of T_xM into ±1-eigenspaces for $T_x s_x$ if and only if the Weyl structure σ satisfies $\Upsilon_1(u_0) = 0$ for each u_0 from the fiber over x.

Lemma. *On a parabolic contact geometry with a symmetry* s_x *at* x, *there are Weyl structures* $\hat{\sigma}$ *such that* $\varphi(\hat{\sigma}(u_0)) = \hat{\sigma}(\varphi_0(u_0)) \exp \hat{\Upsilon}_1(u_0) \exp \Upsilon_2(u_0)$ *holds for suitable* $\hat{\Upsilon}_1$ *such that* $\hat{\Upsilon}_1(u_0) = 0$ *for each* u_0 *from the fiber over* x.

Proof. Consider an arbitrary Weyl structure σ and let (Υ_1, Υ_2) be determined by σ as above. Let us verify that the Weyl structure

$$\hat{\sigma}(u_0) = \sigma(u_0) \exp(-\frac{1}{2}\Upsilon_1(u_0))$$

satisfies the condition: The formula (5) and the Campbell-Baker-Hausdorff formula allow us to write

$$\begin{aligned} \varphi(\hat{\sigma}(u_0)) &= \varphi(\sigma(u_0)) \exp(-\frac{1}{2}\Upsilon_1(u_0)) \\ &= \sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0) \exp(-\frac{1}{2}\Upsilon_1(u_0)) \\ &= \sigma(\varphi_0(u_0)) \exp \frac{1}{2}\Upsilon_1(u_0) \exp \Upsilon_2(u_0). \end{aligned}$$

Equivariancy of Υ_1 gives $\Upsilon_1(\varphi_0(u_0)) = \Upsilon_1(u_0 g_0) = -\Upsilon_1(u_0)$ in the fiber over x for g_0 giving $-\mathrm{id}$ on $\mathfrak{g}_{-1}$ and we can rewrite the above expression as

$$\sigma(\varphi_0(u_0)) \exp(-\frac{1}{2}\Upsilon_1(\varphi_0(u_0))) \exp \Upsilon_2(u_0) = \hat{\sigma}(\varphi_0(u_0)) \exp \Upsilon_2(u_0)$$

in the fiber over x. Thus $\hat{\sigma}$ is the required Weyl structure. □

Let us call each Weyl structure $\hat{\sigma}$ satisfying the condition in the lemma an *almost s_x-invariant Weyl structure* at x. All almost s_x-invariant Weyl structures form a family of Weyl structures which is parametrized over $\mathfrak{g}_2$ at x. Really, all Weyl structures inducing the same isomorphism $T_xM \simeq \mathrm{gr}(T_xM)$ as $\hat{\sigma}$ are of the form $\hat{\sigma} \exp F_1(u_0) \exp F_2(u_0)$ for arbitrary functions $F_2 : \mathcal{G}_0 \to \mathfrak{g}_2$ and $F_1 : \mathcal{G}_0 \to \mathfrak{g}_1$ where $F_1(u_0) = 0$ in the fiber over x, see 1.2.

Let us finally describe the involutivity of our symmetries in the language of Weyl structures.

Proposition. *On a parabolic contact geometry with a symmetry s_x at x, the following facts are equivalent:*

- (a) *the symmetry s_x is involutive,*
- (b) *there exists a Weyl structure σ such that $\varphi(\hat{\sigma}(u_0)) = \sigma(\hat{\varphi}_0(u_0))$ holds in the fiber over x,*
- (c) *there exists a Weyl structure σ_u such that $\varphi(\sigma_u(u_0)) = \sigma_u(\varphi_0(u_0))$ holds over some neighborhood of x.*

Proof. $(a) \Rightarrow (b)$ Let $\hat{\sigma}$ be an arbitrary almost s_x-invariant Weyl structure. The Lemma 2.2 says that the involutivity implies vanishing of Υ_2 in the fiber over x. Thus if s_x is involutive, the almost s_x-invariant Weyl structure $\hat{\sigma}$ has to satisfy (b).

$(b) \Rightarrow (c)$ Let $\hat{\sigma}$ be an arbitrary Weyl structure satisfying $\varphi(\hat{\sigma}(u_0)) = \hat{\sigma}(\varphi_0(u_0))$ in the fiber over x. Consider the normal Weyl structure σ_u such that $\sigma_u(u_0) = \hat{\sigma}(u_0)$ for $p_0(u_0) = x$. The condition of the normality prescribes σ_u uniquely on a normal neighborhood of $x \in M$, see 1.2 for definition. But then, because $\varphi(\sigma_u(u_0)) = \sigma_u(\varphi_0(u_0))$ holds in the fiber over x, it has to hold over some normal neighborhood of x and σ_u satisfies (c).

$(c) \Rightarrow (a)$ Consider an arbitrary Weyl structure σ satisfying (c). This can be equivalently written as $\varphi^{-1}(\sigma(\varphi_0(u_0))) = \sigma(u_0)$ which means that the corresponding Weyl connection is invariant with respect to s_x. Since the isomorphism $T_xM \simeq \mathrm{gr}(T_xM)$ reflects the decomposition of T_xM into ± 1-eigenspaces, we can describe s_x on a neighborhood of x nicely via geodesics of the invariant connection. Indeed, each vector $(\xi_{-2}(x), \xi_{-1}(x)) \in T_xM$ determines uniquely a geodesic at x, and the symmetry s_x maps it on a geodesic at x, which is uniquely determined by a vector $(\xi_{-2}(x), -\xi_{-1}(x))$. This describes s_x on a neighborhood of x and one can see directly that it has to be involutive. □

Let us call each Weyl structure satisfying the condition (b) of the Proposition *s_x-invariant Weyl structure at x* and each Weyl structure satisfying the condition (c) of the Proposition *s_x-invariant Weyl structure on a neighborhood of x.*

3. Symmetries of homogeneous models

In this section, we focus on homogeneous models, which are the simplest examples of parabolic contact geometries. We describe explicitly their symmetries and we give some concrete examples of homogeneous symmetric geometries.

3.1. **Description of symmetries.** Let $(G \to G/P, \omega_G)$ be a homogeneous model of a parabolic contact geometry of type (G, P). It is well known that all automorphisms of the homogeneous model are just left multiplications by elements of G and an analog of the Liouville theorem states that any local automorphism can be uniquely extended to a global one, see [10, 6]. Thus if the homogeneous model is locally symmetric, then it is symmetric. Moreover, because G acts transitively on G/P, it suffices to find a symmetry at the origin to decide whether the homogeneous model is symmetric.

Proposition. *All symmetries of the homogeneous model of a parabolic contact geometry centered at the origin $o = eP$ are given by left multiplications by elements $g_0 \exp Z_1 \exp Z_2 \in P$, where $Z_1 \in \mathfrak{g}_1$ and $Z_2 \in \mathfrak{g}_2$ are arbitrary and $g_0 \in G_0$ is such that $\mathrm{Ad}_{g_0} = -\mathrm{id}$ on $\mathfrak{g}_{-1}$. In particular, if there is one symmetry at a point, then there is an infinite amount of them.*

Proof. For homogeneous models, $T^{-1}(G/P) = G \times_P \mathfrak{g}^{-1}/\mathfrak{p}$. Then we can write each tangent vector $\xi(o) \in T_o^{-1}(G/P)$ as $\xi(o) = [\![e, X]\!]$ for suitable $X \in \mathfrak{g}^{-1}/\mathfrak{p}$. Since automorphisms of the homogeneous model are left multiplications λ_g by elements $g \in G$, all symmetries at the origin are exactly left multiplications λ_g satisfying $\lambda_g(o) = o$ and $T_o\lambda_g.\xi(o) = -\xi(o)$ for all contact vectors $\xi(o)$. The first condition is equivalent to the fact that $g \in P$. Then g can be written as $g = g_0 \exp Z_1 \exp Z_2$ and the second condition means that

$$T_o\lambda_{g_0 \exp Z_1 \exp Z_2}.[\![e, X]\!] = [\![g_0 \exp Z_1 \exp Z_2, X]\!] = [\![e, \underline{\mathrm{Ad}}^{-1}_{g_0 \exp Z_1 \exp Z_2} X]\!]$$

and $-\xi(o) = [\![e, -X]\!]$ coincide for each $X \in \mathfrak{g}^{-1}/\mathfrak{p}$. Thus we look for elements $g \in P$ such that $\underline{\mathrm{Ad}}_{\exp(-Z_2)}\underline{\mathrm{Ad}}_{\exp(-Z_1)}\mathrm{Ad}_{g_0^{-1}}X = -X$ for all $X \in \mathfrak{g}^{-1}/\mathfrak{p}$ and the rest follows immediately from observations in Section 2.2. □

Let us finally discuss involutivity of these symmetries. The symmetry $g_0 \exp Z_1 \exp Z_2$ is involutive if and only if the element $(g_0 \exp Z_1 \exp Z_2)^2$ induces identity on G/P and effectivity says that it has to be equal to e. We have

$$g_0 \exp Z_1 \exp Z_2 g_0 \exp Z_1 \exp Z_2 = e \exp 2Z_2.$$

Thus involutive symmetries at the origin are left multiplications by elements $g_0 \exp Z_1$ where g_0 and Z_1 are as above and Z_2 has to be equal to zero. If Z_2 is non-zero, then the symmetry is not involutive. In particular, there exist non-involutive symmetries on homogeneous models.

Clearly, if g induces (involutive) symmetry at the origin $o = eP$, then hgh^{-1} induces (involutive) symmetry at the point hP.

3.2. **Examples.** Let us introduce here some examples of parabolic contact structures and discuss symmetries on their homogeneous models, see [6] for detailed description.

Lagrangean contact structures. Let us start with $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$, the split real form of $\mathfrak{sl}(n+2, \mathbb{C})$, for $n \geq 1$. This admits a contact grading which is given by the following decomposition into blocks of sizes 1, n and 1:

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^L & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^L & \mathfrak{g}_0 & \mathfrak{g}_1^R \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^R & \mathfrak{g}_0 \end{pmatrix}.$$

The splittings $\mathfrak{g}_{\pm 1} = \mathfrak{g}_{\pm 1}^L \oplus \mathfrak{g}_{\pm 1}^R$ are $\mathfrak{g}_0$-invariant and $\mathfrak{g}_{-1}^L$ and $\mathfrak{g}_{-1}^R$ are isotropic for $[-, -] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$. Let us choose $G = PGL(n+2, \mathbb{R})$, the quotient of $GL(n+2, \mathbb{R})$ by its center. Then P consists of classes of block upper triangular matrices and G_0 of block diagonal matrices. In particular, G_0 coincides by means of the Ad-action with the group of all automorphisms of graded Lie algebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$

which in addition preserve the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}$. Thus for a parabolic contact geometry of type (G, P), the underlying geometry consists of a contact distribution together with a fixed decomposition of the form $T^{-1}M = L \oplus R$ into two subbundles (of rank n), each of which is isotropic with respect to $\mathcal{L}$. These geometries are known as Lagrangean contact structures. The homogeneous model is the flag manifold of lines in hyperplanes in $\mathbb{R}^{n+2}$.

Let us now discuss symmetries at the origin of the homogeneous model. We look for an element $g_0 \in G_0$ such that $\mathrm{Ad}_{g_0} X = -X$ for each $X \in \mathfrak{g}_{-1}$. Elementary matrix computation shows that there is a solution which is represented by the matrix of the form

$$g_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where E is the identity matrix, and thus the homogeneous model is symmetric. All symmetries at the origin are represented by matrices of the form

$$\begin{pmatrix} -1 & -V & \gamma \\ 0 & E & W \\ 0 & 0 & -1 \end{pmatrix},$$

where $V^*, W \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ are arbitrary, and the involutive ones have to satisfy $\gamma = -\frac{1}{2}VW$.

Non-degenerate partially integrable almost CR-structures of hypersurface type. Consider the real form $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$ of $\mathfrak{sl}(n+2, \mathbb{C})$ for $p + q = n \geq 1$. For a suitable choice of Hermitian product, this admits a contact grading of the same block form as the Lagrangean case. Denoting $\mathbb{I}$ the diagonal $n \times n$-matrix with the first p entries equal to 1 and the remaining q entries equal to -1, we write the elements explicitly as

$$\begin{pmatrix} a & Z & iz \\ X & A & -\mathbb{I}Z^* \\ ix & -X^*\mathbb{I} & -\bar{a} \end{pmatrix},$$

where $x, z \in \mathbb{R}$, $a \in \mathbb{C}$, $X, Z^T \in \mathbb{C}^n$, $A \in \mathfrak{u}(n)$ and $a + \mathrm{tr}A - \bar{a} = 0$. The bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is given by $[X, Y] = Y^*\mathbb{I}X - X^*\mathbb{I}Y$, which is twice the imaginary part of the standard Hermitian product of signature (p, q). Choose $G = PSU(p+1, q+1)$, the quotient of $SU(p+1, q+1)$ by its center. Then P consists of classes of block upper triangular matrices, and elements of G_0 are represented by block diagonal matrices. For parabolic contact geometries of type (G, P), the underlying geometry consists of a contact distribution $T^{-1}M$ together with a complex structure J such that $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in \Gamma(T^{-1}M)$. Such geometries are known as partially integrable almost CR-structures of hypersurface type. The homogeneous model is the projectivized null cone of a Hermitian form of signature $(p+1, q+1)$, which is a real hypersurface in $\mathbb{C}P^{n+1}$.

Let us now discuss briefly symmetries of the homogeneous model at the origin. We look for an element $g_0 \in G_0$ such that $\mathrm{Ad}_{g_0} X = -X$ for all $X \in \mathfrak{g}_{-1}$. Elementary computation shows that the solution exists and is given by the same matrix g_0 as in the Lagrangean case. Then all symmetries at the origin are represented by matrices of the form

$$\begin{pmatrix} -1 & -Z & iz \\ 0 & E & -\mathbb{I}Z^* \\ 0 & 0 & -1 \end{pmatrix}$$

where $Z \in \mathbb{C}^{n*}$ and $z \in \mathbb{R}$ are arbitrary, and the involutive ones have to satisfy $iz = Z\mathbb{I}Z^*$.

4. Curvature restrictions

In this section, we discuss restrictions on the curvature of a parabolic contact geometry, which are caused by the existence of a symmetry. We study the torsion of symmetric parabolic contact geometries in detail. We show that there are relations between the curvature of a symmetric geometry and involutivity of its symmetries.

4.1. **Torsion restrictions.** Let us work with normal parabolic contact geometries here. In fact, the normality assumption is only some technical restriction which plays no role, if we understand symmetries as morphisms of the underlying geometry, and this clearly is the most reasonable point of view. For such underlying geometry, there are various non-isomorphic parabolic geometries inducing this underlying structure and it can be proved that the normal one always exists, see [6]. Assuming normal geometry, we can discuss its components of harmonic curvature, which are easily computable and provide information on the whole curvature of the parabolic geometry, see Section 1.1 and Appendix. Let us start with the harmonic torsion $\kappa^{(1)}$, which has its values in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$.

Lemma. *If there is a symmetry s_x at x on a normal parabolic contact geometry, then $\kappa^{(1)}$ vanishes at x.*

Proof. Let φ be as usual. For $u \in p^{-1}(x)$ we have $\varphi(u) = ug_0 \exp Z$ for suitable $g_0 \in G_0$ and $Z \in \mathfrak{p}_+$, see 2.2. Then for each $X, Y \in \mathfrak{g}_{-1}$ we get

$$\kappa^{(1)}(\varphi(u))(X,Y) = \kappa^{(1)}(ug_0 \exp Z)(X,Y) = \exp(-Z)g_0^{-1} \cdot \kappa^{(1)}(X,Y)$$

where $\cdot$ denotes the induced Ad-action on $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$. The action of g_0^{-1} is of the form

$$g_0^{-1} \cdot \kappa^{(1)}(X,Y) = \mathrm{Ad}_{g_0^{-1}}(\kappa^{(1)}(u)(\mathrm{Ad}_{g_0} X, \mathrm{Ad}_{g_0} Y)) =$$

$$-\kappa^{(1)}(u)(-X,-Y) = -\kappa^{(1)}(u)(X,Y)$$

since the element g_0 acts as $-\mathrm{id}$ on $\mathfrak{g}_{-1}$, and the action of $\exp(-Z)$ is trivial. Because automorphisms preserve curvature, $-\kappa^{(1)}(u)(X,Y)$ has to be equal to $\kappa^{(1)}(u)(X,Y)$ in the fiber over x and then it has to vanish at x. □

Proposition. *Each normal symmetric parabolic contact geometry is torsion-free. Moreover, normal symmetric*

- *Lie contact structures,*
- *parabolic contact geometries corresponding to exceptional Lie algebras*

have to be locally isomorphic to the homogeneous models of the same type.

Proof. Thanks to the regularity, the curvature satisfies $\kappa(u)(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+\ell}$ for all $u \in \mathcal{G}$ and for some $\ell \geq 1$ in general. Moreover, it can be proved that the component of degree ℓ mapping $\mathfrak{g}_i \times \mathfrak{g}_j$ to $\mathfrak{g}_{i+j+\ell}$ corresponds to the component of $\kappa_H(u)$ of degree ℓ, see [6]. The above lemma shows that $\ell \geq 2$ for symmetric geometries. Moreover, if the component of degree 2 is non-zero, then the only possibility is that it maps $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1}$ to $\mathfrak{g}_0$. Thus it has its values in $\mathfrak{g}_-^* \wedge \mathfrak{g}_-^* \otimes \mathfrak{p}$. It follows directly from the homogeneity reasons that components of degree ≥ 3 have to have their values in this subbundle, too, and the geometry is torsion-free.

Finally, let us recall that vanishing of the harmonic curvature implies vanishing of the whole curvature, see Theorem 1.1. This applies if κ_H coincides with $\kappa^{(1)}$ which has to vanish for symmetric geometries, and they are locally isomorphic to homogeneous models. Now, it suffices to discuss components of harmonic curvature for concrete geometries, see Appendix. □

4.2. **Obstructions to flatness and involutive symmetries.** One can see from the discussion of the harmonic curvature that among all normal parabolic contact geometries, only

- contact projective structures,
- Lagrangean contact structures,
- partially integrable almost CR-structures of hypersurface type

can carry a symmetry at a point with non-zero harmonic curvature. For each such symmetric geometry, there is exactly one obstruction to being locally isomorphic to the homogeneous model of the same type. For three-dimensional almost CR-structures and three-dimensional Lagrangean contact structures, there is the harmonic curvature $\kappa^{(4)}$ valued in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_1$. For the other ones, we have the harmonic curvature $\kappa^{(2)}$ valued in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$.

Let us first focus on $\kappa^{(2)}$. Let s_x be a symmetry at x on a normal symmetric parabolic contact geometry and let φ be as usual. For $u \in p^{-1}(x)$ and $X, Y \in \mathfrak{g}_{-1}$ we have

$$\kappa^{(2)}(\varphi(u))(X,Y) = \kappa^{(2)}(ug_0 \exp Z)(X,Y) = \exp(-Z)g_0^{-1} \cdot \kappa^{(2)}(u)(X,Y),$$

where $\cdot$ is the induced Ad-action on $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$. For the action of g_0^{-1} we can write

$$g_0^{-1} \cdot \kappa^{(2)}(u)(X,Y) = \mathrm{Ad}_{g_0^{-1}}(\kappa^{(2)}(u)(\mathrm{Ad}_{g_0}X, \mathrm{Ad}_{g_0}Y)) =$$

$$\mathrm{Ad}_{g_0^{-1}}(\kappa^{(2)}(u)(-X,-Y)) = \mathrm{Ad}_{g_0^{-1}}(\kappa^{(2)}(u)(X,Y)).$$

Because $\mathfrak{g}_0$ is a subspace of $L(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \simeq \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$, the element g_0^{-1} has to act trivially on $\mathfrak{g}_0$ and thus on $\kappa^{(2)}(u)(X,Y)$ for each X, Y. Because also $\exp(-Z)$ acts trivially on $\kappa^{(2)}(u)$, we get no additional restriction. In fact, $\kappa^{(2)}$ is a tensor of type $\wedge^2 T^{-1*}M \otimes T^{-1*}M \otimes T^{-1}M$ which is invariant with respect to the symmetry s_x. Let us denote this tensor by W.

We try to differentiate W with respect to various Weyl connections. We focus on connections corresponding to almost s_x-invariant Weyl structures, i.e. Weyl structures σ satisfying $\varphi(\sigma(u_0)) = \sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0)$ for suitable Υ_2 and Υ_1 such that $\Upsilon_1 = 0$ at x, see Section 2.3 for details.

Lemma. *On a symmetric normal parabolic contact geometry with a symmetry s_x at x, let σ be an arbitrary almost s_x-invariant Weyl structure and ∇^σ the corresponding Weyl connection. Then*

(a) *$\nabla^\sigma_\xi W = 0$ holds at x for each ξ from the contact distribution,*
(b) *$\{\Upsilon_2, \xi\} \bullet W = 0$ holds at x for each ξ such that $T_x s_x . \xi(x) = \xi(x)$ and Υ_2 is determined by s_x at x.*

Proof. In general, $\varphi(\sigma(u_0)) = \sigma(\varphi_0(u_0)) \exp \Upsilon_1(u_0) \exp \Upsilon_2(u_0)$ holds for each Weyl structure σ and suitable Υ_1 and Υ_2, and this can be rewritten as

$$\varphi^{-1}(\sigma(\varphi_0(u_0))) = \sigma(u_0) \exp(-\Upsilon_1(u_0)) \exp(-\Upsilon_2(u_0))$$

or simply $\varphi^*\sigma = \sigma \exp(-\Upsilon_1)\exp(-\Upsilon_2)$. For corresponding Weyl connections we then have $\nabla^{\varphi^*\sigma} = \nabla^{\sigma \exp(-\Upsilon_1)\exp(-\Upsilon_2)}$ and if we apply this on W, we get

$$\nabla^{\varphi^*\sigma}_\xi W = \nabla^{\sigma \exp(-\Upsilon_1)\exp(-\Upsilon_2)}_\xi W \tag{6}$$

for each vector field ξ. Because we suppose that σ is almost s_x-invariant Weyl structure, then moreover Υ_1 is equal to zero in the fiber over x.

Let us discuss both sides of Eq. (6) at x in detail. We start with the left hand side. At the point x we have

$$\nabla^{\varphi^*\sigma}_\xi W(\eta,\mu)(\nu) = (s_x^*\nabla^\sigma)_\xi W(\eta,\mu)(\nu) =$$

$$T_x s_x^{-1}.\nabla^\sigma_{T_x s_x.\xi} W(T_x s_x.\eta, T_x s_x.\mu)(T_x s_x.\nu) =$$

$$(-1)^4 \nabla^\sigma_{T_x s_x.\xi} W(\eta,\mu)(\nu) = \nabla^\sigma_{T_x s_x.\xi} W(\eta,\mu)(\nu)$$

for each $\xi \in \mathfrak{X}(M)$ and $\eta, \mu, \nu \in \Gamma(T^{-1}M)$ since $T_x s_x$ gives $-\mathrm{id}$ on $T_x^{-1}M$.

Now we focus on the right hand side of (6). We use the general formula for the change of Weyl connections, see formula (1) in Section 1.2. Because our Weyl structure σ satisfies $\Upsilon_1 = 0$ over x, the right hand side of (6) simplifies to

$$\nabla_\xi^{\sigma \exp(-\Upsilon_1)\exp(-\Upsilon_2)} W = \nabla_\xi^\sigma W + \{\Upsilon_2, \xi_{-2}\} \bullet W$$

in the fiber over x. If we put the above observations together, we see that Eq. (6) can be rewritten as

$$\nabla^\sigma_{T_x s_x.\xi} W = \nabla^\sigma_\xi W + \{\Upsilon_2, \xi_{-2}\} \bullet W \tag{7}$$

in the fiber over x. Let us discuss some concrete choices of the vector $\xi(x)$:

(a) Suppose ξ is contained in the contact distribution $T^{-1}M$. In particular, $\xi(x) = \xi_{-1}(x)$. Then $T_x s_x.\xi(x) = -\xi(x)$ and Eq. (7) simplifies to

$$\nabla^\sigma_{-\xi} W = \nabla^\sigma_\xi W$$

at x. The algebraic bracket simply vanishes because $\xi_{-2}(x) = 0$ in this case. This implies $\nabla^\sigma_\xi W = 0$ at x for each ξ from the contact distribution.

(b) Let us now suppose that $T_x s_x.\xi(x) = \xi(x)$. Such vectors exist and for an almost s_x-invariant Weyl structure σ, these are exactly the vectors satisfying $\xi = \xi_{-2}$ at x, see Section 2.3. Then the Eq. (7) simplifies to

$$\nabla^\sigma_\xi W = \nabla^\sigma_\xi W + \{\Upsilon_2, \xi\} \bullet W$$

and we get the restriction $\{\Upsilon_2, \xi\} \bullet W = 0$ in the fiber over x. $\square$

Part (a) is not surprising. Actually, $\nabla^\sigma W$ defines a tensor of type $T^{-1*}M \otimes \wedge^2 T^{-1*}M \otimes T^{-1*}M \otimes T^{-1}M$, i.e. of odd degree, which is invariant with respect to s_x. The consequences of the part (b) we formulate in the following statement.

Proposition. *On a symmetric normal parabolic contact geometry with a symmetry s_x at x, suppose that W is non-zero at x. Then each almost s_x-invariant Weyl structure has to be s_x-invariant.*

Proof. With the above notation, we will discuss the formula (b) from the lemma for an almost s_x-invariant Weyl structure σ and for some vector field ξ such that $\xi(x)$ is non-zero and satisfies $T_x s_x.\xi(x) = \xi(x)$. Let us point out that such vectors exist and satisfy $\xi(x) = \xi_{-2}(x)$ in the isomorphism $TM \simeq \mathrm{gr}(TM)$ given by σ, see 2.3. In some concrete frame $u = \sigma(u_0)$ from the fiber over x, we can write $\xi(x) = [\![u, X]\!]$ for suitable non-zero $X \in \mathfrak{g}_{-2}$. Similarly, $\Upsilon_2(x) = [\![u, Z]\!]$ for suitable $Z \in \mathfrak{g}_2$ and the algebraic bracket $\{\Upsilon_2, \xi\}$ corresponds to $[\![u, [Z, X]]\!]$ at x. Moreover, if $Z \neq 0$, we can choose X such that $[Z, X]$ is exactly the grading element E, see Section 1.1. Then $\{\Upsilon_2, \xi\}$ corresponds to the grading section $E(x)$, see Section 1.2. In particular, it acts by the algebraic action $\bullet$ on W by its homogeneity. Because W has homogeneity two, we get

$$\{\Upsilon_2, \xi\} \bullet W = 2W$$

in the fiber over x and we have a restriction of the form $2W = 0$ at x. This is a contradiction with the assumption that W is non-zero at x. Thus the only possibility is that $Z = 0$ and thus Υ_2 has to vanish at x. But this means that the almost s_x-invariant Weyl structure is s_x-invariant, see Section 2.3. $\square$

Before we proceed further, let us return to $\kappa^{(4)}$ valued in $\mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-2} \otimes \mathfrak{g}_1$. The discussion of $\kappa^{(4)}$ is parallel to the discussion of $\kappa^{(2)}$ and we summarize it very briefly. If $\kappa^{(4)}$ is non-zero, then it defines a tensor of even degree which has to be invariant with respect to the symmetry s_x. Really, for each $X \in \mathfrak{g}_{-1}$ and $V \in \mathfrak{g}_{-2}$ we have

$$\kappa^{(4)}(\varphi(u))(X, V) = \kappa^{(4)}(u g_0 \exp Z)(X, V) = \exp(-Z) g_0^{-1} \cdot \kappa^{(4)}(u)(X, V) =$$

$$\mathrm{Ad}_{g_0^{-1}}(\kappa^{(4)}(u)(\mathrm{Ad}_{g_0}X, \mathrm{Ad}_{g_0}V)) = -\kappa^{(4)}(u)(-X, V) = \kappa^{(4)}(u)(X, V).$$

Let us denote this tensor by C. Again, we can differentiate C with respect to a Weyl connections corresponding to almost invariant Weyl structures. For an almost s_x-invariant Weyl structure σ we have the equation

$$\nabla_\xi^{\varphi^*\sigma} C = \nabla_\xi^{\sigma \exp(-\Upsilon_1)\exp(-\Upsilon_2)} C$$

for each vector field ξ and suitable Υ_1 and Υ_2 corresponding to σ, where Υ_1 vanishes at x. The left hand side can be rewritten as

$$\begin{aligned}\nabla_\xi^{\varphi^*\sigma} C(\eta, \mu) &= (s_x^* \nabla^\sigma)_\xi C(\eta, \mu) = T_x s_x^{-1}.\nabla^\sigma_{T_x s_x.\xi} C(T_x s_x.\eta, T_x s_x.\mu)\\ &= (-1)^2 \nabla^\sigma_{T_x s_x.\xi} C(\eta, \mu) = \nabla^\sigma_{T_x s_x.\xi} C(\eta, \mu)\end{aligned}$$

for each $\xi \in \mathfrak{X}(M)$, $\eta \in \Gamma(T^{-1}M)$ and $\mu \in \mathfrak{X}(M)$ such that $\mu = \mu_{-2}$ via the isomorphism given by σ at x. Really, $T_x s_x$ gives $-\mathrm{id}$ on $T_x^{-1}M$ and $T_x s_x.\mu(x) = \mu(x)$. Thus we get the restriction of the form

$$\nabla^\sigma_{T_x s_x.\xi} C = \nabla^\sigma_\xi C + \{\Upsilon_2, \xi_{-2}\} \bullet C$$

and we have $\nabla^\sigma_{\xi_{-1}} C = 0$ and $\{\Upsilon_2, \xi_{-2}\} \bullet C = 0$ in the fiber over x. Because C is of homogeneity four, the same arguments as in the proof of the above proposition shows that Υ_2 vanishes at x and then each s_x-invariant Weyl structure has to be s_x-invariant. All these observations together with the last proposition give us the following statement.

Theorem. *On a symmetric normal parabolic contact geometry with a symmetry s_x at x, suppose that its harmonic curvature is non-zero at x. Then s_x is involutive.*

Proof. In such case, almost s_x-invariant Weyl structures have to be s_x-invariant and the rest follows from Section 2.3. □

Corollary. On a symmetric normal parabolic contact geometry with a symmetry s_x at x, suppose that its harmonic curvature is non-zero at x. Then there are admissible affine connections which are invariant with respect to the symmetry s_x: We take Weyl connections corresponding to s_x-invariant Weyl structures.

5. Uniqueness of symmetries

We discuss here the question how many different symmetries can exist at a point with non-zero curvature. We first give one general restriction and then some consequences for concrete geometries.

5.1. **Algebraic restriction.** Let s_x and $\bar{s}_x$ be two different symmetries at x on a symmetric normal parabolic contact geometry with non-zero harmonic curvature at x and denote by φ and $\bar{\varphi}$ corresponding automorphisms of the parabolic geometry. Clearly, $s_x \neq \bar{s}_x$ if and only if $\varphi \neq \bar{\varphi}$. Symmetries s_x and $\bar{s}_x$ are involutive and there exist s_x-invariant and $\bar{s}_x$-invariant Weyl structures, see Sections 4.2 and 2.3.

Lemma. *For each two different involutive symmetries s_x and $\bar{s}_x$ at x on a symmetric parabolic contact geometry, s_x-invariant and $\bar{s}_x$-invariant Weyl structures form two disjoint families of Weyl structures.*

Proof. Suppose there is a Weyl structure σ which is s_x-invariant and $\bar{s}_x$-invariant at x, i.e. $\varphi(\sigma(u_0)) = \sigma(\varphi_0(u_0))$ and simultaneously $\bar{\varphi}(\sigma(u_0)) = \sigma(\bar{\varphi}_0(u_0))$ in the fiber over x. Then, the corresponding Weyl connection ∇^σ is invariant with respect to both symmetries s_x and $\bar{s}_x$. But similarly as in the last part of the proof of Proposition 2.3, the connection ∇^σ determines uniquely the symmetry via behavior of its geodesics at x. Consequently, $s_x = \bar{s}_x$ on a neighborhood of x. □

Let σ be an s_x-invariant Weyl structure and let $\bar{\sigma}$ be $\bar{s}_x$-invariant Weyl structure. Then $\bar{\sigma} = \sigma \exp \Upsilon_1 \exp \Upsilon_2$ holds for suitable $\Upsilon_1 : \mathcal{G}_0 \to \mathfrak{g}_1$ and $\Upsilon_2 : \mathcal{G}_0 \to \mathfrak{g}_2$. The last lemma says that Υ_1 has to be non-zero at x.

Proposition. *Suppose there are two different involutive symmetries at x on a symmetric normal parabolic contact geometry and let σ and $\bar{\sigma}$ are corresponding invariant Weyl structures. For all ξ from the contact distribution, the bracket $\{\Upsilon_1, \xi\}$ acts trivially by the algebraic action on W or Y, respectively, at x.*

Proof. Let us start with W. Let ξ be an arbitrary vector field from the contact distribution, thus $\xi = \xi_{-1}$ for each Weyl structure. The Lemma 4.2 gives $\nabla^{\sigma}_{\xi_{-1}} W = 0$ and $\nabla^{\bar{\sigma}}_{\xi_{-1}} W = 0$ at x. Simultaneously, we have $\bar{\sigma} = \sigma \exp \Upsilon_1 \exp \Upsilon_2$ and the formula (1) from Section 1.2 gives

$$\nabla^{\bar{\sigma}}_{\xi_{-1}} W = \nabla^{\sigma}_{\xi_{-1}} W + \{\xi_{-1}, \Upsilon_1\} \bullet W$$

at x, since $\xi_{-2}(x) = 0$. Because both covariant derivatives vanish at x, we get the restriction of the form $\{\xi_{-1}, \Upsilon_1\} \bullet W = 0$ at x for each ξ from the contact distribution. One can see from Section 4.2 that the same line of arguments works for C and we get the restriction of the form $\{\xi_{-1}, \Upsilon_1\} \bullet C = 0$ at x for each ξ from the contact distribution. □

Remark. *Let us again point out that the existence of a non-involutive symmetry at x causes vanishing of the harmonic curvature at x, see Section 4.2.*

5.2. **Examples.** Let us now discuss the above restrictions for concrete types of geometries. The key point is to find sufficiently nice ξ such that the action of the above algebraic bracket is easily understandable.

Lagrangean contact structures. Let us first point out that we use here the notation from Section 3.2. The decomposition of the contact distribution into two isotropic subbundles $T^{-1}M = L \oplus R$ can be interpreted as a product structure on $T^{-1}M$, which an operator $J : T^{-1}M \to T^{-1}M$ satisfying $J^2 = \text{id}$. The subbundles L and R are simply eigenspaces of J. The Levi bracket $\mathcal{L} : T^{-1}M \times T^{-1}M \to TM/T^{-1}M$ is non-degenerate antisymmetric bilinear map, and then, $\mathcal{L}(-, J-)$ is a non-degenerate symmetric map which defines a conformal class of pseudometrics on $T^{-1}M$ of signature (n, n). We denote the class by g. Each pseudometric is then given by the choice the identification $TM/T^{-1}M \simeq \mathbb{R}$. In particular, the question whether $g(\xi, \eta)$ equals to zero for some $\xi, \eta \in T^{-1}M$ makes sense, because the answer does not depend on the choice of the metric from the class.

Proposition. *Suppose there are two different involutive symmetries at x on a symmetric normal Lagrangean contact structure, and denote by σ and $\sigma \exp \Upsilon_1 \exp \Upsilon_2$ corresponding invariant Weyl structures. Identify Υ_1 with its image in $T^{-1}M = L \oplus R$ via an isomorphism given by a metric from g and denote by Υ_1^L and Υ_1^R corresponding components in L and R. If $g(\Upsilon_1^R, \Upsilon_1^L) \neq 0$ at x, then the harmonic curvature vanishes at x.*

Proof. We discuss the restriction from the Proposition 5.1 for Lagrangean contact structures in detail. Let us write $\Upsilon_1(x) = [\![u, Z]\!]$ for suitable

$$Z = \left(\begin{smallmatrix} 0 & S & 0 \\ 0 & 0 & T \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathfrak{g}_1,$$

which has to be non-zero, see Lemma 5.1. Choose $\xi_{-1} \in \Gamma(T^{-1}M)$ such that $\xi_{-1}(x) = [\![u, X]\!]$ for X of the form

$$X = \left(\begin{smallmatrix} 0 & 0 & 0 \\ T & 0 & 0 \\ 0 & S & 0 \end{smallmatrix}\right) \in \mathfrak{g}_{-1}.$$

The bracket $\{\xi_{-1}, \Upsilon_1\}$ then corresponds to $[\![u, [X, Z]]\!]$ at x, where

$$[X, Z] = \begin{pmatrix} -ST & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & ST \end{pmatrix} \in \mathfrak{g}_0.$$

It is easy to verify that with this choice, $[X, Z]$ is simply a grading element multiplied by a non-zero number $-ST$. Then the bracket $\{\xi_{-1}, \Upsilon_1\}$ is a non-zero multiple of the grading section $E(x)$. Via the identification given by the metric from g, the components S and T correspond to components Υ_1^R and Υ_1^L of Υ_1 in subbundles R and L at x and the fact that $ST \neq 0$ means that $g(\Upsilon_1^R, \Upsilon_1^L) \neq 0$ at x. Because the grading section acts on W by its homogeneity, $\{\xi_{-1}, \Upsilon_1\}$ acts trivially on W if and only W vanishes at x. Clearly, the same arguments work for C. $\square$

Partially integrable almost CR-structures. Let us first point out that we use here the notation from Section 3.2. Moreover, suppose that the geometry is oriented and then, we can speak about the signature of the structure. Using the complex structure J given on $T^{-1}M$, we can define a non-degenerate symmetric mapping $\mathcal{L}(-, J-)$, which defines a conformal class of pseudometrics on $T^{-1}M$. The signature is given by the signature of the structure. Let us denote the class by g. Each pseudometric from the class is given by the choice of the identification $TM/T^{-1}M \simeq \mathbb{R}$. In particular, the question whether $g(\xi, \xi) \neq 0$ for $\xi \in T^{-1}M$ makes sense, because the answer does not depend on the choice of the pseudometric from the class.

Proposition. *Suppose there are two different involutive symmetries at x on a symmetric normal partially integrable almost CR-structure and denote by σ and $\sigma \exp \Upsilon_1 \exp \Upsilon_2$ corresponding invariant Weyl structures. Identify Υ_1 with its image in $T^{-1}M$ via an isomorphism given by a metric from g. If $g(\Upsilon_1, \Upsilon_1) \neq 0$ at x, i.e. if the length of Υ_1 is non-zero at x, then the harmonic curvature vanishes at x.*

Proof. We discuss the restriction from the Proposition 5.1 for CR-structures in detail. Let us write $\Upsilon_1(x) = [\![u, Z]\!]$ for suitable

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ IS^* & 0 & 0 \\ 0 & -S & 0 \end{pmatrix},$$

which has to be non-zero, see Lemma 5.1. Choose $\xi_{-1} \in T^{-1}M$ such that $\xi_{-1}(x) = [\![u, X]\!]$ for X of the form

$$X = \begin{pmatrix} 0 & 0 & 0 \\ S & 0 & 0 \\ 0 & -S^*I & 0 \end{pmatrix} \in \mathfrak{g}_1.$$

The bracket $\{\xi_{-1}, \Upsilon_1\}$ then corresponds to $[\![u, [X, Z]]\!]$ at x, where

$$\begin{pmatrix} -SIS^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & SIS^* \end{pmatrix} \in \mathfrak{g}_0.$$

It is easy to verify that with this choice, $[X, Z]$ is simply a grading element multiplied by a number $-SIS^*$. Then the bracket $\{\xi_{-1}, \Upsilon_1\}$ is a multiple of the grading section $E(x)$. Using the identification $T^{-1}M \simeq T^{-1*}M$ given by a metric from g, $-SIS^*$ corresponds to $g(\Upsilon_1, \Upsilon_1)$ and $SIS^* \neq 0$ means that $g(\Upsilon_1, \Upsilon_1) \neq 0$. Because the grading section acts on W by its homogeneity, $\{\xi_{-1}, \Upsilon_1\}$ acts trivially on W if and only W vanishes at x. Clearly, the same arguments work for Y. $\square$

Corollary. Suppose there are two different involutive symmetries at x on a symmetric normal strictly pseudoconvex partially integrable almost CR-structure. Then the harmonic curvature vanishes at x.

Appendix: Contact gradings and corresponding geometries

Let us sketch here briefly a classification of contact gradings of real semisimple Lie algebras. There is the well know classification of all (complex) semisimple Lie algebras in the language Dynkin diagrams and description of all their real forms in the language of Satake diagrams, see [6, 11]. It can be proved that if a Lie algebra admits a contact grading, then it has to be simple. It turns out that except $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{H})$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$ and some real forms of E_6 and F_4, any non-compact non-complex real simple Lie algebra admits a unique real contact grading, see [6].

Let us start with real classical Lie algebras, i.e. real forms of Lie algebras of type A_ℓ, B_ℓ, C_ℓ and D_ℓ. In the first column of the following table, we indicate a real simple Lie algebra which admits a contact grading. In the second column we specify the geometry, which corresponds to the unique contact grading and in the last column we write its components of harmonic curvature.

real simple $\mathfrak{g}$	contact geometry	components of κ_H
$\mathfrak{sl}(3,\mathbb{R})$	Lagrangean contact structures of dimension 3	$\mathfrak{g}_{-2}\times\mathfrak{g}_{-1}\to\mathfrak{g}_1$ $\mathfrak{g}_{-2}\times\mathfrak{g}_{-1}\to\mathfrak{g}_1$
$\mathfrak{sl}(n+2,\mathbb{R})$ for $n\geq 2$	Lagrangean contact structures of dimension $2n+1$	$\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_0$
$\mathfrak{su}(2,1)$ and $\mathfrak{su}(1,2)$	partially integrable almost CR structures of dimension 3	$\mathfrak{g}_{-2}\times\mathfrak{g}_{-1}\to\mathfrak{g}_1$
$\mathfrak{su}(p+1,q+1)$ for $p+q\geq 2$	partially integrable almost CR structures of dimension $2p+2q+1$	$\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_0$
$\mathfrak{so}(p+2,q+2)$ $p+q\neq 4$	Lie contact structures of dimension $2p+2q+1$	$\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$
$\mathfrak{so}(p+2,q+2)$ for $p+q=4$	Lie contact structures of dimension 9	$\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$ $\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_{-1}$
$\mathfrak{sp}(n+2)$ for $n\geq 1$	contact projective structures	$\mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\mathfrak{g}_0$

Let us also give a brief overview of contact gradings corresponding to exceptional Lie algebras. For types G_2 and F_4, there is exactly one real algebra admitting contact grading, the split real form. For E_6, there are three real forms which admit a contact grading, the split form and two $\mathfrak{su}$-algebras. For E_7, there are three different real forms and for E_8, there are two different real forms admitting a contact grading. The description of corresponding geometries can be found in [6]. All these geometries have harmonic curvatures only of type $\kappa^{(1)}$ valued in $\mathfrak{g}^*_{-1}\wedge\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_{-1}$.

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Chapter 2: Local Generalized Symmetries and Locally Symmetric Parabolic Geometries

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LOCAL GENERALIZED SYMMETRIES AND LOCALLY SYMMETRIC PARABOLIC GEOMETRIES

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Abstract. We investigate (local) automorphisms of parabolic geometries that generalize geodesic symmetries. We show that many types of parabolic geometries admit at most one generalized geodesic symmetry at a point with non–zero harmonic curvature. Moreover, we show that if there is exactly one symmetry at each point, then the parabolic geometry is a generalization of an affine (locally) symmetric space.

1. Introduction

Symmetric spaces are extremely useful geometric objects on smooth manifolds. There are also many generalizations of symmetric spaces appearing in several areas of differential geometry and the theory of Lie groups and algebras. We are interested in generalizations of symmetric spaces in the setting of parabolic geometries, see [3, Section 3.1]. We consider regular normal parabolic geometries $(\mathcal{G} \to M, \omega)$ of type (G, P) on smooth connected manifolds M. We assume that G is a Lie group with a $|k|$–graded simple Lie algebra $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ and P is the parabolic subgroup of G with the Lie algebra $\mathfrak{p} = \oplus_{i=0}^{k} \mathfrak{g}_i$ such that the Klein geometry (G, P) is effective. We fix the reductive Levi decomposition $P = G_0 \rtimes \exp(\mathfrak{p}_+)$, where $\mathfrak{p}_+ := \oplus_{i=1}^{k} \mathfrak{g}_i$ and G_0 is the Lie group of grading preserving elements of P. We write $\mathfrak{g}_- := \oplus_{i=-k}^{-1} \mathfrak{g}_i$.

Regular normal parabolic geometries provide a solution to the equivalence problem for a wide class of geometric structures. In the first step, so called prolongation, one constructs the P–bundle $\mathcal{G}$ over M and the Cartan connection ω, which is a P–equivariant $\mathfrak{g}$–valued absolute parallelism on $\mathcal{G}$ that reproduces the generators of fundamental vector fields of the P–action. The precise process of the prolongation is not directly related to the results presented in this article and will not be reviewed. In the second step, one computes the harmonic curvature κ_H which is the basic invariant of all normal parabolic geometries that (in principle) solves the equivalence problem for normal parabolic geometries. We recall that κ_H is the projection of the curvature $[\omega, \omega] + d\omega$ of the Cartan connection ω viewed as a function $\kappa : \mathcal{G} \to \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ into the cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ of cochains on $\mathfrak{g}_-$ with values in $\mathfrak{g}$.

A (local) automorphism of $(\mathcal{G} \to M, \omega)$ is a (local) P–bundle morphism φ on $\mathcal{G}$ such that $\varphi^*\omega = \omega$ holds. We denote by $\underline{\varphi}$ the underlying (local) diffeomorphism of φ on M. We say that a (local) diffeomorphism f on M preserves the parabolic geometry $(\mathcal{G} \to M, \omega)$ if $f = \underline{\varphi}$ for some (local) automorphism φ of $(\mathcal{G} \to M, \omega)$. Local automorphisms of parabolic geometries are uniquely determined by the underlying diffeomorphisms under our assumption of effectivity of the Klein geometry (G, P). We are interested in a class of (local) diffeomorphisms f on M for which we know a priori the (local) P–bundle morphisms φ on $\mathcal{G}$ covering (local) diffeomorphisms f

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and we ask when they preserve the parabolic geometry, see the Definition 1.1. Let us explain that these diffeomorphisms are closely related to geodesic symmetries.

We recall that a normal coordinate system of a linear connection ∇ on M given by the frame u of T_xM is given by projections of flows $\mathrm{Fl}_t^{B(X)}$ of the standard horizontal vector fields $B(X)$ for $X \in \mathbb{R}^n$ on the first order frame bundle starting at u, see [16, Section III.8.]. Indeed, the projection of $\mathrm{Fl}_t^{B(X)}(u)$ onto M is the geodesic of ∇ going through x in the direction with coordinates X in the frame u. A geodesic symmetry of ∇ at the point x is the unique diffeomorphism with coordinates $-\mathrm{id}_{\mathbb{R}^n}$ in all normal coordinate system given by any frame u of T_xM.

The pair (M, ∇) is an affine locally symmetric space if each geodesic symmetry of ∇ is an affine transformation. In [14] or [1] the authors studied the theory of symmetric spaces, where the geodesic symmetries preserve a geometric structure such as Riemannian metric or quaternionic Kähler structure. The first author classified in [8] all parabolic geometries preserved by all geodesic symmetries on semisimple symmetric spaces. Typical examples of such parabolic geometries are provided by the projective class of ∇ of the affine (locally) symmetric space (M, ∇) or the conformal class of the metric on the Riemannian symmetric space or the (para)–quaternionic geometry given by the (para)–quaternionic Kähler symmetric space.

A normal coordinate system on the parabolic geometry $(p : \mathcal{G} \to M, \omega)$ given by $u \in \mathcal{G}$ is given by projections $p \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ of flows of the constant vector fields $\omega^{-1}(X)$ for coordinates $X \in \mathfrak{g}_-$. If we consider (local) diffeomorphisms f on M that are linear in some normal coordinate system of $(\mathcal{G} \to M, \omega)$, then we know a priori the (local) P–bundle morphisms φ on $\mathcal{G}$ covering (local) diffeomorphisms f and we ask when they preserve the parabolic geometry. The action of G_0 on $\mathcal{G}$ induces a linear change of the normal coordinates, but the change of coordinates induced by the action of $\exp(\mathfrak{p}_+)$ is highly non–linear. Nevertheless, we can consider the class of (local) automorphisms of parabolic geometries with the property that their underlying (local) diffeomorphisms on M, analogously to geodesic symmetries, have the same coordinates in all normal coordinate systems in which the coordinates are linear.

Definition 1.1. For s in the center $Z(G_0)$ of G_0 and $u \in \mathcal{G}$, let s_u be the (local) P–bundle morphism of $\mathcal{G}$ induced by the formula

$$s_u(\mathrm{Fl}_1^{\omega^{-1}(X)}(u)) := \mathrm{Fl}_1^{\omega^{-1}(X)}(us) = \mathrm{Fl}_1^{\omega^{-1}(\mathrm{Ad}(s)(X))}(u)s$$

for all X in a maximal possible neighbourhood of 0 in $\mathfrak{g}_-$ preserved by $\mathrm{Ad}(s)$.

(1) The (local) P–bundle morphism s_u is a *(local) s–symmetry* of the parabolic geometry $(\mathcal{G} \to M, \omega)$ at $x = p(u)$ if $s_u^*\omega = \omega$.
(2) We write $\underline{s}_u$ for the underlying (local) diffeomorphism on M of the P–bundle morphism s_u which has coordinates $\mathrm{Ad}(s) \in Gl(\mathfrak{g}_-)$ in the normal coordinate system given by u.
(3) All (local) s–symmetries at all $x \in M$ for all $s \in Z(G_0)$ together are called *(local) generalized symmetries* of parabolic geometries.
(4) The parabolic geometry is *(locally) s–symmetric* if there is a (local) s–symmetry at each point of M.

Remark 1.2. We always assume that s is not the identity element e in $Z(G_0)$, because $\mathrm{id}_{\mathcal{G}}$ is the unique e–symmetry of each parabolic geometry, and therefore, the results presented in this article are trivial for e–symmetries.

Firstly, let us focus on the automorphisms φ of parabolic geometries such that $\underline{\varphi}$ has coordinates $-\mathrm{id}_{\mathfrak{g}_-}$ in the normal coordinate system given by $u \in \mathcal{G}$. If such an automorphism exists, then there is $m \in Z(G_0)$ such that $\mathrm{Ad}(m) = -\mathrm{id}_{\mathfrak{g}_-}$. Therefore we will always speak about (local) m–symmetries in this case.

The bundle morphisms m_u (and thus diffeomorphisms $\underline{m_u}$) are generally different for different u from the fiber $\mathcal{G}_x$ over x and each of them can be a (local) m–symmetry. In particular, there can be infinitely many (local) m–symmetries at x. In fact, this is the case of all models G/P of AHS–structures, where the bundle maps m_u are m–symmetries for all $u \in \mathcal{G}$. On the other hand, the second author proved in [23, Theorem 2.5] that projective, conformal and (para)–quaternionic geometries are the only types of parabolic geometries allowing m–symmetries at a point x with a non–zero Weyl (harmonic) curvature. Moreover, there is at most one m–symmetry at the point x with a non–zero Weyl curvature.

