MASARYK UNIVERSITY
FACULTY OF SCIENCE

QUALITATIVE THEORY OF FRACTIONAL DIFFERENTIAL SYSTEMS WITH TIME DELAY

HABILITATION THESIS

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Preface

The idea behind derivatives of non-integer orders traces back nearly as far as the classical derivatives themselves. What began in the 17th century as a puzzling thought exercise has evolved into one of the most influential mathematical fields of recent decades, known as fractional calculus. My background in both mathematics and physics continues to feed my interest in this elegant discipline and its impact on both theory and applications.

In the realm of applied sciences, fractional calculus has gained recognition for its linear descriptions of complex systems characterized by nonlocal or memory-based behaviour, traditionally treated within the nonlinear domain. Theoretically, the ability to continuously transition between derivative orders reveals previously unseen connections and enables the study of various phenomena. However, the generalizing nature of fractional calculus also poses a risk: researchers are tempted to deal with artificial, easily solvable problems that neither enrich applications nor contribute much to theory, resulting in mere formalism. With this in mind, I have made it my goal to focus on key problems that help to shed light on deeper mathematical principles and behaviour of complex systems.

This habilitation thesis combines the key results of my seven selected papers [6, 10–13, 15, 37] from 2016-2023. Their unifying theme is the qualitative analysis of fractional delay differential equations, a class of mathematical models involving fractional derivatives and time delays representing inherent lags in the system. The key aim is two-fold: first, to better understand, predict and control the behaviour of such systems, which is essential for numerous applications ranging from physics and biology to engineering and finance. Second, to integrate fractional calculus into the broader landscape of dynamic systems theory, where it can illuminate the intricate interplay of delayed responses and memory effects in shaping system behaviour.

The thesis comprises four chapters that present the main results and provide commentary on key proofs referring the respective papers, and seven appendices containing the full text of the selected published papers. Chapter 1 provides a context of classical and fractional qualitative analysis and introduces known limit cases of our study. Chapter 2 summarizes the original results on one-term fractional delay differential systems. It sets the theoretical foundation for subsequent work and presents key stability and oscillatory conditions, often in non-improvable form. Chapter 3 focuses on two-term fractional delay differential equations. It explores the relationships between derivative order and the stability regions for the system's coefficients. Finally, Chapter 4 offers concluding remarks and reflections on this

research.

I express deep gratitude to Jan Čermák, my former advisor, leader of the scientific group I am proud to be part of and my most frequent co-author. His insights, leadership, continuous support and willingness to share his broad mathematical knowledge were essential for the completion of this work. I also thank my other co-authors and colleagues for their collaboration, namely Zuzana Došlá, Jan Horníček and Luděk Nechvátal who participated in some of the seven papers forming the base of this thesis. I extend my thanks to Alberto Cabada and Matej Dolník who influenced my thinking during joint work on other topics. In addition to the professional support I have received, I am especially grateful to Lída Kiselová, my dear wife, and to my children Bára, Magda and Teodor for their endless patience, encouragement and understanding without which I could not have accomplished this work. I thank my parents, grandparents and entire family as well as my friends for an occasional gentle push that kept me moving forward.

Brno, September 2024 Tomáš Kisela

Contents

Chapter 1

Wider context: classical and fractional qualitative analysis

The study of dynamic systems involving time delays is a classical area of mathematical analysis with significant real-world applications. These systems can accurately model processes that do not respond instantaneously to changes in their state or environment, such as biological systems, economic models, and engineering processes. Despite extensive research efforts, many questions related to the stability and control of these classical systems remain unanswered, primarily due to the inherent challenges of incorporating delayed responses (see, e.g. [21, 22, 38]).

Fractional calculus (extending the concepts of differentiation and integration to non-integer orders) is known for its ability to address the complexities of dynamic systems with nonlocal and memory effects, such as systems where the future state depends on a continuum of past states. The corresponding, so-called fractional, dynamic systems attract a significant attention of scientific community for several decades (see, e.g. [25, 26, 32, 48, 50, 53]). Many qualitative results of classical calculus already found their fractional counterparts (in particular in linear case), many wait for further progress and many may be impossible to generalize.

The developing interest in fractional calculus among scientists and engineers is largely due to its applications and further potential in control theory. Thus, the need to model the fractional systems with delayed feedback with sufficient precision is growing (see [20, 27, 49, 52]). In particular, since there appears to be a clash of two forces: while fractional systems of lower orders typically show larger stability regions, growing delay tends to destabilize the system. That is why this thesis focuses on the domain connecting fractional derivatives and time delays, on the qualitative theory of linear fractional delay differential systems (FDDS). Study of FDDS serves, besides its theoretical value, as a mathematical basis for effective feedback control of complex systems with memory.

This chapter sets the stage by providing the necessary context. We recall basic notions of fractional calculus and qualitative theory, present some classical results serving as comparisons later in the text and outline the basic ideas behind qualitative analysis.

1.1 Basic notions

As mentioned above, the subject of this thesis is a study of systems involving fractional derivative. Throughout the text we utilize the following definitions: Let a be a real number and let f be a real scalar function defined on (a,∞). Its fractional integral of positive real order γ is given by

$$\mathrm{D}_a^{-\gamma} f(t) = \int_a^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)} f(\xi)\mathrm{d}\xi, \qquad t \in (a,\infty)\,.$$

The (Caputo) fractional derivative of positive real order α is given by

$$\mathrm{D}_a^{\alpha} f(t) = \mathrm{D}_a^{-(\lceil\alpha\rceil-\alpha)} \left(\frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} f(t)\right), \qquad t \in (a,\infty) \tag{1.1}$$

where $\lceil\cdot\rceil$ denotes an upper integer part (so-called ceiling function). As it is customary, we put $\mathrm{D}_a^0 f(t) = f(t)$. Besides the Caputo definition (1.1) employed throughout this thesis, some authors use the Riemann-Liouville derivative applying the fractional integral before the integer-order derivative. We mention this approach only occasionally for comparison. For more basics of fractional calculus we refer to [32,53].

If f is a vector function, the corresponding fractional operators are considered component-wise, if f is a complex-valued function, the corresponding fractional operators are introduced for its real and imaginary part separately.

The terminology that we employ is based on the classical qualitative theory:

- A linear differential system is said to be *stable* if all its solutions are bounded as $t \to \infty$.
- A linear differential system is said to be *asymptotically stable* if all its solutions tend to zero as $t \to \infty$.
- The set of all parameters' values for which the differential system is asymptotically stable is called the *stability region*.
- The solution to a differential system is called *oscillatory* if its set of zeros is unbounded.

For more precise large-time solution descriptions we use the following asymptotic notations (K being a suitable positive real):

$$f \sim g \quad \text{as } t \to \infty \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{f(t)}{Kg(t)} = 1\,,$$

$$f \sim_{sup} g \quad \text{as } t \to \infty \qquad \Longleftrightarrow \qquad \limsup_{t\to\infty} \frac{f(t)}{Kg(t)} = 1\,,$$

$$f = \mathcal{O}(g) \quad \text{as } t \to \infty \qquad \Longleftrightarrow \qquad \limsup_{t\to\infty} \frac{|f(t)|}{g(t)} < \infty\,,$$

$$f = o(g) \quad \text{as } t \to \infty \qquad \Longleftrightarrow \qquad \limsup_{t\to\infty} \frac{f(t)}{g(t)} = 0\,.$$

In addition to the *asymptotic equivalence* $\sim$, they enable us to describe asymptotics of a wider class of functions (in particular, unbounded oscillatory functions).

1.2 Classical results of qualitative analysis

In this section, we summarize the most important results regarding qualitative properties of fractional differential systems without delay, ordinary delay differential systems and ordinary differential systems with both delayed and undelayed terms. In particular, we recall the inequalities defining stability regions for the corresponding problems. We also provide commentary to other relevant sources expanding these results.

Fractional differential systems without delay

Let us consider the system

$$\mathrm{D}_0^\alpha y(t) = Ay(t)\,, \quad t \in [0, \infty) \tag{1.2}$$

where A is a constant real $d \times d$ matrix and $\alpha > 0$. The classic stability result comes from [45] and corresponds to the following assertion.

Theorem 1.1. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and let λ_j $(j = 1, 2, \dots, d)$ be all eigenvalues of A. Then (1.2) is asymptotically stable if and only if*

$$0 < \alpha < 2 \qquad \text{and} \qquad |\mathrm{Arg}\,(\lambda_j)| > \alpha\pi/2 \qquad \text{for all } j\,.$$

Moreover, any solution y tends to zero algebraically as $y \sim t^{-\alpha}$ as $t \to \infty$.

Corollary 1.2. *The stability region of (1.2) is given by*

$$\mathcal{S}_\alpha = \{\lambda \in \mathbb{C} : |\mathrm{Arg}\,(\lambda)| > \alpha\pi/2\}\,, \quad 0 < \alpha < 2\,.$$

Remark 1.3. The proof was given in [45] and the used proving technique enables to show several other assertions:

a) Originally, only the order less than one was considered, however, the technique works analogously for higher orders.

b) It was also proven that (1.2) is stable if and only if $0 < \alpha < 2$ and $|\mathrm{Arg}\,(\lambda_j)| \geq \alpha\pi/2$ for all j and those eigenvalues with the principal argument equaling to $\pm\alpha\pi/2$ have geometric multiplicity one.

c) Unbounded solutions of (1.2) follow the asymptotic relation $y \sim_{sup} t^k \exp(\lambda^{1/\alpha} t)$ where k is a suitable nonnegative integer.

d) All solutions of (1.2) tending to zero as $t \to \infty$ are non-oscillatory. For other solutions, oscillatory property might occur.

Figures 1.1 and 1.2 illustrate the evolution of the stability region $\mathcal{S}_\alpha$ for increasing α. We note that for $\alpha \to 1$ the stability boundary coincides with imaginary axis and for $\alpha \to 2^-$ the stability region degenerates into an empty set. That shows the agreement with the classical theory for both first and second order differential systems.

In case of the scalar equation, i.e. $\mathrm{D}_0^\alpha x(t) = \lambda x(t)$ with λ being real, the asymptotic stability and stability occur for $\lambda < 0$ and $\lambda \leq 0$, respectively (provided $\alpha \in (0, 2)$).

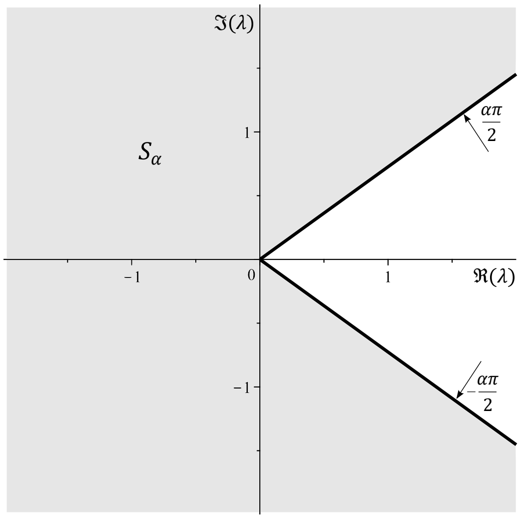

Figure 1.1: Stability region $\mathcal{S}_\alpha$ for (1.2) with $\alpha = 0.4 < 1$.

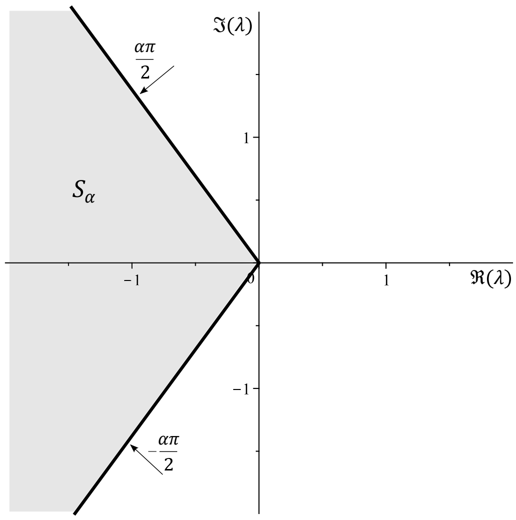

Figure 1.2: Stability region $\mathcal{S}_\alpha$ for (1.2) with $\alpha = 1.4 > 1$.

First-order delay differential system

Let us consider the system with time delay

$$y'(t) = Ay(t - \tau), \qquad t \in [0, \infty), \tag{1.3}$$

where A is a constant real $d \times d$ matrix and $\tau > 0$ is a constant real lag. Its main stability and asymptotic properties were derived in, e.g. [23] and can be formulated as

Theorem 1.4. *Let $A \in \mathbb{R}^{d\times d}$, $\tau \in \mathbb{R}^+$ and let λ_j $(j = 1, 2, \ldots, d)$ be all eigenvalues of A. Then (1.3) is asymptotically stable if and only if*

$$\tau|\lambda_j| < |\mathrm{Arg}\,(\lambda_j)| - \pi/2 \qquad \textit{for all } j\,.$$

Moreover, any solution y tends to zero exponentially as $t \to \infty$.

Corollary 1.5. *The stability region of (1.3) is given by*

$$\mathcal{S}_1^\tau = \left\{\lambda \in \mathbb{C} : |\lambda| < \frac{|\mathrm{Arg}\,(\lambda)| - \pi/2}{\tau}\,,\ |\mathrm{Arg}\,(\lambda)| > \frac{\pi}{2}\right\}, \quad \tau > 0\,.$$

Oscillation properties of (1.3) can be written as follows (see, e.g. [21]).

Theorem 1.6. *Let $A \in \mathbb{R}^{d\times d}$, $\tau \in \mathbb{R}^+$ and let λ_j $(j = 1, 2, \ldots, d)$ be all eigenvalues of A. Then all solutions of* (1.3) *oscillate if and only if*

$$\lambda_j \in \mathbb{C} \setminus [-1/(\tau e), \infty) \qquad \textit{for all } j\,,$$

i.e. A has no real eigenvalues in $[-1/(\tau e), \infty)$.

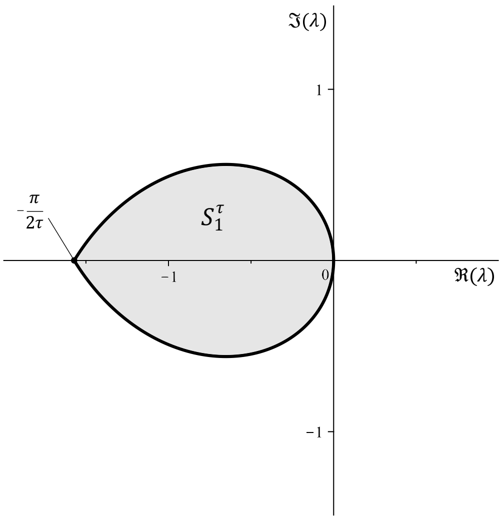

Figure 1.3: Stability region $\mathcal{S}_1^\tau$ for (1.3) depicted for the value $\tau = 1$.

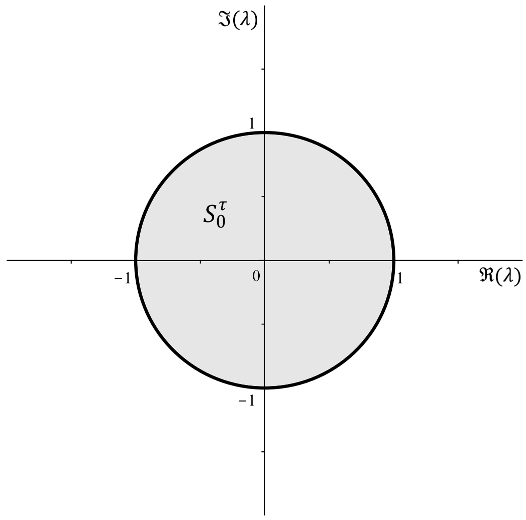

Figure 1.4: Stability region $\mathcal{S}_0^\tau$ for (1.4), independent of τ.

Figure 1.3 shows the stability region for (1.3), highlighting that in the case of scalar equation $x'(t) = \lambda x(t-\tau)$ with λ being real, the asymptotic stability condition reduces to $-\pi/(2\tau) < \lambda < 0$. Also, in scalar case all solutions oscillate if and only if $\lambda < -1/(\tau\mathrm{e})$.

Discrete system

Let us consider the discrete system

$$y(n) = Ay(n-\tau), \qquad t \in \{\tau, 2\tau \dots\}, \tag{1.4}$$

where A is a constant real $d \times d$ matrix and $\tau > 0$ is a constant real lag. This system can be viewed as a modification of (1.3) where the derivative is removed (the derivative order is changed to zero) and the time is discretized into multiples of τ. It is well-known that the corresponding stability region (see Figure 1.4) is given by a unit circle with no dependence on the value of τ, i.e.

$$\mathcal{S}_0^\tau = \{\lambda \in \mathbb{C} : |\lambda| < 1\}\ , \quad \tau > 0\,.$$

First-order differential equation with both delayed and undelayed terms

If an undelayed term is added to the right-hand side of (1.3), we obtain a system for which the stability analysis is quite difficult even in the planar case. Corresponding necessary and sufficient stability conditions given in terms of system parameters are known only in very special cases, e.g. if A, B are simultaneously triangularizable. For more details see [4, 31, 46]. Hence, regarding right-hand side composed of both delayed and undelayed terms, we will focus on scalar equations.

Let us consider the equation

$$y'(t) = ay(t) + by(t-\tau), \qquad t \in [0, \infty), \tag{1.5}$$

where a, b are real and $\tau > 0$ is a constant real lag. Its stability properties can be written as (see [24])

Theorem 1.7. *Let $a, b \in \mathbb{R}$, $\tau \in \mathbb{R}^+$. Then (1.5) is asymptotically stable if and only if either*

$$a \le b < -a \qquad \text{and} \qquad \tau \text{ is arbitrary},$$

or

$$|a| + b < 0 \qquad \text{and} \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}.$$

Corollary 1.8. *The stability region of (1.5) is given by*

$$\mathcal{S}_1^\tau = \{(a, b) \in \mathbb{R}^2 : a - b \le 0 \quad \text{and} \quad a + b < 0\}$$
$$\cup \left\{(a, b) \in \mathbb{R}^2 : |a| + b < 0 \quad \text{and} \quad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}\right\}.$$

Figure 1.5 displays the stability region for (1.5) in the (a, b)–plane. The top part of the stability boundary is formed by the axis of the second and fourth quadrants corresponding to the first condition in Theorem 1.7. The bottom part of the stability boundary representing the second condition in Theorem 1.7 depends on τ as also illustrated by the cusp point coordinates.

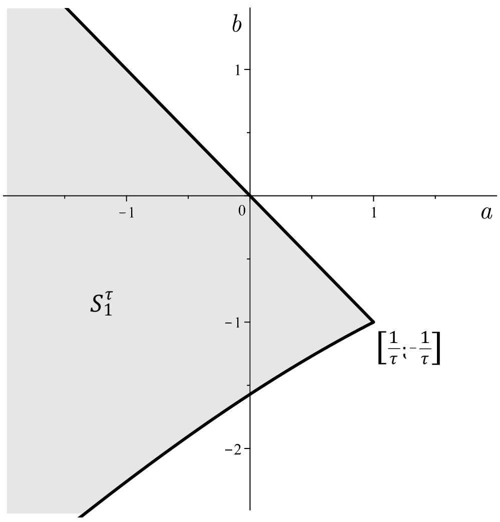

Figure 1.5: Stability region $\mathcal{S}_1^\tau$ for (1.5) depicted for the value $\tau = 1$.

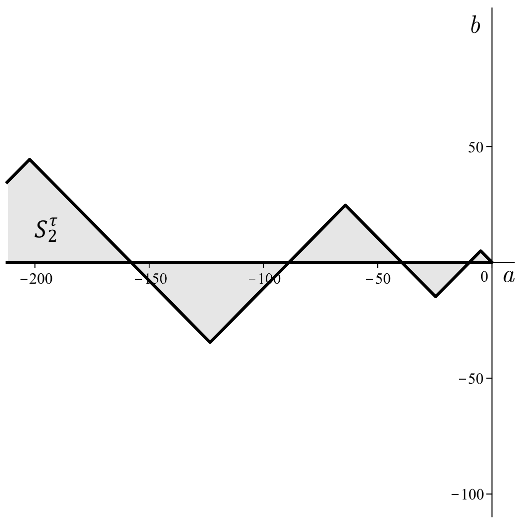

Figure 1.6: Stability region $\mathcal{S}_2^\tau$ for (1.6) depicted for the value $\tau = 1$.

Second-order differential equation with both delayed and undelayed terms

For better comparison, let us also consider the second-order system

$$y''(t) = ay(t) + by(t - \tau), \qquad t \in [0, \infty), \tag{1.6}$$

where a, b are real and $\tau > 0$ is a constant real lag. Its stability properties can be derived from [5] (although the case $b < 0$ was not explicitly discussed there) to obtain

Theorem 1.9. *Let $a, b \in \mathbb{R}$, $\tau \in \mathbb{R}^+$ and $\ell \in \mathbb{Z}_0^+$ be such that*

$$\ell^2 \frac{\pi^2}{\tau^2} < |a| < (\ell+1)^2 \frac{\pi^2}{\tau^2} \,.$$

Then (1.6) is asymptotically stable if and only if $a < 0$ and either

$$0 < b < \min(-\ell^2 \frac{\pi^2}{\tau^2} - a, (\ell+1)^2 \frac{\pi^2}{\tau^2} + a) \qquad \text{for } \ell \text{ being zero or even,}$$

or

$$0 > b > \max(\ell^2 \frac{\pi^2}{\tau^2} + a, -(\ell+1)^2 \frac{\pi^2}{\tau^2} - a) \qquad \text{for } \ell \text{ being odd.}$$

Corollary 1.10. *The stability region of (1.6) is given by*

$$\mathcal{S}_2^\tau = \bigcup_{j=0}^{\infty} \Bigg(\Big\{ (a,b) \in \mathbb{R}^2 : 0 < b < \min(-(2j)^2 \frac{\pi^2}{\tau^2} - a, (2j+1)^2 \frac{\pi^2}{\tau^2} + a) \Big\} \cup \Big\{ (a,b) \in \mathbb{R}^2 : 0 > b > \max((2j+1)^2 \frac{\pi^2}{\tau^2} + a, -(2j+2)^2 \frac{\pi^2}{\tau^2} - a) \Big\} \Bigg) \,.$$

Comparing Theorems 1.7 and 1.9 we see very different stability conditions for the first and second-order equations. This difference is demonstrated in Figures 1.5 and 1.6 in the form of quite distinct shapes of the corresponding stability regions. The transition between them with continuous change of derivative order will be one of the interest of the following chapters.

1.3 Classical characteristic equation approach

The characteristic equation is central to the stability analysis of linear differential systems, including those with delays. The usual approach involves substituting an exponential function with argument st (where s is a complex parameter) as a candidate solution into the system. This substitution transforms the differential system into an algebraic equation with s as the variable, such as

$$\begin{aligned} \det(s\mathrm{I} - A\exp(-s\tau)) &= 0 \qquad \text{for (1.3)}\,, \\ s - a - b\exp(-s\tau) &= 0 \qquad \text{for (1.5)}\,, \\ s^2 - a - b\exp(-s\tau) &= 0 \qquad \text{for (1.6)}\,. \end{aligned}$$

Unlike the characteristic equations of ordinary differential equations, which are polynomial in s, these equations are transcendental, leading to more challenges as they typically have an infinite number of roots.

The system stability is then determined by the location of the characteristic roots in the complex plane due to the well-known behaviour of exponential functions:

- If all roots have negative real parts, the system is asymptotically stable.
- If any root has a positive real part, the system is unstable.
- If the rightmost root, i.e. the root with the largest real part, lies on the imaginary axis, there might be stability or instability based on root multiplicities.

If entry parameters of the system are specified, the position of characteristic roots with respect to imaginary axis can be usually analyzed numerically case by case. However, if we need to design a system or its control, or if there is a risk of parameter uncertainty, this approach is very random and impractical. Thus, the focal point of our effort is a reformulation of stability conditions from terms of characteristic roots into terms of entry parameters.

That is usually done via finding stability boundary in the space of entry parameters. In other words, we are looking for all combinations of entry parameters yielding rightmost roots with zero real part. Due to continuous dependence of characteristic roots on entry parameters, we arrive at a hypersurface in the parameter space where the system transitions from stable to unstable. This approach is often called D-partition method, D-decomposition method or boundary locus method (see, e.g. [24, 30, 39, 46, 47, 54]).

If we consider a system of non-integer order, there is one significant difference: the exponential functions do not longer solve the system. We need to find an alternative way to derive the characteristic equation. The well-established practice is to employ Laplace transform method (for definition we refer to [17]) which leads to the same results for all the classical problems and is successfully used for fractional differential systems as well. In particular, for (1.2) it yields the well-known formula

$$\det(s^{\alpha}\mathrm{I} - A) = 0$$

illustrating that characteristic equations belonging to fractional differential problems typically contain non-analytic functions.

Further, we have to ensure the connection between location of characteristic roots and stability properties of the system other than the exponential argument (as exponentials are no longer solutions, see, e.g. [46, 53]).

As this thesis deals with problems combining both fractional orders and delays, the main challenges addressed in the following chapters are:

- Investigating the properties of roots of characteristic equations that are transcendental and non-analytic.
- Identifying efficient descriptions of stability boundaries in various parameter spaces, clarifying the role of derivative order and delay in shaping the corresponding stability regions.
- Deriving asymptotic expansions of various special functions, often by using the inverse Laplace transform.

Chapter 2

Analysis of one-term fractional delay differential systems

This chapter focuses on the stability, oscillatory and related asymptotic properties derived in author's papers [6, 10, 15, 37] for one-term FDDS

$$\mathrm{D}_0^\alpha y(t) = Ay(t-\tau), \qquad t \in (0, \infty) \tag{2.1}$$

where A is a constant real $d \times d$ matrix and $\alpha, \tau > 0$ are real scalars. The associated initial conditions have typically the form

$$y(t) = \phi(t), \qquad t \in [-\tau, 0]\,, \tag{2.2}$$

$$\lim_{t \to 0^+} y^{(j)}(t) = \phi_j, \qquad j = 0, \ldots, \lceil \alpha \rceil - 1 \tag{2.3}$$

where all components of d-vector function ϕ are absolutely Riemann integrable on $[-\tau, 0]$ and ϕ_j are constant real d-vectors.

The presence of fractional derivative creates room for discussions regarding the proper choice of its lower limit a (see (1.1)) which coincides with the "time origin" of the system. In particular, one might ask why not to put this limit to $a = -\tau$? Similar issues were discussed as the problem of so-called initialization in [44]. Although this discussion is quite interesting, no matter the result it does not significantly affect the qualitative study, because the change of the lower limit is analogous to adding a forcing term on the right-hand side of (2.1). Hence, we adopt the standard approach and consider the lower limit of fractional operators to be zero.

The study of qualitative properties of (2.1) was approached by many authors from different angles. One of the first attempts was [29] dealing with scalar version of (2.1) with real parameter employing the Lambert function to discuss asymptotic properties of solutions. Many authors in 2005-2012, e.g. [16,40,54], studied problems of vector nature, often involving more fractional derivative terms and more time delays. However, the stability criteria were almost exclusively limited on conditions for locations of characteristic roots in complex plane without an explicit link to the entry parameters of the corresponding problem. The difficult practical use and lack of efficiency of such results were often mentioned by authors themselves (see, e.g. [16, 40]). To our knowledge, the first explicit stability criterion was published for scalar version of (2.1) in 2011 by [39].

With this background, we started our work on [6] in 2014 and managed to derive explicit stability criterion and the asymptotics of bounded solutions for (2.1) of low orders (less than one). Three years later, in [10], we expanded our scope to (2.1) of higher orders (greater than one) for which we thoroughly analyzed oscillatory properties which, to our knowledge, were not discussed to that extent in the literature at the time (see, e.g. [3]). In 2020, we consolidated these results in [37], providing a comprehensive summary of the stability and asymptotic properties of (2.1) across all orders. Additionally, we extended our findings to systems involving another, so-called Riemann-Liouville, fractional derivative requiring a different type of initial conditions. It was only in 2023, in [15], when we added an easy-to-use graphical approach to estimate the asymptotic behaviour of unbounded solutions based on properties of Lambert function.

The key results from these four papers serve as the foundation for the following sections. As (2.1) transitions into (1.2) when $\tau \to 0$, and reduces to (1.3) as $\alpha \to 1$, this chapter focuses on comparing the properties of (2.1) with its limit counterparts. In Section 2.1 we establish the structure of solutions to (2.1) and the role of so-called generalized delay exponentials whose asymptotic properties are analyzed in Section 2.2. Section 2.3 describes the decomposition of complex plane naturally imposed by characteristic roots with zero real parts. Finally, Sections 2.4 and 2.5 are devoted to asymptotically stable and unstable systems, respectively, namely to the evolution of stability conditions, asymptotics and oscillatory properties with changes in the derivative order α.

2.1 Structure of solutions

Regarding solving of linear fractional equations, the Laplace transform is one of the most powerful tools. The problem (2.1)-(2.3) is no different. Applying Laplace transform (see [6, 10, 37]), we quickly notice the significance of the following notion:

Definition 2.1. Let $A \in \mathbb{R}^{d\times d}$, let I be the identity $d \times d$ matrix and let $\alpha, \tau \in \mathbb{R}^+$. The matrix function $R : \mathbb{R} \to \mathbb{C}^{d\times d}$ given by

$$R(t) = \mathcal{L}^{-1}\left((s^\alpha I - A\exp\{-s\tau\})^{-1}\right)(t) \tag{2.4}$$

is called the fundamental matrix solution of (2.1). We note that $\mathcal{L}^{-1}$ denotes the standard inverse Laplace transform, i.e. $\mathcal{L}^{-1}(F(s))(t) = (2\pi\,\mathrm{i})^{-1}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F(s)\mathrm{e}^{st}\mathrm{d}s$.

The inverse matrix occurring in (2.4) suggests the well-known characteristic equation associated to (2.1)

$$\det(s^\alpha I - A\exp\{-s\tau\}) = 0\,, \quad \text{i.e.} \quad \prod_{i=1}^{n}(s^\alpha - \lambda_i\exp\{-s\tau\})^{n_i} = 0 \tag{2.5}$$

where λ_i $(i = 1, \dots, n)$ are distinct eigenvalues of A and n_i are their algebraic multiplicities.

The concept of fundamental matrix solution (for integer orders see, e.g. [22]) yields the following solution representation depending on the initial conditions:

Theorem 2.2. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha, \tau \in \mathbb{R}^+$ and R be the fundamental matrix solution of (2.1). Then the solution y of (2.1)–(2.3) is given by*

$$y(t) = \sum_{j=0}^{\lceil\alpha\rceil-1} \mathrm{D}_0^{\alpha-j-1} R(t)\phi_j + \int_{-\tau}^{0} R(t-\tau-u)A\phi(u)\mathrm{d}u\,.$$

Proof. The assertion follows directly from the evaluation of inverse Laplace transform of (2.1)–(2.3), for details see [6, 37]. □

To use these findings for qualitative analysis, we have to find more nuanced description of the solution. Applying the theory of Jordan canonical matrices on the fundamental matrix solution, we discover a key role of the function introduced by

Definition 2.3. Let $\lambda \in \mathbb{C}$, $\eta, \beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+ \cup \{0\}$. The generalized delay exponential function (of Mittag-Leffler type) is introduced via

$$G_{\eta,\beta}^{\lambda,\tau,m}(t) = \sum_{j=0}^{\infty} \binom{m+j}{j} \frac{\lambda^j (t-(m+j)\tau)^{\eta(m+j)+\beta-1}}{\Gamma(\eta(m+j)+\beta)} h(t-(m+j)\tau)$$

where h is the Heaviside step function.

Remark 2.4. We note that special choices of G function parameters yield functions known to solve special cases of (2.1). Indeed,

- $G_{1,1}^{\lambda,0,0}(t)$ reduces to classical exponential $\exp\{\lambda t\}$ solving $y'(t) = \lambda y(t)$,
- $G_{\alpha,1}^{\lambda,0,0}(t)$ coincides with one-parameter Mittag-Leffler function $E_\alpha(\lambda t^\alpha)$ solving the scalar version of (1.2) (see, e.g. [53]),
- $G_{1,1}^{\lambda,\tau,0}(t)$ is the delay exponential solving the scalar version of (1.3) (see, e.g. [2]).

The Laplace transform of the generalized delay exponential function of Mittag-Leffler type is

$$\mathcal{L}(G_{\eta,\beta}^{\lambda,\tau,m}(t))(s) = \frac{s^{\eta-\beta}\exp\{-ms\tau\}}{(s^\eta - \lambda\exp\{-s\tau\})^{m+1}}\,, \tag{2.6}$$

which allows us to detail the fundamental matrix solution as follows.

Lemma 2.5. *The fundamental matrix solution (2.4) can be expressed as $R(t) = T^{-1}\mathcal{G}(t)T$, where T is a regular matrix and $\mathcal{G}$ is a block diagonal matrix with upper-triangular blocks B_j given by*

$$B_j(t) = \begin{pmatrix} G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,1}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,2}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-1}(t) \\ 0 & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,1}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-2}(t) \\ 0 & 0 & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-3}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) \end{pmatrix},$$

where $j = 1,\ldots,J$ ($J \in \mathbb{Z}^+$), r_j is the size of the corresponding Jordan block of A.

Proof. See [6, 10]. □

Summarizing the above-stated results, we arrive at the crucial assertion describing the role of G functions, which serves as foundation for our next analysis.

Theorem 2.6. *Let $R(t)$ be the fundamental matrix solution of (2.1). Further, let λ_i ($i = 1,\ldots,n$) be distinct eigenvalues of A and let p_i be the largest dimension of the Jordan block corresponding to the eigenvalue λ_i. Then the nonzero elements of $R(t)$ are given by linear combinations of the generalized delay exponential functions*

$$G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t), \qquad m = 0,\ldots,p_i-1, \quad i = 1,\ldots,n.$$

2.2 Asymptotics of generalized delay exponentials

As mentioned in Section 1.3, the known asymptotics of exponential functions is underlying most of the qualitative analysis of integer-order problems. Analogously, the asymptotic properties of the generalized delay exponential function of Mittag-Leffler type (Definition 2.3) prove to be crucial in qualitative analysis of (2.1).

First, let us introduce the real-part ordering for the roots of the denominator in (2.6) where, for the sake of simplicity, we set $\eta = \alpha$ (note the link to the characteristic equation (2.5)). Let s_j ($j = 1, 2, \ldots$) be the roots of

$$s^\alpha - \lambda \exp\{-s\tau\} = 0$$

with ordering $\Re(s_j) \geq \Re(s_{j+1})$, particularly s_1 is called the rightmost root. Then we can write the foundational asymptotic result as

Lemma 2.7. *Let $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\beta, \tau \in \mathbb{R}^+$, $m \in \mathbb{Z}_0^+$ and s_j be roots of (2.6) with real-part ordering.*
(i) If $\lambda = 0$, then

$$G_{\alpha,\beta}^{0,\tau,m}(t) = \frac{(t-m\tau)^{m\alpha+\beta-1}}{\Gamma(m\alpha+\beta)} h(t-m\tau).$$

(ii) If $\lambda \neq 0$, then

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \sum_{j=1}^{\infty} \sum_{\ell=0}^{m \cdot k_j} a_{j\ell}(t - m\tau)^{\ell} \exp\{s_j(t - m\tau)\} + P_{\alpha,\beta}^{\lambda,\tau,m}(t) \,,$$

where k_j is a multiplicity of s_j, $a_{j\ell}$ are suitable nonzero complex constants ($\ell = 0, \dots, mk_j$, $j = 1, 2, \dots$) and the term $P_{\alpha,\beta}^{\lambda,\tau,m}$ has the algebraic asymptotic behaviour expressed via

$$\begin{aligned} P_{\alpha,\beta}^{\lambda,\tau,m}(t) = & \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta - \alpha)}(t + \tau)^{\beta-\alpha-1} \\ & + \frac{(-1)^{m+1}(m+1)}{\lambda^{m+2}\Gamma(\beta - 2\alpha)}(t + 2\tau)^{\beta-2\alpha-1} + \mathcal{O}(t^{\beta-3\alpha-1}) \qquad \text{as } t \to \infty. \end{aligned}$$

Proof. For the proof in its complete form see [6], for additional supplementary assertions needed for higher orders see [10].

Its idea is built around evaluation of G for large t through the inverse Laplace transform

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma(R,\frac{\pi}{2}+\delta)} \frac{s^{\alpha-\beta} \exp\{st - ms\tau\}}{(s^{\alpha} - \lambda \exp\{-s\tau\})^{m+1}} \mathrm{d}s \,.$$

The symbol $\gamma\,(R, \pi/2 + \delta)$ denotes the specific oriented piecewise smooth curve (see Figure 2.1) formed by three segments, i.e. $\gamma(\mu, \theta) = \gamma_1 + \gamma_2 + \gamma_3$ where $\mu > 0$, $\theta \in (0, \pi]$ and

$$\begin{aligned} \gamma_1 &= \{s \in \mathbb{C} : \; s = -u \exp\{-\mathrm{i}\theta\}, \, u \in (-\infty, -\mu)\} \,, \\ \gamma_2 &= \{s \in \mathbb{C} : \; s = \zeta \exp\{-\mathrm{i}u\}, \, u \in [-\pi - \theta, \pi + \theta]\} \,, \\ \gamma_3 &= \{s \in \mathbb{C} : \; s = u \exp\{-\mathrm{i}\theta\}, \, u \in (\mu, \infty)\} \,. \end{aligned}$$

The proof, apart from its considerable technical difficulty, utilizes several properties of characteristic roots. In particular, there exists $\delta > 0$ such that all roots s_j of (2.5) satisfy $|\operatorname{Arg}(s_i)| \neq \pi/2 + \delta$ and, moreover, that there are only finitely many of them satisfying $|\operatorname{Arg}(s_i)| < \pi/2 + \delta$. For detail calculations of relevant root properties, see [6]. □

Remark 2.8. Notice that Lemma 2.7 focuses on non-integer values of α. The cases of integer values are already covered by the classical theory and are known to have exponential asymptotics. From the technical standpoint, the difference lies in the fact that for non-integer α the Laplace transform of G contains non-analytic function, while for integer α only analytic functions occur.

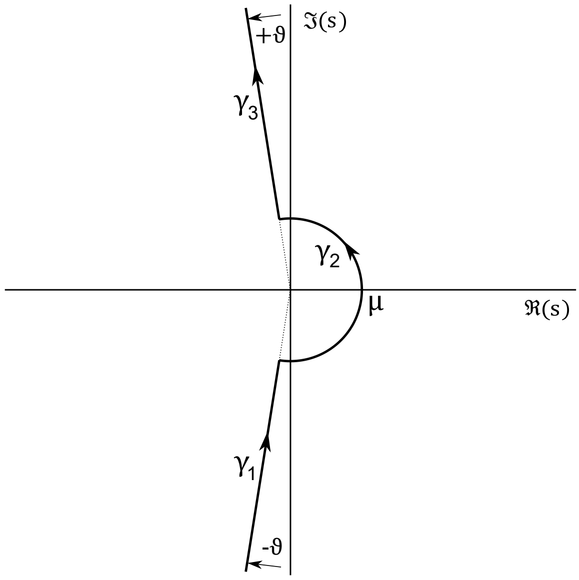

Figure 2.1: The curve $\gamma(\mu, \theta)$ used for evaluation of the inverse Laplace transform in the proof of Lemma 2.7.

2.3 Decomposition of eigenvalues' complex plane

Lemma 2.7 implies that, similarly to integer-order cases, the characteristic roots affect the stability properties primarily depending on the sign of their real parts. Hence, in this section we investigate the relation between locations of system matrix eigenvalues λ for (2.1) and zero real parts of characteristic roots of (2.5). In particular, we decompose the complex plane into regions such that eigenvalues chosen inside these regions guarantee nonzero real parts of the corresponding characteristic roots, and eigenvalues lying on boundaries of these regions imply at least one characteristic root with the zero real part.

Applying the standard approach of substituting $s = \mathrm{i}\,\varphi$ ($\varphi \in \mathbb{R}$) into factors of (2.5), i.e. $s^\alpha - \lambda \exp\{-s\tau\} = 0$, equating real and imaginary parts and rearranging with respect to $|\lambda|$ and $\mathrm{Arg}(\lambda)$. After a tedious calculations (see [10]), we can eliminate the parameter φ and define the regions as follows:

For any $\alpha > 0$ and $m \in \mathbb{Z}^+$ such that $0 < \alpha < 4m + 2$:

$$Q_\alpha^\tau(m) = \Big\{ \lambda \in \mathbb{C} : \, |\lambda| < \Big(\frac{|\mathrm{Arg}(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau} \Big)^\alpha, \\ \frac{\alpha\pi}{2} - 2m\pi < |\mathrm{Arg}(\lambda)| \le \frac{\alpha\pi}{2} - (2m-2)\pi \Big\} \\ \cup \Big\{ \lambda \in \mathbb{C} : \, \Big(\frac{|\mathrm{Arg}(\lambda)| - \frac{\alpha\pi}{2} + (2m-2)\pi}{\tau} \Big)^\alpha < |\lambda| < \Big(\frac{|\mathrm{Arg}(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau} \Big)^\alpha, \\ |\mathrm{Arg}(\lambda)| > \frac{\alpha\pi}{2} - 2m\pi \Big\}$$

where the sets $Q_\alpha^\tau(m)$ ($m \in \mathbb{Z}_0^+$) are defined to be empty whenever $\alpha \ge 4m + 2$.

Further, for $\alpha \in (0,2)$ we add:

$$Q_\alpha^\tau(0) = \left\{ \lambda \in \mathbb{C} : \, |\lambda| < \left(\frac{|\operatorname{Arg}(\lambda)| - \alpha\pi/2}{\tau} \right)^\alpha, \; |\operatorname{Arg}(\lambda)| > \frac{\alpha\pi}{2} \right\}.$$

As illustrated by Figures 2.2-2.5, the sets $Q_\alpha^\tau(m)$ $(m \in \mathbb{Z}_0^+)$ are disjoint and the infinite union of their closures covers the whole complex plane. In the next two section we detail the role of these sets in qualitative properties of (2.1).

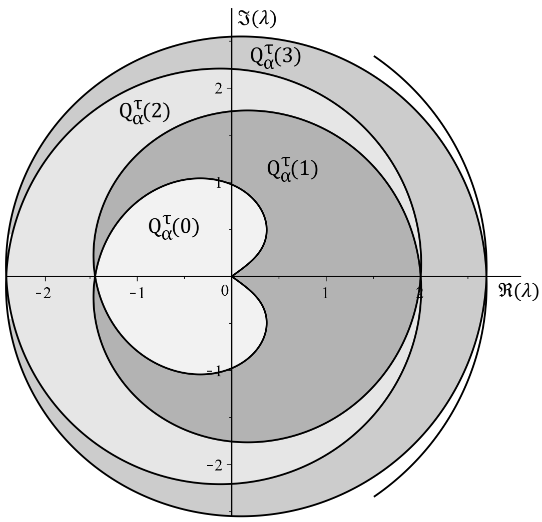

Figure 2.2: Decomposition of eigenvalues' complex plane for $\alpha = 0.4$, $\tau = 1$.

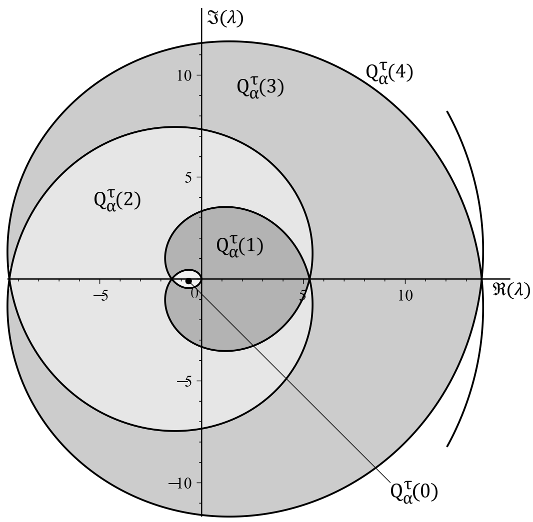

Figure 2.3: Decomposition of eigenvalues' complex plane for $\alpha = 1.1$, $\tau = 1$

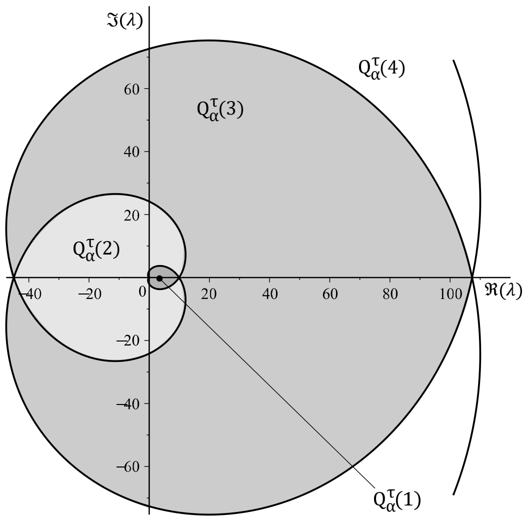

Figure 2.4: Decomposition of eigenvalues' complex plane for $\alpha = 2.1$, $\tau = 1$

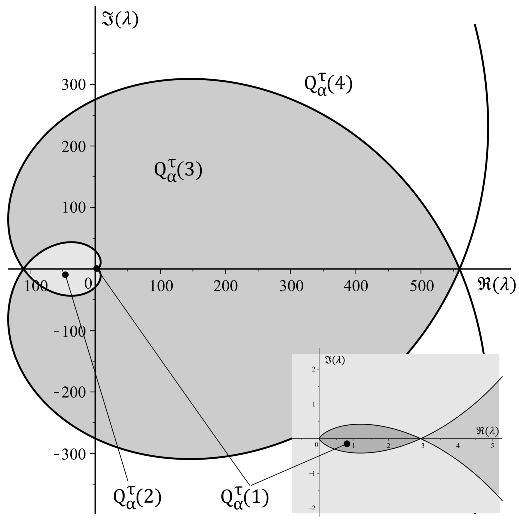

Figure 2.5: Decomposition of eigenvalues' complex plane for $\alpha = 3.1$, $\tau = 1$

2.4 Asymptotically stable systems

The calculations behind the complex plane decomposition from the previous section yield that the set $Q_\alpha^\tau(0)$ contains all the eigenvalues having solely characteristic roots with negative real parts. Thus, it coincides with the stability region for (2.1) which takes the form

$$\mathcal{S}_\alpha^\tau = \left\{ \lambda \in \mathbb{C} : \, |\lambda| < \left(\frac{|\operatorname{Arg}(\lambda)| - \alpha\pi/2}{\tau} \right)^\alpha , \, |\operatorname{Arg}(\lambda)| > \frac{\alpha\pi}{2} \right\} . \qquad (2.7)$$

That enables us to write a fractional counterpart to Theorem 1.4 and simultaneously a delay counterpart to Theorem 1.1 as follows

Theorem 2.9. *Let $A \in \mathbb{R}^{d\times d}$, $\tau \in \mathbb{R}^+$ and $\alpha \in (0,2)$. Then (2.1) is asymptotically stable if and only if all eigenvalues λ_i $(i = 1, \dots, d)$ of A are nonzero and satisfy*

$$\tau |\lambda_i|^{1/\alpha} < |\operatorname{Arg}(\lambda_i)| - \alpha\pi/2 \, .$$

Moreover, if $\alpha \notin \mathbb{Z}^+$, then the convergence to zero is of algebraic type; more precisely, for any solution y of (2.1) there exists a suitable integer $j \in \{0, \dots, \lceil \alpha \rceil\}$ such that $\|y(t)\| \sim t^{j-\alpha-1}$ as $t \to \infty$ (the symbol $\|\cdot\|$ means a norm in $\mathbb{R}^d$).

Proof. The proof is based on Theorems 2.2 and 2.6 combined with Lemma 2.7. Its main challenge lies in asymptotic evaluation of the integral term $\int_{-\tau}^0 R(t-\tau-u)A\phi(u)\mathrm{d}u$. For details see [6, 10, 37]. □

Figures 2.6 and 2.7 illustrate evolution of the stability region for increasing α. For all $\alpha \in (0,1) \cup (1,2)$ the stability boundary has a cusp point at the origin from which it continues symmetrically above and below real axis with tangents $\pm\alpha\pi/2$, respectively. For $\alpha = 1$, the cusp point smoothens as the tangents align with the imaginary axis.

Figures 2.8 and 2.9 show the shape of the stability region for α close to integer-order values one (compare to Figure 1.3) and two (the stability region vanishes). Figure 2.10 outlines the effect of decreasing τ causing expansion of the stability region up to the undelayed case for $\tau \to 0^+$ (compare to Figures 1.1 and 1.2).

The most puzzling insight brought by changing α in (2.1) is depicted in Figure 2.11 where we see shape of stability region for the values α close to zero. Although for all $\alpha > 0$ the positive reals lie outside of stability region, we see that the limit shape for $\alpha \to 0^+$ tends to a circle. As shown in Figure 1.4, the circle is the known stability region for difference equation (1.4) which, in a certain sense, can be seen as (2.1) with $\alpha = 0$. This remarkable connection suggests the potential of fractional-order derivatives to provide transition not just between integer-order differential systems but also between the differential and difference systems (for more comments see [6]).

The approach originating from Lemma 2.7 and Theorems 2.2 and 2.6 allows to address also the boundary of stability region.

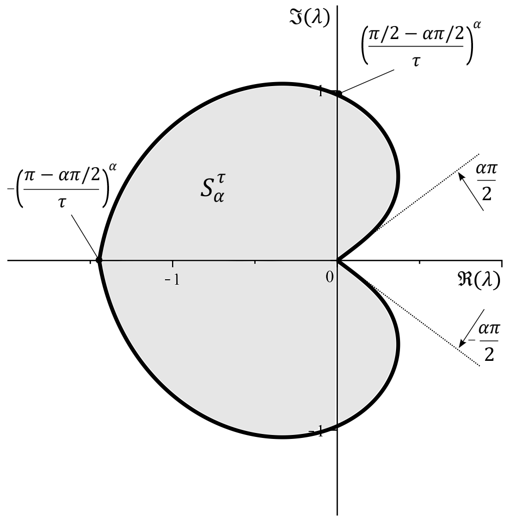

Figure 2.6: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 0.4$, $\tau = 1$.

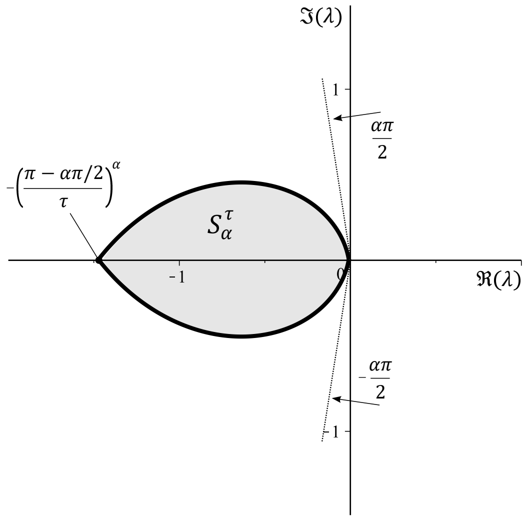

Figure 2.7: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 1.1$, $\tau = 1$.

Theorem 2.10. *Let $A \in \mathbb{R}^{d\times d}$, $\tau \in \mathbb{R}^+$ and $\alpha \in (0,2)$. Then (2.1) is stable if and only if all eigenvalues λ of A belong to $\mathcal{S}_\alpha^\tau$ or its boundary $\partial\mathcal{S}_\alpha^\tau$, and all the ones lying on the boundary have same algebraic and geometric multiplicities.*

Proof. See [6, 37]. □

Remark 2.11. (i) Comparing Theorems 2.9 and 2.10 we see that while presence of an eigenvalue on the stability boundary removes asymptotic stability, it preserves the stability provided it has the same same algebraic and geometric multiplicities. Consequently, for scalar version of (2.1), the system is stable if and only if all eigenvalues lie in the closure of $\mathcal{S}_\alpha^\tau$ (see also [39]).
(ii) Although this thesis deals with fractional derivatives of Caputo type, it is worth noting that (2.1) with a Riemann-Liouville derivative has nearly the same stability properties. The only difference occurs when the zero eigenvalue is present as proved in [37]. Specifically, if $\alpha < 1$ and the maximum size of any Jordan block associated with the zero eigenvalue is less than $1/\alpha$, the asymptotic stability appears.

Oscillatory properties of asymptotically stable systems

Theorem 1.6 implies that in the case of the first-order delay system (1.3), the oscillations occur for almost all λ (more precisely, some solutions of (1.3) do not oscillate, if some eigenvalue lies in $[-1/(\tau e), \infty)$, which is a set of zero measure). In the case of $\alpha \neq 1$, the situation is very different. The following assertion shows that there are no oscillatory solutions tending to zero.

Theorem 2.12. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\tau \in \mathbb{R}^+$ and let (2.1) be stable. If all eigenvalues λ of A belong to $\mathcal{S}_\alpha^\tau \cup \{0\}$, then all nonzero solutions are non-oscillatory.*

Proof. The proof builds on Lemma 2.7 and the fact that in asymptotically stable case the non-oscillating algebraic term $P_{\alpha,\beta}^{\lambda,\tau,m}(t)$ dominates the oscillating exponential

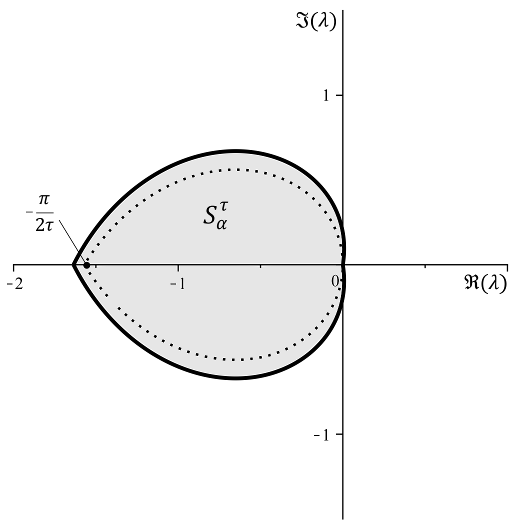

Figure 2.8: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 0.9$, $\tau = 1$ (the corresponding limit case $\alpha \to 1$ is dotted).

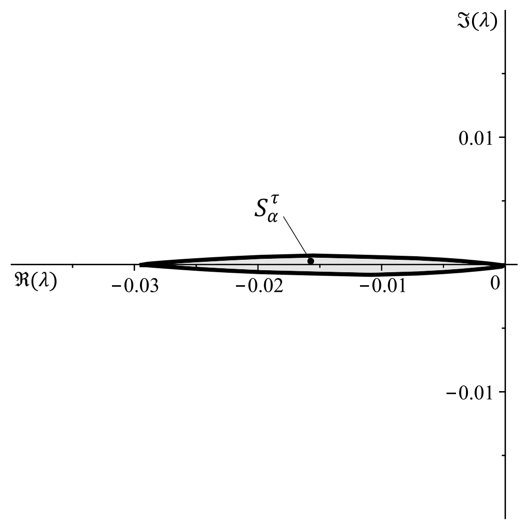

Figure 2.9: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 1.9$, $\tau = 1$ (the corresponding limit case $\alpha \to 2^-$ is an empty set).

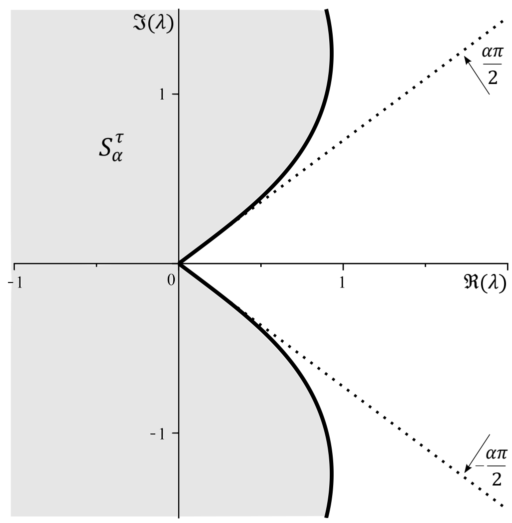

Figure 2.10: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 0.4$, $\tau = 0.1$ (the corresponding limit case $\tau \to 0^+$ is dotted).

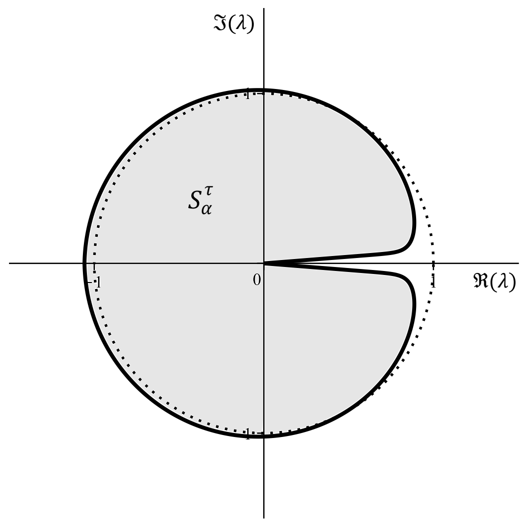

Figure 2.11: Stability region $\mathcal{S}_\alpha^\tau$ for (2.1) depicted for $\alpha = 0.05$, $\tau = 1$ (the corresponding limit case $\alpha \to 0^+$ is dotted).

functions. In case of the zero eigenvalue, the additional term is also non-oscillatory no matter the multiplicity of the zero eigenvalue. □

2.5 Unstable systems

The techniques used in [6] to discuss properties of asymptotically stable systems turn out to be effective also in the case of unstable system. In particular, they enable us to describe the supremum asymptotics of the unbounded solutions as follows.

Theorem 2.13. *Let $A \in \mathbb{R}^{d\times d}$ and $\alpha, \tau \in \mathbb{R}^+$. Let λ_i be all distinct eigenvalues of A ($i = 1, \dots, n$) and let (2.1) is not stable. Then solutions $y(t)$ of (2.1) admit three types of asymptotics:*
(i) Let $\lambda_1 = 0$ be the zero eigenvalue of A with algebraic multiplicity greater than geometric one and let p_1 be the maximal size of Jordan blocks corresponding to λ_1. Further, let $\lambda_i \in \mathcal{S}_\alpha^\tau$ for all $i = 2, \dots, n$. Then

$$\|y(t)\| \sim t^{(p_1-1)\alpha} \qquad \text{as } t \to \infty \qquad \text{for any solution } y(t) \text{ of (2.1).}$$

(ii) Let λ_i ($i = 1, \dots, \ell \le n$) be nonzero eigenvalues of A lying on $\partial\mathcal{S}_\alpha^\tau$ with algebraic multiplicity greater than geometric one and let p_i be the maximal size of Jordan blocks corresponding to λ_i ($i = 1, \dots, \ell$). Further, let $\lambda_i \in \mathcal{S}_\alpha^\tau$ for all $i = \ell+1, \dots, n$ provided $\ell < n$ and $p = \max(p_1, \dots, p_\ell)$. Then

$$\|y(t)\| \sim_{sup} t^{p-1} \qquad \text{as } t \to \infty \qquad \text{for any solution } y(t) \text{ of (2.1).}$$

(iii) Let λ_i ($i = 1, \dots, \ell \le n$) be eigenvalues of A located outside $\mathrm{cl}(\mathcal{S}_\alpha^\tau)$ and let s_1 be the rightmost root of (2.5). Further, let λ_j, $j \in L \subset \{1, \dots, \ell\}$ be eigenvalues of A such that (2.5) with $\lambda = \lambda_j$ has the zero s_1 and let p be the maximal size of Jordan blocks corresponding to λ_j, $j \in L$. Then

$$\|y(t)\| \sim_{sup} t^{p-1} \exp\{\Re(s_1)t\} \qquad \text{as } t \to \infty \qquad \text{for any solution } y(t) \text{ of (2.1).}$$

Proof. The details see in [10]. □

To obtain an actually effective (and non-improvable) asymptotic result for the solutions of (2.1), we have to look at the problem inversely. More precisely, for a given complex $\lambda \notin \mathcal{S}_\alpha^\tau$, we need to find (nonnegative) real values u, v such that the rightmost root s_1 of (2.5) satisfies $\Re(s_1) = u$, $|\Im(s_1)| = v$.

This nontrivial question can be addressed using properties and methods of Lambert function, i.e. the function introduced as a solution of $W(z)\exp(W(z)) = z$, $z \in \mathbb{C}$ (see, e.g. [28]). In [15] we developed a framework allowing to evaluate the precise asymptotic envelop of the unbounded solutions and also the corresponding asymptotic frequency of oscillations. For the sake of simplicity, only scalar version of (2.1) with complex coefficient λ was considered. The findings can be summarized in the following

Theorem 2.14. *Let $\alpha \in (1, \infty)$, $\tau \in \mathbb{R}^+$ and $\lambda \in \mathbb{C}$. If $\lambda \notin \mathcal{S}_\alpha^\tau$, then, for any solution $y(t)$ of $\mathrm{D}_0^\alpha y(t) = \lambda y(t-\tau)$ it holds*

$$y(t) = \exp(ut)(c\exp(\mathrm{i}\,vt) + o(1)) \quad \text{as} \quad t \to \infty$$

where c is a complex constant, $u \geq 0$ is the unique solution of

$$\alpha \arccos\left(\frac{u\exp(\tau u/\alpha)}{|\lambda|^{1/\alpha}}\right) + \frac{\tau\sqrt{|\lambda|^{2/\alpha} - u^2\exp(2\tau u/\alpha)}}{\exp(\tau u/\alpha)} = |\mathrm{Arg}(\lambda)| \,,$$

$v > 0$ is the unique solution of

$$\frac{v^\alpha}{\sin^\alpha\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)}\exp\big(\tau v\cot\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)\big) = |\lambda|, \quad \textit{if } |\mathrm{Arg}(\lambda)| > 0\,,$$

and $v = 0$ if $\mathrm{Arg}(\lambda) = 0$.

Proof. The first and simple part is to express the characteristic roots s_k ($k \in \mathbb{Z}$) of (2.5) in terms of Lambert function, i.e.

$$s_k = \frac{\alpha}{\tau}W_k\left(\frac{\tau}{\alpha}\lambda^{1/\alpha}\right)\,, \qquad k \in \mathbb{Z}\,,$$

where W_k is the kth branch of the Lambert function. The key part is to prove the existence of ordering put on Lambert functions branches, namely that $\Im(W_k(z)) \leq \Im(W_{k+1}(z))$ for all $k \in \mathbb{Z}$ and $z \neq 0$ and $\Re(W_0(z)) \geq \Re(W_k(z))$ for all $k \in \mathbb{Z}$. The proof is then concluded by technically challenging calculations leading to the equations for u, v depending on α, τ, λ. For details we refer to [15]. □

Remark 2.15. (i) Note that Theorem 2.14 is not formulated for $\alpha < 1$. That is a consequence of $1/\alpha$ occurring in the argument of the Lambert function which for $\alpha < 1$ might introduce some additional roots which do not actually solve (2.5). This problem does not seem to be solvable in the framework of Lambert function method and, to the author's knowledge, remains open.
(ii) Figure 2.12 depicts a practical "map" allowing us to quickly estimate asymptotic modulus and oscillation frequency for the solution of $\mathrm{D}_0^\alpha y(t) = \lambda y(t-\tau)$ based on location of λ in the complex plane.
(iii) Theorem 2.14 considers only scalar version of (2.1) with complex coefficient. If we deal with the vector version of (2.1) and the system matrix A has eigenvalues with the same algebraic and geometric multiplicities, we just have to apply Theorem 2.14 for every eigenvalue and combine the results (Figure 2.12 also applies). In case of different algebraic and geometric multiplicities, the estimates for asymptotic frequencies are still valid and the estimates for the modulus have to adjusted by polynomial multiplication.

Oscillatory properties of unstable systems

As in the stable case, (2.1) (for $\alpha \neq 1$) does not have any combination of entry parameters guaranteeing oscillations of all solutions. On the other hand, there are combinations that ensure no oscillatory solutions.

Theorem 2.16. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\tau \in \mathbb{R}^+$ and let (2.1) be unstable. If all eigenvalues λ of A belong to $\mathcal{S}_\alpha^\tau \cup \{0\} \cup (Q_1(\alpha,\tau) \cap \mathbb{R})$, then all nonzero solutions are non-oscillatory.*

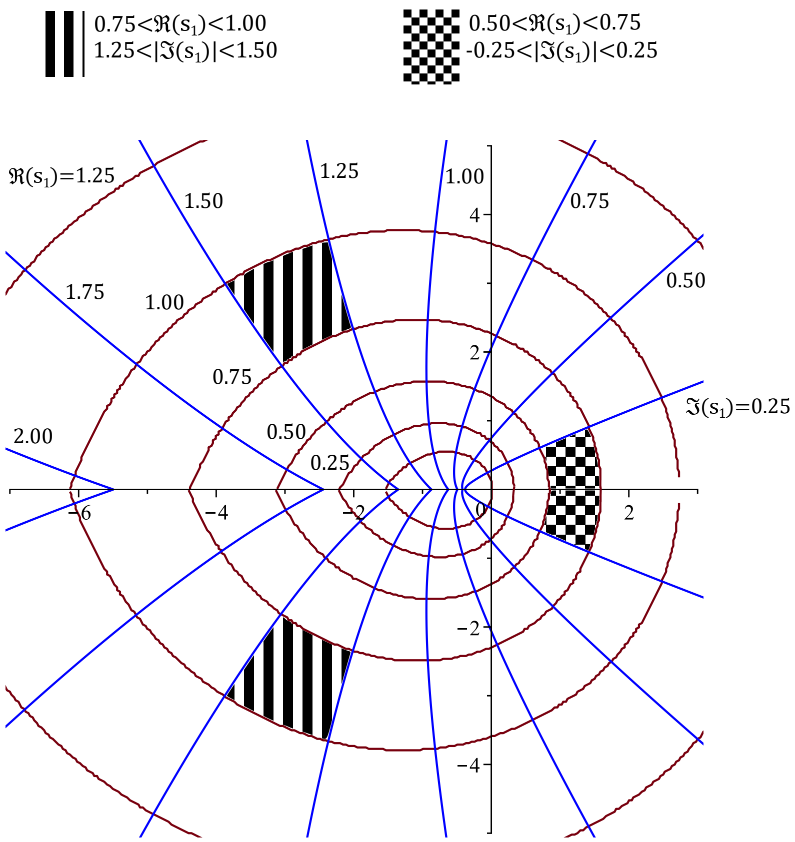

Figure 2.12: The blue curves represent the set of all $\lambda \in \mathbb{C}$ such that the rightmost characteristic root s_1 of (2.5) satisfies $\Im(s_1) = v$, and the particular orange curves represent the set of all $\lambda \in \mathbb{C}$ such that the rightmost characteristic root s_1 of (2.5) satisfies $|\Re(s_1)| = v$ (the scenario corresponds to $\alpha = 1.2$ and $\tau = 1$). As an example, there are highlighted curvilinear rectangles representing sets of all $\lambda \in \mathbb{C}$ yielding $\Re(s_1)$ and $\Im(s_1)$ from a certain range.

Proof. The outline of the prove is following, for details we refer to [10].

It can be seen from Lemma 2.7 that oscillatory solutions can occur only if there is a positive real characteristic root. Further, it is possible to prove that (2.5) has a positive real root if and only if λ is a positive real and this root is simple, unique and it is the rightmost root of (2.5).

Then we employ properties of $Q^\tau_\alpha(m)$ introduced in Section 2.3. In particular, that there exist just m $(m = 0, 1, \dots)$ characteristic roots of (2.5) with a positive real part (while remaining roots have negative real parts) if and only if $\lambda \in Q^\tau_\alpha(m)$. Moreover, (2.5) has a root with the zero real part if $\lambda \in \partial[Q^\tau_\alpha(m)]$ for some $m = 0, 1, \dots$. □

Remark 2.17. Similarly to Theorems 2.12 and 2.16, oscillatory solutions can occur in some cases only for a particular choice of initial conditions (see [10]).

Chapter 3

Analysis of two-term fractional delay differential equations

This chapter summarizes the key findings related to two-term FDDE from author's papers [11–13]. Building on our analysis of one-term FDDS, it would seem natural to turn to

$$\mathrm{D}_0^\alpha y(t) = Ay(t) + By(t-\tau)\,,$$

where A, B are real $d \times d$ matrices and τ is a positive real time delay. Moreover, such a mathematical model would provide a large application potential (see, e.g. [40,43]), especially in control theory regarding stabilization of equilibria of fractional dynamical systems via delayed feedback controls. Although addition of $Ay(t)$ on the right-hand side looks quite straightforward, it highly increases the difficulty of the studied problem. Even the classical case $\alpha = 1$ is still generally unsolved (see, e.g. [4,31,46,55]).

That is why we focus on the proper development of stability theory for the scalar case, namely two-term fractional delay differential equation (FDDE)

$$\mathrm{D}_0^\alpha y(t) = ay(t) + by(t-\tau)\,, \tag{3.1}$$

where a, b are real coefficients, $\tau > 0$ is a real lag and $\alpha \in (0,2)$. Similarly as for (2.1), the associated initial conditions are considered as

$$y(t) = \phi(t)\,, \qquad t \in [-\tau, 0)\,, \tag{3.2}$$

$$\lim_{t\to 0^+} y^{(j)}(t) = \phi_j\,, \qquad j = 0,1 \tag{3.3}$$

where ϕ is absolutely Riemann integrable on $[-\tau, 0)$ and ϕ_j are reals.

The topic of stability and asymptotic analysis of FDDEs attracts the attention of many authors. Before 2016, significant majority of corresponding stability results was derived as parametric equations or implicit relations for the stability boundary or usually as an outcome of the D-decomposition method and an appropriate root locus. For such or similar stability results on (3.1) (with $\alpha \in (0,1)$) we refer to [1,30], but the trend is evident from the literature even for simpler cases (see, e.g. [16, 40, 41, 54]). Hence, we decided to focus on the formulation of explicit stability criteria for (3.1), as they provide much more accessible and practical tool in comparison to the usual parametric or implicit ones. In [13] we succeeded for the derivative order less than one and managed to find the explicit description of stability region in the (a, b)-plane and the formula for the change from stability to instability with respect to increasing τ which is present also in the first-order case.

It is well-known that the integer-order linear delay dynamical systems may change their stability into instability with growing time delay not just once, but repeatedly back and forth. This interesting phenomenon, often referred to as stability switching, is still a current subject of research as exemplified, e.g. by [19,42,47,51] where values of stability switches are described via parameters of the corresponding integer-order system. Thus, the occurrence of stability switching for FDDEs is a natural topic discussed in the second decade of 21st century, e.g. in [56, 57]. Our papers [11,12] are mostly devoted to this area, the former one considering (3.1) with imaginary coefficient a and α less than one, and the latter one with α between one and two. In both the cases we managed to derive explicit values of stability switches as well as conditions for so-called delay-independent stability. Moreover, [12] clarifies the continuous transition between qualitatively very different stability regions for (3.1) with $\alpha = 1$ and $\alpha = 2$.

The following sections are built on the main results of [11–13]. Section 3.1 elaborates on structure and asymptotics of solutions to (3.1). Then, unlike the Chapter 2, we focus less on precision of asymptotics and more on shape of stability regions and their dependence on system parameters. Section 3.2 deals with the case of derivative order less than one, Section 3.3 changes one of the system parameters from real to imaginary and Section 3.4 comes back to real (3.1) with order between one and two. Throughout the chapter, we put stress on the explicit stability conditions which are quite challenging for (3.1), among other things, because of the presence of stability switches (see Sections 3.3 and 3.4).

3.1 Structure and asymptotics of solutions

Because (3.1) shares with (2.1) its linearity, fractional derivative order as well as time delay, the structure of the solution is expected to be similar. The Laplace transform of (3.1)-(3.3) shows that the fundamental solution belongs to the family of functions

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) = \mathcal{L}^{-1}\left(\frac{s^{\alpha-\beta}}{s^\alpha - a - b\exp[-s\tau]}\right)(t) \tag{3.4}$$

where $\alpha, \beta, \tau > 0$ and $a, b \in \mathbb{R}$. We can see that the generalized delay exponential function (with parameter $m = 0$) introduced by Definition 2.3 in the previous chapter is the special case of (3.4). Indeed, $\mathcal{R}^{0,b,\tau}_{\alpha,\beta}(t) = G^{b,\tau,0}_{\alpha,\beta}(t)$ (see (2.6)).

Using the inverse Laplace transform we arrive at the representation of the solution to (3.1)-(3.3) (compare to Theorem 2.2)

$$y(t) = \sum_{j=0}^{\lceil \alpha \rceil} \phi_j \mathcal{R}^{a,b,\tau}_{\alpha,j+1}(t) + b \int_{-\tau}^{0} \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-\tau-u)\phi(u)\mathrm{d}u\,. \tag{3.5}$$

The characteristic equation associated with (3.1) is implied by (3.4) and (3.5) in the expected form

$$s^\alpha - a - b\exp(-s\tau) = 0 \tag{3.6}$$

which has infinitely many complex roots (compare to (2.5) and to characteristic equations for the classical integer-order cases in Section 1.3).

The key auxiliary assertion, the counterpart to Lemma 2.7, deals with asymptotic properties of $\mathcal{R}^{a,b,\tau}_{\alpha,\beta}$ functions.

Lemma 3.1. *Let $\alpha \in (0,1)$, $\beta \in (0,1]$, $a, b \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$ and let s_i be roots of (3.6).*
(i) If $\Re(s_i) < 0$ for all s_i then

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) \sim t^{\beta-\alpha-1} \text{ for } \alpha \neq \beta \quad \text{and} \quad \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t) = \mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t \to \infty\,.$$

(ii) If there exists the zero root of (3.6) and $\Re(s_i) < 0$ otherwise, then

$$\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) \sim 1 \quad \text{and} \quad \mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) = \mathcal{O}(t^{\beta-1}) \text{ for } \beta < 1 \qquad \text{as } t \to \infty\,.$$

(iii) If $\Re(s_i) \le 0$ for all s_i and some of the roots are purely imaginary, then

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) \sim_{sup} 1 \qquad \text{as } t \to \infty\,.$$

(iv) If $\Re(s_i) > 0$ for some s_i then

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) \sim_{sup} (Bt + C)\exp[Mt] \qquad \text{as } t \to \infty$$

where $M = \max_{s_i}(\Re(s_i))$ and reals $B, C \ge 0$ are such that $B + C > 0$.

Proof. The proof utilizes the technique already outlined in the proof of Lemma 2.7, with much higher technical difficulty, more branching to be considered with respect to the parameter values and with several adjustments (see [13, pages 344–349]).

The assumptions for the use of the technique needs the root analyses of (3.6), most importantly showing that for an arbitrary $0 < \omega < \pi$, the characteristic equation has no more than a finite number of roots s such that $|\operatorname{Arg}(s)| \le \omega$. □

Although Lemma 3.1 is formulated for real values of a, b and α less than one, it is only a technical matter to generalize it. The quality of the asymptotic estimates may be affected but the stability implications remains the same (as pointed out in [11, 12]).

Theorem 3.2. *Let $\alpha > 0$, $\tau > 0$ and a, b be complex numbers.*
(i) If all the roots of (3.6) *have negative real parts, then (3.1) is asymptotically stable.*
(ii) If there exists a root of (3.6) *with a positive real part, then (3.1) is not stable.*

Remark 3.3. Theorem 3.2 does not address the stability boundary. As we will discuss later, the stability boundary for (3.1) contains points of asymptotic stability, stability and also instability for various values of system parameters.

3.2 Stability regions for orders less than one

Let us consider (3.1) with $\alpha < 1$ and investigate the boundary locus for the corresponding (3.6), i.e. the set of all real couples (a, b) such that the characteristic equation admits a root with zero real part. Substituting $s = \pm\mathrm{i}\,\varphi$ into (3.6) and considering real and imaginary parts separately yields two qualitatively distinct parts of boundary locus: the line $a + b = 0$ (corresponding to the zero root) and the system of curves (corresponding to purely imaginary roots)

$$a_m(\rho) = \frac{\rho^\alpha \sin(\rho + \alpha\pi/2)}{\tau^\alpha \sin(\rho)}, \qquad b_m(\rho) = -\frac{\rho^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(\rho)}, \tag{3.7}$$

$m\pi < \rho < (m+1)\pi$, $m = 0, 1, \ldots$. The curves forming the boundary locus are depicted in the (a, b)-plane on Figure 3.1 (see also [30] where the authors redundantly considered multi-valued function s^α instead of the single-valued one).

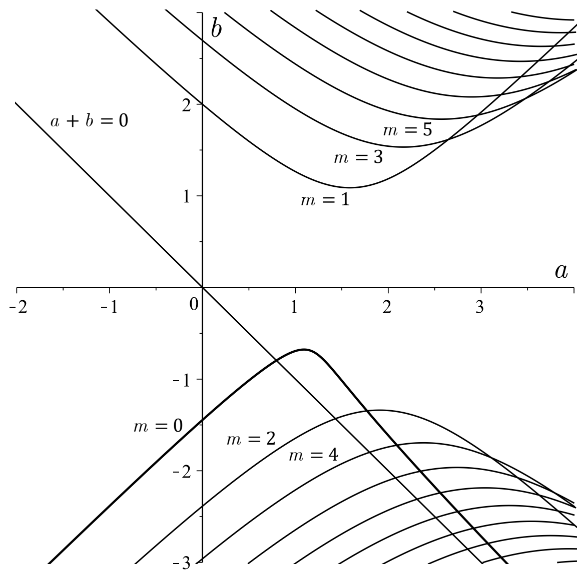

Figure 3.1: Boundary locus of (3.1) for $\alpha = 0.4$, $\tau = 1$.

The necessary link between the boundary locus curves and stability properties of (3.1) is provided by the following

Theorem 3.4. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers. Then all roots of (3.6) have negative real parts if and only if the couple (a, b) is an interior point of the area bounded by the line $a + b = 0$ from above and by the parametric curve*

$$a = \frac{\rho^\alpha \sin(\rho + \alpha\pi/2)}{\tau^\alpha \sin(\rho)}, \quad b = -\frac{\rho^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(\rho)}, \qquad \rho \in ((1-\alpha)\pi, \pi) \tag{3.8}$$

from below.

Proof. It follows from continuous dependence of roots of (3.6) on the coefficients a, b. This property particularly implies that the number of characteristic roots with a positive real part remains unchanged in all open sets whose boundaries are formed by the line $a+b=0$ or by some curves (3.7). Then it is enough to choose representatives of these open sets to specify the number of roots of (3.6) with positive real parts within these sets. For details see [13]. □

Remark 3.5. Theorem 3.4 implies that of all curves in the system (3.7) only a part of the curve characterized by $m=0$ affects the stability boundary (see Figure 3.2). Others lie in the region where (3.1) is not stable. Also, substituting $(1-\alpha)\pi$ into (3.8) enables us to calculate the coordinates of the cusp point (see also Figure 3.2).

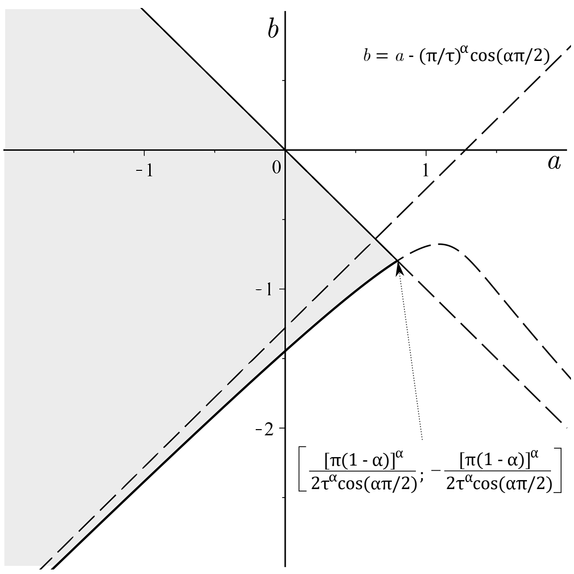

Figure 3.2: The stability region $\mathcal{S}_\alpha^\tau$ for (3.1) depicted for $\alpha=0.4$ and $\tau=1$.

The main objective of our effort is to derive explicit conditions determining the stability region. A related problem has been discussed in [1] where (3.6) is analysed for $a<0$. Here we present the assertion removing the restriction on a and yielding results in a simpler form due to the use of a different computational technique.

Theorem 3.6. *Let $0<\alpha<1$, a, b and $\tau>0$ be real numbers. Then (3.1) is asymptotically stable if and only if it holds either*

$$a \le b < -a \quad \textit{and} \quad \tau \textit{ is arbitrary}, \tag{3.9}$$

or

$$|a|+b<0 \quad \textit{and} \quad \tau<\tau^* = \frac{(1-\alpha)\pi/2+\arccos[(-a/b)\sin(\alpha\pi/2)]}{[a\cos(\alpha\pi/2)+(b^2-a^2\sin^2(\alpha\pi/2))^{1/2}]^{1/\alpha}}\,. \tag{3.10}$$

Proof. The proof is based on rewriting the parametric equations (3.8) into the explicit ones via intersection analysis of the boundary locus curves, elimination of the parameter φ and careful operations with inverse trigonometric functions. For details see [13]. □

Remark 3.7. (i) Clearly, the stability region $\mathcal{S}_\alpha^\tau$ consists of pairs (a, b) such that either (3.9) or (3.10) holds. The condition (3.9) defines the region of delay-independent stability. The second condition, (3.10), shows dependence on the time delay, namely it indicates the one-time loss of stability with increasing τ reaching the value τ^*.
(ii) The delay-dependent part of stability region expands with decreasing time delay. If we consider the limit $\tau \to 0^+$, the stability region simplifies into half-plane $a + b < 0$. That agrees with the stability region for the scalar version of (1.2) with the coefficient $a + b$ which is the limit of (3.1) for $\tau \to 0^+$.
(iii) Considering (3.1) with $a = 0$, we obtain the scalar version of (2.1) with coefficient b which is asymptotically stable for $-(\pi/\tau - \alpha\pi/(2\tau))^\alpha < b < 0$ (see Figures 2.6 and 2.7). That corresponds to Theorem 3.6 as $-(\pi/\tau - \alpha\pi/(2\tau))^\alpha$ is the intersection between b-axis and the lower branch of the stability boundary.

The following theorem shows that for α less than one, the stability boundary fully corresponds to the situation when (3.1) is stable but not asymptotically stable.

Theorem 3.8. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers. Then (3.1) is stable, but not asymptotically stable, if and only if either*

$$a + b = 0, \qquad a \le \frac{[\pi(1-\alpha)]^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)}, \tag{3.11}$$

or

$$|a| + b < 0, \qquad \tau = \tau^*, \quad \tau^* \textit{ being the same expression as in (3.10).} \tag{3.12}$$

Proof. The assertion follows from application of Lemma 3.1, see [13]. □

Remark 3.9. The stability of (3.1) in the cusp point (the intersection of the line $a + b = 0$ and (3.8) is not analogue to the situation known from (1.5). It can be proved by a direct calculation that (1.5) at the cusp point (i.e. $a = -b = 1/\tau$) is not stable.

Due to the scalar nature of (3.1) and quite simple form of the stability boundary for of α less than one, we have quite comprehensive asymptotic description of solutions.

Lemma 3.10. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers and let y be a solution of (3.1).*
(i) Let (3.1) be asymptotically stable. Then

$$y(t) \sim t^{-\alpha} \qquad \textit{or} \qquad y(t) = \mathcal{O}(t^{-\alpha-1}) \qquad \textit{as } t \to \infty\,.$$

(ii) Let (3.1) be stable but not asymptotically stable. If (3.11) *is satisfied, then*

$$y(t) \sim 1 \qquad or \qquad y(t) = \mathcal{O}(t^{\alpha-1}) \qquad as\ t \to \infty\,.$$

If (3.12) holds, then

$$y(t) \sim_{sup} 1 \qquad or \qquad y(t) = \mathcal{O}(1) \qquad as\ t \to \infty\,.$$

(iii) Let (3.1) be unstable. Then $y(t) = \mathcal{O}(t\exp[Mt])$ *as* $t \to \infty$*, where* $M = \max_{s_i}(\Re(s_i))$*,* s_i *being roots of (3.6). Moreover, there exists a solution* y *of (3.1) such that*

$$y(t) \sim_{sup} t\exp[Mt] \qquad or \qquad y(t) \sim_{sup} \exp[Mt] \qquad as\ t \to \infty\,.$$

Proof. The proof is a consequence of Lemma 3.1 and (3.5). See [13] for details. □

Remark 3.11. As usual for asymptotically stable fractional dynamic systems, the decay rate of the solutions is algebraic, while in the classical case (1.5) it is known to be exponential.

3.3 Stability regions for orders less than one and an imaginary coefficient

Let us change the first coefficient of (3.1) into an imaginary one and study the problem

$$\mathrm{D}_0^\alpha y(t) = \mathrm{i}\,ay(t) + by(t-\tau)\,. \tag{3.13}$$

Although (3.13) might look artificially constructed, it actually plays a key role in the stability investigation of a planar fractional delay system

$$\begin{aligned}\mathrm{D}_0^\alpha x_1(t) &= ux_1(t-\tau) + vx_2(t)\\ \mathrm{D}_0^\alpha x_2(t) &= wx_1(t) + ux_2(t-\tau)\end{aligned}$$

with real entries u, v, w ($v, w \neq 0$) which was the focus of [11]. We note that the corresponding classical case ($\alpha = 1$) was studied by [46] due to its stability switching nature.

Finding the boundary locus for the characteristic equation associated with (3.13),

$$s^\alpha - \mathrm{i}\,a - b\exp(s\tau) = 0\,, \tag{3.14}$$

starts in the same manner as in the previous section but soon key differences appear. First, (3.14) does not admit zero root. Second, purely imaginary roots induce the system of curves in a form

$$a_m(\rho) = \pm\frac{\rho^\alpha \sin(\rho + \alpha\pi/2)}{\tau^\alpha \cos(\rho)}, \qquad b_m(\rho) = \frac{\rho^\alpha \cos(\alpha\pi/2)}{\tau^\alpha \cos(\rho)}\,, \tag{3.15}$$

$0 < \rho < \pi/2$ for $m = 0$ and $m\pi - \pi/2 < \rho < m\pi + \pi/2$ for $m \in \mathbb{Z}^+$ while $\rho + \alpha\pi/2 \neq m\pi$ for $m \in \mathbb{Z}_0^+$. Even though (3.15) looks formally similar to (3.7), the description of the stability boundary is now much more complicated as it is formed by parts of all curves (3.15) requiring calculations of infinitely many intersections. For the precise procedure we refer to [11] and state the end result:

Let us define two curves

$$\Gamma^+ = \bigcup_{m=0}^{\infty} \Gamma_{2m} \qquad \text{and} \qquad \Gamma^- = \bigcup_{m=0}^{\infty} \Gamma_{2m+1}$$

composed of the system of curves Γ_m in the (a, b)-plane given by (3.15) with the parameter restriction

$$\frac{(2m+1-\alpha)\pi}{2} - \theta^*_{m-2} < \rho < \frac{(2m+1-\alpha)\pi}{2} + \theta^*_m, \qquad m \in \mathbb{Z}_0^+$$

where $\theta^*_m \in (0, \alpha\pi/2)$ is the unique root of

$$-\frac{\sin(\theta + \alpha\pi/2)}{\sin(\theta - \alpha\pi/2)} = \left(\frac{(2m+3-\alpha)\pi}{\theta + (m + 1/2 - \alpha/2)\pi} - 1\right)^{\alpha}, \quad m \in \mathbb{Z}_0^+ \qquad (3.16)$$

and $\theta^*_{-2} = (\alpha - 1)\pi/2$, $\theta^*_{-1} = \pi/2$. Note that in the first relation of (3.15), both the sign cases have to be considered, and thus any curve Γ_m has two branches symmetric with respect to b-axis.

Using this notation we can write

Lemma 3.12. *Let $0 < \alpha < 1$, $\tau > 0$, $a \neq 0$ and b be real numbers. Then* (3.13) *is asymptotically stable if and only if the couple (a, b) is located inside the area bounded by Γ^+ from above and by Γ^- from below.*

Proof. The proof is using the connection between the growth of $|b|$ and number of characteristic roots with a positive real part, and analysis of intersections among Γ_m. Moreover, it employs appropriate asymptotic properties of the functions $\mathcal{R}^{\pm i a, b, \tau}_{\alpha,\alpha}$, $\mathcal{R}^{\pm i a, b, \tau}_{\alpha,1}$ which, as a by-product, also implies the algebraic decay rate of solutions. For details see [11]. □

Remark 3.13. (i) Lemma 3.12 describes parametrically the stability region $\mathcal{S}^{\tau}_{\alpha}$ of (3.13). Figure 3.3 depicts the known result for $\alpha = 1$ (see [47]) and Figures 3.4, 3.5 and 3.6 show the evolution of the stability region for decreasing derivative order. They illustrate that region of delay-independent stability occurs for any $\alpha < 1$.
(ii) If we consider lines connecting the origin and cusp points of Γ^+, we can prove that they have decreasing tangents with respect to increasing index of a given cusp point starting closest to b-axis (a similar comment is true also for Γ^-). This fact plays a central role in the context of stability switching.
(iii) Γ^+ has the tangent $\pm\cot(\alpha\pi/2)$ at the origin. Let us consider a line connecting

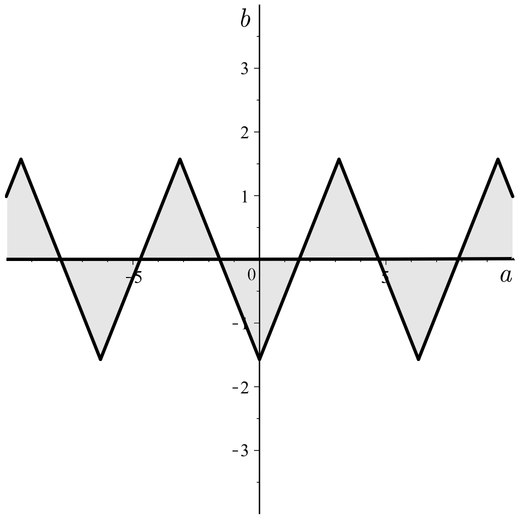

Figure 3.3: A classical result for the stability region $\mathcal{S}_1^\tau$ with $\tau = 1$.