The second author showed in [23, Theorem 3.2] that if a geodesic symmetry at x for some linear connection on M is an automorphism of $(\mathcal{G} \to M, \omega)$, then the geodesic symmetry has coordinates $-\mathrm{id}_{\mathfrak{g}_-}$ in the normal coordinate system given by some $u \in \mathcal{G}_x$. We prove in this article that there is the following characterization of non–flat parabolic geometries which are preserved by all geodesic symmetries on affine (locally) symmetric spaces.

Theorem 1.3. *Suppose there is a parabolic geometry on a smooth connected manifold M with a non–zero harmonic curvature at one point. Then the following claims are equivalent:*

(1) *The parabolic geometry is (locally) m–symmetric, i.e., at each point x of M, there is a (local) automorphism of the parabolic geometry such that the underlying (local) diffeomorphism on M has coordinates $-\mathrm{id}_{\mathfrak{g}_-}$ in the normal coordinate system for some $u \in \mathcal{G}_x$.*

(2) *The parabolic geometry is preserved by each geodesic symmetry on an affine (locally) symmetric space (M, ∇).*

In particular, if one of the above claims is satisfied, then the parabolic geometry is (locally) homogeneous, the affine (locally) symmetric space (M, ∇) from Claim (2) is unique and ∇ is a distinguished (Weyl) connection of the parabolic geometry.

Remark 1.4. Let us emphasize that (local) m–symmetries can appear only on $|1|$–graded parabolic geometries and only the projective, conformal and (para)–quaternionic geometries (and their complexifications) can satisfy the assumptions and conditions of Theorem 1.3.

The global version of this statement was proved in [21] for projective geometries and in [24, Corollary 4.5] for conformal and (para)–quaternionic geometries under the additional assumption of homogeneity or under the assumption that m–symmetries depend smoothly on the point x. In [13, Theorem 1], we proved the global version of Theorem 1.3 for conformal geometries. In this article, we obtain Theorem 1.3 as a special case of Theorem 1.8.

There are many other interesting types of parabolic geometries, e.g., parabolic contact geometries, where there is no element $m \in P$ such that $\mathrm{Ad}(m) = -\mathrm{id}_{\mathfrak{g}_-}$. Thus they cannot be preserved by geodesic symmetries of any affine (locally) symmetric space. On the other hand, there are generalizations of symmetric spaces appearing in the literature that are nearly related to contact geometries. In [2] and [15] the authors study sub–Riemannian and CR geometries preserved by so–called geodesic reflexions on reflexion spaces, see [19]. A geodesic reflexion on a reflexion space is given by an endomorphism $s \in Gl(\mathbb{R}^n)$ such that $s^2 = \mathrm{id}_{\mathbb{R}^n}$ in a normal coordinate system of an admissible linear connection on the reflexion space, see [19].

We studied in [9, 10] parabolic geometries on reflexion spaces preserved by geodesic reflexions. We proved that a geodesic reflexion at x preserving a parabolic geometry $(\mathcal{G} \to M, \omega)$ is given by an endomorphism $\mathrm{Ad}(s) \in Gl(\mathfrak{g}_-)$ for some $s \in G_0$ such that $s^2 = \mathrm{id}$ in a normal coordinate system of the parabolic geometry given by some $u \in \mathcal{G}_x$. However, if $s \in G_0$ is not contained in $Z(G_0)$, then we cannot expect the uniqueness of the automorphisms φ such that $\underline{\varphi}$ has coordinates $\mathrm{Ad}(s) \in Gl(\mathfrak{g}_-)$ in the normal coordinate system given by some $u \in \mathcal{G}_x$. Indeed, if there is an other automorphism ψ such that $\underline{\psi}$ has coordinates $\mathrm{Ad}(g_0) \in Gl(\mathfrak{g}_-)$ for some $g_0 \in G_0$ in the normal coordinate system given by $u \in \mathcal{G}_x$, then $\psi\varphi\psi^{-1}$ is in general a different automorphism such that $\underline{\psi\varphi\psi^{-1}}$ has coordinates $\mathrm{Ad}(s) \in Gl(\mathfrak{g}_-)$ in the normal coordinate system given by $ug_0 \in \mathcal{G}_x$. On the other hand, the second author proved in [25, Section 5] that on some parabolic contact geometries, there is at most one s–symmetry at a point x with a non–zero harmonic curvature for $s \in Z(G_0)$ such that $\mathrm{Ad}(s)|_{\mathfrak{g}_{-1}} = -\mathrm{id}$. We prove in this article that this holds for all parabolic contact geometries.

We classified in [12] all elements $s \in Z(G_0)$ that can appear as coordinates of underlying diffeomorphisms of automorphisms of parabolic geometries in a normal coordinate system at a point with a non–zero harmonic curvature. For example, we have found out that for complex $|1|$–graded parabolic geometries with a harmonic curvature of homogeneity 3, we have to consider elements $s \in Z(G_0)$ such that $s^3 = \mathrm{id}$. Moreover, we constructed in [11, Proposition 6.1] and [12, Proposition 7.2] examples of such parabolic geometries on $\mathbb{Z}_3$–symmetric spaces, which are generalizations of symmetric spaces that are studied in [18].

In fact, there are many known examples of (locally) s–symmetric parabolic geometries. Each locally flat parabolic geometry is locally s–symmetric for each $s \in Z(G_0)$. We classified in [12] the elements $s \in Z(G_0)$ for which all locally s–symmetric parabolic geometries are flat. Further, we showed in [11, Proposition 6.1] that all submaximally symmetric parabolic geometries constructed in [17, Section 4.1] are locally s–symmetric parabolic geometries for elements $s \in Z(G_0)$ that do not impose flatness. Let us emphasize that some of these examples carry more than one s–symmetry at each point and explicit examples can be found in [11, Section 6]. This shows that the results we obtain in this article do not hold for all types of parabolic geometries. There are also further examples of (locally) s–symmetric parabolic geometries in [2, 7, 8, 13, 21].

Let us now summarize our main results for (local) s–symmetries and (locally) s–symmetric parabolic geometries we obtain in this article. The first main result states that there is a large class of types of parabolic geometries whose algebraic structure enforces uniqueness of (local) s–symmetries at points with a non–zero harmonic curvature. We characterize these types in a way that is related to the theory of prolongations of annihilators of the harmonic curvature and the prolongation rigidity from [17, Section 3.4] as follows.

Definition 1.5. Let μ be a component of the harmonic curvature (irreducible as a G_0–submodule of $H^2(\mathfrak{g}_-, \mathfrak{g})$) of regular normal parabolic geometries of type (G, P).

(1) For $\phi \in \mu$, let us denote by

$$ann(\phi) := \{A \in \mathfrak{g}_0 : A.\phi = 0\}$$

the *annihilator* of ϕ in $\mathfrak{g}_0$. We define the *ith prolongation of the annihilator* of ϕ as

$$pr(\phi)_i = \{Z \in \mathfrak{g}_i : \mathrm{ad}(X_1) \dots \mathrm{ad}(X_i)(Z) \in ann(\phi) \text{ for all } \mathrm{X}_1, \dots \mathrm{X}_i \in \mathfrak{g}_{-1}\}.$$

(2) For $s \in Z(G_0)$, we say that the triple $(\mathfrak{g}, \mathfrak{p}, \mu)$ is *prolongation rigid outside of the* 1*–eigenspace of* s if for all weights $\phi \in \mu$, all prolongations of the annihilator of ϕ in $\mathfrak{g}_0$ are contained in the 1–eigenspace of s.

We will see that there are triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s only for some $s \in Z(G_0)$. In particular, such triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ are not prolongation rigid. Indeed, a triple $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid if and only if it is prolongation rigid outside of the 1–eigenspace of s for all $s \in Z(G_0)$.

In Section 3.2, we show how to classify all triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s for some $s \in Z(G_0)$ using the results in [17]. The following Theorem shows that for our purposes, it is enough to carry out the classification only for the components μ that are contained in the 1–eigenspace of s.

Theorem 1.6. *Consider a triple* $(\mathfrak{g}, \mathfrak{p}, \mu)$.

If $s \in Z(G_0)$ *is such that* μ *is not contained in the* 1*–eigenspace of* s*, then there is no (local)* s*–symmetry of a parabolic geometry of type* (G, P) *at each point* x *with a non–zero component of the harmonic curvature in* μ.

If $s \in Z(G_0)$ *is such that* $(\mathfrak{g}, \mathfrak{p}, \mu)$ *is prolongation rigid outside of the* 1*–eigenspace of* s*, then there is at most one (local)* s*–symmetry of a parabolic geometry of type* (G, P) *at the point* x *with a non–zero component of the harmonic curvature in* μ.

We proved Theorem 1.6 in [12, Theorem 1.3] under the assumption that the parabolic geometry is homogeneous, but we can also easily construct non–homogeneous (locally) s–symmetric parabolic geometries of type (G, Q) for certain triples $(\mathfrak{g}, \mathfrak{q}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s. It suffices to consider correspondence spaces for parabolic subgroups $Q \subset P \subset G$ over (locally) s–symmetric parabolic geometries of type (G, P) for $(\mathfrak{g}, \mathfrak{p}, \mu)$ that is prolongation rigid outside of the 1–eigenspace of s, see [12, Proposition 6.1]. We prove Theorem 1.6 in Section 3.1.

Let us now focus on (locally) s–symmetric parabolic geometries. We say that a map S that picks a (local) s–symmetry at each point of M is a system of (local) s–symmetries. In general, systems of (local) s–symmetries are neither smooth nor unique. The conditions in Theorem 1.6 can be used to prove the uniqueness of a system of (local) s–symmetries.

Our second main result concerns the conditions for the smoothness of a system of (local) s–symmetries. We consider the following generalization of affine locally symmetric spaces. There is a class of Weyl connections on each parabolic geometry playing a significant role in the theory of parabolic geometries, see [3, Chapter 5] and Section 2.1. Each Weyl connection is given by a reduction of $\mathcal{G}$ to G_0, i.e., by a smooth G_0–equivariant section σ of the projection from $\mathcal{G}$ to $\mathcal{G}_0 := \mathcal{G}/\exp(\mathfrak{p}_+)$. The sections σ are called Weyl structures and we denote by ∇^σ the Weyl connection given by the Weyl structure σ. Each point of $\sigma(\mathcal{G}_0)_x$ defines a different frame of T_xM. However, the (local) diffeomorphism with coordinates $\mathrm{Ad}(s) \in Gl(\mathfrak{g}_-)$ in a normal coordinate system of a Weyl connection ∇^σ given by a frame $\sigma(u_0) \in \sigma(\mathcal{G}_0)_x$ is independent of the actual choice of $u_0 \in (\mathcal{G}_0)_x$, see Section 3.2. We denote such a (local) diffeomorphism by s_x^σ.

If we choose a class of Weyl connections satisfying $T_x s_x^\sigma = T_x s_x^{\sigma'}$ for all Weyl connections $\nabla^\sigma, \nabla^{\sigma'}$ in the class and all $x \in M$, then the tangent bundle TM has a common decomposition into smooth subbundles according to the eigenvalues of $T_x s_x^\sigma$ for all ∇^σ in the class. We can further consider a subclass $[\nabla]$ of such a class of Weyl connections that restrict to the same partial connection on all smooth subbundles of TM for all eigenvalues of $T_x s_x^\sigma$ different from 1. Such a subclass $[\nabla]$ is equivalently characterized by the condition that the 1–forms Υ measuring the 'differences' (see the formula (2)) between arbitrary two connections in $[\nabla]$ satisfy $(s_x^\sigma)^*\Upsilon(x) = \Upsilon(x)$ for all $x \in M$ and some (and thus all) Weyl connections $\nabla^\sigma \in [\nabla]$. In general, (local) diffeomorphisms s_x^σ are different for different Weyl connections $\nabla^\sigma \in [\nabla]$. Therefore we can consider smooth maps $\underline{S}$ assigning some of these diffeomorphisms to each $x \in M$. Equivalently we can directly assign to each $x \in M$ the Weyl structure σ defining s_x^σ.

Definition 1.7. Let $[\nabla]$ be a maximal subclass of the class of Weyl connections satisfying that

- $T_x s_x^\sigma = T_x s_x^{\sigma'}$ holds for all Weyl connections $\nabla^\sigma, \nabla^{\sigma'} \in [\nabla]$ and all $x \in M$,
- all connections in $[\nabla]$ restrict to the same partial connection on all smooth subbundles of TM for all eigenvalues of $T_x s_x^\sigma$ different from 1.

Let $\underline{S}$ be a smooth map assigning to each $x \in M$ the (local) diffeomorphism s_x^σ for some Weyl connection ∇^σ (depending on x) in $[\nabla]$.

(1) The class $[\nabla]$ is called *$\underline{S}$–invariant* if $\underline{S}(x)^*\nabla^\sigma \in [\nabla]$ and $\underline{S}(x)^*\nabla^\sigma(x) = \nabla^\sigma(x)$ hold for some (and thus each) Weyl connection $\nabla^\sigma \in [\nabla]$ and all $x \in M$.
(2) Weyl connections ∇ in the $\underline{S}$–invariant class $[\nabla]$ are called *almost $\underline{S}$–invariant Weyl connection.*
(3) The almost $\underline{S}$–invariant Weyl connection ∇ is called *invariant at $x \in M$* if $\underline{S}(x)^*\nabla = \nabla$.
(4) The almost $\underline{S}$–invariant Weyl connection ∇ is called *$\underline{S}$–invariant* if $\underline{S}(x)^*\nabla = \nabla$ holds for all $x \in M$.

We show in Section 4.1 that if there is an almost $\underline{S}$–invariant Weyl connection, then each $\underline{S}(x)$ preserves $(\mathcal{G} \to M, \omega)$, i.e., $\underline{S}$ defines a smooth system S of (local) s–symmetries such that $\underline{S}(x)$ are the underlying (local) diffeomorphisms of $S(x)$ for all x. Thus the notation $\underline{S}$ is consistent with the Definition 1.1.

If there is a smooth system S of (local) s–symmetries of $(\mathcal{G} \to M, \omega)$, then we need the prolongation rigidity outside of the 1–eigenspace of s to show the existence of an $\underline{S}$–invariant class of Weyl connections, see Section 4.2. For all $|1|$–graded parabolic geometries and s such that $\mathrm{Ad}(s) = -\mathrm{id}_{\mathfrak{g}_-}$, i.e., $s = m$, we obtain affine (locally) symmetric spaces, because the class $[\nabla]$ consists of a single connection. For all parabolic contact geometries and s such that $\mathrm{Ad}(s)|_{\mathfrak{g}_{-1}} = -\mathrm{id}$ we obtain reflexion spaces, but the $\underline{S}$–invariant class $[\nabla]$ is not the class of admissible connections from [19].

For triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s, we get the following existence result, which in particular implies Theorem 1.3.

Theorem 1.8. *Suppose $s \in Z(G_0)$ is such that $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s. Suppose that the parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) has everywhere non–zero component of the harmonic curvature in μ. Then the following conditions are equivalent:*

(1) *The parabolic geometry is (locally) s–symmetric.*
(2) *There is a smooth system S of (local) s–symmetries.*
(3) *There is an $\underline{S}$–invariant class $[\nabla]$ of Weyl connections.*

Moreover, the smooth system S is unique and $\underline{S}$ consists of the underlying diffeomorphisms of S on M. The equality $S(x) \circ S(y) = S(\underline{S}(x)(y)) \circ S(x)$ holds whenever the compositions are defined. If $\mathrm{Ad}(s) \in Gl(\mathfrak{g}_-)$ has no eigenvalue 1, then $[\nabla]$ consists of a single $\underline{S}$–invariant Weyl connection, which is locally affinely homogeneous.

We prove the claims of Theorem 1.8 except the last one in Section 4. The last claim does not hold without additional assumptions on the 1–eigenspace. We prove the last claim in Section 5, where we study additional properties that follow from assumptions on the position and shape of the 1–eigenspace of s in $\mathfrak{g}_-$.

Outline of the article. We recall basic facts and formulas for Weyl connections in Section 2. In particular, we characterize automorphisms of parabolic geometries with their actions on Weyl structures. We recall the relation between normal coordinates and normal Weyl structures.

In Section 3, we prove Theorem 1.6 and we provide the characterization of the triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s.

In Section 4, we prove Theorem 1.8. We also obtain further properties of (locally) s–symmetric parabolic geometries of type (G, P) that have everywhere non–zero component of the harmonic curvature in μ for the triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s.

In Section 5, we classify all triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s such that μ is in the 1–eigenspace of s. The classification is separated in the tables according to the common properties of the triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ and elements $s \in Z(G_0)$. The notation for the tables and details on the classification can be found in Section 5.1. We show in Section 5.2 that there are triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ for which the $\underline{S}$–invariant class $[\nabla]$ of Weyl connections consists of a single $\underline{S}$–invariant Weyl connection. In particular, such an $\underline{S}$–invariant Weyl connection is always (locally) affinely homogeneous. In Sections 5.3 and 5.5, we show that there are triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ for which the (locally) s–symmetric parabolic geometries are locally correspondence spaces over some other s–symmetric parabolic geometries. In Section 5.4, we prove that there are triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ for which the condition of (local) homogeneity is satisfied for more complicated $\underline{S}$–invariant class $[\nabla]$ of Weyl connections.

In the Appendix A we recall from [11] the construction of (locally) homogeneous s–symmetric parabolic geometries that we need in Section 5.

2. Automorphisms of parabolic geometries

In this Section, we introduce necessary techniques and establish notation from the theory of parabolic geometries that we will use in the article, see [3, Section 5.1]. We focus here on actions of automorphisms on Weyl structures and connections.

2.1. **Weyl structures and connections.** Consider a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P). Many geometric objects on M can be identified with sections of natural bundles $\mathcal{V}$ associated to the P–bundle $\mathcal{G}$ for representations V of P. We can equivalently view the sections of $\mathcal{V}$ as P–equivariant functions $\mathcal{G} \to V$. In other words, the points of $\mathcal{G}$ are (higher order) frames and the P–equivariant functions are the coordinate functions. A crucial tool that allows us to reduce the number and order of the frames are Weyl structures. A *(local) Weyl structure* is a (local) G_0–equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the projection $\pi : \mathcal{G} \to \mathcal{G}_0$, where $\mathcal{G}_0 := \mathcal{G}/\exp(\mathfrak{p}_+)$ and $p_0 : \mathcal{G}_0 \to M$ is a G_0–bundle over M.

Definition 2.1. Assume $\sigma : \mathcal{G}_0 \to \mathcal{G}$ is a Weyl structure. Then for a section τ of a natural bundle $\mathcal{V}$, we denote by $(\tau)_\sigma$ the G_0–equivariant function $\mathcal{G}_0 \to V$ satisfying

$$(\tau)_\sigma := t \circ \sigma,$$

where $t : \mathcal{G} \to V$ is the P–equivariant function corresponding to τ.

In particular, vector fields ξ and 1–forms Υ on M are sections of bundles $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ and $\mathcal{G} \times_P \mathfrak{p}_+$, respectively, and there are corresponding G_0–equivariant functions $(\xi)_\sigma : \mathcal{G}_0 \to \mathfrak{g}_-$ and $(\Upsilon)_\sigma : \mathcal{G}_0 \to \mathfrak{p}_+$.

Weyl structures always exist on parabolic geometries and for each two Weyl structures σ and $\hat{\sigma}$, there exist a 1–form Υ and G_0–equivariant functions $\Upsilon_i : \mathcal{G}_0 \to \mathfrak{g}_i$ for $i = 1, \dots, k$ such that

$$\hat{\sigma} = \sigma \exp(\Upsilon)_\sigma = \sigma \exp(\Upsilon_1) \dots \exp(\Upsilon_k).$$

The G_0–equivariant function $(\Upsilon)_\sigma : \mathcal{G}_0 \to \mathfrak{p}_+$ is related to the functions Υ_i via the Baker–Campbell–Hausdorff (BCH)–formula.

We can decompose the pullback $\sigma^*\omega : T\mathcal{G}_0 \to \mathfrak{g}$ into G_0–equivariant 1–forms $\omega_i^\sigma : T\mathcal{G}_0 \to \mathfrak{g}_i$ according to the grading $\mathfrak{g}_i$ of $\mathfrak{g}$. These forms clearly depend on the choice of the Weyl structure σ. For a Weyl structure $\hat{\sigma} = \sigma \exp(\Upsilon)_\sigma$, there is the following formula describing the change of the forms

$$\omega_l^{\sigma \exp(\Upsilon)_\sigma} = \sum_{|i|+j=l} \frac{(-1)^i}{i!} (\mathrm{ad}(\Upsilon_k)^{i_k} \circ \cdots \circ \mathrm{ad}(\Upsilon_1)^{i_1}) \circ \omega_j^\sigma, \tag{1}$$

where we write $i! = i_1! \dots i_k!$, $|i| = i_1 + 2i_2 + \cdots + ki_k$ and $(-1)^i = (-1)^{i_1+\cdots+i_k}$ for the multi–index $i = (i_1, \dots, i_k)$ with $i_1, \dots, i_k \geq 0$.

The sum ω_-^σ of the forms ω_i^σ for $i < 0$ is called the *soldering form* given by the Weyl structure σ. Suppose $(\xi)_\sigma = \xi_{-k} + \cdots + \xi_{-1}$ holds for the vector field ξ on M and for G_0–equivariant functions $\xi_i : \mathcal{G}_0 \to \mathfrak{g}_i$. If $(\xi)_{\sigma \exp(\Upsilon)_\sigma} = \hat{\xi}_{-k} + \cdots + \hat{\xi}_{-1}$ holds for G_0–equivariant functions $\hat{\xi}_i : \mathcal{G}_0 \to \mathfrak{g}_i$ and the Weyl structure $\sigma \exp(\Upsilon)_\sigma$, then

$$\hat{\xi}_l = \sum_{|i|+j=l} \frac{(-1)^i}{i!} \mathrm{ad}(\Upsilon_k)^{i_k} \circ \cdots \circ \mathrm{ad}(\Upsilon_1)^{i_1}.\xi_j,$$

where . is the algebraic action of the values of functions $\mathcal{G}_0 \to \mathfrak{p}_+$ on the values of the functions $\mathcal{G}_0 \to \mathfrak{g}_-$.

The form ω_0^σ is a principal connection form on $\mathcal{G}_0$. Suppose that the finite–dimensional representation of P on V is completely reducible as a representation of G_0. Then

(1) the form ω_0^σ induces a linear connection ∇^σ on the space of P–equivariant functions $\mathcal{G} \to V$,
(2) for each P–equivariant function $\tau : \mathcal{G} \to V$, the connection ∇^σ preserves the decomposition of $(\tau)_\sigma$ into G_0–equivariant components.

The induced connections ∇^σ on $\mathcal{V}$ are called *Weyl connections*. The Weyl connection $\nabla^{\sigma \exp(\Upsilon)_\sigma}$ on $\mathcal{V}$ is related to the Weyl connection ∇^σ on $\mathcal{V}$ by

$$(\nabla_\xi^{\sigma \exp(\Upsilon)_\sigma} \tau)_\sigma = (\nabla_\xi^\sigma \tau)_\sigma + \sum_{|i|+j=0} \frac{(-1)^i}{i!} (\mathrm{ad}(\Upsilon_k)^{i_k} \circ \cdots \circ \mathrm{ad}(\Upsilon_1)^{i_1}(\xi_j)).(\tau)_\sigma, \tag{2}$$

where τ is a section of $\mathcal{V}$ and . is the algebraic action of the values of functions $\mathcal{G}_0 \to \mathfrak{g}_0$ on the values of the function $(\tau)_\sigma : \mathcal{G}_0 \to V$.

The soldering form ω_-^σ together with the principal connection form ω_0^σ form the Cartan connection $\omega_-^\sigma \oplus \omega_0^\sigma$ on $\mathcal{G}_0$ of a reductive type. In fact, we can view the first order frame bundle $\mathcal{P}^1 M$ as the bundle $\mathcal{G} \times_{\underline{\mathrm{Ad}}} Gl(\mathfrak{g}/\mathfrak{p})$ for the adjoint action $\underline{\mathrm{Ad}}$ of P on $\mathfrak{g}/\mathfrak{p}$. Moreover, each Weyl structure σ provides a reduction $\iota_\sigma : \mathcal{G}_0 \to \mathcal{P}^1 M$ over $\underline{\mathrm{Ad}} : G_0 \to Gl(\mathfrak{g}/\mathfrak{p})$ such that

$$\iota_\sigma^* \theta = \omega_-^\sigma \quad \text{and} \quad \iota_\sigma^* \gamma_\sigma = \omega_0^\sigma$$

hold for the natural soldering form θ on $\mathcal{P}^1 M$ and the principal connection form γ_σ on $\mathcal{P}^1 M$ of the Weyl connection ∇^σ. This allows us to describe explicitly geodesics of Weyl connections. The geodesic of the Weyl connection ∇^σ on TM through x in the direction $\xi(x) \in T_x M$ is the curve

$$p_0 \circ \mathrm{Fl}_t^{(\omega_-^\sigma \oplus \omega_0^\sigma)^{-1}(\xi(x))_\sigma(u_0)}(u_0) \tag{3}$$

for arbitrary $u_0 \in \mathcal{G}_0$ in the fiber over x. Indeed, since $(\omega_-^\sigma \oplus \omega_0^\sigma)^{-1}((\xi(x))_\sigma)$ is contained in the kernel of the connection form $\omega_0^\sigma = \iota_\sigma^* \gamma_\sigma$ and $Tp_0 \circ (\omega_-^\sigma \oplus$

$\omega_0^\sigma)^{-1}((\xi(x))_\sigma)(x) = \xi(x)$, the claimed curve is the projection of a flow of a standard horizontal vector field of γ_σ and therefore a geodesic of ∇^σ.

2.2. **The characterization of automorphisms.** Let $\varphi : \mathcal{G} \to \mathcal{G}$ be a (local) automorphism of the parabolic geometry and denote by $\varphi_0 : \mathcal{G}_0 \to \mathcal{G}_0$ the underlying (local) G_0–bundle morphism. Then for each Weyl structure σ, there is a 1–form $\Upsilon^{\sigma,\varphi}$ on M such that

$$\varphi(\sigma(u_0)) = \sigma(\varphi_0(u_0)) \exp((\Upsilon^{\sigma,\varphi})_\sigma(u_0)) \tag{4}$$

holds for all $u_0 \in \mathcal{G}_0$. Consequently, the pullback of a Weyl structure is again a Weyl structure, i.e.,

$$\varphi^*\sigma = \varphi^{-1} \circ \sigma \circ \varphi_0 = \sigma \exp(-(\Upsilon^{\sigma,\varphi})_\sigma).$$

Lemma 2.2. *Let $\varphi : \mathcal{G} \to \mathcal{G}$ be a (local) automorphism. Then*

$$(\Upsilon^{\sigma \exp(\Upsilon)_\sigma,\varphi})_{\sigma \exp(\Upsilon)_\sigma} = C(-(\Upsilon)_\sigma \circ \varphi_0, C((\Upsilon^{\sigma,\varphi})_\sigma, (\Upsilon)_\sigma)) \tag{5}$$

holds for the Weyl structure $\sigma \exp(\Upsilon)_\sigma$, where C represents the BCH–formula.

Proof. We get immediately from the formula (4) that

$$\begin{aligned}&\varphi(\sigma(u_0)) \exp((\Upsilon)_\sigma(u_0)) = \\ &\sigma(\varphi_0(u_0)) \exp((\Upsilon)_\sigma(\varphi_0(u_0))) \exp((\Upsilon^{\sigma \exp(\Upsilon)_\sigma,\varphi})_{\sigma \exp(\Upsilon)_\sigma}(u_0))\end{aligned}$$

holds for all $u_0 \in \mathcal{G}$. This implies

$$\exp((\Upsilon^{\sigma \exp(\Upsilon)_\sigma,\varphi})_{\sigma \exp(\Upsilon)_\sigma}) = \exp(-(\Upsilon)_\sigma \circ \varphi_0) \exp((\Upsilon^{\sigma,\varphi})_\sigma) \exp((\Upsilon)_\sigma),$$

which gives the formula. □

Therefore if $f = \underline{\varphi}$ for a (local) automorphism φ of the parabolic geometry, then

$$f^*\nabla^\sigma = \nabla^{\sigma \exp(-(\Upsilon^{\sigma,\varphi})_\sigma)}$$

holds for each Weyl connection ∇^σ.

There is a unique lift $\mathcal{P}^1 f$ of each (local) diffeomorphism f on M to the (local) $Gl(\mathfrak{g}/\mathfrak{p})$–bundle morphism on $\mathcal{P}^1 M$ such that $(\mathcal{P}^1 f)^*\theta = \theta$ holds. If $f^*\nabla^\sigma = \nabla^{\sigma'}$ is satisfied for some Weyl connections ∇^σ and $\nabla^{\sigma'}$, then $(\mathcal{P}^1 f)^*\gamma_\sigma = \gamma_{\sigma'}$ holds. However, this does not imply that such f preserves the parabolic geometry. The (local) diffeomorphisms f that preserve the parabolic geometry also satisfy that

$$\mathcal{P}^1 f(\iota_{\sigma'}(\mathcal{G}_0)) = \iota_\sigma(\mathcal{G}_0)$$

holds for reductions $\iota_\sigma(\mathcal{G}_0)$ and $\iota_{\sigma'}(\mathcal{G}_0)$ of $\mathcal{P}^1 M$ and it turns out that this is the crucial property that distinguishes the diffeomorphisms preserving the parabolic geometry among all diffeomorphisms preserving the set of all Weyl connections.

Proposition 2.3. *Let f be a (local) diffeomorphism on M such that for some Weyl structures σ and σ' of the parabolic geometry $(\mathcal{G} \to M, \omega)$*

- *$f^*\nabla^\sigma = \nabla^{\sigma'}$ holds, and*
- *$\mathcal{P}^1 f$ maps a point of $\iota_{\sigma'}(\mathcal{G}_0)$ into the image $\iota_\sigma(\mathcal{G}_0)$.*

Then f preserves the parabolic geometry.

Proof. The assumptions imply that $\varphi_0 := \iota_\sigma^{-1} \circ \mathcal{P}^1 f \circ \iota_{\sigma'}$ is a well–defined (local) G_0–bundle morphism $\varphi_0 : \mathcal{G}_0 \to \mathcal{G}_0$ satisfying $\varphi_0^*\omega_0^\sigma = \omega_0^{\sigma'}$ and $\varphi_0^*\omega_-^\sigma = \omega_-^{\sigma'}$. The associated graded map $(\theta_{-k}, \dots, \theta_{-1}) : T\mathcal{G}_0 \to \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$ corresponding to ω_-^σ is independent of the choice of the Weyl structure according to the formula (1). In fact, the tuple $(p_0 : \mathcal{G}_0 \to M, (\theta_{-k}, \dots, \theta_{-1}))$ is a regular infinitesimal flag structure with a (local) automorphism φ_0, see [3, Section 3.1.6-8]. Therefore the claim of Theorem follows from [3, Theorem 3.1.14] except for projective and contact projective geometries. In the case of projective geometries, the claim trivially follows from the assumption $f^*\nabla^\sigma = \nabla^{\sigma'}$. In the case of contact projective geometries, φ_0 is a (local) automorphism of the regular infinitesimal flag structure if and only if f is a contactomorphism and the claim again follows from $f^*\nabla^\sigma = \nabla^{\sigma'}$, see [3, Section 4.2] for details. □

2.3. **Normal Weyl structures and generalized geodesics.** There is a distinguished class of local Weyl structures, so–called *normal Weyl structures at $x = p(u)$*, each of which is determined by a choice of $u \in \mathcal{G}$. More precisely, we consider local Weyl structures ν_u given by

$$\nu_u(\pi(\mathrm{Fl}_1^{\omega^{-1}(X)}(u))) := \mathrm{Fl}_1^{\omega^{-1}(X)}(u)$$

for X in some neighbourhood of 0 in $\mathfrak{g}_-$. The Weyl structures ν_u for all $u \in \mathcal{G}_x$ exhaust all normal Weyl structures at x, see [3, Section 5.1.12]. These Weyl structures are distinguished by the fact that

$$\varphi(\mathrm{Fl}_1^{\omega^{-1}(X)}(u)) = \mathrm{Fl}_1^{\omega^{-1}(X)}(\varphi(u)) \tag{6}$$

holds for all (local) automorphisms φ of the parabolic geometry and all X in some neighbourhood of 0 in $\mathfrak{g}$. This particularly means that

$$\varphi^*\nu_u = \nu_{\varphi^{-1}(u)}$$

holds for all (local) automorphisms φ of parabolic geometries.

The curves of the form

$$p \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$$

for $X \in \mathfrak{g}_-$ and $u \in \mathcal{G}$ are called *generalized geodesics.* They always provide the normal coordinate system given by u. The crucial observation is that the set of generalized geodesics going through x coincides with the set of geodesics of normal Weyl connections ∇^{ν_u} for all u. Therefore there is the following description of automorphisms of parabolic geometries.

Proposition 2.4. *Let φ be a (local) P–bundle morphism on $\mathcal{G}$ and let $f = \underline{\varphi}$ be its underlying (local) diffeomorphism of M. If φ is a (local) automorphism of the parabolic geometry, then the equality $f^*\nabla^{\nu_u} = \nabla^{\nu_{\varphi^{-1}(u)}}$ holds for all $u \in \mathcal{G}$ and f maps the set of generalized geodesics going through x onto the set of generalized geodesics going through $f(x)$.*

Moreover, if f has coordinates $\mathrm{Ad}(g_0) \in Gl(\mathfrak{g}_-)$ for $g_0 \in G_0$ in the normal coordinate system given by $u \in \mathcal{G}$, then φ is a (local) automorphism of the parabolic geometry if and only if $f^*\nabla^{\nu_u} = \nabla^{\nu_u}$ holds.

Proof. Since $f^*\nabla^\sigma = \nabla^{\varphi^*\sigma}$ holds for all Weyl structures σ and all (local) automorphisms φ of the parabolic geometry, the first claim follows from the formula (3). If f has coordinates $\mathrm{Ad}(g_0) \in Gl(\mathfrak{g}_-)$ in the normal coordinate system given by $u \in \mathcal{G}$, then the second assumption of Proposition 2.3 is satisfied. Then the second claim is a consequence of the first claim and Proposition 2.3, because $\nabla^{\nu_u} = \nabla^{\nu_{ug_0^{-1}}}$ holds. □

3. The uniqueness of s–symmetries and the prolongation rigidity

In this section, we prove Theorem 1.6. We also characterize all triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s.

3.1. **Consequences of the existence of more s–symmetries at one point.** Let us recall that if V is an irreducible G_0–module, then the element $s \in Z(G_0)$ acts on V by a single eigenvalue. In particular, we can decompose each completely reducible G_0–module V into G_0–submodules

$$V^s(a) := \{X \in V : s(X) = aX\}$$

according to the eigenvalues of the action of $s \in Z(G_0)$. In particular, we will often consider the 1–eigenspaces $\mathfrak{g}^s_-(1)$, $\mathfrak{g}^s_i(1)$ and $\mathfrak{p}^s_+(1)$ in $\mathfrak{g}_-$, $\mathfrak{g}_i$ and $\mathfrak{p}_+$, respectively.

The following Proposition is a crucial technical result for the proof of Theorem 1.6.

Proposition 3.1. *Let s_u be a (local) s–symmetry at x for some $u \in \mathcal{G}_x$. Then for each Weyl structure σ, there is a 1–form Υ^{σ,s_u} on M satisfying*

(1) $s_u^*\sigma = \sigma \exp(-(\Upsilon^{\sigma,s_u})_\sigma)$,
(2) $(\Upsilon^{\sigma,s_u})_\sigma(\pi(u)) = C(-\mathrm{Ad}(s)^{-1}(Y), Y)$ *for some* $Y \in \mathfrak{p}_+$, *where* C *represents the BCH–formula on the nilpotent Lie algebra* $\mathfrak{p}_+$, *and*
(3) *if* $(\Upsilon^{\sigma,s_u})_\sigma(\pi(u)) = Z_i + \cdots + Z_k$ *holds for* $Z_j \in \mathfrak{g}_j$, *then the component of* Z_i *contained in* $\mathfrak{g}^s_i(1)$ *is trivial, where* i *is the smallest index* j *such that* $(\Upsilon^{\sigma,s_u})_\sigma(\pi(u))$ *has a non–zero component in* $\mathfrak{g}_i$.

Moreover, if s_v is a (local) s–symmetry at x for some $v \in \mathcal{G}_x$, then $s_u = s_v$ if and only if $\Upsilon^{\sigma,s_u}(x) = \Upsilon^{\sigma,s_v}(x)$ holds.

Proof. The normal Weyl structure ν_u always satisfies $\nu_u(\pi(u)) = u$ and therefore the set of all Weyl structures σ satisfying $\sigma(\pi(u)) = u$ is non–empty. Let s_u be a (local) s–symmetry at x and consider arbitrary Weyl structure σ satisfying $\sigma(\pi(u)) = u$. Then $(\Upsilon^{\sigma,s_u})_\sigma(\pi(u)) = 0$ holds and Lemma 2.2 implies that $(\Upsilon^{\sigma,s_u})_\sigma$ has the claimed properties (1) and (2) for arbitrary Weyl structure. The claimed property (3) holds, because the BCH–formula implies that $C(-\mathrm{Ad}(s)^{-1}(Y), Y)_i = -\mathrm{Ad}(s)^{-1}(Y_i) + Y_i = Z_i$ holds.

If s_v is a (local) s–symmetry at x for some $v \in \mathcal{G}_x$, then $s_u = s_v$ if and only if $us = s_u(u) = s_v(u)$ holds. Thus we need to show that if $\Upsilon^{\sigma,s_u}(x) = \Upsilon^{\sigma,s_v}(x)$ holds, then $s_u = s_v$. We can assume $\sigma(\pi(u)) = u$ for the Weyl structure σ, because the equality $\Upsilon^{\sigma,s_u}(x) = \Upsilon^{\sigma,s_v}(x)$ is preserved if we change the Weyl structure σ. Suppose $g_0 \in G_0$ and $Y \in \mathfrak{p}_+$ are such that $v = ug_0\exp(Y)$ holds. If $\hat\sigma$ is a Weyl structure such that $\hat\sigma(\pi(u)) = ug_0\exp(Y)$, then $\Upsilon^{\sigma,s_u}(x) = 0$, $\Upsilon^{\hat\sigma,s_v}(x) = 0$ and $(\Upsilon^{\sigma,s_v})_\sigma(\pi(u)) = C(-\mathrm{Ad}(s)^{-1}(\mathrm{Ad}(g_0)(Y)), \mathrm{Ad}(g_0)(Y))$ hold. Since $C(-\mathrm{Ad}(s)^{-1}(\mathrm{Ad}(g_0)(Y)), \mathrm{Ad}(g_0)(Y)) = 0$ if and only if $\mathrm{Ad}(s)(Y) = Y$, the element s commutes with $g_0\exp(Y)$ and $s_u = s_v$ holds. □

The harmonic curvature κ_H is preserved by each (local) automorphism of the parabolic geometry. Since κ_H is a section of an associated vector bundle to $\mathcal{G}$ for a representation of P which is trivial on $\exp(\mathfrak{p}_+)$, the function $(\kappa_H)_\sigma$ does not depend on the choice of the Weyl structure σ and we will write $\kappa_H(u)$ instead of $(\kappa_H)_\sigma(\pi(u))$. Consequently, $\kappa_H(p(u)) = 0$ if and only if $\kappa_H(u) = 0$.

If s_u is a (local) s–symmetry at $p(u)$, then $\underline{s}_u^*\kappa_H = \kappa_H$. Thus $s.\kappa_H(u) = \kappa_H(u)$ trivially follows, where we denote by . the tensorial action of $\mathfrak{g}_0$ on κ_H. This proves the first claim of Theorem 1.6.

The second claim of Theorem 1.6 is a consequence of the following Proposition and the Definition 1.5 of the prolongation rigidity.

Proposition 3.2. *Assume there are (local) s–symmetries s_u and s_v at x for some $u, v \in \mathcal{G}_x$. Suppose that $(\Upsilon^{\sigma,s_v})_\sigma(\pi(u)) = 0$ and $(\Upsilon^{\sigma,s_u})_\sigma(\pi(u)) = Z_i + \cdots + Z_k$ hold for some Weyl structure σ. Then $Z_i \in pr(\kappa_H(u))_i$.*

Proof. We show that $\mathrm{ad}(X_1)\dots\mathrm{ad}(X_i)(Z_i).\kappa_H(u)=0$ holds for all $X_1,\dots X_i\in\mathfrak{g}_{-1}$. Consider an arbitrary Weyl structure σ and consider the iterated covariant derivative $(\nabla^\sigma)^j_{\xi^1,\dots,\xi^j}$ for vector fields $\xi^1,\dots,\xi^j$ such that

$$(\xi^b)_\sigma=\xi^b_{-1}:\mathcal{G}_0\to\mathfrak{g}_{-1},$$
$$(\xi^b)_\sigma(\pi(u))=X^b$$

hold for some $X^b\in\mathfrak{g}^s_{-1}(\frac{1}{a_b})$ for some a_b for all $1\le b\le j$. We assume $j\le i$ unless we state otherwise.

We compute

$$\begin{aligned}(\underline{s}_u^*\nabla^\sigma)^j_{\xi^1,\dots,\xi^j}\kappa_H(u)&=\underline{s}_u^*(\nabla^\sigma)^j_{(\underline{s}_u)_*\xi^1,\dots,(\underline{s}_u)_*\xi^j}(\underline{s}_u)_*\kappa_H(u)\\&=(\nabla^\sigma)^j_{(\underline{s}_u)_*\xi^1,\dots,(\underline{s}_u)_*\xi^j}\kappa_H(u).\end{aligned}$$

Since we assume $X^b\in\mathfrak{g}^s_{-1}(\frac{1}{a_b})$, we get

$$\begin{aligned}((\underline{s}_u)_*\xi^b)_{\sigma\exp(\Upsilon^{\sigma,s_u})_\sigma}(\pi(u))&=((\underline{s}_u)_*\xi^b)_\sigma(\pi(u))=(\xi^b)_\sigma(\pi(u)s)\\&=\mathrm{Ad}(s)^{-1}(\xi^b)_\sigma(\pi(u))=a_bX^b.\end{aligned}$$

Thus

$$(\underline{s}_u^*\nabla^\sigma)^j_{\xi^1,\dots,\xi^j}\kappa_H(u)=a_1\cdots a_j(\nabla^\sigma)^j_{\xi^1,\dots,\xi^j}\kappa_H(u).\tag{7}$$

If $(\Upsilon^{\sigma,s_u})_\sigma(\pi(u))=Z_i+\dots+Z_k$ holds for the Weyl structure σ, then the formula (2) together with Proposition 3.1 imply

$$(\underline{s}_u^*\nabla^\sigma)_{\xi^b}\kappa_H(u)=\nabla^{\sigma\exp(-(\Upsilon^{\sigma,s_u})_\sigma)}_{\xi^b}\kappa_H(u)=\nabla^\sigma_\xi\kappa_H(u)+\mathrm{ad}(Z_i)(X^b).\kappa_H(u).$$

In particular, if $i>1$, then

$$(\underline{s}_u^*\nabla^\sigma)_{\xi^b}\kappa_H(u)=\nabla^{\sigma\exp(-(\Upsilon^{\sigma,s_u})_\sigma)}_{\xi^b}\kappa_H(u)=\nabla^\sigma_\xi\kappa_H(u).$$

If we apply the above formulas onto the first connection in $(\underline{s}_u^*\nabla^\sigma)^j_{\xi^1,\dots,\xi^j}\kappa_H(u)$, then we obtain

$$(\underline{s}_u^*(\nabla^\sigma)^j)_{\xi^1,\dots,\xi^j}\kappa_H(u)=\nabla^\sigma_{\xi^1}(\underline{s}_u^*\nabla^\sigma)^{j-1}_{\xi^2,\dots,\xi^j}\kappa_H(u).$$

In the next step, the same formulas for the second connection lead to the formula

$$\begin{aligned}(\underline{s}_u^*\nabla^\sigma)^j_{\xi^1,\dots,\xi^j}\kappa_H(u)&=(\nabla^\sigma)^2_{\xi^1,\xi^2}(\underline{s}_u^*\nabla^\sigma)^{j-2}_{\xi^3,\dots,\xi^j}\kappa_H(u)\\&\quad-\mathrm{ad}(X^2)((\nabla^\sigma)_{\xi^1}(\Upsilon^{\sigma,s_u})_\sigma).(\underline{s}_u^*\nabla^\sigma)^{j-2}_{\xi^1,\dots,\xi^{l-j}}\kappa_H(u).\end{aligned}$$

Thus before we consider the next step, we need to characterize the components of $(\nabla^\sigma_{\xi^b}\Upsilon^{\sigma,s_u})_{\sigma(\pi(u))}$ in $\mathfrak{g}_1\oplus\dots\oplus\mathfrak{g}_j$ for $j<i$. Firstly, let us view $(\Upsilon^{\sigma,s_u})_\sigma$ as a section of the adjoint tractor bundle $\mathcal{G}\times_P\mathfrak{g}$. Observe that the covariant derivative $\nabla^\sigma_{\xi^b}$ coincides with the fundamental derivative on the components in $\mathfrak{g}_-\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\dots\oplus\mathfrak{g}_j$ according to the formula from [3, Proposition 5.1.10]. We know that $(\Upsilon^{\sigma,s_u})_\sigma$ has its values in $\mathfrak{p}_+$ and the components of $(\nabla^\sigma_{\xi^b}\Upsilon^{\sigma,s_u})_{\sigma(\pi(u))}$ in $\mathfrak{g}_1\oplus\dots\oplus\mathfrak{g}_j$ for $j<i$ are tensorial both in ξ^b and Υ^{σ,s_u}. Then, using the formula from [3, Corollary 1.5.8] and the P–equivariancy of ω, we get the following equality on the restriction to $\mathfrak{g}_1\oplus\dots\oplus\mathfrak{g}_j$ for $j<i$

$$(\nabla^\sigma_{\xi^b}\Upsilon^{\sigma,s_u})_{\sigma(\pi(u))}=\omega(\sigma(\pi(u)))([\omega^{-1}(\xi^b),\omega^{-1}(Z_i)])=-\mathrm{ad}(X^b)(Z_i).$$

Therefore

$$(\underline{s}_u^*\nabla^\sigma)^j_{\xi^1,\dots,\xi^j}\kappa_H(u)=(\nabla^\sigma)^2_{\xi^1,\xi^2}(\underline{s}_u^*\nabla^\sigma)^{j-2}_{\xi^3,\dots,\xi^j}\kappa_H(u).$$

If we iterate the computation of $(\nabla^\sigma_{\xi^b}\Upsilon^{\sigma,s_u})_{\sigma(\pi(u))}$ for $j<i$, then we obtain by the same arguments

$$((\nabla^\sigma)^j_{\xi^1,\dots,\xi^j}\Upsilon^{\sigma,s_u})_{\sigma(\pi(u))}=(-1)^j\mathrm{ad}(X^j)\dots\mathrm{ad}(X^1)(Z_i)$$

for the component in $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{i-j}$. Thus for $j < i$, we obtain

$$\begin{aligned}(\underline{s}_u^* \nabla^\sigma)^j_{\xi^1,\dots,\xi^j} \kappa_H(u) &= (\nabla^\sigma)^j_{\xi^1,\dots,\xi^j} \kappa_H(u) - \mathrm{ad}(X^j)((\nabla^\sigma)^{j-1}_{\xi^1,\dots,\xi^{j-1}}(\Upsilon^{\sigma,s_u})_\sigma).\kappa_H(u)\\ &= (\nabla^\sigma)^j_{\xi^1,\dots,\xi^j} \kappa_H(u)\end{aligned}$$

and for $j = i$, we obtain

$$\begin{aligned}(\underline{s}_u^* \nabla^\sigma)^i_{\xi^1,\dots,\xi^i} \kappa_H(u) &= (\nabla^\sigma)^i_{\xi^1,\dots,\xi^i} \kappa_H(u)\\ &\quad + (-1)^i \mathrm{ad}(X^i)\dots\mathrm{ad}(X^1)(Z_i).\kappa_H(u).\end{aligned} \tag{8}$$

If we compare the formulas (7) and (8) for $(\underline{s}_u^* \nabla^\sigma)^j_{\xi^1,\dots,\xi^j} \kappa_H(u)$, we obtain

$$(-1)^j (a_1 \cdots a_j - 1)(\nabla^\sigma)^j_{\xi^1,\dots,\xi^j} \kappa_H(u) = \mathrm{ad}(X^j)\dots\mathrm{ad}(X^1)(Z_i).\kappa_H(u) \tag{9}$$

for all $j \leq i$.

If the Weyl structure σ satisfies $(\Upsilon^{\sigma,s_v})_\sigma(\pi(u)) = 0$, then we simultaneously have

$$(a_1 \cdots a_j - 1)(\nabla^\sigma)^j_{\xi^1,\dots,\xi^j} \kappa_H(u) = 0$$

for all $j \leq i$ if we follow the proof for s_v instead of s_u. Thus if $a_1 \cdots a_j - 1 \neq 0$, then $\mathrm{ad}(X^i)\dots\mathrm{ad}(X^1)(Z_i).\kappa_H(u) = 0$. But since Z_i has a trivial component in $\mathfrak{g}_i^s(1)$, we know that $\mathrm{ad}(X^i)\dots\mathrm{ad}(X^1)(Z_i) \neq 0$ implies $a_1 \cdots a_j - 1 \neq 0$ and the claim of Proposition holds due to the linearity. □

If we follow the computations from the proof of Proposition 3.2 for a Weyl structure σ satisfying $\Upsilon^{\sigma,s_u}(x) = 0$, then most of the assumptions on the vector fields ξ^b are vacuous and $(\underline{s}_u^* \nabla^\sigma)_\xi \kappa_H(u) = \nabla^\sigma_\xi \kappa_H(u)$ holds for arbitrary vector field ξ. Therefore we obtain the following corollary using the formula (7) for ξ from particular eigenspaces of $T_x \underline{s}_u$.

Corollary 3.3. *Let s_u be a (local) s–symmetry at $x = p(u)$ on a parabolic geometry and assume $\Upsilon^{\sigma,s_u}(x) = 0$. Then we get*

$$\nabla^\sigma_\xi \kappa_H(x) = \nabla^\sigma_{\xi_{fix}} \kappa_H(x),$$

where $\xi_{fix} \in T_xM$ is the component of $\xi \in T_xM$ such that $(\xi_{fix})_\sigma(\pi(u)) \in \mathfrak{g}^s_-(1)$. In particular, if $\mathfrak{g}^s_-(1) = 0$, then $\nabla^\sigma_\xi \kappa_H(x) = 0$ holds for all $\xi \in T_xM$.

Remark 3.4. The authors showed in [22] and [5] that there are projective and conformal geometries satisfying $\nabla^\sigma \kappa_H(x) = 0$ for all $x \in M$ for a suitable Weyl connection ∇^σ, but (M, ∇^σ) are not an affine locally symmetric spaces. Therefore Theorem 1.3 implies that the condition $\nabla^\sigma \kappa_H = 0$ is necessarily satisfied on (locally) m–symmetric parabolic geometries, but is not sufficient to distinguish the (locally) m–symmetric parabolic geometries among the geometries satisfying $\nabla^\sigma \kappa_H = 0$.