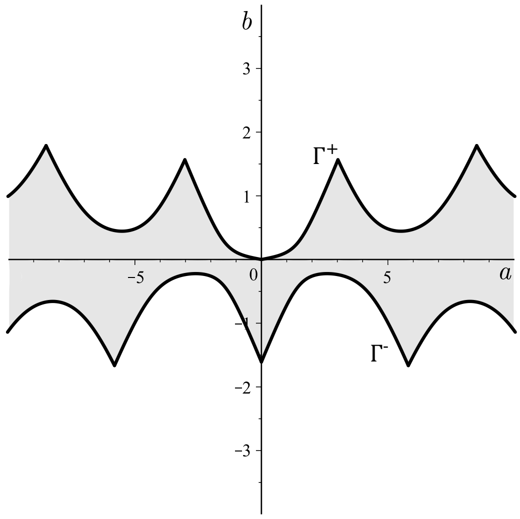

Figure 3.4: The stability region $\mathcal{S}_\alpha^\tau$ for $\alpha = 0.95$, $\tau = 1$.

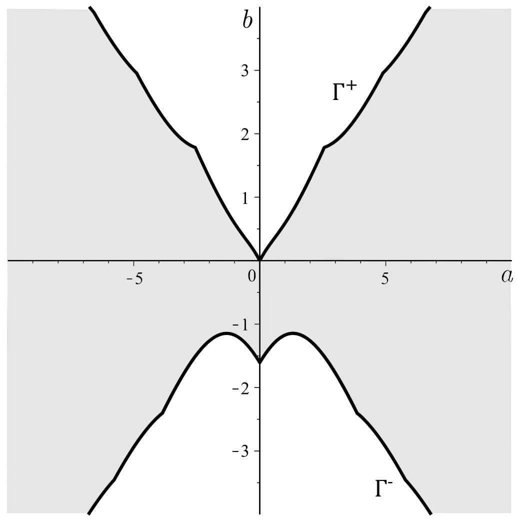

Figure 3.5: The stability region $\mathcal{S}_\alpha^\tau$ for $\alpha = 0.6$, $\tau = 1$.

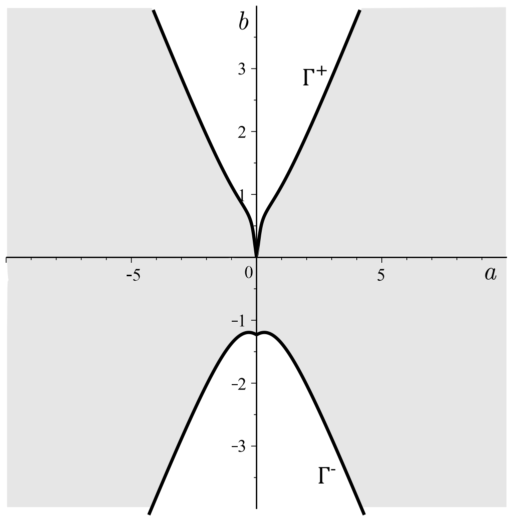

Figure 3.6: The stability region $\mathcal{S}_\alpha^\tau$ for $\alpha = 0.2$, $\tau = 1$.

origin with the first cusp point of Γ^+ (i.e. the endpoint of Γ_0). It can be calculated that the tangent of this line is equal to $\pm\cot(\alpha\pi/2)$ if and only if $\alpha = \alpha^*$ where

$$\alpha^* \approx 0.6150768144 \,. \tag{3.17}$$

If $\alpha > \alpha^*$, then for some values of b/a the first stability switch changes instability into stability. If $\alpha < \alpha^*$, the first stability switch is always from stability to instability. For the full calculation we refer to [11].

The shape of the stability boundary for (3.13) is quite complex and its proper reformulation into the explicit form of stability conditions brings many challenges.

It seems that the most useful perspective is provided by considering the ratio b/a as presented in the main result of this section:

Theorem 3.14. *Let $0 < \alpha < 1$, $\tau > 0$ and a, b be real numbers, let α^* be given by (3.17) and let $\theta_m^* \in (0, \alpha\pi/2)$ be the unique root of (3.16) for $m \in \mathbb{Z}_0^+$. Further, assuming $|b|/|a| \geq \cos(\alpha\pi/2)$, let $n \geq 0$ be an even integer (if $b \geq 0$), or an odd integer (if $b < 0$), uniquely determined by*

$$\frac{\cos(\alpha\pi/2)}{\cos(\theta_n^*)} \leq \frac{|b|}{|a|} < \frac{\cos(\alpha\pi/2)}{\cos(\theta_{n-2}^*)}, \quad n \geq 2, \qquad \text{or} \qquad \frac{\cos(\alpha\pi/2)}{\cos(\theta_n^*)} \leq \frac{|b|}{|a|}, \quad n \in \{0, 1\}. \tag{3.18}$$

The zero solution to (3.13) is asymptotically stable if and only if any of the following conditions holds:

$$\frac{|b|}{|a|} < \cos(\alpha\pi/2)\,; \tag{3.19}$$

$$\cos(\alpha\pi/2) \leq \frac{b}{|a|} < \cot(\alpha\pi/2) \quad \text{and} \quad \tau \in \bigcup_{j=-1}^{n/2-1} (\tau_{2j,-1}^*, \tau_{2j+2,1}^*)\,; \tag{3.20}$$

$$\alpha > \alpha^*, \quad \cot(\alpha\pi/2) \leq \frac{b}{|a|} < \frac{\cos(\alpha\pi/2)}{\cos(\theta_0^*)} \quad \text{and} \quad \tau \in \bigcup_{j=0}^{n/2-1} (\tau_{2j,-1}^*, \tau_{2j+2,1}^*)\,; \tag{3.21}$$

$$\frac{b}{|a|} \leq -\cos(\alpha\pi/2) \quad \text{and} \quad \tau \in \bigcup_{j=-1}^{(n-1)/2-1} (\tau_{2j+1,-1}^*, \tau_{2j+3,1}^*) \tag{3.22}$$

where $\tau_{i,\kappa}^* = 0$ for negative integers i and $\tau_{i,\kappa}^* = \tau_{i,\kappa}^*(a, b)$ where $i \in \mathbb{Z}_0^+$, $\kappa = \pm 1$ and

$$\tau_{i,\kappa}^*(a, b) = \frac{(i + (1 - \alpha)/2)\,\pi - \kappa \arccos\left(|a/b| \cos(\alpha\pi/2)\right)}{\left(\kappa\sqrt{b^2 - a^2\cos^2(\alpha\pi/2)} + |a| \sin(\alpha\pi/2)\right)^{1/\alpha}}.$$

Proof. The proof of this theorem requires many preliminary assertions such as guaranteeing existence, uniqueness and ordering of θ_m^* ($m \in \mathbb{Z}_0^+$), derivation of α^* and analysis of the ratio b/a for points belonging to Γ^+ and Γ^-. In particular, for a given ratio $|b/a|$, an analytical description of delays τ such that $(a, b) \in \Gamma^+ \cup \Gamma^-$ has to be given. For details see [11, pages 7-13]. □

Remark 3.15. (i) We note that the nonlinear inequalities (3.18) enable us to determine the exact number of stability switches. For (3.20) then there are $n+1$ switches (odd number), for (3.21) there are n switches (even number) and for (3.22) there are n switches (odd number).
(ii) The condition (3.19) describes the region of delay-independent stability symmetric with respect to both axes. Decreasing α expands this region towards the cone $b = |a|$ which is the limit case for $\alpha \to 0^+$.
(iii) For $b/|a| > 0$ we can see in Figures 3.8, 3.9 and 3.10 the role of the value $\alpha = \alpha^*$. It separates two qualitatively different patterns of stability switching, namely for $\alpha \leq \alpha^*$ it always starts with the first switch changing stability into instability.

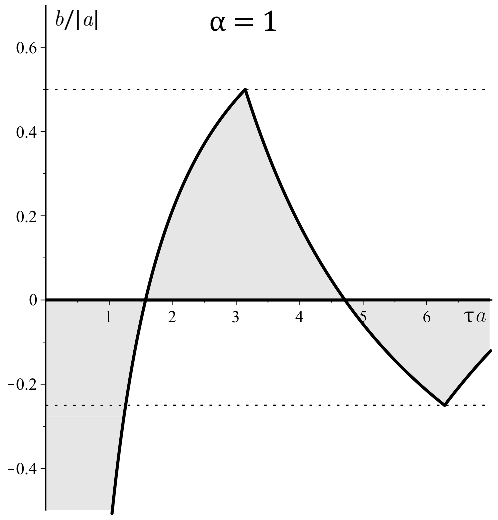

Figure 3.7: A classical result for the stability region $\mathcal{S}_1^\tau$ ($\alpha = 1$) in the $(\tau a, b/|a|)$-plane, see [46].

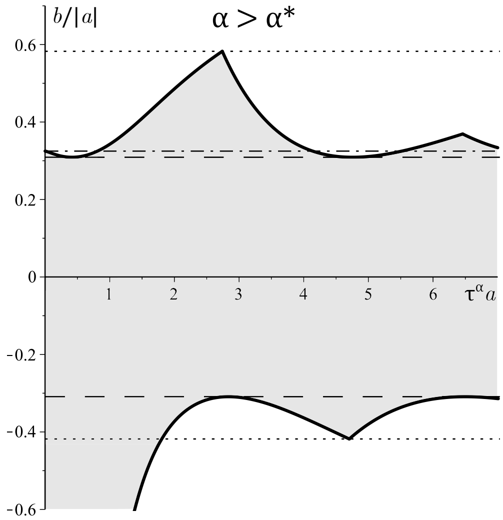

Figure 3.8: The stability region $\mathcal{S}_\alpha^\tau$ in the $(\tau^\alpha a, b/|a|)$-plane for $\alpha = 0.8$ (i.e. $\alpha > \alpha^*$).

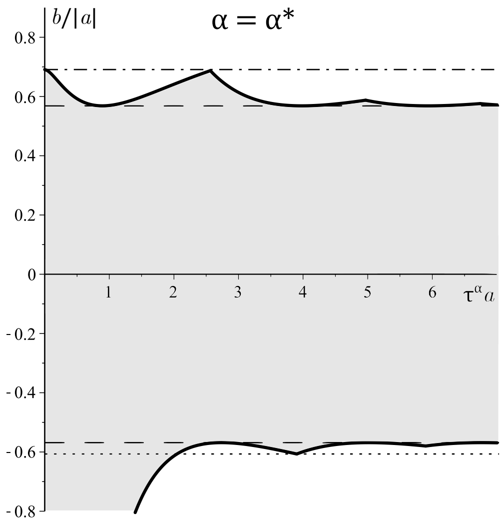

Figure 3.9: The stability region $\mathcal{S}_\alpha^\tau$ in the $(\tau^\alpha a, b/|a|)$-plane for $\alpha = \alpha^*$.

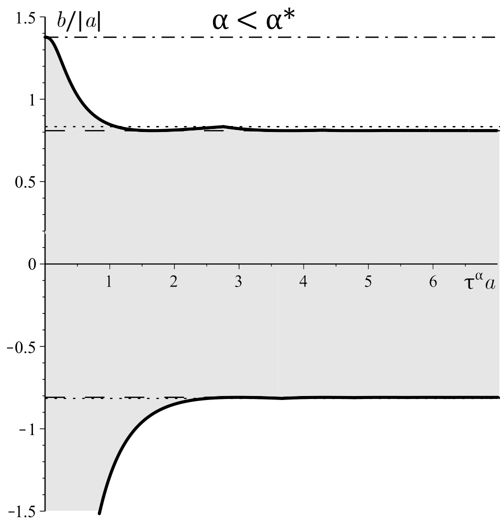

Figure 3.10: The stability region $\mathcal{S}_\alpha^\tau$ in $(\tau^\alpha a, b/|a|)$-plane for $\alpha = 0.4 < \alpha^*$.

3.4 Stability regions for orders from one to two

Let us consider (3.1) with $\alpha \in (1,2)$. The boundary locus formulas have the same form as for the case $\alpha \in (0,1)$, i.e. it is formed by the line $a + b = 0$ and the system of curves Γ_m in the (a,b)-plane given by (3.7). Properties of these curves for higher α significantly differ from the case $\alpha \in (0,1)$ and are actually qualitatively more similar to (3.15) discussed in the previous section.

In [12] it proved to be useful for lucidity, to think of this system of curves from two perspectives: their asymptotes and intersections.

Lemma 3.16. *Let* $\alpha \in (1,2)$, $\tau \in \mathbb{R}^+$ *and let* $\Gamma_m = \{(a,b) \in \mathbb{R}^2 : a = a_m(\rho),\ b = b_m(\rho),\ \rho \in (m\pi, m\pi+\pi)\}$ *(*$m = 0, 1\dots$*) be the curves defined by (3.7). Then it holds:*
(i) The line $a + b = 0$ *is tangent to the curve* Γ_0 *at the origin, and the line*

$$p_0^- : b = a - \left(\frac{\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)$$

is the asymptote to Γ_0 *as* $\rho \to \pi^-$*. Moreover,* $b_0(\rho) < a_0(\rho) - (\pi/\tau)^\alpha \cos(\alpha\pi/2)$, $b_0(\rho) < 0$ *and* $b_0(\rho) < -a_0(\rho)$ *for all* $\rho \in (0,\pi)$.
(ii) If m *is a positive odd integer, then* Γ_m *has asymptotes* p_m^+ *(as* $\rho \to m\pi^+$*) and* p_m^- *(as* $\rho \to (m+1)\pi^-$*) given by*

$$p_m^+ : b = a - \left(\frac{m\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \qquad \text{and} \qquad p_m^- : b = -a + \left(\frac{m\pi+\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Moreover, $b_m(\rho) > 0$, $b_m(\rho) > a_m(\rho) - (m\pi/\tau)^\alpha \cos(\alpha\pi/2)$ *and* $b_m(\rho) > -a_m(\rho) + ((m\pi+\pi)/\tau)^\alpha \cos(\alpha\pi/2)$ *for all* $\rho \in (m\pi, (m+1)\pi)$.
(iii) If m *is a positive even integer, then* Γ_m *has asymptotes* p_m^+ *(as* $\rho \to m\pi^+$*) and* p_m^- *(as* $\rho \to (m+1)\pi^-$*) given by*

$$p_m^+ : b = -a + \left(\frac{m\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \qquad \text{and} \qquad p_m^- : b = a - \left(\frac{m\pi+\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Moreover, $b_m(\rho) < 0$, $b_m(\rho) < -a_m(\rho) + (m\pi/\tau)^\alpha \cos(\alpha\pi/2)$ *and* $b_m(\rho) < a_m(\rho) - ((m\pi+\pi)/\tau)^\alpha \cos(\alpha\pi/2)$ *for all* $\rho \in (m\pi, (m+1)\pi)$.

Proof. The proof is of a technical nature utilizing limits calculations for (3.7). For details see [12]. □

Lemma 3.17. *Let* $\alpha \in (1,2)$, $\tau \in \mathbb{R}^+$ *and let et* $\Gamma_m = \{(a,b) \in \mathbb{R}^2 : a = a_m(\rho),\ b = b_m(\rho),\ \rho \in (m\pi, m\pi+\pi)\}$ *(*$m = 0, 1\dots$*) be the curves defined by (3.7). Further, let* $X_{m,n} = (a_{m,n}, b_{m,n})$ *be intersections of* Γ_m *and* Γ_n *(if they exist). Then it holds:*
(i) The intersection $(a_{m,n}, b_{m,n})$ *exists (and it is unique) if and only if* m, n *have the same parity.*
(ii) $a_{m,m+2k} < 0$ *for all* $k \in \mathbb{Z}$ *such that* $k > -m/2$.
(iii) $a_{m,m+2k} > a_{m,m+2(k+1)}$ *for all* $k \in \mathbb{Z}$ *such that* $k > -m/2$.
(iv) $a_{m,m+2k} > a_{m+2\ell,m+2k+2\ell}$ *for all* $k \in \mathbb{Z}$ *such that* $k > -m/2$ *and* $\ell = 1, 2\dots$.

Proof. The question of analysing intersections of Γ_m, Γ_n is transformed into root study of an equation involving transcendental expression similar to (3.16). For detail see [12]. □

Remark 3.18. (i) Lemma 3.16 says that each curve Γ_m ($m = 0, 1\dots$) is contained in an infinite trapezoid consisting of its asymptotes and the a-axis. Each pair Γ_m, Γ_{m+1} shares a common asymptote as depicted in Figure 3.11.

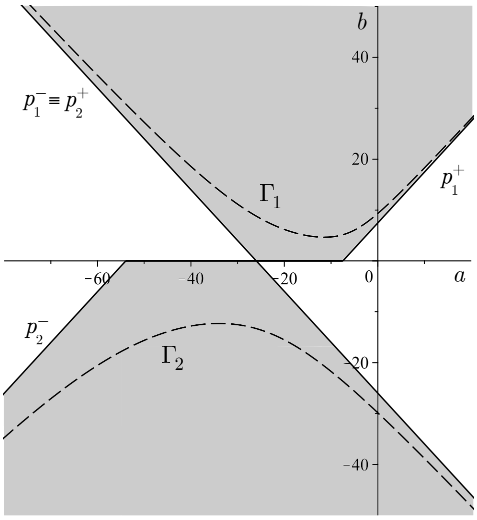

Figure 3.11: The common asymptote to Γ_1 and Γ_2 and the corresponding trapezoids for $\alpha = 1.8$ and $\tau = 1$.

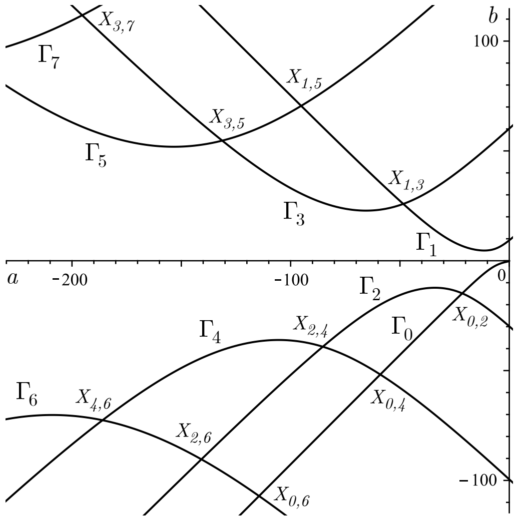

Figure 3.12: Some intersections $X_{m,n}$ for $\alpha = 1.8$, $\tau = 1$ and $m, n \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

(ii) Figure 3.12 demonstrates the locations and ordering of intersections $X_{m,n}$ described in Lemma 3.17.
(iii) A similar asymptotes and intersections analyses might be useful also in the case of (3.15), however it was not the point of study in [11].

In order to describe the stability region, we introduce the following notation. Let P be the line segment

$$a = -\rho, \quad b = \rho, \qquad \rho \in \left(0, \frac{(3\pi - \alpha\pi)^\alpha}{2\tau^\alpha |\cos(\alpha\pi/2)|}\right)$$

and let $\tilde{\Gamma}_m$ ($m = 0, 1, \dots$) be the parts of Γ_m with the endpoints $X_{m,m-2}$ and $X_{m,m+2}$ given by its intersections with the neighbouring curves Γ_{m-2}, Γ_{m+2} (see Figure 3.12), by origin for $m = 0$ and by the second endpoint of P for $m = 1$. Further, we put

$$\Gamma^{AS} = \bigcup_{m=0}^{\infty} \tilde{\Gamma}_m \cup P\,.$$

Using this notation, the geometric description of the stability region is provided by the following assertion (compare to Theorem 3.4 and Lemma 3.12).

Theorem 3.19. *Let $\alpha \in (1, 2)$, $\tau \in \mathbb{R}^+$ and $a, b \in \mathbb{R}$. Then (3.1) is asymptotically stable, if the couple (a, b) is located inside the area containing the negative part of a-axis and bounded by Γ^{AS}.*
Moreover, (3.1) is not stable, if (a, b) lies inside the area containing the positive part of a-axis and bounded by Γ^{AS}.

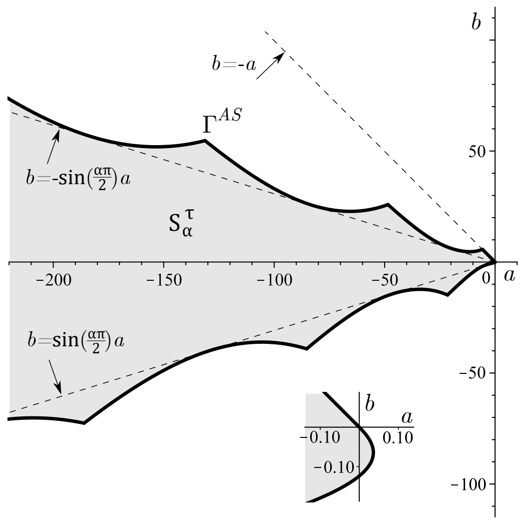

Figure 3.13: Stability boundary Γ^{AS} and stability region $\mathcal{S}_\alpha^\tau$ of (3.1) for $\alpha = 1.8$ and $\tau = 1$.

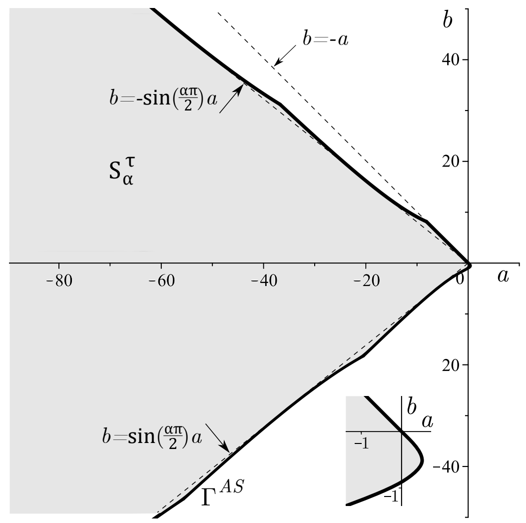

Figure 3.14: Stability boundary Γ^{AS} and stability region $\mathcal{S}_\alpha^\tau$ of (3.1) for $\alpha = 1.4$ and $\tau = 1$.

Proof. In order to complete this proof, the preconditions similar to the case of $\alpha \in (0, 1)$ have to validated and recalculated (see proofs of Theorem 3.4 and Lemma 2.7). For details we refer to [12] and highlight here only the most interesting partial results such as:

If $(a, b) \in \Gamma_m$ (for unique m) and $|b|$ increases, then a new characteristic root with a positive real part appears.

If $a < 0$ and $b \in \mathbb{R}$, then there exists $\delta = \delta(\alpha, a) > 0$ such that all characteristic roots have negative real parts whenever $|b| < \delta$.

The nonzero characteristic roots depend on a, b continuously.

The case $\alpha \in (1, 2)$ stands out mainly due to the need to consider the occurrence of multiple roots. We proved that a characteristic root has multiplicity greater than one if and only if either it is zero or there exists an integer k such that $\alpha\rho - \rho + \tau r \sin(\rho) = k\pi$ and $\tau r \sin(\alpha\rho) + \alpha \sin(\alpha\rho - \rho) = 0$. Moreover, any characteristic root has multiplicity at most three. □

Remark 3.20. (i) Although Theorem 3.19 gives only sufficient conditions for asymptotic stability, its second part makes them near-optimal. The only additional points where the stability might occur, lie on the stability boundary given by Γ^{AS}.
(ii) The stability region $\mathcal{S}_\alpha^\tau$ described in Theorem 3.19 is depicted in Figures 3.13 and 3.14 including a detail of the situation near the origin. Comparing these details to Figure 1.5 suggests the limit transition for $\alpha \to 1^+$, as the rightmost point of Γ_1 changes into the cusp point appearing for $\alpha \leq 1$.

Our main goal is to obtain the explicit stability conditions, not just a geometric description of the stability boundary as in Theorem 3.19. In the sequel, we provide an alternative stability criterion for the case $a < 0$ that better agrees with the form of the conditions of Theorems 1.7 and 1.9. We do not consider the case $a > 0$ because the corresponding stability conditions are quite straightforward (see Figures 3.13 and 3.14).

Theorem 3.21. *Let $\alpha \in (1,2)$, $\tau > 0$, $a < 0$ and b be real numbers and τ_ℓ^+, τ_ℓ^- be defined as*

$$\tau_\ell^\pm = \frac{(\ell + \frac{1\mp 1}{2})\pi + \frac{(2-\alpha)\pi}{2} \pm \arcsin\left(\left|\frac{a}{b}\right| \sin(\frac{\alpha\pi}{2})\right)}{\left(a\cos(\frac{\alpha\pi}{2}) \pm \sqrt{b^2 - a^2\sin^2(\frac{\alpha\pi}{2})}\right)^{1/\alpha}}, \qquad \ell \in \mathbb{Z}_0^+ .$$

(i) If $-\sin(\alpha\pi/2) < b/a < \sin(\alpha\pi/2)$, then (3.1) is asymptotically stable.
(ii) If $b/a > \sin(\alpha\pi/2)$, then there exists an integer $N_1 \geq 0$ such that (3.1) is asymptotically stable for any $\tau \in (\tau_{2k-2}^-, \tau_{2k}^+)$, and it is not stable for any $\tau \in (\tau_{2k}^+, \tau_{2k+2}^-)$ where $k = 0, \ldots, N_1$ (here we set $\tau_{-2}^- = 0$, $\tau_{2N_1+2}^- = \infty$).
(iii) If $-1 < b/a < -\sin(\alpha\pi/2)$, then there exists an integer $N_2 \geq 0$ such that (3.1) is asymptotically stable for any $\tau \in (\tau_{2k-1}^-, \tau_{2k+1}^+)$, and it is not stable for any $\tau \in (\tau_{2k+1}^+, \tau_{2k+3}^-)$ where $k = 0, \ldots, N_2$ (here we set $\tau_{-1}^- = 0$, $\tau_{2N_2+3}^- = \infty$).
(iv) If $b/a < -1$, then (3.1) is not stable.

Proof. See [12]. □

Remark 3.22. (i) Comparing to Theorems 3.21 and 3.14 we notice two obvious differences: In the real case (Theorem 3.21), there is no special value α^* changing the switching pattern, and the real case is not symmetric with respect to b-axis (see also Figures 3.13 and 3.14). Theorem 3.21 is formulated without the exact condition for calculation for the number of stability switches, however these conditions (in form

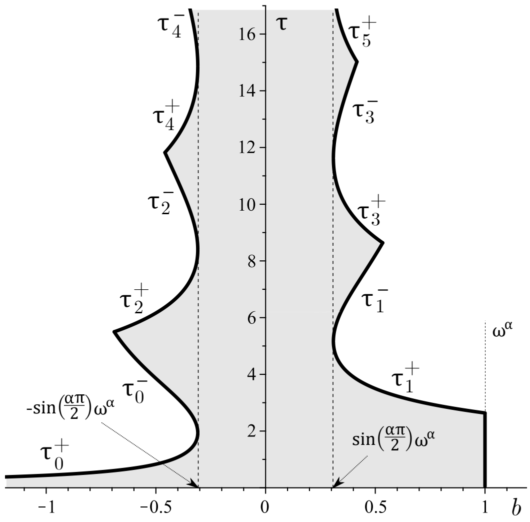

Figure 3.15: The stability region in (b, τ)-plane for $\alpha = 1.8$ and $a = -1$.

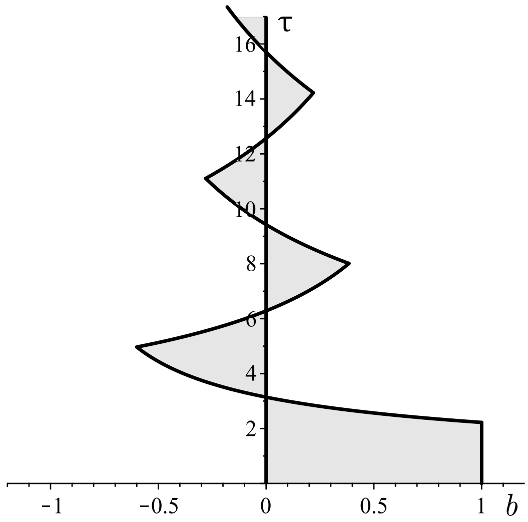

Figure 3.16: The stability region in (b, τ)-plane for $\alpha = 2$ and $a = -1$.

of a system of nonlinear inequalities) were given as a part of an example in [12].
(ii) Figure 3.15 depicts the stability region in the (b,τ)-plane showing clearly the delay-independent stability region, the stability switching property as well as the role of $\tau_j^{\pm}$ $(j \in \mathbb{Z}_0^+)$. Compare it to the Figures 3.8-3.10.

Let us consider $\alpha \to 2^-$ and compare it to the known results for (1.6). The asymptotes $p^+_{\alpha,m}$, $p^-_{\alpha,m}$ tend to the lines $b = -a - (m\pi)^2/\tau^2$ and $b = a + (m\pi)^2/\tau^2$, which are the lines forming the stability boundary of (1.6) (see Theorem 1.9). Taking $\alpha \to 2^-$ in Theorem 3.21, we obtain the limit for the stability region in the form: $0 < |b| < -a$ and

$$\frac{\ell\pi}{\sqrt{-a-|b|}} < \tau < \frac{(\ell+1)\pi}{\sqrt{-a+|b|}}$$

where ℓ is a nonnegative integer that is even for $b > 0$ and odd for $b < 0$ (see Figure 3.16). This form of conditions seems to be more effective compared to that of Theorem 1.9, especially with respect to explicit evaluations of stability switches for a varying delay parameter.

Chapter 4

Conclusions

In this thesis, we presented an in-depth exploration of the stability, asymptotic behaviour, and oscillatory properties of several linear fractional differential problems with a time delay. The focus was placed on optimal or near-optimal nature of achieved results and on their explicit form in terms of system parameters whenever possible. The introduced findings extend our understanding of nuances of fractional and classical dynamics and in many cases they pioneered the qualitative theory of fractional delay differential problems as outlined below.

We derived optimal stability conditions for the one-term FDDS of an arbitrary order (2.1), including a comprehensive analysis at the stability boundary (see Theorems 2.9 and 2.10). Rather unconventionally, we also dealt with detailed asymptotic description of unbounded solutions based on the location of system eigenvalues (see Theorems 2.13 and 2.14). While stability properties display a smooth transition across derivative orders, the asymptotic behaviour shows a striking contrast, as can be expected based on theory of undelayed fractional differential equations. In asymptotically stable cases, fractional derivatives lead to algebraic decay rates, as opposed to the exponential decay seen in classical systems. As a consequence, solutions to FDDS tend toward dominantly non-oscillatory behaviour which is a clear difference from their integer-order counterparts (see Theorem 2.12 and 2.16).

For the two-term FDDE (3.1) of orders less than two, we described the stability regions and grasped their evolution as the derivative order increases, passing through classical integer cases (see Theorems 3.4, 3.19). Moreover, we provided several insights into the emergence mechanism of stability switching phenomenon. Our particular focus was on providing stability criteria in practical form, i.e. in terms of entry coefficients, often in non-improvable versions (see Theorems 3.6, 3.14 and 3.21).

These theoretical results naturally transfer to practical applications, particularly to control theory. They outline effective design strategies for stabilization or destabilization of fractional systems via delayed feedback loops, as well as more nuanced prediction of large-time behaviour or such system (see [11,12]). In the future, we can expect emergence of other applications, e.g. in theory of complex systems where nonlocal and memory-based nature of fractional derivatives in combination with time lagging seems to be a promising direction.

Regarding future research, there are several promising directions extending stability and oscillatory analyses to more general cases. In particular, there are significant opportunities in areas where the author already has substantial experience in the undelayed context, such as discrete settings (see [7, 14, 34]), time-scale calculus (see [18, 33, 36]), nonlinear dynamics and variable coefficients (see [8, 35]), or problems including multiple fractional operators (see [9]).

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Appendices

Appendix A

Paper on lower-order one-term FDDS [6] (CNSNS, 2016)

Until 2016, I had already published nearly a dozen papers on fractional differential and difference equations. However, it was the paper [6] (co-authors: J. Čermák, J. Horníček; my author's share 45 %) that expanded my scope to include equations involving time delay. To this day, it remains my most cited work across all databases.

We focused on basics which were not sufficiently covered at the time. The stability and asymptotic properties of autonomous linear FDDS of order less than one. The main result was the formulation of necessary and sufficient conditions for stability via the location of system matrix eigenvalues in complex plane. We also derived algebraic decay rate of solutions tending to zero.

This paper laid the groundwork for techniques that we later employed for more advanced and technically challenging problems. Notably, we have re-established the fundamental solution for FDDS and introduced a generalized delay exponential function of Mittag-Leffler type. Most importantly, we adopted a technique utilizing the inverse Laplace transform and root analysis of the characteristic equation to derive asymptotic behaviour of solutions. This approach proved crucial for our subsequent research, as the presence of fractional derivatives disallows the direct use of the link between the real part of characteristic roots and argument of exponentials (since they do not belong among the solutions of fractional problems).

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ABSTRACT

The paper investigates stability and asymptotic properties of autonomous fractional differential systems with a time delay. As the main result, necessary and sufficient stability conditions are formulated via eigenvalues of the system matrix and their location in a specific area of the complex plane. These conditions represent a direct extension of Matignon's stability criterion for fractional differential systems with respect to the inclusion of a delay. For planar systems, our stability conditions can be expressed quite explicitly in terms of entry parameters. Applicability of these results is illustrated via stability investigations of the fractional delay Duffing's equation.

1. Introduction

Many problems described via ordinary differential equations require the inclusion of time delay terms. There are various types of these delays (technological, transport, incubation and others) appearing in many science areas. Delays are often involved in chemical processes (behaviours in chemical kinetics), technical processes (electric, pneumatic and hydraulic networks), biosciences (heredity in population dynamics), economics (dynamics of business cycles) and other branches. In general, the distinguishing feature of corresponding mathematical models is that the evolution rate of these processes depends on the past history. The differential equations modelling these problems are called delay differential equations (DDEs). Basic qualitative theory of these equations is well-established, especially in the linear case (for general references see [1] and [2]).

In the last decades, mathematical descriptions via fractional differential equations (FDEs) involving derivatives of non-integer orders turned out to be a very useful tool in the modelling of various phenomena of viscoelasticity, anomalous diffusion, control theory and other areas. Because of more degrees of freedom, fractional-order models are highly successful in situations where non-Gaussian and non-Markovian processes occur. A survey of interesting applications as well as basic qualitative properties of FDEs can be found, e.g. in the monographs [3–5].

Both these types of differential equations have a similar historical background. Although basic notions and properties related to these equations were discussed already by the founders of classical calculus, a systematic study of these equations was missing. It was only a rich application potential which gave rise to an enormous interest of mathematicians, engineers and other scientists in the qualitative and numerical investigations of DDEs and FDEs (starting in sixties of the last century).

The unification of DDEs and FDEs is provided by fractional delay differential equations (FDDEs), involving both the delay and non-integer derivative terms and disposing of great complexity. In technical applications, this approach is useful in creating strongly realistic models of certain processes and systems with memory and heredity. It is involved, among others, in analysis

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of various time delay systems whose stabilisation and control is realised via a state feedback. Following the recent trend, these controllers are often proposed using the fractional-order integro-differentiation method which gives rise to various types of FDDEs (e.g., for a fractional delay state space model of PD^{α} control of Newcastle robot we refer to [6]). This originally purely theoretical idea of fractional-order controllers was recently supported and justified by several papers studying experimentally collected impedance data of supercapacitors whose underlying electrochemical dynamics can be described and captured using of fractional-order models (see, e.g. [7] and the papers cited therein).

The stability issue is standardly of the main interest in the control theory and other areas where DDEs and FDEs play a significant role. The presence of time delay terms in feedback control systems results in characteristic equations involving transcendental terms of the exponential type. The corresponding stability polynomials (usually called quasi-polynomials) have infinitely many isolated zeros and analysis of their location (necessary for stability analysis of time invariant delay systems) is often a complicated matter. If these systems involve non-integer derivative terms, then the situation becomes even more difficult. As noted by several authors (see, e.g. [8]), the existing stability conditions for FDDEs do not provide effective algebraic criteria or algorithms for testing of stability of FDDEs and they are difficult to use in practice (see, e.g. [9]). From this viewpoint, the lack of such algebraic algorithms has hindered the advance of FDDEs for designs of control systems. Therefore, the main goal of this paper is to fill in this gap and present a simple and easily verifiable criterion for stability testing of linear time invariant fractional delay systems, including its rigorous mathematical justification (which seems to be extendable also to more general types of FDDEs).

The paper is organised as follows. In Section 2, we formulate our main result, namely effective stability criterion for a linear system of FDDEs. A brief survey of existing results on this topic is included as well. Section 3 shortly recapitulates a necessary background, mainly concerning fractional calculus and the Laplace transform. Section 4 discusses some properties of the studied system of FDDEs which are important in our investigations. In Section 5, we consider the characteristic equation for this system and analyse locations of its zeros in the complex plane. Using previous auxiliary assertions, Section 6 completes the proof of our main stability criterion. In Section 7, we reformulate this criterion for the corresponding planar system in terms of trace and determinant of the system matrix. Also, we apply the obtained results to stability investigations of the fractional delay Duffing's model of a nonlinear oscillator. Some remarks concerning future perspectives conclude the paper.

2. The main result

In the linear case, the fractional delay system

$$\mathrm{D}_0^{\alpha} y(t) = Ay(t-\tau), \quad t \in (0,\infty), \tag{2.1}$$

where D_0^{α} is the Caputo derivative of a real order $0 < \alpha < 1$, $A \in \mathbb{R}^{d\times d}$ is a constant real $d \times d$ matrix, $y : (-\tau,\infty) \to \mathbb{R}^d$ and $\tau > 0$ is a constant real lag, may serve as the basic prototype of FDDEs. The standard initial condition associated with (2.1) is

$$y(t) = \phi(t), \quad t \in [-\tau, 0] \tag{2.2}$$

where $\phi(t) \in L^1([-\tau,0])$, i.e. all components of $\phi(t)$ are absolutely Riemann integrable on $[-\tau, 0]$.

Contrary to DDEs and FDEs, a general qualitative theory of FDDEs is just at the beginning. During the past years, several pioneering works on this topic appeared, with an emphasis put on stability issues. We recall that the zero solution of (2.1) is said to be stable (asymptotically stable) if for any $\phi(t) \in L^1([-\tau,0])$ the solution $y(t)$ of (2.1), (2.2) is bounded (tends to zero as $t \to \infty$). Sufficient asymptotic stability conditions for (2.1) with multiple fractional orders and multiple delays have been formulated in [9] and [10] via the zeros location of associated characteristic equations. In the scalar case, asymptotic stability properties of

$$\mathrm{D}_0^{\alpha} y(t) = \lambda y(t-\tau), \quad t \in (0,\infty), \tag{2.3}$$

where λ is a real number, have been discussed in [11] by use of a transcendent inequality involving the fractional Lambert function. The bounded input–bounded output (BIBO) stability regions for (2.3), involving also the non-delayed term $y(t)$ on its right-hand side, have been investigated in [12] where the *D*-decomposition method was employed to describe the stability boundary of the studied equation in the form of parametric equations.

The first explicit stability criterion for (2.3) appeared in Theorem 5.1 of [13]. We recall here the relevant result in the following assertion (the symbol $\sim$ used here means an asymptotic equivalence in the sense that the ratio of both involved terms tends to a nonzero finite constant).

Theorem 1. *Let $\lambda \in \mathbb{R}$, $0 < \alpha < 1$ and $\tau > 0$.*
(i) The zero solution of (2.3) is asymptotically stable if and only if

$$-\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^{\alpha} < \lambda < 0. \tag{2.4}$$

Moreover, $y(t) \sim t^{-\alpha}$ as $t \to \infty$ for any solution $y(t)$ of (2.3).
(ii) The zero solution of (2.3) is stable if and only if

$$-\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^{\alpha} \le \lambda \le 0. \tag{2.5}$$

Note that for $\alpha = 1$, the condition (2.4) becomes $-\pi/2 < \lambda\tau < 0$ which is the classical asymptotic stability criterion for the first order delay equation $y'(t) = \lambda y(t-\tau)$.

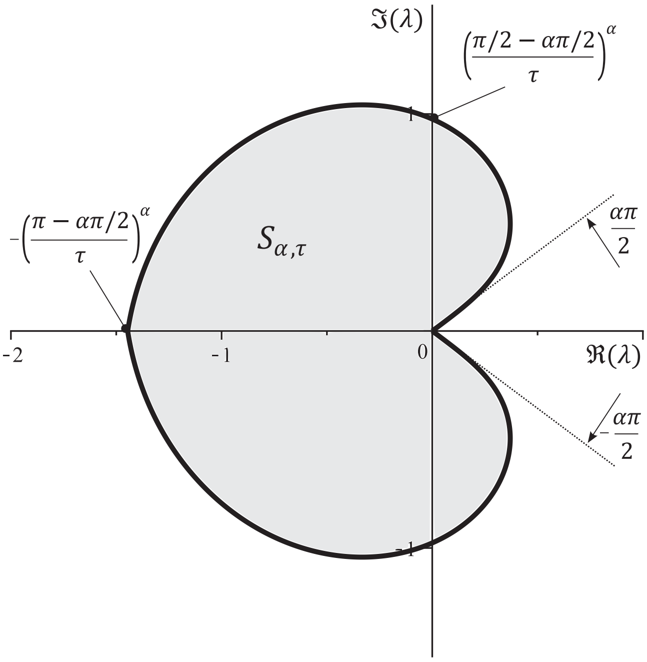

Fig. 1. The stability region $\mathcal{S}_{\alpha,\tau}$ for the values $\alpha = 0.4$ and $\tau = 1$ (including tangents of the stability boundary at the origin).

The main goal of this paper is to follow the previous works and, in particular, to provide a vector extension of Theorem 1. Doing this, we introduce the stability set

$$\mathcal{S}_{\alpha,\tau} = \left\{ \lambda \in \mathbb{C} : |\lambda| < \left(\frac{|\arg(\lambda)| - \alpha\pi/2}{\tau} \right)^{\alpha}, \; |\arg(\lambda)| > \frac{\alpha\pi}{2} \right\}$$

where we assume $-\pi < \arg(\cdot) \le \pi$. Throughout this paper, we utilise the usual notation for its closure $\mathrm{cl}(\mathcal{S}_{\alpha,\tau})$ and its boundary $\partial\mathcal{S}_{\alpha,\tau}$. Then we have the following generalisation of Theorem 1 (the symbol $\|\cdot\|$ means a norm in $\mathbb{R}^d$).

Theorem 2. *Let $A \in \mathbb{R}^{d\times d}$, $0 < \alpha < 1$ and $\tau > 0$.*
(i) The zero solution of (2.1) is asymptotically stable if and only if all eigenvalues λ of A are located inside $\mathcal{S}_{\alpha,\tau}$. Moreover, $\|y(t)\| \sim t^{-\alpha}$ as $t \to \infty$ for any solution $y(t)$ of (2.1).
(ii) The zero solution of (2.1) is stable if and only if all eigenvalues λ of A belong to $\mathrm{cl}(\mathcal{S}_{\alpha,\tau})$ and all eigenvalues lying on $\partial\mathcal{S}_{\alpha,\tau}$ have the same algebraic and geometric multiplicities.

Remark 1. (a) Theorem 2 immediately implies that if $A = \lambda$ is a real scalar, then the corresponding stability and asymptotic stability conditions become (2.5) and (2.4), respectively. In this connection, we consider two other important limit cases of (2.1), namely for $\alpha \to 1^-$ and $\tau \to 0^+$ when (2.1) becomes the first order linear DDE

$$y'(t) = Ay(t-\tau), \quad t \in (0,\infty) \tag{2.6}$$

and the α-order linear FDE

$$\mathrm{D}_0^{\alpha} y(t) = Ay(t), \quad t \in (0,\infty), \tag{2.7}$$

respectively. If we set $\alpha = 1$ in Theorem 2, then we get just stability conditions formulated in Theorem 3.4 of [14] for the delay system (2.6). Similarly, if $\tau \to 0^+$, then the conditions of Theorem 2 are reduced to the known Matignon's stability condition $|\arg(\lambda)| > \alpha\pi/2$ which is taken for the starting point in stability analysis of FDEs (see [15]).

(b) The stability regions $\mathcal{S}_{\alpha,\tau}$ for (2.1) and their dependance on parameters α, τ are depicted in Figs. 1–4. In particular, we can observe here that $\mathcal{S}_{\alpha,\tau}$ approaches the unit circle $|\lambda| < 1$ as $\alpha \to 0^+$ which is the well-known stability area for the discrete system $y_n = Ay_{n-\tau}$, $n = \tau, 2\tau, \dots$ (see Fig. 4). From this stability viewpoint, (2.1) provides a bridge between the delay differential system (2.6) and this discrete system.

3. Some preliminaries

Let $f(t)$ be a real function. Throughout this paper, we employ the standard definitions of the fractional integral

$$\mathrm{D}_a^{-\gamma} f(t) = \int_a^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)} f(\xi)\,\mathrm{d}\xi, \quad \gamma > 0, \; a \in \mathbb{R}, \; t \ge a$$

and the Caputo fractional derivative

$$\mathrm{D}_a^{\alpha} f(t) = \mathrm{D}_a^{-(1-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t} f(t) \right), \quad 0 < \alpha < 1, \; a \in \mathbb{R}, \; t \ge a$$

where we put $\mathrm{D}_a^0 f(t) = f(t)$ (for more information on fractional operators we refer, e.g. to [3,5]).