3.2. **The characterization of triples that are prolongation rigid outside of the 1–eigenspace of s.** We can estimate the dimension of $pr(\kappa_H(u))_i$ in the following way: The result of [17, Proposition 3.1.1] states that the dimension of $ann(\kappa_H(u))$ is bounded by the dimension of the annihilator $\mathfrak{a}_0 := \cap_{\phi_0} ann(\phi_0)$ of all minus lowest weights ϕ_0 in (the complexification of) all irreducible $\mathfrak{g}_0$–modules in which $\kappa_H(u)$ has a non–zero component. Moreover, the dimension of $pr(\kappa_H(u))_i$ is bounded by the dimension of the prolongation $\mathfrak{a}_i := \cap_{\phi_0} pr(\phi_0)_i$ of $\mathfrak{a}_0$. The main result of [17, Theorem 3.3.3 and Recipe 7] states that there is a semisimple Lie subalgebra $\bar{\mathfrak{g}}$ of $\mathfrak{g}$ and a parabolic subalgebra $\bar{\mathfrak{p}}$ of $\bar{\mathfrak{g}}$ such that $\mathfrak{a}_i = \bar{\mathfrak{g}}_i$ for $i > 0$.

Let us prove that these estimates are compatible with the decomposition of $\mathfrak{g}_i$ into $\mathfrak{g}_0$–submodules, which allows us to characterize the triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s.

Proposition 3.5. *Suppose $Z \in pr(\kappa_H(u))_i$ decomposes as $Z = Z_a + Z_b$ for Z_a, Z_b in different $\mathfrak{g}_0$–submodules of $\mathfrak{g}_i$. Then $Z_a \in pr(\kappa_H(u))_i$ and $Z_b \in pr(\kappa_H(u))_i$.*

Therefore the triple $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s if and only if $\mathfrak{a}_i$ corresponding to μ is a subspace of $\mathfrak{g}_i^s(1)$ for all i.

Proof. Let $(\alpha_1, \dots, \alpha_j)$ be an ordering of simple positive roots of $\mathfrak{g}$ such that the root space $\mathfrak{g}_{\alpha_r}$ satisfies $\mathfrak{g}_{\alpha_r} \in \mathfrak{g}_1$. Then we can uniquely assign a j–tuple $(a_1, \dots, a_j)$ to each irreducible $\mathfrak{g}_0$–component of $\mathfrak{g}_i$, where a_ℓ is the height of all root spaces in the $\mathfrak{g}_0$–component with respect to α_ℓ. This defines a multigrading of $\mathfrak{g}$ and the Lie bracket in $\mathfrak{g}$ is multigraded.

Let us decompose the element $Z \in pr(\kappa_H(u))_i$ as the sum of the elements $\sum Z_{(b_1,\dots,b_j)}$ over all possible j–tuples with respect to this multigrading. Similarly, let us decompose the module $\otimes^i \mathfrak{g}_{-1}$ as the sum of modules $\oplus \mathfrak{n}_{(a_1,\dots,a_j)}$ over all possible j–tuples with respect to this multigrading. The multigrading of $\mathfrak{g}_0$ is of the form $(0, \dots, 0)$, and therefore,

$$\mathrm{ad}^i(X)(Z) = \sum \mathrm{ad}^i(X_{(a_1,\dots,a_j)})(\sum Z_{(b_1,\dots,b_j)}) = \sum \mathrm{ad}^i(X_{(-b_1,\dots,-b_j)})(Z_{(b_1,\dots,b_j)})$$

holds for all $X = \sum X_{(a_1,\dots,a_j)} \in \oplus \mathfrak{n}_{(a_1,\dots,a_j)}$. Thus we get that

$$\mathrm{ad}^i(X_{(-b_1,\dots,-b_j)})(Z_{(b_1,\dots,b_j)}) \in ann(\kappa_H(u))$$

holds for all $X = X_{(-b_1,\dots,-b_j)} \in \mathfrak{n}_{(-b_1,\dots,-b_j)}$. Thus $Z_{(b_1,\dots,b_j)} \in pr(\kappa_H(u))_i$ follows from the linearity for all components $Z_{(b_1,\dots,b_j)}$ of Z.

The first claim implies that the proof of [17, Proposition 3.1.1] can be carried separately for each component of $pr(\kappa_H(u))_i$ in $\mathfrak{g}_0$–submodule in $\mathfrak{g}_i$ and thus the second claim follows from [17, Theorem 3.3.3]. □

One can find in [12, Appendix C] tables containing the classification of the triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ such that μ is contained in the 1–eigenspace of s for some $s \in Z(G_0)$ (different from identity), the classification of the modules $\mathfrak{a}_i$ and the classification of the 1–eigenspaces of s in $\mathfrak{p}_+$. This allows us to classify all triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s such that μ is contained in the 1–eigenspace of s.

We would like to present the classification together with additional properties of the corresponding (locally) s–symmetric parabolic geometries. Therefore we postpone the classification to Section 5.1 and continue by looking on geometric properties of generic (locally) s–symmetric parabolic geometries.

4. Geometric properties of parabolic geometries of general types

We present here geometric properties that are common for (locally) s–symmetric parabolic geometries for triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s. In particular, we prove Theorem 1.8. In order to prove that Claim (3) implies Claim (2), we discuss in Section 4.1 when a geodesic transformation s_x^σ of a Weyl connection ∇^σ preserves the parabolic geometry. The Claim (1) follows trivially from Claim (2) and we discuss the remaining implication in Section 4.2.

4.1. **Automorphisms and normal coordinate systems of Weyl connections.**

Let us describe the (local) diffeomorphisms s_x^σ in detail. We know from the formula (3) that the (local) diffeomorphism s_x^σ of M defined by the formula

$$\begin{aligned} s_x^\sigma(p_0 \circ \mathrm{Fl}_1^{(\omega_-^\sigma \oplus \omega_0^\sigma)^{-1}(\xi(x))_\sigma(u_0)}(u_0)) :&= p_0 \circ \mathrm{Fl}_1^{(\omega_-^\sigma \oplus \omega_0^\sigma)^{-1}\mathrm{Ad}(s)(\xi(x))_\sigma(u_0)}(u_0) \\ &= p_0 \circ \mathrm{Fl}_1^{(\omega_-^\sigma \oplus \omega_0^\sigma)^{-1}(\xi(x))_\sigma(u_0)}(u_0 s) \end{aligned} \tag{10}$$

for some $u_0 \in (\mathcal{G}_0)_x$ does not depend on the choice of $u_0 \in (\mathcal{G}_0)_x$. So s_x^σ is the unique (local) diffeomorphism with coordinates $\mathrm{Ad}(s) \in Gl(\mathfrak{g}_-)$ for $s \in Z(G_0)$ in the normal coordinate system for the Weyl connection ∇^σ given by some $u_0 \in \mathcal{G}_0$.

We also know from Proposition 2.4 that for a normal Weyl structure ν_u for $u \in \mathcal{G}_x$, the equality

$$s_x^{\nu_u} = \underline{s}_u$$

holds. Thus $s_x^{\nu_u}$ preserves the parabolic geometry (and therefore s_u is a (local) automorphism of the parabolic geometry) if and only if $(s_x^{\nu_u})^*\nabla^{\nu_u} = \nabla^{\nu_u}$ holds.

The situation is different for a general Weyl structure σ and the following Proposition gives a sufficient condition for s_x^σ to be a (local) s–symmetry.

Proposition 4.1. *Assume the (local) diffeomorphism s_x^σ satisfies*

- $(s_x^\sigma)^*\nabla^\sigma = \nabla^{\sigma \exp(\Upsilon)_\sigma}$ *for some 1–form Υ on M, and*
- $\Upsilon(x) = 0$.

Then $s_{\sigma(u_0)}$ is a (local) s–symmetry at x for all u_0 in the fiber over x such that $\Upsilon^{\sigma, s_{\sigma(u_0)}} = -\Upsilon$, and $\underline{s}_{\sigma(u_0)} = s_x^\sigma$, i.e., s_x^σ preserves the parabolic geometry.

Proof. Suppose $(s_x^\sigma)^*\nabla^\sigma = \nabla^{\sigma \exp(\Upsilon)_\sigma}$ holds for Υ such that $\Upsilon(x) = 0$. Then the inclusions ι_σ and $\iota_{\sigma \exp(\Upsilon)_\sigma}$ of $\mathcal{G}_0$ into $\mathcal{P}^1M$ coincide in the fiber over x by the assumption $\Upsilon(x) = 0$. Thus the formula (10) implies that $\mathcal{P}^1 s_x^\sigma$ maps the frames $\iota_\sigma(u_0) = \iota_{\sigma \exp(\Upsilon)_\sigma}(u_0)$ in the fiber over x onto frames $\iota_\sigma(u_0 s) = \iota_{\sigma \exp(\Upsilon)_\sigma}(u_0 s)$. Therefore the conditions of Proposition 2.3 are satisfied and s_x^σ preserves the parabolic geometry. Since $\Upsilon(x) = 0$, it follows from Proposition 3.1 that the covering of s_x^σ maps $\sigma(u_0)$ onto $\sigma(u_0)s$ and thus coincides with $s_{\sigma(u_0)}$ due to the formula (6). □

In particular, if there is an $\underline{S}$–invariant class of Weyl connections, then all (local) diffeomorphisms $\underline{S}(x)$ for all $x \in M$ satisfy the conditions of Proposition 4.1 and therefore Claim (3) of Theorem 1.8 implies Claim (2) of Theorem 1.8.

A consequence of Propositions 4.1 and 3.1 is that the condition $\Upsilon^{\sigma, s_u}(p(u)) = 0$ is necessary for the equality $\underline{s}_u = s_{p(u)}^\sigma$ to hold for s–symmetry s_u at $p(u)$. On the other hand, it is clear that the condition $\Upsilon^{\sigma, s_u}(p(u)) = 0$ is far from being sufficient. There is the following consequence of the fact that the affine maps are determined by the image of a single point in $\iota_\sigma(\mathcal{G}_0) \subset \mathcal{P}^1M$.

Corollary 4.2. *Let s_u be a (local) s–symmetry at x and assume $\Upsilon^{\sigma, s_u} \equiv 0$ holds for some Weyl structure σ. Then $\underline{s}_u = s_x^\sigma$.*

4.2. **The prolongation rigidity for s–symmetric parabolic geometries.** Let $(\mathfrak{g}, \mathfrak{p}, \mu)$ be prolongation rigid outside of the 1–eigenspace of s. Let $U \subset M$ be the open subset of M consisting of points x such that $\kappa_H(x)$ has a non–zero component in the $\mathfrak{g}_0$–module given by μ. If the parabolic geometry is (locally) s–symmetric, then there is a unique (local) s–symmetry s_u at each point of U, i.e., there is the unique system S of (local) s–symmetries on U. This means that if there is an almost $\underline{S}$–invariant Weyl connection on U, then the system $\underline{S}$ coincides (due to uniqueness) with the system of (local) diffeomorphisms $\underline{s}_u$. We call a Weyl structure σ (almost) S–invariant (at x) if ∇^σ is (almost) $\underline{S}$–invariant Weyl connection (at x).

The uniqueness of s–symmetries on U has the following consequences in the case $U = M$.

Proposition 4.3. *Assume $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s and $\kappa_H(x)$ has a non–zero component in the $\mathfrak{g}_0$–module given by μ at all $x \in M$. Let S be the unique system of (local) s–symmetries on the (locally) s–symmetric parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P). Then:*

(1) *There exists an almost S–invariant Weyl structure σ and the map S is smooth.*
(2) *If σ is an almost S–invariant Weyl structure, then $\sigma\exp(\Upsilon)_\sigma$ is an almost S–invariant Weyl structure if and only if $(\Upsilon)_\sigma$ has its values in $\mathfrak{p}_+^s(1)$.*
(3) *For each $x \in M$, there is a local almost S–invariant Weyl structure σ, which is invariant at x, and $\underline{S}(x) = s_x^\sigma$ holds.*
(4) *The equality $S(p_0(u_0)) = s_{\sigma(u_0)}$ holds for each almost S–invariant Weyl structure σ for all $u_0 \in \mathcal{G}_0$.*
(5) *The equality $S(x) \circ S(y) \circ S(x)^{-1} = S(\underline{S}(x)(y))$ holds for $x, y \in M$, where the compositions are defined.*
(6) *For each eigenvalue a, the union of the a–eigenspaces $T_xM^s(a)$ of $T_x\underline{S}(x)$ in T_xM over all $x \in M$ defines a distribution $TM^s(a)$ on M that is preserved by all (local) s–symmetries for each a.*
(7) *The equality $TM^s(a) = Tp_0 \circ (\omega_-^\sigma + \omega_0^\sigma)^{-1}(\mathfrak{g}_-^s(a))$ holds for each almost S–invariant Weyl structure σ.*
(8) *The decomposition $TM = \oplus_a TM^s(a)$ is preserved by all almost $\underline{S}$–invariant Weyl connections ∇^σ.*
(9) *All almost $\underline{S}$–invariant Weyl connections restrict to the same partial linear connection on TM corresponding to the distribution $\oplus_{a\neq 1} TM^s(a)$.*

We show that Claim (1) of Theorem 1.8 implies Claim (3) of the Theorem 1.8 and simultaneously obtain all the claims of Proposition.

Proof. Let us pick an arbitrary Weyl structure $\hat\sigma$ and consider the G_0–equivariant function $(S)_{\hat\sigma} : \mathcal{G}_0 \to \{C(-\mathrm{Ad}(s)^{-1}(Y), Y), Y \in \mathfrak{p}_+\}$ defined by

$$S(p_0(u_0))^*\hat\sigma(u_0) = \hat\sigma(u_0)\exp(-(S)_{\hat\sigma}(u_0))$$

for all $u_0 \in \mathcal{G}_0$. We show that $(S)_{\hat\sigma}$ is smooth.

We decompose

$$(S)_{\hat\sigma} = \sum_a \tau_i(a) + \cdots + \sum_a \tau_k(a)$$

according to the grading and the eigenvalues a of $\mathrm{Ad}(s)$. It follows from Claim (3) of Proposition 3.1 that $\tau_i(1) \equiv 0$. Thus the formula (9) from the proof of Proposition 3.2 that holds under our assumptions at each point of M implies that each $\tau_i(a)$ is smooth.

The formula (5) from Lemma 2.2 gives

$$(S)_{\hat\sigma\exp(\Upsilon)_{\hat\sigma}} = C(-\mathrm{Ad}(s)^{-1}(\Upsilon)_{\hat\sigma}, C((S)_{\hat\sigma}, (\Upsilon)_{\hat\sigma})).$$

If we take $\Upsilon = r\tau_i(a)$ for arbitrary $r \in \mathbb{R}$, then

$$\begin{aligned} C(-\mathrm{Ad}(s)^{-1}(r\tau_i(a)), C((S)_{\hat\sigma}, r\tau_i(a)))_i(a) &= C(-\frac{r}{a}\tau_i(a), C(\tau_i(a), r\tau_i(a)))_i(a) \\ &= \frac{r(1-a)+a}{a}\tau_i(a) \end{aligned}$$

holds for the component of the BCH–formula in $\mathfrak{g}_i(a)$, while the components of the BCH–formula in $\mathfrak{g}_i(b)$ for the other eigenvalues $b \neq a$ of $\mathrm{Ad}(s)$ remain $\tau_i(b)$. Consequently, if we take

$$\Upsilon_i := \sum_{a\neq 1} \frac{a}{a-1}\tau_i(a)$$

and consider the Weyl structure $\hat\sigma\exp(\Upsilon_i)$ instead of $\hat\sigma$, then we get

$$(S)_{\hat\sigma\exp(\Upsilon_i)} = \sum_a \tilde\tau_{i+1}(a) + \cdots + \sum_a \tilde\tau_k(a).$$

By induction, we obtain in finitely many steps a Weyl structure σ such that $(S)_\sigma \equiv 0$ holds. Since $(S)_\sigma \equiv 0$ and all the changes we made are smooth, the function $(S)_{\hat\sigma}$

and the Weyl structure $\sigma = \hat{\sigma}\exp(\Upsilon_i)\cdots\exp(\Upsilon_k)$ are smooth. Let $[\nabla^\sigma]$ be the class consisting of all Weyl connections for Weyl structures σ constructed for all Weyl structures $\hat{\sigma}$. We complete the proof by showing that $[\nabla^\sigma]$ is an $\underline{S}$–invariant class of Weyl connections and thus Claim (1) holds.

It is clear from the construction of σ that if we start with $\hat{\sigma}\exp(\Upsilon)_\sigma$ for $(\Upsilon)_\sigma$ with values in $\mathfrak{p}_+^s(1)$, then we get $\sigma\exp(\Upsilon)_\sigma$. Thus the class $[\nabla^\sigma]$ satisfies Claim (2) and Claims (6), (7), (8) and (9) are then consequences of Claim (2) and the formulas for the change of Weyl structures and connections. In particular, the class $[\nabla^\sigma]$ is a maximal subclass of the class of Weyl connections that satisfy $T_x s_x^\sigma = T_x s_x^{\sigma'}$ for all Weyl connections $\nabla^\sigma, \nabla^{\sigma'} \in [\nabla^\sigma]$ and all $x \in M$, and that all connections in $[\nabla^\sigma]$ restrict to the same partial connection on all smooth subbundles of TM for all eigenvalues of $T_x s_x^\sigma$ different from 1.

If $\hat{\sigma} = \nu_u$ is the normal Weyl structure for $u \in \mathcal{G}_x$ satisfying $S(x)(u) = us$, then

$$\begin{aligned}\sigma\exp(-(\Upsilon^{\sigma,S(x)})_\sigma) =&\nu_u \exp(S(x)^*\Upsilon_i)\cdots\exp(S(x)^*\Upsilon_k)\\ =&\sigma\exp(-\Upsilon_k)\cdots\exp(C(-\Upsilon_i, S(x)^*\Upsilon_i))\cdots\exp(S(x)^*\Upsilon_k).\end{aligned}$$

Since the component of $C(-\Upsilon_i, S(x)^*\Upsilon_i)$ contained in $\mathfrak{g}_i$ has a trivial component in $\mathfrak{p}_+^s(1)$ and $(\Upsilon^{\sigma,S(x)})_\sigma$ has its values in $\mathfrak{p}_+^s(1)$, the equality $\Upsilon_i = S(x)^*\Upsilon_i$ holds. Thus we get $0 = C(-\Upsilon_i, S(x)^*\Upsilon_i)$. Therefore $\sigma\exp(-(\Upsilon^{\sigma,S(x)})_\sigma) = \sigma$ follows by induction, and thus $S(x)^*\sigma = \sigma$. The Corollary 4.2 and the last claim of Proposition 3.1 implies that

$$\underline{S}(x) = \underline{s}_u = s_x^{\nu_u\exp(\Upsilon_i)\cdots\exp(\Upsilon_k)} = \underline{s}_{\sigma(\pi(u))}$$

holds for all $x \in M$, all $u \in \mathcal{G}_x$ satisfying $S(x)(u) = us$ and arbitrary σ such that $\nabla^\sigma \in [\nabla^\sigma]$. In particular, $\underline{S}$ and S are smooth, because σ is smooth. Therefore Claims (3) and (4) hold.

Since

$$S(x)\circ S(y)\circ S(x)^{-1}(S(x)(\sigma(u_0))) = S(x)(\sigma(u_0))s$$

holds for u_0 in the fiber over x, the composition $S(x)\circ S(y)\circ S(x)^{-1}$ is an s–symmetry at the point $\underline{S}(x)(y)$. The equality $S(x)\circ S(y)\circ S(x)^{-1} = S(\underline{S}(x)(y))$ then follows from the uniqueness of s–symmetries. Therefore Claim (5) holds.

In particular, $\underline{S}(x)\circ\underline{S}(y)(y) = \underline{S}(\underline{S}(x)(y))\circ\underline{S}(x)(y)$ holds. This implies that

$$\begin{aligned}\sigma'(v_0)\exp((\Upsilon^{\sigma',S(x)})_{\sigma'}(v_0 s)) = (S(x)\circ S(y))^*\sigma'(v_0) = (S(\underline{S}(x)(y))\circ S(x))^*\sigma'(v_0)\\ = \sigma'(v_0)\exp((\Upsilon^{\sigma',S(x)})_{\sigma'}(v_0))\end{aligned}$$

holds for v_0 in the fiber over y for arbitrary σ' such that $\nabla^{\sigma'} \in [\nabla^\sigma]$. Thus

$$\mathrm{Ad}(s)(\Upsilon^{\sigma,S(x)})_\sigma(v_0) = (\Upsilon^{\sigma,S(x)})_\sigma(v_0)$$

holds and thus $[\nabla^\sigma]$ is an $\underline{S}$–invariant class of Weyl connections. □

5. Geometric properties of parabolic geometries of distinguished types and classification

In this section, we study properties of (locally) s–symmetric parabolic geometries of particular types (G,P) for triples $(\mathfrak{g},\mathfrak{p},\mu)$ that are prolongation rigid outside of the 1–eigenspace of s for μ in the 1–eigenspace of s. The properties follow from the position and shape of $\mathfrak{g}_-^s(1)$ inside of $\mathfrak{g}_-$. We classify all triples $(\mathfrak{g},\mathfrak{p},\mu)$ where $\mathfrak{g}_-^s(1)$ has such a position and shape for generic s.

5.1. **Classification results and notation.** Let us use the characterization from Section 3.2 for the classification of the triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of $s \in Z(G_0)$ such that μ is in the 1–eigenspace of s. We separate the classification into the series of tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 and 13. The main reason for such a separation is that parabolic geometries from different tables have different geometric properties and we divide the tables according to these properties.

Theorem 5.1. *Let $(\mathfrak{g}, \mathfrak{p}, \mu)$ be a triple obtained from one of the tables 1-13 in the following way:*

- *The Lie algebra $\mathfrak{g}$ is a simple Lie algebra of the (complex) rank n that is at least A_4, B_4, C_4, D_5 or some explicit Lie algebra of lower rank from the column $\mathfrak{g}$.*
- *The parabolic subalgebra $\mathfrak{p}$ is the parabolic algebra from* [3, Section 3.2.9] *for the set Σ in the column Σ.*
- *The component of the harmonic curvature μ is specified by an ordered pair of simple roots of $\mathfrak{g}$ from the column μ that provides the highest weight of μ by the affine action of corresponding elements of the Weyl group, see* [3, Theorem 3.3.5].
- *The component μ is contained in the 1–eigenspace of s for the elements $s \in Z(G_0)$ that have the eigenvalues j_{i_a} from the columns j_{i_a} on the irreducible $\mathfrak{g}_0$–components that are determined by the i_ath element of the set Σ.*

Then $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s if the eigenvalues j_{i_a} of s satisfy the condition in the column PR.

The tables 1-13 contain the complete classification of triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ that are prolongation rigid outside of the 1–eigenspace of s for μ in the 1–eigenspace of s (except the cases that are conjugated by an outer automorphism of $\mathfrak{g}$ to one of the listed entries).

The remaining notation we will use in the tables is the following:

We characterize the real form of $\mathfrak{g}$ by a number q and a field $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

The set Σ characterizes the set of crossed nodes in the Dynkin or Satake diagram that provides the parabolic subalgebra $\mathfrak{p}$. We use the ordering of nodes which is consistent with [3, Appendix B] and we will not add the conjugated crossed nodes to Σ in the case of complex Lie algebras, $\mathfrak{su}(q, n+1-q)$ and $\mathfrak{so}(3, 5)$. We distinguish the complex conjugated simple roots by $'$.

If the column for the eigenvalue j_{i_a} is blank, then the value of j_{i_a} is generic. If the eigenvalue $j_{i_a} \notin \mathbb{R}$ and $ln(j_{i_a}) = r_{i_a} + i\phi_{i_a}$, then either $r_{i_a} = 0$ or $\phi_{i_a} = 0$ and we specify only the non–zero one in the table.

If the column PR is missing or the condition is blank, then the triple $(\mathfrak{g}, \mathfrak{p}, \mu)$ is either prolongation rigid or the condition that μ is contained in the 1–eigenspace of s is sufficient for $\mathfrak{a}_i$ corresponding to μ to be a subspace of $\mathfrak{g}_i^s(1)$ for all i.

The classification tables are presented in the following subsections and the triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ are obtained from the tables according to Theorem 5.1.

5.2. **Parabolic geometries with $\mathfrak{g}_-^s(1) = 0$.** The Table 1 contains all triples $(\mathfrak{g}, \mathfrak{p}, \mu)$ with the property that if $s \in Z(G_0)$ is such that $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s, then $\mathfrak{g}_-^s(1) = 0$. In particular, all AHS–structures that have a component of the harmonic curvature in the 1–eigenspaces of some $s \in Z(G_0)$ are prolongation rigid outside of the 1–eigenspace of s and thus are contained in this table.

Example 5.2. Before we formulate the general result, let us demonstrate how the results for (locally) symmetric conformal geometries that we presented in [13] can be obtained from the Table 1 and Theorem 5.3:

TABLE 1. Theorem 5.3

$\mathfrak{g}$	Σ	j_{i_1}	μ
$\mathfrak{sl}(3,\mathbb{C})$	$\{1\}$	ϕ_1	$(\alpha_1,\alpha_{1'})$
$\mathfrak{sl}(3,\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	$(\alpha_{1'},\alpha_{2'})$
$\mathfrak{sl}(4,\{\mathbb{R},\mathbb{C}\})$	$\{1\}$	$\sqrt{1}$	(α_1,α_2)
$\mathfrak{sl}(4,\mathbb{C})$	$\{1\}$	ϕ_1	$(\alpha_1,\alpha_{1'})$
$\mathfrak{sl}(4,\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	$(\alpha_{1'},\alpha_{2'})$
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1\}$	$\sqrt{1}$	(α_1,α_2)
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1\}$	ϕ_1	$(\alpha_1,\alpha_{1'})$
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C},\mathbb{H}\})$	$\{2\}$	$\sqrt{1}$	(α_2,α_1)
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{p\}$	$\sqrt[3]{1}$	$(\alpha_{p'},\alpha_{p+1'})$
$\mathfrak{so}(1,5),\mathfrak{so}(2,4),\mathfrak{so}(3,3),\mathfrak{so}(6,\mathbb{C})$, $\mathfrak{so}(1,6)$, $\mathfrak{so}(2,5)$, $\mathfrak{so}(3,4)$, $\mathfrak{so}(7,\mathbb{C})$, $\mathfrak{so}(1,7)$, $\mathfrak{so}(2,6)$, $\mathfrak{so}(3,5)$, $\mathfrak{so}(4,4)$, $\mathfrak{so}(8,\mathbb{C})$	$\{1\}$	$\sqrt{1}$	(α_1,α_2)
$\mathfrak{so}(6,\mathbb{C})$, $\mathfrak{so}(7,\mathbb{C})$, $\mathfrak{so}(8,\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	$(\alpha_{1'},\alpha_{2'})$
$\mathfrak{so}(7,\mathbb{C})$	$\{3\}$	$\sqrt[3]{1}$	(α_3,α_2)
$\mathfrak{so}(q,n-q)$, $\mathfrak{so}(n,\mathbb{C})$	$\{1\}$	$\sqrt{1}$	(α_1,α_2)
$\mathfrak{so}(n,\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	$(\alpha_{1'},\alpha_{2'})$
$\mathfrak{so}(2n,\mathbb{C})$	$\{n\}$	$\sqrt[3]{1}$	$(\alpha_{n'},\alpha_{n-2'})$
$\mathfrak{so}(2n+1,\mathbb{C})$	$\{n\}$	$\sqrt[5]{1}$	$(\alpha_{n'},\alpha_{n-1'})$
$\mathfrak{sp}(4,\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	(α_1,α_2)
$\mathfrak{sp}(4,\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	$(\alpha_{1'},\alpha_{2'})$
$\mathfrak{sp}(4,\mathbb{C})$	$\{2\}$	$\sqrt[3]{1}$	$(\alpha_{2'},\alpha_{1'})$
$\mathfrak{sp}(6,\mathbb{C})$	$\{2\}$	$\sqrt[5]{1}$	$(\alpha_{2'},\alpha_{3'})$
$\mathfrak{sp}(6,\mathbb{C})$	$\{3\}$	$\sqrt[3]{1}$	$(\alpha_{3'},\alpha_{2'})$
$\mathfrak{sp}(2n,\mathbb{C})$	$\{n-1\}$	$\sqrt[5]{1}$	$(\alpha_{n-1'},\alpha_{n'})$
$\mathfrak{sp}(2n,\mathbb{C})$	$\{n\}$	$\sqrt[3]{1}$	$(\alpha_{n'},\alpha_{n-1'})$
$\mathfrak{e}_6(\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	$(\alpha_{1'},\alpha_{2'})$
$\mathfrak{e}_7(\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	$(\alpha_{1'},\alpha_{2'})$

There are rows with $\mathfrak{g}=\mathfrak{so}(q,n-q)$ and $\Sigma=\{1\}$ in the Table 1 and the triples $(\mathfrak{so}(q,n-q),\mathfrak{p}_{\{1\}},\mu_{(\alpha_1,\alpha_2)})$ are prolongation rigid outside of the 1–eigenspace of s for $n>5$ and $q>0$. We read of the corresponding line that the eigenvalue $j_{i_1}=\sqrt{1}$. Thus $s=m$ and the m–symmetries in question are the symmetries of conformal geometries presented in [13]. We get immediately from Theorem 5.3 that Theorem 1.3 holds for conformal geometries.

In the following theorem, we summarize geometric properties of geometries from the Table 1 and prove the last claim of Theorem 1.8.

Theorem 5.3. *Assume $(\mathfrak{g},\mathfrak{p},\mu)$ is prolongation rigid outside of the 1–eigenspace of s for $s\in Z(G_0)$ such that $\mathfrak{g}^s_-(1)=0$ holds. If the harmonic curvature κ_H of the (locally) s–symmetric parabolic geometry $(\mathcal{G}\to M,\omega)$ of type (G,P) has a non–zero component in μ at some x, then:*

(1) The parabolic geometry is (locally) homogeneous, $\kappa_H(x)\neq 0$ at all $x\in M$ and there is a unique smooth system of (local) s–symmetries S on M.

(2) There is a unique distinguished Weyl structure σ which is uniquely characterized by one of the following equivalent properties:

(a) *The equalities* $\nabla^\sigma T^\sigma = 0$, $s.(T^\sigma)_\sigma = (T^\sigma)_\sigma$, $\nabla^\sigma R^\sigma = 0$ *and* $s.(R^\sigma)_\sigma = (R^\sigma)_\sigma$ *hold for the torsion and the curvature of the Weyl connection* ∇^σ.
(b) *The Weyl connection* ∇^σ *is* $\underline{S}$*–invariant.*
(c) *All (local) automorphisms of the parabolic geometry cover affine transformations of* ∇^σ.
(d) *All (local) diffeomorphisms* s_x^σ *are affine transformations of* ∇^σ.
(e) *All (local)* P*–bundle morphisms* $s_{\sigma(u_0)}$ *are (local)* s*–symmetries.*

(3) The pseudo–group generated by all local s*–symmetries is transitive on* M *and its connected component of identity is generated by the flows of the Lie algebra* $\mathfrak{l}$, *which is the vector subspace of* $\mathfrak{g}_- \oplus \mathfrak{g}_0$, *generated by* $\mathfrak{g}_-$ *by the bracket* $(T^\sigma + R^\sigma)_\sigma$ *on* $\wedge^2\mathfrak{g}_-^* \otimes \mathfrak{l}$ *and the natural bracket on the rest of* $\mathfrak{l}$.

(4) The equalities

$$\underline{S}(x) = \underline{s}_{\sigma(u_0)} = s_x^\sigma$$

hold for the Weyl structure σ *from Claim (2). In particular,*

- *the maps* $\underline{S}(x)$ *can be extended to a larger neighbourhood of* x *as long as the corresponding geodesic transformations of* ∇^σ *are defined,*
- $\underline{S}(x) \circ \underline{S}(y) \circ \underline{S}(x)^{-1}(z) = \underline{S}(\underline{S}(x)(y))(z)$ *holds for* (x, y, z) *in some neighbourhood of the diagonal in* $M \times M \times M$, *and*
- *for each eigenvalue* a, *the distribution* $TM^s(a)$ *is preserved by all (local) automorphisms of the parabolic geometry.*

Proof. Let $U \subset M$ be the set of points x such that $\kappa_H(x)$ has a non–zero component in μ. Then there is a unique system of (local) s–symmetries on U due to the prolongation rigidity of the triple $(\mathfrak{g}, \mathfrak{p}, \mu)$ outside of the 1–eigenspace of s. It suffices to prove Theorem under the assumption $U = M$, because if we prove Claim (3) on U, then the equality $U = M$ follows from the (local) homogeneity, i.e., Claim (1) follows from Claim (3). Then Claim (4) follows from Claim (2) due to Claims (5) and (7) of Proposition 4.3.

Therefore, it suffices to prove Claims (2) and (3) under the assumption $U = M$ to complete the proof. If $\mathfrak{g}_-^s(1) = 0$, then $\mathfrak{p}_+^s(1) = 0$ and Proposition 4.3 implies that there is a unique S–invariant Weyl structure σ. It follows from Proposition 3.1 and Proposition 4.1 that the Weyl structure σ satisfies (2b) if and only if it satisfies (2e). Further, Proposition 4.1 and the Corollary 4.2 imply that the Weyl structure σ satisfies (2e) if and only if it satisfies (2d).

We show now that (2b) implies (2a). The torsion and the curvature of $\underline{S}$–invariant Weyl connection ∇^σ are $\underline{S}$–invariant. In particular,

$$s.(T^\sigma)_\sigma(u_0) = (\underline{S}(p_0(u_0))^* T^\sigma(p_0(u_0)))_\sigma(u_0) = (T^\sigma)_\sigma(u_0)$$

and

$$s.(R^\sigma)_\sigma(u_0) = (\underline{S}(p_0(u_0))^* R^\sigma(p_0(u_0)))_\sigma(u_0) = (R^\sigma)_\sigma(u_0)$$

hold for all $u_0 \in \mathcal{G}_0$ for the natural action . of G_0 on the values of $(T^\sigma)_\sigma$ and $(R^\sigma)_\sigma$. Since the same arguments can be applied on $\nabla^\sigma T^\sigma$ and $\nabla^\sigma R^\sigma$, it follows that $(\nabla^\sigma_\xi T^\sigma)_\sigma = s.(\nabla^\sigma_\xi T^\sigma)_\sigma = a(\nabla^\sigma_\xi T^\sigma)_\sigma$ and $(\nabla^\sigma_\xi R^\sigma)_\sigma = s.(\nabla^\sigma_\xi R^\sigma)_\sigma = a(\nabla^\sigma_\xi R^\sigma)_\sigma$ hold for any vector field ξ on M such that $(\xi)_\sigma(u_0) \in \mathfrak{g}_-^s(a^{-1})$ for all $u_0 \in \mathcal{G}_0$. Thus (2b) implies (2a), because $\mathfrak{g}_-^s(1) = 0$.

The Claim (2a) implies that ∇^σ is a locally affinely homogeneous connection. Therefore, according to [9, Section 1.5], the affine geometry (M, ∇^σ) can be encoded as a locally homogeneous Cartan geometry of type $(\mathfrak{g}_- \rtimes Gl(\mathfrak{g}_-), Gl(\mathfrak{g}_-))$ on the first order frame bundle $\mathcal{P}^1 M$. Moreover, the assumptions of [11, Lemma 2.2] are satisfied, because $(T^\sigma + R^\sigma)_\sigma(\iota_\sigma(u_0))$ is the bracket of the infinitesimal affine transformation at $\iota_\sigma(u_0) \in \iota_\sigma(\mathcal{G}_0) \subset \mathcal{P}^1 M$. Thus there is a (local) affine transformation A of ∇^σ such that $(A)_\sigma(u_0) = s$. Therefore Claim (2d) follows from Claim (2a) and

Proposition 4.1 due to the uniqueness of s–symmetries. In particular, if we consider a (local) one–parameter subgroup $\exp(t\xi)$ for an infinitesimal affine transformation ξ, then $\exp(t\xi)\underline{S}(x)\exp(-t\xi)$ is the (local) s–symmetry at $\exp(t\xi)(x)$ and the map $\frac{d}{dt}|_{t=0}\exp(t\xi)\underline{S}(x)\exp(-t\xi)\underline{S}(x)^{-1}$ maps ξ into $\mathfrak{l}$. If $(\mathcal{P}^1\xi)_\sigma(u_0)=X$, then the element $X-\mathrm{Ad}(s)(X)$ is contained in $\mathfrak{l}$. Thus $\mathfrak{g}_-\subset\mathfrak{l}$ as a vector subspace. Thus the flows of the Lie algebra $\mathfrak{l}$ generate a sub–pseudo–group, which is the connected component of identity of the pseudo–group generated by local s–symmetries. Since $\mathrm{Ad}(s)$ preserves $\mathfrak{l}$, Claim (3) follows.

We can use the results from [12, Theorem 1.3] due to the local homogeneity and (2c) follows from (2a). Clearly (2c) implies (2d), which completes the proof. □

5.3. **Parabolic geometries with distinguished parabolic subalgebras $\mathfrak{g}^s_-(1)+\mathfrak{p}$.** There are triples $(\mathfrak{g},\mathfrak{p},\mu)$ that are prolongation rigid outside of the 1–eigenspace of s which admit 1–eigenspace in $\mathfrak{g}_-$ for some s such that $\mathfrak{q}:=\mathfrak{g}^s_-(1)+\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ such that the harmonic curvature in μ vanishes on insertions of elements of $\mathfrak{q}/\mathfrak{p}$ at all points of M. These are listed in the Tables 2, 3 and 4 due to [12, Proposition 6.2 and Proposition A.2].

Example 5.4. To demonstrate our results, let us look in the Table 2 on the row $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbb{R})$ and $\Sigma=\{1,2\}$ which corresponds to generalized path–geometries (for systems of second order ODEs), see [3, Sections 4.4.3-5]. These parabolic geometries generally have two harmonic curvatures, one torsion $\kappa_{(\alpha_1,\alpha_2)}$ and one curvature $\kappa_{(\alpha_2,\alpha_1)}$. However, they fall in the Table 2 only when the torsion $\kappa_{(\alpha_1,\alpha_2)}$ vanishes and the harmonic curvature consists only of the curvature $\kappa_{(\alpha_2,\alpha_1)}$ corresponding to $\mu_{(\alpha_2,\alpha_1)}$. There are many $s\in Z(G_0)$ that act trivially on $\mu_{(\alpha_2,\alpha_1)}$, but the triple $(\mathfrak{sl}(n+1,\mathbb{R}),\mathfrak{p}_{\{1,2\}},\mu_{(\alpha_2,\alpha_1)})$ is prolongation rigid outside of the 1–eigenspace of s only for $s\in Z(G_0)$ with eigenvalues $j_1=1, j_2=-1$. In such case, $\mathfrak{q}=\mathfrak{p}_{\{2\}}$ is the parabolic subalgebra of $\mathfrak{g}$ corresponding to $\Sigma=\{2\}$.

The torsion–freeness of generalized path–geometries implies that the space of local solutions of the corresponding ODEs carries a Grassmanian structure, which is a parabolic geometry on the local leaf space of type (G,Q) from Theorem 5.5. Therefore if $(\mathcal{G}\to M,\omega)$ is a (locally) s–symmetric torsion–free generalized path–geometry with a non–zero harmonic curvature, then we conclude from Theorem 5.5 that the space of local solutions N is a locally symmetric space $(N,\underline{S})$, while M together with the system of (local) s–symmetries S is a reflexion space $(M,\underline{S})$ over $(N,\underline{S})$, see [6]. Let us emphasize that due to dimensional reasons and the formula [3, Theorem 5.2.9], the pseudo–group generated by all local s–symmetries is locally transitive at $x\in M$ if and only if the Rho–tensor $\mathsf{P}^\sigma(n(x))$ of the S–invariant Weyl structure σ on N does not vanish on $T_{n(x)}N$.

We summarize geometric properties of geometries from the Tables 2, 3 and 4 in the following theorem.

Theorem 5.5. *Assume $(\mathfrak{g},\mathfrak{p},\mu)$ is prolongation rigid outside of the 1–eigenspace of s for $s\in Z(G_0)$ such that $\mathfrak{q}=\mathfrak{g}^s_-(1)+\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{q}/\mathfrak{p}$ inserts trivially into the harmonic curvature κ_H of the (locally) s–symmetric parabolic geometry $(\mathcal{G}\to M,\omega)$ of type (G,P). If κ_H has a non–zero component in μ at some x, then:*

(1) The inequality $\kappa_H\neq 0$ holds in an open dense subset of M, and there is a unique smooth system of (local) s–symmetries S on M.

(2) There are

- *a parabolic subgroup Q of G with the Lie algebra $\mathfrak{q}$ such that $P\subset Q$,*
- *a neighbourhood U_x of each $x\in M$ with the local leaf space $n:U_x\to N$ for the foliation given by the integrable distribution $Tp\circ\omega^{-1}(\mathfrak{q})$, and*

TABLE 2. Theorem 5.5, part with $|\Sigma| = 2$.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	μ	PR
$\mathfrak{sl}(3,\mathbb{C})$	$\{1,2\}$		$2r_1$	$(\alpha_1,\alpha_{1'})$	
$\mathfrak{sl}(3,\mathbb{C})$	$\{1,2\}$	$-\frac{2}{3}\phi_2$		$(\alpha_{1'},\alpha_{2'})$	$r_2=0,\phi_2=2\pi$
$\mathfrak{sl}(3,\mathbb{C})$	$\{1,2\}$		$-\frac{2}{3}\phi_1$	$(\alpha_{2'},\alpha_{1'})$	$r_1=0,\phi_1=2\pi$
$\mathfrak{sl}(4,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$	j_2^{-2}		(α_2,α_1)	$j_2=-1$
$\mathfrak{sl}(4,\mathbb{R})$	$\{1,2\}$	j_2^2		(α_2,α_3)	
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,2\}$		$2r_1$	$(\alpha_1,\alpha_{1'})$	
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,2\}$	$-\frac{2}{3}\phi_2$		$(\alpha_{1'},\alpha_{2'})$	$r_2=0,\phi_2=2\pi$
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,2\}$		$-\frac{2}{3}\phi_1$	$(\alpha_{2'},\alpha_{1'})$	$r_1=0,\phi_1=2\pi$
$\mathfrak{sl}(4,\mathbb{R})$	$\{1,3\}$		j_1^2	(α_1,α_2)	
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,3\}$		$2r_1$	$(\alpha_1,\alpha_{1'})$	
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$	j_2^{-2}		(α_2,α_1)	$j_2=-1$
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,2\}$		$2r_1$	$(\alpha_1,\alpha_{1'})$	
$\mathfrak{sl}(n+1,\mathbb{R})$	$\{1,3\}$		j_1^2	(α_1,α_2)	
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,p\}$ $2<p<n$		$2r_1$	$(\alpha_1,\alpha_{1'})$	$r_1=0$
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,p\}$ $3<p<n$		j_1^2	(α_1,α_2)	$j_1=-1$
$\mathfrak{sl}(n+1,\mathbb{R})$	$\{1,n\}$		j_1^2	(α_1,α_2)	
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,n\}$		$2r_1$	$(\alpha_1,\alpha_{1'})$	
$\mathfrak{sl}(n+1,\mathbb{R})$	$\{2,3\}$		j_2^2	(α_2,α_1)	
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C},\mathbb{H}\})$	$\{2,p\}$ $3<p<n$		j_2^2	(α_2,α_1)	$j_2=-1$
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{H}\})$	$\{2,n\}$		j_2^2	(α_2,α_1)	
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{p,p+1\}$		$-\frac{2}{3}\phi_p$	$(\alpha_{p+1'},\alpha_{p'})$	$r_p=0,\phi_p=2\pi$
$\mathfrak{so}(2,5),\mathfrak{so}(3,4),\mathfrak{so}(7,\mathbb{C})$, $\mathfrak{so}(2,6),\mathfrak{so}(3,5)$, $\mathfrak{so}(4,4),\mathfrak{so}(8,\mathbb{C})$	$\{1,2\}$	$\sqrt{1}$		(α_1,α_2)	$j_2=1$
$\mathfrak{so}(4,4)$	$\{1,4\}$		j_1^2	(α_1,α_2)	
$\mathfrak{so}(q,n-q),\mathfrak{so}(n,\mathbb{C})$	$\{1,2\}$	$\sqrt{1}$		(α_1,α_2)	$j_2=1$
$\mathfrak{so}(n,n),\mathfrak{so}(2n,\mathbb{C})$	$\{1,n\}$		j_1^2	(α_1,α_2)	$j_1=-1$

TABLE 3. Theorem 5.5, part with $|\Sigma| = 3$.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	j_{i_3}	μ	PR
$\mathfrak{sl}(4,\mathbb{R})$	$\{1,2,3\}$			$j_1j_2^2$	(α_2,α_1)	$j_1=1$
$\mathfrak{sl}(n+1,\mathbb{R})$	$\{1,2,3\}$			$j_1j_2^2$	(α_2,α_1)	$j_1=1$
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,p\}$ $3<p<n$			$j_1j_2^2$	(α_2,α_1)	$j_1=1,j_2=-1$
$\mathfrak{sl}(n+1,\mathbb{R})$	$\{1,2,n\}$			$j_1j_2^2$	(α_2,α_1)	$j_1=1$
$\mathfrak{so}(4,4)$	$\{1,2,4\}$			j_1^2	(α_1,α_2)	$j_2=1$
$\mathfrak{so}(n,n)$, $\mathfrak{so}(4n,\mathbb{C})$	$\{1,2,n\}$			j_1^2	(α_1,α_2)	$j_1=-1,j_2=1$

- *a (locally) s–symmetric parabolic geometry $(\mathcal{G}' \to N, \omega')$ of type (G, Q) satisfying the assumptions of Theorem 5.3*

such that $(\mathcal{G}|_{U_x} \to U_x, \omega|_{U_x})$ is isomorphic to an open subset of $(\mathcal{G}' \to \mathcal{G}'/P, \omega')$ for each x. In particular, there is a unique s–symmetry $\underline{S}(n(y))$ on $(\mathcal{G}' \to N, \omega')$ at

TABLE 4. Theorem 5.5, part with $|\Sigma| = 4$.

$\mathfrak{g}$	Σ	eigenvalues	μ	PR
$\mathfrak{sl}(n+1,\mathbb{R})$	$\{1,2,3,p\}$ $3<p<n$	$j_p = j_1 j_2^2 j_3^{-1}$	(α_2,α_1)	$j_1 = 1, j_3 = j_2^2$
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,p,q\}$ $3<p,q<n$	$j_q = j_1 j_2^2 j_p^{-1}$	(α_2,α_1)	$j_1 = 1, j_2 = -1, j_p = 1$
$\mathfrak{sl}(n+1,\mathbb{R})$	$\{1,2,p,n\}$ $3<p<n$	$j_n = j_1 j_2^2 j_p^{-1}$	(α_2,α_1)	$j_1 = 1, j_p = 1$

each $n(y) \in N$ such that $n \circ \underline{S}(y) = \underline{S}(n(y)) \circ n$ holds for all $y \in U_x$ in the fiber over $n(y)$.

(3) The connected component of identity of the pseudo–group generated by all local s–symmetries is generated by the flows of the Lie algebra $\mathfrak{l}$, which is the vector subspace of $\mathfrak{q}^{op}$, generated by $\mathfrak{q}_+^{op}$ by the bracket $(T^\sigma(n(x))+R^\sigma(n(x)))_\sigma$ on $\wedge^2(\mathfrak{q}_+^{op})^*\otimes\mathfrak{l}$ and the natural bracket on the rest of $\mathfrak{l}$ for the S–invariant Weyl structure σ on $(\mathcal{G}' \to N, \omega')$, where $\mathfrak{q}^{op}$ is the opposite parabolic subalgebra of $\mathfrak{g}$ to $\mathfrak{q}$.

The pseudo–group generated by all local s–symmetries is locally transitive at x if and only if $\mathfrak{q}/\mathfrak{p} \subset \mathfrak{l}/(\mathfrak{l}\cap\mathfrak{p})$, i.e., if and only if $(R^\sigma(n(x)))_\sigma$ spans the whole $\mathfrak{q}/\mathfrak{p}$.

(4) There is a bijection between

- *the almost S–invariant Weyl structures on U_x, and*
- *the reductions of the image in $\mathcal{G}'$ of the (unique) S–invariant Weyl structure σ on N (that exists due to Theorem 5.3) to $\exp(\mathfrak{g}_-^s(1)) \rtimes G_0 \subset Q_0$.*

A reduction corresponds to an S–invariant Weyl structure on U_x if and only if it is a holonomy reduction of ∇^σ.

(5) In particular,

- *the maps $\underline{S}(x)$ can be extended to a larger neighbourhood of x as long as the corresponding geodesic transformations of ∇^σ on N are defined,*
- *$\underline{S}(x) \circ \underline{S}(y) \circ \underline{S}(x)^{-1}(z) = \underline{S}(\underline{S}(x)(y))(z)$ holds for (x,y,z) in some neighbourhood of the diagonal in $M \times M \times M$,*
- *the distribution $TM^s(1)$ is the vertical distribution of the local leaf space $n : U_x \to N$,*
- *for each eigenvalue a, $Tn(T_xM^s(a))$ is the a–eigenspace of $T_{n(x)}\underline{S}(n(x))$ in $T_{n(x)}N$, and*
- *for each eigenvalue a, the distribution $TM^s(a)$ is preserved by all (local) automorphisms of the parabolic geometry.*

Proof. The Claim (1) is a direct consequence of Claims (2) and (3), because $\kappa_H \neq 0$ holds for the harmonic curvature of $(\mathcal{G}' \to N, \omega')$ and thus $\kappa_H = 0$ can hold only in the subset of the fiber corresponding to a (Zariski) closed subset of Q. The Claim (2) follows from [4, Theorem 3.3] and the fact that $(\mathfrak{q}_+^{op})^s(1) = 0$. Then Claim (3) is a clear consequence of Theorem 5.3. The Claim (4) follows from the comparison of images in $\mathcal{G}'$ of the S–invariant Weyl structure on N and the almost S–invariant Weyl structures on U_x, because they intersect precisely in a reduction to $\exp(\mathfrak{g}_-^s(1)) \rtimes G_0 \subset Q_0$, i.e., in a subbundle with the structure group $\exp(\mathfrak{g}_-^s(1)) \rtimes G_0$. The Claim (5) is a consequence of Claim (4) of Theorem 5.3 and Claim (2). $\square$

5.4. **Parabolic geometries with $\mathfrak{g}_{-1}^s(1) = 0$.** There are triples $(\mathfrak{g},\mathfrak{p},\mu)$ that are prolongation rigid outside of the 1–eigenspace of s which admit a 1–eigenspace in

$\mathfrak{g}_-$ for some s such that $\mathfrak{g}^s_{-1}(1) = 0$ holds, but which do not generically satisfy $\mathfrak{g}^s_-(1) = 0$. These are contained in the Tables 5 and 6.

TABLE 5. Theorem 5.7, part with $|\Sigma| = 1$.