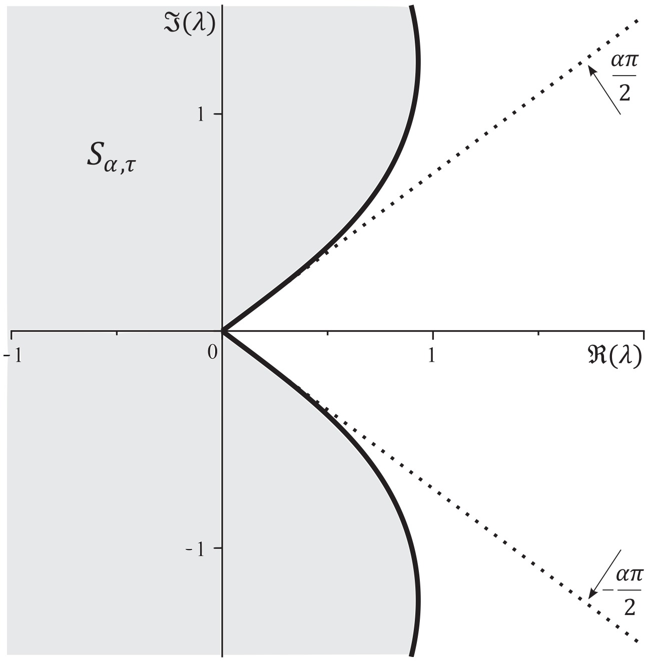

Fig. 2. The stability region $\mathcal{S}_{\alpha,\tau}$ for the values $\alpha = 0.4$ and $\tau = 0.1$ (the corresponding limit case as $\tau \to 0^+$ is dotted).

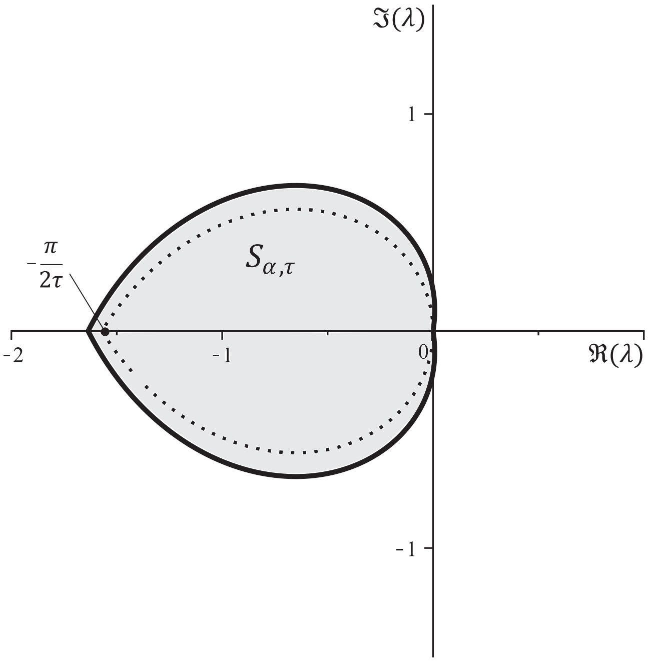

Fig. 3. The stability region $\mathcal{S}_{\alpha,\tau}$ for the values $\alpha = 0.9$ and $\tau = 1$ (the corresponding limit case as $\alpha \to 1^-$ is dotted).

In this connection, we note that the notion of FDDEs opens a space for discussions concerning the proper choice of lower limit of the fractional derivative. In particular, it is worth to consider the possibility of putting this limit $a = -\tau$. This issue is closely related to the problem of initialisation introduced and developed by Lorenzo and Hartley (see, e.g. [16]). Although a deeper analysis of this issue extends the scope of this paper, we note that such an adjustment can be viewed as adding a forcing term on the right-hand side of (2.1), which does not affect stability and asymptotic behaviour of the solution due to its decay rate $t^{-\alpha}$. In this paper, we adopt the standard approach and consider the lower limit of fractional operators to be zero, i.e. $a = 0$.

In our next considerations, we utilise the well-known Laplace transform of $f(t)$ introduced as

$$\mathcal{L}(f(t))(s) = \int_0^\infty \exp\{-st\} f(t)\mathrm{d}t, \quad s \in \mathbb{C}$$

provided the integral converges. For the sake of lucidity, we recall some relevant basic formulae, namely

$$\mathcal{L}(f(t-\tau)h(t-\tau))(s) = \exp\{-s\tau\}\mathcal{L}(f(t))(s), \quad \tau > 0, \tag{3.1}$$

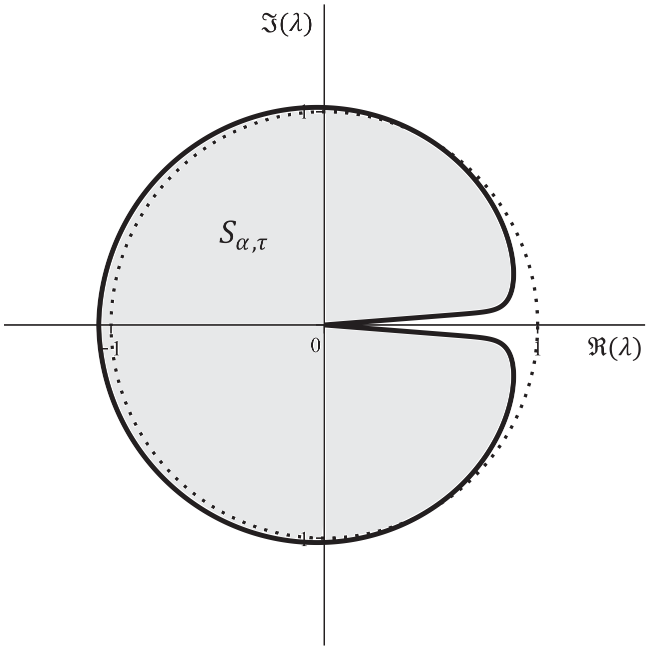

Fig. 4. The stability region $\mathcal{S}_{\alpha,\tau}$ for the values $\alpha = 0.05$ and $\tau = 1$ (the corresponding limit case as $\alpha \to 0^+$ is dotted).

$$\mathcal{L}(f(t-\tau))(s) = \exp\{-s\tau\}\mathcal{L}(f(t))(s) + \exp\{-s\tau\}\int_{-\tau}^{0}\exp\{-su\}f(u)\mathrm{d}u, \quad \tau > 0 \tag{3.2}$$

where $h(\cdot)$ is the Heaviside step function defined as $h(\xi) = 1$ for $\xi \geq 0$ and $h(\xi) = 0$ for $\xi < 0$. Furthermore, we also recall the well-known convolution formula for real functions $f(t)$, $g(t)$

$$\mathcal{L}\left(\int_0^t f(t-\xi)g(\xi)\mathrm{d}\xi\right)(s) = \mathcal{L}(f(t))(s)\cdot\mathcal{L}(g(t))(s), \tag{3.3}$$

which is useful, among others, in procedures related to the Laplace inverse $\mathcal{L}^{-1}$.

In the frame of fractional calculus, a key relationship is provided by the Laplace transform of a power function

$$\mathcal{L}\left(\frac{t^\eta}{\Gamma(\eta+1)}\right)(s) = s^{-\eta-1}, \quad \eta > -1. \tag{3.4}$$

This along with (3.3) leads to the Laplace transforms of fractional operators in the form

$$\mathcal{L}(\mathrm{D}_0^{-\gamma}f(t))(s) = s^{-\gamma}\mathcal{L}(f(t))(s), \quad \gamma > 0, \tag{3.5}$$

$$\mathcal{L}(\mathrm{D}_0^{\alpha}f(t))(s) = s^{\alpha}\mathcal{L}(f(t))(s) - s^{\alpha-1}f(0), \quad 0 < \alpha < 1 \tag{3.6}$$

(see, e.g. [5]). Note also that the Laplace transform of vector and matrix functions is considered componentwise.

In stability analysis of (2.1), it is convenient to employ the Jordan canonical form of the system matrix A. On this account, we recall that every $d \times d$ matrix A is similar to a $d \times d$ matrix Λ with the Jordan blocks on its diagonal, i.e.

$$A = T\Lambda T^{-1}, \quad \Lambda = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_q \end{pmatrix}, \quad J_k = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & & \lambda_i & 1 \\ 0 & \cdots & & 0 & \lambda_i \end{pmatrix}, \quad k = 1, \dots, q \tag{3.7}$$

where T is an invertible matrix and λ_i $(i = 1, \dots, n)$ are distinct eigenvalues of A. The number of Jordan blocks corresponding to λ_i is called the geometric multiplicity of λ_i. The sum of the sizes of all Jordan blocks corresponding to λ_i is called the algebraic multiplicity of λ_i.

Besides the symbol $\sim$ for the asymptotic equivalence of given functions (recalled in the introductory part), we employ also another asymptotic notation

$$f \sim_{sup} g \quad \text{as } t \to \infty \quad \Longleftrightarrow \quad \limsup_{t\to\infty}\frac{f(t)}{g(t)} = K \neq 0$$

which enables to describe asymptotics of a wider class of functions (in particular, unbounded oscillatory functions).

4. Solution representation for (2.1) and its properties

In this section, we discuss a suitable representation of the solution of (2.1), (2.2). Similarly to the integer-order case (see, e.g. [1,2]), the essential role in this matter is played by the so-called fundamental matrix solution.

Definition 1. Let $A \in \mathbb{R}^{d\times d}$ and let I be the identity $d \times d$ matrix. The matrix function $R : \mathbb{R} \to \mathbb{C}^{d\times d}$ given by

$$R(t) = \mathcal{L}^{-1}\Big((s^\alpha I - A\exp\{-s\tau\})^{-1}\Big)(t)$$

is called the fundamental matrix solution of (2.1).

Remark 2. In the integer-order case, the fundamental matrix solution is usually considered to be a matrix function whose ith column solves the correspoding delay system supplied with the initial condition $\phi(t) = 0$ for $t \in [-\tau, 0)$ and $\phi(0) = e_i$, where e_i is the standard basis vector in $\mathbb{R}^d$. Such an introduction of the fundamental matrix solution is possible also in the fractional-order case (see, e.g. [13]). In our investigations, it does not seem to be convenient especially due to some difficulties connected with the variation of constants formula. On this account, we prefer Definition 1 which enables us to simplify the main parts of the proofs. We stress that Definition 1 for $\alpha \to 1^-$ coincides with the standard integer-order definition of the fundamental matrix solution.

The notion of fundamental matrix solution leads to the following solution representation.

Theorem 3. *The solution $y(t)$ of (2.1), (2.2) is given by*

$$y(t) = \mathrm{D}_0^{-(1-\alpha)}R(t)\phi(0) + \int_{-\tau}^{0} R(t-\tau-u)h(t-\tau-u)A\phi(u)\mathrm{d}u\,, \quad t > 0\,, \tag{4.1}$$

where $h(\xi)$ is the Heaviside step function.

Proof. Applying (3.2) and (3.6), we easily obtain the Laplace image of the solution of (2.1), (2.2) in the form

$$\begin{aligned}\mathcal{L}(y(t))(s) &= (s^\alpha I - A\exp\{-s\tau\})^{-1}\left[s^{\alpha-1}\phi(0) + \exp\{-s\tau\}\int_{-\tau}^{0}\exp\{-su\}A\phi(u)\mathrm{d}u\right]\\ &= s^{\alpha-1}\mathcal{L}(R(t))(s)\phi(0) + \int_{-\tau}^{0}\exp\{-s(\tau+u)\}\mathcal{L}(R(t))(s)A\phi(u)\mathrm{d}u.\end{aligned}$$

Then, using (3.5) for the first term and employing linearity of the Laplace transform and (3.1) with respect to the expression $\exp\{-s(\tau+u)\}\mathcal{L}(R(t))(s)$ for the second term, we get the assertion. □

Remark 3. We point out that (4.1) is equivalent to the expressions obtained in [12] and [13] using a different definition of the fundamental matrix solution.

It is well-known that the solution of (2.7) can be expressed via functions of Mittag–Leffler type (see, e.g. [5]). Following this, we introduce

Definition 2. Let $\lambda \in \mathbb{C}$, $\eta, \beta, \tau \in \mathbb{R}$ and $m \in \mathbb{Z}$ be such that $\eta, \beta, \tau > 0$, $m \ge 0$. The generalised delay exponential function (of Mittag–Leffler type) is given by

$$G_{\eta,\beta}^{\lambda,\tau,m}(t) = \sum_{j=0}^{\infty}\binom{m+j}{j}\frac{\lambda^j(t-(m+j)\tau)^{\eta(m+j)+\beta-1}}{\Gamma(\eta(m+j)+\beta)}h(t-(m+j)\tau), \quad t > 0 \tag{4.2}$$

where $h(\xi)$ is the Heaviside step function.

Remark 4. We note that the sum (4.2) is infinite only formally. In fact, for every fixed $t > 0$, all terms satisfying $m + j > t/\tau$ are equal to zero due to the occurrence of the Heaviside step function.

Lemma 1. *Let $\lambda \in \mathbb{C}$, $\eta, \beta, \tau \in \mathbb{R}$ and $m \in \mathbb{Z}$ be such that $\eta, \beta, \tau > 0$, $m \ge 0$. Then it holds*

$$\mathcal{L}(G_{\eta,\beta}^{\lambda,\tau,m}(t))(s) = \tfrac{s^{\eta-\beta}\exp\{-ms\tau\}}{(s^\eta-\lambda\exp\{-s\tau\})^{m+1}}.$$

Proof. Using linearity of the Laplace transform and the formulae (3.1) and (3.4), we get

$$\mathcal{L}(G_{\eta,\beta}^{\lambda,\tau,m}(t))(s) = \sum_{j=0}^{\infty}\lambda^j\binom{m+j}{j}\exp\{-(m+j)s\tau\}\frac{1}{s^{\eta(m+j)+\beta}} = \exp\{-m\tau s\}s^{-m\eta-\beta}\sum_{j=0}^{\infty}\binom{m+j}{j}\left(\frac{\lambda}{s^\eta\exp\{s\tau\}}\right)^j.$$

Then we can employ the property $\binom{m+j}{j} = (-1)^j\binom{-m-1}{j}$, which is valid for all nonnegative integers m, j, to obtain

$$\mathcal{L}(G_{\eta,\beta}^{\lambda,\tau,m}(t))(s) = \exp\{-m\tau s\}s^{-m\eta-\beta}\sum_{j=0}^{\infty}(-1)^j\binom{-m-1}{j}\left(\frac{\lambda}{s^\eta\exp\{s\tau\}}\right)^j.$$

Furthermore, application of the binomial formula yields

$$\mathcal{L}(G_{\eta,\beta}^{\lambda,\tau,m}(t))(s) = \frac{\exp\{-m\tau s\}s^{-m\eta-\beta}}{\left(1-\frac{\lambda}{s^\eta \exp\{s\tau\}}\right)^{m+1}} = \frac{s^{\eta-\beta}\exp\{-ms\tau\}}{(s^\eta - \lambda\exp\{-s\tau\})^{m+1}},$$

which concludes the proof. □

Now, we are in a position to formulate the main result of this section.

Theorem 4. *Let $R(t)$ be the fundamental matrix solution of (2.1). Furthermore, let λ_i $(i = 1, \dots, n)$ be distinct eigenvalues of A and let p_i be the largest dimension of the Jordan block corresponding to the eigenvalue λ_i. Then the nonzero elements of $R(t)$ are given by linear combinations of the generalised delay exponential functions*

$$G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t), \quad m = 0, \dots, p_i - 1, \quad i = 1, \dots, n.$$

Proof. Using (3.7) we can rewrite the fundamental matrix solution of (2.1) as

$$R(t) = \mathcal{L}^{-1}\left((s^\alpha I - A\exp\{-s\tau\})^{-1}\right)(t) = T^{-1}\mathcal{L}^{-1}\left((s^\alpha I - \Lambda\exp\{-s\tau\})^{-1}\right)(t)T \tag{4.3}$$

where T is a constant invertible matrix and Λ is the Jordan canonical form of A. Thus, $R(t)$ is formed by linear combinations of elements of the fundamental matrix solution of (2.1) with A replaced by Λ. Since Λ is the block diagonal matrix, $(s^\alpha I - \Lambda\exp\{-s\tau\})^{-1}$ is block diagonal as well. Moreover, its blocks are upper triangular strip matrices of the type

$$(s^\alpha I - J_k e^{-s\tau})^{-1} = \begin{pmatrix} (s^\alpha - \lambda_i e^{-s\tau})^{-1} & e^{-s\tau}(s^\alpha - \lambda_i e^{-s\tau})^{-2} & \cdots & e^{-(r_k-1)s\tau}(s^\alpha - \lambda_i e^{-s\tau})^{-r_k} \\ 0 & (s^\alpha - \lambda_i e^{-s\tau})^{-1} & \ddots & e^{-(r_k-2)s\tau}(s^\alpha - \lambda_i e^{-s\tau})^{-(r_k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (s^\alpha - \lambda_i e^{-s\tau})^{-1} \end{pmatrix} \tag{4.4}$$

where J_k $(k = 1, \dots, q)$ is the kth block of Λ and r_k is its dimension. Lemma 1 implies that all involved elements are Laplace transforms of the functions stated in the assertion. □

5. The characteristic equation and its analysis

It is known from theories of DDEs and FDEs that positions of singular points of the Laplace image of the solutions indicate stability properties (and in some cases also a decay rate of the solutions) of the original equation. In Section 6, we show that a similar connection occurs in the case of FDDEs. As a preliminary step, this section is devoted to the pole analysis of the Laplace image of the fundamental matrix solution of (2.1). In other words, we discuss locations of zeros of

$$\det(s^\alpha I - A\exp\{-s\tau\}) = 0 \tag{5.1}$$

which is called the characteristic equation of (2.1). We can see from (4.3) and (4.4) that, instead of (5.1), it is sufficient to investigate the zeros of the equation

$$p(s) \equiv s^\alpha - \lambda\exp\{-s\tau\} = 0 \tag{5.2}$$

where λ is a complex parameter.

The standard way to prove stability criteria for linear autonomous DDEs is based on the *D*-decomposition method applied to the corresponding characteristic quasi-polynomials. Although this procedure is applicable also in the fractional-order case, we prefer an alternative method based on a direct zero analysis of (5.2).

The basic zero properties of (5.2) can be collected in the following

Proposition 1.

(i) The complex number s is a zero of (5.2) if and only if its complex conjugate s^* is a zero of

$$s^\alpha - \lambda^*\exp\{-s\tau\} = 0$$

where λ^* is complex conjugate to λ.

(ii) There exists $\delta > 0$ such that (5.2) has only a finite number of zeros s with $|\arg(s)| < \pi/2 + \delta$.

(iii) All zeros of (5.2) have negative real parts if and only if $\lambda \in \mathcal{S}_{\alpha,\tau}$.

(iv) If s_1, s_2 are zeros of (5.2) satisfying $\Re(s_1) = \Re(s_2)$, then $|s_1| = |s_2|$.

(v) Let s be a zero of (5.2), i.e. $p(s) = 0$. Then $p'(s) \neq 0$ for all s with $\Im(s) \neq 0$ or $\Re(s) > 0$.

Proof. We assume $\lambda \neq 0$ (the opposite case is trivial). Let $s = r\exp\{i\varphi\}$, $\lambda = \varrho\exp\{i\psi\}$ for $r = |s|$, $\varrho = |\lambda|$ and suitable $-\pi < \varphi \le \pi$, $-\pi < \psi \le \pi$. Then (5.2) takes the form

$$r^\alpha\exp\{i\alpha\varphi\} - \varrho\exp\{i\psi\}\exp\{-r\tau\exp\{i\varphi\}\} = 0.$$

We equate the real and imaginary parts to obtain

$$r^\alpha \cos(\alpha\varphi) - \varrho \exp\{-r\tau\cos(\varphi)\}\cos(\psi - r\tau\sin(\varphi)) = 0, \tag{5.3}$$

$$r^\alpha \sin(\alpha\varphi) - \varrho \exp\{-r\tau\cos(\varphi)\}\sin(\psi - r\tau\sin(\varphi)) = 0. \tag{5.4}$$

If $\varphi \neq 0$, then $0 < \alpha|\varphi| < \pi$, hence $\psi - r\tau\sin(\varphi) \neq k\pi$ for any $k \in \mathbb{Z}$. Dividing (5.3) over (5.4) we obtain

$$\operatorname{cotan}(\alpha\varphi) = \operatorname{cotan}(\psi - r\tau\sin(\varphi)), \quad \text{i.e.} \quad \alpha\varphi = \psi - r\tau\sin(\varphi) + k\pi, \quad k \in \mathbb{Z}. \tag{5.5}$$

If we substitute (5.5) into (5.4), we get

$$r^\alpha = (-1)^k \varrho \exp\{-r\tau\cos(\varphi)\}, \quad k \in \mathbb{Z}$$

where k must be even due to positivity of r and ϱ.

If $\varphi = 0$, then $\psi = 0$ due to (5.4) and (5.3), i.e. $\lambda = \varrho$ and (5.2) is reduced to

$$r^\alpha = \varrho \exp\{-r\tau\}.$$

Consequently, summarising this part, s is a zero of (5.2) if and only if its modulus r and argument φ solve the system

$$\alpha\varphi = \psi - r\tau\sin(\varphi) + 2k\pi, \tag{5.6}$$

$$r^\alpha = \varrho \exp\{-r\tau\cos(\varphi)\} \tag{5.7}$$

for a suitable $k \in \mathbb{Z}$. This reformulation turns out to be useful in our analysis of (i)–(iv). While the properties (i) and (iv) follow from it directly, the proof of (ii) and (iii) requires some additional steps.

If $0 < |\varphi| < \pi$, then (5.6) yields

$$r = \frac{\psi - \alpha\varphi + 2k\pi}{\tau\sin(\varphi)} > 0 \tag{5.8}$$

and, substituting (5.8) into (5.7), we get

$$\left(\frac{\psi - \alpha\varphi + 2k\pi}{\tau\sin(\varphi)}\right)^\alpha = \varrho \exp\{(-\psi + \alpha\varphi - 2k\pi)\operatorname{cotan}(\varphi)\}.$$

To analyse this equality, we set

$$g_k(\varphi) = \left(\frac{\psi - \alpha\varphi + 2k\pi}{\tau\sin(\varphi)}\right)^\alpha, \quad h_k(\varphi) = \varrho \exp\{(-\psi + \alpha\varphi - 2k\pi)\operatorname{cotan}(\varphi)\}$$

and discuss intersections of $g_k(\varphi)$ and $h_k(\varphi)$ in $(0, \pi/2]$. Obviously, the function $g_k(\varphi)$ is decreasing in $(0, \pi/2]$ with

$$\lim_{\varphi\to 0^+} g_k(\varphi) = \infty \quad \text{and} \quad g_k(\pi/2) = \left(\frac{\psi - \alpha\pi/2 + 2k\pi}{\tau}\right)^\alpha.$$

Similarly, $h_k(\varphi)$ is increasing in $(0, \pi/2]$. Indeed, using (5.8) we get

$$\tfrac{d}{d\varphi}[(-\psi + \alpha\varphi - 2k\pi)\operatorname{cotan}(\varphi)] = \alpha\operatorname{cotan}(\varphi) + \tfrac{\psi - \alpha\varphi + 2k\pi}{\sin^2(\varphi)} = \alpha\operatorname{cotan}(\varphi) + \tfrac{r\tau}{\sin(\varphi)} > 0$$

in $(0, \pi/2]$. Moreover,

$$\lim_{\varphi\to 0^+} h_k(\varphi) = 0 \quad \text{and} \quad h_k(\pi/2) = \varrho.$$

Consequently, if m_1 is the smallest non-negative integer such that

$$\varrho < \left(\frac{\psi - \alpha\pi/2 + 2m_1\pi}{\tau}\right)^\alpha,$$

then, because of the continuity of $g_k(\varphi)$ and $h_k(\varphi)$ in $(0, \pi)$, the system (5.6) and (5.7) has just m_1 solutions (r, φ) with $\varphi \in (0, \pi/2 + \delta)$, $\delta > 0$ being sufficiently small. Equivalently, (5.2) has m_1 zeros s with $0 < \arg(s) < \pi/2 + \delta$. Analogously, (5.2) has m_2 zeros s with $-(\pi/2 + \delta) < \arg(s) < 0$ where m_2 is the smallest non-negative integer such that

$$\varrho < \left(\frac{-\psi - \alpha\pi/2 + 2m_2\pi}{\tau}\right)^\alpha.$$

Furthermore, as noted above, (5.2) has a (unique) real zero if and only if λ is a positive real, i.e. $\lambda = \varrho$. This proves (ii).

To complete the proof of (iii), it is enough to repeat the previous argumentation. In particular, the functions $g_k(\varphi)$ and $h_k(\varphi)$ have no intersections in $(0, \pi/2]$ for any $k = 0, 1, \ldots$ if and only if $g_k(\pi/2) > h_k(\pi/2)$, i.e.

$$\varrho < \left(\frac{\psi - \alpha\pi/2 + 2k\pi}{\tau}\right)^\alpha$$

for any $k = 0, 1, \ldots$. Taking into account positivity of r in (5.8), this inequality holds if and only if

$$\varrho < \left(\frac{\psi - \alpha\pi/2}{\tau}\right)^{\alpha} \quad \text{and} \quad \psi > \frac{\alpha\pi}{2}.$$

Using $\varrho = |\lambda|$, $\psi = \arg(\lambda)$ and the property (i), these conditions imply $\lambda \in S_{\alpha,\tau}$.

Finally, the proof of (v) follows from a direct calculation. Indeed, the relations $p(s) = p'(s) = 0$ imply

$$s^{\alpha} - \lambda \exp\{-s\tau\} = \alpha s^{\alpha-1} + \lambda\tau \exp\{-s\tau\} = 0$$

which can be considered as the system of two equations with unknown s and λ. Its simple analysis yields (v). □

Remark 5. The property (i) implies, among others, the symmetry of the area $\mathcal{S}_{\alpha,\tau}$ with respect to the real axis. More precisely, the asymptotic stability boundary, when two zeros of (5.2) are purely imaginary and all the remaining zeros have negative real parts, is the curve symmetric with respect to the line $\Im(\lambda) = 0$ and given via the parametric equations in the complex λ-plane

$$\Re(\lambda) = |\omega|^{\alpha} \cos\left(|\omega|\tau + \frac{\alpha\pi}{2}\right), \quad \Im(\lambda) = \operatorname{sgn}(\omega)|\omega|^{\alpha} \sin\left(|\omega|\tau + \frac{\alpha\pi}{2}\right), \quad -\frac{\pi - \alpha\pi/2}{\tau} \le \omega \le \frac{\pi - \alpha\pi/2}{\tau}.$$

6. Proof of Theorem 2

We start this section with the following auxiliary, but essential result describing the asymptotic behaviour of the generalised delay exponential functions.

Lemma 2. *Let $\lambda \in \mathbb{C}$, $\alpha, \beta, \tau \in \mathbb{R}$ and $m \in \mathbb{Z}$ be such that $0 < \alpha < 1$, $\beta, \tau > 0$, $m \ge 0$. Furthermore, let s_i $(i = 1, 2, \ldots)$ be the zeros of (5.2) with ordering $\Re(s_i) \ge \Re(s_{i+1})$ (in particular, s_1 is the zero with the largest real part).*

(i) If $\lambda = 0$, then

$$G_{\alpha,\beta}^{0,\tau,m}(t) = \frac{(t - m\tau)^{m\alpha+\beta-1}}{\Gamma(m\alpha + \beta)} h(t - m\tau).$$

(ii) If $\lambda \in \mathcal{S}_{\alpha,\tau}$, then

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta - \alpha)}(t + \tau)^{\beta-\alpha-1} + \frac{(-1)^{m+1}(m+1)}{\lambda^{m+2}\Gamma(\beta - 2\alpha)}(t + 2\tau)^{\beta-2\alpha-1} + \mathcal{O}(t^{\beta-3\alpha-1}) \quad \text{as } t \to \infty.$$

(iii) If $\lambda \notin \mathcal{S}_{\alpha,\tau} \cup \{0\}$, then

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \sum_{j=0}^{m} (t - m\tau)^{j}\left(a_j \exp\{s_1(t - m\tau)\} + b_j \exp\{s_2(t - m\tau)\}\right) + \begin{cases} \mathcal{O}(t^{m} \exp\{\Re(s_3)t\}), & \text{if } \Re(s_3) \ge 0, \\ \mathcal{O}(t^{\beta-\alpha-1}), & \text{if } \Re(s_3) < 0 \end{cases} \quad \text{as } t \to \infty$$

where a_j, b_j are suitable nonzero complex constants $(j = 0, \ldots, m)$.

Proof. If $\lambda = 0$, then Definition 2 implies directly the assertion (i). Let $\lambda \ne 0$. By Lemma 1, it holds

$$\mathcal{L}(G_{\alpha,\beta}^{\lambda,\tau,m}(t))(s) = \frac{s^{\alpha-\beta} \exp\{-ms\tau\}}{(s^{\alpha} - \lambda \exp\{-s\tau\})^{m+1}}.$$

We can see that $\mathcal{L}(G_{\alpha,\beta}^{\lambda,\tau,m}(t))(s)$ has poles s_i $(i = 1, 2, \ldots)$ corresponding to zeros of (5.2) and, if $\beta > \alpha$, then there is also a pole at $s_0 = 0$. Furthermore, the occurrence of the power function implies that the negative real axis consists of singular points only (zero is a branch point). The inverse Laplace transform formula yields

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \exp\{st\} \frac{s^{\alpha-\beta} \exp\{-ms\tau\}}{(s^{\alpha} - \lambda \exp\{-s\tau\})^{m+1}} \mathrm{d}s,$$

where $c \in \mathbb{R}$ satisfies $c > \max(0, \Re(s_1))$, i.e. all singularities of the previous integrand are located to the left of the line $\Re(s) = c$. Obviously, instead of the line $\Re(s) = c$, we can use an arbitrary curve with the same property (we note that the idea of some next proof procedures originates from the method presented in [5] for the asymptotic description of Mittag–Leffler functions). In particular, we are going to employ oriented contours formed by three segments given via

$$\gamma(\zeta, \theta) = \left\{s \in \mathbb{C} : s = -u \exp\{-\mathrm{i}\theta\},\ u \in (-\infty, -\zeta) \text{ or } s = \zeta \exp\{-\mathrm{i}u\},\ u \in [-\theta, \theta] \text{ or } s = u \exp\{-\mathrm{i}\theta\},\ u \in (\zeta, \infty)\right\}.$$

Proposition 1 (ii) implies that there exists $\delta > 0$ such that all zeros s_i of (5.2) satisfy $|\arg(s_i)| \ne \pi/2 + \delta$ and, moreover, that there are only finitely many of them satisfying $|\arg(s_i)| < \pi/2 + \delta$. Hence, there exist $R > \varepsilon > 0$ such that all s_i lie to the left of $\gamma(R, \pi/2 + \delta)$ and those satisfying $|\arg(s_i)| < \pi/2 + \delta$ are located to the right of $\gamma(\varepsilon, \pi/2 + \delta)$. Consequently, for every $t > 1$, we can split the inverse Laplace transform formula into

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma(R, \frac{\pi}{2}+\delta)} \frac{s^{\alpha-\beta} \exp\{st - ms\tau\}}{(s^{\alpha} - \lambda \exp\{-s\tau\})^{m+1}} \mathrm{d}s = I_1(t) + I_2(t)$$

where $I_1(t)$ and $I_2(t)$ denote the integrals over $\gamma(\varepsilon/t, \pi/2 + \delta)$ and $\gamma(R, \pi/2 + \delta) - \gamma(\varepsilon/t, \pi/2 + \delta)$, respectively.

First, we analyse the integral $I_1(t)$. Employing the change of variables $s = u^{1/\alpha}/t$, which transforms the contour $\sigma = \gamma(\varepsilon/t, \pi/2+\delta)$ into $\sigma' = \gamma(\varepsilon^\alpha, \alpha\pi/2+\alpha\delta)$, and the identity

$$\left(\frac{1}{\xi - z}\right)^{m+1} = \left(-\frac{1}{z} - \frac{\xi}{z^2} + \frac{\xi^2}{z^2(\xi - z)}\right)^{m+1} = \frac{(-1)^{m+1}}{z^{m+1}} + (m+1)\frac{(-1)^{m+1}\xi}{z^{m+2}} + \sum_{(k_1,k_2,k_3)\in K} \frac{(m+1)!}{k_1!k_2!k_3!}\frac{(-1)^{k_1+k_2}\xi^{k_2+2k_3}}{z^{k_2+k_3+m+1}(\xi - z)^{k_3}},$$

where $K = \{(k_1,k_2,k_3) \in (\mathbb{Z}_0^+)^3 : k_1+k_2+k_3 = m+1,\ (k_1,k_2,k_3) \neq (m+1,0,0) \text{ and } (k_1,k_2,k_3) \neq (m,1,0)\}$ (for the multinomial theorem see, e.g. [17]), we obtain

$$\begin{aligned} I_1(t) &= \frac{1}{2\pi \mathrm{i}}\int_\sigma \frac{s^{\alpha-\beta}\exp\{st - ms\tau\}}{(s^\alpha - \lambda\exp\{-s\tau\})^{m+1}}\mathrm{d}s = \frac{t^{m\alpha+\beta-1}}{2\pi\alpha \mathrm{i}}\int_{\sigma'} \frac{u^{(1-\beta)/\alpha}\exp\{(1+\tau/t)u^{1/\alpha}\}}{(u\exp\{u^{1/\alpha}\tau/t\} - \lambda t^\alpha)^{m+1}}\mathrm{d}u \\ &= \frac{t^{m\alpha+\beta-1}}{2\pi\alpha \mathrm{i}}\Bigg((-1)^{m+1}\int_{\sigma'}\frac{u^{(1-\beta)/\alpha}\exp\{(1+\tau/t)u^{1/\alpha}\}}{(\lambda t^\alpha)^{m+1}}\mathrm{d}u + (-1)^{m+1}(m+1)\int_{\sigma'}\frac{u^{(1-\beta)/\alpha+1}\exp\{(1+2\tau/t)u^{1/\alpha}\}}{(\lambda t^\alpha)^{m+2}}\mathrm{d}u \\ &\quad + \sum_{(k_1,k_2,k_3)\in K}\frac{(m+1)!}{k_1!k_2!k_3!}\int_{\sigma'}\frac{(-1)^{k_1+k_2}}{(\lambda t^\alpha)^{k_2+k_3+m+1}}\frac{u^{(1-\beta)/\alpha+k_2+2k_3}\exp\{(1+(1+k_2+2k_3)\tau/t)u^{1/\alpha}\}}{(u\exp\{u^{1/\alpha}\tau/t\} - \lambda t^\alpha)^{k_3}}\mathrm{d}u\Bigg). \end{aligned}$$

Furthermore, we use the change of variables $v = (1+k\tau/t)^\alpha u$, transforming σ' into $\sigma''_k = \gamma((1+k\tau/t)^\alpha\varepsilon^\alpha, \alpha\pi/2+\alpha\delta)$ $(k = 1, 2)$, and the repeated application of the integral representation of the reciprocal Euler Γ-function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\alpha \mathrm{i}}\int_{\gamma(\zeta,\theta)}\exp\{\xi^{1/\alpha}\}\xi^{1/\alpha - z/\alpha - 1}\mathrm{d}\xi, \quad \zeta > 0,\ \pi\alpha/2 < \theta < \alpha\pi$$

(see [5]) to get

$$\begin{aligned} I_1(t) &= (-1)^{m+1}\frac{(t+\tau)^{\beta-\alpha-1}}{2\pi\alpha\lambda^{m+1}\mathrm{i}}\int_{\sigma''_1} v^{(1-\beta)/\alpha}\exp\{v^{1/\alpha}\}\mathrm{d}v + (-1)^{m+1}(m+1)\frac{(t+2\tau)^{\beta-2\alpha-1}}{2\pi\alpha\lambda^{m+2}\mathrm{i}}\int_{\sigma''_2} v^{(1-\beta)/\alpha+1}\exp\{v^{1/\alpha}\}\mathrm{d}v \\ &\quad + \sum_{(k_1,k_2,k_3)\in K}\frac{t^{\beta-(k_2+k_3+1)\alpha-1}}{2\pi\alpha\lambda^{k_2+k_3+m+1}\mathrm{i}}\frac{(m+1)!}{k_1!k_2!k_3!}\int_{\sigma'}(-1)^{k_1+k_2}\frac{u^{(1-\beta)/\alpha+k_2+2k_3}\exp\{(1+(1+k_2+2k_3)\tau/t)u^{1/\alpha}\}}{(u\exp\{u^{1/\alpha}\tau/t\} - \lambda t^\alpha)^{k_3}}\mathrm{d}u \\ &= \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta-\alpha)}(t+\tau)^{\beta-\alpha-1} + \frac{(-1)^{m+1}(m+1)}{\lambda^{m+2}\Gamma(\beta-2\alpha)}(t+2\tau)^{\beta-2\alpha-1} + I_3(t) \end{aligned}$$

where $I_3(t)$ denotes the last term in the previous expression for $I_1(t)$. In order to discuss the asymptotic behaviour of $I_3(t)$, we first formulate the following estimate. Since the contour σ does not contain any zero s_i of (5.2), there exist $\delta_k > 0$ $(k = 0, \dots, m+1)$ such that $|s^\alpha - \lambda\exp\{-s\tau\}|^k \geq \delta_k$ for all $s \in \sigma$. Consequently, we have

$$|u\exp\{u^{1/\alpha}\tau/t\} - \lambda t^\alpha|^k \geq \delta_k t^{k\alpha}|\exp\{u^{1/\alpha}\tau/t\}|^k \quad \text{for all } u \in \sigma'.$$

It enables us to write

$$\begin{aligned} |I_3(t)| &\leq \sum_{(k_1,k_2,k_3)\in K}\frac{t^{\beta-(k_2+k_3+1)\alpha-1}}{2\pi\alpha|\lambda|^{k_2+k_3+m+1}}\frac{(m+1)!}{k_1!k_2!k_3!}\int_{\sigma'}\left|\frac{u^{(1-\beta)/\alpha+k_2+2k_3}\exp\{(1+(1+k_2+2k_3)\tau/t)u^{1/\alpha}\}}{(u\exp\{u^{1/\alpha}\tau/t\} - \lambda t^\alpha)^{k_3}}\right|\mathrm{d}u \\ &\leq \sum_{(k_1,k_2,k_3)\in K}\frac{t^{\beta-(k_2+2k_3+1)\alpha-1}}{2\pi\alpha|\lambda|^{k_2+k_3+m+1}\delta_{k_3}}\frac{(m+1)!}{k_1!k_2!k_3!}\int_{\sigma'}\left|u^{(1-\beta)/\alpha+k_2+2k_3}\exp\{(1+(1+k_2+k_3)\tau/t)u^{1/\alpha}\}\right|\mathrm{d}u \\ &\leq \sum_{(k_1,k_2,k_3)\in K}\frac{t^{\beta-(k_2+2k_3+1)\alpha-1}}{2\pi\alpha|\lambda|^{k_2+k_3+m+1}\delta_{k_3}}\frac{(m+1)!}{k_1!k_2!k_3!}\Bigg(\int_{-\alpha\pi/2-\alpha\delta}^{\alpha\pi/2+\alpha\delta}\varepsilon^{(1-\beta)+(1+k_2+2k_3)\alpha} \\ &\quad \times\exp\{(1+(1+k_2+k_3)\tau/t)\varepsilon\cos(\varphi/\alpha)\}\mathrm{d}\varphi \\ &\quad + 2\int_{\varepsilon^\alpha}^\infty r^{(1-\beta)/\alpha+k_2+2k_3}\exp\{(1+(1+k_2+k_3)\tau/t)r^{1/\alpha}\cos(\pi/2+\delta)\}\mathrm{d}r\Bigg). \end{aligned}$$

We can see that the last inequality contains real integrals of two types, namely the integrals of functions continuous on the compact integration domain $[-\alpha\pi/2-\alpha\delta, \alpha\pi/2+\alpha\delta]$ (which implies their convergence) and the integrals over the unbounded domain $(\varepsilon^\alpha, \infty)$ which converge due to $\cos(\pi/2+\delta) < 0$.

Now we turn our attention to the second term $I_2(t)$. Clearly, the contour $\gamma(R, \pi/2+\delta) - \gamma(\varepsilon/t, \pi/2+\delta)$ is a simple positively oriented closed curve. Proposition 1 (ii) implies that the integrand has only finitely many poles s_i $(i = 1, \dots, N)$ lying in the interior of $\gamma(R, \pi/2+\delta) - \gamma(\varepsilon/t, \pi/2+\delta)$. Moreover, it follows from Proposition 1 (v) that s_i are poles of order $m+1$. Employing the residue theorem, we get

$$I_2(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma(R,\pi/2+\delta)-\gamma(\varepsilon/t,\pi/2+\delta)} \frac{s^{\alpha-\beta}\exp\{st - ms\tau\}}{(s^\alpha - \lambda\exp\{-s\tau\})^{m+1}}\mathrm{d}s = \sum_{i=1}^{N} \mathrm{Res}_{s=s_i}\left(\frac{s^{\alpha-\beta}\exp\{st - ms\tau\}}{(s^\alpha - \lambda\exp\{-s\tau\})^{m+1}}\right).$$

At every pole s_i we can utilise the Laurent expansions

$$\frac{s^{\alpha-\beta}}{(s^\alpha - \lambda\exp\{-s\tau\})^{m+1}} = c_{-m-1}(s-s_i)^{-m-1} + c_{-m}(s-s_i)^{-m} + c_{-m+1}(s-s_i)^{-m+1} + \dots,$$

$$\exp\{st - ms\tau\} = \exp\{s_i(t - m\tau)\}\left(1 + (t-m\tau)(s-s_i) + \frac{(t-m\tau)^2}{2!}(s-s_i)^2 + \frac{(t-m\tau)^3}{3!}(s-s_i)^3 + \dots\right)$$

where c_j $(j = -m-1, -m, \dots)$ are complex constants independent of t and $c_{-m-1} \neq 0$. Thus, the Cauchy product of these expansions enables us to write the sum of residues (i.e. the coefficients at $(s-s_i)^{-1}$) as

$$I_2(t) = \textstyle\sum_{i=1}^{N}\sum_{j=0}^{m} d_{i,j}(t-m\tau)^j \exp\{s_i(t-m\tau)\}$$

where $d_{i,j}$ are suitable complex constants and $d_{i,m} \neq 0$ $(i = 1, \dots, N,\ j = 0, \dots, m)$.

Combining the obtained results, we get

$$G_{\alpha,\beta}^{\lambda,\tau,m}(t) = \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta-\alpha)}(t+\tau)^{\beta-\alpha-1} + \frac{(-1)^{m+1}(m+1)}{\lambda^{m+2}\Gamma(\beta-2\alpha)}(t+2\tau)^{\beta-2\alpha-1}$$
$$+ \sum_{i=1}^{N}\sum_{j=0}^{m} d_{i,j}(t-m\tau)^j \exp\{s_i(t-m\tau)\} + \mathcal{O}(t^{\beta-3\alpha-1})$$

as $t \to \infty$. Proposition 1 (iii) implies that for $\lambda \in \mathcal{S}_{\alpha,\tau}$ the exponential terms vanish as $t \to \infty$ and the part (ii) of the assertion is proved. If $\lambda \notin \mathcal{S}_{\alpha,\tau}$, then the exponential terms play the dominant role in asymptotic investigations. Moreover, Proposition 1 (iv) and (v) yields that there are at most two poles s_i of the maximal real part, i.e. it holds $\Re(s_3) < \Re(s_1)$. This concludes the proof of the part (iii). □

Now we have all the partial results necessary to prove the main result.

Proof of Theorem 2. Theorems 3 and 4 imply that every coordinate of the solution $y(t)$ of (2.1), (2.2) consists of a linear combination of terms

$$\mathrm{D}_0^{-(1-\alpha)} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t), \quad \int_{-\tau}^{0} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t-\tau-u)\phi(u)\mathrm{d}u, \quad m = 0, \dots, p_i - 1, \quad i = 1, \dots, n$$

where λ_i are distinct eigenvalues of A and p_i are the maximal sizes of Jordan blocks corresponding to λ_i.

We analyse the stability properties of these terms. First, we consider the $(1-\alpha)$-integral of $G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t)$ to get

$$\mathrm{D}_0^{-(1-\alpha)} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t) = \mathrm{D}_0^{-(1-\alpha)} \sum_{j=0}^{\infty} \binom{m+j}{j} \frac{\lambda_i^j (t-(m+j)\tau)^{(m+j)\alpha+\alpha-1}}{\Gamma((m+j)\alpha+\alpha)} h(t-(m+j)\tau)$$
$$= \sum_{j=0}^{\infty} \binom{m+j}{j} \lambda_i^j \mathrm{D}_{(m+j)\tau}^{-(1-\alpha)} \frac{(t-(m+j)\tau)^{(m+j)\alpha+\alpha-1}}{\Gamma((m+j)\alpha+\alpha)} h(t-(m+j)\tau)$$
$$= \sum_{j=0}^{\infty} \binom{m+j}{j} \lambda_i^j \frac{(t-(m+j)\tau)^{(m+j)\alpha}}{\Gamma((m+j)\alpha+1)} h(t-(m+j)\tau) = G_{\alpha,1}^{\lambda_i,\tau,m}(t)$$

(for the fractional power rule see [5]).

Furthermore, we employ the condition $\phi(t) \in L^1([-\tau, 0])$ and Lemma 2 to estimate the influence of the initial function as

$$\left|\int_{-\tau}^{0} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t-\tau-u)\phi(u)\mathrm{d}u\right| \le \sup_{t-\tau<u<t} |G_{\alpha,\alpha}^{\lambda_i,\tau,m}(u)| \int_{-\tau}^{0} |\phi(u)|\mathrm{d}u \le \begin{cases} K_1 t^{-\alpha-1}, & \lambda_i \in \mathcal{S}_{\alpha,\tau}, \\ K_2 t^{m\alpha+\alpha-1}, & \lambda_i = 0, \\ K_3 t^m, & \lambda_i \in \partial\mathcal{S}_{\alpha,\tau} \setminus \{0\}, \\ K_4 t^m \exp\{|s_1|(t-m\tau)\}, & \lambda_i \notin \mathrm{cl}\mathcal{S}_{\alpha,\tau} \end{cases}$$

for large t, where K_1, K_2, K_3, K_4 are suitable positive real constants. Note that if algebraic and geometric multiplicities of λ_i are equal, then $m = 0$. Consequently, applying Lemma 2 to $G_{\alpha,1}^{\lambda_i,\tau,m}(t)$ we get the assertion of Theorem 2.

We point out that the proof of Theorem 2 actually enables us to specify the asymptotic behaviour of solutions $y(t)$ of (2.1) also in the unstable case. We can summarise it into the following

Theorem 5. *Let $A \in \mathbb{R}^{d\times d}$, $0 < \alpha < 1$ and $\tau > 0$. Let λ_i be all distinct eigenvalues of A $(i = 1, \dots, n)$ and let the condition of Theorem 2 (ii) be not satisfied (i.e. the zero solution of (2.1) is not stable). Then solutions $y(t)$ of (2.1) admit three types of asymptotics:*

(i) *Let* $\lambda_1 = 0$ *be the zero eigenvalue of A with algebraic multiplicity greater than geometric one and let* p_1 *be the maximal size of Jordan blocks corresponding to* λ_1*. Furthermore, let* $\lambda_i \in \mathcal{S}_{\alpha,\tau}$ *for all* $i = 2, \dots, n$*. Then*

$$\|y(t)\| \sim t^{(p_1-1)\alpha} \quad \text{as } t \to \infty \qquad \text{for any solution } y(t) \text{ of (2.1)}.$$

(ii) *Let* λ_i $(i = 1, \dots, \ell \le n)$ *be nonzero eigenvalues of A lying on* $\partial\mathcal{S}_{\alpha,\tau}$ *with algebraic multiplicity greater than geometric one and let* p_i *be the maximal size of Jordan blocks corresponding to* λ_i $(i = 1, \dots, \ell)$*. Furthermore, let* $\lambda_i \in \mathcal{S}_{\alpha,\tau}$ *for all* $i = \ell+1, \dots, n$ *provided* $\ell < n$ *and* $p = \max(p_1, \dots, p_\ell)$*. Then*

$$\|y(t)\| \sim_{sup} t^{p-1} \quad \text{as } t \to \infty \qquad \text{for any solution } y(t) \text{ of (2.1)}.$$

(iii) *Let* λ_i $(i = 1, \dots, \ell \le n)$ *be eigenvalues of A located outside* $\mathrm{cl}(\mathcal{S}_{\alpha,\tau})$ *and let* s_1 *be a zero of (5.2) such that* $\Re(s_1) = \max(\Re(s) : s \text{ is a zero of (5.2) for some } \lambda = \lambda_i,\ i = 1, \dots, \ell)$*. Furthermore, let* $\lambda_j, j \in L \subset \{1, \dots, \ell\}$ *be eigenvalues of A such that (5.2) with* $\lambda = \lambda_j$ *has the zero* s_1 *and let p be the maximal size of Jordan blocks corresponding to* $\lambda_j, j \in L$*. Then*

$$\|y(t)\| \sim_{sup} t^{p-1} \exp\{\Re(s_1)t\} \quad \text{as } t \to \infty \qquad \text{for any solution } y(t) \text{ of (2.1)}.$$

7. Some consequences

In this section, we consider the system (2.1) in the planar form, i.e. when $d = 2$ and reformulate the stability conditions of Theorem 2 quite explicitly in terms of trace of A ($\mathrm{tr}\,A$) and determinant of A ($|A|$). If $d = 2$, then the characteristic equation of A becomes

$$\lambda^2 - (\mathrm{tr}\,A)\,\lambda + |A| = 0.$$

In such a case, an explicit reformulation of conditions of Theorem 2 is connected with tedious, but essentially straightforward calculations. On this account, we omit these technical procedures and present only the relevant conclusion on the asymptotic stability property of the zero solution.

Corollary 1. *Let* $A \in \mathbb{R}^{2\times 2}$, $0 < \alpha < 1$ *and* $\tau > 0$*. The zero solution of (2.1) is asymptotically stable if and only if*

$$0 < \tau^{2\alpha}|A| < (\pi - \alpha\pi/2)^{2\alpha}$$

and

$$-\left(\frac{\tau}{\pi - \alpha\pi/2}\right)^{\alpha}|A| - \left(\frac{\pi - \alpha\pi/2}{\tau}\right)^{\alpha} < \mathrm{tr}\,A < 2|A|^{1/2}\cos\left(\tau|A|^{1/(2\alpha)} + \frac{\alpha\pi}{2}\right).$$

Remark 6. The previous conditions enable various interpretations with respect to parameters $\mathrm{tr}\,A$, $|A|$, α and τ. In particular, for a changing τ (and with the remaining parameters being fixed), we get the following consequence:

If $\mathrm{tr}\,A \le -2|A|^{1/2}$, then the zero solution of the planar system (2.1) is asymptotically stable if and only if $\tau < \tau^*$ where

$$\tau^* = (\pi - \alpha\pi/2)\left(\frac{-\mathrm{tr}\,A - \left((\mathrm{tr}\,A)^2 - 4|A|\right)^{1/2}}{2|A|}\right)^{1/\alpha}.$$

Similarly, if $\mathrm{tr}\,A \ge -2|A|^{1/2}$, then the zero solution of the planar system (2.1) is asymptotically stable if and only if $\tau < \tau^{**}$ where

$$\tau^{**} = \left(\arccos\frac{\mathrm{tr}\,A}{2|A|^{1/2}} - \alpha\pi/2\right) \Big/ \ |A|^{1/(2\alpha)}.$$

From this viewpoint, the values τ^* and τ^{**} represent stability switches for the planar system (2.1), i.e. the critical values of a delay τ when (2.1) loses its asymptotic stability property (for similar results on related integro-differential systems we refer to [18]). Notice that the largest value of a delay τ, when asymptotic stability of the zero solution of (2.1) turns into instability, can be achieved for $\mathrm{tr}\,A = -2|A|^{1/2}$. In this case, both the values τ^* and τ^{**} become

$$\tau^{***} = \frac{\pi - \alpha\pi/2}{|A|^{1/(2\alpha)}}.$$

In other words, if $\mathrm{tr}\,A = -2|A|^{1/2}$, then the zero solution of the planar system (2.1) is asymptotically stable if and only if $\tau < \tau^{***}$. For a better clarity, the conditions of Corollary 1 are depicted in Fig. 5.

Example 1. As an illustration of Corollary 1, we consider Duffing's model of an unforced nonlinear oscillator

$$\begin{aligned} y_1'(t) &= y_2(t), \\ y_2'(t) &= y_1(t) - (y_1(t))^3 + \delta y_2(t) \end{aligned} \tag{7.1}$$

depending on the damping parameter δ (we admit that δ is arbitrary real number). Note that involvement of other standard parameters to this model has no contribution to our discussion and therefore we omit them.

Local stability analysis of (7.1) usually originates from the linearisation method. Application of this method to the nonlinear system (7.1) along its equilibrium points (0, 0) and (± 1, 0) and analysis of the corresponding Jacobi matrices imply that the

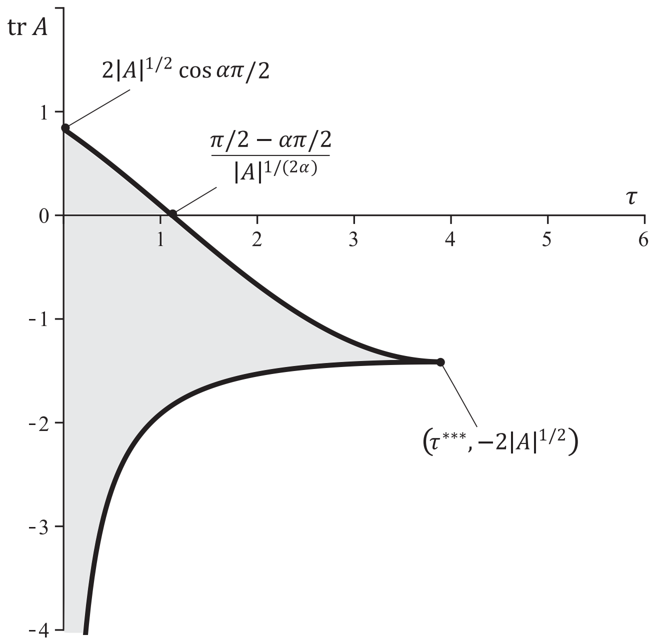

Fig. 5. The relationship between tr A and delay τ (with the fixed parameters $\alpha = 0.6$ and $|A| = 0.5$).

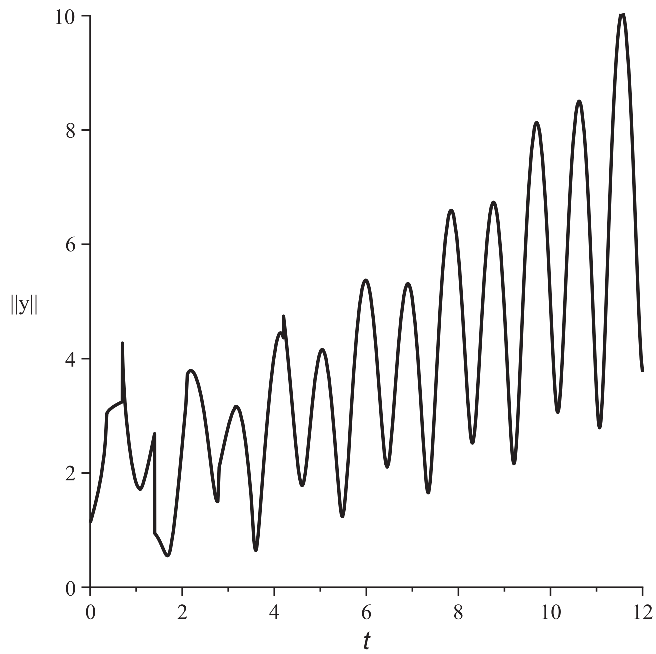

Fig. 6. The norm of the solution $y(t)$ of (7.3) for the values $\alpha = 0.25$, $\tau = 0.35$, $\delta = 0.01$ and the initial function $\phi(t) \equiv (1, 1)^T$ on $[-0.35, 0]$.

equilibrium point (0, 0) is not stable, while the equilibrium points $(\pm 1, 0)$ are asymptotically stable for $\delta < 0$, stable (but not asymptotically) for $\delta = 0$ and unstable for $\delta > 0$.

If we replace the conventional derivatives in (7.1) by the fractional ones and the current time t on the right-hand side of (7.1) by the delayed time $t - \tau$, then we get the fractional delay Duffing's model

$$\begin{aligned} \mathrm{D}_0^\alpha y_1(t) &= y_2(t-\tau), \\ \mathrm{D}_0^\alpha y_2(t) &= y_1(t-\tau) - (y_1(t-\tau))^3 + \delta y_2(t-\tau) \end{aligned}$$

where $0 < \alpha < 1$ and $\tau > 0$. This system has the same equilibrium points (0, 0) and $(\pm 1, 0)$. The corresponding linearised models along these points are

$$\begin{aligned} \mathrm{D}_0^\alpha y_1(t) &= y_2(t-\tau), \\ \mathrm{D}_0^\alpha y_2(t) &= y_1(t-\tau) + \delta y_2(t-\tau) \end{aligned} \tag{7.2}$$

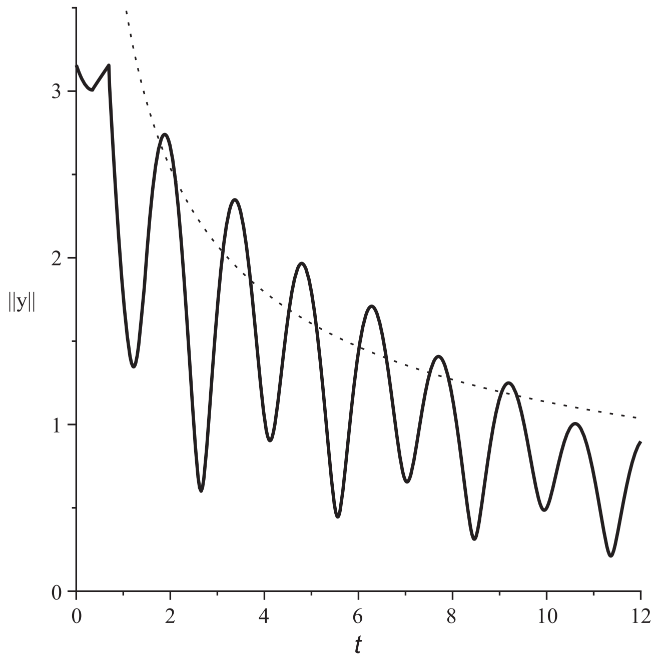

Fig. 7. The norm of the solution $y(t)$ of (7.3) for the values $\alpha = 0.5$, $\tau = 0.35$, $\delta = 0.01$ and the initial function $\phi(t) \equiv (1, 1)^T$ on $[-0.35, 0]$.

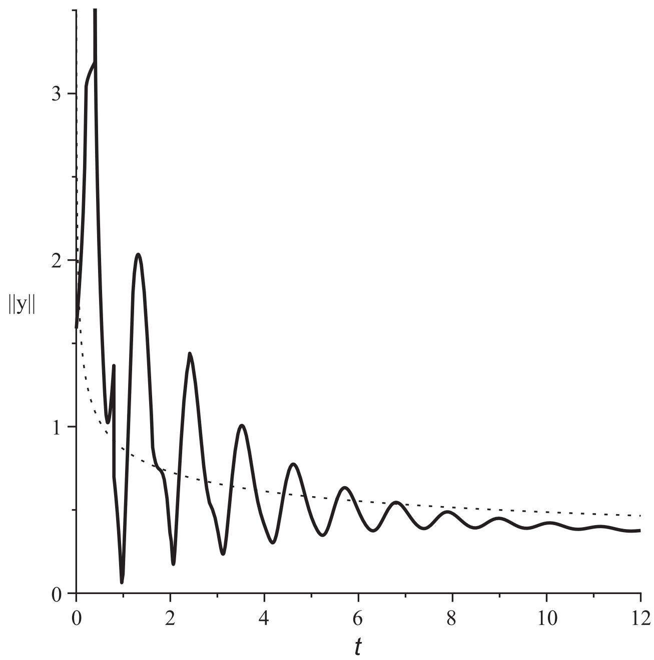

Fig. 8. The norm of the solution $y(t)$ of (7.3) for the values $\alpha = 0.25$, $\tau = 0.2$, $\delta = 0.01$ and the initial function $\phi(t) \equiv (1, 1)^T$ on $[-0.2, 0]$.

and

$$\begin{aligned} \mathrm{D}_0^\alpha y_1(t) &= y_2(t-\tau), \\ \mathrm{D}_0^\alpha y_2(t) &= -2y_1(t-\tau) + \delta y_2(t-\tau), \end{aligned} \tag{7.3}$$

respectively (for a linearisation theorem for fractional dynamical systems we refer to [19] and for its application to a two-term fractional differential equation see also [20]). Applying Corollary 1 to (7.2), we get instability of the equilibrium point (0, 0) due to $|A| = -1$ (hence, the involvement of fractional derivative order α and delay τ has no influence on its stability behaviour). Regarding (7.3), the situation is more interesting. In this case, $|A| = 2$, $\operatorname{tr} A = \delta$ and thus it is easy to rewrite the stability conditions of Corollary 1. In particular, we can observe the following influence of parameters α and τ on the stability interval for the damping parameter δ.

If $\tau = 0$, then the zero solution of (7.3) is asymptotically stable if and only if $\delta < 8^{1/2}\cos(\alpha\pi/2)$. If $\tau > 0$, then the stability interval for damping parameter δ becomes more restrictive, namely

$$-2\left(\frac{\tau}{\pi-\alpha\pi/2}\right)^{\alpha} - \left(\frac{\pi-\alpha\pi/2}{\tau}\right)^{\alpha} < \delta < 8^{1/2}\cos\left(\tau 2^{1/(2\alpha)} + \alpha\pi/2\right) \quad (7.4)$$

where $0 < \tau < (\pi - \alpha\pi/2)/2^{1/(2\alpha)}$ (see also Fig. 5 with respect to $\mathrm{tr}A = \delta$). In particular, if $\tau < (\pi/2 - \alpha\pi/2)/2^{1/(2\alpha)}$, then (contrary to the classical model) the zero solution of (7.3) is asymptotically stable for $\delta = 0$ as well as for appropriate positive values of δ. The critical value of the stability switch τ^{***} now equals $(\pi - \alpha\pi/2)/2^{1/(2\alpha)}$.

To support these theoretical conclusions by a numerical experiment, we depict a norm of the vector solution $y(t)$ of (7.3) for three particular choices of entry parameters α, τ and δ. First let $\alpha = 0.25$, $\tau = 0.35$ and $\delta = 0.01$. We can easily check that this triplet does not satisfy the stability condition (7.4). If the value α is increased to $\alpha = 0.5$ and the remaining parameters τ, δ are unchanged, then (7.4) is already satisfied (it might be interesting to note that further increase of the value α leads again to the loss of stability). Similarly, if the value τ is decreased to $\tau = 0.2$ (with the original values $\alpha = 0.25$ and $\delta = 0.01$), then the stability condition (7.4) is also met. The whole situation is depicted in Figs. 6–8 illustrating previous theoretical results. Figs. 7 and 8, corresponding to the asymptotically stable case, contain also asymptotic algebraic upper bounds (the dotted curves) for the solutions $y(t)$ of (7.3) following from Theorem 2 and Lemma 2.

8. Concluding remarks

In this paper, we have fully described the stability region for the fractional delay system (2.1), including the description of asymptotics of its solutions. The utilised proof method was based on the Laplace transform method and analysis of zeros of the corresponding characteristic equation. This approach has a wider usage and can be applied also in the study of some other qualitative properties of FDDEs, e.g. their oscillatory behaviour (for some preliminary results in this direction we refer to [21] whose results seem to be extendable just by our approach).

The assertion of Theorem 2 might be the starting point for stability analysis of other FDDEs. In the linear case, a natural extension consists in the involvement of a non-delayed term on the right-hand side of (2.1), i.e. we can study the fractional delay system

$$D_0^{\alpha} y(t) = Ay(t-\tau) + By(t), \quad t \in (0, \infty)$$

under various assumptions on matrices A, B (we emphasise that the problem of necessary and sufficient stability conditions for this system with general $d \times d$ matrices A, B seems to be extremely complicated and it is still open even in the integer-order case $\alpha = 1$). In the nonlinear case, we have already outlined applicability of Theorem 2 and Corollary 1 in local stability investigations of Duffing's model of a nonlinear oscillator where we have described the influence of the fractional order α and the time delay τ on its stability behaviour. These observations can be extended to local stability investigations of other nonlinear models which enables to describe their dynamics in the neighbourhood of equilibrium points. Another potential applicability of our results consists in the synchronisation process between a drive system (described via a nonlinear FDDE) and the corresponding response system involving some control parameters. Then the synchronisation between both the systems is equivalent to the asymptotic stability of the zero solution of the corresponding error system consisting of linear FDDEs (see, e.g. [9]). Finally, (2.1) may serve as the test system either for numerical analysis of FDDEs (for a similar situation in the fractional non-delayed case we refer to [22,23]), or for stability investigations of FDDEs via other methods (for basic principles of Lyapunov methods for FDDEs see, e.g. [24]).

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Appendix B

Paper on higher-order one-term FDDS [10] (EJDE, 2019)

We do not find direct technical generalizations of previous papers particularly interesting, which likely led us to postpone the work on higher-order one-term FDDS as it seemed like a simple follow-up on [6]. However, our hesitation proved unnecessary. While the stability results were indeed expectedly straightforward generalizations of our previous findings, [10] (co-author: J. Čermák; my author's share 50 %) shifted our focus towards the oscillatory properties - a challenge introduced by higher-order systems.

In this paper, we conducted a deeper analysis of the locations of characteristic roots depending on location of eigenvalues. Unlike common practice, we were not only interested in the case of negative real parts. We detailed the occurrence and conditions of roots with positive real parts, including their number. That started our interest in the properties of unbounded solutions, which we revisited also in subsequent papers.

Ultimately, [10] addresses FDDS of all positive non-integer orders, revealing predominantly non-oscillatory behaviour. To better discuss the mechanism by which initial conditions influence the oscillatory and stability properties of the given solution, we introduced the terms *major* and *n-minor* solutions. That allowed us to explore the effects of initial conditions more comprehensively.

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OSCILLATORY AND ASYMPTOTIC PROPERTIES OF FRACTIONAL DELAY DIFFERENTIAL EQUATIONS

JAN ČERMÁK, TOMÁŠ KISELA

Abstract. This article discusses the oscillatory and asymptotic properties of a test delay differential system involving a non-integer derivative order. We formulate corresponding criteria via explicit necessary and sufficient conditions that enable direct comparisons with the results known for classical integer-order delay differential equations. In particular, we shall observe that oscillatory behaviour of solutions of delay system with non-integer derivatives embodies quite different features compared to the classical results known from the integer-order case.

1. Introduction and preliminaries

Basic qualitative properties of the delay differential equation

$$y'(t) = Ay(t - \tau), \quad t \in (0, \infty), \tag{1.1}$$

where A is a constant real $d \times d$ matrix and $\tau > 0$ is a constant real lag, are well described in previous numerous investigations. While stability and asymptotic properties of (1.1) were reported in [8], answers to various oscillation problems regarding (1.1) were surveyed in [7].

A crucial role in these investigations was played by the associated characteristic equation

$$\det(sI - A\exp\{-s\tau\}) = 0, \tag{1.2}$$

where I is the identity matrix. More precisely, appropriate properties of (1.1) were first described via location of all roots of (1.2) in a specific area of the complex plane. Then, efficient criteria guaranteeing such root locations were formulated in terms of conditions imposed directly on the eigenvalues of A.

We recall some of relevant statements (reformulated in the above mentioned sense) along with their consequences to the scalar case when (1.1) becomes

$$y'(t) = ay(t - \tau), \quad t \in (0, \infty) \tag{1.3}$$

where a is a real number. Since we are primarily interested in discussions of oscillatory properties of appropriate fractional extensions of (1.1), we first state (see [7]) oscillation conditions for (1.1) (as it is customary, we say that a solution of (1.1) is oscillatory if every its component has arbitrarily large zeros; otherwise the solution is called non-oscillatory).

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Theorem 1.1. *Let $A \in \mathbb{R}^{d\times d}$ and $\tau \in \mathbb{R}^+$. Then the following statements are equivalent:*

(a) *All solutions of* (1.1) *oscillate;*
(b) *The characteristic equation* (1.2) *has no real roots;*
(c) *A has no real eigenvalues in $[-1/(\tau\mathrm{e}), \infty)$.*

Corollary 1.2. *Let $a \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$. All solutions of* (1.3) *oscillate if and only if*

$$a < -\frac{1}{\tau\mathrm{e}}\,.$$

As we shall see later, oscillatory properties of the corresponding fractional delay system are closely related to convergence of all its solutions to the zero solution. In the first-order case (1.1), this property was characterized in [8] via

Theorem 1.3. *Let $A \in \mathbb{R}^{d\times d}$ and $\tau \in \mathbb{R}^+$. Then the following statements are equivalent:*

(a) *Any solution y of* (1.1) *tends to zero as $t \to \infty$;*
(b) *The characteristic equation* (1.2) *has all roots with negative real parts;*
(c) *All eigenvalues λ_i $(i = 1, \dots, d)$ of A satisfy*

$$\tau|\lambda_i| < |\arg(\lambda_i)| - \pi/2\,.$$

Moreover, the convergence of y to zero is of exponential type.

Remark 1.4. The condition (c) can be equivalently expressed via the requirement that all eigenvalues λ_i $(i = 1, \dots, d)$ of A have to be located inside the region bounded by the curve

$$\Re(\lambda) = \omega\cos(\omega\tau), \quad \Im(\lambda) = -\omega\sin(\omega\tau), \quad -\frac{\pi}{2\tau} \le \omega \le \frac{\pi}{2\tau}$$

in the complex plane.

Corollary 1.5. *Let $a \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$. Any solution y of* (1.3) *tends to zero as $t \to \infty$ if and only if*

$$-\frac{\pi}{2\tau} < a < 0\,.$$

Extensions of previous results to the n-th order equation (n is a positive integer)

$$y^{(n)}(t) = Ay(t-\tau), \quad t \in (0,\infty) \tag{1.4}$$

yield different conclusions. In this case, the characteristic equation becomes

$$\det(s^n I - A\exp\{-s\tau\}) = 0\,. \tag{1.5}$$

If $n \ge 2$, then there is no analogue to Theorem 1.3. More precisely, the convergence of all solutions of (1.4) to zero is not possible whenever $n \ge 2$ (see, e.g. [6]). Regarding oscillatory properties of (1.4), equivalency of conditions (a) and (b) (with (1.2) replaced by (1.5)) of Theorem 1.1 remains preserved, but their conversion into an explicit form depends on parity of n (see [7]).

The main goal of this article is to discuss these oscillatory and related asymptotic properties of (1.1) with respect to their possible extension to the fractional delay differential equation

$$\mathrm{D}_0^{\alpha} y(t) = Ay(t-\tau), \quad t \in (0,\infty) \tag{1.6}$$

where $\alpha > 0$ is a real scalar and the symbol D_0^α is the Caputo derivative of order α introduced in the following way: First let y be a real scalar function defined on $(0,\infty)$. For a positive real γ, the fractional integral of y is given by

$$\mathrm{D}_0^{-\gamma} y(t) = \int_0^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)} y(\xi)\mathrm{d}\xi, \quad t \in (0,\infty)$$

and, for a positive real α, the Caputo fractional derivative of y is given by

$$\mathrm{D}_0^\alpha y(t) = \mathrm{D}_0^{-(\lceil\alpha\rceil-\alpha)}\Big(\frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} y(t)\Big), \quad t \in (0,\infty)$$

where $\lceil\cdot\rceil$ means the upper integer part. As it is customary, we put $\mathrm{D}_0^0 y(t) = y(t)$ (for more on fractional calculus, see, e.g. [10, 15]). If y is a real vector function, the corresponding fractional operators are considered component-wise (similarly, if y is a complex-valued function, then these fractional operators are introduced for its real and imaginary part separately). We add that the initial conditions associated to (1.6) are

$$y(t) = \phi(t), \quad t \in [-\tau, 0]\,, \tag{1.7}$$

$$\lim_{t\to 0^+} y^{(j)}(t) = \phi_j, \quad j = 0,\dots,\lceil\alpha\rceil - 1 \tag{1.8}$$

where all components of ϕ are absolutely Riemann integrable on $[-\tau,0]$ and ϕ_j are real scalars. In the frame of our oscillatory and asymptotic discussions on (1.6), we are going not only to extend previous results to (1.6) but also discuss a dependence of relevant conditions on changing derivative order α (with a special attention to the case when α is crossing integer values).

The structure of this paper is following: Section 2 recalls some related special functions as well as the characteristic equation associated with (1.6). Some asymptotic expansions of the studied special functions are described as well. In Section 3, we discuss in detail distribution of roots of the characteristic equation in specific areas of the complex plane. Using these auxiliary statements, Sections 4 and 5 formulate a series of results describing oscillation and asymptotic properties of (1.6) in the vector and scalar case. More precisely, Section 4 presents analogues of Theorems 1.1 and 1.3, and Section 5 contains some additional oscillation results in the scalar case. Discussions on non-consistency of the obtained results with the above recalled classical properties of (1.1) and (1.3) are subject of Section 6 concluding the paper.

2. Special functions and their properties

In this section, we recall and extend some notions and formulae introduced in [3] in the frame of stability analysis of (1.6) with $0 < \alpha < 1$. As we shall see later, these tools turn out to be very useful also in oscillatory investigations of (1.6) with arbitrary real $\alpha > 0$. Since the proofs of auxiliary statements stated below are (essentially) analogous to the proofs of appropriate assertions from [3], we omit them.

In the sequel, the symbols $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and inverse Laplace transform of appropriate functions, respectively.

Definition 2.1. Let $A \in \mathbb{R}^{d\times d}$, let I be the identity $d\times d$ matrix and let $\alpha,\tau \in \mathbb{R}^+$. The matrix function $R : \mathbb{R} \to \mathbb{C}^{d\times d}$ given by

$$R(t) = \mathcal{L}^{-1}\left((s^\alpha I - A\exp\{-s\tau\})^{-1}\right)(t) \tag{2.1}$$

is called the fundamental matrix solution of (1.6).

Theorem 2.2. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha, \tau \in \mathbb{R}^+$ and let R be the fundamental matrix solution of* (1.6)*. Then the solution y of* (1.6)*–*(1.8) *is given by*

$$y(t) = \sum_{j=0}^{\lceil\alpha\rceil-1} \mathrm{D}_0^{\alpha-j-1} R(t)\phi_j + \int_{-\tau}^{0} R(t-\tau-u)A\phi(u)\mathrm{d}u.$$

Remark 2.3. Theorem 2.2 along with Definition 2.1 imply that the poles of the Laplace image of solution of (1.6) coincide with roots of

$$\det(s^\alpha I - A\exp\{-s\tau\}) = 0, \quad \text{equivalently} \quad \prod_{i=1}^{n}(s^\alpha - \lambda_i \exp\{-s\tau\})^{n_i} = 0\,, \tag{2.2}$$

where λ_i $(i = 1, \dots, n)$ are distinct eigenvalues of A and n_i are their algebraic multiplicities. This confirms the well-known fact that (2.2) is the characteristic equation associated to (1.6) (see, e.g. [5, 9, 11]).

The following notion of a generalized delay exponential function plays an important role in description of asymptotic expansions of the fundamental matrix solution of (1.6).

Definition 2.4. Let $\lambda \in \mathbb{C}$, $\eta, \beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+ \cup \{0\}$. The generalized delay exponential function (of Mittag-Leffler type) is introduced via

$$G_{\eta,\beta}^{\lambda,\tau,m}(t) = \sum_{j=0}^{\infty} \binom{m+j}{j} \frac{\lambda^j (t-(m+j)\tau)^{\eta(m+j)+\beta-1}}{\Gamma(\eta(m+j)+\beta)} h(t-(m+j)\tau)$$

where h is the Heaviside step function.

The relationship between the fundamental matrix solution R and the generalized delay exponential functions $G_{\eta,\beta}^{\lambda,\tau,m}$ can be specified via the following lemma.

Lemma 2.5. *The fundamental matrix solution* (2.1) *can be expressed as $R(t) = T^{-1}\mathcal{G}(t)T$, where T is a regular matrix and $\mathcal{G}$ is a block diagonal matrix with upper-triangular blocks B_j given by*

$$B_j(t) = \begin{pmatrix} G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,1}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,2}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-1}(t) \\ 0 & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & G_{\alpha,\alpha}^{\lambda_i,\tau,1}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-2}(t) \\ 0 & 0 & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,r_j-3}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & G_{\alpha,\alpha}^{\lambda_i,\tau,0}(t) \end{pmatrix},$$

where $j = 1, \dots, J$ $(J \in \mathbb{Z}^+)$ and r_j is the size of the corresponding Jordan block of A.

As a next key auxiliary result, we describe asymptotic behaviour of $G_{\eta,\beta}^{\lambda,\tau,m}$ functions.

Lemma 2.6. *Let $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+ \cup \{0\}$. Further, let s_i $(i = 1, 2, \dots)$ be the roots of*

$$s^\alpha - \lambda \exp\{-s\tau\} = 0 \tag{2.3}$$

with ordering $\Re(s_i) \ge \Re(s_{i+1})$ $(i = 1, 2, \dots$; in particular, s_1 is the rightmost root).

(i) *If* $\lambda = 0$, *then*

$$G^{0,\tau,m}_{\alpha,\beta}(t) = \frac{(t-m\tau)^{m\alpha+\beta-1}}{\Gamma(m\alpha+\beta)} h(t-m\tau).$$

(ii) *If* $\lambda \neq 0$, *then*

$$G^{\lambda,\tau,m}_{\alpha,\beta}(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{m\cdot k_i} a_{ij}(t-m\tau)^j \exp\{s_i(t-m\tau)\} + S^{\lambda,\tau,m}_{\alpha,\beta}(t)\,,$$

where k_i *is a multiplicity of* s_i, a_{ij} *are suitable nonzero complex constants* ($j = 0,\dots,mk_i$, $i = 1,2,\dots$) *and the term* $S^{\lambda,\tau,m}_{\alpha,\beta}$ *has the algebraic asymptotic behaviour expressed via*

$$\begin{aligned} S^{\lambda,\tau,m}_{\alpha,\beta}(t) &= \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta-\alpha)}(t+\tau)^{\beta-\alpha-1} \\ &\quad + \frac{(-1)^{m+1}(m+1)}{\lambda^{m+2}\Gamma(\beta-2\alpha)}(t+2\tau)^{\beta-2\alpha-1} + \mathcal{O}(t^{\beta-3\alpha-1}) \quad \text{as } t\to\infty. \end{aligned}$$

3. Distribution of characteristic roots

The aim of this section is to analyse (2.2) with respect to existence of its real roots as well as number of its roots with positive real parts. Doing this, it is enough to consider its partial form (2.3).

First, we characterize the set of all roots of (2.3) in terms of their magnitudes and arguments (we assume here $\lambda \neq 0$, i.e. $s \neq 0$). Using the goniometric forms of s and λ we obtain that (2.3) is equivalent to

$$\begin{aligned} &|s|^\alpha \cos[\alpha \arg(s)] - |\lambda| \exp\{-|s|\tau\cos[\arg(s)]\} \cos[\arg(\lambda) - |s|\tau \sin(\arg(s))] \\ &= 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &|s|^\alpha \sin[\alpha \arg(s)] - |\lambda| \exp\{-|s|\tau\cos[\arg(s)]\} \sin[\arg(\lambda) - |s|\tau \sin(\arg(s))] \\ &= 0. \end{aligned} \tag{3.2}$$

To solve (2.3), we consider (3.1)–(3.2) as a system with unknowns $|s|$ and $\arg(s)$. If $\alpha\arg(s) = \ell_1\pi$ for some integer ℓ_1, then $\arg(\lambda) - |s|\tau\sin[\arg(s)] = \ell_2\pi$ for some integer ℓ_2 and (3.1) yields

$$|s|^\alpha(-1)^{\ell_1} - |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\}(-1)^{\ell_2} = 0,$$

i.e.

$$|s|^\alpha = (-1)^\ell |\lambda| \exp\{-|s|\tau\cos[\arg(s)]\} = 0 \quad \text{for some integer } \ell. \tag{3.3}$$

Thus (3.1)–(3.2) can be reduced to

$$\alpha\arg(s) - \arg(\lambda) - |s|\tau\sin[\arg(s)] = 2k\pi \quad \text{for some integer } k, \tag{3.4}$$

$$|s|^\alpha = |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\}. \tag{3.5}$$

If $\alpha\arg(s) \neq \ell_1\pi$ for any integer ℓ_1, then $\arg(\lambda) - |s|\tau\sin[\arg(s)] \neq \ell_2\pi$ for any integer ℓ_2 and division (3.1) over (3.2) yields

$$\alpha\arg(s) = |\lambda|\exp\{-|s|\tau\cos[\arg(s)]\} + \ell\pi \quad \text{for some integer } \ell.$$

This, after substitution into (3.1), yields (3.3). Now, the same argumentation as above shows equivalency of (2.3) and (3.4)–(3.5).

Using the previous process, we can derive the following characterization of possible real roots of (2.3).

Proposition 3.1. *Let $\lambda \in \mathbb{C}$ and $\alpha, \tau \in \mathbb{R}^+$.*

(i) *The characteristic equation (2.3) has a positive real root if and only if λ is a positive real. This root is simple, unique and it is the rightmost root of (2.3).*

(ii) *The characteristic equation (2.3) has a negative real root if and only if*

$$0 < |\lambda| \le (\frac{\alpha}{\tau e})^{\alpha} \quad \text{and} \quad \arg(\lambda) = (\alpha - 2k)\pi \quad \text{for some } k \in \mathbb{Z}.$$

More precisely, if

$$0 < |\lambda| = (\frac{\alpha}{\tau e})^{\alpha} \quad \text{and} \quad \arg(\lambda) = (\alpha - 2k)\pi \quad \text{for some } k \in \mathbb{Z},$$

then $s_{1,2} = -\alpha/\tau$ is double and the rightmost root of (2.3) (remaining roots of (2.3) are not real). If

$$0 < |\lambda| < \big(\frac{\alpha}{\tau e}\big)^{\alpha} \quad \text{and} \quad \arg(\lambda) = (\alpha - 2k)\pi \text{ for some } k \in \mathbb{Z},$$

then (2.3) has a couple of simple real negative roots, the greater of them being rightmost (remaining roots of (2.3) are not real).

(iii) *The characteristic equation (2.3) has the zero root if and only if $\lambda = 0$.*

Furthermore, using (3.4)–(3.5) we can specify the distribution of characteristic roots of (2.3) with respect to the imaginary axis. Before doing this, we introduce the following areas in the complex plane.

For real parameters $0 < \alpha < 2$ and $\tau > 0$, we define the set $Q_0(\alpha, \tau)$ of all complex λ such that

$$|\arg(\lambda)| > \frac{\alpha\pi}{2} \quad \text{and} \quad |\lambda| < \Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2}}{\tau}\Big)^{\alpha}.$$

Further, for any positive integer m and real parameters $0 < \alpha < 4m+2$ and $\tau > 0$, we define the sets $Q_m(\alpha, \tau)$ of all complex λ such that either

$$\frac{\alpha\pi}{2} - 2m\pi < |\arg(\lambda)| \le \frac{\alpha\pi}{2} - (2m-2)\pi \quad \text{and} \quad |\lambda| < \Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau}\Big)^{\alpha},$$

or $|\arg(\lambda)| > \frac{\alpha\pi}{2} - 2m\pi$ and

$$\Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2} + (2m-2)\pi}{\tau}\Big)^{\alpha} < |\lambda| < \Big(\frac{|\arg(\lambda)| - \frac{\alpha\pi}{2} + 2m\pi}{\tau}\Big)^{\alpha}.$$

We add that the sets $Q_m(\alpha, \tau)$ $(m = 0, 1, \dots)$ are defined to be empty whenever $\alpha \ge 4m+2$.

Now, we can describe the location of the roots of (2.3) with respect to the imaginary axis in terms of the sets $Q_m(\alpha, \tau)$ (we utilize here the standard notation $\partial[Q_m(\alpha, \tau)]$ for their boundaries).

Proposition 3.2. *Let $\lambda \in \mathbb{C}$ and $\alpha, \tau \in \mathbb{R}^+$. Then there exist just m $(m = 0, 1, \dots)$ characteristic roots of (2.3) with a positive real part (while remaining roots have negative real parts) if and only if $\lambda \in Q_m(\alpha, \tau)$. Moreover, (2.3) has a root with the zero real part if $\lambda \in \partial[Q_m(\alpha, \tau)]$ for some $m = 0, 1, \dots$.*

The appropriate regions $Q_m(\alpha, \tau)$ are depicted in Figures 1 and 2.

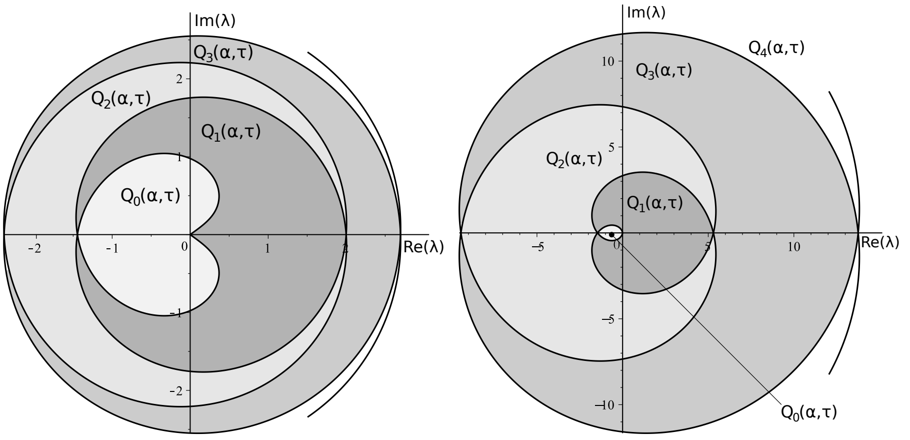

FIGURE 1. $\alpha = 0.4$ and $\tau = 1$ (left). $\alpha = 1.1$ and $\tau = 1$ (right)

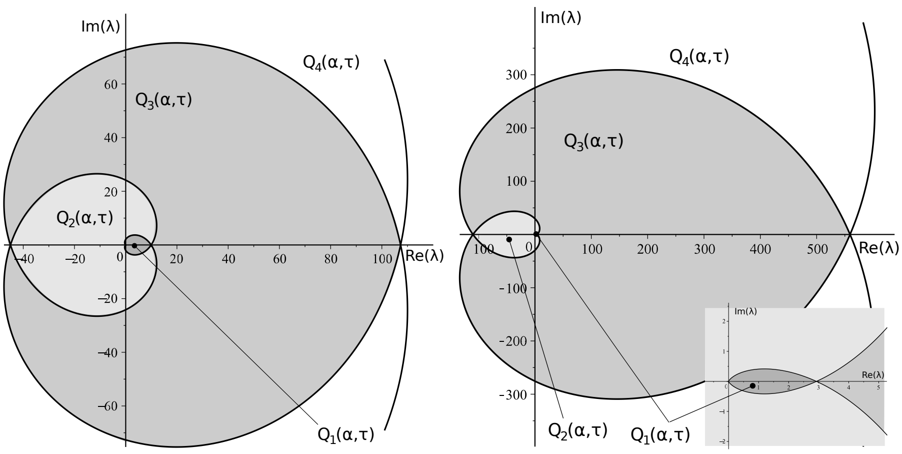

FIGURE 2. $\alpha = 2.1$ and $\tau = 1$ (left). $\alpha = 3.1$ and $\tau = 1$ (right)

Proof. We start with the proof of Proposition 3.1 and consider the characterization of roots s of (2.3) via (3.4)–(3.5). Obviously, (2.3) has a positive real root if $\arg(\lambda) = 0$ (i.e. λ is a positive real). In this case, the characteristic function

$$F(s) = s^\alpha - \lambda \exp\{-s\tau\}$$

is strictly increasing for all $s \geq 0$ with $F(0) = -\lambda < 0$ and $F(\infty) = \infty$, hence there is a unique positive real root s_1 of (2.3). To show its dominance, we consider remaining roots s_i of (2.3) with a positive real parameter λ. Then (3.5) yields

$$(s_1)^\alpha = \lambda \exp\{-s_1\tau\}, \quad |s_i|^\alpha = \lambda \exp\{-|s_i|\tau \cos[\arg(s_i)]\}.$$

From here, we obtain

$$\Big(\frac{s_1}{|s_i|}\Big)^\alpha = \exp\{(-s_1 + |s_i| \cos[\arg(s_i)])\tau\}. \tag{3.6}$$

Assume that s_1 is not the rightmost root of (2.3), i.e. $|s_i|\cos[\arg(s_i)] \ge s_1$ for some root s_i of (2.3). Then

$$\frac{s_1}{|s_i|} < 1 \quad \text{and} \quad \exp\{(-s_1 + |s_i|\cos[\arg(s_i)])\tau\} \ge 1$$

which contradicts (3.6). This proves Proposition 3.1(i).