$\mathfrak{g}$	Σ	j_{i_1}	μ
$\mathfrak{su}(1,2)$	$\{1\}$	$\sqrt[4]{1}$	(α_1, α_2)
$\mathfrak{su}(1,3)$, $\mathfrak{su}(2,2)$	$\{1\}$	ϕ_1	(α_1, α_3)
$\mathfrak{su}(1,3)$, $\mathfrak{su}(2,2)$	$\{1\}$	$\sqrt[3]{1}$	(α_1, α_2)
$\mathfrak{su}(q, n+1-q)$	$\{1\}$	ϕ_1	(α_1, α_n)
$\mathfrak{su}(q, n+1-q)$	$\{1\}$	$\sqrt[3]{1}$	(α_1, α_2)
$\mathfrak{su}(q, n+1-q)$	$\{2\}$	$\sqrt[3]{1}$	(α_2, α_1)
$\mathfrak{so}(3,5)$	$\{3\}$	$\sqrt[3]{1}$	(α_3, α_2)
$\mathfrak{sp}(4, \mathbb{C})$	$\{1\}$	ϕ_1	$(\alpha_1, \alpha_{1'})$
$\mathfrak{sp}(6, \{\mathbb{R}, \mathbb{C}\})$	$\{1\}$	$\sqrt{1}$	(α_1, α_2)
$\mathfrak{sp}(6, \mathbb{C})$	$\{1\}$	ϕ_1	(α_1, α'_1)
$\mathfrak{sp}(1,2), \mathfrak{sp}(6, \{\mathbb{R}, \mathbb{C}\})$	$\{2\}$	$\sqrt{1}$	(α_2, α_1)
$\mathfrak{sp}(2n, \{\mathbb{R}, \mathbb{C}\})$	$\{1\}$	$\sqrt{1}$	(α_1, α_2)
$\mathfrak{sp}(2n, \mathbb{C})$	$\{1\}$	ϕ_1	(α_1, α'_1)
$\mathfrak{sp}(q, n-q), \mathfrak{sp}(2n, \{\mathbb{R}, \mathbb{C}\})$	$\{2\}$	$\sqrt{1}$	(α_2, α_1)
$\mathfrak{g}_2(\{2, \mathbb{C}\})$	$\{1\}$	$\sqrt[4]{1}$	(α_1, α_2)

TABLE 6. Theorem 5.7, part with $|\Sigma| = 2$.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	μ
$\mathfrak{sl}(3, \{\mathbb{R}, \mathbb{C}\})$	$\{1,2\}$	$\sqrt[4]{1}$	$\sqrt[4]{1}^3$	(α_1, α_2)
$\mathfrak{sl}(3, \mathbb{C})$	$\{1,2\}$	$\sqrt[5]{1}$	$\sqrt[5]{1}^3$	(α_1, α_2)
$\mathfrak{sl}(4, \{\mathbb{R}, \mathbb{C}\})$	$\{1,3\}$		j_1^{-1}	(α_1, α_3)
$\mathfrak{sl}(n+1, \{\mathbb{R}, \mathbb{C}\})$	$\{1,n\}$		j_1^{-1}	(α_1, α_n)
$\mathfrak{so}(2,3), \mathfrak{so}(5, \mathbb{C})$	$\{1,2\}$	$\sqrt[4]{1}$	$\sqrt[4]{1}^3$	(α_1, α_2)
$\mathfrak{so}(5, \mathbb{C})$	$\{1,2\}$	$j_1^5 = 1$ or $j_1^7 = 1$	j_1^3	(α_1, α_2)
$\mathfrak{so}(3,4)$	$\{1,3\}$	j_3^3		(α_3, α_2)

Example 5.6. We see that partially integrable almost CR–structures of hypersurface type are contained in the Table 5, i.e., $\mathfrak{g} = \mathfrak{su}(q, n+1-q), q > 0, n > 1$ and $\Sigma = \{1\}$. With the exception of the case $n = 2$, there are two possible components of the harmonic curvature such that the triple $(\mathfrak{su}(q, n+1-q), \mathfrak{p}_{\{1\}}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s for $s \in Z(G_0)$ with the specified eigenvalue. Moreover, $\mathfrak{g}^s_-(1) = \mathfrak{g}_{-2}$ holds in all the cases when $(\mathfrak{su}(q, n+1-q), \mathfrak{p}_{\{1\}}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s. Let us emphasize that the possibility $s^3 = \mathrm{id}$ is available for both components of the harmonic curvature. Since $\mathfrak{g}^s_-(1) = \mathfrak{g}_{-2}$, we need some additional assumptions in Theorem 5.7 to show that (M, S) is (locally, under these assumptions) either a (locally) homogeneous one–dimensional fiber bundle over (reduced) $\mathbb{S}^1$–space, or a $\mathbb{Z}_3$–space or a symmetric space (due to [12, Proposition 7.3], see also [20]) that carries some $\underline{S}$–invariant Weyl connection on TM. In particular, all such parabolic geometries can be classified using [9, Theorem 5.1.4] and Theorem 5.7, if one knows the classification of $\mathbb{S}^1$–spaces, $\mathbb{Z}_3$–spaces and symmetric spaces. Let us emphasize that a part of the classification is done in [7].

As mentioned in the example, we need an additional assumption on where the local s–symmetries are defined for parabolic geometries in question.

Theorem 5.7. *Let $(\mathfrak{g}, \mathfrak{p}, \mu)$ be prolongation rigid outside of the 1–eigenspace of s for $s \in Z(G_0)$ such that $\mathfrak{g}^s_{-1}(1) = 0$ holds. Assume that for the (locally) s–symmetric parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), the open subset U of M containing the points at which κ_H has a non–zero component in μ is non–trivial, and the maps $\underline{S}(x)(y)$ and $\underline{S}(x) \circ \underline{S}(y)^{-1}(z)$ are defined on neighbourhoods of diagonals in $U \times U$ and $U \times U \times U$ for the unique system S of (local) s–symmetries on U. Then:*

(1) The parabolic geometry is (locally) homogeneous and $U = M$, i.e., $\kappa_H(x) \neq 0$ at all $x \in M$ and there is a unique smooth system of (local) s–symmetries S on M.

(2) There is a class of distinguished Weyl structures characterized by one of the following equivalent properties for each Weyl structure σ in the class:

- (a) *The equalities $\nabla^\sigma T^\sigma = 0$, $s.(T^\sigma)_\sigma = (T^\sigma)_\sigma$, $\nabla^\sigma R^\sigma = 0$ and $s.(R^\sigma)_\sigma = (R^\sigma)_\sigma$ hold for the torsion and the curvature of the Weyl connection ∇^σ.*
- (b) *The Weyl connection ∇^σ is $\underline{S}$–invariant.*
- (c) *All (local) automorphisms of the parabolic geometry cover affine transformations of ∇^σ.*
- (d) *All (local) diffeomorphisms s^σ_x are affine transformations of ∇^σ.*

Two Weyl structures σ and $\sigma \exp(\Upsilon)_\sigma$ from the class differ by a G_0–equivariant function $(\Upsilon)_\sigma : \mathcal{G}_0 \to \mathfrak{p}^s_+(1)$ which is invariant with respect to all (local) automorphisms of the parabolic geometry and is provided by an invariant element of $\mathfrak{p}^s_+(1)$.

(3) The pseudo–group generated by all local s–symmetries is transitive on M and its connected component of identity is generated by the flows of the Lie algebra $\mathfrak{l}$, which is the vector subspace of $\mathfrak{g}_- \oplus \mathfrak{g}_0$, generated by $\mathfrak{g}_-$ by the bracket $(T^\sigma + R^\sigma)_\sigma$ on $\wedge^2 \mathfrak{g}^*_- \otimes \mathfrak{l}$ and the natural bracket on the rest of $\mathfrak{l}$.

(4) The equalities

$$\underline{S}(x) = \underline{s}_{\sigma(u_0)} = s^\sigma_x$$

hold for any Weyl structure σ from (2). In particular,

- *the maps $\underline{S}(x)$ can be extended to a larger neighbourhood of x as long as the corresponding geodesic transformations of ∇^σ are defined,*
- *for each eigenvalue a, the distribution $TM^s(a)$ is preserved by all (local) automorphisms of the parabolic geometry.*

(5) The distribution $TM^s(1)$ is integrable and for each $x \in M$, the leaf $\mathcal{F}_x$ of the foliation $\mathcal{F}$ of $TM^s(1)$ through x is a totally geodesic submanifold for arbitrary Weyl structure.

Let $n : U_x \to N$ be a sufficiently small local leaf space of $TM^s(1)$.

- (a) *There is a unique local diffeomorphism $\underline{S}(n(y))$ of the local leaf space N at each $n(y) \in N$ such that $\underline{S}(n(y)) \circ n = n \circ \underline{S}(x)$ holds for all $y \in U_x$, and*
- (b) *for each eigenvalue a, $T_y n(T_y M^s(a))$ is the a–eigenspace of $T_{n(y)}\underline{S}(n(x))$ in $T_{n(y)}N$ for all $y \in U_x$.*

Proof. The proof is similar to the proof of Theorem 5.3. However, we need a different method to prove the local homogeneity in Claim (3), because the existence of some S–invariant Weyl structure does not follow from Proposition 4.3 anymore. Therefore we need an additional assumption on the system S on U in order to apply the following lemma. Nevertheless, the fact from Proposition 4.3 that $S(x) = s_{\sigma(u_0)}$ holds for any almost S–invariant Weyl structure σ implies that the system S is smooth on U.

Lemma 5.8. *Suppose the smooth system of (local) s–symmetries S on M satisfies that the maps $\underline{S}(x)(y)$ and $\underline{S}(x) \circ \underline{S}(y)^{-1}(z)$ are defined on neighbourhoods of diagonals in $M \times M$ and $M \times M \times M$.*

- *If $c(t)$ is a curve in M such that $c(0) = x$ and $\xi := \frac{d}{dt}|_{t=0} c(t)$, then the vector field*
$$L_\xi(y) := \frac{d}{dt}|_{t=0} \underline{S}(c(t)) \circ \underline{S}(x)^{-1}(y)$$
is defined for y in some neighbourhood of x in M.
- *Then $L_\xi(y)$ is an infinitesimal automorphism of the parabolic geometry.*
- *If ξ is contained in the a–eigenspace of $T_x\underline{S}(x)$, then $L_\xi(x) = (1-a)\xi$.*
- *The map $\xi \mapsto L_\xi$ for $\xi \in T_xM$ is a linear map onto the Lie algebra of local infinitesimal automorphisms of the parabolic geometry. Its kernel consists of the 1–eigenspace of $T_x\underline{S}(x)$ in T_xM, and it is injective on the sum of the remaining eigenspaces in T_xM.*

Proof of Lemma 5.8. Since $S(c(0)) \circ S(x)^{-1} = \mathrm{id}_{\mathcal{G}}$, there is a natural lift of $L_\xi(y)$ onto the P–invariant vector field $\frac{d}{dt}|_{t=0} S(c(t)) \circ S(x)^{-1}(u)$ for $u \in \mathcal{G}$ in the fiber over y. Since $S(c(t)) \circ S(x)^{-1}$ is an automorphism, the vector field is P–invariant and $\frac{d}{dt}|_{t=0}(S(c(t)) \circ S(x)^{-1})^*\omega = 0$. Thus $L_\xi(y)$ is an infinitesimal automorphism.

Since $\underline{S}(c(t))(c(t)) = c(t)$, we conclude that $L_\xi(x) + (\underline{S}(x))_*(\xi) = \xi$. Thus $L_\xi(x) = \xi - (\underline{S}(x))_*(\xi)$ and the claim follows due to the linearity of $T_x\underline{S}(x)$. □

Let us continue in the proof of Theorem 5.7. Since the map $\xi \mapsto L_\xi$ from Lemma 5.8 is injective on the bracket generating distribution given by $\mathfrak{g}_{-1}$ due to the assumption $\mathfrak{g}^s_{-1}(1) = 0$, the local homogeneity follows from the regularity of the parabolic geometry. This implies Claim (1). Then Claim (4) follows again from Claim (2).

Since we are on a (locally) homogeneous (locally) s–symmetric parabolic geometry, the parabolic geometry can be described as in Theorem A.1. It follows from [12, Theorem 1.3] that there is a K–invariant Weyl connection ∇ on the K–homogeneous parabolic geometry described Theorem A.1 such that all local automorphisms of the parabolic geometry are affine transformations of ∇. Therefore it follows from the last claim of Theorem A.1 that the pullback of ∇ to M does not depend on the local isomorphism with the K–homogeneous parabolic geometry. Therefore we obtain a Weyl structure σ that satisfies (2c), which implies the remaining parts (2a), (2b) and (2d). It is clear that the K–invariant Weyl connection ∇ from [12, Theorem 1.3] is not unique and the difference between two such Weyl structures is the claimed Υ provided by a K–invariant element of $\mathfrak{p}^s_+(1)$.

The Proposition 4.1 implies that the Weyl structure σ satisfies (2b) if and only if it satisfies (2d). Again, results in [12, Theorem 1.3] imply that (2b) implies (2c) and the same arguments as in the proof of Theorem 5.3 show that (2b) implies (2a) and (2a) implies (2d).

To prove Claim (5), we use the fact that $s.(T^\sigma(x))_\sigma = (T^\sigma(x))_\sigma$ holds for the torsion of the $\underline{S}$–invariant Weyl connection ∇^σ. Thus $TM^s(1)$ is involutive, because each (almost) $\underline{S}$–invariant Weyl connection ∇^σ preserves $TM^s(1)$. Moreover, the formula for the difference between ∇^σ and arbitrary Weyl connection implies that the difference in the parallel transport is an element of $TM^s(1)$ at each point of $\mathcal{F}_x$. Thus $\mathcal{F}_x$ is a totally geodesic submanifold for any Weyl connection.

We know that $\underline{S}(x) = s^\sigma_x$ and this implies $\underline{S}(x)|_{\mathcal{F}_x} = s^\sigma_x|_{\mathcal{F}_x} = \mathrm{id}_{\mathcal{F}_x}$. If $v = \mathrm{Fl}^{\omega^{-1}(X)}_1(u)$ for $X \in \mathfrak{g}^s_-(1)$, then $S(x)v = vs$ holds and $y = p \circ \mathrm{Fl}^{\omega^{-1}(X)}_1(u) \in \mathcal{F}_x$, because $\mathcal{F}_x$ is a totally geodesic submanifold. Thus $\underline{S}(x)$ is covered by the s–symmetry at y and $S(x) = S(y)$ holds in some neighbourhood of x due to the uniqueness of s–symmetries. Consequently, Claim (5a) holds on a sufficiently small local leaf space and Claim (5b) is a clear consequence of Claim (4). □

5.5. **Parabolic geometries with $\mathfrak{g}^s_{-1}(1)+\mathfrak{p}$ in a distinguished parabolic subalgebra.** There are triples $(\mathfrak{g},\mathfrak{p},\mu)$ that are prolongation rigid outside of the 1–eigenspace of s that admit a 1–eigenspace in $\mathfrak{g}_-$ for some s such that $\mathfrak{g}^s_{-1}(1)+\mathfrak{p}\subset\mathfrak{q}\subset\mathfrak{g}^s_{-}(1)+\mathfrak{p}$ holds for some parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ such that the harmonic curvature vanishes on insertions of elements of $\mathfrak{q}/\mathfrak{p}$ at all points of M. These are listed in the Tables 7, 8 and 9, due to [12, Proposition 6.2 and Proposition A.2].

Table 7. Theorem 5.10, part with $|\Sigma|=2$.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	μ	PR
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,2\}$		j_2^2	(α_2,α_3)	
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,3\}$		j_1^2	(α_1,α_2)	
$\mathfrak{su}(2,2)$	$\{1,2\}$		$2r_1$	(α_1,α_3)	$r_1=0$
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,3\}$		j_1^2	(α_1,α_2)	
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{2,3\}$		j_2^2	(α_2,α_1)	
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,n\}$		j_1^2	(α_1,α_2)	
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{2,n\}$		j_2^2	(α_2,α_1)	
$\mathfrak{su}(n,n)$	$\{1,n\}$		$2r_1$	(α_1,α_{2n-1})	$r_1=0$
$\mathfrak{so}(7,\mathbb{C})$	$\{1,3\}$	j_3^3		(α_3,α_2)	
$\mathfrak{so}(8,\mathbb{C})$	$\{1,3\}$		j_1^2	(α_1,α_2)	
$\mathfrak{sp}(4,\mathbb{C})$	$\{1,2\}$		$2r_1$	$(\alpha_1,\alpha_{1'})$	$r_1=0$
$\mathfrak{sp}(6,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$	j_2^{-2}		(α_2,α_1)	$j_2=-1$
$\mathfrak{sp}(6,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$		j_1^2	(α_1,α_2)	
$\mathfrak{sp}(6,\{\mathbb{R},\mathbb{C}\})$	$\{1,3\}$		j_1^2	(α_1,α_2)	
$\mathfrak{sp}(6,\mathbb{C})$	$\{1,3\}$		$2r_1$	$(\alpha_1,\alpha_{1'})$	$r_1=0$
$\mathfrak{sp}(6,\{\mathbb{R},\mathbb{C}\})$	$\{2,3\}$		j_2^2	(α_2,α_1)	
$\mathfrak{sp}(2n,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$	j_2^{-2}		(α_2,α_1)	$j_2=-1$
$\mathfrak{sp}(2n,\{\mathbb{R},\mathbb{C}\})$	$\{1,n\}$		j_1^2	(α_1,α_2)	$j_1=-1$
$\mathfrak{sp}(\frac{n}{2},\frac{n}{2}),\mathfrak{sp}(2n,\{\mathbb{R},\mathbb{C}\})$	$\{2,n\}$		j_2^2	(α_2,α_1)	$j_2=-1$
$\mathfrak{sp}(2n,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$		j_1^2	(α_1,α_2)	
$\mathfrak{sp}(2n,\mathbb{C})$	$\{1,n\}$		$2r_1$	$(\alpha_1,\alpha_{1'})$	$r_1=0$

Table 8. Theorem 5.10, part with $|\Sigma|=3$.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	j_{i_3}	μ	PR
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,2,3\}$			$j_1j_2^2$	(α_2,α_1)	$j_1=1$
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,2,3\}$			$j_1j_2^2$	(α_2,α_1)	$j_1=1$
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,2,n\}$			$j_1j_2^2$	(α_2,α_1)	$j_1=1$
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,p,n\}$, $p>2$		j_1j_n		(α_1,α_n)	$j_1=j_n^{-1}$
$\mathfrak{so}(8,\mathbb{C})$	$\{1,2,4\}$			j_1^2	(α_1,α_2)	$j_2=1$
$\mathfrak{sp}(6,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,3\}$			$j_1j_2^2$	(α_2,α_1)	$j_1=1$
$\mathfrak{sp}(2n,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,p\}$, $p<n$			$\sqrt{j_1j_2^2}$	(α_2,α_1)	$j_1=1,j_2=-1,j_p=1$
$\mathfrak{sp}(2n,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,n\}$			$j_1j_2^2$	(α_2,α_1)	$j_1=1,j_2=-1$

Example 5.9. Let us focus on Lagrangean complex contact geometries, i.e., $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbb{C})$ and $\Sigma=\{1,n\}$. If we consider the triple $(\mathfrak{sl}(n+1,\mathbb{C}),\mathfrak{p}_{\{1,n\}},\mu_{(\alpha_1,\alpha_2)})$

TABLE 9. Theorem 5.10, part with $|\Sigma| = 4$.

$\mathfrak{g}$	Σ	eigenvalues	μ	PR
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,2,3,q\}$, $q<n$	$j_q = j_1 j_2^2 j_p^{-1}$	(α_2,α_1)	$j_1 = 1, j_p = j_2^2$
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,2,p,n\}$, $3<p$	$j_n = j_1 j_2^2 j_p^{-1}$	(α_2,α_1)	$j_1 = 1, j_p = 1$

from the Table 7 that is prolongation rigid outside of the 1–eigenspace of s, then different situations arise depending on the choice of $s \in Z(G_0)$. If $j_1 = -1$, then $\mathfrak{q} = \mathfrak{g}^s_-(1) + \mathfrak{p}$ is a parabolic subalgebra satisfying the assumptions of Theorem 5.5. If $j_1 = \sqrt[3]{1}$, then $\mathfrak{g}^s_-(1) = \mathfrak{g}_{-2}$ and we need the assumptions of Theorem 5.7 to state the results. We can apply Theorem 5.3 for the other values j_1.

In general, $\mathfrak{q}$ can be a proper subspace of $\mathfrak{g}^s_-(1) + \mathfrak{p}$ and we can (locally) apply the general result for parabolic geometries from [4] to obtain the following theorem.

Theorem 5.10. *Assume $(\mathfrak{g},\mathfrak{p},\mu)$ is prolongation rigid outside of the 1–eigenspace of s for $s \in Z(G_0)$ such that $\mathfrak{q}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}^s_{-1}(1) + \mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{g}^s_-(1) + \mathfrak{p}$ and $\mathfrak{q}/\mathfrak{p}$ inserts trivially into the harmonic curvature of the (locally) s–symmetric parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G,P). Assume the open subset U of M containing all points at which κ_H has a non-zero component in μ is non–trivial, and the maps $\underline{S}(x)(y)$ and $\underline{S}(x) \circ \underline{S}(y)^{-1}(z)$ are defined on neighbourhoods of diagonals in $U \times U$ and $U \times U \times U$ for the unique system S of (local) s–symmetries on U. Then:*

(1) The set U is an open dense subset of M and there is a unique smooth system of (local) s–symmetries S on M.

(2) There are

- *a parabolic subgroup Q of G with the Lie algebra $\mathfrak{q}$ such that $P \subset Q$,*
- *a neighbourhood U_x of each $x \in M$ with a local leaf space $n : U_x \to N$ for the foliation given by the integrable distribution $Tp \circ \omega^{-1}(\mathfrak{q})$, and*
- *a (locally) s–symmetric parabolic geometry $(\mathcal{G}' \to N, \omega')$ of type (G,Q) satisfying the assumptions of Theorem 5.7*

such that $(\mathcal{G}|_{U_x} \to U_x, \omega|_{U_x})$ is isomorphic to an open subset of $(\mathcal{G}' \to \mathcal{G}'/P, \omega')$ for each x. In particular, there is a unique s–symmetry $\underline{S}(n(y))$ of $(\mathcal{G}' \to N, \omega')$ at each $n(y) \in N$ such that $n \circ \underline{S}(y) = \underline{S}(n(y)) \circ n$ holds for all $y \in U_x$ in the fiber over $n(y)$.

(3) The connected component of identity of the pseudo–group generated by all local s–symmetries is generated by the flows of the Lie algebra $\mathfrak{l}$, which is the vector subspace of $\mathfrak{q}^{op}$, generated by $\mathfrak{q}^{op}_+$ by the bracket $(T^\sigma(n(x)) + R^\sigma(n(x)))_\sigma$ on $\wedge^2(\mathfrak{q}^{op}_+)^* \otimes \mathfrak{l}$ and the natural bracket on the rest of $\mathfrak{l}$ for arbitrary S–invariant Weyl structure σ on $(\mathcal{G}' \to N, \omega')$.

The pseudo–group generated by all local s–symmetries is locally transitive at x if and only if $\mathfrak{q}/\mathfrak{p} \subset \mathfrak{l}/(\mathfrak{l} \cap \mathfrak{p})$, i.e., if and only if $(R^\sigma(n(x)))_\sigma$ spans the whole $\mathfrak{q}/\mathfrak{p}$.

(4) There is a class of almost S–invariant Weyl structures on U_x given by reductions of the images in $\mathcal{G}'$ of the S–invariant Weyl structures on N (that exist due to Theorem 5.7) to $\exp(\mathfrak{g}^s_-(1)) \rtimes G_0 \subset Q_0$. A reduction corresponds to an S–invariant Weyl structure on U_x if and only if it is a holonomy reduction.

(5) We get that

- *the maps $\underline{S}(x)$ can be extended to a larger neighbourhood of x as long as the corresponding geodesic transformations of ∇^σ on N are defined,*

- *the space $TM^s(1)$ is integrable, it contains the vertical space of the local leaf space $n : U_x \to N$, and $Tn(T_xM^s(a))$ is the a–eigenspace of $T_{n(x)}\underline{S}(n(x))$ in $T_{n(x)}N$,*
- *for each eigenvalue a, the distribution $TM^s(a)$ is preserved by all (local) automorphisms of the parabolic geometry, and*
- *all almost $\underline{S}$–invariant Weyl connections from Claim (4) restrict to the same partial linear connection on TM corresponding to the distribution $\mathcal{G}_0 \times_{G_0} \mathfrak{q}_+^{op}$, which is preserved by $\underline{S}(x)$ for all $x \in M$.*

Proof. The Claim (1) is a direct consequence of Claims (2) and (3). The Claim (2) follows from [4] and the fact that $(\mathfrak{q}_1^{op})^s(1) = 0$ holds. Then Claim (3) is a clear consequence of Theorem 5.7. The Claim (4) follows from the comparison of images in $\mathcal{G}'$ of the S–invariant Weyl structure on N and the almost S–invariant Weyl structures on U_x, because they intersect precisely in the reduction to $\exp(\mathfrak{g}_-^s(1)) \rtimes G_0 \subset Q_0$. The Claim (5) is a consequence of Claim (5) of Theorem 5.7 and the properties of Weyl structures from Claim (4). □

5.6. **Parabolic geometries with $\mathfrak{g}_{-1}^s(1)$ that inserts non–trivially into the harmonic curvature.** There are also some remaining parabolic geometries, which can have a part of $\mathfrak{g}_{-1}^s(1)$ that inserts non–trivially into the harmonic curvature. These are contained in the Tables 10, 11 and 12.

Example 5.11. Let us continue in the discussion of generalized path geometries from the Example 5.4. The case when the harmonic curvature $\kappa_{(\alpha_2,\alpha_1)}$ vanishes and the harmonic torsion $\kappa_{(\alpha_1,\alpha_2)}$ does not vanish can be found in the Table 10. There are several possible situations depending on the eigenvalues of $s \in Z(G_0)$.

If $j_1 = 1$, then we are precisely in the situation which is not covered by any of the previous theorems and we can apply only the results of Propositions 4.3 and 5.12.

If $j_1 = -1$ and $j_2 = 1$, then we can apply Theorem 5.5 and we are in the situation of a generalized path geometry on the projectivized cotangent space of an affine locally symmetric space.

If $j_1 = -1$ and $j_2 = -1$, then $\mathfrak{g}_-^s(1) = \mathfrak{g}_{-2}$ and we need the assumptions of Theorem 5.7 to show that we are in the situation of a generalized path geometry on a (locally) homogeneous $(n-1)$–dimensional fiber bundle over an affine locally symmetric space.

Finally, if $j_1 = -1$ and $j_2 \neq \sqrt{1}$, then we can apply Theorem 5.3.

The properties of these geometries are as follows.

Proposition 5.12. *Assume $(\mathfrak{g}, \mathfrak{p}, \mu)$ is prolongation rigid outside of the 1–eigenspace of s for some $s \in Z(G_0)$. Assume the harmonic curvature κ_H of a (locally) s–symmetric parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) has a non–zero component in μ at all $x \in M$ and S is the unique system of (local) s–symmetries on M. Then the distribution $TM^s(1)$ is integrable and for each $x \in M$, the leaf $\mathcal{F}_x$ of the foliation $\mathcal{F}$ of $TM^s(1)$ through x is a totally geodesic submanifold for arbitrary Weyl connection.*

Let $n : U_x \to N$ be a sufficiently small local leaf space of $TM^s(1)$.

- *There is a unique local diffeomorphism $\underline{S}(n(y))$ of the local leaf space N at each $n(y) \in N$ such that $\underline{S}(n(y)) \circ n = n \circ \underline{S}(x)$ holds for all $y \in U_x$, and*
- *for each eigenvalue a, $T_yn(T_yM^s(a))$ is the a–eigenspace of $T_{n(y)}\underline{S}(n(y))$ for all $y \in U_x$.*

Proof. The proof is analogous to the proof of Claim (6) of Theorem 5.7, but, instead of an S–invariant Weyl structure σ, we need to consider some almost S–invariant Weyl structure invariant at x from Claim (3) of Proposition 4.3 for each $x \in M$. □

TABLE 10. Theorem 5.12, part with $|\Sigma| = 2$.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	μ
$\mathfrak{sl}(3,\mathbb{C})$	$\{1,2\}$	$2\phi_2$		$(\alpha_1,\alpha_{2'})$
$\mathfrak{sl}(4,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$	$\sqrt{1}$		(α_1,α_2)
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,2\}$	$2\phi_2$		$(\alpha_1,\alpha_{2'})$
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,3\}$	$2\phi_3$		$(\alpha_1,\alpha_{3'})$
$\mathfrak{su}(2,2)$	$\{1,2\}$	r_1	$\sqrt{1}$	(α_2,α_1)
$\mathfrak{su}(2,2)$	$\{1,2\}$	$\sqrt[3]{1}$		(α_1,α_2)
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$	$\sqrt{1}$		(α_1,α_2)
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,p\}$, $2<p<n$	1		(α_1,α_p)
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,p\}$	$2\phi_p$		$(\alpha_1,\alpha_{p'})$
$\mathfrak{su}(q,n-q+1)$	$\{1,2\}$		$-\frac{2}{3}\phi_1$	(α_2,α_1)
$\mathfrak{so}(3,4),\mathfrak{so}(7,\mathbb{C})$	$\{2,3\}$		$\sqrt[3]{1}$	(α_3,α_2)
$\mathfrak{so}(2,5)$, $\mathfrak{so}(3,4)$, $\mathfrak{so}(7,\mathbb{C})$ $\mathfrak{so}(2,6)$, $\mathfrak{so}(3,5)$, $\mathfrak{so}(4,4)$, $\mathfrak{so}(7,\mathbb{C})$	$\{1,2\}$		1	(α_2,α_1)
$\mathfrak{so}(3,5)$	$\{2,3\}$		$\sqrt[3]{1}$	(α_3,α_2)
$\mathfrak{so}(q,n-q)$, $\mathfrak{so}(n,\mathbb{C})$	$\{1,2\}$		1	(α_2,α_1)
$\mathfrak{sp}(4,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$	$\sqrt[3]{1}$		(α_1,α_2)
$\mathfrak{sp}(4,\mathbb{C})$	$\{1,2\}$	$2\phi_2$		$(\alpha_1,\alpha_{2'})$
$\mathfrak{sp}(4,\mathbb{C})$	$\{1,2\}$	$-\frac{2}{5}\phi_2$		$(\alpha_{1'},\alpha_{2'})$
$\mathfrak{sp}(6,\{\mathbb{R},\mathbb{C}\})$	$\{1,3\}$	1		(α_1,α_3)
$\mathfrak{sp}(6,\{\mathbb{R},\mathbb{C}\})$	$\{2,3\}$	1		(α_2,α_3)
$\mathfrak{sp}(6,\mathbb{C})$	$\{1,3\}$	$2\phi_3$		$(\alpha_1,\alpha_{3'})$
$\mathfrak{sp}(6,\mathbb{C})$	$\{2,3\}$	$-\frac{2}{5}\phi_3$		$(\alpha_{2'},\alpha_{3'})$
$\mathfrak{sp}(2n,\{\mathbb{R},\mathbb{C}\})$	$\{1,n\}$	1		(α_1,α_n)
$\mathfrak{sp}(2n,\{\mathbb{R},\mathbb{C}\})$	$\{n-1,n\}$	1		(α_{n-1},α_n)
$\mathfrak{sp}(2n,\mathbb{C})$	$\{1,n\}$	$2\phi_n$		$(\alpha_1,\alpha_{n'})$
$\mathfrak{sp}(2n,\mathbb{C})$	$\{n-1,n\}$	$-\frac{2}{5}\phi_n$		$(\alpha_{n-1'},\alpha_{n'})$
$\mathfrak{g}_2(\{2,\mathbb{C}\})$	$\{1,2\}$	$\sqrt[4]{1}$		(α_1,α_2)

TABLE 11. Theorem 5.12, part with $|\Sigma| = 3$.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	j_{i_3}	μ	PR
$\mathfrak{sl}(4,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,3\}$			j_1^2	(α_1,α_2)	
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,3\}$			j_1^2	(α_1,α_2)	
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,p\}$, $3<p<n$			j_1^2	(α_1,α_2)	$j_1=\sqrt{1}$
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,n\}$			j_1^2	(α_1,α_2)	
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,n\}$		$j_1 j_n$		(α_1,α_n)	
$\mathfrak{so}(3,4)$, $\mathfrak{so}(7,\mathbb{C})$	$\{1,2,3\}$	j_3^3			(α_3,α_2)	

TABLE 12. Theorem 5.12, part with $|\Sigma| = 4$.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	j_{i_3}	j_{i_4}	μ	PR
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,3,n\}$				$j_1 j_2^2 j_p^{-1}$	(α_2,α_1)	$j_1=1$

5.7. **Parabolic geometries that do not admit non–flat examples.** There are triples $(\mathfrak{g},\mathfrak{p},\mu)$ that are prolongation rigid outside of the 1–eigenspace of s for some

$s \in Z(G_0)$, but they admit only flat (locally) s–symmetric parabolic geometries due to the structure of the harmonic curvature and [11, Lemma 2.2]. These are contained in the Table 13.

TABLE 13. Flat geometries.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	μ
$\mathfrak{sl}(3,\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	1	(α_1,α_2)
$\mathfrak{sl}(3,\{\mathbb{R},\mathbb{C}\})$	$\{1,2\}$	$j_1^4 \neq 1, j_1^5 \neq 1$	j_1^3	(α_1,α_2)
$\mathfrak{so}(5,\mathbb{C})$	$\{1\}$	$\sqrt[3]{1}$	1	(α_1,α_2)
$\mathfrak{so}(2,3),\mathfrak{so}(5,\mathbb{C})$	$\{1,2\}$	$j_1^4 \neq 1, j_1^5 \neq 1, j_1^7 \neq 1$	j_1^3	(α_1,α_2)

5.8. **Parabolic geometries with more non–zero components of the harmonic curvature.** Let us also look at the parabolic geometries that allow a harmonic curvature κ_H with several non–zero components μ_i such that for each μ_i the triple $(\mathfrak{g},\mathfrak{p},\mu_i)$ is not prolongation rigid outside of the 1–eigenspace of s. In the Table 14, we present the complete classification of all triples $(\mathfrak{g},\mathfrak{p},\mu_i)$ that are not prolongation rigid outside of the 1–eigenspace of s for the same $s \in Z(G_0)$, but for which $\mathfrak{a}_i$ in Proposition 3.5 is contained in the 1–eigenspaces of s when the harmonic curvature has non–zero component in each μ_i. Geometric properties of the geometries from the Table 14 can be deduced from the previous sections depending on the position and shape of $\mathfrak{g}_-^s(1)$ inside of $\mathfrak{g}_-$.

TABLE 14. More non–zero components of the harmonic curvature.

$\mathfrak{g}$	Σ	eigenvalues	μ
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,2\}$	$j_1 = \sqrt[4]{1}^2, j_2 = \sqrt[4]{1}$	(α_2,α_3), (α_2,α_1)
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{1,n-1\}$	$j_1 = \sqrt[3]{1}, j_{n-1} = \sqrt[3]{1}^2$	(α_1,α_2), (α_{n-1},α_n)
$\mathfrak{sl}(n+1,\mathbb{C})$	$\{2,n-1\}$	$j_2 = \sqrt[3]{1}, j_{n-1}\sqrt[3]{1}^2$	(α_2,α_1), (α_{n-1},α_n)
$\mathfrak{sl}(4,\mathbb{C})$	$\{1,2,3\}$	$j_2 = \sqrt[4]{1}, j_3 = j_1(\sqrt[4]{1})^2$	(α_2,α_1), (α_2,α_3)
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{1,2,n-1,n\}$	$j_{n-1} = j_2^{-1}, j_n = j_1 j_2^3$	(α_2,α_1), (α_{n-1},α_n)

5.9. **Remaining parabolic geometries with μ in the 1–eigenspace of s.** For the sake of completeness, let us remark that there are triples $(\mathfrak{g},\mathfrak{p},\mu)$ that are not prolongation rigid outside of the 1–eigenspace of s for any s such that μ is in the 1–eigenspace of s. These are contained in the Table 15.

TABLE 15. Remaining parabolic geometries with μ in the 1–eigenspace of s.

$\mathfrak{g}$	Σ	j_{i_1}	j_{i_2}	μ
$\mathfrak{sl}(n+1,\{\mathbb{R},\mathbb{C}\})$	$\{p,p+1\}$, $n-1>p>1$		1	(α_{p+1},α_p)
$\mathfrak{so}(q,n-q),\mathfrak{so}(n,\mathbb{C})$	$\{2,3\}$		1	(α_3,α_2)

Appendix A. A construction of locally homogeneous locally s–symmetric parabolic geometries

It is proved in [11, Section 2] how to algebraically construct and classify all homogeneous s–symmetric parabolic geometries. Part of the classification is done in [7, 8] using the classification of semisimple symmetric spaces. There is the result from [9, Section 1.3] and [11, Lemma 2.2] stating that for the construction and the classification of locally homogeneous locally s–symmetric parabolic geometries, it is sufficient to find the following data:

- an extension (α, i) of the Klein geometry (K, H) to (G, P) such that the action of s preserves $\alpha(\mathfrak{k}) \subset \mathfrak{g}$, and s acts trivially on the tensor $[\cdot,\cdot] - \alpha([\alpha^{-1}(\cdot), \alpha^{-1}(\cdot)])$ in $\wedge^2\mathfrak{g}/\mathfrak{p}^* \otimes \mathfrak{g}$, and
- the subset $\mathcal{A}$ of P consisting of elements $g \in P$, which act as local automorphisms on the parabolic geometry $(K \times_i P \to K/H, \omega_\alpha)$ of type (G, P) given by the extension (α, i).

If U and V are open subsets of K/H such that there are $k \in K$, $g \in \mathcal{A}$ and a maximal open subset W of U such that $kg(W) \subset V$, then we can glue $K \times_i P|_U \to U$ with $K \times_i P|_V \to V$ by identifying $w \in W \subset U$ with $kg(w) \in V$, and glue the Cartan connection $\omega_\alpha|_U$ with the pullback connection $(kg)^*\omega_\alpha|_V = \omega_\alpha|_{(kg)^*(V)}$. Of course, we can without loss of generality assume that U, V and W are simply connected, because we can always choose coverings of our manifolds by open sets satisfying this condition. Therefore, we can also assume that the automorphism k is given by the flow of a local infinitesimal automorphism of $(K \times_i P \to K/H, \omega_\alpha)$. Then we obtain the following result as a consequence of the construction in [12, Section 3] and [9, Section 1.3].

Theorem A.1. *Let $(\mathcal{G} \to M, \omega)$ be a locally homogeneous locally s–symmetric parabolic geometry, let $\mathfrak{k}$ be the Lie algebra of the local infinitesimal automorphisms and denote by α the inclusion of $\mathfrak{k}$ into $\mathfrak{g}$ given by $\omega(u)$ at some $u \in \mathcal{G}$. Then:*

(1) $\mathrm{Ad}(s)(\mathfrak{k}) \subset \mathfrak{k}$ *is an automorphism of the Lie algebra $\mathfrak{k}$,*
(2) *there exist (see* [12, Section 3] *for the explicit construction)*
  - *a Klein geometry (K, H) such that $\mathfrak{k}$ is the Lie algebra of K,*
  - *an extension (α, i) of (K, H) to (G, P),*
  - *an open covering U_a of M, and*
  - *isomorphisms $\phi_a : U_i \to K/H$ of parabolic geometries $(\mathcal{G}|_{U_a} \to U_i, \omega|_{U_b})$ and $(K \times_i P|_{\phi_a(U_a)} \to \phi_a(U_a), \omega_\alpha|_{\phi_a(U_a)})$ of type (G, P) such that $\phi_a \circ \phi_b^{-1}$ is the restriction of the left action of some element of K for each a, b.*

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Chapter 3: On Symmetric CR Geometries of Hypersurface Type

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ON SYMMETRIC CR GEOMETRIES OF HYPERSURFACE TYPE

JAN GREGOROVIČ AND LENKA ZALABOVÁ

Abstract. We study non–degenerate CR geometries of hypersurface type that are symmetric in the sense that, at each point, there is a CR transformation reversing the CR distribution at that point. We show that such geometries are either flat or homogeneous. We show that non–flat non–degenerate symmetric CR geometries of hypersurface type are covered by CR geometries with a compatible pseudo–Riemannian metric preserved by all symmetries. We construct examples of simply connected flat non–degenerate symmetric CR geometries of hypersurface type that do not carry a pseudo–Riemannian metric compatible with the symmetries.

1. Introduction

In [10], Kaup and Zaitsev generalized Riemannian symmetric spaces to the setting of CR geometries, i.e., smooth manifolds with so–called CR distribution endowed with a formally integrable complex structure. They consider Riemannian metrics, whose restrictions to the CR distribution is Hermitian. Such manifold is symmetric in the sense of [10] if, at each point, there is an isometric CR transformation that preserves the point and which, at that point, acts as $-\mathrm{id}$ on the CR distribution [10, Definition 3.5.]. They show that such isometric CR transformations are uniquely determined by the tangent action on the CR distribution [10, Theorem 3.3]. They also show that such CR geometries are homogeneous [10, Proposition 3.6]. In fact, these CR geometries may be considered as reflexion spaces in the sense of [12]. In [1], the authors study these CR geometries in the setting of so–called CR algebras.

We studied in [9] filtered geometric structures that carry an automorphism at each point that fixes the point and acts as $-\mathrm{id}$ on a distinguished part of the filtration at the point. Let us point out that the non–degenerate CR geometries of hypersurface type, i.e., those with CR distribution of codimension 1, are among these geometries. We answered the question whether these filtered geometries are homogeneous and can be considered as reflexion spaces. However, our result [9, Theorem 5.7.] holds under weaker conditions than the result of [10] for non–degenerate CR geometries of hypersurface type. In particular, the sufficient condition for such non–degenerate CR geometry of hypersurface type to be homogeneous is that it is non–flat at one point.

In this paper, we study the case of non–degenerate CR geometries of hypersurface type in more detailed way. We consider point preserving CR transformations which, at that point, induce $-\mathrm{id}$ on the CR distribution. We say that a non–degenerate CR geometry of hypersurface type is symmetric (in our sense) if there exists a symmetry at each point, see Definition 1. In particular, our definition does not require the existence of a metric compatible with the CR geometry. We adapt and significantly

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improve general results of [5, 8, 9] for our particular class of CR geometries. Let us emphasize that every non–degenerate CR geometry of hypersurface type that is symmetric in the sense of [10] is symmetric (in our sense).

Let us say that [9, Theorem 5.7.] is formulated in the general setting of parabolic geometries. We provide here the particular results of this theorem for CR geometries. We also provide new direct proofs, because we will need the presented ideas to explain new results, see Lemmas 2, 3 and Propositions 1, 2. This allows us to compare our results with results of [10] and [1].

We prove in Theorem 2 that non–flat non–degenerate CR geometries of hypersurface type that are symmetric (in our sense) are covered by symmetric non–degenerate CR geometries of hypersurface type that carry a pseudo–Riemannian metric compatible with the CR geometry that is preserved by all our symmetries. In the Riemannian signature, these coverings are symmetric in the sense of [10], see Theorem 3. Moreover, we show in Theorem 4 that it is always possible to embed the CR geometry on these coverings into a complex manifold. In the Riemannian signature, this embedding is provided by a different construction than the one in [10, Proposition 7.3].

Finally, we construct examples of non–homogeneous symmetric (in our sense) flat non–degenerate CR geometries of hypersurface type. These examples do not admit a pseudo–Riemannian metric that would be preserved by some symmetry at each point and in particular, they are not symmetric in the sense of [10]. We also discuss examples of homogeneous CR geometries on orbits of real forms in complex flag manifolds. In particular, we show that there are homogeneous CR geometries which are locally symmetric but not globally symmetric. In fact, Proposition 8 provides complete description of all possible cases.

2. CR geometries of hypersurface type

2.1. **CR geometries.** Let M be a smooth manifold of dimension $2n+1$ for $n>1$ together with a distribution $\mathcal{H} \subset TM$ of dimension $2n$ and a *complex structure* J on $\mathcal{H}$, i.e., $J : \mathcal{H} \to \mathcal{H}$ is an endomorphism with the property that $J^2 = -\mathrm{id}$. The triple $(M, \mathcal{H}, J)$ is called a *CR geometry of hypersurface type* if the i–eigenspace $\mathcal{H}^{1,0}$ of J in the complexification of $\mathcal{H}$ is integrable, i.e., $[\mathcal{H}^{1,0}, \mathcal{H}^{1,0}] \subset \mathcal{H}^{1,0}$. A CR geometry $(M, \mathcal{H}, J)$ is called *non–degenerate* if $\mathcal{H}$ is completely non–integrable.

On $\mathcal{H}$ there exists a symmetric bilinear form h with values in the line bundle $TM/\mathcal{H}$ given by $h(\xi,\eta) = \frac{1}{2}\pi([\xi, J\eta])$ for all $\xi, \eta \in \Gamma(M, \mathcal{H})$, where $\pi : TM \to TM/\mathcal{H}$ is a natural projection. Let us recall that h is the real part of the Levi form $\tilde{h}$ of $(M, \mathcal{H}, J)$ whose imaginary part is the map given by $\frac{1}{2}\pi([\xi,\eta])$. We assume that M is orientable and denote by (p,q) the signature of the Levi form, where our convention is $p \le q$, $p+q = n$. Then the signature of h is $(2p, 2q)$.

The homogeneous space $PSU(p+1, q+1)/P$ is usually called the *standard model* of a non–degenerate CR geometry of hypersurface type of signature (p,q), where P in $PSU(p+1, q+1)$ is the stabilizer of the complex line generated by the first basis vector in the standard basis of $\mathbb{C}^{n+2}$. We use the convention that elements of $PSU(p+1, q+1)$ are represented (up to a multiple) by matrices preserving the Hermitian form given for $u, v \in \mathbb{C}^{n+2}$ by

$$m(u,v) = u^T \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{pmatrix} \overline{v},$$

where I is a square matrix of order n defining a Hermitian inner product of signature (p,q). In [2], the matrix I is chosen to be diagonal with the first p entries equal to 1 and the remaining q entries equal to -1.

The standard model $PSU(p+1,q+1)/P$ is a smooth real hypersurface in $\mathbb{C}P^{n+1}$ that can be also viewed as the projectivization of the null cone of m in $\mathbb{C}^{n+2}$.

In the rest of the paper, by a CR geometry we mean a non–degenerate CR geometry of hypersurface type of signature (p,q) for $p \leq q$. Such CR geometries can be equivalently described as parabolic geometries modeled on standard models $PSU(p+1,q+1)/P$. This description can be found in [2, Section 4.2.4]. We only use several consequences of this description later in the text.

2.2. **Distinguished connections.** There exist many admissible connections, i.e., connections preserving $\mathcal{H}$ and J, on CR geometries. In particular, there are several distinguished classes of admissible connections given by a particular normalization condition on the torsion of admissible connections in the class. The most commonly used are Tanaka–Webster [2, Section 5.2.12] and Weyl [2, Sections 5.1.2 and 5.2.13] connections. In this paper we will use the latter, since in some arguments we need to relate CR and geodesic transformations for the so–called normal Weyl connection [2, Section 5.1.12].

In fact, Tanaka–Webster connections and Weyl connections induce the same class of distinguished partial connections ∇ on $\mathcal{H}$. Two distinguished partial connections ∇ and $\hat{\nabla}$ are related by

$$\hat{\nabla}_\xi(\eta) = \nabla_\xi(\eta) + F(\xi)\eta + F(\eta)\xi - h(\xi,\eta)h^{-1}(F) \text{ for all } \xi,\eta \in \Gamma(\mathrm{M},\mathcal{H}), \tag{1}$$

where F is a one–form on $\mathcal{H}$ and h is the duality map associated to the real part of the Levi form. Note that while the duality map depends on the choice of a local trivialization of $TM/\mathcal{H}$, the composition $h(\xi,\eta)h^{-1}(F)$ does not depend on this choice. We will write for short "$\hat{\nabla} = \nabla + F$" for the partial connection described by (1).

Each Weyl connection D is associated to the choice of decompositions $TM \simeq \mathcal{H} \oplus \ell$, $T^*M \simeq \mathcal{H}^* \oplus \ell^*$, for a one dimensional complementary distribution ℓ of $\mathcal{H}$ and with $\mathcal{H}^*, \ell^*$ being the annihilator bundles of $\ell, \mathcal{H}$, respectively. The one-form F in $\mathcal{H}^*$ from formula (1) then describes the difference of the two decompositions of TM (and T^*M) associated with D and $\hat{D}$. The precise formula for the change of decompositions can be easily computed using [2, Section 5.1.5].

The explicit formula for the relation between two arbitrary Weyl connections D and $\hat{D}$ is complicated and can be computed using [2, Section 5.1.6]. This relation is given by an action of a one–form $\Upsilon = \Upsilon_1 + \Upsilon_2$ in $T^*M = \mathcal{H}^* \oplus \ell^*$ on the space of Weyl connections, which we write as $\hat{D} = D + \Upsilon_1 + \Upsilon_2$, because this action is linear in Υ. Let us point out that the decomposition $\Upsilon_1 + \Upsilon_2$ of Υ is the decomposition associated with the Weyl connection D and that Υ_1 coincides with F from formula (1) for the corresponding partial connections ∇, $\hat{\nabla}$ determined by D, $\hat{D}$.

Let us finally point out that admissible connections provide the fundamental invariant W of CR geometries which is known as *Chern–Moser* or *Weyl tensor* and coincides with the totally trace–free part of the curvature of arbitrary Weyl or Tanaka–Webster connections. Vanishing of this invariant implies that a CR geometry is *flat*, meaning that is locally equivalent to the standard model $PSU(p+1,q+1)/P$.

3. Symmetries of CR geometries

3.1. **Definition of symmetries.** A *CR transformation* of a CR geometry $(M,\mathcal{H},J)$ is a diffeomorphism of M whose tangent map preserves the *CR distribution* $\mathcal{H}$ and whose restriction to $\mathcal{H}$ is complex linear. We study the following CR transformations.

Definition 1. A *symmetry* at $x \in M$ on a CR manifold $(M, \mathcal{H}, J)$ is a CR transformation S_x of M such that:

(1) $S_x(x) = x$,
(2) $T_x S_x = -\mathrm{id}$ on $\mathcal{H}$.

We say that a CR geometry is *symmetric* if there is a symmetry at each point $x \in M$. A *system of symmetries* on M is a choice of a symmetry S_x at each $x \in M$. We call the system *smooth*, if the map $S : M \times M \to M$ given by $S(x, y) = S_x(y)$ is smooth.

Let us show that the standard model $PSU(p+1, q+1)/P$ is symmetric. The Lie group $PSU(p+1, q+1)$ is the group of all CR transformations of the standard model $PSU(p+1, q+1)/P$, where we consider left action. Direct computation shows that all symmetries of the standard model $PSU(p+1, q+1)/P$ at the origin eP are represented by $(1, n, 1)$–block matrices of the form

$$s_{Z,z} = \begin{pmatrix} -1 & -Z & iz + \frac{1}{2} ZIZ^* \\ 0 & E & -IZ^* \\ 0 & 0 & -1 \end{pmatrix}, \tag{2}$$

where $Z \in \mathbb{C}^{n*}$, $z \in \mathbb{R}^*$ are arbitrary and E is the identity matrix of the rank $n = p + q$.

Lemma 1. *There are infinitely many symmetries at each point kP of $PSU(p+1, q+1)/P$ given by matrices of the form $k s_{Z,z} k^{-1}$ for all $Z \in \mathbb{C}^{n*}$ and $z \in \mathbb{R}^*$. In particular:*

(1) *There are infinitely many involutive symmetries at each point characterized by the condition $z = 0$. For each such symmetry, there is a different metric preserved by this symmetry compatible with the CR geometry.*
(2) *There are infinitely many non–involutive symmetries at each point characterized by the condition $z \neq 0$. They do not preserve any metric compatible with the CR geometry.*

Proof. An element $s_{Z,z}$ belongs to a compact subgroup of $PSU(p+1, q+1)$ if and only if $z = 0$. □

The standard model $PSU(p+1, q+1)/P$ is endowed with a pseudo–Riemannian metric compatible with the CR geometry given by a choice of a maximally compact subgroup of $PSU(p+1, q+1)$, which always acts transitively on the standard model. Moreover, there is exactly one involutive symmetry at each point of this model that is contained in the chosen maximal compact subgroup. These symmetries preserve the corresponding pseudo–Riemannian metric and form a smooth system. This means that in the Riemannian signature, the standard model $PSU(1, n+1)/P$ is symmetric in the sense of [10].

On flat CR geometries, local symmetries are defined as in the standard model, but may not extend globally. This means that on flat CR geometries, there is always locally a pseudo–Riemannian metric compatible with some local CR symmetry at each point. We show in Example 1 that such pseudo–Riemannian metrics, compatible with a system of CR symmetries, may not exist globally.

3.2. **Involutive and non–involutive symmetries.** Suppose that there is a symmetry S_x at x on a CR geometry $(M, \mathcal{H}, J)$. If D is a Weyl connection, then $S_x^* D$ is a Weyl connection, too. Therefore, there is a one–form $\Upsilon_1 + \Upsilon_2 \in \mathcal{H}^* \oplus \ell^*$ such that

$$S_x^* D = D + \Upsilon_1 + \Upsilon_2. \tag{3}$$

Lemma 2. *Suppose S_x is a symmetry at $x \in M$. Let D be an arbitrary Weyl connection and let $\Upsilon_1 + \Upsilon_2 \in \mathcal{H}^* \oplus \ell^*$ be the one–form from formula* (3)*. Then the following claims are equivalent:*

(1) *the symmetry S_x is involutive,*
(2) $\Upsilon_2(x) = 0$*, and*
(3) *the diffeomorphism S_x is linear in the normal coordinates given by the normal Weyl connection $\bar{D}$ at x that is uniquely determined by the property that $\bar{D}$ coincides with the Weyl connection $D + \frac{1}{2}\Upsilon_1$ at x.*

Moreover, the partial connection ∇^{S_x} induced by the Weyl connection $D^{S_x} := D + \frac{1}{2}\Upsilon_1$ does not depend on the choice of D at x and satisfies

- $S_x^*(\nabla^{S_x}) = \nabla^{S_x}$ *at x, and*
- $\nabla^{S_x} W(x) = 0$.

Proof. Iterating the formula (3) we compute

$$S_x^* S_x^* D = D + \Upsilon_1 + S_x^*(\Upsilon_1) + \Upsilon_2 + S_x^*(\Upsilon_2).$$

The component of the (dual) action of $T_x S_x$ on $T_x^* M$ preserving the decomposition $T_x^* M = \mathcal{H}^*(x) \oplus \ell^*(x)$ is $-\mathrm{id} \oplus \mathrm{id}$, and the component that maps $\mathcal{H}^*(x)$ into $\ell^*(x)$ depends linearly on Υ_1 and is antisymmetric as a map $\mathcal{H}^*(x) \otimes \mathcal{H}^*(x) \to \ell^*(x)$. Therefore, $S_x^*(\Upsilon_1)(x) = -\Upsilon_1(x)$ and $S_x^*(\Upsilon_2)(x) = \Upsilon_2(x)$.

If the symmetry S_x is involutive, i.e., $S_x^2 = \mathrm{id}$, then

$$0 = \Upsilon_2(x) + S_x^*(\Upsilon_2)(x) = 2\Upsilon_2(x)$$

and thus $\Upsilon_2(x) = 0$.

If $\Upsilon_2(x) = 0$, then the normal Weyl connection $\bar{D}$ that coincides with the Weyl connection $D + \frac{1}{2}\Upsilon_1$ at x satisfies

$$S_x^*(D + \frac{1}{2}\Upsilon_1) = D + \Upsilon_1 + S_x^*(\frac{1}{2}\Upsilon_1).$$

At the point x, we get

$$\Upsilon_1(x) + S_x^*(\frac{1}{2}\Upsilon_1)(x) = \Upsilon_1(x) - \frac{1}{2}\Upsilon_1(x) = \frac{1}{2}\Upsilon_1(x)$$

and thus $S_x^* \bar{D} = \bar{D}$ follows from the normality [2, Section 5.1.12]. Thus S_x is an affine map, which is linear in the normal coordinates.

If the symmetry S_x at x is linear in the normal coordinates of a Weyl connection, then its (dual) tangent action preserves the decomposition $T_x^* M = \mathcal{H}^*(x) \oplus \ell^*(x)$ and therefore $(T_x S_x)^2 = \mathrm{id}$. Then it follows from the linearity that S_x is involutive.

Finally, the last claim follows, because $(\nabla^{S_x} W)(x)$ is a tensor of type $\otimes^4 \mathcal{H}_x^* \otimes \mathcal{H}_x$ invariant with respect to S_x. □

Lemma 3. *Suppose that there is a symmetry S_x at $x \in M$. Let D be an arbitrary Weyl connection and let $\Upsilon_1 + \Upsilon_2 \in \mathcal{H}^* \oplus \ell^*$ be the one–form from formula* (3)*. If $W(x) \neq 0$, then $\Upsilon_2(x) = 0$ and the symmetry S_x is involutive.*

Proof. Consider the covariant derivative of W with respect to $D + \frac{1}{2}\Upsilon_1$ in the direction ℓ and compute $S_x^*(D + \frac{1}{2}\Upsilon_1)_r W(x)$ for $r \in \ell(x)$. We know that $W(x)$ is S_x–invariant and thus

$$S_x^*(D + \frac{1}{2}\Upsilon_1)_r W(x) = (D + \frac{1}{2}\Upsilon_1)_{S_x^*(r)} W(x) = (D + \frac{1}{2}\Upsilon_1)_r W(x).$$

On the other hand, it generally holds that $S_x^*(D + \frac{1}{2}\Upsilon_1) = D + \frac{1}{2}\Upsilon_1 + \Upsilon_2$ and $\Upsilon_2(x) = a\theta(x)$ for a covector $\theta \in \ell^*(x)$ such that $\theta(r) = 1$. Then

$$S_x^*(D + \frac{1}{2}\Upsilon_1)_r W(x) = (D + \frac{1}{2}\Upsilon_1)_r W(x) + 2aW(x)$$

and thus $a = 0$ which implies $\Upsilon_2(x) = 0$. □

3.3. **Smooth systems of involutive symmetries.** Let us show that the assumption that W does not vanish at any point not only implies that all symmetries are involutive, but also that at each point of M there is exactly one symmetry S_x and the S_x's make a smooth system on M:

Proposition 1. *Suppose that $(M, \mathcal{H}, J)$ is a symmetric CR geometry and that $W(x) \neq 0$ for all $x \in M$. Then*

(1) *there is a unique symmetry S_x at each $x \in M$,*
(2) *the map $S : x \mapsto S_x$ is smooth, and*
(3) *$S_x \circ S_y \circ S_x = S_{S_x(y)}$ holds for all $x, y \in M$.*

In particular, (M, S) is a reflexion space, i.e., $S : M \times M \to M$ is a smooth map that for all $x, y, z \in M$ satisfies that

- *$S(x, x) = x$,*
- *$S(x, S(x, y)) = y$, and*
- *$S(x, S(y, z)) = S(S(x, y), S(x, z))$.*

Proof. We show that if there are two different symmetries at x on a CR geometry $(M, \mathcal{H}, J)$, then W vanishes at x. Consider two different symmetries S_x and S'_x at x (both must be involutive). We know from Lemma 2 that $\nabla^{S_x} W(x) = 0$ and $\nabla^{S'_x} W(x) = 0$ hold for partial connections $\nabla^{S_x}, \nabla^{S'_x}$. These partial connections are different (at x) due to the claim (3) of Lemma 2, i.e., $\nabla^{S'_x} = \nabla^{S_x} + F$ holds according to the formula (1) for $F(x) \neq 0$. This means that the linear map $\mathcal{H}_x \to \mathcal{H}_x$ given by

$$\eta \mapsto (F(\xi)\eta + F(\eta)\xi - \tilde{h}(\xi, \eta)\tilde{h}^{-1}(F))(x) \tag{4}$$

defines a non–zero element $\xi(F)(x)$ of a Lie algebra $\mathfrak{csu}(p, q)$ for each $\xi \in \mathcal{H}_x$, where we identify $\mathfrak{csu}(p, q)$ with

$$\{X \in \mathfrak{gl}(\mathcal{H}_x) : [X, J_x] = 0, h_x(X(\xi), \nu) + h_x(\xi, X(\nu)) = a \cdot h_x(\xi, \nu), a \in \mathbb{R}\}.$$

Moreover, the element $\xi(F)(x)$ of $\mathfrak{csu}(p, q)$ has to act trivially on $W(x)$ for all vectors ξ. Let us denote by $\mathfrak{ann}(W_x)$ the set of all $A \in \mathfrak{csu}(p, q)$ such that A acts trivially on $W(x)$. Then we get

$$F(x) \in \mathfrak{ann}(W_x)^{(1)} := \{F : \xi(F)(x) \in \mathfrak{ann}(W_x) \text{ for all } \xi \in \mathcal{H}_x\}.$$

The result of [11] states that if $W(x)$ is non–trivial, then $\mathfrak{ann}(W_x)^{(1)} = 0$, and thus $\xi(F)(x) = 0$ for all $\xi \in \mathcal{H}_x$. Since $\xi(-)(x) : \mathcal{H}^*_x \to \mathfrak{csu}(p, q)$ is a linear map at each $x \in M$, this implies $F(x) = 0$, which is a contradiction. This proves the uniqueness of symmetries at x in the case $W(x) \neq 0$.

Since $S_x \circ S_y \circ S_x$ is a symmetry at $S_x(y)$, the condition $S_x \circ S_y \circ S_x = S_{S_x(y)}$ trivially follows from the uniqueness of symmetries. Thus it remains to prove the smoothness of S.

Let us fix a partial Weyl connection ∇. For each $y \in M$, there is $F(y)$ such that $(\nabla^{S_y} - \nabla)(y) = F(y)$ by the formula (1), which is well–defined due to the uniqueness of ∇^{S_y} at y. Thus $\nabla W(y)$ is given by the algebraic action (4) of $\xi(F(y))$ on $W(y)$ for each $\xi \in \mathcal{H}_y$. Since $\nabla W(y)$ is smooth, the image of $\xi(F(y))$ in $\mathfrak{csu}(p, q)$ depends smoothly on y for each $\xi \in \mathcal{H}_y$. Since the kernel of the action coincides with $\mathfrak{ann}(W_y)^{(1)}$, we conclude that $F(y)$ depends smoothly on y.

Let D be an arbitrary Weyl connection inducing the partial Weyl connection ∇. Then S_y is linear in the normal coordinates of the normal Weyl connection $\bar{D}$ constructed for $D + \frac{1}{2}F(y)$ due to the claim (3) of Lemma 2. Since $\bar{D}$ depends smoothly on y, we get that S is smooth.

It clearly holds that $S_x(x) = x$ and $S_x^2 = \mathrm{id}$ for all $x \in M$. We have proved that S is smooth and satisfies $S_x \circ S_y = S_{S_x(y)} \circ S_x^{-1} = S_{S_x(y)} \circ S_x$ for all $x, y \in M$. Thus it follows that (M, S) satisfies the conditions of the reflexion space. □

Proposition 1 has the following consequence.

Proposition 2. *Suppose that $(M,\mathcal{H},J)$ is a symmetric CR geometry. Then either*

(1) *$W=0$ and the CR geometry is locally equivalent to the standard model, or*
(2) *$W\neq 0$ and the CR symmetries generate a transitive Lie group of CR transformations of M. In particular, the CR manifold $(M,\mathcal{H},J)$ and the reflection space (M,S) from Proposition 1 are homogeneous.*

Proof. Suppose that $U\subset M$ consists of all points with non–trivial W. It suffices to prove that the group generated by symmetries at points in U acts transitively on U to obtain the claim of the Theorem, because then W is constant on U due to the homogeneity. The fact that the group generated by symmetries on a reflexion space is a Lie group can be found in [12].

Let $c(t)$ be a curve in U such that $c(0)=x$ and $\frac{d}{dt}|_{t=0}c(t)=X\in\mathcal{H}_x$. Then $\frac{d}{dt}|_{t=0}S_{c(t)}(x)$ is tangent to the orbit of the action of the group generated by symmetries at points in U. Differentiation of the equality $c(t)=S_{c(t)}c(t)$ gives

$$X=\frac{d}{dt}|_{t=0}S_{c(t)}(c(t))=\frac{d}{dt}|_{t=0}S_{c(t)}(x)+T_xS_x.X,$$

and we get

$$\frac{d}{dt}|_{t=0}S_{c(t)}(x)=X-T_xS_x.X=2X.$$

Thus at all $x\in U$, the CR distribution $\mathcal{H}$ is tangent to the orbit of the group generated by symmetries at points in U. Therefore the group generated by symmetries at points in U acts transitively on U. $\square$

Flat symmetric CR geometries do not have to be homogeneous. We construct an explicit example in Section 6.

4. Non–flat symmetric CR geometries

4.1. **Homogeneous CR geometries and their symmetries.** There are several possible ways of describing a homogeneous CR geometry. We will use the description from [2, Section 1.5.15] that is closely related to the setting of Cartan geometries, but as we show in this section, it can be treated independently of the general theory. We need only to recall that the Lie algebra $\mathfrak{su}(p+1,q+1)$ of $PSU(p+1,q+1)$ consists of the $(1,n,1)$–block matrices

$$\begin{pmatrix} a & Z & iz\\ X & A & -IZ^*\\ ix & -X^*I & -\bar{a}\end{pmatrix},$$

where $\mathfrak{csu}(p,q)=\{(a,A):a\in\mathbb{C},\ A\in\mathfrak{u}(p,q),\ a+tr(A)-\bar{a}=0\}$, $X\in\mathbb{C}^n$, $Z\in\mathbb{C}^{n*}$, $x\in\mathbb{R}$ and $z\in\mathbb{R}^*$. This means that we have the following decomposition

$$\mathfrak{su}(p+1,q+1)=\mathbb{R}\oplus\mathbb{C}^n\oplus\mathfrak{csu}(p,q)\oplus\mathbb{C}^{n*}\oplus\mathbb{R}^*.$$

The Lie algebra $\mathfrak{p}$ of P corresponds to $(1,n,1)$–block upper triangular part and decomposes as $\mathfrak{p}=\mathfrak{csu}(p,q)\oplus\mathbb{C}^{n*}\oplus\mathbb{R}^*$. In fact, $P\cong CSU(p,q)\exp(\mathbb{C}^{n*}\oplus\mathbb{R}^*)$, where $CSU(p,q)$ consists of all elements of P whose adjoint action preserves the above decomposition.

Lemma 4. *Let K be an arbitrary transitive Lie group of CR transformations of a homogeneous CR geometry $(M,\mathcal{H},J)$ and let $L\subset K$ be the stabilizer of a point. Then there is a pair of maps (α,i) such that i is an injective Lie group homomorphism $i:L\to P$ and α is a linear map $\alpha:\mathfrak{k}\to\mathfrak{su}(p+1,q+1)$ satisfying the following conditions:*

(1) *$\alpha:\mathfrak{k}\to\mathfrak{su}(p+1,q+1)$ is a linear map extending $T_ei:\mathfrak{l}\to\mathfrak{p}$,*

(2) *α induces an isomorphism $\underline{\alpha}: \mathfrak{k}/\mathfrak{l} \to \mathfrak{su}(p+1, q+1)/\mathfrak{p}$ of vector spaces,*
(3) *$\mathrm{Ad}(i(l)) \circ \alpha = \alpha \circ \mathrm{Ad}(l)$ holds for all $l \in L$,*
(4) *the linear map $\wedge^2\mathfrak{k} \to \mathfrak{su}(p+1, q+1)$ given on $X \wedge Y$ by $[\alpha(X), \alpha(Y)] - \alpha([X, Y])$ for all $X, Y \in \mathfrak{k}$ takes values in $\mathfrak{p}$ and defines a K–invariant two–form κ on M with values in $K \times_{\mathrm{Ad}\circ i} \mathfrak{p}$,*
(5) *the component of κ in $K \times_{\underline{\mathrm{Ad}}\circ i} \mathfrak{csu}(p, q)$ is a tensor that coincides with W, where $\underline{\mathrm{Ad}}$ is the induced action of P on $\mathfrak{csu}(p, q) \cong \mathfrak{p}/(\mathbb{C}^{n*} \oplus \mathbb{R}^*)$.*

Conversely, suppose that (α, i) is such pair of maps from (K, L) to $(PSU(p+1, q+1), P)$. Then there is a K–homogeneous CR geometry $(K/L, \mathcal{H}, J)$ satisfying $\mathcal{H}_{eL} = \alpha^{-1}(\mathbb{C}^n \oplus \mathfrak{p})/\mathfrak{l}$ and $J_{eL} = \underline{\alpha}^*(J)$, where J is the complex structure on $\mathbb{C}^n$.

A pair (α, i) satisfying conditions (1)–(3) of Lemma 4 is usually called an *extension* of (K, L) to $(PSU(p+1, q+1), P)$. The two–form κ from condition (4) is the curvature of the Cartan connection given by the extension (α, i). Finally, the normalization condition (5) on the curvature κ can also be expressed by the vanishing ($\partial^*\kappa = 0$) of Kostant's codifferential [2, Section 3.1.11].

Proof. It is shown in [2, Section 1.5.15] that each homogeneous Cartan (and thus parabolic) geometry can be described by a particular extension and that each extension determines a homogeneous Cartan geometry. The formula for κ in condition (4) is obtained from [2, Section 1.5.16]. Therefore, it follows from the description of CR geometries in [2, Section 4.2.4] that conditions (4) and (5) on the curvature κ have to be satisfied. □

Definition 2. The pair (α, i) from Lemma 4 is called a *normal extension* of (K, L) to $(PSU(p+1, q+1), P)$ describing the homogeneous CR geometry $(M, \mathcal{H}, J)$.

Examples of normal extensions describing homogeneous CR geometries and the explicit formula from condition (5) of Lemma 4 can be found in [4].

It is clear from the second part of Lemma 4 that the maps i and $\underline{\alpha}$ are sufficient to determine a CR geometry. This means that there are many normal extensions (α, i) of (K, L) to $(PSU(p+1, q+1), P)$ describing the same CR geometry. The other parts of α are completely determined by condition (5) from Lemma 4 and carry the information about Weyl connections. The remaining freedom (for fixed i) is in the choice of a complex basis of $\underline{\alpha}^{-1}(\mathbb{C}^n)$. In general, if $h \in P$, then the pair $(\mathrm{Ad}(h)\circ\alpha, conj(h)\circ i)$ is also a normal extension of (K, L) to $(PSU(p+1, q+1), P)$ describing the same CR geometry.

Let us summarize the results characterizing symmetric non–flat homogeneous CR geometries following from [7, 8].

Proposition 3. *Let K be the Lie group of all CR transformations of a non–flat homogeneous CR geometry $(M, \mathcal{H}, J)$. Then the following is equivalent:*

(1) *There is a (unique) symmetry at each point.*
(2) *There is $s \in L$ such that the triple (K, L, s) is a (non–prime) homogeneous reflexion space, i.e.,*
   - *s commutes with all elements of L,*
   - *$s^2 = e$, where e is the identity element of L, and*
   - *all symmetries are of the form $S_{kL} = ksk^{-1}$ for $k \in K$.*
(3) *There is a normal extension (α, i) of (K, L) to $(PSU(p+1, q+1), P)$ describing $(M, \mathcal{H}, J)$ such that $i(L) \subset CSU(p, q)$ and $s_{0,0} \in i(L)$ (see the formula (2)).*
(4) *For each normal extension (α, i) of (K, L) to $(PSU(p+1, q+1), P)$ describing $(M, \mathcal{H}, J)$, there is a (unique) $Z \in \mathbb{C}^{n*}$ such that $\mathrm{Ad}(\exp(Z))\alpha(\mathfrak{k})$ is preserved by $\mathrm{Ad}(s_{0,0})$, and the Lie algebra automorphism of $\mathfrak{k}$ given by $\mathrm{Ad}(s_{0,0})$ defines an automorphism of the Lie group K.*

The condition (3) of Proposition 3 immediately implies that there are K–invariant Weyl connections on a symmetric non–flat CR geometry $(M,\mathcal{H},J)$. According to [2, Proposition 1.4.8], a K–invariant connection on $T(K/L)$ can be described by a map $\gamma:\mathfrak{k}\to\mathfrak{gl}(\mathfrak{k}/\mathfrak{l})$ such that

- $\gamma|_{\mathfrak{l}}=\underline{\mathrm{ad}}$, and
- $\gamma(\mathrm{Ad}(h)(X))=\underline{\mathrm{Ad}}(h)\circ\gamma(X)\circ\underline{\mathrm{Ad}}(h)^{-1}$

hold for all $X\in\mathfrak{k}$ and $h\in L$, where $\underline{\mathrm{Ad}}:L\to Gl(\mathfrak{k}/\mathfrak{l})$ is induced by the adjoint representation.

Proposition 4. *Let K be the Lie group of all CR transformations of a non–flat symmetric CR geometry $(M,\mathcal{H},J)$. Let (α,i) be a normal extension of (K,L) to $(PSU(p+1,q+1),P)$ describing $(M,\mathcal{H},J)$ such that $i(L)\subset CSU(p,q)$ and $s_{0,0}\in i(L)$. Then $\gamma:=\underline{\alpha}^*(\underline{\mathrm{ad}}\circ r_0)$ describes a K–invariant Weyl connection, where $r_0:\mathfrak{su}(p+1,q+1)\to\mathfrak{csu}(p,q)$ is the projection along $\mathbb{R}\oplus\mathbb{C}^n\oplus\mathbb{C}^{n*}\oplus\mathbb{R}^*$.*

In particular, there is a bijection between the set of K–invariant Weyl connections on M and the set of $z\in\mathbb{R}^*$ such that $conj(\exp(z))\circ i(L)\subset CSU(p,q)$ holds for the extension (α,i).

Proof. We proved the existence of K–invariant Weyl connections on non–flat symmetric CR geometries in [8]. Therefore it is enough to check that they can be described by the functions γ. Since $i(L)\subset CSU(p,q)$, the projection r_0 is $i(L)$–equivariant and $\gamma|_{\mathfrak{l}}=\underline{\mathrm{ad}}$ holds. Therefore each γ describes a K–invariant connection. The fact that this is a Weyl connection follows directly from condition (5) in Lemma 4.

It is clear that the one–form $\Upsilon_1+\Upsilon_2$ measuring the "difference" between two K–invariant Weyl connections is given by an $i(L)$–invariant element of $\mathbb{C}^{n*}\oplus\mathbb{R}^*$. Since $s_{0,0}\in i(L)$, it has to be an element of $\mathbb{R}^*$. It is clear that $z\in\mathbb{R}^*$ is $i(L)$–invariant element if and only if $conj(\exp(z))\circ i(L)\subset CSU(p,q)$ holds. $\square$

4.2. **Groups generated by symmetries.** The following Theorem significantly improves the characterization of non–flat symmetric homogeneous CR geometries given by Propositions 2 and 3.

Theorem 1. *Let K be the Lie group generated by all symmetries of a non–flat symmetric CR geometry $(M,\mathcal{H},J)$. Let (α,i) be a normal extension of (K,L) to $(PSU(p+1,q+1),P)$ describing the CR geometry that satisfies $i(s)=s_{0,0}$ and $i(L)\subset CSU(p,q)$. Denote by $\mathfrak{h}$ the 1–eigenspace of s in $\mathfrak{k}$ and by $\mathfrak{m}$ the -1–eigenspace of s in $\mathfrak{k}$. Then:*

1. *The following conditions hold*
   - $\alpha(\mathfrak{l})\subset\mathfrak{u}(p,q)$,
   - $\alpha(\mathfrak{m})\subset\mathbb{C}^n\oplus\mathbb{C}^{n*}$, *and*
   - $\alpha(\mathfrak{h})\subset\mathbb{R}\oplus\mathfrak{csu}(p,q)\oplus\mathbb{R}^*$ *is a Lie subalgebra.*
2. *There is a basis of $\mathfrak{h}/\mathfrak{l}\oplus\mathfrak{m}$ such that for a vector in $\mathfrak{h}/\mathfrak{l}\oplus\mathfrak{m}$ with coordinates (x,X) holds*
$$\alpha((x,X)+\mathfrak{l})=\mathrm{Ad}(\exp(z))\circ\begin{pmatrix} aix & \mathsf{P}_1(X) & \mathsf{P}_2(x)i\\ X & Ax & -I\mathsf{P}_1(X)^*\\ xi & -X^*I & aix\end{pmatrix}+\alpha(\mathfrak{l}),$$
where $z\in\mathbb{R}^*$, $\mathsf{P}_1:\mathbb{C}^n\to\mathbb{C}^{n*}$, $\mathsf{P}_2:\mathbb{R}\to\mathbb{R}^*$ and $(a,A)\in\mathfrak{u}(p,q)$ normalizes $\alpha(\mathfrak{l})$.
3. *The maps $\mathsf{P}_1,\mathsf{P}_2$ and the matrix (a,A) are completely determined by condition (5) from Lemma 4.*

Proof. We know from Proposition 3 that there exists a normal extension (α,i) of (K,L) to $(PSU(p+1,q+1),P)$ satisfying our assumptions.

Consider the canonical decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is 1–eigenspace of s and $\mathfrak{m}$ is -1–eigenspace of s. Then $\alpha(\mathfrak{m}) \subset \mathbb{C}^n \oplus \mathbb{C}^{n*}$ and $\alpha(\mathfrak{h}) \subset \mathbb{R} \oplus \mathfrak{csu}(p,q) \oplus \mathbb{R}^*$ follow from the assumption $i(s) = s_{0,0}$ and $\alpha(\mathfrak{h})$ is a Lie subalgebra, because $dim(\mathfrak{h}/\mathfrak{l}) = 1$. We can identify $\mathfrak{m}$ with $\mathbb{C}^n$ via α, because the restriction of α to the map $\mathfrak{m} \to \mathbb{C}^n$ is injective. Indeed, if the restriction is not injective, then the elements in its kernel would be another symmetries at eL, but we know that there is only one symmetry. This identification uniquely determines the map $i : L \to CSU(p,q)$.

Further, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ holds and we have the corresponding symmetric space K/H^0, where H^0 is the connected component of identity of the fixed point set of the conjugation by s. Therefore $\exp([X,Y]) \in H^0$ for each $X, Y \in \mathfrak{m}$. The map $\mathrm{Ad} : H^0 \to GL(\mathfrak{m})$ can be restricted to the connected component of identity L^0 of L and the restriction coincides with i. Therefore, it suffices to show that the element $\mathrm{ad}([X,Y]) \in \mathfrak{gl}(\mathfrak{m})$ belongs to $\mathfrak{sl}(\mathfrak{m})$ for all $X, Y \in \mathfrak{m}$. But we have $\mathrm{ad}([X,Y]) = \mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)$ and the trace equals to

$$tr(\mathrm{ad}([X,Y])) = tr(\mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)) = B(X,Y) - B(Y,X),$$

where B denotes the Killing form, which is symmetric. Therefore $i(L^0) \subset U(p,q)$ and $T_e i(\mathfrak{l}) \subset \mathfrak{u}(p,q)$. In particular, the claim (1) holds. The map α can be expressed as in the claim (2), because there is always $z \in \mathbb{R}^*$ such that the extension $(\mathrm{Ad}(\exp(-z)) \circ \alpha, conj(\exp(-z)) \circ i)$ satisfies

$$\mathrm{Ad}(\exp(-z)) \circ \alpha((x,0) + \mathfrak{l}) = \begin{pmatrix} aix & 0 & \mathsf{P}_2(x)i \\ 0 & Ax & 0 \\ xi & 0 & aix \end{pmatrix} + \mathrm{Ad}(\exp(-z)) \circ \alpha(\mathfrak{l}).$$

Since the CR geometry $(M, \mathcal{H}, J)$ does not depend on parts $\mathsf{P}_1, \mathsf{P}_2$ and (a, A) of α, these parts are completely determined by condition (5) from Lemma 4. □

Let us remark that although the Lie algebra homomorphism i is uniquely determined by the isomorphism $\mathfrak{m} \cong \mathbb{C}^n$ given by α, the converse is not true. See [4] for examples of non–equivalent CR geometries described by extensions with the same Lie group homomorphism i.

Let us further remark that we are not aware of any example of an extension (α, i) of (K, L) to $(PSU(p+1, q+1), P)$ where $z \in \mathbb{R}^*$ from the claim (2) of Theorem 1 does not correspond to an invariant Weyl connection. The main reason for this is the following result.

Proposition 5. *Suppose that* $\mathrm{Ad}(L^0)|_{\mathfrak{h}/\mathfrak{l}} = \mathrm{Ad}(L)|_{\mathfrak{h}/\mathfrak{l}}$. *Then* $i(L) \subset U(p,q)$ *and there is a bijection between* $\mathbb{R}^*$ *and the set of* K*–invariant Weyl connections. In particular, there is a unique* K*–invariant Weyl connection corresponding to the normal extension* (α, i) *satisfying* $i(L) \subset U(p,q)$, $i(s) = s_{0,0}$ *and*

$$\alpha((x,X) + \mathfrak{l}) = \begin{pmatrix} aix & \mathsf{P}_1(X) & \mathsf{P}_2(x)i \\ X & Ax & -I\mathsf{P}_1(X)^* \\ xi & -X^*I & aix \end{pmatrix} + \alpha(\mathfrak{l}). \tag{5}$$

This particularly holds when the transitive group K *is semisimple.*

Proof. If $\mathrm{Ad}(L^0)|_{\mathfrak{h}/\mathfrak{l}} = \mathrm{Ad}(L)|_{\mathfrak{h}/\mathfrak{l}}$, then $i(L) \subset U(p,q)$ holds and the claim follows. It follows from the classification of semisimple symmetric spaces that H is reductive and there is a complement to $\mathfrak{l}$ in the center of $\mathfrak{h}$. Consequently $\mathrm{Ad}(L^0)|_{\mathfrak{h}/\mathfrak{l}} = \mathrm{Ad}(L)|_{\mathfrak{h}/\mathfrak{l}}$. □

4.3. **Relations to CR algebras.** We explain here relations between our concept and the concept of CR algebras introduced in [1]. We denote here by $\mathfrak{n}_{\mathbb{C}}$ the complexification of a Lie algebra $\mathfrak{n}$.

Let (α, i) be an extension of (K, L) to $(PSU(p+1, q+1), P)$. We complexify the linear map α to obtain a map

$$\alpha_{\mathbb{C}} : \mathfrak{k}_{\mathbb{C}} \to \mathfrak{sl}(n+2, \mathbb{C}).$$

The Lie algebra $\mathfrak{sl}(n+2, \mathbb{C})$ decomposes as

$$\mathfrak{sl}(n+2, \mathbb{C}) = \mathbb{C} \oplus (\mathbb{C}^n \oplus \mathbb{C}^{n*}) \oplus (\mathfrak{gl}(n, \mathbb{C}) \oplus \mathbb{C}) \oplus (\mathbb{C}^{n*} \oplus \mathbb{C}^n) \oplus \mathbb{C},$$

where $\mathfrak{p}_{\mathbb{C}} = (\mathfrak{gl}(n, \mathbb{C}) \oplus \mathbb{C}) \oplus (\mathbb{C}^{n*} \oplus \mathbb{C}^n) \oplus \mathbb{C}$. The subspace $\mathbb{C}^{n*} \oplus \mathfrak{p}_{\mathbb{C}}$ is a parabolic subalgebra of $\mathfrak{sl}(n+2, \mathbb{C})$ that satisfies

$$\mathcal{H}^{0,1}_{eL} = \alpha_{\mathbb{C}}^{-1}(\mathbb{C}^{n*} \oplus \mathfrak{p}_{\mathbb{C}})/\mathfrak{l}_{\mathbb{C}}.$$

Therefore, the preimage $\mathfrak{q}$ of $\mathcal{H}^{0,1}_{eL}$ in $\mathfrak{k}_{\mathbb{C}}$ is a Lie subalgebra of the form

$$\mathfrak{q} = \alpha_{\mathbb{C}}^{-1}(\mathbb{C}^{n*} \oplus \mathfrak{p}_{\mathbb{C}}).$$

The pair $(\mathfrak{k}, \mathfrak{q})$ satisfies conditions of a *CR algebra* from [1, Section 1.2.]. It is proved in [1] that this is the minimal set of data describing a CR geometry on the homogeneous space K/L. However, CR algebras do not provide as much information as the extension (α, i). In particular, we cannot obtain directly the curvature κ of the corresponding Cartan connection from the CR algebra. Therefore, it is not easy to distinguish whether two CR algebras correspond to equivalent CR geometries.

There are conditions in [1, Section 1.4] that characterize CR algebras of CR geometries that are symmetric in the sense of [10]. One of these conditions ensures that there is a Riemannian metric compatible with a CR geometry. Other conditions are analogous to condition (4) of Proposition 3 which says that the Lie algebra automorphism of $\mathfrak{k}$ given by $\mathrm{Ad}(s_{0,0})$ defines an automorphism of the Lie group K.

There is the following method to check whether CR geometries corresponding to CR algebras $(\mathfrak{k}, \mathfrak{q})$ are symmetric (in our sense) and to construct the normal extensions (α, i) that describe them.

(1) We consider $\mathfrak{l} = \mathfrak{k} \cap \mathfrak{q} \cap \bar{\mathfrak{q}}$ and $\mathcal{H}_{eL} = (\mathfrak{k}/\mathfrak{l}) \cap ((\mathfrak{q} + \bar{\mathfrak{q}})/(\mathfrak{q} \cap \bar{\mathfrak{q}}))$, where $\bar{\mathfrak{q}}$ is the subalgebra conjugated to $\mathfrak{q}$ in $\mathfrak{k}_{\mathbb{C}}$.

(2) We choose a complex basis of $\mathcal{H}_{eL}$. This choice defines a Lie algebra homomorphism $\mathfrak{l} \to \mathfrak{csu}(p, q)$ and the following facts hold:
$(2a)$ If this homomorphism is not injective, then the CR geometry is flat (we will discuss this situation later).
$(2b)$ If this homomorphism is injective and the CR geometry is symmetric, then it coincides with the restriction of α to $\mathfrak{l}$ for some normal extension (α, i) describing the CR geometry.
$(2c)$ If this homomorphism is injective and the CR geometry is not symmetric, then the homomorphism corresponds only to associated graded map corresponding to the restriction of α to $\mathfrak{l}$ for some normal extension (α, i) describing the CR geometry.

(3) Each choice of representatives (in $\mathfrak{k}$) of the complex basis of $\mathcal{H}_{eL}$ from (2) together with a choice of an element of $\mathfrak{k}$ complementary to $\mathcal{H}_{eL}$ allows us to define
$(3a)$ a linear map α of the form (5) from Proposition 5 for (at this point) unknown linear maps $a, A, \mathsf{P}_1, \mathsf{P}_2$,
$(3b)$ a linear map $\tau : \wedge^2\mathfrak{k} \to \mathfrak{su}(p+1, q+1)$ given for all $X, Y \in \mathfrak{k}$ by the formula

$$\tau(X, Y) := [\alpha(X), \alpha(Y)] - \alpha([X, Y]),$$

$(3c)$ a linear map $\nu : \mathfrak{k} \to \mathfrak{k}$ such that ν equals to

- $-\mathrm{id}$ on the representatives (in $\mathfrak{k}$) of complex basis of $\mathcal{H}_{eL}$, and
- id on the element of $\mathfrak{k}$ complementary to $\mathcal{H}_{eL}$ and on $\mathfrak{l}$.

Moreover, we consider only the choices that satisfy the equivalent conditions of the following statement.

Proposition 6. *The map ν is a Lie algebra automorphism of $\mathfrak{k}$ if and only if the components*

$$(\mathbb{R} \oplus \alpha(\mathfrak{l})) \otimes \mathbb{C}^n \to \mathbb{R} \oplus \mathfrak{csu}(p,q) \oplus \mathbb{R}^*, \quad \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^{n*}$$

of τ vanish for all linear maps $a, A, \mathsf{P}_1, \mathsf{P}_2$.

Proof. A consequence of the formula for τ is that ν is a Lie algebra automorphism of $\mathfrak{k}$ if and only if

$$\mathrm{Ad}(s_{0,0})\tau(\nu(X), \nu(Y)) = \tau(X, Y), \quad \mathrm{Ad}(s_{0,0})[\alpha(\nu(X)), \alpha(\nu(X))] = [\alpha(X), \alpha(Y)]$$

hold for all $X, Y \in \mathfrak{k}$. If α is of the form (5), then

$$\mathrm{Ad}(s_{0,0})[\alpha(\nu(X)), \alpha(\nu(X))] = [\alpha(X), \alpha(Y)]$$

holds for all $X, Y \in \mathfrak{k}$ and all linear maps $a, A, \mathsf{P}_1, \mathsf{P}_2$, and

$$\mathrm{Ad}(s_{0,0})\tau(\nu(X), \nu(Y)) = \tau(X, Y)$$

holds for all $X, Y \in \mathfrak{k}$ if and only if the claimed components vanish. □

(4) There are the following possibilities for the choice in the step (3).
(4*a*) If there is no choice such that ν is a Lie algebra automorphism of $\mathfrak{k}$, then the CR geometry corresponding to the CR algebra $(\mathfrak{k}, \mathfrak{q})$ is not symmetric.
(4*b*) If there is a choice such that ν is a Lie algebra automorphism of $\mathfrak{k}$, then the CR geometry corresponding to the CR algebra $(\mathfrak{k}, \mathfrak{q})$ is symmetric if and only if ν induces a Lie group automorphism of K and L is contained in fixed point set of ν.

(5) We require from now that the CR geometry corresponding to the CR algebra $(\mathfrak{k}, \mathfrak{q})$ is symmetric. The remaining step is to determine the choice of an element of $\mathfrak{k}$ complementary to $\mathcal{H}_{eL}$ and $i : L \to P$ such that (α, i) is a normal extension describing the CR geometry. We know that there is a choice such that $(\mathrm{Ad}(\exp(z)) \circ \alpha, i')$ is an extension for some $z \in \mathbb{R}^*$, where the Lie group homomorphisms $i' : L \to P$ is induced by (adjoint) action of L on $\mathcal{H}_{eL}$ and $\alpha(\mathfrak{l})$. Thus it suffices to check the vanishing of components

$$\alpha(\mathfrak{l}) \otimes \mathbb{R} \to \mathfrak{su}(p+1, q+1), \quad \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{R}$$

of τ for all linear maps $a, A, \mathsf{P}_1, \mathsf{P}_2$. The condition (5) of Lemma 4 provides linear equations that determine uniquely the linear maps $a, A, \mathsf{P}_1, \mathsf{P}_2$ for which the extension (α, i) is normal.

4.4. **Example of non–flat symmetric CR geometries.** Consider the Lie group $E(2) = \mathbb{R}^2 \rtimes SO(2)$ of isometries of Euclidean plane. There is the following normal extension (α, i) of $(E(2), \{\mathrm{id}\})$ to $(PSU(1,2), P)$ of the form

$$\alpha \begin{pmatrix} 0 & 0 & 0 \\ \frac{x}{2} & 0 & -X_1 \\ X_2 & X_1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{ix}{16} & -\frac{5}{16}X_1 - \frac{3i}{16}X_2 & -\frac{15ix}{256} \\ X_1 + iX_2 & -\frac{ix}{8} & \frac{5}{16}X_1 - \frac{3i}{16}X_2 \\ ix & -X_1 + iX_2 & \frac{ix}{16} \end{pmatrix}, \tag{6}$$

where the choice of the basis of the Lie algebra of $\mathbb{R}^2 \oplus \mathfrak{so}(2) = \langle x, X_2 \rangle \oplus \langle X_1 \rangle$ reflects the convention from Section 4.2, i.e., $(x, (X_1, X_2))$ are the distinguished coordinates from Theorem 1. Indeed, since i is trivial and

$$\tau\left((x, (X_1, X_2)), (y, (Y_1, Y_2))\right) =$$
$$\begin{pmatrix} 0 & \frac{3i}{32}yX_1 - \frac{3}{32}yX_2 - \frac{3i}{32}xY_1 + \frac{3}{32}xY_2 & 0 \\ 0 & 0 & \frac{3i}{32}yX_1 + \frac{3}{32}yX_2 - \frac{3i}{32}xY_1 - \frac{3}{32}xY_2 \\ 0 & 0 & 0 \end{pmatrix}$$

holds for the linear map τ determining the curvature κ, it follows that (α, i) is a normal extension describing a non–flat symmetric CR geometry.

In fact, any invertible linear map $B : \mathbb{R}^2 \oplus \mathfrak{so}(2) \to \mathbb{R} \oplus \mathbb{C}$ defines a CR algebra $(\mathfrak{k}, \mathfrak{q})$ for
$$\mathfrak{q} = B_{\mathbb{C}}^{-1}(\mathbb{C}^* \oplus \mathfrak{p}_{\mathbb{C}})$$
and we ask the following question: Which maps B correspond to non–equivalent non–degenerate symmetric CR geometries of hypersurface type on the Lie group $E(2)$ of isometries of Euclidean plane?

We give the answer to this question (using the algorithm from previous section and [4, Lemma 3.5]) in the following statement.

Proposition 7. *The normal extension* (α, i) *of the form* (6) *describes the unique (up to equivalence) non–degenerate symmetric CR geometry of hypersurface type on the Lie group* $E(2)$.

Proof. Consider an invertible linear map $B : \mathbb{R}^2 \oplus \mathfrak{so}(2) \to \mathbb{R} \otimes \mathbb{C}^2$. The construction of the objects from the algorithm is clear in this case. We need to find for which maps B the components
$$\mathbb{R} \otimes \mathbb{C} \to \mathbb{R} \oplus \mathfrak{csu}(1) \oplus \mathbb{R}^*, \quad \mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}^*, \quad \mathbb{C} \otimes \mathbb{C} \to \mathbb{R}$$
of τ vanish for all linear maps $a, A, \mathsf{P}_1, \mathsf{P}_2$. In fact, this provides three equations on the entries of the matrix B that can be solved explicitly. In the standard basis of $\mathbb{R}^2 \oplus \mathfrak{so}(2)$ and $\mathbb{R} \otimes \mathbb{C}$, the inverses of matrices B that satisfy these equations define the following subvariety:
$$\begin{pmatrix} \frac{p_1 p_2 - p_3 p_4}{2} & p_5 & \frac{p_5 p_3 - 2p_6}{2} \\ p_6 & p_4 & p_2 \\ 0 & p_1 & p_3 \end{pmatrix}. \tag{7}$$
Thus it remains to check the action of morphisms from [4, Lemma 3.5] that determine which extensions define equivalent CR geometries. In particular, there are

- four–dimensional Lie group of derivations of $\mathbb{R}^2 \oplus \mathfrak{so}(2)$ that in addition contains the homothethies, and
- two–dimensional Lie subgroup that forms center of $CSU(p, q)$.

We compute that the induced action of these morphisms on the six–dimensional variety (7) is transitive and the matrix
$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
corresponds to the extension (6). □

5. Metrizability and CR embeddings

In this section, we always consider the K–invariant Weyl connection D corresponding to a normal extension (α, i) describing a homogeneous CR geometry $(M, \mathcal{H}, J)$ that satisfies $i(L) \subset U(p, q)$, $i(s) = s_{0,0}$ and
$$\alpha((x, X) + \mathfrak{l}) = \begin{pmatrix} aix & \mathsf{P}_1(X) & \mathsf{P}_2(x) \\ X & Ax & -I\mathsf{P}_1(X)^* \\ x & -X^* I & aix \end{pmatrix} + \alpha(\mathfrak{l}).$$
Moreover, we always assume $\mathrm{Ad}(L^0)|_{\mathfrak{h}/\mathfrak{l}} = \mathrm{Ad}(L)|_{\mathfrak{h}/\mathfrak{l}}$, where L^0 is the identity component of L. This gives almost no restriction, because this condition is always satisfied on the symmetric CR geometry on the covering $K^0/L^0 \to K/L$, where K^0 is the identity component of K.

5.1. **Distinguished metrics compatible with the CR geometry.** The symmetric bilinear form h generally does not define a pseudo–Riemannian metric on $\mathcal{H}$, because there is no natural way, how to measure the length of elements of $TM/\mathcal{H}$. The situation is different, if there is a Weyl connection preserving not only the decomposition $\mathcal{H} \oplus \ell$, but also a non–zero vector field r in ℓ. Such Weyl connection is called *exact* and the vector field r is called the *Reed field.* Equivalently, each exact Weyl connection corresponds to the *contact form* θ that annihilates $\mathcal{H}$ and satisfies $\theta(r) = 1$ for the Reeb field r. If there is an exact Weyl connection, then $\theta \circ h$ is a pseudo–Riemannian metric on $\mathcal{H}$. This metric is compatible with the CR–structure, because the form h satisfies $h(J\xi, J\nu) = h(\xi, \nu)$ for all sections ξ, ν of $\mathcal{H}$. The exact Weyl connection preserves this metric and the Reeb field can be used to construct a pseudo–Riemannian metric on TM, for which the connection is a metric connection. This metric is usually called a *Webster metric.* However, the Webster metric neither has to exist nor has to be compatible with the symmetries. Therefore, if we want to find a metric compatible with the CR geometry that is preserved by all symmetries, we need to show that the distinguished Weyl connection D is exact.

Theorem 2. *Let K be the Lie group generated by all symmetries of a non–flat symmetric CR geometry $(M, \mathcal{H}, J)$. Suppose that $\mathrm{Ad}(L^0)|_{\mathfrak{h}/\mathfrak{l}} = \mathrm{Ad}(L)|_{\mathfrak{h}/\mathfrak{l}}$. The distinguished Weyl connection D is exact and furthermore, there exists*

- *a K–invariant contact form θ,*
- *a K–invariant pseudo–Riemannian metric $\bar{g} := \theta \circ h$ on $\mathcal{H}$, and*
- *a K–invariant Webster metric $g := \theta \circ h + \theta \otimes \theta$ on TM*

such that

1. *$D\bar{g} = 0, Dg = 0$,*
2. *$g|_{\mathcal{H}} = \bar{g}$ and the Reeb field of D is orthogonal to $\mathcal{H}$ and has length 1,*
3. *choosing the Reeb field of D as a trivialization of $(TM/\mathcal{H}) \otimes \mathbb{C}$, the pseudo–Riemannian metric $\bar{g}$ on $\mathcal{H}$ coincides with the real part of the Levi form up to a constant multiple,*
4. *the symmetry at x is linear in geodesic coordinates of D at x, reverses the directions of $\mathcal{H}_x$ and preserves the direction of the Reeb field of D at x.*

Proof. The image of α is contained in $\mathbb{R} \oplus \mathbb{C}^n \oplus \mathfrak{u}(p,q) \oplus \mathbb{C}^{n*} \oplus \mathbb{R}^*$ and thus γ describing the corresponding K–invariant Weyl connection has values in $\underline{\mathrm{ad}}(\mathfrak{u}(p,q))$. Furthermore, the assumption $\mathrm{Ad}(L^0)|_{\mathfrak{h}/\mathfrak{l}} = \mathrm{Ad}(L)|_{\mathfrak{h}/\mathfrak{l}}$ implies that $i(L) \subset U(p,q)$ and therefore the maps $\underline{\mathrm{ad}}^{-1} \circ \gamma$ and i satisfy all conditions of [2, Theorem 1.4.5]. This means that the Weyl connection D is an associated connection to a K–invariant principal connection on the bundle $K \times_{i(L)} U(p,q) \to K/L$. Therefore it is an exact Weyl connection, because its holonomy is contained in $U(p,q)$. The remaining claims then follow from general theory. □

In the Riemannian signature, Theorem 2 particularly allows to compare symmetric CR geometries (in our sense) with the symmetric CR geometries in the sense of [10], because we have found a metric compatible with the CR geometry that is preserved by all symmetries.

Theorem 3. *Suppose that $p = 0$. Then each non–flat symmetric CR geometry is covered by a symmetric CR geometry in the sense of* [10]*, where the covering is a CR map that intertwines the symmetries.*

5.2. **CR embeddings.** Consider the fiber bundle $K \times_i CSU(p,q)/U(p,q) \to K/L$. If $\mathrm{Ad}(L^0)|_{\mathfrak{h}/\mathfrak{l}} = \mathrm{Ad}(L)|_{\mathfrak{h}/\mathfrak{l}}$ holds, then this bundle is trivial, i.e.,

$$K \times_i CSU(p,q)/U(p,q) = K/L \times \mathbb{R}.$$

Let us prove the following statement:

Theorem 4. *Let K be the Lie group generated by all symmetries of a non–flat symmetric CR geometry $(M, \mathcal{H}, J)$. Suppose that $\mathrm{Ad}(L^0)|_{\mathfrak{h}/\mathfrak{l}} = \mathrm{Ad}(L)|_{\mathfrak{h}/\mathfrak{l}}$. Then:*

1. *the manifold $K/L \times \mathbb{R}$ is a complex manifold, and*
2. *the inclusion $K/L \to K/L \times \mathbb{R}$ given as a zero section is a CR embedding.*

Proof. We need some more details from the theory of Cartan geometries from [2, Sections 1.5.13 and 3.1.2] to proceed with the proof. First, there is a natural complement of $\mathfrak{u}(p,q)$ in $\mathfrak{csu}(p,q)$ given by so–called grading element, which is the unique element $Z \in \mathfrak{csu}(p,q)$ acting by -2 on $\mathbb{R}$, -1 on $\mathbb{C}^n$, 0 on $\mathfrak{csu}(p,q)$, 1 on $\mathbb{C}^{n*}$ and 2 on $\mathbb{R}^*$. Furthermore, there is a Cartan connection on $K/L \times \mathbb{R}$ induced by the CR geometry, where we identify $\mathbb{R}$ (via exp) with the multiples of the grading element Z. Then the Weyl connection D provides a reduction of this Cartan connection to $U(p,q)$, which allows us to identify the tangent space of $K/L\times\mathbb{R}$ with the fiber bundle $(K\times\mathbb{R})\times_i(\mathbb{R}\oplus\mathbb{C}^n\oplus\mathfrak{csu}(p,q)/\mathfrak{u}(p,q))$. We can extend the complex structure on $\mathbb{C}^n$ to $\mathbb{R}\oplus\mathbb{C}^n\oplus\mathfrak{csu}(p,q)/\mathfrak{u}(p,q)$ by declaring $\mathbb{R}$ to be the imaginary part of $\mathbb{C}$ and the multiples of the grading element in $\mathfrak{csu}(p,q)/\mathfrak{u}(p,q)$ to form the real part of $\mathbb{C}$. This definition is clearly $U(p,q)$–invariant (and thus K–invariant) and defines an almost complex structure J on $K\times_i CSU(p,q)/U(p,q)$.