Similarly, (3.4)–(3.5) imply that (2.3) has a negative real root s if and only if

$$\arg(\lambda) = (\alpha - 2k)\pi \quad \text{for some } k \in \mathbb{Z}$$

and

$$|s|^\alpha = |\lambda| \exp\{|s|\tau\}.$$

Put $r = |s|$ and $G(r) = r^\alpha - |\lambda|\exp\{r\tau\}$, $r \ge 0$. Then $G(0) = -|\lambda| < 0$, $G(\infty) = -\infty$ and G is increasing in $(0, r^*)$ and decreasing in (r^*, ∞) for a suitable $r^* > 0$. Thus G has (one or two) positive roots if and only if $G(r^*) \ge 0$. In particular, G has a unique positive root r^* if and only if $G(r^*) = G'(r^*) = 0$, i.e.

$$(r^*)^\alpha - |\lambda|\exp\{r^*\tau\} = \alpha(r^*)^{\alpha-1} - |\lambda|\tau\exp\{r^*\tau\} = 0\,.$$

From here, we obtain

$$r^* = \frac{\alpha}{\tau} \quad \text{and} \quad |\lambda| = \Big(\frac{\alpha}{\tau \mathrm{e}}\Big)^\alpha.$$

Obviously, if

$$|\lambda| < \Big(\frac{\alpha}{\tau \mathrm{e}}\Big)^\alpha,$$

then G has two real positive roots $r_1 < r_2$. We show that $s_1 = -r_1$ is the rightmost root of (2.3), i.e $s_1 > |s_i|\cos[\arg(s_i)]$ for all remaining roots s_i $(i = 2, 3, \dots)$ of (2.3). Indeed, by (3.5),

$$|s_1|^\alpha = |\lambda|\exp\{|s_1|\tau\} \quad \text{and} \quad |s_i|^\alpha = |\lambda|\exp\{-|s_i|\tau\cos[\arg(s_i)]\}\,.$$

Then $|s_1| < |s_i|$, i.e. $|s_1| + |s_i|\cos[\arg(s_i)] < 0$. Analogously we can show the dominance of a double real root $s_{1,2}$ (if exists). This proves Proposition 3.1 (ii). The assertion of Proposition 3.1(iii) is trivial.

Now, we show the validity of Proposition 3.2. Since the case of real characteristic roots of (2.3) has been discussed previously, we first search the roots s with $0 < \arg(s) \le \pi/2$. Then (3.4)–(3.5) can be reduced to

$$|s| = \frac{\arg(\lambda) - \alpha\arg(s) + 2k\pi}{\tau\sin[\arg(s)]}, \tag{3.7}$$

$$\begin{aligned}\Big(\frac{\arg(\lambda) - \alpha\arg(s) + 2k\pi}{\tau\sin[\arg(s)]}\Big)^\alpha - |\lambda|\exp\big\{(-\arg(\lambda)\\ + \alpha\arg(s) - 2k\pi)\,\mathrm{cotan}[\arg(s)]\big\} = 0\,.\end{aligned} \tag{3.8}$$

We denote the left-hand side of (3.8) by $H_k = H_k(\arg(s))$. Then

$$H_k(0^+) = \infty, \quad H_k(\pi/2) = \Big(\frac{\arg(\lambda) - \alpha\pi/2 + 2k\pi}{\tau}\Big)^\alpha - |\lambda|$$

and $H_k(\arg(s))$ decreases as $\arg(s)$ increases from 0 to $\pi/2$. This implies that (3.7)–(3.8) has just m couples of solutions with $|s| > 0$ and $0 < \arg(s) \le \pi/2$ if and only if either

$$\frac{\alpha\pi}{2} - 2m\pi < \arg(\lambda) \le \frac{\alpha\pi}{2} - (2m-2)\pi \quad \text{and} \quad H_m(\pi/2) > 0\,,$$

or

$$\arg(\lambda) > \frac{\alpha\pi}{2} - 2m\pi \quad \text{and} \quad H_m(\pi/2) > 0 > H_{m-1}(\pi/2)\,.$$

If $-\pi/2 \le \arg(s) < 0$, then we obtain the same conclusion with $\arg(\lambda)$ replaced by $-\arg(\lambda)$. This implies the main part of the assertion. The supplement on existence of purely imaginary roots of (2.3) follows from continuous dependence of roots s on parameter λ. Alternatively, it can be obtained via the standard D-decomposition method. □

4. Main results

In this section, we derive and formulate fractional-order analogues to Theorems 1.1 and 1.3.

Theorem 4.1. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and $\tau \in \mathbb{R}^+$. Then the following statements are equivalent:*

(a) *All non-trivial solutions of* (1.6) *are non-oscillatory;*
(b) *The characteristic equation* (2.2) *admits only real roots or roots with a negative real part;*
(c) *A has all eigenvalues lying in $Q_0(\alpha,\tau) \cup (Q_1(\alpha,\tau)\cap\mathbb{R}) \cup \{0\}$.*

Proof. Theorem 2.2 and Lemma 2.5 imply that every solution of (1.6)–(1.8) can be expressed as

$$y(t) = T^{-1}\sum_{j=0}^{\lceil\alpha\rceil-1} \mathrm{D}_0^{\alpha-j-1}\mathcal{G}(t)T\phi_j + T^{-1}\int_{-\tau}^{0}\mathcal{G}(t-\tau-u)JT\phi(u)\mathrm{d}u\,, \tag{4.1}$$

where $\mathcal{G}$ is a matrix function introduced in Lemma 2.5, J is a Jordan form of the system matrix A and T is the corresponding regular projection matrix, i.e. $A = TJT^{-1}$. Employing (4.1) and Lemma 2.5, we can see that every component of y is a linear combination of terms derived from elements of $\mathcal{G}$. We distinguish two cases with respect to (non)zeroness of eigenvalues λ_i of A.

First, let $\lambda_i \neq 0$ for all $i = 1,\dots,n$ (n being the number of distinct eigenvalues of A). Then the elements of matrices $\mathrm{D}_0^{\alpha-j-1}\mathcal{G}(t)$ ($j = 0,\dots,\lceil\alpha\rceil-1$) can be asymptotically expanded via the relation

$$\begin{aligned} &\mathrm{D}_0^{\alpha-j-1}G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t) = G_{\alpha,j+1}^{\lambda_i,\tau,m}(t)\\ &= \sum_{w=1}^{N}\sum_{\ell=0}^{mk_w} t^\ell \exp\{s_w t\} b_{w,\ell}\big(1-\frac{m\tau}{t}\big)^\ell \exp\{-s_w m\tau\}\\ &\quad + t^{j-\alpha}\frac{(-1)^{m+1}(1+\tau/t)^{j-\alpha}}{\lambda_i^{m+1}\Gamma(j-\alpha+1)} + \mathcal{O}(t^{j-2\alpha}) \quad \text{as } t\to\infty\,, \end{aligned} \tag{4.2}$$

where s_w ($w = 1,2,\dots,N$) are roots of (2.3) with the largest real parts ordered as $\Re(s_w) \ge \Re(s_{w+1})$, N is any positive integer satisfying $\Re(s_N) < 0$, k_w is multiplicity of s_w and $b_{w,\ell}$ are suitable real constants (see $a_{i,j}$ in Lemma 2.6(ii)). Similarly, the elements of the matrix $\int_{-\tau}^{0} \mathcal{G}(t-\tau-u)JT\phi(u)\mathrm{d}u$ have the expansions

$$\begin{aligned}
&\int_{-\tau}^{0} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t-\tau-u)\hat{\phi}^p(u)\mathrm{d}u \\
&= \sum_{w=1}^{N}\sum_{\ell=0}^{mk_w} t^\ell \exp\{s_w t\} c_{w,\ell}\lambda_i \int_{-\tau}^{0}\Big(1-\frac{(m+1)\tau}{t}-\frac{u}{t}\Big)^\ell \mathrm{e}^{-s_w((m+1)\tau+u)}\hat{\phi}^p(u)\mathrm{d}u \\
&\quad + t^{-\alpha-1}\int_{-\tau}^{0}\frac{(-1)^{m+1}(m+1)(1+\tau/t-u/t)^{-\alpha-1}}{\lambda_i^{m+1}\Gamma(-\alpha)}\hat{\phi}^p(u)\mathrm{d}u + \mathcal{O}(t^{-2\alpha-1})
\end{aligned} \tag{4.3}$$

as $t\to\infty$, where $\hat{\phi}^p(u)$ is pth row of the vector $JT\phi(u)$ and $c_{w,\ell}$ are suitable real constants (see $a_{i,j}$ in Lemma 2.6(ii)).

If $\lambda_i = 0$ for some $i = 1,\dots,n$, then the appropriate analogues of (4.2)–(4.3) involve only algebraic terms (see Lemma 2.6(i)). Now, we can prove the presented equivalencies:

(a)⇔(b): The property (a) holds if and only if, for any choice of ϕ, the dominating terms involved in (4.2) and (4.3) are non-oscillatory. We can see that all the algebraic terms from (4.2) and (4.3) are non-oscillatory and eventually dominating with respect to all exponential terms with negative real parts of their arguments. Contrary, an exponential term is eventually dominating provided its argument has a non-negative real part. Clearly, if such a case does occur, the solution y of (1.6) is non-oscillatory only if the imaginary parts of the corresponding arguments are zero. By (4.2) and (4.3), the discussed arguments of the exponential terms are expressed via roots of (2.2), which yields equivalency of (a) and (b).

(b)⇔(c): This equivalency follows immediately from Propositions 3.1 and 3.2. □

In the scalar case, when (1.6) becomes

$$\mathrm{D}_0^\alpha y(t) = ay(t-\tau), \quad t\in(0,\infty)\,, \tag{4.4}$$

a being a real scalar, we obtain the following explicit characterization of non-existence of a non-trivial oscillatory solution.

Corollary 4.2. *Let $a\in\mathbb{R}$, $\alpha\in\mathbb{R}^+\setminus\mathbb{Z}^+$ and $\tau\in\mathbb{R}^+$. All non-trivial solutions y of (4.4) are non-oscillatory if and only if*

$$0<\alpha<2 \quad \textit{and} \quad -\Big(\frac{(2-\alpha)\pi}{2\tau}\Big)^\alpha < a < \Big(\frac{(4-\alpha)\pi}{2\tau}\Big)^\alpha,$$

or

$$2<\alpha<4 \quad \textit{and} \quad 0<a<\Big(\frac{(4-\alpha)\pi}{2\tau}\Big)^\alpha.$$

Remark 4.3. In the first-order case, the value $a=-1/(\tau\mathrm{e})$ is of a particular importance: crossing this value, the (negative) real roots of the associated characteristic equation disappear and all solutions of (1.3) become oscillatory for $a<-1/(\tau\mathrm{e})$. In the fractional-order case, the (negative) real roots disappear for $a<-(\alpha/(\tau\mathrm{e}))^\alpha$. However, such roots have no impact on oscillatory behaviour of the solutions of (4.4) because the exponential terms with negative arguments involved in the formulae (4.1)–(4.3) are eventually suppressed by algebraic terms.

By Theorem 4.1, if all roots of (2.2) have negative real parts, then all non-trivial solutions of (1.6) are non-oscillatory. Therefore, we give an explicit characterization of this assumption and thus provide a fractional-order analogue to Theorem 1.3.

Theorem 4.4. *Let $A \in \mathbb{R}^{d\times d}$ and $\alpha, \tau \in \mathbb{R}^+$. Then the following statements are equivalent*

(a) *Any solution y of* (1.6) *tends to zero as $t \to \infty$;*
(b) *The characteristic equation* (2.2) *has all roots with negative real parts;*
(c) *All eigenvalues λ_i $(i = 1, \dots, d)$ of A are nonzero and satisfy*

$$\tau|\lambda_i|^{1/\alpha} < |\arg(\lambda_i)| - \alpha\pi/2 .$$

Moreover, if $\alpha \notin \mathbb{Z}^+$, then the convergence to zero is of algebraic type; more precisely, for any solution y of (1.6) *there exists a suitable integer $j \in \{0, \dots, \lceil\alpha\rceil\}$ such that $|y(t)| \sim t^{j-\alpha-1}$ as $t \to \infty$ (the symbol $\sim$ stands for asymptotic equivalency).*

Proof. (a)⇔(b): If $\lambda_i = 0$ for some $i = 1, \dots, d$, then the appropriate analogues of (4.2) and (4.3) yield that there is always a constant term involved in these expansions (this constant is nonzero if ϕ_0 is nonzero), hence the property (a) is not true. Obviously, the property (b) cannot occur as well provided $\lambda_i = 0$ for some $i = 1, \dots, d$. Thus, without loss of generality, we may assume $\lambda_i \neq 0$ for all $i = 1, \dots, d$.

The statement (a) is valid if and only if (4.2) and (4.3) do not contain any terms with a non-negative real part of the argument, which directly yields the equivalency (see also [11]).

(b)⇔(c): It is a direct consequence of Proposition 3.2.

Consequently, since all the exponential terms in (4.2) and (4.3) have a negative argument, they are suppressed by the algebraic terms. The presence of the term behaving like $t^{j-\alpha-1}$ for $j = 1, \dots, \lceil\alpha\rceil$ as $t \to \infty$ is determined by values ϕ_{j-1}. If $\phi_{j-1} = 0$ for all $j = 1, \dots, \lceil\alpha\rceil$, the integral term (4.3) becomes dominant. The integrability of ϕ enables us to write

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t^{-\alpha-1}} \Big| \int_{-\tau}^{0} G_{\alpha,\alpha}^{\lambda_i,\tau,m}(t-\tau-u)\hat{\phi}^p(u)\mathrm{d}u\Big| \\
&= \Big| \int_{-\tau}^{0} \lim_{t\to\infty} \frac{(-1)^{m+1}(m+1)(1+\tau/t-u/t)^{-\alpha-1}}{\lambda_i^{m+1}\Gamma(-\alpha)}\hat{\phi}^p(u)\mathrm{d}u\Big| \\
&= K\Big| \int_{-\tau}^{0} \hat{\phi}^p(u)\mathrm{d}u\Big|
\end{aligned}$$

for a suitable real K, therefore the integral term behaves like $t^{-\alpha-1}$ as $t \to \infty$. This completes the proof. □

For the case of scalar equation (4.4), we obtain the following result.

Corollary 4.5. *Let $a \in \mathbb{R}$ and $\alpha, \tau \in \mathbb{R}^+$. All solutions y of* (4.4) *tend to zero if and only if*

$$\alpha < 2 \quad \textit{and} \quad -\Big(\frac{(2-\alpha)\pi}{2\tau}\Big)^{\alpha} < a < 0 .$$

In particular, an interesting link between Theorems 4.1 and 4.4 is provided by the following assertion.

Corollary 4.6. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and $\tau \in \mathbb{R}^+$. If* (1.6) *has a nontrivial oscillatory solution, then it has also a solution which does not tend to zero as $t \to \infty$.*

Remark 4.7. In fact, formulae (4.1)–(4.3) reveal that any non-trivial solution of (1.6) tending to zero is non-oscillatory. Moreover, the solutions tending to zero pose an algebraic decay (there is no solution with an exponential decay).

5. Other oscillatory properties of (4.4)

In the classical integer-order case, oscillation argumentation often uses the fact that $\exp(s_w t)$ is a solution of (1.3) for any root s_w of the corresponding characteristic equation

$$s - a\exp\{-s\tau\} = 0\,. \tag{5.1}$$

In particular, if (5.1) admits a real root, then (1.3) has (via appropriate choice of ϕ) a non-oscillatory solution. In the fractional-order case, no such a direct connection for the influence study of characteristic roots of

$$s^\alpha - a\exp\{-s\tau\} = 0 \tag{5.2}$$

on the oscillatory behaviour of (4.4) is available. Nevertheless, as we can see from (4.1)–(4.3), the exponential functions generated by characteristic roots of (5.2) again play an important role in qualitative analysis of solutions of (4.4). Using this fact, we are able to describe some oscillatory properties of (4.4) with respect to asymptotic relationship between the studied solutions and the corresponding exponential functions. To specify this relationship, we introduce the following asymptotic classifications of solutions of (4.4).

Definition 5.1. Let $a \in \mathbb{R}$ and $\alpha, \tau \in \mathbb{R}^+$. The solution y of (4.4) is called major solution, if it satisfies the asymptotic relationship

$$\limsup_{t\to\infty} \Big|\frac{y(t)}{t^{k_1}\exp\{s_1 t\}}\Big| > 0\,,$$

where s_1 is the rightmost root of (5.2) and k_1 its algebraic multiplicity.

Definition 5.2. Let $a \in \mathbb{R}$, $\alpha, \tau \in \mathbb{R}^+$, s_w $(w = 1, 2, \dots)$ be roots of (5.2) with ordering $\Re(s_w) \geq \Re(s_{w+1})$ and let k_w $(w = 1, 2, \dots)$ be the corresponding algebraic multiplicities. The solution y of (4.4) is called m-minor solution, if it satisfies the asymptotic relationships

$$\limsup_{t\to\infty} \Big|\frac{y(t)}{t^{k_m}\exp\{s_m t\}}\Big| = 0 \quad \text{and} \quad \limsup_{t\to\infty} \Big|\frac{y(t)}{t^{k_{m+1}}\exp\{s_{m+1} t\}}\Big| > 0\,.$$

Remark 5.3. The notions of the major and m-minor solutions are not just theoretical, but such solutions can be constructively obtained via appropriate choice of the initial function ϕ. For example, if s_1 is simple with a non-negative real part, then, by (4.1)–(4.3), the major solution occurs if ϕ meets the condition

$$\sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j b_{1,j} + a c_{1,0} \int_{-\tau}^{0} \phi(u)\exp\{-s_1(\tau+u)\}\mathrm{d}u \neq 0$$

where $b_{1,j}$, $c_{1,0}$ have the same meaning as in (4.2)–(4.3). Clearly, such a condition is satisfied by infinitely many initial functions, e.g. by $\phi(u) = 1$, $\phi_j = 0$ $(j = 1, \dots, \lceil\alpha\rceil - 1)$ and $\phi_0 \neq -a c_{1,0}(1 - \exp\{-s_1\tau\})/(b_{1,0}s_1)$. Similarly, m-minor solution is characterized by the conditions

$$\sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j b_{w,j} + a c_{w,0} \int_{-\tau}^{0} \phi(u)\exp\{-s_w(\tau+u)\}\mathrm{d}u = 0 \quad \text{for } w = 1, \dots, m\,,$$

$$\sum_{j=0}^{\lceil\alpha\rceil-1} \phi_j b_{m+1,j} + a c_{m+1,0} \int_{-\tau}^{0} \phi(u) \exp\{-s_{m+1}(\tau+u)\} \mathrm{d}u \neq 0$$

provided s_w $(w = 1,\dots,m+1)$ are simple roots and $b_{w,j}$, $c_{w,0}$, $b_{m+1,j}$, $c_{m+1,0}$ have the same meaning as in (4.2)–(4.3).

Using the notions of major and m-minor solutions, we can formulate in a more detail assertions revealing the relation between oscillatory properties of (4.4) and location of roots of (5.2) in the complex plane.

Lemma 5.4. *Let $a \in \mathbb{R} \setminus (Q_0(\alpha,\tau) \cup \{0\})$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $\tau \in \mathbb{R}^+$ and let s_w $(w = 1,2,\dots)$ be roots of (5.2) with ordering $\Re(s_w) \geq \Re(s_{w+1})$. Then the major solutions of (4.4) do not tend to zero and there exists $M > 0$ such that all m-minor solutions of (4.4) are non-oscillatory and tend to zero as $t \to \infty$ for all $m \geq M$. Furthermore, it holds:*

- (i) *If $a \leq -((2-\alpha)\pi/(2\pi))^\alpha$ for $\alpha < 2$ or $a < 0$ for $\alpha > 2$, then all major solutions of (4.4) are oscillatory.*
- (ii) *If $\alpha < 4$ and $0 < a < ((4-\alpha)\pi/(2\pi))^\alpha$, then all non-trivial solutions of (4.4) are non-oscillatory.*
- (iii) *If $\alpha < 4$ and $a = ((4-\alpha)\pi/(2\pi))^\alpha$, then all major solutions of (4.4) are non-oscillatory. Moreover, all 1-minor solutions are oscillatory and bounded.*
- (iv) *If $a > ((4-\alpha)\pi/(2\pi))^\alpha$ for $\alpha < 4$ or $a > 0$ for $\alpha > 4$, then all major solutions of (4.4) are non-oscillatory. Moreover, all 1-minor solutions are oscillatory and unbounded.*

Proof. The first part of the assertion follows from the expansion of solution y of (4.4) based on (4.2)–(4.3). By Proposition 3.2, the rightmost root s_1 has a non-negative real part, therefore the major solutions involve, as a dominant term, an exponential function which does not tend to zero. Using a technique similar to that in Remark 5.3 we can always eliminate all terms in the asymptotic expansion of y corresponding to the characteristic roots with a non-negative real part, and, thus, construct non-oscillatory m-minor solutions algebraically tending to zero. Further utilization of this arguments enables us to obtain even more detailed results:

(i) The value $a \leq -((2-\alpha)\pi/(2\pi))^\alpha$ for $\alpha < 2$ or $a < 0$ for $\alpha > 2$ guarantees that the rightmost root s_1 has a non-negative real part and non-zero imaginary part (see Propositions 3.1 and 3.2), therefore the major solutions are oscillatory.

(ii)–(iv) If $a > 0$, Proposition 3.1(i) implies that the rightmost root s_1 is a positive real, therefore the major solutions are non-oscillatory. Eliminating the rightmost root s_1 as in Remark 5.3, the terms corresponding to s_2 become dominant and, again using Proposition 3.2, we obtain the parts (ii)–(iv). □

Remark 5.5. For $a = 0$, (5.2) has the only root $s_1 = 0$ with multiplicity $\lceil\alpha\rceil$ and the qualitative behaviour is implied directly by Lemma 2.6(i). In particular, if $\alpha < 1$, then all non-trivial solutions of (4.4) are constant, i.e. they are bounded and non-oscillatory. If $\alpha > 1$, then all non-trivial solutions of (4.4) are non-oscillatory. Moreover, if $\phi_j = 0$ for all $j = 1,\dots,\lceil\alpha\rceil - 1$, then the solutions are bounded, otherwise being unbounded.

It is of a particular interest to emphasize that unlike the integer-order case, there is no combination of entry parameters such that all the solutions of (4.4) are oscillatory. In fact, (4.4) has always infinitely many non-oscillatory solutions.

6. Concluding remarks

We have discussed oscillatory and related asymptotic properties of solutions of the fractional delay differential system (1.6) as well as of the corresponding scalar equation (4.4). The obtained oscillation results qualitatively differ from those known from the classical oscillation theory of (integer-order) delay differential equations. We survey here the most important notes related to this phenomenon.

First, while the appropriate criteria from the classical theory (such as Theorem 1.1) formulate necessary and sufficient conditions for oscillation of all solutions, their fractional counterparts (Theorem 4.1) present conditions for non-oscillation of all non-trivial solutions. In particular, our analysis shows that (1.6) cannot admit only oscillatory solutions. Secondly, considering (1.6), one can observe a close resemblance between non-oscillation of all non-trivial solutions and convergence to zero of all solutions (this property defines asymptotic stability of the zero solution of (1.6)). The latter property is sufficient for non-oscillation of all non-trivial solutions of (1.6) and, moreover, it is not far from being also a necessary one. These features (along with some other precisions made in Section 5) demonstrate that (non)oscillatory properties of (1.6) qualitatively depend on the fact if the value α is integer or non-integer. In particular, Corollary 4.2 implies that the endpoints of corresponding non-oscillation intervals depend continuously on changing non-integer derivative order α; when α is crossing the integer-order value, a sudden change in oscillatory behaviour occurs (see Corollary 1.2). Note that despite of some introductory papers on oscillation of (1.6) and other related fractional delay differential equations (see, e.g. [1, 17]), these properties have not been reported yet.

On the other hand, one can observe that dependence of stability areas of (1.6) on changing derivative order is "continuous". As illustrated via Figures 1–4, this area is continuously becoming smaller, starting from the circle (corresponding to the non-differential case when $\alpha = 0$) to the empty set (when $\alpha = 2$). We add that the way to stability remains closed for all real $\alpha \geq 2$. From this viewpoint, considerations of (1.6) with non-integer derivative order enable a better understanding of classical stability results on (1.6) with integer α.

The method utilized in our oscillation analysis indicates that the main reason of a rather strange oscillatory behaviour of (1.6) with non-integer α is hidden in the algebraic rate of convergence of its solutions to zero (compared to the exponential rate known from the integer-order case). Since this type of convergence has been earlier described not only for other types of fractional delay equations (see [2, 9, 11, 12]), but also for fractional equations without delay (see [4, 13, 14, 16]), the above described oscillatory behaviour might be typical for a more general class of fractional differential equations.

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Appendix C

Paper on overview for one-term FDDS [37] (Math Appl, 2020)

Our work on [6,10] brought us significant insights into the stability and asymptotics of one-term FDDS, but not all findings aligned with the concepts of previous papers. Consequently, [37] (my author's share 100 %) aimed to consolidate the topic and to extend it.

In this paper, we provided a comprehensive overview of the stability and asymptotics theory for one-term FDDS, considering the two most common definitions of fractional derivatives: Caputo and Riemann-Liouville. Building on techniques adopted in our prior research, we derived optimal stability conditions based on the position of eigenvalues in the complex plane. We elaborated on the implications of using different definitions of fractional derivative, detailing distinctions on the stability boundary and in overall asymptotic behaviour.

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ON STABILITY OF DELAYED DIFFERENTIAL SYSTEMS OF ARBITRARY NON-INTEGER ORDER

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Abstract. This paper summarizes and extends known results on qualitative behavior of solutions of autonomous fractional differential systems with a time delay. It utilizes two most common definitions of fractional derivative, Riemann–Liouville and Caputo one, for which optimal stability conditions are formulated via position of eigenvalues in the complex plane. Our approach covers differential systems of any non-integer orders of the derivative. The differences in stability and asymptotic properties of solutions induced by the type of derivative are pointed out as well.

1. Introduction

In many areas of science and technology we often meet problems which are well described by differential systems with a time delay. Examples of such situations might be reaction time of technical and chemical systems or heredity in population dynamics. Qualitative theory for these equations is summarized in, e.g. [2,5]. The study of delayed systems involving viscoelasticity, anomalous diffusion or control theory naturally suggests to enrich our models with derivatives of non-integer order which proved to be very effective in these areas (see, e.g. [4,8]).

This is the main motivation for our study of two delayed systems which can be written as

$$\mathbf{D}_0^{\alpha} y(t) = Ay(t-\tau)\,, \quad t \in (0,\infty),\ \alpha \in \mathbb{R}^+ \setminus \mathbb{Z}, \tag{1.1}$$

$$y(t) = \phi(t)\,, \quad t \in [-\tau, 0], \tag{1.2}$$

$$\mathbf{D}_0^{\alpha-k} y(t)\big|_{t=0} = y_{\alpha-k}\,, \quad k = 1,\dots,\lceil\alpha\rceil \tag{1.3}$$

and

$${}^{C}\mathrm{D}_0^{\alpha} y(t) = Ay(t-\tau)\,, \quad t \in (0,\infty),\ \alpha \in \mathbb{R}^+ \setminus \mathbb{Z}, \tag{1.4}$$

$$y(t) = \phi(t)\,, \quad t \in [-\tau, 0], \tag{1.5}$$

$$y^{(\lceil\alpha\rceil-k)}(0) = y_{\lceil\alpha\rceil-k}\,, \quad k = 1,\dots,\lceil\alpha\rceil\,, \tag{1.6}$$

where $\mathbf{D}_0^{\alpha}$ and ${}^{C}\mathrm{D}_0^{\alpha}$ denote the so-called Riemann–Liouville and Caputo differential operators of order α, respectively. Further, $A \in \mathbb{R}^{d\times d}$ is a constant $d \times d$ matrix, $y_{\cdot} \in \mathbb{R}^d$ are constant vectors and $\tau > 0$ is a constant delay. As usual for delayed

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equations, the initial condition is given by $\phi \in L^1[-\tau, 0]$ (componentwise) and the use of fractional derivatives allows us to prescribe also initial values for $t = 0$ separately. We intentionally leave out the integer-order values of α since they coincide with the known classical cases.

A serious qualitative analysis of such equations is being performed less than two decades. It spans across scalar and vector cases, various methods like D-decomposition or Laplace transform are used. For more details we refer to [1,3,6,9] which are the main sources for this paper.

The paper is organized as follows. In Section 2 we outline some basic preliminary results useful in our further considerations. Section 3 is devoted to the discussion of solution representations and their comparison. The main results are concentrated in Section 4 where we summarize known facts as well as derive some original ones. Section 5 concludes the paper with a few final remarks.

2. Preliminaries

Let f be a real function. We use the standard definition of fractional integral of order $\gamma > 0$

$$\mathrm{I}_0^{\gamma} f(t) = \int_0^t \frac{(t-\xi)^{\gamma-1}}{\Gamma(\gamma)} f(\xi) \mathrm{d}\xi, \quad t \geq 0\,.$$

We employ both the wide used definitions of fractional derivative of order $\alpha > 0$ called the Riemann-Liouville and Caputo derivative introduced as

$$\mathbf{D}_0^{\alpha} f(t) = \frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} \left(\mathrm{I}_0^{\lceil\alpha\rceil-\alpha} f(t) \right), \quad t \geq 0\,,$$
$${}^{C}\mathrm{D}_0^{\alpha} f(t) = \mathrm{I}_0^{\lceil\alpha\rceil-\alpha} \left(\frac{\mathrm{d}^{\lceil\alpha\rceil}}{\mathrm{d}t^{\lceil\alpha\rceil}} f(t) \right), \quad t \geq 0\,,$$

respectively. Additionally, we put ${}^{C}\mathrm{D}_0^{0} f(t) = \mathbf{D}_0^{0} f(t) = f(t)$ (for more information on fractional operators we refer, e.g. to [4,8]).

The key tool, utilized throughout this paper, is the Laplace transform which is, for f, introduced as

$$\mathcal{L}(f(t))(s) = \int_0^{\infty} \exp\{-st\} f(t) \mathrm{d}t, \quad s \in \mathbb{C}$$

provided the integral converges. To perform the transform of (1.1) and (1.4), we need a clear view on Laplace transform of a function with shifted (delayed) argument which is given by

$$\begin{aligned} &\mathcal{L}(f(t-\tau)h(t-\tau))(s) = \exp\{-\tau s\} \mathcal{L}(f(t))(s), \quad \tau > 0, \\ &\mathcal{L}(f(t-\tau))(s) = \exp\{-\tau s\} \mathcal{L}(f(t))(s) + \exp\{-\tau s\} \int_{-\tau}^{0} \exp\{-st\} f(t) \mathrm{d}t, \quad \tau > 0\,. \end{aligned} \tag{2.1}$$

Also, using the formulae for Laplace transform of convolution and power function

$$\mathcal{L}\left(\int_0^t f(t-\xi)g(\xi)\mathrm{d}\xi\right)(s) = \mathcal{L}(f(t))(s)\cdot\mathcal{L}(g(t))(s),$$
$$\mathcal{L}\left(\frac{t^\eta}{\Gamma(\eta+1)}\right)(s) = s^{-\eta-1}, \quad \eta > -1,$$

we can see the origin of Laplace transforms of fractional operators

$$\mathcal{L}(\mathrm{I}_0^\gamma f(t))(s) = s^{-\gamma}\mathcal{L}(f(t))(s), \quad \gamma > 0,$$
$$\mathcal{L}(\mathbf{D}_0^\alpha f(t))(s) = s^\alpha\mathcal{L}(f(t))(s) - \sum_{k=1}^{\lceil\alpha\rceil} s^{k-1}\mathbf{D}_0^{\alpha-k}f(t)\big|_{t=0}, \quad \alpha > 0, \tag{2.2}$$
$$\mathcal{L}({}^C\mathrm{D}_0^\alpha f(t))(s) = s^\alpha\mathcal{L}(f(t))(s) - \sum_{k=1}^{\lceil\alpha\rceil} s^{\alpha-k}f^{(k-1)}(0), \quad \alpha > 0. \tag{2.3}$$

The symbol h denotes the Heaviside step function defined as $h(\xi) = 1$ for $\xi \geq 0$ and $h(\xi) = 0$ for $\xi < 0$. When applied on a vector function, the Laplace transform is considered componentwise.

We note that the system matrix A of (1.1) and (1.4) can be rewritten with the use of a matrix Λ in a Jordan canonical form with the Jordan blocks on its diagonal as $A = T\Lambda T^{-1}$, where T is a regular real $d\times d$ matrix,

$$\Lambda = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_q \end{pmatrix}, \quad J_k = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & & \lambda_i & 1 \\ 0 & \cdots & & 0 & \lambda_i \end{pmatrix}, \quad k = 1,\dots,q,$$

and λ_i $(i = 1,\dots,n)$ are distinct eigenvalues of A. The number of Jordan blocks corresponding to λ_i is called geometric multiplicity of λ_i. The sum of the sizes of all Jordan blocks corresponding to λ_i is called algebraic multiplicity of λ_i.

Before we proceed to the next section, we recall the stability notions related to our linear fractional differential systems with a delay. The zero solution is said to be stable (asymptotically stable) if the solution of the system is bounded (tends to zero as $t \to \infty$) for any initial function $\phi \in L^1([-\tau, 0])$.

3. Solution representations for (1.1) and (1.4)

As in the integer-order case (see, e.g. [2,5]), an essential role is played by analogue of the fundamental matrix solution also for (1.1) and (1.4) (see, e.g. [1]). In order to simplify the notation dealing with the orders α greater than one, we introduce its generalization in form of the following functions

$$R_{\alpha,\beta}^{A,\tau}(t) = \mathcal{L}^{-1}\left((s^\alpha I - A\exp\{-s\tau\})^{-1}s^{\alpha-\beta}\right)(t), \quad \alpha\in\mathbb{R}^+\setminus\mathbb{Z},\ \beta\in\mathbb{R}^+$$

where $A \in \mathbb{R}^{d\times d}$ and I is the identity $d\times d$ matrix. Employing these R-functions, we arrive at the following solution representations.

Theorem 3.1. *The solution y_{RL} of (1.1)–(1.3) is given by*

$$y_{RL}(t) = \sum_{k=1}^{\lceil\alpha\rceil} R^{A,\tau}_{\alpha,\alpha-k+1}(t) y_{\alpha-k} + \int_{-\tau}^{0} R^{A,\tau}_{\alpha,\alpha}(t-\tau-u) A\phi(u)\mathrm{d}u.$$

Proof. Applying (2.1), (2.2) on (1.1)–(1.3), we get

$$\begin{aligned}
&\mathcal{L}(y(t))(s) \\
&= (s^\alpha I - A\exp\{-s\tau\})^{-1}\left[\sum_{k=1}^{\lceil\alpha\rceil} s^{k-1} y_{\alpha-k} + \int_{-\tau}^{0} \exp\{-s(t+\tau)\}A\phi(t)\mathrm{d}t\right] \\
&= \sum_{k=1}^{\lceil\alpha\rceil} \mathcal{L}(R^{A,\tau}_{\alpha,\alpha-k+1}(t))(s) y_{\alpha-k} + \int_{-\tau}^{0} \exp\{-s(t+\tau)\}\mathcal{L}(R^{A,\tau}_{\alpha,\alpha}(t))(s)A\phi(t)\mathrm{d}t
\end{aligned}$$

which yields the assertion. □

Theorem 3.2. *The solution y_C of (1.4)–(1.6) is given by*

$$y_C(t) = \sum_{k=1}^{\lceil\alpha\rceil} R^{A,\tau}_{\alpha,k}(t) y_{k-1} + \int_{-\tau}^{0} R^{A,\tau}_{\alpha,\alpha}(t-\tau-u) A\phi(u)\mathrm{d}u.$$

Proof. Analogously as above, applying (2.1), (2.3) on (1.4)–(1.6), we obtain

$$\begin{aligned}
&\mathcal{L}(y(t))(s) \\
&= (s^\alpha I - A\exp\{-s\tau\})^{-1}\left[\sum_{k=1}^{\lceil\alpha\rceil} s^{\alpha-k} y_{k-1} + \int_{-\tau}^{0} \exp\{-s(t+\tau)\}A\phi(t)\mathrm{d}t\right] \\
&= \sum_{k=1}^{\lceil\alpha\rceil} \mathcal{L}(R^{A,\tau}_{\alpha,k}(t))(s) y_{k-1} + \int_{-\tau}^{0} \exp\{-s(t+\tau)\}\mathcal{L}(R^{A,\tau}_{\alpha,\alpha}(t))(s)A\phi(t)\mathrm{d}t
\end{aligned}$$

which again concludes the proof. □

Remark 3.3. We can see that the integral terms involving the initial function ϕ are for y_{RL} and y_C identical. The difference occurs in the terms involving the local initial conditions. Although the Caputo case is more studied in the literature, in particular of order $\alpha \in (0,1]$ (see, e.g. [1,3,6]), the Riemann-Liouville one actually appears to be structurally closer to the classical case. Indeed, $R^{A,\tau}_{\alpha,\alpha}$ seems to be playing practically the same role as the fundamental matrix solution in integer-order delay differential equations.

It might look like Theorems 3.1 and 3.2 are not that much explicit since the R-functions are defined via the inverse Laplace transform. Now we show that these functions can be actually evaluated pretty straighforwardly.

Applying the Jordan canonical form theory, we can write

$$\mathcal{L}(R^{A,\tau}_{\alpha,\beta}(t))(s) = (s^\alpha I - A\exp\{-s\tau\})^{-1}s^{\alpha-\beta} = T(s^\alpha I - \Lambda\exp\{-s\tau\})^{-1}s^{\alpha-\beta}T^{-1}\,.$$

Clearly, the matrix $(s^\alpha I - \Lambda \exp\{-s\tau\})^{-1} s^{\alpha-\beta}$ is block diagonal with the blocks given by upper triangular strip matrices of the form

$$(s^\alpha I - J_k \mathrm{e}^{-s\tau})^{-1} s^{\alpha-\beta} = \begin{pmatrix} \frac{s^{\alpha-\beta}}{s^\alpha - \lambda_i \mathrm{e}^{-s\tau}} & \frac{\mathrm{e}^{-s\tau} s^{\alpha-\beta}}{(s^\alpha - \lambda_i \mathrm{e}^{-s\tau})^2} & \cdots & \frac{\mathrm{e}^{-(r-1)s\tau} s^{\alpha-\beta}}{(s^\alpha - \lambda_i \mathrm{e}^{-s\tau})^{r_k}} \\ 0 & \frac{s^{\alpha-\beta}}{s^\alpha - \lambda_i \mathrm{e}^{-s\tau}} & \ddots & \frac{\mathrm{e}^{-(r-2)s\tau} s^{\alpha-\beta}}{(s^\alpha - \lambda_i \mathrm{e}^{-s\tau})^{r_k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{s^{\alpha-\beta}}{s^\alpha - \lambda_i \mathrm{e}^{-s\tau}} \end{pmatrix}, \tag{3.1}$$

where J_k $(k = 1, \dots, q)$ is the k-th block of Λ and r_k is its size. It was proven in [1] that the elements of this matrix can be expressed as

$$\frac{\exp\{-ms\tau\} s^{\alpha-\beta}}{(s^\alpha - \lambda \exp\{-s\tau\})^{m+1}} = \mathcal{L}(G^{\lambda,\tau,m}_{\alpha,\beta}(t))(s)$$

where

$$G^{\lambda,\tau,m}_{\alpha,\beta}(t) = \sum_{j=0}^{\lceil t/\tau - m - 1 \rceil} \binom{m+j}{j} \frac{\lambda^j (t - (m+j)\tau)^{\alpha(m+j)+\beta-1}}{\Gamma(\alpha(m+j)+\beta)}, \quad t > 0\,.$$

To summarize the previous considerations, we can write the following assertion.

Lemma 3.4. *Let $A \in \mathbb{R}^{d\times d}$, λ_i $(i = 1, \dots, n)$ be distinct eigenvalues of A and let p_i be the largest size of the Jordan block corresponding to the eigenvalue λ_i. Then the non-zero elements of matrix function $R^{A,\tau}_{\alpha,\beta}$ are linear combinations of scalar functions*

$$G^{\lambda_i,\tau,m}_{\alpha,\beta}(t), \quad m = 0, \dots, p_i - 1, \quad i = 1, \dots, n\,.$$

4. Main results

It is well known from the basic theory of the Laplace transform method that if all poles of the Laplace image of solutions (roots of the so-called characteristic equation) have negative real parts, then the zero solution of the studied equation is asymptotically stable (and their non-zero solutions tend to zero in an exponential rate). On the other hand, if there exists a pole with a positive real part, the corresponding zero solution is not stable (its absolute value tends to infinity exponentially). In the fractional case, it usually occurs a more complex situation, involving also singular points and poles with the zero real parts, which require a deeper analysis.

For our fractional problems (1.1) and (1.4), as it can be seen from the proof of Theorems 3.1 and 3.2, the characteristic equation takes the form

$$\det(s^\alpha I - A \exp\{-s\tau\}) = 0 \quad \text{or} \quad \prod_{i=1}^{n} (s^\alpha - \lambda_i \exp\{-s\tau\})^{w_i} = 0\,, \tag{4.1}$$

where λ_i $(i = 1, \dots, n)$ are distinct eigenvalues of A and w_i are the corresponding algebraic multiplicities. As we can see from (4.1) and (3.1), for further eigenvalues considerations it is sufficient to investigate the roots of the equation

$$p(s; \lambda) \equiv s^\alpha - \lambda \exp\{-s\tau\} = 0 \tag{4.2}$$

where λ is a complex parameter. Now, we perform a direct root analysis of (4.2). In particular, we formulate the optimal conditions on λ ensuring that (4.2) does not have any root with positive real part.

Lemma 4.1. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}$, $\tau > 0$ and $\lambda \in \mathbb{C}$. Then all the roots of (4.2) have negative real parts if and only if*

$$\alpha \in (0,2)\,, \quad |\mathrm{Arg}\,(\lambda)| > \frac{\alpha\pi}{2} \quad \textit{and} \quad |\lambda| < \left(\frac{|\mathrm{Arg}\,(\lambda)| - \alpha\pi/2}{\tau}\right)^{\alpha} \tag{4.3}$$

where $\mathrm{Arg}\,(\lambda) \in (-\pi, \pi]$ *is the principal argument of* λ.

Proof. The case $\lambda = 0$ is trivial since then (4.2) has only the zero solution which does not satisfy (4.3). Let $\lambda \neq 0$ and put

$$s = r\exp\{\mathrm{i}\varphi\}\,, \quad \lambda = \varrho\exp\{\mathrm{i}\psi\}$$

where $r = |s|$, $\varrho = |\lambda|$ and $\varphi, \psi \in (-\pi, \pi]$ are principal arguments of s, λ, respectively. Then we can write (4.2) for real and imaginary parts as a system of two equations in the form

$$r^\alpha \cos(\alpha\varphi) - \varrho\exp\{-r\tau\cos(\varphi)\}\cos(\psi - r\tau\sin(\varphi)) = 0, \tag{4.4}$$

$$r^\alpha \sin(\alpha\varphi) - \varrho\exp\{-r\tau\cos(\varphi)\}\sin(\psi - r\tau\sin(\varphi)) = 0. \tag{4.5}$$

Now, let us assume that (4.2) has a root with a non-negative real part, i.e. $|\varphi| \leq \pi/2$.

For $\varphi = 0$, we have $\psi = 0$ (i.e. $\lambda = \varrho$) from (4.5). Further, (4.4) implies, for r and ϱ, the relation $r^\alpha = \varrho\exp\{-r\tau\}$ which always allows to find an appropriate r to a given ϱ. Hence, (4.2) has a non-negative real root if and only if λ is a non-negative real.

Let $|\varphi| \in (0, \pi/2] \setminus \{\pi/\alpha\}$. Since $|\varphi| \neq \pi/\alpha$, we have $\psi - r\tau\sin(\varphi) \neq k\pi$ for any $k \in \mathbb{Z}$ and, by dividing and rearranging (4.4) and (4.5), we arrive at a new reformulation of (4.4), (4.5) in the form

$$\alpha\varphi = \psi - r\tau\sin(\varphi) + 2k\pi, \tag{4.6}$$

$$r^\alpha = \varrho\exp\{-r\tau\cos(\varphi)\} \tag{4.7}$$

for a suitable $k \in \mathbb{Z}$ (the replacement of $k\pi$ by $2k\pi$ is implied by positivity of r and ϱ). Further, by eliminating r from (4.6), (4.7), we get the equation for φ as

$$\left(\frac{\psi - \alpha\varphi + 2k\pi}{\tau\sin(\varphi)}\right)^{\alpha} = \varrho\exp\{(\alpha\varphi - \psi - 2k\pi)\cot(\varphi)\}\,.$$

As proven in [1] for $\alpha \in (0,1)$, the left-hand side is decreasing with respect to φ on $(0, \pi/2]$ with the lowest value at $\varphi = \pi/2$ for any k. The right-hand side is increasing with respect to φ on $(0, \pi/2]$ with the largest value at $\varphi = \pi/2$ for any k. It can be easily checked that the situation for $\alpha \geq 1$ is the same provided we put the left-hand side equal to zero for φ such that $\psi - \alpha\varphi + 2k\pi < 0$. Obviously, the existence of a root $\varphi \in (0, \pi/2]$ for at least one k is ensured if and only if

$$|\psi| \leq \frac{\alpha\pi}{2} \quad \text{or} \quad \varrho \geq \left(\frac{|\psi| - \alpha\pi/2}{\tau}\right)^{\alpha}. \tag{4.8}$$

We can see that $(4.8)_1$ is automatically satisfied for $\alpha \geq 2$, hence for $\alpha \geq 2$ there is always a root of (4.2) with a non-negative real part. We can see that, for $\alpha \in (0, 2)$, (4.8) is a complement of (4.3).

So far, we have not investigated the situation $|\varphi| = \pi/\alpha \leq \pi/2$. However, it can occur only for $\alpha \geq 2$ and in that case we already know that there is always a root of (4.2) with a non-negative real part.