Let us compute the Nijenhuis tensor $[\xi,\eta]-[J\xi,J\eta]+J([J\xi,\eta]+[\xi,J\eta])$ of J for $\xi,\eta \in T(K/L\times\mathbb{R})$. For each $x \in K/L\times\mathbb{R}$, there are vector fields $\tilde{\xi},\tilde{\eta}$ such that $\tilde{\xi}(x)=\xi(x), \tilde{\eta}(x)=\eta(x)$ and that the element $[\tilde{\xi},\tilde{\eta}](x)$ is identified with the element

$$[X,Y]-[\alpha(X+\mathfrak{h}),\alpha(Y+\mathfrak{h})]+\alpha([X+\mathfrak{h},Y+\mathfrak{h}]) \mod \mathfrak{u}(p,q)\oplus\mathbb{C}^{n*}\oplus\mathbb{R}^*,$$

where $\xi(x),\eta(x)$ are identified with $X,Y \in \mathbb{R}\oplus\mathbb{C}^n\oplus\mathfrak{csu}(p,q)/\mathfrak{u}(p,q)$. This identification can be obtained using the technique analogous to [2, Proposition 3.1.8] for $T(K/L\times\mathbb{R})$ instead of $T(K/L)$. Indeed, the Cartan connection in the background remains the same and we only need to restrict ourselves to normal Weyl connections that coincide with D at x and project the results given by the Cartan connection to $T(K/L\times\mathbb{R})$ instead of $T(K/L)$. However,

$$[X,Y]-[\alpha(X+\mathfrak{h}),\alpha(Y+\mathfrak{h})]+\alpha([X+\mathfrak{h},Y+\mathfrak{h}])=[X,Y] \mod \mathfrak{u}(p,q)\oplus\mathbb{C}^{n*}\oplus\mathbb{R}^*$$

due to condition (5) from Lemma 4. Therefore we have

$$([\xi,\eta]-[J\xi,J\eta]+J([J\xi,\eta]+[\xi,J\eta]))(x) = [X,Y]-[JX,JY]+J([JX,Y]+[X,JY]).$$

Let us now discuss possible values of this expression for all possible incomes:

- For $X,Y\in\mathbb{C}^n$ we have $[X,Y]-[JX,JY]+J([JX,Y]+[X,JY])=0$.
- For $X\in\mathbb{C}^n$ and $Y=JZ\in\mathbb{R}$ we have $[X,Y]-[JX,JY]+J([JX,Y]+[X,JY])=[JX,Z]-J([X,Z])=0$.
- For $X\in\mathbb{C}^n$ and $Y=Z$ we have $[X,Y]-[JX,JY]+J([JX,Y]+[X,JY])=[X,Z]+J([JX,Z])=0$.
- For $X=JZ\in\mathbb{R}$ and $Y=Z$ we have $[X,Y]-[JX,JY]+J([JX,Y]+[X,JY])=[JZ,Z]+[Z,JZ]=0$.

The remaining possibilities vanish trivially. Thus the complex structure is integrable. Then the zero section is a CR embedding, because it is a closed orbit. □

In holomorphic coordinates on $U\subset K/L\times\mathbb{R}$, the hypersurface $K/L\cap U\subset\mathbb{C}^{n+1}$ may be described as a zero set of a function $F: U\to\mathbb{R}$. Theorem 4 and Lemma 2 provide distinguished holomorphic coordinates in which the function F has a specific form.

Corollary 1. *Let K be the Lie group generated by all symmetries of a non–flat symmetric CR geometry $(M, \mathcal{H}, J)$. Suppose that $\mathrm{Ad}(L^0)|_{\mathfrak{h}/\mathfrak{l}} = \mathrm{Ad}(L)|_{\mathfrak{h}/\mathfrak{l}}$. Then for every point $x \in M$, there is a holomorphic coordinate system on $U \subset K/L \times \mathbb{R}$ centred at x such that the function $F(z, w)$ defining M satisfies $F(z, w) = F(-z, w)$.*

6. Locally flat CR symmetric spaces

Locally flat CR geometries are always locally symmetric (in our sense). Therefore, the following question appears: Which local symmetries are globally defined? The answer depends on the topology of the manifold. We show on series of examples that various situations are possible. There are two sources of examples that we study here that are related to flag manifolds. The first series of examples follows the construction from [15, 6] that we apply to CR geometries. The second series of examples involves CR geometries on orbits of real forms in flag manifolds from [1].

In the case $p, q > 1$, it is convenient for the presentation of examples to consider the Hermitian form m given for $u, v \in \mathbb{C}^{n+2}$ by

$$m(u, v) = u^T \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I' & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \overline{v},$$

where the matrix I' is a diagonal square matrix of order $n-2$ with the first $p-1$ entries equal to 1 and the remaining $q-1$ entries equal to -1.

6.1. **Non–homogeneous symmetric CR geometries.** Let us apply the construction from [15, 6] to CR geometries. We start with the standard model $PSU(p+1, q+1)/P$. Consider the CR manifold $M := PSU(p+1, q+1)/P - \{\langle u \rangle, \langle v \rangle\}$, where $u, v \in \mathbb{C}^{n+2}$ are arbitrary non–zero null vectors of m. The group $K(u, v)$ of CR transformations of the flat CR geometry on M has two connected components. The identity component of $K(u, v)$ is the intersection of stabilizers of $\langle u \rangle$ and $\langle v \rangle$. Let us remark that if $m(u, v) \neq 0$, then the identity component of $K(u, v)$ is isomorphic to the group $CSU(p, q)$. If $m(u, v) = 0$, then $p, q > 1$ and there is $g \in PSU(p+1, q+1)$ such that gu and gv are the first two basis elements of $\mathbb{C}^{n+2}$ and thus elements of Lie algebra of $K(u, v)$ are g–conjugated to

$$\begin{pmatrix} a_1 & 0 & Z & z_n & iz \\ 0 & a_2 & B & ib & -\overline{z_n} \\ 0 & 0 & A & -I'B^* & -I'Z^* \\ 0 & 0 & 0 & -\overline{a_2} & 0 \\ 0 & 0 & 0 & 0 & -\overline{a_1} \end{pmatrix}$$

for some $a_1, a_2, z_n \in \mathbb{C}$, $b, z \in \mathbb{R}^*$, $B, Z \in \mathbb{C}^{p+q-2*}$ and $A \in \mathfrak{u}(p-1, q-1)$. The other connected component of $K(u, v)$ contains the elements that swap $\langle u \rangle$ and $\langle v \rangle$.

We check whether there is a symmetry at each $K(u, v)$–orbit on M. Let us emphasize that if all symmetries at one point of a $K(u, v)$–orbit preserve or swap the points $\langle u \rangle$ and $\langle v \rangle$, then all symmetries at all points of the whole orbit have the same property. The orbits of the action of $K(u, v)$ on M are characterized by the fact that the action preserves

- the subspace $\langle u, v \rangle$, and
- the (non)–isotropy with respect to the Hermitian form m.

Moreover, the action of $K(u, v)$ on $\langle u, v \rangle$ depends on whether $\langle u, v \rangle$ is isotropic subspace or not.

Example 1. Assume that $p, q > 1$, i.e., not the Riemannian signature. Consider the CR manifold $M = PSU(p+1, q+1)/P - \{\langle u\rangle, \langle v\rangle\}$ for arbitrary non–zero null vectors $u, v \in \mathbb{C}^{n+2}$ isotropic with respect to m, i.e., $m(u, v) = 0$. Then $\langle u, v\rangle - \{\langle u\rangle, \langle v\rangle\}$ consists of a single orbit of $K(u, v)$. Furthermore, $K(u, v)$–orbits of points $\langle x\rangle$ such that $x \notin \langle u, v\rangle - \{\langle u\rangle, \langle v\rangle\}$ depend only on the (non)–isotropy of x with respect to u, v.

We show that there exist symmetries at all points of each orbit of $K(u, v)$. Instead of fixing $\langle u\rangle, \langle v\rangle$ and discussing symmetries at various points $\langle x\rangle$, we fix the point $\langle x\rangle$ as the point $\langle e_0\rangle$ given by the first vector of the standard basis $e_0, \dots, e_{n+1}$ of $\mathbb{C}^{n+2}$ and we choose admissible $\langle u\rangle$ and $\langle v\rangle$ such that $\langle e_0\rangle$ lies in the correct orbit. Then we find all symmetries at $\langle e_0\rangle$. Let us recall that all symmetries of the standard model at the origin $\langle e_0\rangle$ are of the form

$$s_{Z,z} = \begin{pmatrix} -1 & -Z & iz + \frac{1}{2}ZIZ^* \\ 0 & E & -IZ^* \\ 0 & 0 & -1 \end{pmatrix},$$

where $I = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & I' & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$ and $Z = (z_1, \dots, z_n) \in \mathbb{C}^{n*}$ and $z \in \mathbb{R}^*$ are arbitrary. Involutive symmetries are those satisfying $z = 0$.

(1) Let us start with the orbit corresponding to the case $m(e_0, u) \neq 0$ and $m(e_0, v) \neq 0$. We choose $u = ie_0 + e_1 + e_n - ie_{n+1}$ and $v = ie_0 - e_1 + e_n + ie_{n+1}$. Direct computation gives that there is exactly one symmetry $s_{Z,z}$, where $Z = (-2i, 0, \dots, 0)$ and $z = 0$. This symmetry is involutive and swaps $\langle u\rangle$ and $\langle v\rangle$. There is no symmetry preserving them.

(2) Let us now consider the orbit for the case $m(e_0, u) = 0$ and $m(e_0, v) \neq 0$ (which is the same orbit as the orbit for the case $m(e_0, u) \neq 0$ and $m(e_0, v) = 0$). We choose $u = \sqrt{2}e_1$ and $v = ie_{n+1}$. Direct computation gives that there is exactly one symmetry $s_{Z,z}$, where $Z = (0, \dots, 0)$ and $z = 0$. This symmetry is involutive and preserves $\langle u\rangle$ and $\langle v\rangle$. There is no symmetry swapping them.

(3) The next possibility is the orbit for the case $m(e_0, u) = m(e_0, v) = 0$ and $e_0 \in \langle u, v\rangle$. We choose $u = \sqrt{2}e_1$ and $v = e_0 + \sqrt{2}e_1$. Computation gives that there are (many) symmetries $s_{Z,z}$, where $Z = (z_1, \dots, z_n)$ with components $z_k = a_k + ib_k$ for $k = 1, \dots, n$ satisfies $\sqrt{2}a_1 + 1 = 0$ and $b_1 = 0$, and a_k, b_k for $k = 2, \dots, n-1$ and z are arbitrary. All these symmetries swap $\langle u\rangle$ and $\langle v\rangle$, and there are no symmetries preserving them. In particular, there are also non–involutive symmetries for $z \neq 0$.

In fact, this covers all possible orbits for the case $p = 1$ or $q = 1$, i.e., the Lorentzian signature. In the other cases, there is one more orbit.

(4) Consider the orbit for the case $m(u, e_0) = m(v, e_0) = 0$ and $e_0 \notin \langle u, v\rangle$. We choose $u = \sqrt{2}e_1$ and $v = e_2 + e_{n-1}$. Computation gives that there are (many) symmetries $s_{Z,z}$, where $Z = (z_1, \dots, z_n)$ satisfies $a_1 = 0$, $b_1 = 0$, $a_2 + a_{n-1} = 0$ and $b_2 + b_{n-1} = 0$ and a_k, b_k for $k = 3, \dots, n-2$ and z are arbitrary. All these symmetries preserve $\langle u\rangle$ and $\langle v\rangle$ and there are no symmetries swapping them. In particular, there are also non–involutive symmetries for $z \neq 0$.

Altogether, symmetries at different orbits behave differently. Therefore, there is no smooth system of symmetries. In particular, there is no pseudo–Riemannian metric compatible with the CR geometry that would be preserved by some symmetry at every point. ♢

Let us show that this principle does not work if we remove two points corresponding to non–isotropic vectors.

Example 2. Consider the manifold $M = PSU(p+1,q+1)/P - \{\langle u\rangle, \langle v\rangle\}$ for arbitrary non–zero null vectors $u, v \in \mathbb{C}^{n+2}$ that are non–isotropic for m, i.e. $m(u,v) \neq 0$. We choose $u = e_{n+1}$ and $v = e_0 + (2+i)e_1 - ie_n$. Computation gives that there is no symmetry at $\langle e_0\rangle$ preserving or swapping $\langle u\rangle$ and $\langle v\rangle$. Let us remark that the identity component of $K(u,v)$ is isomorphic to the group $CSU(p,q)$ and $K(u,v)$ does not act transitively on $\langle u,v\rangle - \{\langle u\rangle, \langle v\rangle\}$. ♢

6.2. **Flat homogeneous symmetric CR geometries and orbits of real forms in complex flag manifolds.** It follows from Lemma 4 that an extension (α, i) of (K, L) to $(PSU(p+1,q+1), P)$ corresponds to a flat CR geometry if and only if α is a Lie algebra homomorphisms. Therefore, we can present examples of extensions describing flat homogeneous symmetric CR geometries just by specifying the Lie subalgebra of $\mathfrak{su}(p+1,q+1)$ that coincides with the image of α. In general, the group K does not have to contain symmetries. Moreover, symmetries do not have to preserve $\alpha(\mathfrak{k})$. This is satisfied if K is the group generated by symmetries or the full group of CR automorphisms.

Example 3. Consider the orbits of $PSp(1,1)$ on $\mathbb{C}P^4$ given by inclusion $PSp(1,1) \subset PSp(4,\mathbb{C}) \subset PGl(4,\mathbb{C})$. Due to the isomorphisms $PSp(1,1) \cong PO(1,4)$, these orbits can also be interpreted as orbits in the flag manifold of 2–planes in quadric in $\mathbb{C}P^5$. There is a normal extension given by identifying the following Lie subalgebra of $\mathfrak{su}(2,2)$ with the image of $\alpha(\mathfrak{sp}(1,1))$:

$$\begin{pmatrix} l_1+il_2 & -X_1+iX_2 & -X_3+l_4+iX_4+il_5 & i\,(l_3+x) \\ X_1+iX_2 & l_1-il_2 & -i(x+l_3) & X_3-l_4+iX_4+il_5 \\ X_3+iX_4 & -ix & -l_1-il_2 & X_1+iX_2 \\ ix & -X_3+iX_4 & -X_1+iX_2 & -l_1+il_2 \end{pmatrix},$$

where l_i–entries generate the Lie algebra of the stabilizer $L = CSO(2) \rtimes S^2\mathbb{R}^2$ of a point in the minimal orbit. Precisely, $\langle l_1, l_2\rangle = \mathfrak{cso}(2)$ and $\langle l_3, l_4, l_5\rangle = S^2\mathbb{R}^2$.

Example 4. Consider the orbits of $PSp(4,\mathbb{R})$ on $\mathbb{C}P^4$ given by inclusion $PSp(4,\mathbb{R}) \subset PSp(4,\mathbb{C}) \subset PGl(4,\mathbb{C})$. Due to the isomorphisms $PSp(4,\mathbb{R}) \cong PO(2,3)$, these orbits can again be interpreted as orbits in the flag manifold of 2–planes in quadric in $\mathbb{C}P^5$. There is a normal extension given by identifying the following Lie subalgebra of $\mathfrak{su}(2,2)$ with the image of $\alpha(\mathfrak{sp}(n+2,\mathbb{R}))$:

$$\begin{pmatrix} l_1+il_2 & X_1-iX_2 & X_3+l_4-iX_4+il_5 & i\,(l_3+x) \\ X_1+iX_2 & l_1-il_2 & i\,(l_3+x) & -X_3-l_4-iX_4+il_5 \\ X_3+iX_4 & ix & -l_1-il_2 & -X_1-iX_2 \\ ix & -X_3+iX_4 & -X_1+iX_2 & -l_1+il_2 \end{pmatrix},$$

where l_i–entries generate the Lie algebra of the stabilizer $L = CSO(2) \rtimes S^2\mathbb{R}^2$ of a point in 5–dimensional orbit (which is not minimal). Precisely, $\langle l_1, l_2\rangle = \mathfrak{cso}(2)$ and $\langle l_3, l_4, l_5\rangle = S^2\mathbb{R}^2$.

In both examples, $\mathfrak{k}$ is simple and $\mathfrak{q}$ is a parabolic subgalgebra of $\mathfrak{k}_{\mathbb{C}}$. In [1], the authors discuss which CR algebras $(\mathfrak{k}, \mathfrak{q})$ for simple Lie algebras $\mathfrak{k}$ and parabolic subalgebras $\mathfrak{q}$ of the complexification of $\mathfrak{k}$ are symmetric. In fact, they correspond to orbits of real forms in complex flag varieties. Therefore, symmetric CR algebras of these types generalize bounded symmetric domains.

We show that if a CR algebra $(\mathfrak{k}, \mathfrak{q})$ for a simple Lie algebra $\mathfrak{k}$ and a parabolic subalgebra $\mathfrak{q}$ of the complexification $\mathfrak{k}_{\mathbb{C}}$ of $\mathfrak{k}$ corresponds to a non–degenerate symmetric CR geometry of hypersurface type, then the geometry is necessarily flat. Therefore, we can use the results of [13] to classify all possible cases.

Proposition 8. *Let $(\mathfrak{k}, \mathfrak{q})$ be a CR algebra such that $\mathfrak{k}$ is simple and $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{k}_{\mathbb{C}}$ and the corresponding CR geometry is non–degenerate and of hypersurface type. Then the following statements hold:*

(1) *If the CR geometry is symmetric, then the CR geometry is flat.*
(2) *If the CR geometry is flat, then it corresponds to one of the following possibilities:*
  (a) $\mathfrak{k} = \mathfrak{su}(p+1, q+1)$ *and* $\mathfrak{l} = \mathfrak{p}$,
  (b) $\mathfrak{k} = \mathfrak{sp}(\frac{p+1}{2}, \frac{q+1}{2})$ *and* $\mathfrak{l} = \mathfrak{co}(2) \oplus \mathfrak{sp}(\frac{p-1}{2}, \frac{q-1}{2}) \oplus (\mathbb{R}^2 \otimes \mathbb{R}^{n-2*}) \oplus S^2\mathbb{R}^2$,
  (c) $\mathfrak{k} = \mathfrak{sp}(n+2, \mathbb{R})$ *and* $\mathfrak{l} = \mathfrak{co}(2) \oplus \mathfrak{sp}(n-2, \mathbb{R}) \oplus (\mathbb{R}^2 \otimes \mathbb{R}^{n-2*}) \oplus S^2\mathbb{R}^2$, *where* $(\mathbb{R}^2 \otimes \mathbb{R}^{n-2*}) \oplus S^2\mathbb{R}^2$ *is the positive part of the parabolic subalgebra corresponding to the stabilizer of a Lagrangian 2–plane in* $\mathbb{R}^{n+2}$.
(3) *If the CR geometry is flat and* $\mathfrak{k}_{\mathbb{C}} = \mathfrak{sp}(n+2, \mathbb{C})$ *is the full Lie algebra of complete infinitesimal automorphism and* $n > 2$*, then the corresponding CR geometry is not symmetric.*
(4) *If the CR geometry is flat and corresponds to a* $(2n+1)$*–dimensional orbit of the real form of* $\mathfrak{sp}(n+2, \mathbb{C})$ *in* $\mathbb{C}P^{n+2}$*, then the corresponding CR geometry is symmetric if and only if* $n = 2$ *or the orbit is minimal, i.e., if* $\mathfrak{k} \neq \mathfrak{sp}(n+2, \mathbb{R})$.

Proof. If such symmetric CR geometry is non–flat, then K has to be generated by symmetries and it follows from [4, Theorem 3.1] that the complexification of $\mathfrak{l}$ does not contain a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. On the other hand, if $\mathfrak{k}$ is simple and $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{k}_{\mathbb{C}}$, then $\mathfrak{q} \cap \bar{\mathfrak{q}}$ contains a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. This is a contradiction and therefore, the claim (1) holds.

If such symmetric CR geometry is flat, then $\mathfrak{k}_{\mathbb{C}}$ is isomorphic to a Lie subalgebra of $\mathfrak{sl}(n+2, \mathbb{C})$, $\mathfrak{q} = \mathfrak{k}_{\mathbb{C}} \cap (\mathbb{C}^* \oplus \mathfrak{p}_{\mathbb{C}})$ is a parabolic subalgebra of $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}/\mathfrak{q} = \mathfrak{sl}(n+2, \mathbb{C})/(\mathbb{C}^* \oplus \mathfrak{p}_{\mathbb{C}})$. All such cases are classified in [13] and it follows that $\mathfrak{k}_{\mathbb{C}} = \mathfrak{sl}(n+2, \mathbb{C})$ or $\mathfrak{k}_{\mathbb{C}} = \mathfrak{sp}(2n+2, \mathbb{C})$. The first case corresponds to the standard model. The remaining cases correspond to the symmetric pair $(\mathfrak{sl}(n+2, \mathbb{C}), \mathfrak{sp}(2n+2, \mathbb{C}))$. Real forms of this symmetric pair are well–known and correspond to suitable inclusions $\mathfrak{sp}(\frac{p+1}{2}, \frac{q+1}{2}) \subset \mathfrak{su}(p+1, q+1)$ or $\mathfrak{sp}(n+2, \mathbb{R}) \subset \mathfrak{su}(n+1, n+1)$. If such inclusion provides an extension, then it is unique (up to equivalence). Therefore, it suffices to show that the cases in the claim (2) correspond to non–degenerate CR geometries of hypersurface type. This follows from the fact that $\mathfrak{co}(2) \cong \mathbb{C}$ defines a complex structure on the whole $\mathfrak{k}/\mathfrak{l}$ with the exception of the trace part of $(S^2\mathbb{R}^2)^*$.

If $\mathfrak{k}_{\mathbb{C}} = \mathfrak{sp}(n+2, \mathbb{C})$ is the full Lie algebra of complete infinitesimal automorphism and the corresponding CR geometry is symmetric, then $\mathrm{Ad}(s_{0,0})$ induces an involution of $\mathfrak{sp}(n+2, \mathbb{R})$. It follows from the description of $\mathfrak{l}$ that the stabilizer has to have the form $\mathfrak{gl}(2, \mathbb{C}) \oplus \mathfrak{sp}(n-2, \mathbb{C})$. Therefore the claim (3) follows from the fact that this stabilizer does not appear in the classification of simple symmetric spaces if $n > 2$.

Since $\mathfrak{sp}(n+2, \mathbb{C})$ is maximal subalgebra of $\mathfrak{sl}(n+2, \mathbb{C})$, the only possibility for the orbit to be symmetric is to be equivalent to standard model which is compact. Since the orbit is compact if only if the orbit is minimal, the claim (4) follows. It follows from [1] that the orbit is minimal if and only if $\mathfrak{k} \neq \mathfrak{sp}(n+2, \mathbb{R})$. □

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Chapter 4: Submaximally Symmetric Almost Quaternionic Structures

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SUBMAXIMALLY SYMMETRIC ALMOST QUATERNIONIC STRUCTURES

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Abstract. The symmetry dimension of a geometric structure is the dimension of its symmetry algebra. We investigate symmetries of almost quaternionic structures of quaternionic dimension n. The maximal possible symmetry is realized by the quaternionic projective space $\mathbb{H}P^n$, which is flat and has the symmetry algebra $\mathfrak{sl}(n+1,\mathbb{H})$ of dimension $4n^2+8n+3$. For non-flat almost quaternionic manifolds we compute the next biggest (submaximal) symmetry dimension. We show that it is equal to $4n^2-4n+9$ for $n>1$ (it is equal to 8 for $n=1$). This is realized both by a quaternionic structure (torsion–free) and by an almost quaternionic structure with vanishing quaternionic Weyl curvature.

1. Introduction

An *almost quaternionic structure* on a manifold M is a rank three subbundle $Q \subset \mathrm{End}(TM)$ such that locally (in a neighbourhood of each point) we can find a basis I, J, K of Q with $I^2 = J^2 = K^2 = -\mathbb{1}$ and $IJ = K$. A manifold M with a fixed almost quaternionic structure Q is called an *almost quaternionic manifold.* A (local) automorphism of (M,Q) is a (local) diffeomorphism of M that preserves Q. There exists a class of the so–called *Oproiu connections* $[\nabla^{Op}]$ on (M,Q) that preserve Q and share the same minimal torsion T_∇, which equals to the structure torsion of Q [1]. If ∇^{Op} is torsion–free, then (M,Q) is a *quaternionic manifold.*

An almost quaternionic manifold (M,Q) can be equivalently described as a normal parabolic geometry $(\mathcal{G}\to M,\omega)$ of type $PGL(n+1,\mathbb{H})/P$, where P is the stabilizer of a quaternionic line in $\mathbb{H}^{n+1}$ [4]. The fundamental invariant of each parabolic geometry is its *harmonic curvature* κ_H, which has two components in the almost quaternionic case: the *torsion* κ_1 of homogeneity 1 and the *quaternionic Weyl curvature* κ_2 of homogeneity 2. In particular, κ_1 coincides with the torsion T_∇ of arbitrary ∇^{Op} and vanishes for quaternionic geometries.

The quaternionic projective space $\mathbb{H}P^n$ is the set of quaternionic lines in $\mathbb{H}^{n+1}$, and the group $PGL(n+1,\mathbb{H})$ acts transitively on $\mathbb{H}P^n$ as automorphisms of the natural quaternionic structure. The subgroup P consists of transformations preserving the first basis line in $\mathbb{H}^{n+1}$. Then, $\mathbb{H}P^n = PGL(n+1,\mathbb{H})/P$ is the *flat model* of (almost) quaternionic geometry. The flat model has vanishing harmonic curvature and conversely, each almost quaternionic geometry such that $\kappa_H \equiv 0$ is locally equivalent to the flat model. In particular, every local automorphism of $\mathbb{H}P^n$ uniquely extends to a global one, and it is exactly the left multiplication by an element of $PGL(n+1,\mathbb{H})$. The space $\mathbb{H}P^n$ has maximal possible dimension of the symmetry algebra among all (almost) quaternionic manifolds with fixed quaternionic dimension n, that is $\dim \mathfrak{sl}(n+1,\mathbb{H}) = 4(n+1)^2-1$ for $\dim M = 4n$.

For curved almost quaternionic structures, local automorphisms generally do not extend to global ones. We consider infinitesimal symmetries, which correspond to local automorphisms. We focus on the problem of establishing the *submaximal symmetry dimension*, i.e. the maximal dimension of the symmetry algebra of an almost quaternionic structure with $\kappa_H \not\equiv 0$ and fixed quaternionic dimension. Specifically, we answer the following question:

When an almost quaternionic manifold (M^{4n}, Q) is not everywhere flat, what is the maximal dimension $\mathfrak{S}$ of its Lie algebra of infinitesimal symmetries?

Remark 1. The submaximal dimension of the automorphism groups (without the requirement $\kappa_H \not\equiv 0$) is $\dim P = 4n^2 + 4n + 3$. This is achieved on the flat manifold $M = \mathbb{H}P^n \setminus \{p\}$ for some $p \in \mathbb{H}P^n$. However the symmetry algebra of this (M, Q) is of maximal dimension $4n^2 + 8n + 3$.

From the point of view of parabolic geometry, a model with the symmetry algebra of submaximal dimension typically has exactly one non-zero component of its harmonic curvature [12]. Sometimes, the same submaximal bound is achieved for different non-zero components of κ_H. We will show that this is the case with almost quaternionic structures. Our main result is the following.

Theorem 1. *The maximal dimension of the symmetry algebra of almost quaternionic structures (M, Q) with $\dim M = 4n > 4$ and $\kappa_H = (\kappa_1, \kappa_2) \not\equiv 0$ is*

$$\mathfrak{S} = 4n^2 - 4n + 9.$$

This is realized in both cases, when $\kappa_1 \equiv 0$ and when $\kappa_2 \equiv 0$.

We exclude the case $n = 1$ due to the exceptional isomorphism $\mathfrak{sl}(2, \mathbb{H}) \simeq \mathfrak{so}(1, 5)$. In this case the geometry $PGL(2, \mathbb{H})/P$ can be interpreted as a four-dimensional Riemannian conformal geometry and κ_H has two components of homogeneity 2, which are the self–dual and anti–self–dual parts of the Weyl curvature. The submaximal symmetry dimension is 8 and is achieved by $M = \mathbb{C}P^2$ [7, 12].

Thus for $n > 1$ there is a gap $6(2n - 1) = \dim \mathfrak{sl}(n + 1, \mathbb{H}) - \mathfrak{S}$ in the set of all possible symmetry dimensions. If $\kappa_1 \neq 0$ and $\kappa_2 \neq 0$, the symmetry dimension is strictly smaller than $\mathfrak{S}$, so the submaximal symmetry dimension is achieved only when $\kappa_1 \otimes \kappa_2 \equiv 0$. This is explained in the beginning of Section 5.

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2. Background on almost quaternionic and related geometries

Almost quaternionic geometries are closely related to projective and c–projective geometries, so we recall basic concepts common to these. Two (real) affine connections on a manifold M of dimension n are *projectively equivalent* if their unparameterized geodesics, i.e. curves satisfying $\nabla_{\dot\gamma}\dot\gamma \in \langle\dot\gamma\rangle$, coincide. Here $\langle - \rangle$ denotes the linear span over $C^\infty(M)$. Projectively equivalent connections do not necessarily have the same torsion, but any connection ∇ is projectively equivalent to a torsion–free connection $\nabla - \frac{1}{2}T_\nabla$. Two connections ∇ and $\hat\nabla$ with the same torsion are projectively equivalent if and only if there is a one-form $\Upsilon \in \Omega^1(M)$ such that

$$\nabla - \hat\nabla = \mathbb{1} \otimes (\Upsilon \circ \mathbb{1}) + (\Upsilon \circ \mathbb{1}) \otimes \mathbb{1}.$$

A fixed class of torsion–free projectively equivalent connections $[\nabla]$ on a manifold M is a *projective structure* on M. It is proven in [6] that the submaximal symmetry dimension in the class of projective structures of dimension $n > 2$ is equal to $(n - 1)^2 + 4$ (for $n = 2$ it is 3), see also [12].

A generalization of this concept to almost complex manifolds leads to almost c–projective structures. A connection ∇ on M of dimension $2n > 2$ with almost complex structure J is called *complex* if $\nabla J = 0$. Each almost complex manifold (M, J) admits complex connections, because for arbitrary ∇ the connection $\frac{1}{2}(\nabla - J\nabla J)$ is complex. A complex connection ∇ can be chosen *minimal* meaning $T_\nabla = \frac{1}{4}N_J$, where

$$N_J(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y]$$

is the Nijenhuis tensor.

A curve γ on M is *J-planar* if $\nabla_{\dot\gamma}\dot\gamma \in \langle \dot\gamma, J\dot\gamma\rangle$ for a complex connection ∇. Two complex connections on (M, J) are *c–projectively equivalent* if they share the same J–planar curves. Two complex connections ∇ and $\hat\nabla$ with the same torsion are c–projectively equivalent if and only if there is a one-form $\Upsilon \in \Omega^1(M)$ such that

$$\nabla - \hat\nabla = \sum_{A\in\{\mathbb{1},J\}} A^2\big(A \otimes (\Upsilon\circ A) + (\Upsilon\circ A)\otimes A\big).$$

An *almost c–projective structure* on (M, J) is a class of c–projectively equivalent complex connections $[\nabla]$ sharing the same fixed torsion. It is proven in [11] that the submaximal dimension in the class of almost c–projective structures of (complex) dimension n is equal to $2n^2 - 2n + 4$ for $n \neq 3$ and 18 for $n = 3$.

Let us return to almost quaternionic structures. Consider an almost quaternionic manifold (M, Q) of dimension $4n$. Analogously to the almost complex case, this admits a quaternionic connection. Indeed, for any local basis $b = (I, J, K)$ of Q and a linear connection ∇, the connection $\nabla_b := \frac{1}{4}(\nabla - I\nabla I - J\nabla J - K\nabla K)$ is quaternionic. Any other choice $\hat b = (\hat I, \hat J, \hat K)$ is related to b via a transformation from $SO(Q)$, so $\nabla_{\hat b} = \frac{1}{4}(\nabla - \hat I\nabla\hat I - \hat J\nabla\hat J - \hat K\nabla\hat K)$ coincides with ∇_b. Denote $B := \frac{1}{6}(N_I + N_J + N_K)$. The canonical *structure tensor* of Q is given by

$$T_Q := B + \delta(\tau_I\otimes I) + \delta(\tau_J \otimes J) + \delta(\tau_K\otimes K),$$

where $\tau_A(X) = \frac{1}{4n-2}Tr(AB(X))$ for $A = I, J, K$ and $\delta : T^*M\otimes T^*M\otimes TM \to \wedge^2T^*M\otimes TM$ denotes the Spencer operator of alternation [1]. A quaternionic connection can be chosen minimal meaning its torsion coincides with T_Q. An almost quaternionic structure Q is a *quaternionic structure* if T_Q vanishes.

A curve γ is called *Q–planar* if $\nabla_{\dot\gamma}\dot\gamma \in \langle\dot\gamma, I\dot\gamma, J\dot\gamma, K\dot\gamma\rangle$ for a quaternionic connection ∇. Two quaternionic connections ∇ and $\hat\nabla$ on (M,Q) with the same torsion share the same Q–planar curves if and only if there is a one-form $\Upsilon \in \Omega^1(M)$ such that

$$\nabla - \hat\nabla = \sum_{A\in\{\mathbb{1},I,J,K\}} A^2\Big(A\otimes(\Upsilon\circ A) + (\Upsilon\circ A)\otimes A\Big).$$

Analogously to the c–projective case, we fix the class of connections $[\nabla]$ sharing the same Q–planar curves and with the minimal torsion $T_\nabla = T_Q$. These are called *Oproiu connections.* The Q–planar curves are the (unparameterized) geodesics of all Oproiu connections [8]. Given an arbitrary quaternionic connection, one can construct an Oproiu connection by an explicit formula [1, §3.11].

An almost quaternionic structure is quaternionic if and only if some (and thus any) Oproiu connection ∇ is torsion–free. In that case, the curvature R_∇ of an Oproiu connection ∇ decomposes as $R_\nabla = W_\nabla + P_\nabla$, where the totally trace-free part W_∇ is the *(quaternionic) Weyl tensor of* R_∇ and P_∇ is the *Ricci part of* R_∇[1]. Let us remark that almost quaternionic structures can be viewed as first–order

$G_0 = Sp(1)GL(n,\mathbb{H})$–structures. Then the above decomposition of R_∇ is just the decomposition with respect to the action of this structure group.

It turns out that the Weyl part W_∇ of R_∇ does not depend on the choice of Oproiu connection and is a complete obstruction to the flatness of a quaternionic structure.

Remark 2. The equivalence class of Oproiu connections is already determined by the quaternionic structure on the quaternionic manifold. But the complex structure alone does not determine a c–projective structure. The choice of the class of c–projectively equivalent connections on a complex manifold is an additional choice.

Remark 3. All three geometries discussed in this section can be described as parabolic geometries of type $PGL(n+1,\mathbb{K})/P$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and P is the stabilizer of a $\mathbb{K}$-line in $\mathbb{K}^{n+1}$ [4]. This explains many similarities between them.

3. Parabolic geometry setup and the gap problem

In this section, we summarize basic facts about almost quaternionic structures from the parabolic viewpoint. We will need the notations describing parabolic subalgebras $\mathfrak{p}$ of a real semi-simple Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. The conjugacy classes of such are in bijection with some subsets of the Satake diagram corresponding to a fixed choice of (maximally non-compact) Cartan subalgebra. These will be denoted by crossing out certain white nodes on the Satake diagram, cf. [4]. Let Σ be the set of crossed out nodes. We denote by $\mathfrak{p}_\Sigma$ the standard parabolic subalgebra corresponding to Σ. The semi-simple Levi factor $\mathfrak{p}_\Sigma^{ss}$ is given by the Satake diagram with Σ removed. A Lie subgroup $P_\Sigma \subset G$ is parabolic if its Lie algebra $\mathfrak{p}_\Sigma$ is such. We will use the standard Bourbaki enumeration of the nodes of the Dynkin diagram, and encode parabolic subgroups P_Σ respectively.

Consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{H})$, which is a real form of $A_{2n+1} = \mathfrak{sl}(2n+2,\mathbb{C})$. The parabolic subalgebra $\mathfrak{p} = \mathrm{Lie}(P)$, for $P = P_2$, corresponding to a $|1|$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, is encoded by the following Satake diagram:

$$\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\cdots-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\bullet$$

This grading can be viewed via the matrix $(1,n)$-block decomposition which is given by the standard representation of $\mathfrak{g}$ on $\mathbb{H}^{n+1} = \mathbb{H}\times\mathbb{H}^n$. Thus $\mathfrak{g}$ has the matrix form $\begin{pmatrix} a & p \\ v & A \end{pmatrix}$, where $\mathfrak{g}_{-1} = \{v \in \mathbb{H}^n\}$, $\mathfrak{g}_1 = \{p \in \mathbb{H}^{*n}\}$ and $\mathfrak{g}_0 = \{(a,A) \in \mathbb{H} \oplus \mathfrak{gl}(n,\mathbb{H}) : \mathrm{Re}(a) + \mathrm{Re}(\operatorname{tr} A) = 0\}$. In particular, the real part of $a \in \mathbb{H}$ is determined by $\operatorname{tr} A$ and the imaginary part belongs to $\mathfrak{sp}(1)$. Thus the reductive Lie algebra $\mathfrak{g}_0$ can be equivalently viewed as $\mathfrak{sp}(1) + \mathfrak{gl}(n,\mathbb{H})$, and this further decomposes as $\mathfrak{g}_0 = \mathfrak{sp}(1) + \mathbb{R}Z + \mathfrak{sl}(n,\mathbb{H})$, where the semisimple part is $\mathfrak{g}_0^{ss} = \mathfrak{sp}(1) + \mathfrak{sl}(n,\mathbb{H})$ and the grading element $Z = \mathrm{diag}\big(\frac{n}{n+1}, \frac{-1}{n+1}, \dots, \frac{-1}{n+1}\big)$ generates the center $\mathfrak{z}(\mathfrak{g}_0)$. The Lie algebra $\mathfrak{g}_0^{ss}$ is encoded by the Satake diagram produced by removing the crossed node and adjacent edges:

$$\bullet \qquad\qquad \bullet\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\cdots-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\bullet$$

A fundamental invariant of a regular normal parabolic geometry is the *harmonic curvature* κ_H, taking values in the G_0–module $H^2(\mathfrak{g}_-,\mathfrak{g})$ (that is the Lie algebra cohomology of $\mathfrak{g}_-$ with values in $\mathfrak{g}$; in the quaternionic case the regularity requirement is vacuous, i.e. $H^2 = H^2_+$ has positive homogeneity because the geometry is $|1|$-graded). This is a completely reducible module, and its two irreducible components

H^2_1 and H^2_2 (the subscript denotes homogeneity of the cohomology with respect to Z) yield the corresponding decomposition of κ_H into two summands:

- the *torsion* κ_1 of homogeneity 1 valued in $H^2_1(\mathfrak{g}_{-1}, \mathfrak{g})$, and
- the *quaternionic Weyl curvature* κ_2 of homogeneity 2 valued in $H^2_2(\mathfrak{g}_{-1}, \mathfrak{g})$.

The harmonic curvature κ_1 coincides with the torsion $T_\nabla = T_Q$ of an arbitrary Oproiu connection, and if the torsion vanishes, then the harmonic curvature κ_2 coincides with Weyl tensor W_∇ of an arbitrary Oproiu connection. For an almost quaternionic structure that is not quaternionic $\kappa_H = \kappa_1 + \kappa_2$ and both components are non-vanishing in general.

To compute the structure of these modules, where κ_1 and κ_2 have their values, we invoke the complexification: the corresponding parabolic subalgebra $\mathfrak{p}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ induces a $|1|$-grading of $\mathfrak{g}^{\mathbb{C}}$ and $H^2(\mathfrak{g}_-, \mathfrak{g}) \otimes \mathbb{C} \simeq H^2(\mathfrak{g}^{\mathbb{C}}_-, \mathfrak{g}^{\mathbb{C}})$. Explicit algorithmic description of the $G^{\mathbb{C}}_0$–module structure of the latter follows from Kostant's version of the Bott-Borel-Weil theorem [4]. The real curvature module is then a real completely reducible $\mathfrak{p}$-submodule of its complexification, which is a $\mathfrak{p}^{\mathbb{C}}$-module.

In the case of almost quaternionic structures, the submodules corresponding to the quaternionic torsion κ_1 and curvature κ_2 are encoded by minus lowest weights of the complexified modules (adapting the convention of [2]) as follows, where the number over the i'th node is the coefficient of the fundamental weight ω_i:

$$\kappa_1 : \quad \overset{3}{\bullet}-\overset{-3}{\times}-\overset{0}{\bullet}-\overset{1}{\circ}-\cdots-\overset{0}{\circ}-\overset{1}{\bullet} \qquad \kappa_2 : \quad \overset{0}{\bullet}-\overset{-4}{\times}-\overset{3}{\bullet}-\overset{0}{\circ}-\cdots-\overset{0}{\circ}-\overset{1}{\bullet}$$

Remark 4. Let us point out that $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ is a real $\mathfrak{g}_0$–module that we identify with a real $\mathfrak{g}_0$–submodule of $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \otimes \mathbb{C}$. Note also that minus the lowest weight is equal to the highest weight of the dual module.

Let us recall how to get a universal upper bound $\mathfrak{U}$ on the submaximal symmetry dimension $\mathfrak{S}$ (the gap problem), and explain the role of the G_0-module $H^2(\mathfrak{g}_-, \mathfrak{g})$. Each element ξ of the infinitesimal symmetry algebra $\mathfrak{inf}$ has to preserve (both components of) κ_H, and thus the isotropy subalgebra of $\mathfrak{inf}$ embeds into the annihilator of κ_H in $\mathfrak{p}$. Moreover, for arbitrary fixed $u \in \mathcal{G}$ there is the inclusion $\mathfrak{inf} \hookrightarrow \mathfrak{g}$ of filtered vector spaces, given by $\xi \mapsto \omega(\xi(u))$. Its image $\mathfrak{f} \simeq \mathfrak{inf}$ is a filtered Lie algebra with the bracket obtained by a deformation of the Lie bracket $[\cdot,\cdot]$ of $\mathfrak{g}$. The associated graded algebra $\mathfrak{s} = \mathrm{gr}(\mathfrak{f})$ is a graded subalgebra of $\mathfrak{g}$ [3, 12]. The grading $\mathfrak{s} = \oplus_i \mathfrak{s}_i$ satisfies

$$\mathfrak{s}_0 \subset \mathfrak{a}_0 = \{\phi \in \mathfrak{g}_0 : \phi \cdot \kappa_H = 0\}.$$

Furthermore it is proven in [12] that $\mathfrak{S} \leq \mathfrak{U}$ for

$$\mathfrak{U} = \max\{\dim(\mathfrak{a}^\psi) : 0 \neq \psi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})\},$$

where the graded subalgebra $\mathfrak{a}^\psi \subset \mathfrak{g}$ is the *Tanaka prolongation* of the pair $(\mathfrak{g}_-, \mathfrak{a}^\psi_0)$, and $\mathfrak{a}^\psi_0$ is the annihilator of ψ in $\mathfrak{g}_0$. Moreover, $\mathfrak{S} \leq \mathfrak{U} \leq \mathfrak{U}^{\mathbb{C}}$, where $\mathfrak{U}^{\mathbb{C}}$ is the universal upper bound for the complexified geometry, and the universal upper bound is realized by the stabilizer of minus the lowest weight vector in the complex case.

By [12, Corollary 3.4.8] the parabolic structures of type A_{2n+1}/P_2 are *prolongation–rigid*, i.e. the Tanaka prolongation $\mathfrak{a}^\psi_+ = 0$ for any $\psi \neq 0$. This implies the corresponding statement for real geometries [10, Proposition 3]. Thus almost quaternionic structures are prolongation–rigid $\mathfrak{a}^\psi_1 = 0$, and so $\mathfrak{a}^\psi = \mathfrak{g}_{-1} \oplus \mathfrak{a}^\psi_0$ for each non-zero element $\psi \in H^2_+(\mathfrak{g}_{-1}, \mathfrak{g})$.

Let $\mathfrak{S}_i$ be the maximal symmetry dimension of the almost quaternionic geometry of $\dim_{\mathbb{H}} = n$ in the case $\kappa_i \not\equiv 0$. We are going to bound this $\mathfrak{S}_i \leq \mathfrak{U}_i = \max\{\dim(\mathfrak{a}^\psi) : 0 \neq \psi \in H^2_i(\mathfrak{g}_{-1}, \mathfrak{g})\}$ and prove that the submaximal symmetry dimension is

$$\mathfrak{U}_1 = \mathfrak{S}_1 = \mathfrak{S} = \mathfrak{S}_2 = \mathfrak{U}_2.$$

We, however, cannot directly apply the methods from complex parabolic geometry. It turns out that the corresponding upper bounds are strictly less than the upper bounds for the complexification: $\mathfrak{U}_i < \mathfrak{U}_i^{\mathbb{C}}$ and thus $\mathfrak{U} < \mathfrak{U}^{\mathbb{C}}$.

A similar phenomenon was noticed for Lorentzian conformal geometries in [5], where the submaximal symmetry dimension was computed by listing all subalgebras of high dimensions that stabilize a non-zero element in the harmonic curvature module. In this paper, we choose a different approach by identifying a real analogue to the lowest weight vector in the real harmonic curvature module.

4. Minimal orbits

Recall that in the case of complex parabolic geometries, obtaining the upper symmetry bound is based on the Borel fixed point theorem, which states that there is a unique closed orbit, which is of minimal dimension, in the projectivization of $H^2(\mathfrak{g}_-, \mathfrak{g})$. Then the upper bound is given by dimension of the stabilizer of a weight vector corresponding to minus the lowest weight (generating the minimal orbit). The Borel theorem cannot be applied in the case of almost quaternionic structures, but we still consider the $\mathfrak{g}_0$–orbits in the projectivization of $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ to find one of the minimal dimension. The following statement is immediate.

Lemma 2. *The annihilator of $0 \neq \kappa_i \in H^2_i$ $(i = 1, 2)$ is of maximal dimension in $\mathfrak{g}_0$ if and only if the G_0–orbit through κ_i has minimal dimension in the projectivization of $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$.*

We will need the following result on existence of closed orbits. Here we denote by $[v]$ the projection of a non-zero vector $v \in \mathbb{V}$ to the projective space $P\mathbb{V}$.

Lemma 3. *Let $\mathbb{V}$ be a real, finite-dimensional, irreducible L–module for a real connected Lie group L, such that the center of L acts by real scalars. Then there exists $0 \neq v \in \mathbb{V}$ such that for $[v] \in P\mathbb{V}$ the orbit $L \cdot [v] \subset P\mathbb{V}$ is closed and of minimal dimension. In fact, any orbit in $P\mathbb{V}$ of minimal dimension is closed.*

Proof. We may quotient the group by the kernel of the representation, to obtain a faithful representation, and this will not affect the (projective) orbits. Therefore we assume without loss of generality that the representation is faithful.

Because the representation is irreducible, $\mathfrak{l} = \mathrm{Lie}(L)$ is the direct sum of a semi-simple ideal and a central subalgebra [9, Chapter 3]. Our assumption on the center action means that the module $\mathbb{V}$ is a tensor product of an irreducible module $\mathbb{V}_0$ over the semi-simple ideal, and a one-dimensional module $\mathbb{R}$ over the center.

Let $L \cdot [v]$ be an orbit of minimal dimension $d = \dim L \cdot [v]$. We want to prove it is closed. Consider the complexification of the group, the action and the representation. The element $v \in \mathbb{V} + 0 \cdot i \subset \mathbb{V}^{\mathbb{C}}$ determines the complex orbit $L^{\mathbb{C}} \cdot [v] \subset P\mathbb{V}^{\mathbb{C}}$ of the same *complex* dimension d (because the annihilator of v in $\mathfrak{l}^{\mathbb{C}}$ intersects $\mathfrak{l}$ by the annihilator of v in $\mathfrak{l}$). If the closure of the orbit $L \cdot [v]$ contains another orbit $L \cdot [v']$ (necessarily of the same dimension d), then the closure of the complex orbit $L^{\mathbb{C}} \cdot [v]$ contains the complex orbit $L^{\mathbb{C}} \cdot [v']$ (again of the same complex dimension d). To exclude the latter note that the action of $L^{\mathbb{C}}$ on $P\mathbb{V}^{\mathbb{C}}$ is algebraic, because both the semi-simple action on $\mathbb{V}_0^{\mathbb{C}}$ and the central action on $\mathbb{C}$ are algebraic [13, Chapter 3]. But then the boundary of any orbit can only contain orbits of strictly smaller dimensions, which are less than d, cf. proof of Corollary in [13, III;§1.5]. Since such are non-existent in the real case, this proves the claim. □

Let's consider the minimal orbits in the projectivizations PH_1^2 and PH_2^2 of both irreducible components of $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$.

Conjugacy classes of parabolic subalgebras are in bijection with conjugacy classes of $\mathbb{Z}$-gradings of semi-simple Lie algebras $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_k$. If $\mathfrak{g}$ has a proper non-trivial parabolic subalgebra $\mathfrak{p}$, then there exists an element $Z \in \mathfrak{g}$ which acts diagonalizably on $\mathfrak{g}$, with an eigenspace decomposition of the form $\mathfrak{ad}_Z|_{\mathfrak{g}_n} = n\,\mathrm{Id}_{\mathfrak{g}_n}$, such that the direct sum of the non-negative eigenspaces equals the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_{\geq 0}$. This Z is called a grading element for $\mathfrak{p}$. Changing the grading element Z to $\mathrm{Ad}_p(Z)$ for any p in $P \subset G$ we get another grading element and another grading of $\mathfrak{g}$. Moreover, given a grading of $\mathfrak{g}$, the subalgebra $\mathfrak{p} = \mathfrak{g}_{\geq 0}$ of non-negative gradation is parabolic (all parabolics arise in this way [4]).

Recall that in the quaternionic case $\mathfrak{g}_0 = \mathfrak{sp}(1) + \mathbb{R}Z + \mathfrak{sl}(n, \mathbb{H})$. Let $\mathbb{V}$ be a $\mathfrak{g}_0$-module which is irreducible under the restricted representation of $\mathfrak{sl}(n, \mathbb{H})$. The group $\tilde{G}_0 = PGL(n, \mathbb{H})$ acts effectively on $P\mathbb{V}$, and the Lie algebra $\tilde{\mathfrak{g}}_0 = \mathfrak{sl}(n, \mathbb{H})$ is simple. Consider the following parabolic subgroups of $\tilde{G}_0$: $H = \tilde{P}_2$ in the case $n = 2$, and $H = \tilde{P}_{2,2n-2}$ in the case $n > 2$ (tilde in P indicates that numeration of the parabolics is with respect to $\tilde{G}_0$, the indices indicating the crossed nodes on the Satake diagram of $\tilde{\mathfrak{g}}_0$). This parabolic determines the grading on $\tilde{\mathfrak{g}}_0$ with respect to which the parabolic $(\tilde{\mathfrak{g}}_0)_{\geq 0}$ is equal to $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_+$ as a vector space, where

$$\mathfrak{h}_0 = \mathfrak{sp}(1) \oplus \mathfrak{gl}(n-2, \mathbb{H}) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}Z',$$
$$\mathfrak{h}_+ = \mathfrak{h}_1 \oplus \mathfrak{h}_2 = \mathfrak{heis}(8n-12, \mathbb{H}).$$

Here $\mathfrak{heis}(8n-12, \mathbb{H})$ is the quaternionification of the real (nilpotent) Heisenberg algebra $\mathfrak{heis}(2n-3)$, and Z' is a grading element in $\tilde{\mathfrak{g}}_0$. The action of $\mathfrak{heis}(8n-12, \mathbb{H})$ on $\mathbb{H}^n$ is given by $n \times n$ quaternionic matrices with zeroes everywhere except for the first row and the last column, and with zeroes on the diagonal.

To distinguish the summand $\mathfrak{sp}(1)$ in $\mathfrak{g}_0$ from those in $\mathfrak{h}_0$ we will use the notations $\mathfrak{sp}(1)_{\mathrm{left}}$ and $\mathfrak{sp}(1)_{\mathrm{right}}$ for the latter (marking them in the appearing order).

The grading element $Z' \in \tilde{\mathfrak{g}}_0$ acts on $\mathbb{V}$, and $\mathbb{V}$ decomposes as $\mathbb{V} = \oplus_i \mathbb{V}_{\theta_i}$ with respect to the action of Z', where $Z'|_{\mathbb{V}_{\theta_i}} = \theta_i \mathrm{Id}_{\mathbb{V}_{\theta_i}}$, and θ_i is real for all i. Indeed, the weights λ of the representation $\mathbb{V}$ are rational linear combinations of the roots of $\mathfrak{g}_0$, hence the eigenvalues $\lambda(Z')$ of Z' on $\mathbb{V}$ can be rationally expressed via eigenvalues of Z' on $\mathfrak{g}_0$, which are integers. Therefore, $\exp(tZ') = \oplus_i e^{t\theta_i}\mathrm{Id}_{\mathbb{V}_{\theta_i}}$. For $p \in \mathfrak{h}_1$, we have $[Z', p] = p = Z'p - pZ'$. This implies that for all $v \in \mathbb{V}_{\theta_i}$ we have

$$Z' \cdot (p \cdot v) = (p + pZ') \cdot v = (1 + \theta_i)(p \cdot v),$$

which implies $p \cdot v \in \mathbb{V}_{\theta_i+1}$.

Lemma 4. *Let $\mathbb{V}$ be an irreducible $\tilde{\mathfrak{g}}_0$-module and $0 \neq v \in \mathbb{V}$. Suppose that the orbit $\tilde{G}_0 \cdot [v] \subset P\mathbb{V}$ is closed. Then there exists $0 \neq w \in \mathbb{V}_{\theta_{max}} = \ker \mathfrak{h}_+$, $\theta_{max} = \max_i\{\theta_i\}$, such that $[w] \in \tilde{G}_0 \cdot [v]$.*

Proof. Decompose $v = \sum_{\theta_i} v_{\theta_i}$ into Z'-eigenvectors as above. Let v_{θ_j} be the non-zero component of the greatest index. If $\theta_j < \theta_{max}$ then, due to irreducibility, there exists $p \in \mathfrak{h}_1$ such that $p \cdot v_{\theta_j} \neq 0$ (in the opposite case v generates a proper submodule). Then $w_0 = \exp(\tau p)v$ for small $\tau > 0$ has a non-zero component in $\mathbb{V}_{\theta_j+1}$. Repeat this procedure for $w_0, w_1, \dots, w_{k-1}$ until w_k has non-zero component in $\mathbb{V}_{\theta_{max}}$ (if $\theta_j = \theta_{max}$ then $w_k = v$). This takes a finite number of operations, because θ_{max} is finite, and the index $\theta_i \in [-\theta_{\max}, \theta_{\max}]$ is incremented by a fixed amount on each iteration. Since the greatest eigenvalue dominates and the orbit of v is closed, the limit $\langle w\rangle = \lim\limits_{t\to+\infty} e^{tZ'}\langle w_k\rangle \in P\mathbb{V}_{\theta_{max}}$ exists. Here $\langle w_k\rangle = \mathbb{R}\cdot w_k$. Moreover, there exists an element w in $\mathbb{V}_{\theta_{max}}$ which is projected to the limit $\langle w\rangle$. □

Note that the proof of Lemma 4 would work if we switched out $\mathfrak{h}$ for any parabolic subalgebra in $\tilde{\mathfrak{g}}_0$, but the particular choice $\mathfrak{h}$ will turn out to be well adapted to describing the minimal orbits in those modules we are interested in.

4.1. **Minimal orbits in the curvature module.** The irreducible G_0-submodule $H^2_2 \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ will be denoted in this subsection by $\mathbb{V}^{\mathrm{II}}$ to indicate its homogeneity 2. Since the grading element Z acts on it by multiplication by 2, it cannot be in the annihilator of κ_2. It follows from the description of the (complexified) curvature module in the previous section that the action of $\mathfrak{sp}(1)$ preserves κ_2 and is always contained in the annihilator. Therefore we can restrict our attention to the action of $\tilde{\mathfrak{g}}_0 = \mathfrak{sl}(n,\mathbb{H})$. With respect to it the curvature module has the highest weight $\omega_1 + 3\omega_{2n-1}$ (we flip the Satake diagram), and hence can be identified with an irreducible real $\tilde{G}_0$–module

$$\mathbb{V}^{\mathrm{II}} = S^3_{\mathbb{C}}\mathbb{H}^{*n} \odot \mathbb{H}^n.$$

Here $\odot$ denotes the Cartan product (kernel of the tensor contractions), and we use the complex notations (the complex tensor products are taken with respect to an arbitrary invariant complex structure, say $I \in \operatorname{Im}\mathbb{H} \subset \operatorname{End}(\mathbb{H}^n)$, whose choice is inessential). For real description we refer to [4, Proposition 4.1.8], see also Remark 5, but we use the complex notations (even in describing the real objects).

We would like to find a $\tilde{G}_0$-orbit of minimal dimension (closed by Lemma 3) in $P\mathbb{V}^{\mathrm{II}}$. Due to Lemma 4 we can assume it is represented by a non-zero element $w \in \mathbb{V}^{\mathrm{II}}_{\theta_{max}} = \ker \mathfrak{h}_+$. The element w has pure grading with respect to Z' and hence its annihilator in $\tilde{\mathfrak{g}}_0$ is also graded: $\mathfrak{ann}(w) = \bigoplus\limits_{s=-2}^{2} \mathfrak{ann}(w)_s$ $\big(= \bigoplus\limits_{s=-1}^{1} \mathfrak{ann}(w)_s$ for $n = 2\big)$. We already know that $\mathfrak{ann}(w)_+ = \mathfrak{h}_+$.

Lemma 5. *We have:* $\mathfrak{ann}(w) \cap (\tilde{\mathfrak{g}}_0)_- = 0$, *i.e.* $\mathfrak{ann}(w)_- = 0$.

Proof. Let us consider the case $n > 2$ (The case $n = 2$ is a simple adaptation). The $\mathfrak{h}_0$ module $(\tilde{\mathfrak{g}}_0)_{-1}$ is reducible – it is the sum of two irreps: $(\tilde{\mathfrak{g}}_0)'_{-1} \equiv$ (the first column in the matrix from $\mathfrak{sl}(n,\mathbb{H})$ with the first and last entries zero) and $(\tilde{\mathfrak{g}}_0)''_{-1} \equiv$ (the last row in the matrix from $\mathfrak{sl}(n,\mathbb{H})$ with the first and last entries zero). This is also true when we restrict to $\mathfrak{sl}(n-2,\mathbb{H}) \subset \mathfrak{h}_0$.

Let $q = q' + q'' \in \mathfrak{ann}(w)_{-1}$ be non-zero. Due to the highest weight of the curvature module, $\mathfrak{sl}(n-2,\mathbb{H}) \subset \mathfrak{ann}(w)$, and both $(\tilde{\mathfrak{g}}_0)'_{-1}$ and $(\tilde{\mathfrak{g}}_0)''_{-1}$ are $\mathfrak{sl}(n-2,\mathbb{H})$-modules, so we conclude that at least one of $(\tilde{\mathfrak{g}}_0)'_{-1}$ and $(\tilde{\mathfrak{g}}_0)''_{-1}$ is entirely in the annihilator. But then, since $\mathfrak{h}_1 \subset \mathfrak{ann}(w)$ (and computing the brackets), we conclude that at least one of $\mathfrak{sp}(1)_{\mathrm{left}}$ and $\mathfrak{sp}(1)_{\mathrm{right}}$ is entirely in $\mathfrak{ann}(w)$, which is impossible.

Thus $\mathfrak{ann}(w)_{-1} = 0$. If there is $0 \neq q \in \mathfrak{ann}(w)_{-2}$, then taking brackets with $\mathfrak{h}_1 \subset \mathfrak{ann}(w)$ we get a non-zero element in $\mathfrak{ann}(w)_{-1}$, which is impossible by the above. This contradiction proves the claim. □

As already noted in the above proof, due to the highest weight, $\mathfrak{sl}(n-2,\mathbb{H})$ acts trivially on $\mathbb{V}^{\mathrm{II}}$, and so from $\mathfrak{h}_0^{ss}$ only $\mathfrak{sp}(1)^2=\mathfrak{sp}(1)_{\mathrm{left}}\oplus\mathfrak{sp}(1)_{\mathrm{right}}$ acts non-trivially. With respect to this algebra the module $\mathbb{V}^{\mathrm{II}}_{\theta_{max}}$ has highest weight $\omega_1+3\omega_2$, and as an irreducible *real* module it has real dimension 8. We want to maximize the annihilator of an element w.

Lemma 6. *Dimension of the $\mathfrak{sp}(1)^2$-orbit through a non-zero element $w\in\mathbb{V}^{\mathrm{II}}_{\theta_{max}}$ is either* 5 *or* 6. *Thus nontrivial annihilator can be only $\mathfrak{so}(2)\subset\mathfrak{sp}(1)^2$ of* $\dim=1$.

Proof. The complex $\mathfrak{sp}(1)^2$-module $\mathbb{V}^{\mathrm{II}}_{\theta_{max}}\otimes\mathbb{C}$ of the highest weight $\omega_1+3\omega_2$ is the outer product of the irreducible $\mathfrak{sp}(1)$–modules $\mathbb{C}^2$ and $S^3_{\mathbb{C}}\mathbb{C}^2$. The algebra $\mathfrak{sp}(1)^2$ is a compact real form of the rank 2 algebra $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$, therefore the subalgebra of dimension 2 is a Cartan subalgebra $\mathfrak{t}^2$, which is unique up to conjugation. Any subalgebra of dimension >2 contains a Cartan subalgebra, but $\mathfrak{t}^2$ does not annihilate any vector in the module. Therefore the maximal possible annihilator dimension is 1, and any subalgebra of dimension 1 is isomorphic to $\mathfrak{so}(2)$. This is realized by annihilator of the highest weight vector, and its real part has the same annihilator. This annihilator is generated by $3e_{\mathrm{left}}-e_{\mathrm{right}}$, where e_{left} and e_{right} are generators of Cartan subalgebras in the two ideals of $\mathfrak{sp}(1)^2$. ☐

Corollary 7. *The largest annihilator of a non-zero $w\in\mathbb{V}^{\mathrm{II}}_{\theta_{max}}$ with respect to the action of $\mathfrak{g}_0$ is $\mathfrak{sp}(1)\oplus\big(\mathfrak{so}(2)\oplus\mathbb{R}\oplus\mathfrak{gl}(n-2,\mathbb{H})\big)\ltimes\mathfrak{h}_+$, where $\mathbb{R}$ is generated by a suitable linear combination of the grading elements Z and Z' of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}_0$.*

We realize this annihilator in complex notations as follows. Let $\mathbb{H}^n=\mathbb{H}_1\oplus\ldots\oplus\mathbb{H}_n$ and v_m be the standard basis of $\mathbb{H}_m$ ($v=1,i,j,k$), v_m^* be the real dual basis, $m=1,\ldots,n$. Denote by $q_{r,s}\in\mathfrak{sl}(n,\mathbb{H})$ the matrix that contains q on the r-th row and s-th column, and that contains zeros elsewhere. The action on $\mathbb{H}^n$ is $q_{r,s}\cdot v_t=(qv)_r\,\delta_{s,t}$ and the action on $\mathbb{H}^{*n}$ is minus the transpose.

Let $w=1_n^{*3}\otimes 1_1\in\mathbb{V}^{\mathrm{II}}$ (this element is contained in the Cartan product because $\langle 1_n^*,1_1\rangle=0$, so the tensor contractions yield zero). Then $\mathfrak{ann}(w)$ in $\tilde{\mathfrak{g}}_0$ is generated by $q_{r,s}$ for $1\le r<n$, $1<s\le n$ ($q=1,i,j,k$), where if $r=s$ and q real we have to compensate by $1_{n,n}+3\cdot 1_{1,1}$, and the element $i_{n,n}+3\cdot i_{1,1}$. To get the annihilator in $\mathfrak{g}_0$ we add $\mathfrak{sp}(1)$ and the element $Z+1_{1,1}-1_{n,n}$.

Remark 5. The element $w=1_n^{*3}\otimes 1_1\in\mathbb{V}^{\mathrm{II}}$ is actually written in complex tensor notation. To get it as a real tensor, one should project the corresponding real tensor product to its complex linear submodule, and then take another projection to a self-conjugate submodule with respect to an invariant complex conjugation. We note that the first projection depends on the choice of invariant complex structure. We choose $i\in\mathrm{Im}(\mathbb{H})$. Then

$$\mathrm{proj}_i(1_n^{*3}\otimes 1_1)=\tfrac{1}{8}(1_n^{*3}\otimes 1_1+3\cdot 1_n^{*2}i_n^*\otimes i_1-3\cdot 1_n^*i_n^{*2}\otimes 1_1-i_n^{*3}\otimes i_1).$$

A complex conjugation can be chosen as (right) multiplication by j of all tensor factors of a monomial, extended by linearity. A stable element is then given by the projector $\mathrm{proj}_{\mathrm{sc}}$ to the self-conjugate submodule.

$$\begin{aligned}\mathrm{proj}_{\mathrm{sc}}(\mathrm{proj}_i(1_n^{*3}\otimes 1_1))=\tfrac{1}{16}(&1_n^{*3}\otimes 1_1+3\cdot 1_n^{*2}i_n^*\otimes i_1-3\cdot 1_n^*i_n^{*2}\otimes 1_1-i_n^{*3}\otimes i_1+\\&+j_n^{*3}\otimes j_1+3\cdot j_n^{*2}k_n^*\otimes k_1-3\cdot j_n^*k_n^{*2}\otimes j_1-k_n^{*3}\otimes k_1).\end{aligned}$$

Note that the symmetric tensor products come with factors of $\frac{1}{3}$, which will cancel out the factors of 3 in our formula, so that e.g. the coefficient of $1_n^*\otimes 1_n^*\otimes i_n^*\otimes i_1$ is $\frac{1}{16}$.

The tensor $\mathrm{proj}_{\mathrm{sc}}(\mathrm{proj}_i(1_n^{*3} \otimes 1_1))$ has the required annihilator, index symmetries, vanishing contraction, and so can serve as a generator of the real curvature module under the action of $\mathfrak{sl}(n, \mathbb{H})$.

Taking the semi-direct product of the annihilator in $\mathfrak{g}_0$ and the Abelian algebra $\mathbb{H}^n$, we get the graded algebra $\mathfrak{a}^{\psi_{\mathrm{II}}}$ of maximal dimension provided $0 \neq \psi_{\mathrm{II}} \in \mathbb{V}^{\mathrm{II}}$:

$$\mathfrak{a}^{\psi_{\mathrm{II}}} = \Big(\mathfrak{sp}(1) \oplus \big(\mathfrak{so}(2) \oplus \mathbb{R} \oplus \mathfrak{gl}(n-2, \mathbb{H})\big) \ltimes \mathfrak{heis}(8n-12, \mathbb{H})\Big) \ltimes \mathbb{H}^n.$$

This will be shown to be associated to the filtration on the symmetry algebra $\mathfrak{s}$ of a geometry with $\kappa_2 \neq 0$ in the next section.

4.2. **Minimal orbits in the torsion module.** The irreducible G_0-submodule $H_1^2 \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ will be denoted in this subsection by $\mathbb{V}^{\mathrm{I}}$ to indicate its homogeneity 1. It is a quaternionic module. From the weighted Dynkin diagram (our Satake diagram in Section 3 with all nodes white) we see that the the complexification $\mathbb{V}^{\mathrm{I}} \otimes \mathbb{C}$ is an outer product of the $\mathfrak{sp}(1)$-module $S^3_{\mathbb{C}}\mathbb{H}$ and the $\mathfrak{sl}(n, \mathbb{H})$-module $\Lambda^2_{\mathbb{C}}\mathbb{H}^{*n} \odot \mathbb{H}^n$ (this Cartan product is the kernel of the contraction $\Lambda^2_{\mathbb{C}}\mathbb{H}^{*n} \otimes_{\mathbb{C}} \mathbb{H}^n \to \mathbb{H}^{*n}$). We refer to [4, Proposition 4.1.8] for the description of $\mathbb{V}^{\mathrm{I}}$ as the real module.

One could expect that a generator of a minimal orbit can be realized as the tensor product of such generators in each factor, and this is indeed the case. A minimal non-zero $\mathfrak{sp}(1)$-orbit in $S^3_{\mathbb{C}}\mathbb{H}$ can have dimension no less than 2, because the maximal proper subalgebra of $\mathfrak{sp}(1)$ is of dimension 1. This means that any element of the torsion module which achieves maximal stabilizer in $\mathfrak{sl}(n, \mathbb{H})$ and a stabilizer of dimension 1 in $\mathfrak{sp}(1)$ generates a minimal orbit.

Thus, we analyze the torsion module under the action of $\tilde{\mathfrak{g}}_0 = \mathfrak{sl}(n, \mathbb{H})$ alone, which yields

$$\mathbb{V}^{\mathrm{I}} = \mathbb{C}^4 \otimes_{\mathbb{C}} \Lambda^2_{\mathbb{C}}\mathbb{H}^{*n} \odot \mathbb{H}^n,$$

where the first factor $\mathbb{C}^4$ is a trivial module. This decomposes as a direct sum of modules isomorphic to $\Lambda^2_{\mathbb{C}}\mathbb{H}^{*n} \odot \mathbb{H}^n$. One can always find a minimal orbit in a completely reducible module which is contained in an irreducible summand. Moreover this orbit is closed by Lemma 3 and we again utilize Lemma 4 to ensure that the minimal orbit has an element in $\ker \mathfrak{h}_+$. Using the grading element Z' of $\tilde{\mathfrak{g}}_0$ we identify $\mathbb{V}^{\mathrm{I}}_{\theta_{max}} = \ker \mathfrak{h}_+$.

Since the module $\Lambda^2_{\mathbb{C}}\mathbb{H}^{*n} \odot \mathbb{H}^n$ has highest weight $\omega_1 + \omega_{2n-2}$ and is quaternionic, so $\dim_{\mathbb{R}} \mathbb{V}^{\mathrm{I}}_{\theta_{max}} = 4$. Taking $0 \neq w \in \mathbb{V}^{\mathrm{I}}_{\theta_{max}}$ (of pure grade), its annihilator is a graded algebra (containing $\mathfrak{h}_+$).

Lemma 8. *We have:* $\mathfrak{ann}(w) \cap (\tilde{\mathfrak{g}}_0)_- = 0$*, i.e.* $\mathfrak{ann}(w)_- = 0$.

Proof. Let us consider the case $n > 2$ (The case $n = 2$ is a simple adaptation). First, we show that the evaluation map $(\tilde{\mathfrak{g}}_0)_{-1} \otimes \mathbb{V}^{\mathrm{I}}_{\theta_{max}} \to \mathbb{V}^{\mathrm{I}}$ is injective. Take any element $q = q' + q'' \in (\tilde{\mathfrak{g}}_0)_{-1} = (\tilde{\mathfrak{g}}_0)'_{-1} \oplus (\tilde{\mathfrak{g}}_0)''_{-1}$, where the latter splitting into irreps is the same as in the proof of Lemma 5. Using the same argument as in this proof, given a non-zero annihilator element in one of these submodules, we conclude (because $\mathfrak{sl}(n-2, \mathbb{H})$ is in the annihilator) that the whole submodule is in the annihilator. So it is enough to check injectivity of the action on the two elements only, which are $q'_{s,1}$ and $q''_{n,s}$ for $1 < s < n$.

Notice that $\mathbb{V}^{\mathrm{I}}_{\theta_{max}} = \{1_n^* \wedge j_n^* \otimes v_1 : v \in \mathbb{H}\}$. If $q = q'_{s,1}$, then the action is $q \cdot (1_n^* \wedge j_n^* \otimes v_1) = 1_n^* \wedge j_n^* \otimes (qv)_s \neq 0$, and if $q = q''_{n,s}$, then the action is $q \cdot (1_n^* \wedge j_n^* \otimes v_1) = -(q_s''^* \wedge j_n^* + 1_n^* \wedge (q''j)_s^*) \otimes v_1 \neq 0$. Thus $\mathfrak{ann}(w) \cap (\tilde{\mathfrak{g}}_0)_{-1} = 0$.

The rest mimics the proof of Lemma 5: if a non-zero annihilator element exists in $(\tilde{\mathfrak{g}}_0)_{-2}$, then bracketing with $\mathfrak{h}_1$ we obtain a non-zero annihilator element in $(\tilde{\mathfrak{g}}_0)_{-1}$, which is a contradiction. $\square$

Thus it remains to consider the action of $\mathfrak{h}_0$ on $\mathbb{V}^{\mathrm{I}}_{\theta_{max}}$. Since in the semi-simple part $\mathfrak{h}_0^{ss} = \mathfrak{sp}(1)_{\text{left}} \oplus \mathfrak{sl}(n-2,\mathbb{H}) \oplus \mathfrak{sp}(1)_{\text{right}}$ the last two summands are in the annihilator (because of the weight of the representation), this reduces to considering $\mathbb{V}^{\mathrm{I}}_{\theta_{max}}$ as $\mathfrak{sp}(1) = \mathfrak{sp}(1)_{\text{left}}$-module (of the highest weight ω_1). This is the standard left action of $\mathfrak{sp}(1)$ on $\mathbb{H}$, any element of $\mathfrak{sp}(1)$ acts as a complex structure, and so this part gives no contribution to the annihilator of any $0 \neq w \in \mathbb{V}^{\mathrm{I}}_{\theta_{max}}$. Also, similar to the curvature module, a combination of the grading elements acts trivially.

Corollary 9. *The largest annihilator of a non-zero $w \in \mathbb{V}^{\mathrm{I}}_{\theta_{max}}$ with respect to the action of $\mathfrak{g}_0$ is $\mathfrak{so}(2) \oplus \big(\mathbb{R} \oplus \mathfrak{gl}(n-2,\mathbb{H}) \oplus \mathfrak{sp}(1)_{\text{right}}\big) \ltimes \mathfrak{h}_+$, where $\mathbb{R}$ is generated by a suitable linear combination of the grading elements Z and Z' of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}_0$.*

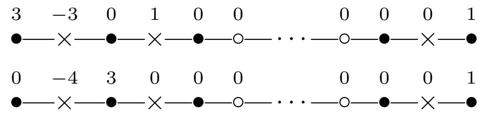
Let us give the generators of this annihilator in the complex tensor notations. Fixing $w = 1_n^* \wedge j_n^* \otimes 1_1$ (again this element is contained in the Cartan product) we conclude that $\mathfrak{ann}(w)$ in $\tilde{\mathfrak{g}}_0$ is generated by the elements $v_{n,n}$ $(v = i, j, k)$ and the elements $q_{r,s}$ for $1 \leq r < n$, $1 < s \leq n$ $(q = 1, i, j, k)$; if $r = s$ and q is real, then $q_{r,s}$ is compensated by $1_{n,n} + 2 \cdot 1_{1,1}$ to belong to $\tilde{\mathfrak{g}}_0$. To get the annihilator in $\mathfrak{g}_0$ we add one element from $\mathfrak{sp}(1)$ and the element $Z + 1_{1,1} - 1_{n,n}$.

Taking the semi-direct product of this annihilator and the Abelian algebra $\mathbb{H}^n$, we get the graded algebra $\mathfrak{a}^{\psi_{\mathrm{I}}}$ of maximal dimension provided $0 \neq \psi_{\mathrm{I}} \in \mathbb{V}^{\mathrm{I}}$:

$$\mathfrak{a}^{\psi_{\mathrm{I}}} = \Big(\mathfrak{so}(2) \oplus \big(\mathbb{R} \oplus \mathfrak{gl}(n-2,\mathbb{H}) \oplus \mathfrak{sp}(1)_{\text{right}}\big) \ltimes \mathfrak{heis}(8n-12,\mathbb{H})\Big) \ltimes \mathbb{H}^n.$$

This will be shown to be associated to the filtration on the symmetry algebra $\mathfrak{s}$ of a geometry with $\kappa_1 \neq 0$ in the next section.

Remark 6. The annihilator algebras from Corollaries 7 and 9 are very similar but not isomorphic. The following is an explanation of this phenomenon. We reduce the curvature- and torsion-modules via the parabolic subalgebra $\mathfrak{p}_{2,2n-2}$ (or $\mathfrak{p}_2$ for $n = 2$) of $\mathfrak{sl}(n,\mathbb{H})$. This yields the diagrams



We note that the numbers above connected pieces correspond to the action of a semi-simple subalgebra, and the numbers above crosses only affects the scaling factors of the center of $\mathfrak{g}_0$. One can express this by the diagram with crosses removed

that is the same for the two modules, after a permutation. Hence the contribution from $\mathfrak{g}_0^{ss}$ to the annihilator must be abstractly isomorphic in the two cases. The difference then comes from the action of $\mathfrak{g}_0^{ss}$ on $\mathfrak{h}_+$.

5. Realizations of sub-maximal models

In the previous section we found the annihilator algebras $\mathfrak{a}^{\psi_{\mathrm{I}}}$ and $\mathfrak{a}^{\psi_{\mathrm{II}}}$ of maximal dimension that is $\mathfrak{U} = 4n^2 - 4n + 9$ in both cases. To prove this is realizable, we follow the idea of [12, §4.2] and deform the graded bracket structure on $\mathfrak{a}$ to obtain a new filtered Lie algebra $\mathfrak{f}$. We use the real highest weight vector in our modules (that correspond to the minus lowest weight vectors of the duals - note that we used flip of the Satake diagram in our construction).

This is expected to correspond to the symmetry algebra of a submaximally symmetric model, which is (non-flat) homogeneous with the isotropy being $\mathfrak{f}_{\geq 0} = \mathfrak{a}_0$ (because of the prolongation-rigidity), and we show it is the case.

To do this we follow the approach in [12, §4.1] that allows to establish an abstract model, basing on the extension functor construction. However we also provide explicit matrix models of the corresponding almost quaternionic manifolds (M, Q), for which the direct computation confirms the amount of symmetry is submaximal $\mathfrak{S} = \mathfrak{U}$. We consider the curvature and torsion cases separately. The corresponding theorems imply the main result of this paper.

We consider the cases when non-zero $\kappa = (\kappa_1, \kappa_2)$ is either $(\psi_{\mathrm{I}}, 0)$ or $(0, \psi_{\mathrm{II}})$. One could also question if the submaximal symmetry dimension can be achieved when both torsion and curvature are non-zero, but even though abstractly the maximal annihilator algebras $\mathfrak{a}^{\psi_{\mathrm{I}}} \simeq \mathfrak{a}^{\psi_{\mathrm{II}}}$ the discussion in Remark 6 shows that the answer to the above question is negative.

5.1. **Realization of the curvature model.** Consider the case of non-vanishing curvature and vanishing torsion first. In order to realize the symmetry algebra, the minimal orbit in the abstract curvature module $\mathbb{V}^{\mathrm{II}}$ needs to be reinterpreted as a deformation of the graded algebra $\mathfrak{a}^{\psi_{\mathrm{II}}}$. This can be done by finding a G_0-equivariant map

$$b : \mathbb{V}^{\mathrm{II}} = S^3_{\mathbb{C}}\mathbb{H}^{*n} \odot \mathbb{H}^n \to \mathbb{B} = \Lambda^2\mathbb{H}^{*n} \otimes \mathfrak{g}_0,$$

and using the Lie bracket deformation given by the image $b(v)$ of a generator $0 \neq v \in \mathbb{V}^{\mathrm{II}}$ of the minimal orbit $G_0 \cdot [v]$.

Lemma 10. *The equivariant map b exists and is unique up to scale.*

Proof. The real module $\mathbb{B}$ is completely reducible, and we compute its decomposition into irreducible submodules by finding an $\mathfrak{sl}(n, \mathbb{H})$-invariant real subspace in the complexification after applying standard methods from the complex representation theory of $A_{2n-1} = \mathfrak{sl}(n, \mathbb{H}) \otimes \mathbb{C}$. We have (in complexification $\mathbb{H}^n = \mathbb{C}^{2n}$):

$$(\Lambda^2_{\mathbb{R}}\mathbb{H}^{*n})^{\mathbb{C}} = \Lambda^2_{\mathbb{C}}(2 \cdot \mathbb{H}^{*n}) = 3 \cdot \Lambda^2_{\mathbb{C}}\mathbb{H}^{*n} \oplus S^2_{\mathbb{C}}\mathbb{H}^{*n}.$$

Here $2 \cdot \mathbb{H}^{*n}$ means the direct sum of 2 copies of $\mathbb{H}^{*n}$, and similarly for $3 \cdot \Lambda^2_{\mathbb{C}}\mathbb{H}^{*n}$. The Cartan product $S^2_{\mathbb{C}}\mathbb{H}^{*n} \odot \mathrm{ad}^{\mathbb{C}}_{\mathfrak{sl}(n,\mathbb{H})}$ has the same highest weight as $\mathbb{V}^{\mathrm{II}}$, and so is isomorphic to it as a complex A_{2n-1}-module, with the isomorphism mapping the $\mathfrak{sl}(n, \mathbb{H})$-invariant real submodules into each other. This is the unique submodule in $\mathbb{B}$ of the required isomorphism type, so the map b is defined and is unique up to scalar multiplication (since $\mathrm{End}_{\mathfrak{g}_0}(\mathbb{V}^{\mathrm{II}}) = \mathbb{R}$, this scalar is a real number). $\square$

We construct b in the complex tensor notations as in the previous section.

Proposition 11. *The bracket deformation on an extremal generator $w \in \mathbb{V}^{\mathrm{II}}_{\theta_{max}}$, corresponding to a minimal orbit $G_0 \cdot [w]$ in $P\mathbb{V}^{\mathrm{II}}$, is given by the formula:*

$$\begin{aligned} b(w) =& (i^*_n \wedge j^*_n - 1^*_n \wedge k^*_n) \otimes (i^*_n \otimes 1_1 - 1^*_n \otimes j_1 + j^*_n \otimes k_1 - k^*_n \otimes j_1) \\ & - (1^*_n \wedge j^*_n + i^*_n \wedge k^*_n) \otimes (1^*_n \otimes 1_1 + i^*_n \otimes i_1 + j^*_n \otimes j_1 + k^*_n \otimes k_1). \end{aligned}$$

Define the deformed Lie bracket on the space of $\mathfrak{a}^{\psi_{\mathrm{II}}}$ via $b(w)$:

$$[\,,\,]_{\mathfrak{f}_{\mathrm{II}}} = [\,,\,]_{\mathfrak{a}^{\psi_{\mathrm{II}}}} + b(w)(\,,\,).$$

Similarly to [12, Lemma 4.1.1] one can check that this is a Lie bracket (the Jacobi identity holds), and the space $\mathfrak{a}^{\psi_{\mathrm{II}}}$ equipped with it is a new (now filtered) Lie algebra $\mathfrak{f}_{\mathrm{II}}$.

This deformation changes the Lie brackets of the previously Abelian subalgebra $\mathbb{H}^n$, and this subspace $\mathbb{H}_n \subset \mathbb{H}^n$ becomes non-Abelian. The new bracket component takes values in the center of the subalgebra $\mathfrak{heis}(8n-12,\mathbb{H})$:

$$[\mathbb{H}_n, \mathbb{H}_n]_{\mathfrak{f}_{\mathrm{II}}} \subset \mathfrak{z}(\mathfrak{heis}(8n-12,\mathbb{H})).$$

Notice that the semi-simple Levi factor of the symmetry algebra is unchanged by this deformation: $\mathfrak{f}_{\mathrm{II}}^{ss} = (\mathfrak{a}^{\psi_{\mathrm{II}}})^{ss} = \mathfrak{sp}(1) \oplus \mathfrak{sl}(n-2,\mathbb{H})$. Due to the presence of the subalgebra $\mathfrak{sl}(n-2,\mathbb{H}) \ltimes \mathbb{H}^{n-2}$ in $\mathfrak{f}_{\mathrm{II}}$, a sub-maximal model can be realized as a direct product (although this notion is coordinate dependent) of the flat structure $\mathbb{H}^{n-2}$ and a sub-maximal structure of dimension 2.

Although the symmetry algebra is not solvable, its solvable radical acts locally transitively, which allows us to integrate the algebra and produce a coordinate description of the model. For $n = 2$ the operator I on $\mathbb{H}^2 = \mathbb{R}^8(h_1,\dots,h_8)$ is given by the matrix

$$I = \begin{pmatrix} A_I & C_I \\ 0 & B_I \end{pmatrix},\quad A_I = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$B_I = \frac{1}{\alpha^2}\begin{pmatrix} 0 & 2h_2^2-\alpha^2 & -2\,h_2\,h_4 & 2\,h_2\,h_3 \\ \alpha^2-2h_2^2 & 0 & 2\,h_2\,h_3 & 2\,h_2\,h_4 \\ 2\,h_2\,h_4 & -2\,h_2\,h_3 & 0 & 2h_2^2-\alpha^2 \\ -2\,h_2\,h_3 & -2\,h_2\,h_4 & \alpha^2-2h_2^2 & 0 \end{pmatrix}$$

$$\begin{aligned} C_I^t = &\frac{1}{2\alpha^2}\begin{pmatrix} 0 & h_2(2\alpha^2-3h_2^2) & h_3(\alpha^2-3h_2^2) & h_4(\alpha^2-3h_2^2) \\ h_2(4\alpha^2-3h_2^2) & 0 & -h_4(h_2^2+\alpha^2) & h_3(h_2^2+\alpha^2) \\ h_4(3h_2^2-\alpha^2) & h_3(3h_2^2+\alpha^2) & h_2(3h_3^2+h_4^2) & 2h_2h_3h_4 \\ h_3(\alpha^2-3h_2^2) & h_4(3h_2^2+\alpha^2) & 2h_2h_3h_4 & h_2(h_3^2+3h_4^2) \end{pmatrix} \\ &+\frac{1}{\alpha^2}\begin{pmatrix} h_2h_5+h_3h_7+h_4h_8 & -h_2h_6-h_3h_8+h_4h_7 & 0 & 0 \\ h_2h_6-h_3h_8+h_4h_7 & h_2h_5-h_3h_7-h_4h_8 & 0 & 0 \\ h_2h_7-h_3h_5-h_4h_6 & -h_2h_8+h_3h_6-h_4h_5 & 0 & 0 \\ h_2h_8+h_3h_6-h_4h_5 & h_2h_7+h_3h_5+h_4h_6 & 0 & 0 \end{pmatrix} \end{aligned}$$

(note the transpose). Here $\alpha^2 = h_2^2 + h_3^2 + h_4^2$. The operator J is given by

$$J = \begin{pmatrix} A_J & C_J \\ 0 & B_J \end{pmatrix},\quad A_J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$B_J = \frac{1}{\alpha^2}\begin{pmatrix} 0 & -2h_2h_3 & 2h_3h_4 & \alpha^2-2h_3^2 \\ 2h_2h_3 & 0 & \alpha^2-2h_3^2 & -2h_3h_4 \\ -2h_3h_4 & 2h_3^2-\alpha^2 & 0 & -2h_2h_3 \\ 2h_3^2-\alpha^2 & 2h_3h_4 & 2h_2h_3 & 0 \end{pmatrix}$$

$$C_J^t = \frac{1}{4\alpha^2}\begin{pmatrix} 3h_4\alpha^2 & h_3(6h_2^2-\alpha^2) & h_2(6h_3^2-\alpha^2) & 6h_2h_3h_4 \\ h_3(6h_2^2-5\alpha^2) & -3h_4\alpha^2 & 2h_2h_3h_4 & h_2(3\alpha^2-2h_3^2) \\ -6h_2h_3h_4 & 3h_2(\alpha^2-2h_3^2) & h_3(\beta^2-4h_3^2) & h_4(3\alpha^2-4h_3^2) \\ h_2(6h_3^2-\alpha^2) & -6h_2h_3h_4 & -h_4(3\alpha^2+4h_3^2) & h_3(\beta^2-4h_4^2) \end{pmatrix}$$
$$+\frac{1}{\alpha^2}\begin{pmatrix} h_2h_7-h_3h_5+h_4h_6 & 0 & h_2h_6+h_3h_8-h_4h_7 & 0 \\ -h_2h_8-h_3h_6-h_4h_5 & 0 & -h_2h_5+h_3h_7+h_4h_8 & 0 \\ -h_2h_5-h_3h_7+h_4h_8 & 0 & h_2h_8-h_3h_6+h_4h_5 & 0 \\ h_2h_6-h_3h_8-h_4h_7 & 0 & -h_2h_7-h_3h_5-h_4h_6 & 0 \end{pmatrix}$$

(note the transpose). Here $\beta^2 = h_2^2 - h_3^2 - h_4^2$. Then we let $K = IJ$.

To get the quaternionic structure for general quaternionic dimension n, re-denote the above operators for $n = 2$ by $I_{(2)}$ and $J_{(2)}$. Now let I be given as the block matrix with $I_{(2)}$ in the top 8×8 block, A_I on the following diagonal 4×4 blocks and zeroes elsewhere. Similarly let J be given as the block matrix with $J_{(2)}$ in the top 8×8 block, A_J on the following diagonal 4×4 blocks and zeroes elsewhere. Define $K = IJ$. Denote the obtained quaternionic structure (I, J, K) by $Q_{\rm II}$.

Theorem 12. *The quaternionic structure $(M, Q_{\rm II})$ has symmetry algebra $\mathfrak{s}_{\rm II}$ of submaximal dimension $\mathfrak{S}_2 = 4n^2 - 4n + 9$.*

Proof. The proof of [12, Lemma 4.1.4] gives the abstract parabolic model via the extension functor construction. The symmetry algebra $\mathfrak{s}$ of this model contains (by construction) the deformed algebra $\mathfrak{f}_{\rm II}$ constructed above. Thus we already have at least $\mathfrak{S}_2$ symmetries. Since this coincides with the universal upper bound $\mathfrak{U}_2 = \mathfrak{U}$, there can be no more symmetries: $\mathfrak{s}_{\rm II} = \mathfrak{f}_{\rm II}$. □

Remark 7. There is a reductive decomposition $\mathfrak{s} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{g}_{-1} = \mathbb{H}^n$. Moreover, we have $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Thus $(\mathfrak{h}, \mathfrak{m})$ is a symmetric pair. This reflects the fact that the submaximally symmetric quaternionic geometry is a locally affine symmetric space in the sense of [14]. Direct computations (in Maple's DifferentialGeometry package, for $n = 2$) gives locally a unique quaternionic invariant connection and this connection has vanishing torsion and parallel curvature. Thus the connection corresponds to the canonical connection on the symmetric space for the pair $(\mathfrak{h}, \mathfrak{m})$. This local connection is hypercomplex for the above I, J, K, and it is the unique Obata connection [1]. Because the structure is torsion–free, the connection is also one of the Oproiu connections and determines the class of Oproiu connections on the submaximal model. Moreover, the connection is Ricci–flat and its curvature coincides with the quaternionic Weyl curvature (and in particular it is harmonic). However, Oproiu connections in the class are not Ricci-flat in general. Finally, direct computation shows that there is no invariant metric (of any signature) on the submaximal model.

5.2. **Realization of the torsion model.** The case of non-vanishing torsion and vanishing curvature can be treated similarly. In this case, we immediately interpret the element

$$w = 1_n^* \wedge j_n^* \otimes 1_1$$

as a deformation to the graded algebra $\mathfrak{a}^{\psi_{\rm I}}$, so the deformed Lie bracket on the space of $\mathfrak{a}^{\psi_{\rm I}}$ is

$$[\,,\,]_{\mathfrak{f}_{\rm I}} = [\,,\,]_{\mathfrak{a}^{\psi_{\rm I}}} + w(\,,\,).$$

As in the curvature case, the previously Abelian subalgebra $\mathbb{H}^n$ will become non-Abelian, but in this case we have

$$[\mathbb{H}_1, \mathbb{H}_1]_{\mathfrak{f}_{\rm I}} \subset \mathbb{H}_n \subset \mathbb{H}^n,$$

which means that $\mathbb{H}^n$ remains a subalgebra. It is a nilpotent ideal of the algebra $\mathfrak{f}_{\mathrm{I}}$ acting locally transitively on the corresponding (local) homogeneous model F/K (cf. [12, Lemma 4.1.4]). Therefore, the minimal model is locally equivalent to a left invariant structure on the nilpotent Lie group corresponding to $(\mathbb{H}^n, [\,,\,]_{\mathfrak{f}_{\mathrm{I}}})$.

As in the curvature case, the semi-simple Levi factor of the symmetry algebra is unchanged by the deformation, and due to the presence of $\mathfrak{sl}(n-2,\mathbb{H})$ the model can once again be realized as a direct product of a submaximal structure in quaternionic dimension 2 and a flat structure in quaternionic dimension $n-2$.

The matrices for $n=2$ in the torsion case turn out to be considerably simpler than in the curvature case. Namely, the operators I and J on $\mathbb{H}^2 = \mathbb{R}^8(h_1,\dots,h_8)$ are given by the following matrices and $IJ = K$.

$$I = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & h_7 & 0 & 0 \\ 1 & 0 & 0 & 0 & h_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -h_7 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -h_7 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

To get the quaternionic structure for general quaternionic dimension n, we again extend the above I by the 4×4 block-matrices that form the block-diagonal of the 8×8 matrix, and do similarly for J; then we define $K = IJ$. Denote the obtained quaternionic structure (I,J,K) by Q_{I}.

Theorem 13. *The quaternionic structure* (M, Q_{I}) *has symmetry algebra* $\mathfrak{s}_{\mathrm{I}}$ *of submaximal dimension* $\mathfrak{S}_1 = 4n^2 - 4n + 9$.

Proof. The proof mimics that of Theorem 12, and we conclude $\mathfrak{S}_1 = \mathfrak{U}_1 = \mathfrak{U}$, and therefore $\mathfrak{s}_{\mathrm{I}} = \mathfrak{f}_{\mathrm{I}}$. □

Remark 8. The submaximally symmetric almost quaternionic geometry of torsion type is locally representable as a group. Such structures always have at least a one-parameter family of invariant connections [14]. Direct computations (in Maple's DifferentialGeometry package) shows that (for $n = 2$) there is a six-parameter family of invariant connections, each with vanishing curvature and parallel torsion. However, only a two-parametric sub-family is quaternionic, and all the invariant quaternionic connections are hypercomplex. Among all these connections, we can find exactly one connection such that its torsion coincides with the structure torsion of the hypercomplex structure, and this is the Obata connection of the hypercomplex structure [1]. However, the structure torsion of the hypercomplex structure differs from that of the almost quaternionic structure, because the submaximal model has non-vanishing (harmonic) torsion. There is no invariant quaternionic connection such that its torsion coincides with the structure torsion of the almost quaternionic structure. Thus, no Oproiu connection is invariant. Clearly, the class of Oproiu connections is invariant, but unlike the quaternionic submaximal model (torsion–free with curvature), there is no fixed point in the class.

Remark 9. Suppose the almost quaternionic structure Q is induced by an almost hypercomplex structure I, J, K. Then the hypercomplex symmetry algebra consists of the quaternionic symmetries that preserve each of I, J, K by itself. In particular, the almost hypercomplex structure inducing the sub-maximal quaternionic structure Q_{II} has hypercomplex symmetry algebras of dimension $4n^2 - 4n + 6$, while that for the almost quaternionic structure Q_{I} has that dimension $4n^2 - 4n + 8$.

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Chapter 5: Local geometric control of a certain mechanism with the growth vector $(4, 7)$

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LOCAL GEOMETRIC CONTROL OF A CERTAIN MECHANISM WITH THE GROWTH VECTOR $(4,7)$

JAROSLAV HRDINA AND LENKA ZALABOVÁ

Abstract. We study local control of the mechanism with the growth vector $(4,7)$. We study controllability and extremal trajectories on the nilpotent approximation as an example of the control theory on Lie group. We give solutions of the system an show examples of local extremal trajectories.

1. Introduction

Originally, the general trident snake robot has been introduced in [11]. Let us recall that the trident robot is a mechanism composed of three snake–legs, each connected to an equilateral triangle root block in its vertices [10, 11, 12, 21] for further details. Generally, the branches can be multi–link, assumed that each link has its own passive wheel, which provides footing for the robot. Active elements, which affect controllability, are placed on branches. Its simplest non–trivial version, corresponding to one–links, has been mainly discussed, see e.g. [12, 10, 21]. In this case, the control distribution is that of the growth vector $(3,6)$ [20].

We are interested in the modification corresponding to one or more prismatic joints such that the control distribution will be a that of the growth vector $(4,7)$. Local controllability of such robot is given by the appropriate Pfaff system of ODEs. The solution space gives a control system $\dot{q} = \sum u_i X_i$ where the vector fields X_1, X_2, X_3, X_4 describe the horizontal distribution and $u : \mathbb{R} \to \mathbb{R}^4$ is the control of the system. Consequently, the system is controllable by Chow–Rashevsky theorem [1, 14, 5], see Section 2.

We construct a nilpotent approximation to get nilpotent Lie algebra and corresponding Lie group to study the controllability of approximated left invariant control system, see Section 3. We study geometric properties and symmetries of the nilpotent approximation in Section 4. We use the theory of Hamiltonians and Pontryagin's maximum principle to study local control and extremal trajectories, see Section 5. In particular, we provide analysis of the system and present explicit solutions.

2. Analysis of the mechanisms

In this Section we describe a mechanism that is a modification of the trident snake robot (for more details see [10, 11, 12, 21]). We present our mechanisms as a new example of non-holonomic systems with multi–generators and discuss local controllability of our mechanism based on the principles of non-holonomic mechanics.

2.1. **Description of the mechanism and its movement.** In the sequel, we study a generalized trident mechanism which consists of a root block in the shape of an equilateral triangle with three 1–link branches with variable length. Each of

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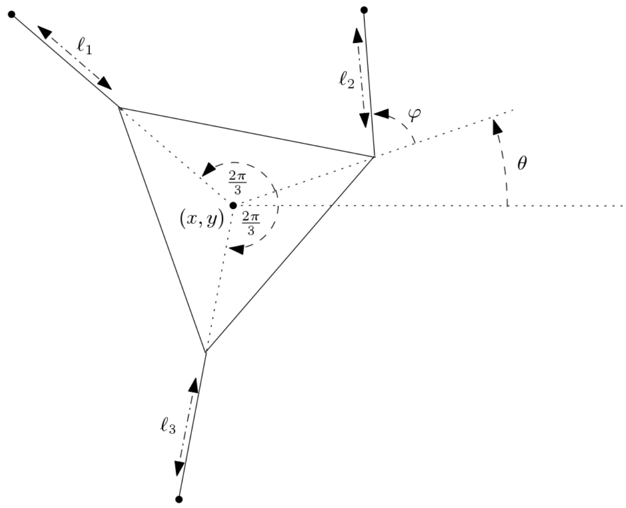

FIGURE 1. Description of the mechanism

the branch is connected to one vertex of the root block and they form prismatic joints. Second leg is in addition connected to the root block via the revolute joint, connection of the two remaining is fixed. Each link has a passive wheel on its branches, which is assumed neither to slip, nor slide sideways.

The configuration space of the planar mechanism in question corresponds to a manifold that locally coincides with $\mathbb{R}^7$ (but not globally). Since we study local problems, we can consider the configuration space to be $\mathbb{R}^7$ with the coordinates $(x, y, \theta, \varphi, \ell_1, \ell_2, \ell_3)$. With respect to a fixed coordinate system, the first three coordinates describe completely the global position of the mechanism in the plane. The coordinates x and y give the position of the centre of mass of the root block in $\mathbb{R}^2$ and θ gives the amount of the counter-clockwise rotation. Remaining four coordinates represent the input for the mechanism's active elements. Thus as active elements we consider the revolute joint of the branch ℓ_2 with the root block, namely φ, and prismatic joints, which can change absolute lengths of branches ℓ_1, ℓ_2 and ℓ_3, see Figure 1.

To provide the description of the robot's movement, we shall only point out that we do not get any singularities as long as the robot's configuration satisfies that $0 < \ell_i$ and φ is not too far from 0.

Using the method of moving frame, the kinematic model is the set of equations for i–th wheel position in the form

$$\begin{aligned} x_i &= x + \cos(\theta + \alpha_i) + \ell_i \cos(\theta + \alpha_i), \\ y_i &= y + \sin(\theta + \alpha_i) + \ell_i \sin(\theta + \alpha_i), \end{aligned} \tag{1}$$

for $i = 1, 3$, where $\alpha_1 = -\frac{2}{3}\pi$, $\alpha_3 = \frac{2}{3}\pi$, and

$$\begin{aligned} x_2 &= x + \cos(\theta) + \ell_2 \cos(\theta + \varphi), \\ y_2 &= y + \sin(\theta) + \ell_2 \sin(\theta + \varphi). \end{aligned} \tag{2}$$

Consequently, we differentiate the position equations (1) and (2) with respect to time t and obtain the velocity equations as follows

$$\begin{aligned} \dot{x}_i &= \dot{x} - \sin(\theta + \alpha_i)\dot{\theta} - \ell_i \sin(\theta + \alpha_i)\dot{\theta} + \cos(\theta + \alpha_i)\dot{\ell}_i, \\ \dot{y}_i &= \dot{y} + \cos(\theta + \alpha_i)\dot{\theta} + \ell_i \cos(\theta + \alpha_i)\dot{\theta} + \sin(\theta + \alpha_i)\dot{\ell}_i, \end{aligned}$$

for $i = 1, 3$, where $\alpha_1 = -\frac{2}{3}\pi$, $\alpha_3 = \frac{2}{3}\pi$, and

$$\dot{x}_2 = \dot{x} - \sin(\theta)\dot{\theta} - \ell_2 \sin(\theta+\varphi)(\dot{\theta}+\dot{\varphi}) + \cos(\theta+\varphi)\dot{\ell}_2,$$
$$\dot{y}_2 = \dot{y} + \cos(\theta)\dot{\theta} + \ell_2 \cos(\theta+\varphi)(\dot{\theta}+\dot{\varphi}) + \sin(\theta+\varphi)\dot{\ell}_2.$$

The conditions preventing slipping lead to the non-holonomic constraints of the form

$$0 = (-\sin(\theta+\alpha_i), \cos(\theta+\alpha_i)) \cdot (\dot{x}_i, \dot{y}_i),$$
$$0 = (-\sin(\theta+\varphi), \cos(\theta+\varphi)) \cdot (\dot{x}_2, \dot{y}_2),$$

where $i = 1, 3$, $\alpha_1 = -\frac{2}{3}\pi$, $\alpha_3 = \frac{2}{3}\pi$ and $\cdot$ is the Riemannian scalar product on the Euclidean space $\mathbb{R}^2$. We obtain the following differential kinematic equations, so the movement of the mechanism is described by the Pfaff system of three nonlinear homogeneous equations

$$\begin{aligned} 0 &= -\sin(\theta - \frac{2\pi}{3})\mathrm{dx} + \cos(\theta - \frac{2\pi}{3})\mathrm{dy} + (1+\ell_1)\mathrm{d}\theta, \\ 0 &= -\sin(\theta+\varphi)\mathrm{dx} + \cos(\theta+\varphi)\mathrm{dy} + (-\cos(\varphi)+\ell_2)\mathrm{d}\theta - \ell_2\mathrm{d}\varphi, \\ 0 &= -\sin(\theta + \frac{2\pi}{3})\mathrm{dx} + \cos(\theta + \frac{2\pi}{3})\mathrm{dy} + (1+\ell_3)\mathrm{d}\theta. \end{aligned} \tag{3}$$

2.2. **Local controllability of the system.** The space of solutions to the system (3) forms four dimensional distribution on the configuration space, so–called horizontal distribution. It follows from (3) that the solution space always contains the vector fields $X_2 := \partial_{\ell_1}$, $X_3 := \partial_{\ell_2}$ and $X_4 := \partial_{\ell_3}$ as generators. In the case $\ell_2 \neq 0$ the last generating vector field X_1 is a combination of $\partial_x, \partial_y, \partial_\theta$ and ∂_φ which is generically complicated and we do not need to write it here in the biggest generality while $X_1 = \partial_\varphi$ in the case $\ell_2 = 0$. In fact, from the mechanical point of view, zero length of each leg makes no sense, so we suppose in the following that $\ell_i > 0$.