Summarizing the previous arguments, we can conclude the proof. □

Lemma 3.4 shows that functions of the type $G_{\alpha,\beta}^{\lambda,\tau,m}$ play, for (1.1) and (1.4), an analogous role as exponential functions for integer-order systems. Hence, it is crucial to have a good uderstanding of asymptotic behavior of $G_{\alpha,\beta}^{\lambda,\tau,m}$ and its relation to (4.2) which is provided by the following assertion which slightly extends the result presented in [1].

Lemma 4.2. *Let $\lambda \in \mathbb{C}$, $\alpha, \beta, \tau \in \mathbb{R}^+$ and $m \in \mathbb{Z}$ be such that $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}$, $m \geq 0$. Further, let s_i $(i = 1, 2, \dots)$ be the roots of (4.2) with ordering $\Re(s_i) \geq \Re(s_{i+1})$ (in particular, s_1 is the zero with the largest real part).*

(i) *If $\lambda = 0$, then*

$$G_{\alpha,\beta}^{0,\tau,m}(t) = \frac{(t - m\tau)^{m\alpha+\beta-1}}{\Gamma(m\alpha + \beta)} h(t - m\tau).$$

(ii) *If λ is such that s_1 has negative real part, then*

$$\begin{aligned} G_{\alpha,\beta}^{\lambda,\tau,m}(t) = & \frac{(-1)^{m+1}}{\lambda^{m+1}\Gamma(\beta - \alpha)} (t + \tau)^{\beta-\alpha-1} \\ & + \frac{(-1)^{m+1}(m + 1)}{\lambda^{m+2}\Gamma(\beta - 2\alpha)} (t + 2\tau)^{\beta-2\alpha-1} + \mathcal{O}(t^{\beta-3\alpha-1}) \qquad \text{as } t \to \infty. \end{aligned}$$

(iii) *If λ is such that s_1 is purely imaginary or it has positive real part, then*

$$\begin{aligned} G_{\alpha,\beta}^{\lambda,\tau,m}(t) = & \sum_{j=0}^{m} (t - m\tau)^j (a_j \exp\{s_1(t - m\tau)\} \\ & + b_j \exp\{s_2(t - m\tau)\}) + \begin{cases} \mathcal{O}(t^m \exp\{\Re(s_3)t\}), & \text{if } \Re(s_3) \geq 0, \\ \mathcal{O}(t^{\beta-\alpha-1}), & \text{if } \Re(s_3) < 0 \end{cases} \\ & \text{as } t \to \infty \end{aligned}$$

where a_j, b_j are suitable nonzero complex constants $(j = 0, \dots, m)$.

Proof. The assertion was proved in [1] for the case $\alpha \in (0, 1)$. The generalization for $\alpha > 1$ is a tedious but direct analogue. □

Now we are in a position to formulate the main results of this paper. For the sake of lucidity, we introduce the following subset of complex numbers motivated by (4.3) as

$$\mathcal{S}_{\alpha,\tau} = \left\{ \lambda \in \mathbb{C} : |\lambda| < \left(\frac{|\mathrm{Arg}\,(\lambda)| - \alpha\pi/2}{\tau} \right)^{\alpha}, \ |\mathrm{Arg}\,(\lambda)| > \frac{\alpha\pi}{2} \right\},$$

which we call the stability region of (1.1) and (1.4).

Theorem 4.3. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $\tau > 0$. Further, let $p_0 \in \mathbb{Z}$ be the largest size of the Jordan block corresponding to the zero eigenvalue of A, where we put $p_0 = 0$ if A has only non-zero eigenvalues.*

(i) *The zero solution of (1.1) is asymptotically stable if and only if $\alpha \in (0, 2)$, all non-zero eigenvalues of A belong to $\mathcal{S}_{\alpha,\tau}$ and $p_0 < 1/\alpha$.*

(ii) *The zero solution of (1.1) is stable if and only if $\alpha \in (0, 2)$, all eigenvalues of A belong to $\mathrm{cl}\,(\mathcal{S}_{\alpha,\tau})$, all non-zero eigenvalues of A lying on $\partial\mathcal{S}_{\alpha,\tau}$ have the same algebraic and geometric multiplicities and $p_0 \le 1/\alpha$.*

Proof. Theorem 3.1 and Lemma 3.4 imply that the solution components of (1.1) are formed as linear combinations of functions

$$G^{\lambda_i,\tau,m}_{\alpha,\alpha-k+1}(t) \quad \text{and} \quad \int_{-\tau}^{0} G^{\lambda_i,\tau,m}_{\alpha,\alpha}(t-\tau-u)\phi_j(u)\mathrm{d}u\,, \tag{4.9}$$

where $k = 1, \dots, \lceil\alpha\rceil$, λ_i $(i = 1, \dots, n)$ are eigenvalues of A, m is a non-negative integer as specified in Lemma 3.4 and ϕ_j $(j = 1, \dots, d)$ are components of the initial function.

Lemma 4.1 implies that all roots of (4.1) have negative real part if and only if $\alpha \in (0, 2)$ and all eigenvalues belong to $\mathcal{S}_{\alpha,\tau}$. Moreover, (4.1) has at least one root with zero real part and other roots with a negative real part if and only if at least one eigenvalue lies on the boundary of $\mathcal{S}_{\alpha,\tau}$.

Thus, the asymptotic behavior of the solution can be derived from Lemma 4.2. The functions $(4.9)_1$ are described directly, we just point out that for $\lambda_i \in \mathcal{S}_{\alpha,\tau}$ the first term in the expansion cancels out due to the negative integer argument in the Gamma function, so that we obtain

$$G^{\lambda_i,\tau,m}_{\alpha,\alpha-k+1}(t) = \frac{(-1)^{m+1}(m+1)}{\lambda_i^{m+2}\Gamma(-\alpha-k+1)}(t+2\tau)^{-\alpha-k} + \mathcal{O}(t^{-2\alpha-k}) \qquad \text{as } t \to \infty.$$

Now, we investigate $(4.9)_2$. Employing the assuption $\phi \in L^1[-\tau, 0]$ and Lemma 4.2, we can distinguish several cases:

Let $\alpha \in (0, 2)$ and $\lambda_i \in \mathcal{S}_{\alpha,\tau}$. The second mean value theorem implies

$$\begin{aligned}\int_{-\tau}^{0} G^{\lambda_i,\tau,m}_{\alpha,\alpha}(t-\tau-u)\phi(u)\mathrm{d}u &= G^{\lambda_i,\tau,m}_{\alpha,\alpha}(t)\int_{-\tau}^{\xi}\phi(u)\mathrm{d}u\\ &= K_1(t+2\tau)^{-\alpha-1} + \mathcal{O}(t^{-2\alpha-1}) \qquad \text{as } t \to \infty\,,\end{aligned}$$

where $K_1 \in \mathbb{R}$ is non-zero and $\xi \in (-\tau, 0]$.

Now, let $\lambda_i = 0$. By the same approach we arrive at

$$\int_{-\tau}^{0} G^{0,\tau,m}_{\alpha,\alpha}(t-\tau-u)\phi(u)\mathrm{d}u = K_2(t-m\tau)^{(m+1)\alpha-1}\,,$$

where $K_2 \in \mathbb{R}$ is non-zero. This expression vanishes for $t \to \infty$, if and only if $m + 1 = p_0 < 1/\alpha$.

The cases for $\lambda_i \in \partial\mathcal{S}_{\alpha,\tau} \setminus \{0\}$ and $\lambda_i \notin \mathrm{cl}\,(\mathcal{S}_{\alpha,\tau})$ can be handled similarly. We arrive at the conclusion that $(4.9)_2$ is bounded, when the non-zero eigenvalue lying on the boundary of stability region has the same algebraic and geometric multiplicity. Otherwise the absolute value of $(4.9)_2$ increases polynomially (when the eigenvalue lies on the boundary) or exponentially. □

Theorem 4.4. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{Z}$ and $\tau > 0$. Further, let $p_0 \in \mathbb{Z}$ be the largest size of the Jordan block corresponding to the zero eigenvalue of A, where we put $p_0 = 0$ if A has only non-zero eigenvalues.*

(i) *The zero solution of (1.4) is asymptotically stable if and only if $\alpha \in (0, 2)$ and all eigenvalues of A belong to $\mathcal{S}_{\alpha,\tau}$.*

(ii) *The zero solution of (1.4) is stable if and only if $\alpha \in (0, 2]$, all eigenvalues of A belong to $\operatorname{cl}(\mathcal{S}_{\alpha,\tau})$, all non-zero eigenvalues of A lying on $\partial\mathcal{S}_{\alpha,\tau}$ have the same algebraic and geometric multiplicities and $p_0 \leq 2 - \lceil\alpha\rceil$.*

Proof. The idea of the proof is equivalent to that one of Theorem 4.3. In particular, the solution components of (1.4) are given by linear combinations of

$$G^{\lambda_i,\tau,m}_{\alpha,k}(t) \quad \text{and} \quad \int_{-\tau}^{0} G^{\lambda_i,\tau,m}_{\alpha,\alpha}(t-\tau-u)\phi_j(u)\mathrm{d}u\,, \tag{4.10}$$

where $k = 1, \ldots, \lceil\alpha\rceil$, λ_i $(i = 1, \ldots, n)$ are eigenvalues of A, m is a non-negative integer as specified in Lemma 3.4 and ϕ_j $(j = 1, \ldots, d)$ is a component of the initial function. Thus, we see that $(4.10)_2$ is the same as $(4.9)_2$ while $(4.10)_1$ differs with respect to $(4.9)_1$ due to the change of index. This causes only a different decay rate for $\lambda_i \in \operatorname{cl}(\mathcal{S}_{\alpha,\tau})$.

Overall, there is only one difference in stability behavior which occurs for $\lambda_i = 0$ when we have

$$G^{0,\tau,m}_{\alpha,k}(t) = \frac{(t-m\tau)^{m\alpha+k-1}}{\Gamma(m\alpha+k)}.$$

We can see that this function never tends to zero with $t \to \infty$ and it is bounded if and only if $m\alpha + k - 1 = 0$ which means $\lceil\alpha\rceil = 1$ (i.e. $k = 1$) and $p_0 = 1$ (i.e. $m = 0$). □

Remark 4.5. (i) Theorems 4.3 and 4.4 show that $\mathcal{S}_{\alpha,\tau}$ is the stability region for delayed fractional differential systems for Riemann-Liouville and Caputo derivative, i.e. for (1.1) and (1.4), respectively. Figure 1 represents the situation for $\alpha \in (0, 1)$ when the stability region includes also points with positive real part. We can see in Figure 2 how the region is transformed for $\alpha \in (1, 2)$, and it is apparent how the stability region vanishes for $\alpha \to 2$. Also, for $\tau \to 0$, $\mathcal{S}_{\alpha,\tau}$ tends to the stability region known from theory of fractional differential equations without delay (see, e.g. [7,9]).

(ii) From the stability viewpoint, the only difference between (1.1) and (1.4) occurs if there is a zero eigenvalue and the order of derivatives is less than 1. In this case, the zero solution to (1.1) can be asymptotically stable, stable or unstable, depending on the particular value of α and multiplicities of the zero eigenvalue. The zero solution of (1.4) is stable if algebraic and geometric multiplicities of the zero eigenvalue are equal, otherwise it is unstable (i.e. it does not depend on the particular value of α).

The proof technique used for Theorems 4.3 and 4.4 actually reveals more than the stability properties. Due to its constructive nature we can actually derive also the asymptotic behavior of the solutions to (1.1) and (1.4). We summarize the comparisons of the two cases in the following assertions dealing with the asymptotic

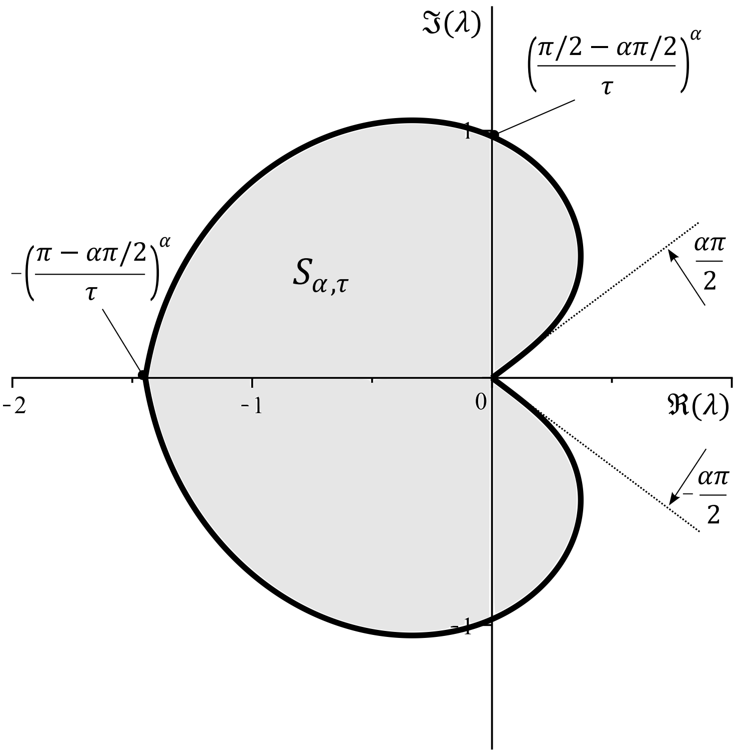

Figure 1. The stability region $\mathcal{S}_{\alpha,\tau}$ for the values $\alpha = 0.4$ and $\tau = 1$.

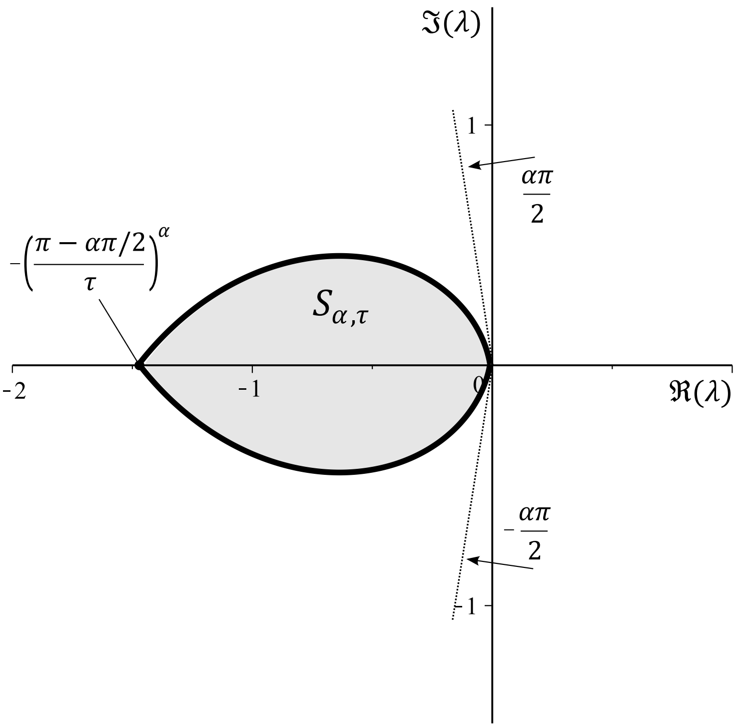

Figure 2. The stability region $\mathcal{S}_{\alpha,\tau}$ for the values $\alpha = 1.1$ and $\tau = 1$.

equivalence (denoted by the symbol $\sim$) relationships for norms of solutions (we use the symbol $\|\cdot\|$ for Euclidean norm in $\mathbb{R}^d$).

Theorem 4.6. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in (0,2)$, $\tau > 0$ and let all the eigenvalues of A belong to $\mathcal{S}_{\alpha,\tau}$. Further, we denote by y_{RL} and y_C the solutions of (1.1)–(1.3) and (1.4)–(1.6), respectively. Then it holds*

$$\|y_{RL}(t)\| \sim t^{-\alpha-1} \quad \text{and} \quad \|y_C(t)\| \sim t^{\lceil\alpha\rceil-\alpha-1} \qquad \text{as } t\to\infty \tag{4.11}$$

for almost all choices of initial conditions. If y_{RL} and y_C do not follow (4.11), then their norms tend to zero with a faster decay rate.

Proof. Theorems 3.1 and 3.2 indicate some particular choices of initial conditions, e.g. $y_0 = 0$, which can remove the dominating terms from y_{RL} and y_C and therefore affect the decay rate. The particular asymptotic properties are then implied by Lemma 4.2. □

Theorem 4.7. *Let $A \in \mathbb{R}^{d\times d}$, $\alpha \in (0,2)$ and $\tau > 0$. Let A has the zero eigenvalue and denote p_0 the size of the largest Jordan block corresponding to this zero eigenvalue. Let all non-zero eigenvalues of A belong to $\mathcal{S}_{\alpha,\tau}$. Further, we denote y_{RL} and y_C the solutions of (1.1)–(1.3) and (1.4)–(1.6), respectively. Then it holds*

$$\|y_{RL}(t)\| \sim t^{p_0\alpha-1} \quad \text{and} \quad \|y_C(t)\| \sim t^{(p_0-1)\alpha+\lceil\alpha\rceil-1} \qquad \text{as } t\to\infty \tag{4.12}$$

for almost all choices of initial conditions. If y_{RL} and y_C do not follow (4.12), then their norms are even smaller for t large enough.

Proof. The idea of the proof is analogous to the previous case. □

Remark 4.8. (i) We can observe an interesting distinction between the way how the asymptotic behavior of y_{RL} and y_C depends on α. While in the Riemann–Liouville case we see the algebraic decay rate depending directly on α, in the Caputo case the decay rate is driven by the decimal part of α, i.e. by the difference $\lceil\alpha\rceil - \alpha$. Indeed, if we consider for example $\alpha_1 = 0.4$ and $\alpha_2 = 1.4$, then the solutions of (1.4) follow essentially the same asymptotic relations, while the Riemann–Liouville ones do not.

(ii) We can employ a similar analysis also in the cases that are not covered by Theorems 4.6 and 4.7, i.e. when there is a non-zero eigenvalue on the boundary or outside the closure of the stability region. We note that if there is a non-zero eigenvalue lying outside the closure of the stability region, the norms of non-zero solutions increase exponentially for both (1.1) and (1.4).

(iii) We point out that the asymptotic results obtained for the delayed fractional differential systems actually mirror the well-known results for fractional differential systems without a delay.

5. Conclusions

We have summarized and extended the results on qualitative behavior of solutions of delayed fractional differential systems (1.1) and (1.4) of arbitrary order.

We have shown that the stability of the zero solution occurs only if the order of derivatives is less than 2. Further, we have derived the precise description of the stability region which is for both (1.1) and (1.4) identical. The only difference regarding the stability occurs when the system matrix A has a zero eigenvalue. Then we observe the asymptotic stability property for (1.1) only if $\alpha < 1$ and the maximum size of the Jordan block corresponding to the zero eigenvalue being less than $1/\alpha$. In the Caputo case (1.4), the asymptotic stability does not appear and the zero solution is stable (but not asymptotically stable) only if $\alpha < 1$ and algebraic multiplicity of the zero eigenvalue being equal to the geometric one.

The asymptotic behavior displays more diversity. If the system matrix A has all eigenvalues lying in the stability region, i.e. the zero solutions of both (1.1) and (1.4) are asymptotically stable, we can generally say that the solutions of (1.1) go to zero as $t \to \infty$ faster that solutions of (1.4). Moreover, unlike the Riemann-Liouville case, the decay rate of solutions to (1.4) does not depend on the value α itself, but on its decimal part only.

The area of qualitative analysis of fractional differential equations with a time delay, especially with higher-order derivative, provides a lot of open problems. Our research may serve as one of the prerequisites to studies of more complex systems, such as $\mathbf{D}_0^\alpha y(t) = ay(t) + by(t-\tau)$ or its vector analogues.

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Appendix D

Paper on Lambert function and one-term FDDE [15] (FCAA, 2023)

The topic of one-term FDDS seemed, in principle, mostly complete to us, at least regarding stable solutions. We revisited it only recently in a re-union of the author's trio, which was primarily active before 2016. The inspiration for [15] (co-authors: J. Čermák, L. Nechvátal; my author's share 33 %) came from discussions nearly nine years ago about a possible fractional generalization of the classical Lambert function technique known from the stability analysis of ordinary delay differential equations.

As it turned out, this generalization is not only possible but, in some cases, easier than other known approaches. We managed to re-derive fully explicit criteria that had not been reached by this technique before, contributing to a better understanding of unbounded solutions for higher-order FDDS. As a by-product, we created an "asymptotic map" for unbounded solutions, showing their large-time exponential modulus growth and frequency of oscillations based on the location of system matrix eigenvalue.

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ORIGINAL PAPER

The Lambert function method in qualitative analysis of fractional delay differential equations

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Abstract

We discuss an analytical method for qualitative investigations of linear fractional delay differential equations. This method originates from the Lambert function technique that is traditionally used in stability analysis of ordinary delay differential equations. Contrary to the existing results based on such a technique, we show that the method can result into fully explicit stability criteria for a linear fractional delay differential equation, supported by a precise description of its asymptotics. As a by-product of our investigations, we also state alternate proofs of some classical assertions that are given in a more lucid form compared to the existing proofs.

Keywords Fractional delay differential equation (primary) · Lambert function · Stability · Asymptotic behavior

Mathematics Subject Classification (Primary) 34K37 · 33E30 · 33E12 · 34K20 · 34K25

1 Introduction

The paper discusses an analytical method for qualitative investigations of fractional delay differential equations (FDDEs). These equations are currently very intensively studied due to their importance in various application areas, with a special emphasis to control theory. Indeed, presence of both the time lag as well as non-integer derivative

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order as control or tunning parameters in studied models provides a very efficient tool for various control processes such as stabilization or destabilization of the particular solutions of these models (for a pioneering work in this direction we refer to [17]).

Systematic investigations of FDDEs were initiated in the paper [9]. Here, stability properties of

$$\mathrm{D}^{\alpha}x(t) = \lambda x(t-\tau), \tag{1}$$

where $\alpha, \tau \in \mathbb{R}^+$, $\lambda \in \mathbb{R}$ and D^{α} is a fractional differential operator, were analyzed using the fact that (1) is asymptotically stable (i.e., any its solution is eventually tending to zero) if and only if all the roots of the characteristic equation

$$s^{\alpha} - \lambda \exp(-s\tau) = 0 \tag{2}$$

have negative real parts. To explore such a location of characteristic roots with respect to the imaginary axis, the Lambert function technique was utilized. The essence of the method consists in a representation formula for characteristic roots in terms of appropriate branches of this multi-valued function (for some precisions concerning the correct use of the Lambert function technique in stability analysis of (1), we refer also to [12]). A certain general disadvantage of this approach consists in its (seeming) disability to provide stability criteria in an explicit form depending on entry parameters only (i.e., on α, λ and τ in the case of (1)).

As the other papers on stability and asymptotic properties of (1) followed, the Lambert function method was replaced by some alternate classical tools of stability investigations (such as D–partition method or τ-decomposition method) modified to the fractional case. Using these approaches, effective and non-improvable stability conditions for (1), supported by some asymptotic bounds, were derived in [16] (the case $\lambda \in \mathbb{R}$, $0 < \alpha < 1$), [6] (the case $\lambda \in \mathbb{C}$, $0 < \alpha < 1$), and partially also in [7] (the case $\lambda \in \mathbb{C}$, $\alpha > 0$). Some of the mentioned results can be extended also to the case of a two–term FDDE

$$\mathrm{D}^{\alpha}x(t) = \mu x(t) + \lambda x(t-\tau). \tag{3}$$

In this respect, we refer to [2, 5, 15] (the case $\mu, \lambda \in \mathbb{R}$, $0 < \alpha < 1$) and [8] (the case $\mu, \lambda \in \mathbb{R}$, $1 < \alpha < 2$). Following the integer-order case (see, e.g., [1]), (1) and (3) may serve as test equations for numerical analysis of FDDEs. From this point of view, it is very important to describe their basic qualitative properties in the strongest possible form. Then, when analyzing appropriate numerical schemes applied to these test equations, the ability to keep the key qualitative properties of the underlying exact equations is of basic importance. For some other recent advances in qualitative theory of FDDEs, we refer, e.g., to [3, 10, 11, 18–20, 23].

Following the above outlines, the aim of this paper is twofold. First, we deepen the existing knowledge on some qualitative properties of (1) with the Caputo fractional derivative. Second, perhaps a more important aspect of the paper consists in the way how we aim to do it. We come back to the Lambert function method used in [9] and show that this approach can offer more than formulae depending on the use of supporting software packages. In fact, this technique can result into actually effective stability and asymptotic criteria.

The paper is organized as follows. Section 2 recalls some existing findings on (1) and essentials of the Lambert function theory. In Sect. 3, we explore the Lambert function method in details. In particular, we give an alternate proof of the classical assertion saying that the characteristic root generated by the principal branch of the Lambert function has the largest real part, and formulate a criterion that enables to localize values of the principal branch in the complex plane. Section 4 presents applications of these results to (1). Here, we extend the existing stability criteria for (1) to arbitrary (positive) real values of α, and formulate sharp asymptotic estimates for the solutions of (1). Some final remarks in Sect. 5 conclude the paper.

2 Basic mathematical background

In this section, we summarize some known facts relevant to our next investigations. First, we recall a close relationship between stability and asymptotic properties of (1), and distribution of the characteristic roots of (2). Then, we recall some basics of the Lambert function and its use in stability analysis of FDDEs.

It was shown in [7] that any solution x of (1) with the Caputo fractional derivative (and a generally complex λ) can be written using the Mittag-Leffler type function

$$G^{\lambda,\tau}_{\alpha,\beta}(t) = \sum_{j=0}^{\lceil t/\tau \rceil - 1} \frac{\lambda^j (t - j\tau)^{\alpha j + \beta - 1}}{\Gamma(\alpha j + \beta)}, \qquad \alpha, \beta > 0, \tag{4}$$

where $\lceil \cdot \rceil$ denotes the upper integer part. More precisely, if ϕ is a continuous initial (complex-valued) function on $[-\tau, 0]$, $\phi_0 = \phi(0)$ and ϕ_j, $j = 1, \ldots, \lceil \alpha \rceil - 1$, are (complex) constants (considered when $\alpha > 1$), then

$$x(t) = \sum_{j=0}^{\lceil \alpha \rceil - 1} \phi_j G^{\lambda,\tau}_{\alpha,j+1}(t) + \lambda \int_{-\tau}^{0} G^{\lambda,\tau}_{\alpha,\alpha}(t - \tau - \xi)\phi(\xi)\,\mathrm{d}\xi \tag{5}$$

is the solution of (1) satisfying $x(t) = \phi(t)$ for all $t \in [-\tau, 0]$, and $\lim_{t\to 0^+} x^{(j)}(t) = \phi_j$, $j = 1, \ldots, \lceil \alpha \rceil - 1$.

Based on some asymptotic results on (4), the solution (5) can be rewritten by the use of the characteristic roots having non-negative real parts. We recall that (2) admits countably many roots, and only a finite number of them is lying right to any line $\Re(s) = p$, $p \in \mathbb{R}$ (throughout the paper, the symbol $\Re(z)$ and $\Im(z)$ stands for the real and imaginary part of $z \in \mathbb{C}$, respectively). If we denote by S the set of all roots of (2) having non-negative real parts (note that S must be a finite set), then, for a non-integer α, (5) can be rewritten as

$$x(t) = \sum_{s\in S} c_s \exp(st) + \mathcal{O}(t^{j-\alpha}) \quad \text{as } t \to \infty \tag{6}$$

where c_s are complex coefficients depending on $\alpha, \tau, \lambda, \phi$, and $j \in \{-1, 0, \dots, \lceil\alpha\rceil - 1\}$ (the particular value of j depends on limit behavior of ϕ at $t = 0$). Notice that $j - \alpha < 0$, i.e., the function $t^{j-\alpha}$ always tends to zero.

By (6), the roots of (2) play an essential role in qualitative behavior of the solutions of (1). Following the classical integer-order pattern, the authors in [9] used the following chain of steps

$$s^\alpha \exp(s\tau) = \lambda \quad \to \quad s \exp\Big(\frac{\tau}{\alpha}s\Big) = \lambda^{\frac{1}{\alpha}} \quad \to \quad \frac{\tau}{\alpha}s \exp\Big(\frac{\tau}{\alpha}s\Big) = \frac{\tau}{\alpha}\lambda^{\frac{1}{\alpha}} \tag{7}$$

to express the roots of (2) via the Lambert function introduced as the solution of

$$W(z)\exp(W(z)) = z, \quad z \in \mathbb{C}. \tag{8}$$

Before we recall the root formula for (2) based on this special function, some of its basic properties might be collected. The Lambert function is a multi-valued function (except at $z = 0$) with infinitely many (single-valued) branches W_k, $k \in \mathbb{Z}$. Neither of them can be expressed in terms of elementary functions. In particular, W_0 is called a principal branch. For any $z \in \mathbb{C}$, $\Im(W_0(z))$ is between $-\pi$ and π. The other branches are numbered so that $\Im(W_k(z))$ is between $(2k-2)\pi$ and $(2k+1)\pi$ while $\Im(W_{-k}(z))$ is between $-(2k+1)\pi$ and $-(2k-2)\pi$ for any $z \in \mathbb{C}$ and $k = 1, 2, \dots$. More precisely, the ranges of $W_{\pm k}$ and $W_{\pm(k+1)}$, $k = 0, 1, \dots$, are separated by the curves

$$\{w = x + \mathrm{i}\, y \in \mathbb{C} : x = -y\cot(y),\ 2k\pi < |y| < (2k+1)\pi\}$$

and the ranges of W_1 and W_{-1} are separated by the half-line

$$\{w = x + \mathrm{i}\, y \in \mathbb{C} : -\infty < x \le -1,\ y = 0\}.$$

These separating curves correspond to the branch cuts in the z-plane defined as

$$\{z = \xi + \mathrm{i}\,\eta \in \mathbb{C} : -\infty < \xi \le -\exp(-1),\ \eta = 0\}$$

in the case of W_0, and

$$\{z = \xi + \mathrm{i}\,\eta \in \mathbb{C} : -\infty < \xi \le 0,\ \eta = 0\}$$

in the case of W_k, $k \neq 0$. Conventionally, the branch cut (having the argument π in the z-plane) is mapped by W_k on its upper boundary in the w-plane. Only the branches W_0 and W_{-1} take on real values for a real $z \in [-\exp(-1), \infty)$ and a real $z \in [-\exp(-1), 0)$, respectively. Further details on the Lambert function (including some historical remarks) can be found in [4], for other comments, see also [13] and [22].

Now, following (7), all the roots of (2) can be expressed in the form

$$s_k = \frac{\alpha}{\tau} W_k\left(\frac{\tau}{\alpha}\lambda^{\frac{1}{\alpha}}\right), \quad k \in \mathbb{Z}. \tag{9}$$

By (6), a crucial role in analysis of (1) is played by the rightmost characteristic root (i.e., the root of (2) with the largest real part). The following classical assertion says that this root is just s_0.

Lemma 1 *Let $z \in \mathbb{C}$. Then $W_0(z)$ has the largest real part $\Re(W_0(z))$ among all the other real parts $\Re(W_k(z))$, $k \in \mathbb{Z}$.*

The original proof of Lemma 1 is pretty long (see [22]). As a by-product of our next procedures, we are going to present an alternate (and more simple) way how to prove this assertion.

Remark 1 As pointed out in [12], the expression (9) is not quite correct for some complex values of λ. More precisely, (7) contains taking the $1/\alpha$-power which means that the roots given by (9) are identical to those of (2) only in the case

$$|\mathrm{Arg}(\lambda)| \leq \alpha\pi$$

(we recall that $-\pi < \mathrm{Arg}(\cdot) \leq \pi$). This inequality is satisfied trivially when $\alpha \geq 1$ but makes a restriction when $0 < \alpha < 1$. In other words, if $|\mathrm{Arg}(\lambda)| > \alpha\pi$, then the representation (9) can produce some superfluous roots that are actually not the true roots of (2). As an example, we can consider, e.g., the case $\lambda = -1$, $\alpha = 1/2$ and $\tau = 1$ when (2) has the rightmost root $s_0 \approx -0.4172 - \mathrm{i}\,2.2651$ (i.e., (1) is asymptotically stable) while (9) produces $s_0 \approx 0.4263 > 0$. On this account, we discuss qualitative properties of (1) for $\alpha > 1$. Comments to the case $0 < \alpha < 1$ are provided in the final section.

3 Some advances on the Lambert *W* function

This section contains several key results on the Lambert function which proved to be useful in qualitative investigations of (1). To obtain an actually effective and strong asymptotic description of the solutions of (1), we need to effectively localize the position of the rightmost characteristic root in the complex plane. More precisely, by (6) and (9), we need to derive effective expressions of the real and imaginary parts of $W_0(z)$ in terms of z. Thus, keeping in mind intended stability and asymptotic analysis of (1), we can pose the following problems: *For given $p \in \mathbb{R}$ and $z \in \mathbb{C}$, is it possible to characterize the properties $\Re(W_0(z)) < p$ and $\Re(W_0(z)) = p$ directly in terms of z and p, i.e., without an evaluation of the principal branch of the Lambert function? Further, for given $q \in \mathbb{R}$ and $z \in \mathbb{C}$, is it possible to similarly elaborate on the properties $|\Im(W_0(z))| > q$ and $|\Im(W_0(z))| = q$?* The following result yields an affirmative answer to these questions.

Theorem 1 *Let $p, q \in \mathbb{R}$, $p > -1$, $0 < q < \pi$, and $z \in \mathbb{C}$, $z \neq 0$. Then*
(i) *$\Re(W_0(z)) < p$ if and only if either $|z| < p\exp(p)$ or*

$$|z| \geq |p|\exp(p) \quad \textit{and} \quad \arccos\left(\frac{p\exp(p)}{|z|}\right) + \frac{\sqrt{|z|^2 - p^2\exp(2p)}}{\exp(p)} < |\mathrm{Arg}(z)|\,; \tag{10}$$

(ii) $\Re(W_0(z)) = p$ *if and only if*

$$|z| \geq |p|\exp(p) \quad \textit{and} \quad \arccos\left(\frac{p\exp(p)}{|z|}\right) + \frac{\sqrt{|z|^2 - p^2\exp(2p)}}{\exp(p)} = |\mathrm{Arg}(z)|\,; \tag{11}$$

(iii) $|\Im(W_0(z))| > q$ *if and only if*

$$|\mathrm{Arg}(z)| > q \quad \textit{and} \quad \frac{q}{\sin(|\mathrm{Arg}(z)| - q)}\exp\big(q\cot(|\mathrm{Arg}(z)| - q)\big) < |z|; \tag{12}$$

(iv) $|\Im(W_0(z))| = q$ *if and only if*

$$|\mathrm{Arg}(z)| > q \quad \textit{and} \quad \frac{q}{\sin(|\mathrm{Arg}(z)| - q)}\exp\big(q\cot(|\mathrm{Arg}(z)| - q)\big) = |z|. \tag{13}$$

Proof (i) We write $z = |z|\exp(\mathrm{i}\,\mathrm{Arg}(z))$ and put $x_k = \Re(W_k(z))$, $y_k = \Im(W_k(z))$ where W_k, $k \in \mathbb{Z}$ are particular branches of the Lambert function. Substitution into (8) yields

$$\exp(x_k)(x_k\cos(y_k) - y_k\sin(y_k)) = |z|\cos(\mathrm{Arg}(z)), \tag{14}$$
$$\exp(x_k)(x_k\sin(y_k) + y_k\cos(y_k)) = |z|\sin(\mathrm{Arg}(z)). \tag{15}$$

If we solve (14)–(15) with respect to unknowns $x_k\exp(x_k)$ and $y_k\exp(x_k)$, then

$$x_k\exp(x_k) = |z|\cos(\mathrm{Arg}(z) - y_k), \tag{16}$$
$$y_k\exp(x_k) = |z|\sin(\mathrm{Arg}(z) - y_k). \tag{17}$$

To show that $x_0 = \Re(W_0)(z)) < p$ whenever $|z| < p\exp(p)$, we consider (16) implying

$$x_0\exp(x_0) \leq |z| < p\exp(p).$$

Then the monotony property of the function $g(p) = p\exp(p)$ on $(-1, \infty)$ actually implies $x_0 < p$.

Now we assume that $|z| \geq |p|\exp(p)$. Squaring and adding (16) and (17) we get

$$|z|^2 = ((x_k)^2 + (y_k)^2)\exp(2x_k),$$

i.e.,

$$|y_k| = \frac{\sqrt{|z|^2 - (x_k)^2\exp(2x_k)}}{\exp(x_k)}. \tag{18}$$

For the principal branch, it holds $x_0 \geq -y_0 \cot(y_0)$, $|y_0| < \pi$, i.e., $x_0 \sin(y_0) + y_0 \cos(y_0) \geq 0$ whenever $y_0 \geq 0$. Multiplying this by $\exp(x_0)$ and using (15), one gets

$$|z| \sin(\operatorname{Arg}(z)) = \exp(x_0)(x_0 \sin(y_0) + y_0 \cos(y_0)) \geq 0$$

which implies $\operatorname{Arg}(z) \geq 0$ for $y_0 \geq 0$. If $y_0 < 0$, the same argumentation leads to $\operatorname{Arg}(z) \leq 0$, hence $\operatorname{Arg}(z) y_0 \geq 0$, i.e., $|\operatorname{Arg}(z) - y_0| \leq \pi$. Then (16) with $k = 0$ is equivalent to

$$\arccos(x_0 \exp(x_0)/|z|) = |\operatorname{Arg}(z) - y_0| . \tag{19}$$

Moreover, sign analysis of (17) with respect to $\operatorname{Arg}(z) y_0 \geq 0$ yields $|\operatorname{Arg}(z)| \geq |y_0|$, i.e.,

$$|\operatorname{Arg}(z) - y_0| = |\operatorname{Arg}(z)| - |y_0|. \tag{20}$$

Then, using (18), (19) and (20), we are able to set up an implicit dependence between $x_0 = \Re(W_0(z))$ and z in the form $f(x_0, z) = 0$ where f is defined via

$$f(p, z) = \arccos\left(\frac{p \exp(p)}{|z|}\right) - |\operatorname{Arg}(z)| + \frac{\sqrt{|z|^2 - p^2 \exp(2p)}}{\exp(p)}$$

for all $p > -1$ and $z \in \mathbb{C}$ such that $|p| \exp(p) \leq |z|$. Let z be fixed. Then

$$\frac{\mathrm{d}f}{\mathrm{d}p}(p, z) = -\frac{(2p + 1)\exp(3p) + |z|^2 \exp(p)}{\exp(2p)\sqrt{|z|^2 - p^2 \exp(2p)}} \leq -\frac{(p + 1)^2 \exp(p)}{\sqrt{|z|^2 - p^2 \exp(2p)}} \leq 0,$$

hence, f is decreasing in p if $|p| \exp(p) \leq |z|$. Therefore,

$$f(p, z) < f(x_0, z) = 0$$

whenever

$$p > x_0 = \Re(W_0(z)) \quad \text{and} \quad |p| \exp(p) \leq |z|.$$

(ii) The property follows directly from the proof of (i) using the fact that $f(p, z) = 0$ if and only if $p = x_0$ due to monotony of f with respect to p.

(iii) Since W_0 is symmetric in the sense $W_0(\overline{z}) = \overline{W_0(z)}$ for all $z \in \mathbb{C}$ except those lying on the branch cut along the negative real axis between $-\infty$ and $-\exp(-1)$, it suffices to assume the case $y_0 = \Im(W_0(z)) > q > 0$. We have already observed that $\operatorname{Arg}(z) \geq y_0$. In addition, a stronger property holds, namely $\operatorname{Arg}(z) > y_0$. Indeed, possible equality $\operatorname{Arg}(z) = y_0$ implies $y_0 = 0$ (due to (17)) which contradicts the assumption $y_0 > q > 0$. Hence, it must be $0 < \operatorname{Arg}(z) - y_0 < \pi$ as well as $0 < \operatorname{Arg}(z) - q < \pi$. We divide (16) by (17) and put $k = 0$ to get

$$x_0 = y_0 \cot(\operatorname{Arg}(z) - y_0). \tag{21}$$

Taking the logarithm of (17) with $k = 0$, we also have

$$x_0 = \ln\bigl(|z| \sin(\mathrm{Arg}(z) - y_0)\bigr) - \ln(y_0). \tag{22}$$

Combining (21) and (22), we arrive at

$$\ln\bigl(|z| \sin(\mathrm{Arg}(z) - y_0)\bigr) - \ln(y_0) - y_0 \cot(\mathrm{Arg}(z) - y_0) = 0$$

representing again an implicit dependence, now between $\Im(W_0(z))$ and z. If we denote

$$h(q, z) = \ln\bigl(|z| \sin(\mathrm{Arg}(z) - q)\bigr) - \ln(q) - q \cot(\mathrm{Arg}(z) - q),$$

then we have

$$\frac{\mathrm{d}h}{\mathrm{d}q}(q, z) = \frac{-q \sin(2(\mathrm{Arg}(z) - q)) - \sin^2(\mathrm{Arg}(z) - q) - q^2}{q \sin^2(\mathrm{Arg}(z) - q)}.$$

While the denominator is positive, the numerator

$$N(q, z) = -q \sin(2(\mathrm{Arg}(z) - q)) - \sin^2(\mathrm{Arg}(z) - q) - q^2$$

is negative for each $0 \le q \le \mathrm{Arg}(z)$. Indeed, we have

$$\frac{\mathrm{d}N}{\mathrm{d}q}(q, z) = 2q(\cos^2(\mathrm{Arg}(z) - q) - 1) \le 0$$

which implies that $N(\cdot, z)$ is non-increasing and, together with $N(0, z) = -\sin^2(\mathrm{Arg}(z)) < 0$, negative on $[0, \mathrm{Arg}(z)]$. Consequently, $h(\cdot, z)$ is decreasing and therefore $h(q, z) > h(y_0, z) = 0$ whenever $0 < q < y_0 < \mathrm{Arg}(z)$. Taking into account the above mentioned symmetry, we arrive (after some elementary algebra) at (12).

(iv) The required property is again a consequence of monotony of the function h from the previous part. □

Remark 2 (a) The properties (ii) and (iv) of Theorem 1 provide a new tool for evaluations of the principal branch of the Lambert function. Let $z \neq 0$ be a fixed complex number. Then the left-hand side of (11) is decreasing for all $p \in [a, W_0(|z|)]$ ($a = -1$ if $|z| \ge \exp(-1)$ and $a = W_0(-|z|)$ if $|z| < \exp(-1)$) from π to the zero value. Hence, (11) has a unique root p^* lying in this interval, and this root equals just $\Re(W_0(z))$. Similarly, the left-hand side of (13) is increasing for all $q \in (0, \mathrm{Arg}(z))$ from the zero value to infinity, i.e., (13) admits a unique positive root q^* which is just $\Im(W_0(z))$.

To illustrate this evaluation technique, we compute $W_0(z)$ for $z = \frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}$. Then $|z| = 1$, $\mathrm{Arg}(z) = \pi/3$ and the standard Newton method returns $\Re(z) = p^* \approx 0.4843$ in 5 iterations with the initial value $p_0 = 0.5$ and the stopping criterion taken as $|p_{k+1} - p_k| \le 10^{-16}$. The same method gives $\Im(z) = q^* \approx 0.3808$ in 7 iterations with the initial value $q_0 = 0.5$ and the same precision as in the case of the real part.

In fact, the value $p^* + \mathrm{i}\, q^*$ matches the value produced by the MATLAB command `lambertw(1/2+sqrt(3)/2*1i)` to all the 15 digits behind the decimal point. Standardly, the Newton or Halley method is applied directly to the equation $w \exp(w) - z = 0$ using the complex arithmetic. MATLAB employs the latter method with some advanced guess of the starting point. For computing the values of the Lambert function with arbitrary precision, we refer to the recent paper [14].

(b) Using a different approach, the property (i) of Theorem 1 was also discussed in [21].

In the sequel, we clarify ordering of the real as well as imaginary parts of the particular branches of the Lambert function. This ordering may be useful in a deeper asymptotic analysis of (1), and, moreover, results into an alternate proof of Lemma 1.

Following the proof of Theorem 1, we introduce the functions

$$G_z(x, y) = x \sin(y) + y \cos(y) - |z| \sin(\mathrm{Arg}(z)) \exp(-x) \quad \text{and}$$
$$f_z(x) = \sqrt{|z|^2 \exp(-2x) - x^2}.$$

In view of (15) and (18), the couples (x_k, y_k), where $x_k = \Re(W_k(z))$, $y_k = \Im(W_k(z))$, have to meet the relations $G_z(x, y) = 0$ and $y = \pm f_z(x)$, respectively.

The following assertion specifies ordering of imaginary parts of the branches of the Lambert function.