We can equivalently rewrite the solution space of the Pfaff system (3) in the following form

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q) + u_3 X_3(q) + u_4 X_4(q), \tag{4}$$

where $q = (x, y, \theta, \varphi, \ell_1, \ell_2, \ell_3)$. This is a 4–input symmetric affine control system. In general, controllability of symmetric affine systems is completely characterized by the controllability Lie algebra by Chow–Rashevsky's theorem [14, 23, 19]. Control system (4) satisfies the Chow's condition at the point q if $Lie(X_1, X_2, X_3, X_4)(q) = T_q\mathbb{R}^7$, where the controllability Lie algebra $Lie(X_1, X_2, X_3, X_4)$ is the Lie algebra generated by X_1, X_2, X_3, X_4. In this case point q is regular and can be connected with any point in a suitable neighbourhood of q by a horizontal trajectory. If the Chow's condition is satisfied at all points (of a connected space) then any two points can be joined by a horizontal trajectory and the system is locally controllable.

In our case, mechanical description leads to an observation that local controllability depends on the shape of the mechanism only, not on its configuration in the plane. In other words, regular points have to be invariant with respect to the rigid body (Euclidean) transformations of the plane. So we choose $x = y = 0$ and $\theta = \frac{\pi}{2}$ without loss of generality. In particular, for points of the form $q_0 = (0, 0, \frac{\pi}{2}, \varphi, \ell_1, \ell_2, \ell_3)$ the vector fields

$$X_1 = \partial_x + \frac{(\ell_1 - \ell_3)\sqrt{3}}{3L}\partial_y - \frac{1}{L}\partial_\theta + \frac{\sin(\varphi)\sqrt{3}(\ell_1-\ell_3) + 3\cos(\varphi)(L+1) + 3\ell_2}{3\ell_2 L}\partial_\varphi$$
$$X_2 = \partial_{\ell_1}, \quad X_3 = \partial_{\ell_2}, \quad X_4 = \partial_{\ell_3},$$

generate the solution space of our Pfaff system, where we denote $L = \ell_1 + \ell_3 + 2$. Moreover, at q_0, the controllability Lie algebra is obtained by the Lie bracket operation as

$$X_{12} := [X_1, X_2] = \frac{-2(\ell_3+1)}{\sqrt{3}L^2}\partial_y - \frac{1}{L^2}\partial_\theta + \frac{-2\sin(\varphi)(\ell_3+1) + \sqrt{3}\cos(\varphi) + \sqrt{3}\ell_2}{\sqrt{3}\ell_2 L^2}\partial_\varphi,$$
$$X_{13} := [X_1, X_3] = \frac{\sin(\varphi)(\ell_1-\ell_3) + \sqrt{3}\cos(\varphi)(L+1)}{\sqrt{3}\ell_2^2 L}\partial_\varphi,$$
$$X_{14} := [X_1, X_4] = \frac{2\ell_1+2}{\sqrt{3}L^2}\partial_y - \frac{1}{L^2}\partial_\theta + \frac{2\sin(\varphi)(\ell_1+1) + \sqrt{3}\cos(\varphi) + \sqrt{3}\ell_2}{\sqrt{3}\ell_2 L^2}\partial_\varphi,$$

and remaining brackets are trivial. Then the matrix $\bar{G}$ consisting of coordinates of vector fields $X_1, X_2, X_3, X_4, X_{12}, X_{13}, X_{14}$ spans full tangent space $\mathbb{R}^7$ as long as $\det(\bar{G}(q_0)) \neq 0$ and the system is locally controllable at q_0. Thus our system is locally controllable at q_0 (and without loss of generality everywhere because of mechanical meaning) because $\ell_2 \neq 0$ and $L \neq 0$.

We consider the filtration $\Delta^1 = \langle X_1, \dots, X_4\rangle \subset \Delta^2 = \langle X_1, \dots, X_4, X_{12}, X_{13}, X_{14}\rangle$. We have $\dim \Delta^1(q) = 4$ and $\dim \Delta^2(q) = 7$ at all points and we have filtration with the growth vector $(4, 7)$.

2.3. **Remark on corresponding dynamical systems.** If we restrict our considerations only to nontrivial movements of the root block, i.e. movements in the X_1 direction or in the direction of iterated bracket of X_1 and X_i for $i = 2, 3, 4$, we can use methods of dynamic pairs.

In general, each control affine system

$$\dot{x} = X(x) + \sum_{j=1}^{m} u_j Y_j(x)$$

on a manifold M, where $X, Y_1, \dots, Y_m$ are smooth vector fields on M and $u = (u_1, \dots, u_m)^T$ are controls, defines a dynamic pair (X, V), where V is a distribution spanned by $Y_1, ..., Y_m$. Then there is a sequence of distributions defined inductively, using Lie bracket, by $V^0 := V,\ V^{i+1} := V^i + [X, V^i]$ and one imposes the regularity conditions [13, 8]:

$$\operatorname{rk} V^i = (i+1)m, \quad \text{for } i = 0, \dots, k,$$
$$V^k \oplus \langle X\rangle = TM.$$

Then our control system (4) can be adapted to this situation as $X := f(q)X_1$, where f is an arbitrary non–zero function and $Y_1 := X_2, Y_2 := X_3, Y_3 := X_4$, and the regularity conditions are satisfied. Indeed,

$$V^0 = \langle Y_1, Y_2, Y_3\rangle,$$
$$V^1 = \langle Y_1, Y_2, Y_3, [X, Y_1], [X, Y_2], [X, Y_3]\rangle,$$

such that $\operatorname{rk} V^0 = 3 = m$, $\operatorname{rk} V^1 = 6 = 2m$ and $V^1 \oplus \langle X\rangle = TM$. In fact, each $f(q)$ defines a dynamical system with specific drift and all of them are regular.

2.4. **Remark on dual curvature.** Following [18, 7], curvature of a distribution H on a manifold Q is a linear bundle map $F : \wedge^2 H \to TQ/H$ defined by $F(X, Y) = -[X, Y] \bmod H$. Denote by $H^\perp$ the bundle of covectors that annihilate H. Since the curvature F is a linear bundle map, the dual of curvature is a linear map $F^* : H^\perp \to \wedge^2 H^*$, called the dual curvature. Because our distribution is equipped with the growth vector $(4, 7)$, the space $\operatorname{Im}(F^*)$ is three–dimensional subspace of $\wedge^2 H^*$. We can define the Pfaffian $H^\perp \to \wedge^4 H^*$ as $\mu \mapsto F^*(\mu) \wedge F^*(\mu)$. One can see that $\wedge^4 H_q^*$, $q \in Q$, is one–dimensional vector space and the Pfaffian may be understood as a real valued quadratic form on $H^\perp$ by choosing a volume form. Then possible signatures for the Pfaffian are $(3,0),(2,1),(2,0),(1,1),(1,0),(0,0)$. Note that the signatures (p,r) and (r,p) must be considered as identical because $\wedge^4 H_q^*$ is not oriented.

From parabolic geometry point of view, the generic distribution with growth vector $(4,7)$ corresponds to quaternionic contact structures and split quaternionic contact structures for signature $(3,0)$ and $(2,1)$, respectively. The distribution corresponding to our mechanism has signature $(0,0)$ and we will see that it is a parabolic geometry called generalized path geometry [6]. Let us point out that there exist different modifications of trident snake robot that lead to the growth vector $(4,7)$. However, it turned out that all of them have non-regular signature. We have not found a mechanism with signature $(3,0)$ or $(2,1)$, yet.

3. Nilpotent approximation

We recall a constructive method to approximate vector fields of a nonlinear control system by a similar system on the same configuration space. The method leads to an approximate distribution which has a nilpotent basis. The techniques of nilpotent approximation have been developed by various researchers, see e.g. [2, 9]. We recall the following concept of orders of functions or vector fields and distribution weights. Let X_i, $i = 1,\dots,m$ denote a family of smooth vector fields on a manifold M and $C^\infty(p)$ the set of germs of smooth functions at $p \in M$. For $f \in C^\infty(p)$ we say that Lie derivatives $X_i f, X_i X_j f, \dots$ are non–holonomic derivatives of f of order $1,2,\dots$. The non–holonomic derivative of order 0 of f at p is $f(p)$. Then the non–holonomic order $\mathrm{ord}_p(f)$ of f at p is the biggest integer k such that all non–holonomic derivatives of f of order smaller than k vanish at p, i.e.

$$\mathrm{ord}_p(f) = \min\Big\{ s \in \mathbb{N} : \exists i_1 \dots i_s \in \{1,\dots,m\} \text{ s.t.} (X_{i_1} \cdots X_{i_s} f)(p) \neq 0 \Big\}.$$

Denote by $VF(p)$ the set of germs of smooth vector fields at $p \in M$. Then the notion of non–holonomic order extends to vector fields as follows: For $X \in VF(p)$ the non–holonomic order $\mathrm{ord}_p(X)$ of X at p is a real number defined by

$$\mathrm{ord}_p(X) = \sup\Big\{ \sigma \in \mathbb{R} : \mathrm{ord}_p(Xf) \geq \sigma + \mathrm{ord}_p(f), \forall f \in C^\infty(p) \Big\}.$$

Note that $\mathrm{ord}_p(X) \in \mathbb{Z}$. Moreover, the zero vector field $X \equiv 0$ has infinite order, i.e. $\mathrm{ord}_p(0) = \infty$. Furthermore, $X_1,\dots,X_m$ are of order ≥ -1, $[X_i, X_j]$ of order ≥ -2, etc.

3.1. **Construction.** We construct a nilpotent approximation of the distribution with respect to the given filtration at point $q_0 = (0,0,\frac{\pi}{2},0,1,1,1)$. We use Bellaïche algorithm, which may be found in [4]. Let us point out that all constructions are local in the neighbourhood of q_0. In our case, as the first step of Bellaïche algorithm, the adapted frame

$$X_1, X_2, X_3, X_4, X_{12}, X_{13}, X_{14} \tag{5}$$

was chosen. Then we use four local coordinates $(x, \ell_1, \ell_2, \ell_3)$ as first four adapted coordinates. The others can be obtained from the original coordinate system by an affine change in the form

$$
\begin{aligned}
y_1 &= -2x - 2\sqrt{3}y - 8\theta, \\
y_2 &= \frac{4}{5}\varphi - \frac{4}{5}x + \frac{8}{5}\theta, \\
y_3 &= -2x + 2\sqrt{3}y - 8\theta,
\end{aligned} \tag{6}
$$

where we use the following conditions:

$$
\begin{aligned}
&\partial_x|_{q_0} = X_1(q_0),\ \partial_{\ell_1}|_{q_0} = X_2(q_0),\ \partial_{\ell_2}|_{q_0} = X_3(q_0),\ \partial_{\ell_3}|_{q_0} = X_4(q_0), \\
&\partial_{y_1}|_{q_0} = X_{12}(q_0),\ \partial_{y_2}|_{q_0} = X_{13}(q_0),\ \partial_{y_3}|_{q_0} = X_{23}(q_0).
\end{aligned}
$$

Finally, note that the adapted coordinates $(x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3)$ are privileged coordinates in the case of adapted frame with growth vector $(4, 7)$ [14].

Following [14], we group together the monomial vector fields in Taylor expansions of the same weighted degree and thus we express X_i as a series

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \cdots ,$$

for $i = 1, 2, 3, 4$, where $X_i^{(s)}$ is a homogeneous vector field of order s. By [14, Proposition 2.3.] we set $N_i := X_i^{(-1)}$ for $i = 1, 2, 3, 4$. The family of vector fields (N_1, N_2, N_3, N_4) forms called first order approximation of (X_1, X_2, X_3, X_4) at q_0 and generates a nilpotent Lie algebra of step $r = 2$, i.e. all brackets of length greater than 2 are zero. The family (N_1, N_2, N_3, N_4) is called the (homogeneous) nilpotent approximation of (X_1, X_2, X_3, X_4) at q_0 associated with coordinates $(x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3)$.

In the sequel, by the above algorithm, we obtain the following vector fields:

$$
\begin{aligned}
&N_1 = \partial_x - \left(\frac{-\sqrt{3}}{2}x + \ell_1 - 1\right)\partial_{y_1} - (\ell_2 - 1)\partial_{y_2} - \left(\frac{\sqrt{3}}{2}x + \ell_3 - 1\right)\partial_{y_3}, \\
&N_2 = \partial_{\ell_1},\quad N_3 = \partial_{\ell_2},\quad N_4 = \partial_{\ell_3}.
\end{aligned}
$$

In particular, the family of vector fields (N_1, N_2, N_3, N_4) is the nilpotent approximation of vector fields (X_1, X_2, X_3, X_4) at $(0, 0, \frac{\pi}{2}, 0, 1, 1, 1)$ in the coordinates $(x, y, \theta, \varphi, \ell_1, \ell_2, \ell_3)$, while it is the point $(0, 1, 1, 1, -4\pi, \frac{4}{5}\pi, -4\pi)$ in the coordinates $(x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3)$.

The remaining three vector fields N_{12}, N_{13}, N_{14} are generated by Lie brackets of (N_1, N_2, N_3, N_4) as

$$N_{12} = [N_1, N_2] = \partial_{y_1},\quad N_{13} = [N_1, N_3] = \partial_{y_2},\quad N_{14} = [N_1, N_4] = \partial_{y_3}.$$

Note that due to the linearity of coefficients of (N_1, N_2, N_3, N_4), the coefficients of (N_{12}, N_{13}, N_{14}) must be constant.

To show how the nilpotent approximation affects integral curves of the distribution and the resulting control, we compute the Lie brackets of relevant vector fields. In Fig. 2, there is a comparison of the Lie bracket motions in the original distribution (red line) and in the nilpotent approximation (blue line). The following figures show the trajectories of the root centre point, vertices and wheels when a particular Lie bracket motion is realized. To simulate the bracket motion of the nilpotent approximation, we choose the initial state q_0 and apply the periodic input on couples (N_1, N_2), (N_1, N_3) and (N_1, N_4) to receive the displacement approximately parallel to $[N_1, N_2]$, $[N_1, N_3]$ and $[N_1, N_4]$ respectively [19]. More precisely, for one cycle we apply

$$u_1(t) = -A\omega\sin(\omega t),\quad u_i(t) = A\omega\cos(\omega t),\quad u_j(t) = 0,\quad u_k(t) = 0$$

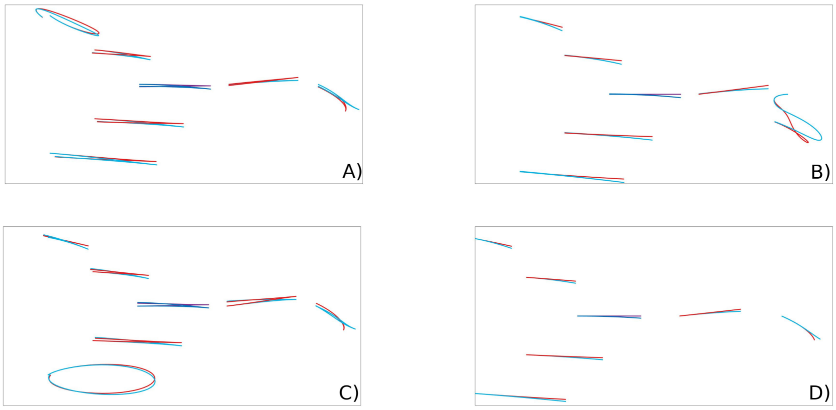

FIGURE 2. Motions of the mechanism in directions: A) N_{12}, X_{12}, B) N_{13}, X_{13}, C) N_{14}, X_{14}, D) N_1, X_1,

for $i \in \{2,3,4\}$ and $j,k \in \{2,3,4\} - \{i\}$, respectively, all with respect to the control system (10), amplitude $A = 0.4$ and angular speed $\omega = \frac{2\pi}{50}$. Then we apply the same process to original vector fields X_1, X_2, X_3 and X_4.

3.2. **Control theory on nilpotent Lie groups.** The above construction led to vector fields $N_1, N_2, N_3, N_4, N_{12}, N_{13}, N_{14}$ which determine a 7–dimensional nilpotent Lie algebra $\mathfrak{n}$. The computation gives that

$$[N_1, N_2] = N_{12}, \quad [N_1, N_3] = N_{13}, \quad [N_1, N_4] = N_{14}$$

and the remaining brackets are trivial. The corresponding connected simply connected nilpotent Lie group $N \simeq \mathbb{R}^7$ with the nilpotent Lie algebra $\mathfrak{n}$ is then endowed, in new coordinates $(x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3)$, with the following group structure

$$\begin{pmatrix} x \\ \ell_1 \\ \ell_2 \\ \ell_3 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \times \begin{pmatrix} \bar{x} \\ \bar{\ell}_1 \\ \bar{\ell}_2 \\ \bar{\ell}_3 \\ \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix} = \begin{pmatrix} x + \bar{x} \\ \ell_1 + \bar{\ell}_1 \\ \ell_2 + \bar{\ell}_2 \\ \ell_3 + \bar{\ell}_3 \\ y_1 + \bar{y}_1 + \frac{\sqrt{3}}{2} x\bar{x} - \ell_1 \bar{x} \\ y_2 + \bar{y}_2 - \ell_2 \bar{x} \\ y_3 + \bar{y}_3 - \frac{\sqrt{3}}{2} x\bar{x} - \ell_3 \bar{x} \end{pmatrix} \tag{7}$$

and the vector fields N_1, N_2, N_3, N_4, N_{12}, N_{13}, N_{14} are left–invariant with respect to the left action given by the group structure. In particular, the vector fields N_i for $i = 1,2,3,4$ determine a left–invariant distribution $\mathcal{D}$ on N, which has the growth vector $(4,7)$ everywhere.

Altogether, $\mathcal{D} \subset TN$ defines an invariant 4–input symmetric affine control system

$$\dot{q} = u_1 N_1 + u_2 N_2 + u_3 N_3 + u_4 N_4, \tag{8}$$

with $q = (x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3)$, that approximates the original control system. It clearly satisfies the Chow's condition and is controllable.

4. INFINITESIMAL SYMMETRIES

We focus on basic geometric properties and infinitesimal symmetries of the nilpotent approximation. By infinitesimal symmetries we mean vector fields such that their flows preserve the geometric structure at any time [22]. In our case, infinitesimal symmetries preserve the distribution and also the metric.

4.1. **Generalized path geometries and their symmetries.** Let us discuss the geometric structure that occurs behind the control problem (8). In the previous section, we constructed a nilpotent Lie group N with a filtered nilpotent Lie algebra $\mathfrak{n}$ with the growth vector $(4,7)$, where the 4–dimensional distribution $\mathcal{D}$ is generated by the left–invariant fields N_1, N_2, N_3 and N_4. Consider subbundles $E = \langle N_1 \rangle$, $V = \langle N_2, N_3, N_4 \rangle$ in TN. One can see from the structure of Lie brackets that the following holds:

(1) $E \cap V = 0$,
(2) Lie bracket of two sections of V is a section of $E \oplus V$, and
(3) for sections $\xi \in \Gamma(E)$ and $\nu \in \Gamma(V)$ and a point $q \in N$, the equation $[\xi, \nu](q) \in E_q \oplus V_q$ implies that $\xi(q) = 0$ or $\nu(q) = 0$.

Such geometric structures are usually called *(generalized) path geometries* (in dimension 7) [6, Section 4.4.3].

General theory [6] says that generalized path geometries have finite–dimensional Lie algebras of symmetries and in the case of 7–dimensional manifolds, the maximal possible dimension is 24. This is the case of generalized path geometries that are locally equivalent to a generalized flag manifold $PSL(5,\mathbb{R})/P_{1,2}$, where by $PSL(5,\mathbb{R})$ we denote the projectivised special linear group with Lie algebra $\mathfrak{sl}(5,\mathbb{R})$, and by $P_{1,2}$ the stabiliser of the flag of a line in a plane for the projectivised standard action of $PSL(5,\mathbb{R})$. In particular, the Lie algebra of symmetries for such generalized path geometry is exactly the simple Lie algebra $\mathfrak{sl}(5,\mathbb{R})$ and the symmetries with a fixed point form its 17–dimensional parabolic subalgebra.

There is a general method to find all infinitesimal symmetries of a nilpotent filtered structure [22]. One can apply this method to our structure $(N, E \oplus V \subset TN)$ and it turns out that the symmetry algebra is of dimension 24 and is exactly $\mathfrak{sl}(5,\mathbb{R})$. In other words, symmetries of $(N, E \oplus V \in TN)$ are left multiplications by elements of a Lie group with the Lie algebra $\mathfrak{sl}(5,\mathbb{R})$ and the structure is left–invariant with respect to this action. From this point of view, $N \simeq PSL(P,\mathbb{R})/P_{1,2}$. Altogether, nilpotent approximation forms a flat generalized path geometry.

Remark 1. *The concept of Cartan geometries* [24] *generalizes the concept of Klein geometries* [15, 16] *and generalized flag manifolds are special cases of Klein geometries for the case of parabolic subgroups in semisimple groups. The Cartan's generalization then leads to a wide theory of parabolic geometries* [6]*, that are curved versions of flag manifolds. Generalized path geometries are examples of such parabolic geometries* [6, Definition 4.4.3.].

4.2. **Sub–Riemannian structure and its symmetries.** To study extremal trajectories in the next section, we need the sub–Riemannian structure on the nilpotent approximation. We consider a control metric g in $\mathcal{D} = \langle N_1, N_2, N_3, N_4 \rangle$ such that the fields N_i for $i = 1,2,3,4$ are orthogonal and have the length one with respect to g. This clearly determines a left–invariant sub–Riemannian structure g of $\mathcal{D}$ (with respect to the action given by group structure (7) on N).

Let us now focus on the symmetries of the nilpotent control problem $(M, \mathcal{D} = E + V, g)$. Thus we are interested in symmetries that preserve not only the flat generalized path geometry, but also the control metric. The symmetry algebra $\mathfrak{k}$ of $(M, E + V, g)$ clearly is a subalgebra of $\mathfrak{sl}(5,\mathbb{R})$. In fact, both $\mathfrak{k}$ and $\mathfrak{sl}(5,\mathbb{R})$ contain the same nilpotent subalgebra that reflects the nilpotent group structure and that acts effectively and transitively on N. It is generated by vector fields

$$\begin{aligned}
w_1 &:= -\partial_x - \frac{\sqrt{3}}{2}\partial_{\ell_1} + \frac{\sqrt{3}}{2}x\partial_{y_3},\\
w_2 &:= \partial_{\ell_1} - x\partial_{y_1},\\
w_3 &:= \partial_{\ell_2} - x\partial_{y_2},\\
w_4 &:= \partial_{\ell_3} - x\partial_{y_3},\\
w_{12} &:= \partial_{y_1},\\
w_{13} &:= \partial_{y_2},\\
w_{14} &:= \partial_{y_3}.
\end{aligned}$$

In general, $\mathfrak{k}$ also contains symmetries preserving arbitrary fixed point. Since all points are equivalent, we can fix the origin $o = (0,0,0,0,0,0,0)$. If we study symmetries preserving the origin, intuition suggests that the sub–Riemannian metric g shall be preserved by an orthogonal algebra $\mathfrak{so}(4)$. However, each such symmetry shall also preserve the control distribution $\mathcal{D}$ and its decomposition into E and V. Thus it acts trivially on the 1–dimensional subspace E_o and restricts to the action of $\mathfrak{so}(3)$ on the 3–dimensional subspace V_o. Direct computation gives that there really is the symmetry algebra $\mathfrak{so}(3) \subset \mathfrak{k}$ preserving the origin generated by fields

$$\begin{aligned}
v_1 &:= -\ell_3\partial_{\ell_2} + \ell_2\partial_{\ell_3} - (\frac{\sqrt{3}x^2}{4} - x + y_3)\partial_{y_2} - (x - y_2)\partial_{y_3},\\
v_2 &:= \ell_3\partial_{\ell_1} - \ell_1\partial_{\ell_3} + (\frac{\sqrt{3}x^2}{4} - x + y_3)\partial_{y_1} + (\frac{\sqrt{3}x^2}{4} + x - y_1)\partial_{y_3},\\
v_3 &:= -\ell_2\partial_{\ell_1} + \ell_1\partial_{\ell_2} + (x - y_2)\partial_{y_1} - (\frac{\sqrt{3}x^2}{4} + x - y_1)\partial_{y_2}.
\end{aligned} \tag{9}$$

We can write it also in a 'vector–matrix–like' notation as

$$\begin{aligned}
\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix} &= \begin{pmatrix} 0 & -\ell_3 & \ell_2\\ \ell_3 & 0 & -\ell_1\\ -\ell_2 & \ell_1 & 0\end{pmatrix}\begin{pmatrix}\partial_{\ell_1}\\ \partial_{\ell_2}\\ \partial_{\ell_3}\end{pmatrix}\\
&+ \begin{pmatrix} 0 & -(\frac{\sqrt{3}x^2}{4} - x + y_3) & -(x - y_2)\\ \frac{\sqrt{3}x^2}{4} - x + y_3 & 0 & \frac{\sqrt{3}x^2}{4} + x - y_1\\ x - y_2 & -(\frac{\sqrt{3}x^2}{4} + x - y_1) & 0\end{pmatrix}\begin{pmatrix}\partial_{y_1}\\ \partial_{y_2}\\ \partial_{y_3}\end{pmatrix}.
\end{aligned}$$

One can verify directly that $\mathcal{L}_{v_i}V \subset V$ and $\mathcal{L}_{v_i}N_1 = 0$ and $\mathcal{L}_{v_i}g = 0$ for $i = 1,2,3$, where $\mathcal{L}$ denotes the Lie derivative, and that $[v_1, v_2] = -v_3$, $[v_1, v_3] = v_2$ and $[v_2, v_3] = -v_1$.

4.3. **Properties of the $\mathfrak{so}(3)$–action.** The action of $\mathfrak{so}(3)$ is clear if the action of the generators v_1, v_2, v_3 is explained: the action decomposes into two independent actions first of which is on $\mathbb{R}^3$ given by ∂_{ℓ_1}, ∂_{ℓ_2} and ∂_{ℓ_3} and the second one on $\mathbb{R}^4$ given by ∂_x, ∂_{y_1}, ∂_{y_2} and ∂_{y_3}. Moreover, the algebra $\mathfrak{so}(3)$ acts trivially on ∂_x, which then defines a one–dimensional invariant subspace. The action of the algebra $\mathfrak{so}(3)$ on $(\partial_{\ell_1}, \partial_{\ell_2}, \partial_{\ell_3}) \in \mathbb{R}^3$ corresponds to the rotations around all axes passing through $(0,0,0)$ and this does not depend on the remaining variables x, y_1, y_2 and y_3. In particular, v_i for $i = 1,2,3$ corresponds to the rotation around the axis generated by ∂_{ℓ_i}. Analogously, for arbitrary fixed x, the algebra $\mathfrak{so}(3)$ acts on $(\partial_{y_1}, \partial_{y_2}, \partial_{y_3})$ via rotations around axes going through $(x + \frac{\sqrt{3}x^2}{4}, x, x - \frac{\sqrt{3}x^2}{4})$. In particular, v_i for $i = 1,2,3$ corresponds to the rotation around the axis generated by ∂_{y_i}.

One can also see from the shape of the generators that the rotation around ∂_{ℓ_i} is tied to the rotation around ∂_{y_i} for $i = 1,2,3$. Thus the action on $(\partial_{\ell_1}, \partial_{\ell_2}, \partial_{\ell_3})$

determines the action on $(\partial_{y_1}, \partial_{y_2}, \partial_{y_3})$ and vice versa. In fact, x parametrizes (in the coordinates $(x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3)$) the curve $(x, 0, 0, 0, x + \frac{\sqrt{3}x^2}{4}, x, x - \frac{\sqrt{3}x^2}{4}) \subset N$ which can be viewed as a curve of centres of the above 'double–rotations'.

In particular, the nilpotent sub–Riemannian structure $(N, \mathcal{D} = V + E, g)$ is invariant with respect to the action, and we can study its action on curves passing through the origin. Assume that $c(t)$ is a (parametrized) curve such that $c(0) = o$. Consider the flow Fl^t_v of the infinitesimal symmetry $v := a_1v_1 + a_2v_2 + a_3v_3 \in \mathfrak{so}(3)$ for some a_1, a_2, a_3. It clearly preserves the origin o. Assume that the point $c(t_0) \neq o$ for some $t_0 \neq 0$ is preserved by the action of Fl^t_v. Then either the curve $c(t)$ is preserved by this action on $[0, t_0]$, or the action determines a family of curves of the same length from o to $c(t_0)$ on $[0, t_0]$. In particular, if such a curve $c(t)$ that is not invariant with respect to the action of Fl^s_v is an extremal curve for the invariant control system, then it is no more minimiser after it reaches the point. Moreover, if one finds one such point then its orbit with respect to the action of $\mathfrak{so}(3)$ consists of such points. Indeed, a family of curves from o to $c(t_0)$ is mapped to the family of curves of the same length from o to $\hat{c}(t_0)$, where $\hat{c}(t_0)$ is image of $c(t)$ with respect to the action of $\mathfrak{so}(3)$.

We can describe explicitly the set of such points that are fixed for the action of (the flow of) some infinitesimal symmetry $a_1v_1 + a_2v_2 + a_3v_3$. First, one can check that each point of the curve $(x, 0, 0, 0, x + \frac{\sqrt{3}x^2}{4}, x, x - \frac{\sqrt{3}x^2}{4})$ is preserved by each such symmetry. Then the fixed points of any symmetry are given by axes of the corresponding 'double–rotations'. Explicitly, the fixed points of the symmetry $a_1v_1 + a_2v_2 + a_3v_3$ form the set

$$\left\{(x, ka_1, ka_2, ka_3, x + \frac{\sqrt{3}x^2}{4} + ka_1, x + ka_2, x - \frac{\sqrt{3}x^2}{4} + ka_3) : k \in \mathbb{R}\right\}.$$

Let us finally say that for each k_i, $i = 2, 3, 4$ it holds that $[N_1, k_2N_2 + k_3N_3 + k_4N_4] = k_2N_{12} + k_3N_{13} + k_4N_{14}$. So the triple $(N_1, k_2N_2 + k_3N_3 + k_4N_4, k_2N_{12} + k_3N_{13} + k_4N_{14})$ determines a subalgebra which has the structure of the Heisenberg algebra. One can see from the above mentioned that the action of the symmetry algebra $\mathfrak{so}(3)$ simply maps each such Heisenberg subalgebra to another Heisenberg subalgebra.

5. Pontryagin's maximum principle

We study local control of particular mechanisms. We use Hamiltonian formalism and Pontryagin's maximum principle to find local length minimisers. We study the corresponding Hamiltonian system of ODEs in detail. Finally, we model several explicit movements of the mechanism.

5.1. **Formulation of the problem.** Consider two configurations q_1, q_2 in the nilpotent approximation N. Among all admissible curves $c(t)$, i.e. locally Lipschitz curves such that $c(0) = q_1$ and $c(T) = q_2$ that are tangent to $\mathcal{D}$ for almost all $t \in [0, T]$, we would like to find length minimisers with respect to g.

We would like to minimize the length l among all the horizontal curves c, where the length is given by $l(c) = \int_0^T \sqrt{g(\dot{c}, \dot{c})}dt$ for the control metric g. Let us recall that the distance between two points $q_1, q_2 \in N$ is defined as $d : M \times M \to [0, \infty]$, $d(q_1, q_2) = \inf_{\{c \in \mathcal{S}_{q_1,q_2}\}} l(c)$, where $\mathcal{S}_{q_1,q_2} = \{c : c(0) = q_1, c(T) = q_2, c \text{ admissible}\}$ [1, 5, 14]. However, since minimizing of the energy of a curve implies minimizing of its length, we will rather minimize energy of curves.

We know from Chow–Rashevsky theorem that the control system of the nilpotent approximation is controllable, see Section 2.2. In particular, any two points can be joined by a horizontal curve and the distance of arbitrary two points is finite [1].

We study the following nilpotent control problem

$$\frac{d}{dt}q = u_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \frac{\sqrt{3}}{2}x - \ell_1 + 1 \\ -\ell_2 + 1 \\ -\frac{\sqrt{3}}{2}x - \ell_3 + 1 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + u_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + u_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

for $q \in N$ and the control $u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ with the boundary condition $q(0) = (x_0, \ell_{10}, \ell_{20}, \ell_{30}, y_{10}, y_{20}, y_{30})$ and $q(T) = (x_1, \ell_{11}, \ell_{21}, \ell_{31}, y_{11}, y_{21}, y_{31})$ arbitrary but fixed, and we minimize

$$\frac{1}{2}\int_0^T (u_1^2 + u_2^2 + u_3^2 + u_4^2)\ dt. \tag{11}$$

Without loss of generality, we choose the origin $o = (0, 0, 0, 0, 0, 0, 0)$ as the initial point $q(0) = q_1$. Since we solve nilpotent control problem, we get curves starting at different points using the left action coming from the multiplication in N, see Section 3.2.

5.2. **Hamiltonian formalism.** Let us consider a cotangent bundle $T^*N \to N$ and the coordinate functions $h_i = \langle \lambda, N_i \rangle$, where $\lambda \in T^*N$. Then we consider the Hamiltonian of the maximum principle

$$H(\lambda, \nu) = u_1h_1 + u_2h_2 + u_3h_3 + u_4h_4 + \frac{\nu}{2}(u_1^2 + u_2^2 + u_3^2 + u_4^2),$$

which is a family of smooth functions affine on fibres that is parametrized by controls $(u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ and a real number $\nu \le 0$. The Pontryagin's maximum principle can be formulated as follows [1]: Assume $(\bar{u}(t), c(t))$ to be a pair such that $c(t)$ is a length minimizer for (10), (11) with control function $u = \bar{u}(t)$. Then there exist a Lipschitz curve $\lambda(t) \in T^*_{c(t)}M$ and a number $\nu \le 0$ such that

$$(\lambda(t), \nu) \neq 0,\ \ \dot{\lambda}(t) = \vec{H}_{\bar{u}(t)}(\lambda(t))$$

for Hamiltonian vector field $\vec{H}$ corresponding to H and $H_{\bar{u}(t)} = \max H(\lambda(t), \nu)$.

If $\lambda(t)$ satisfies the principle for $\nu = 0$, then it is called abnormal and it is called normal otherwise. The abnormal minimiser is strictly abnormal, if it is not normal. It follows from the Goh condition [17, 1] that there are no strictly abnormal minimisers in the case of 2–step distributions. Thus we will focus on the case $\nu < 0$ and we can normalize it in such a way that $\nu = -1$.

The extreme is achieved when $\frac{\partial H_u}{\partial u_i} = h_i - u_i = 0$ for $i = 1, 2, 3, 4$ and this implies for the controls that $u_i = h_i$ for $i = 1, 2, 3, 4$. In this case, the Hamiltonian of the maximum principle is of the form

$$H = \frac{1}{2}(h_1^2 + h_2^2 + h_3^2 + h_4^2).$$

Then the corresponding Hamiltonian system associated with H is the following

$$\dot{q} = h_1N_1(q) + h_2N_2(q) + h_3N_3(q) + h_4N_4(q), \tag{12}$$

$$\dot{h}_i = \{H, h_i\}, \tag{13}$$

where N_i are the generators of the sub–Riemannian structure, $q \in N$ and $\{\ ,\ \}$ is the usual Lie–Poisson bracket [1]. Using the fact that $\{h_i, h_j\} = \langle \lambda, [N_i, N_j] \rangle$ we conclude that the fibre system (13) is of the form

$$\begin{aligned}
&\frac{d}{dt}h_1(t) = -h_5(t)h_2(t) - h_6(t)h_3(t) - h_7(t)h_4(t)\\
&\frac{d}{dt}h_2(t) = h_5(t)h_1(t)\\
&\frac{d}{dt}h_3(t) = h_6(t)h_1(t)\\
&\frac{d}{dt}h_4(t) = h_7(t)h_1(t)\\
&\frac{d}{dt}h_5(t) = \frac{d}{dt}h_6(t) = \frac{d}{dt}h_7(t) = 0.
\end{aligned}\tag{14}$$

The equations are clearly independent of the horizontal coordinates q. The base system has the form

$$\begin{aligned}
&\frac{d}{dt}x(t) = h_1(t)\\
&\frac{d}{dt}\ell_1(t) = h_2(t)\\
&\frac{d}{dt}\ell_2(t) = h_3(t)\\
&\frac{d}{dt}\ell_3(t) = h_4(t)\\
&\frac{d}{dt}y_1(t) = \left(1 + \frac{\sqrt{3}}{2}x(t) - \ell_1(t)\right) h_1(t)\\
&\frac{d}{dt}y_2(t) = (1 - \ell_2(t))\, h_1(t)\\
&\frac{d}{dt}y_3(t) = \left(1 - \frac{\sqrt{3}}{2}x(t) - \ell_3(t)\right) h_1(t).
\end{aligned}\tag{15}$$

5.3. **Analysis of the fibre system.** Let us first discuss the fibre system (14). Obviously, functions h_5, h_6 and h_7 are constant. If they are all zero, then the functions h_1, h_2, h_3 and h_4 are constant, too.

Let us denote the solution constants corresponding to h_5, h_6 and h_7 by C_5, C_6 and C_7, respectively, and assume that at least one of them is non–zero. Define $K \equiv \sqrt{C_5^2 + C_6^2 + C_7^2}$. Then we get

$$\ddot{h}_1 = -C_5\dot{h}_2 - C_6\dot{h}_3 - C_7\dot{h}_4 = -(C_5^2 + C_6^2 + C_7^2)\, h_1 = -K^2 h_1.\tag{16}$$

Since $K^2 > 0$, the solution of (16) is

$$h_1 = C_{11}\cos(Kt) + C_{12}\sin(Kt),\tag{17}$$

for some constants C_{11} and C_{12}. Now, if $C_5 \neq 0$, we have

$$\dot{h}_2 = C_5 h_1 = C_5(C_{11}\cos(Kt) + C_{12}\sin(Kt))$$

and hence

$$h_2 = \frac{C_5}{K}(C_{11}\sin(Kt) - C_{12}(\cos(Kt)) + C_{13}.\tag{18}$$

This analogously holds for h_6 and h_7 if $C_6 \neq 0$ and $C_7 \neq 0$, respectively. In the same way we get h_3 and h_4. Indeed, only the equation for function h_1 merges everything together.

We know from the above mentioned, that the functions h_i for $i = 1, 2, 3, 4$ equal to the controls u_i of the system. Thus we control each vector field N_i with a function u_i which is either a constant or which oscillates. In fact, N_i, $i = 2, 3, 4$

corresponds to ∂_{ℓ_j}, $j=1,2,3$ and reflects the movement of legs which is natural from the mechanical point of view. The field N_1 is the crucial field for the movement and if it is controlled by a constant, then the remaining fields N_i are controlled by constant, too.

Let us point out that the choice of h_5, h_6 and h_7 to be zero or non–zero, in fact, corresponds to the choice of constants in the solution. Since that corresponds to the choice of initial conditions, we can interpret the choice of zero or non–zero solution h_5, h_6 and h_7 as the choice of initial conditions, equivalently. Moreover, we are interested in the solution of the whole system $(14,15)$ and we are basically interested in the curves, that are images in N with respect to the canonical projection $T^*N \to N$. There can be curves of that type in N that differ only by the parametrization. It is reasonable to consider the curves parametrized by the arc–length, only. These are exactly the images of restrictions of the canonical projection to the solutions of $(14,15)$ with initial conditions satisfying $h_1^2+h_2^2+h_3^2+h_4^2=1$ at the initial point (and thus everywhere).

Finally, let us remark that the choice $h_6=h_7=0$ implies that h_3,h_4 are constants and the remaining equations result in the system $\dot{h}_1=-h_5h_2, \dot{h}_2=h_5h_1, \dot{h}_5=0$, which is the fibre system of the control problem on the Heisenberg group.

5.4. **Analysis of the base system.** Let us first say that it is enough to study solutions with the initial condition $q(0)=o$. Then one can use the action given by the multiplication on N to get a solution starting at an arbitrary point in N.

The first four equations of the system depend on h_i, $i=1,2,3,4$ only and can be computed by direct integration. Thus the equation $\dot{x}=h_1$ gives that either $x=C_1t$ for some constant C_1 in the case that h_1 is constant, or

$$x=\frac{C_{11}}{K}\sin(Kt)-\frac{C_{12}}{K}\cos(Kt)+\frac{C_{12}}{K}$$

in the case that h_1 is of the form (17), where K, C_{11}, C_{12} are from the previous section. Then we compute directly from (18) that

$$\ell_1=\frac{C_5}{K^2}(C_{11}-C_{11}\cos(Kt)-C_{12}\sin(Kt))+C_{13}t.$$

Analogous observation can be made for ℓ_2 and ℓ_3.

From mechanical point of view, this simply means that each branch can either prolong or shorten or does not change its length in the first case, or oscillates in the second case. In particular, the choice $h_6=h_7=0$ reflects the situation when lengths ℓ_2 and ℓ_3 of the second and third branch are constant and the robot uses only the first branch of the length ℓ_1. The same principles work for the remaining two branches and the corresponding choices of h_i.

The equations for y_i depend on x and ℓ_i for $i=1,2,3$, only. So we can find y_i by considering closed subsystems for x, ℓ_i, y_i for $i=1,2,3$. Let us remark that y_i, $i=1,2,3$ have no evident mechanical meaning. One should use the transformation between these coordinates and the original ones to get some information about the behaviour of y, ϕ, θ. We do not mention all the solutions explicitly. We rather provide several examples to demonstrate the explicit paths and movements of the mechanism in the following section.

Let us finally remark that in the case that $h_5=h_6=h_7=0$, we can find for each solution an infinitesimal symmetry such that the solution is contained in the fixed points set of the symmetry. The situation is more complicated in the case when some of h_5, h_6, h_7 is non–zero.

5.5. **Examples of solutions.** Let us present several particular solutions (satisfying all initial conditions of the system and also constraints coming from the mechanical setting). To get an information about the movement of our mechanism, we transform the solutions into the original coordinates which express the kinematics of the mechanism. So we use the transformation between coordinates $(x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3)$ and $(x, y, \theta, \varphi, \ell_1, \ell_2, \ell_3)$ in the form

$$\begin{aligned}\varphi &= \frac{5}{4}y_2 + \frac{3}{2}x + \frac{1}{8}y_1 + \frac{1}{8}y_3,\\ \theta &= -\frac{1}{16}y_1 - \frac{1}{16}y_3 - \frac{1}{4}x,\\ y &= -\frac{1}{12}\sqrt{3}(y_1 - y_3),\end{aligned}$$

which is the inverse transformation to the transformations (6). We also add illustrative graphs of behaviour of the control parameters which are prismatic joints ℓ_i and the revolute joint φ.

Example 1. *In the case that* $h_5 = h_6 = h_7 = 0$*, we can choose* $h_1 = \frac{7}{10}$*,* $h_2 = h_3 = \frac{1}{2}$*,* $h_4 = \frac{1}{10}$ *and, with suitable choice of constants, we get the solution in the form*

$$\begin{aligned}x &= \frac{7t}{10}, \quad y = (\frac{7\sqrt{3}}{600} - \frac{49}{800})t^2, \quad \theta = \frac{21t^2}{1600} - \frac{21t}{80},\\ \varphi &= -\frac{49t^2}{200} + \frac{21t}{10}, \quad \ell_1 = \ell_2 = \frac{t}{2}, \quad \ell_3 = \frac{t}{10}.\end{aligned}$$

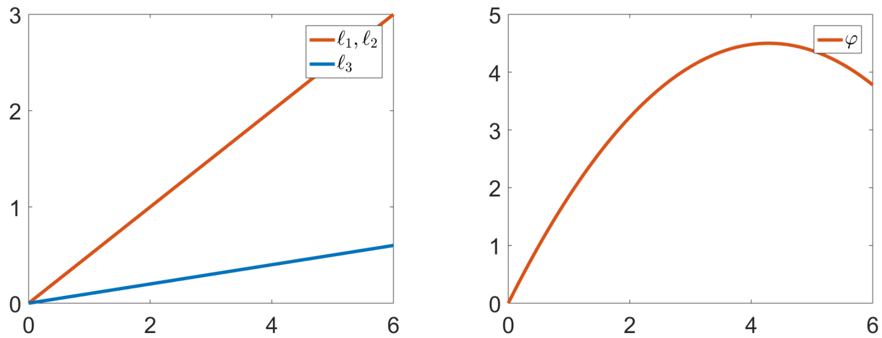

FIGURE 3. The graphs of control parameters: Example 1

Example 2. *In the case that* $h_5 \neq 0$ *and* $h_6 = h_7 = 0$*, let us choose the constants in such a way that* $h_5 = 1$ *and* $h_1 = -\frac{1}{2}\sin(t) + \frac{1}{2}\cos(t)$*,* $h_2 = \frac{1}{2}\sin(t) + \frac{1}{2}\cos(t)$ *and* $h_3 = h_4 = \frac{1}{2}$*. Then we get the solution in the form*

$$\begin{aligned}x(t) &= \frac{1}{2}\sin(t) + \frac{1}{2}\cos(t) - \frac{1}{2},\\ y(t) &= -\frac{\sqrt{3}}{48}\left(\sqrt{3}(\sin(t) - 1)(\cos(t) - 1) + \cos(t)^2 + t\cos(t) + (t-2)\sin(t) + t - 1\right),\\ \theta(t) &= -\frac{1}{64}\cos(t)^2 + \frac{t-10}{64}\cos(t) + \frac{t-12}{64}\sin(t) - \frac{t}{64} + \frac{11}{64},\\ \varphi(t) &= \frac{1}{32}\cos(t)^2 + \frac{36-11t}{32}\cos(t) + \frac{58-11t}{32}\sin(t) + \frac{t}{32} - \frac{37}{32}.\\ \ell_1(t) &= \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t) + \frac{1}{2}, \quad \ell_2 = \ell_3 = \frac{t}{2}\end{aligned}$$

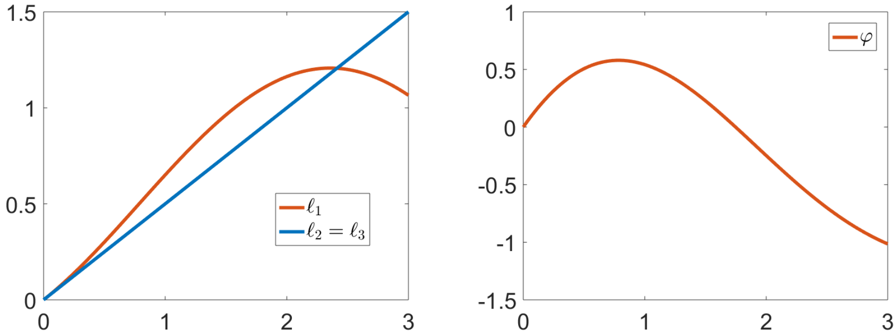

FIGURE 4. The graphs of control parameters: Example 2

Example 3. *Let us finally discuss the case that all* h_5, h_6, h_8 *are non–zero. Let us choose the constants in such a way that* $h_1 = -\frac{\sqrt{10}}{4}\cos(t)$, $h_2 = -\frac{\sqrt{30}}{12}\sin(t) + \frac{1}{2}$, $h_3 = h_4 = \frac{\sqrt{30}}{12}\sin(t) + \frac{1}{4}$, $h_5 = \frac{\sqrt{3}}{3}$, $h_6 = h_7 = -\frac{\sqrt{3}}{3}$. *We get the solution in the form*

$$x = -\frac{\sqrt{10}}{4}\sin(t)$$
$$y = \frac{\sqrt{30}}{192}(1 - t\sin(t) - \cos(t)) + \frac{5}{64}\cos(t)^2 - \frac{5}{96}\sin(t)\cos(t) - \frac{5t}{96} + \frac{5}{48}\sin(t) - \frac{5}{64}$$
$$\theta = -\frac{3\sqrt{10}}{256}\left((t-8)\sin(t) + \cos(t) - 1\right)$$
$$\varphi = \frac{13\sqrt{3}}{384}(((t - \frac{96}{13})\sin(t) + \cos(t) - 1)\sqrt{10}\sqrt{3} + (\frac{100}{13} - \frac{50}{13}\cos(t))\sin(t) - \frac{50}{13}t)$$
$$\ell_1 = \frac{\sqrt{30}}{12}(\cos(t) - 1) + \frac{t}{2} \qquad \ell_2 = \ell_3 = \frac{\sqrt{30}}{12}(1 - \cos(t)) + \frac{t}{4}$$

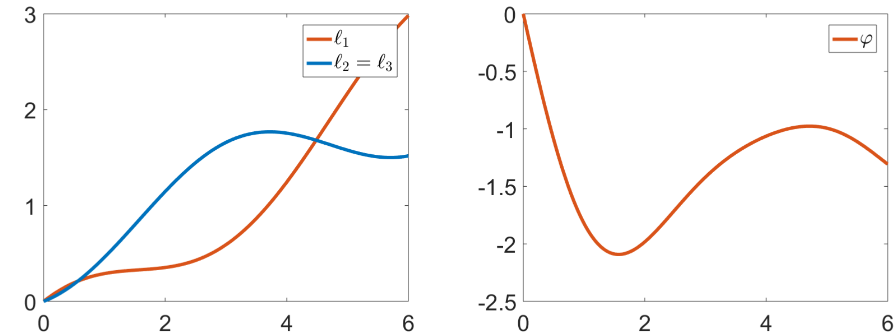

FIGURE 5. The graphs of control parameters: Example 3

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