Lemma 2 *Let $z \in \mathbb{C} \setminus \{0\}$. Then $\Im(W_k(z)) \le \Im(W_{k+1}(z))$ for all $k \in \mathbb{Z}$. In fact, all the inequalities are strict with the only exception: If $z \in [-\exp(-1), 0)$, then we have $\Im(W_{-1}(z)) = \Im(W_0(z)) = 0$.*

Proof For the sake of formal simplicity, we identify complex numbers $w = x + \mathrm{i}\, y$ with couples $(x, y) \in \mathbb{R}^2$. First, let $z \in \mathbb{C} \setminus \{0\}$ be such that $0 \le \mathrm{Arg}(z) \le \pi$ and define sets S_j^z, $j \in \mathbb{Z}$, as

$$S_j^z = \{(x, y) \in \mathbb{R}^2 : G_z(x, y) = 0,\ (2j - 1)\pi < y < (2j + 1)\pi\} \qquad \text{for } j = 1, 2, \dots;$$
$$S_j^z = \{(x, y) \in \mathbb{R}^2 : G_z(x, y) = 0,\ 0 \le y < \pi\} \qquad \text{for } j = 0;$$
$$S_j^z = \{(x, y) \in \mathbb{R}^2 : G_z(x, y) = 0,\ -2\pi < y \le 0\} \qquad \text{for } j = -1;$$
$$S_j^z = \{(x, y) \in \mathbb{R}^2 : G_z(x, y) = 0,\ 2j\pi < y < (2j + 2)\pi\} \qquad \text{for } j = -2, -3, \dots$$

(note that the equation $G_z(x, y) = 0$ has no solution for $y = (2j - 1)\pi$, $j = 1, 2, \dots,$ and for $y = 2j\pi$, $j = -1, -2, \dots$). We wish to show that S_j^z is a part of the range of W_k just when $j = k$.

Let $k \ge 1$ be arbitrary. Then, by the definition of W_k (see also Sect. 2),

$$(2k - 2)\pi < y_k < (2k + 1)\pi. \tag{23}$$

Let j be such that $(x_k, y_k) \in S_j^z$. We distinguish the following cases with respect to j.

If $j > k$, then $y_k > (2k+1)\pi$ which contradicts (23). Let $j = k$. The ranges of W_k and W_{k+1}, $k = 0, 1, \ldots,$ are separated by the curve

$$\gamma_k = \{(u, v) \in \mathbb{R}^2 : u = -v\cot(v),\ 2k\pi < v < (2k+1)\pi\}$$

(see Sect. 2). The equation $G_z(x, y) = 0$, $(2k-1)\pi < y < (2k+1)\pi$, is equivalent to

$$x = -y\cot(y) + |z|\sin(\mathrm{Arg}(z))\exp(-x) \tag{24}$$

provided $y \neq 2k\pi$. To estimate the y-coordinate of a point $(x, y) \in S_k^z$, we put $u = x$ in γ_k.

First, let $2k\pi < y < (2k+1)\pi$. Then any point (x, y) of S_k^z together with the corresponding point (x, v) of γ_k have to fulfill the formula

$$y\cot(y) - v\cot(v) = \frac{|z|\sin(\mathrm{Arg}(z))}{\sin(y)}\exp(-x) \tag{25}$$

due to $x = -v\cot(v)$ and (24). Since $0 \leq \mathrm{Arg}(z) \leq \pi$, the right-hand side of (25) is non-negative, hence, we have $y\cot(y) \geq v\cot(v)$ implying $y \leq v$. In other words, any point $(x, y) \in S_k^z$ is located below or on the curve γ_k separating W_k and W_{k+1}. Second, let $(2k-1)\pi < y \leq 2k\pi$. Then the points $(x, y) \in S_k^z$ lie below γ_k trivially (note that the equation $G_z(x, y) = 0$ has a unique solution x for $y = 2k\pi$, $0 < \mathrm{Arg}(z) < \pi$, and has no solution for $y = 2k\pi$, $\mathrm{Arg}(z) = 0$ or $\mathrm{Arg}(z) = \pi$). On the other hand, any point of S_k^z lies above the upper bound $(2k-1)\pi$ of γ_{k-1}, hence, we have proven that S_k^z is contained in the range of W_k.

Finally, if $j < k$, then we can similarly verify that any point of S_j^z is already located below or on γ_{k-1}. Thus, to summarize the previous observations, S_j^z is contained in the range of W_k $(k = 1, 2, \ldots)$ just when $j = k$; otherwise, S_j^z and the range of W_k are disjoint. Consequently, $(2k-1)\pi < y_k < (2k+1)\pi$, hence, $y_k < y_{k+1}$ for all $k = 1, 2, \ldots$.

The same line of arguments can be used in the case $k \leq -1$ to obtain $2k\pi < y_k < (2k+2)\pi$, and, in the remaining case $k = 0$, to obtain $0 \leq y_0 < \pi$ (and therefore, $y_0 < y_1$). In this respect, a real z is mapped by W_0 to γ_0 (hence, $y_0 > 0$) if $z < -\exp(-1)$, and is mapped by W_0 to reals if $z \geq -\exp(-1)$ (hence, $y_0 = 0$). Also W_{-1} takes real values just when $-\exp(-1) \leq z < 0$ (hence, $y_{-1} = y_0 = 0$).

The procedure can be analogously applied to the case $-\pi < \mathrm{Arg}(z) < 0$ (definition of the sets S_j^z now differ by shifting the particular y-domains vertically down by π). □

Lemma 2 is useful also for ordering of the real parts of the branches of the Lambert function. In particular, it enables to prove the classical assertion of Lemma 1 in a more lucid way compared to the existing proof techniques.

Proof of Lemma 1 We recall that the couple (x_k, y_k), $k \in \mathbb{Z}$, satisfies $|y| = f_z(x)$, see the text preceding Lemma 2. Put $\zeta_1 = W_{-1}(-|z|)$, $\zeta_2 = W_0(-|z|)$, $\zeta_3 = W_0(|z|)$.

Then, we can easily observe that f_z is defined on $(-\infty, \zeta_3]$ and $(-\infty, \zeta_1] \cup [\zeta_2, \zeta_3]$ if $|z| \geq \exp(-1)$ and $0 < |z| < \exp(-1)$, respectively. Moreover, f_z has a (unique) root $\zeta_3 > 0$ if $|z| > \exp(-1)$, a couple of roots $\zeta_1 = \zeta_2 = -1$ and $\zeta_3 > 1$ if $|z| = \exp(-1)$, and a triple of roots $\zeta_1 < -1, -1 < \zeta_2 < 0, \zeta_3 > 0$ if $0 < |z| < \exp(-1)$. Otherwise, f_z is positive at all other points of its domain.

Further, we have

$$f_z'(x) = \frac{-|z|^2 \exp(-2x) - x}{\sqrt{|z|^2 \exp(-2x) - x^2}}.$$

If $|z| \geq \frac{\sqrt{2}}{2} \exp(-1/2)$, then f_z' is negative, hence f_z is decreasing on $(-\infty, \zeta_3)$. If $\frac{\sqrt{2}}{2} \exp(-1/2) > |z| > \exp(-1)$, then f_z has a local minimum at $\zeta_4 = \frac{1}{2} W_{-1}(-2|z|^2)$ and a local maximum at $\zeta_5 = \frac{1}{2} W_0(-2|z|^2)$ (note that $\zeta_4 < \zeta_5$). Consequently, f_z is decreasing on $(-\infty, \zeta_4)$, increasing on (ζ_4, ζ_5) and again decreasing on (ζ_5, ζ_3).

If $|z| = \exp(-1)$, then f_z is decreasing on $(-\infty, -1)$, increasing on $(-1, \zeta_5)$ and decreasing on (ζ_5, ζ_3). Finally, if $\exp(-1) > |z| > 0$, then f_z is decreasing on $(-\infty, \zeta_1)$ increasing on (ζ_2, ζ_5) and decreasing on (ζ_5, ζ_3).

Thus, f_z is decreasing on its domain up to "a small part" which is, however, lying within the range of W_0 (we again identify $w = x + \mathrm{i}\, y$ with $(x, y) \in \mathbb{R}^2$). Indeed, if $\frac{\sqrt{2}}{2} \exp(-1/2) > |z| > \exp(-1)$, we have to show that the graph of f_z between the points $[\zeta_4, f_z(\zeta_4)]$ and $[\zeta_5, f_z(\zeta_5)]$ is contained in the range of W_0. Since

$$\zeta_4^2 + f_z^2(\zeta_4) = -\frac{1}{2} W_{-1}(-2|z|) < 1 \quad \text{and} \quad \zeta_5^2 + f_z^2(\zeta_5) = -\frac{1}{2} W_0(-2|z|^2) < \frac{1}{2}$$

for any $\frac{\sqrt{2}}{2} \exp(-1/2) > |z| > \exp(-1)$, both the endpoints of the graph belong to the range of W_0 (note that the open unit disk is a part of the W_0 range). Also, if $\zeta_4 \leq x \leq \zeta_5$, then $x^2 + f_z^2 = |z| \exp(-2x)$ is decreasing in x, hence, we have $|z| \exp(-2x) < 1$ for any $\zeta_4 \leq x \leq \zeta_5$ and any $\frac{\sqrt{2}}{2} \exp(-1/2) > |z| > \exp(-1)$ meaning that the graph of the increasing part of f_z is again lying in the range of W_0.

Similarly, we can show that, for $\exp(-1) \geq |z| > 0$, the graph of f_z between the relevant points is contained in the range of W_0 as well.

Collecting the above monotony properties together with Lemma 2, we can conclude that $x_{\pm(k+1)} < x_{\pm k}$ for any $k = 1, 2, \ldots$. If $k = 0$, we have $x_1 < x_0$ but $x_{-1} \leq x_0$ because of $W_0(z) = \overline{W_{-1}(z)}$ for any $z \in \mathbb{R}, z \leq -\exp(-1)$. □

Remark 3 We have actually proven monotony of the real parts of $W_k(z)$ with respect to k which is a slightly stronger result than that stated in Lemma 1.

4 Applications towards qualitative properties of (1) with the Caputo derivative

In this section, we apply our previous observations on the principal branch of the Lambert function to describe important qualitative properties of (1), including their

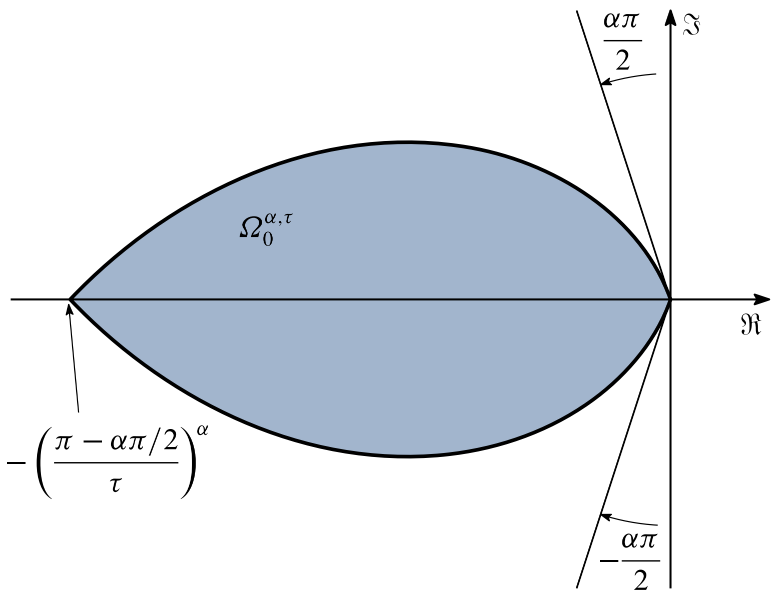

Fig. 1 The set $\Omega_0^{\alpha,\tau}$ as the stability region for (1) (the figure corresponds to $\alpha = 1.2$)

dependence on the parameter λ. We remind that here we restrict ourselves to the case $\alpha > 1$ due to the reason mentioned in Remark 1.

We start with a basic stability criterion for (1). We introduce a (parametric) set $\Omega_0^{\alpha,\tau}$ given as

$$\Omega_0^{\alpha,\tau} = \left\{ z \in \mathbb{C} : |z| < \left(\frac{|\mathrm{Arg}(z)| - \alpha\pi/2}{\tau} \right)^{\alpha}, \quad |\mathrm{Arg}(z)| > \frac{\alpha\pi}{2} \right\}, \tag{26}$$

see Fig. 1. This set already appeared in [6, Thm. 2] as the asymptotic stability region for (1) with $0 < \alpha < 1$. As indicated in [7], the D-subdivision method (combined with some tools of fractional calculus) used in [6] is extendable also to the case $\alpha > 1$. In the following assertion, we confirm validity of this stability result for $\alpha > 1$. Contrary to the existing techniques, we are able to prove this result (as a consequence of Theorem 1(i)) in an almost elementary way.

Theorem 2 *Let $\alpha > 1$, $\tau > 0$ and $\lambda \in \mathbb{C}$. Then (1) is asymptotically stable if and only if $\lambda \in \Omega_0^{\alpha,\tau}$.*

Proof By (9) and Lemma 1, we need to analyze $\Re(s_0) = \frac{\alpha}{\tau}\Re\big(W_0\big(\frac{\tau}{\alpha}\lambda^{1/\alpha}\big)\big) < 0$. Using (10) with $z = \tau\lambda^{1/\alpha}/\alpha$ and $p = 0$, this inequality can be converted into

$$\left|\frac{\tau}{\alpha}\lambda^{\frac{1}{\alpha}}\right| + \frac{\pi}{2} < \left|\mathrm{Arg}(\lambda^{\frac{1}{\alpha}})\right|,$$

i.e.,

$$\frac{\tau}{\alpha}|\lambda|^{\frac{1}{\alpha}} < \left|\mathrm{Arg}(\lambda^{\frac{1}{\alpha}})\right| - \frac{\pi}{2}.$$

Taking into account $\left|\mathrm{Arg}(\lambda^{1/\alpha})\right| = \frac{1}{\alpha}\left|\mathrm{Arg}(\lambda)\right|$, this is equivalent to the condition defining $\Omega_0^{\alpha,\tau}$ in (26). □

Remark 4 If $\lambda \in \partial\Omega_0^{\alpha,\tau}$, then (1) is stable but not asymptotically stable. If $\lambda \notin \mathrm{cl}\Omega_0^{\alpha,\tau}$, then (1) is unstable. Obviously, $\Omega_0^{\alpha,\tau}$ becomes empty for any $\alpha \geq 2$, hence, (1) cannot be asymptotically stable for any complex λ whenever $\alpha \geq 2$.

Theorem 1 can be used in a more subtle way to bring a deeper insight into behavior of (1). More precisely, relations (10)–(13) enable to reveal a relationship between the position of the rightmost characteristic root s_0 and the value of λ. Then, by (6), such a relationship easily results into a precise asymptotic description of the solutions of (1) in the unstable case ($\Re(s_0) > 0$). Indeed, while in the asymptotically stable case ($\Re(s_0) < 0$) the decay rate of solutions is algebraic and independent of the particular value of s_0, in the unstable case ($\Re(s_0) > 0$), the growth rate is exponential and governed by the real part of the rightmost characteristic root s_0 (it is well known that this is true also for integer values of α). In addition, the imaginary part of s_0 is related to the frequency characteristics describing an oscillatory behavior of (1).

Thus, for given $u, v \geq 0$, we need to find a region of all complex λ such that the rightmost characteristic root s_0 is lying on or left to the line $u + \mathrm{i}\,\omega$, $\omega \in \mathbb{R}$, and on or above (below) the line $\omega + \mathrm{i}\,v$ (the line $\omega - \mathrm{i}\,v$), $\omega \in \mathbb{R}$. On this account, similarly as in the proof of Theorem 2, we put $z = \tau\lambda^{1/\alpha}/\alpha$, $p = \tau u/\alpha$, $q = \tau v/\alpha$ and consider $u \geq 0$, $\alpha\pi/\tau > v > 0$. Then, Theorem 1 immediately implies that

(i) $\Re(s_0) \leq u$ if and only if either

$$|\lambda| < u^{\alpha}\exp(\tau u) \tag{27}$$

or

$$|\lambda| \geq u^{\alpha}\exp(\tau u) \quad \text{and} \quad \alpha \arccos\left(\frac{u\exp(\tau u/\alpha)}{|\lambda|^{1/\alpha}}\right) + \frac{\tau\sqrt{|\lambda|^{2/\alpha} - u^2\exp(2\tau u/\alpha)}}{\exp(\tau u/\alpha)} \leq |\mathrm{Arg}(\lambda)|\,; \tag{28}$$

(ii) $|\Im(s_0)| \geq v$ if and only if

$$|\mathrm{Arg}(\lambda)| > \tau v/\alpha \quad \text{and} \quad \frac{v^{\alpha}}{\sin^{\alpha}\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)} \times \exp\Big(\tau v\cot\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)\Big) \leq |\lambda|. \tag{29}$$

Thus, if we introduce the set $\overline{\Omega}_u^{\alpha,\tau}$ as a set of all $\lambda \in \mathbb{C}$ such that either (27) or (28) holds, and $\overline{\Psi}_v^{\alpha,\tau}$ as a set of all $\lambda \in \mathbb{C}$ such that (29) holds (we put $\overline{\Psi}_v^{\alpha,\tau} = \emptyset$ for $\alpha\pi/\tau \leq v < \pi$ and $\overline{\Psi}_v^{\alpha,\tau} = \mathbb{C}$ for $v = 0$), then we can rewrite our previous observations into the following assertion:

Lemma 3 *Let $\alpha > 1$, $\tau > 0$, $u \geq 0$, $\pi > v \geq 0$, $\lambda \in \mathbb{C}$ and let s_0 be given by (9). Then*

(i) $\Re(s_0) \le u$ *if and only if* $\lambda \in \overline{\Omega}_u^{\alpha,\tau}$;
(ii) $|\Im(s_0)| \ge v$ *if and only if* $\lambda \in \overline{\Psi}_v^{\alpha,\tau}$ ($\overline{\Psi}_v^{\alpha,\tau}$ *is non-empty whenever* $0 \le v < \alpha\pi/\tau$).

Remark 5 (a) The set $\overline{\Omega}_u^{\alpha,\tau}$ contains the origin ($\lambda = 0$) which is excluded by Theorem 1. However, admitting $\lambda = 0$, we have the only characteristic root $s_0 = 0$ of (2), hence, Lemma 3 remains true.

(b) It is easy to check that $\Omega_0^{\alpha,\tau}$ introduced in (26) coincides with int $\overline{\Omega}_0^{\alpha,\tau}$.

The part (i) of Lemma 3 immediately implies

Corollary 1 *Let* $u \ge 0$ *be fixed. Then* $\overline{\Omega}_u^{\alpha,\tau}$ *is the set of all* $\lambda \in \mathbb{C}$ *such that* $x(t) = \mathcal{O}(\exp(ut))$ *as* $t \to \infty$ *for any solution* x *of* (1).

To obtain an actually effective (and non-improvable) asymptotic result for the solutions of (1), we have to look at the problem inversely. More precisely, for a given complex $\lambda \notin \Omega_0^{\alpha,\tau}$, we need to find (non-negative) real values u_0, v_0 such that the rightmost root s_0 of (2) satisfies $\Re(s_0) = u_0$, $|\Im(s_0)| = v_0$.

The way how to do it easily follows from Theorems 1(ii) and 1(iv), respectively, taking into account the above introduced substitutions $z = \tau\lambda^{1/\alpha}/\alpha$, $p = \tau u/\alpha$, $q = \tau v/\alpha$. Then the corresponding relations (11) and (13) become

$$|\lambda| \ge u^\alpha \exp(\tau u) \quad \text{and} \quad \alpha \arccos\left(\frac{u\exp(\tau u/\alpha)}{|\lambda|^{1/\alpha}}\right) + \frac{\tau\sqrt{|\lambda|^{2/\alpha} - u^2\exp(2\tau u/\alpha)}}{\exp(\tau u/\alpha)}$$
$$= |\mathrm{Arg}(\lambda)| \tag{30}$$

and

$$|\mathrm{Arg}(\lambda)| > \tau v/\alpha \quad \text{and} \quad \frac{v^\alpha}{\sin^\alpha\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)} \exp\big(\tau v \cot\big((|\mathrm{Arg}(\lambda)| - \tau v)/\alpha\big)\big)$$
$$= |\lambda|, \tag{31}$$

respectively (see also (28) and (29)). Thus, the unique solution of $(30)_2$ defines the value u_0, while the unique positive solution of $(31)_2$ defines the value v_0 (provided $\mathrm{Arg}(\lambda) > 0$). If $\mathrm{Arg}(\lambda) = 0$, then s_0 is real, and we set $v_0 = 0$.

Notice that $(30)_2$, defining the relation between the modulus and argument of λ that is explicit with respect to the argument, forms the boundary of $\overline{\Omega}_u^{\alpha,\tau}$. Its left-hand side, considered as a function of $|\lambda|$, is continuous, increasing and unbounded on $[u^\alpha \exp(\tau u), \infty)$, and its graph in the complex plane creates a Jordan curve symmetric with respect to the real axis, see Fig. 2.

Similarly, $(31)_2$ provides the relation between the modulus and argument of λ that is explicit with respect to the modulus. The equality $(31)_2$ is the boundary of $\overline{\Psi}_v^{\alpha,\tau}$, and its left-hand side, as a function of $|\mathrm{Arg}(\lambda)|$, is continuous on $(\tau v/\alpha, \pi]$ and unbounded in a right neighborhood of the point $\tau v/\alpha$ (for $|\mathrm{Arg}(\lambda)| = \pi$, it takes a value on the negative real axis). This implies that $\overline{\Psi}_v^{\alpha,\tau}$ is unbounded for any $0 < v < \alpha\pi/\tau$ and its boundary splits the complex plane into two parts, see Fig. 2.

Now we are in a position to formulate a complete asymptotic description for the solutions of (1).

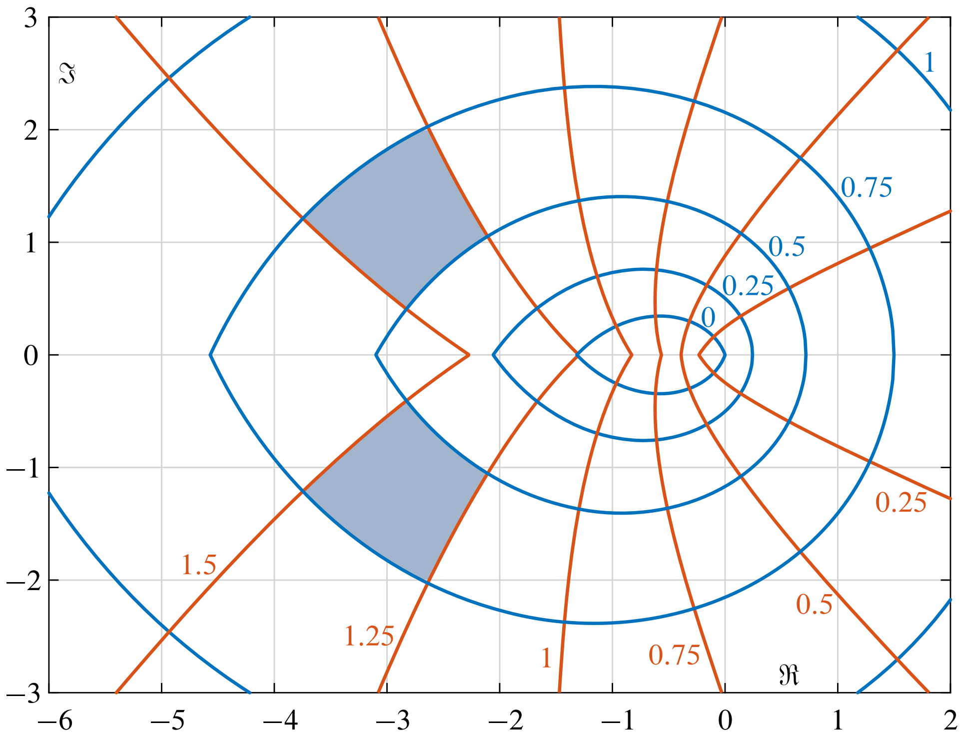

Fig. 2 The figure depicts the boundaries of the sets $\overline{\Omega}_u^{\alpha,\tau}$ (blue) and $\overline{\Psi}_v^{\alpha,\tau}$ (orange) for several values of u and v, respectively (the scenario corresponds to $\alpha = 1.2$ and $\tau = 1$). The particular blue curves represent the set of all $\lambda \in \mathbb{C}$ such that the rightmost characteristic root s_0 of (2) satisfies $\Re(s_0) = u$, and the particular orange curves represent the set of all $\lambda \in \mathbb{C}$ such that the rightmost characteristic root s_0 of (2) satisfies $|\Im(s_0)| = v$. As an example, the blueish curvilinear rectangles then represent the set of all $\lambda \in \mathbb{C}$ such that $0.5 < \Re(s_0) < 0.75$ and $1.25 < |\Im(s_0)| < 1.5$

Theorem 3 *Let $\alpha > 1$, $\tau > 0$ and $\lambda \in \mathbb{C}$.*

(i) *If $\lambda \in \Omega_0^{\alpha,\tau}$, then, for any solution x of (1),*

$$x(t) = \mathcal{O}(t^{1-\alpha}) \quad as \quad t \to \infty.$$

Moreover the algebraic decay order $1-\alpha$ cannot be improved;

(ii) *If $\lambda \notin \Omega_0^{\alpha,\tau}$, then, for any solution x of (1),*

$$x(t) = \exp(u_0 t)(c \exp(\mathrm{i}\, v_0 t) + o(1)) \quad as \quad t \to \infty$$

where c is a complex constant, $u_0 \geq 0$ is the unique solution of $(30)_2$, $v_0 > 0$ is the unique solution of $(31)_2$ if $|\mathrm{Arg}(\lambda)| > 0$, and $v_0 = 0$ if $\mathrm{Arg}(\lambda) = 0$.

Proof (i) The property is a direct consequence of (6) as the set S is empty.

(ii) Let $s_0 = u_0 + \mathrm{i}\, v_0$ be the rightmost characteristic root, and S_0 be the set of the remaining characteristic roots s with a non-negative real part (we remind that $\Re(s) < \Re(s_0)$ for any $s \in S_0$). Then, if α is a non-integer, we can write (6) as

$$\begin{aligned} x(t) &= \exp(u_0 t)\Big(c \exp(\mathrm{i}\, v_0 t) + \sum_{s \in S_0} c_s \exp((s - u_0)t)\Big) + \mathcal{O}(t^{1-\alpha}) \\ &= \exp(u_0 t)\big(c \exp(\mathrm{i}\, v_0 t) + o(1) + \mathcal{O}(o(1))\big) = \exp(u_0 t)\big(c \exp(\mathrm{i}\, v_0 t) + o(1)\big) \end{aligned}$$

as $\quad t \to \infty,$

where $c = c_{s_0}$ is the complex constant from (6) corresponding to the rightmost characteristic root s_0. If α is an integer, then the dominating role of s_0 in asymptotic behavior of (1) is well known. In this case, the assertion of (ii) holds as well. □

Remark 6 (a) The asymptotic formula from Theorem 3(ii) immediately implies

$$x(t) = \mathcal{O}(\exp(u_0 t)) \quad \text{as} \quad t \to \infty \tag{32}$$

for any solution x of (1), and the constant u_0 is non-improvable. Moreover, for large t, the roots of the real and imaginary parts of x tend to the roots of $\cos(v_0 t)$ and $\sin(v_0 t)$, respectively. In both the cases, the distance between the subsequent roots tends to π/v_0. These properties are illustrated by Example 1.

(b) The asymptotic behavior of (1) significantly depends on stability of (1). In particular, the exponential terms in (6) are vanishing in the asymptotically stable case $\Re(s_0) < 0$. However, the situation changes in the limit case $\alpha = 1$ when, in accordance with the first-order theory, the rightmost characteristic root s_0 determines an exponential decay rate of the solutions also in the asymptotically stable case. Since the above argumentation can be extended to this problem as well, our results provide a contribution also to the corresponding classical first-order theory.

Example 1 Let $\alpha = 1.2$, $\tau = 1$, and consider (1) along with the initial conditions $\phi(t) = 1$ $(-1 \le t \le 0)$, $\phi_0 = \phi(0) = 1$, and $\phi_1 = \lim_{t\to 0+} x'(t) = 0$. We compare the corresponding (numerical) solutions of (1) for two distinct values of λ, namely $\lambda_1 = -2 + \mathrm{i}$ and $\lambda_2 = -3 + \mathrm{i}\,0.1$. As indicated by Fig. 2, both the values λ_1, λ_2 lie in the instability region. In particular, the real parts u_0 of the corresponding rightmost roots are approximately 0.4721 and 0.4917, and their imaginary parts v_0 are 1.2321 and 1.5844, respectively.

The real parts of the solutions of (1) with two above specified sets of entries, along with the growth-rate functions $\exp(u_0 t)$, are depicted in Figs. 3 and 4. The graphs suggest that the modulus of constant c introduced in Theorem 3(ii) is less than one for $\lambda = \lambda_1$, and greater than one for $\lambda = \lambda_2$.

To illustrate behavior of the solutions x in better detail, Figs. 5 and 6 depict the ratio $\Re(x(t))/\exp(u_0 t)$ for λ_1 and λ_2, respectively. The resulting functions are bounded, but do not tend to zero which is a consequence of non-improvability of the constant u_0 in (32).

As mentioned in Remark 6(a), the distance between the subsequent roots of $\Re(x(t))$ tends to π/v_0. Figs. 7 and 8 illustrate this fact. We can see that while in the case of λ_1 the convergence is rather fast and the distance seems to be somewhat stabilized around the seventh root, in the case of λ_2, the stabilization occurs around the hundredth root.

5 Concluding remarks

The aim of the paper was to develop the Lambert function theory, and then apply the obtained results in qualitative investigations of (1). Using this approach, we were able

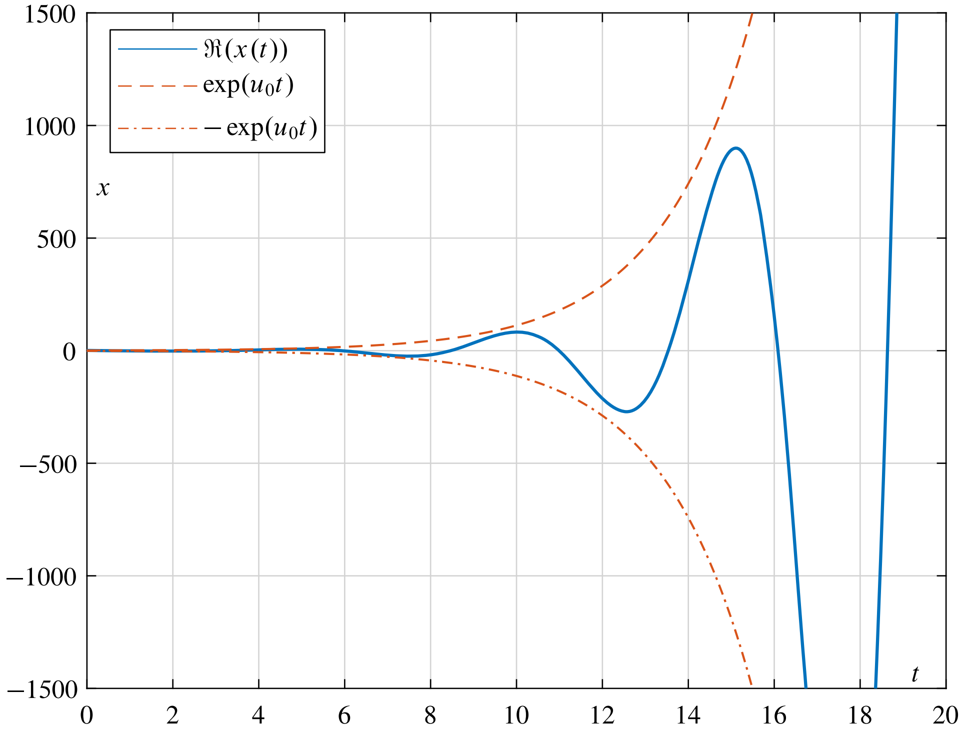

Fig. 3 The real part of the solution x of (1) for $\alpha = 1.2$, $\tau = 1$ and $\lambda_1 = -2 + \mathrm{i}$, along with the corresponding growth-rate functions $\pm\exp(0.4721t)$

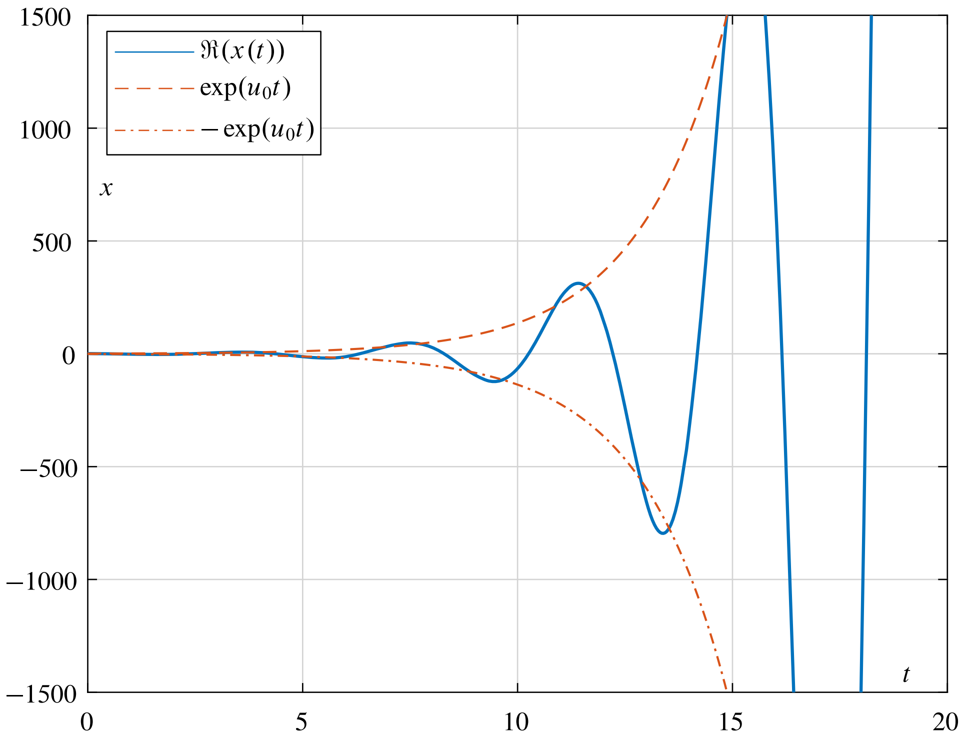

Fig. 4 The real part of the solution x of (1) for $\alpha = 1.2$, $\tau = 1$ and $\lambda_2 = -3 + \mathrm{i}\,0.1$, along with the corresponding growth-rate functions $\pm\exp(0.4917t)$

to formulate a precise asymptotic description of the solutions of (1). Particularly, in addition to an algebraic decay rate of the solutions in the stable case (described in some earlier papers), we could observe an exponential growth of the solutions in the

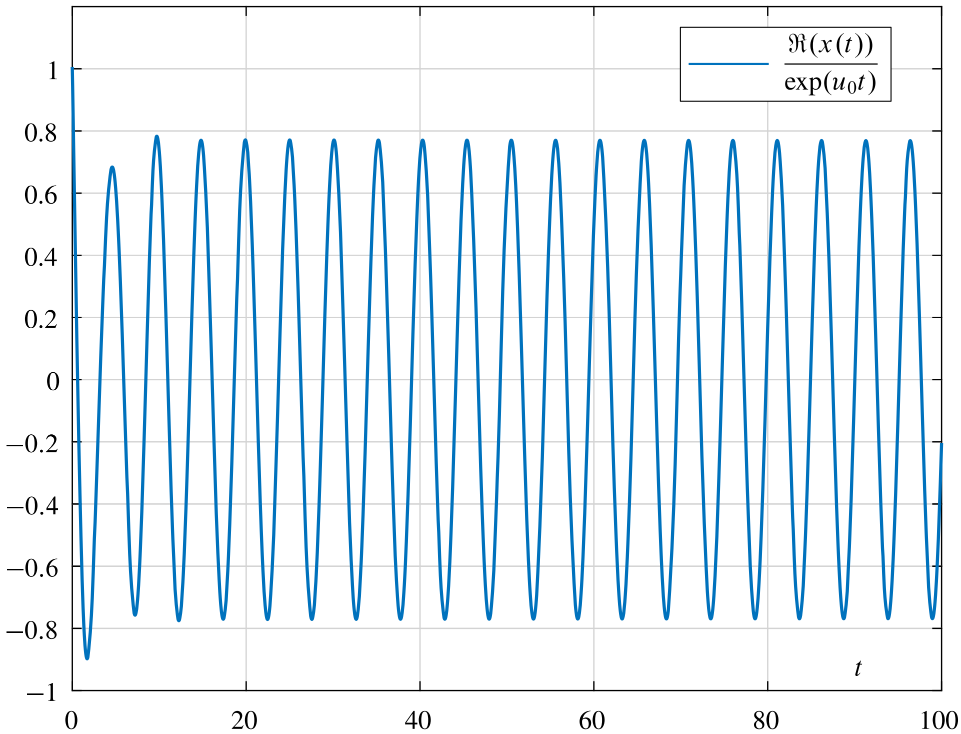

Fig. 5 The real part of the solution x of (1) for $\alpha = 1.2$, $\tau = 1$ and $\lambda_1 = -2 + \mathrm{i}$, divided by its growth-rate function $\exp(0.4721t)$

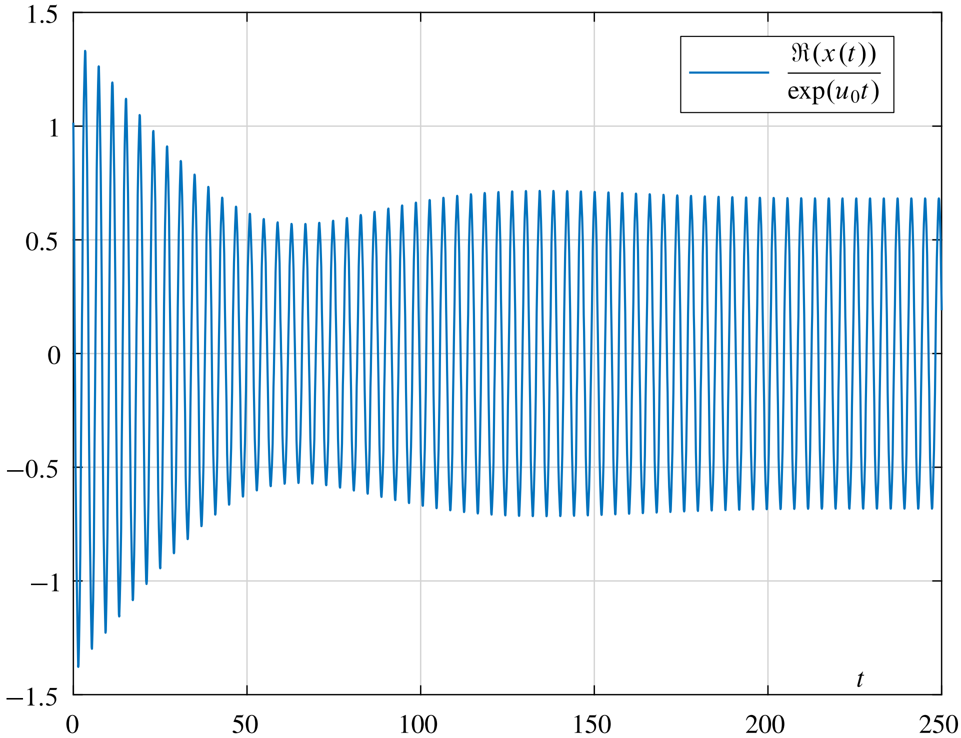

Fig. 6 The real part of the solution x of (1) for $\alpha = 1.2$, $\tau = 1$ and $\lambda_2 = -3 + \mathrm{i}\,0.1$, divided by its growth-rate function $\exp(0.4917t)$

unstable case; the rate of this growth was determined as a (unique) real root of an auxiliary transcendental equation.

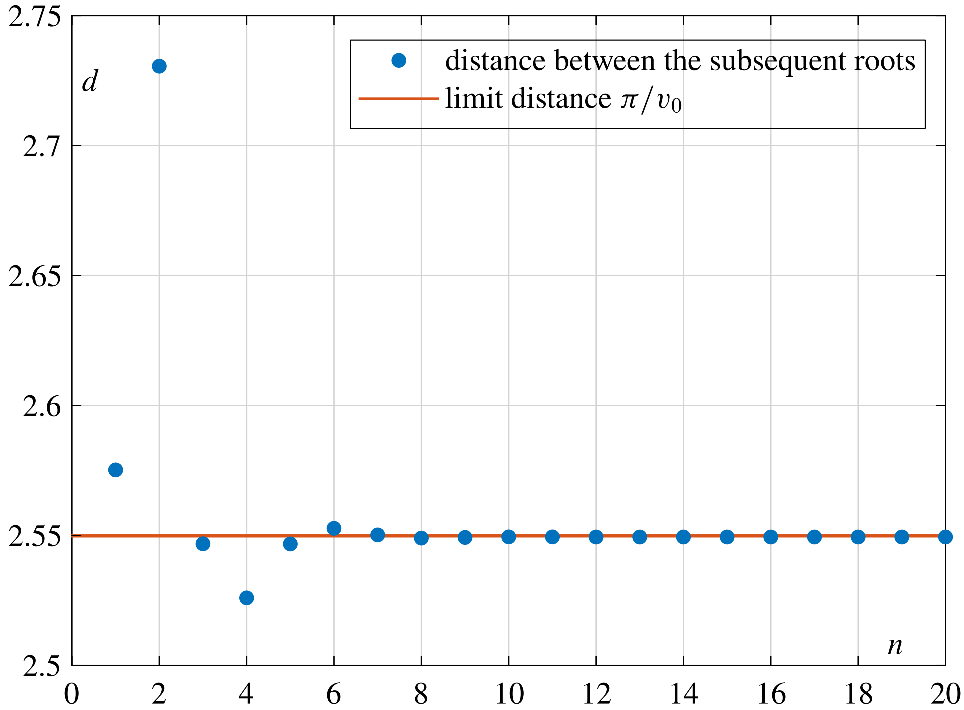

Fig. 7 The distance between the subsequent roots of $\Re(x(t))$ for $\alpha = 1.2$, $\tau = 1$ and $\lambda_1 = -2+\mathrm{i}$ is tending to $\pi/1.2321$

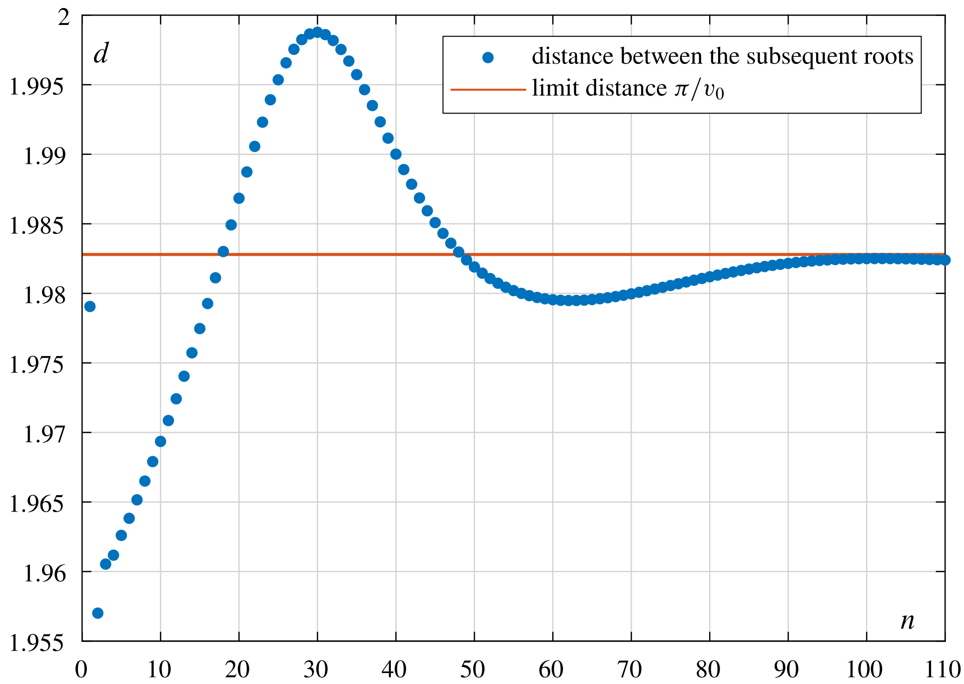

Fig. 8 The distance between the subsequent roots of $\Re(x(t))$ for $\alpha = 1.2$, $\tau = 1$ and $\lambda_2 = -3 + \mathrm{i}\,0.1$ is tending to $\pi/1.5844$

However, the impact of the presented results is not limited to the theory of FDDEs only. Our approach offers an alternate way how to prove (and also strengthen) some classical assertions of the Lambert function theory. Moreover, to the best of our knowledge, the derived asymptotic formulae are new also in the first-order case. Here, contrary to the fractional case, our results can be applied also in the stable case where a (non-improvable) rate of exponential decay of the solutions can be determined.

Since we have formulated our results for (1) with a complex coefficient λ, their extension to the vector case is nearly straightforward provided the eigenvalues of a (real) system matrix are simple. Regarding eigenvalues with higher multiplicities, some additional argumentation seems to be necessary. Based on related cases discussed in earlier papers, one can expect a slight modification of the solutions growth, but no impact on the asymptotic frequency.

Our final remark concerns the case $0 < \alpha < 1$ not involved among the assumptions of the assertions of Sect. 4. The procedure of computing the characteristic roots uses the law of exponents which is, in general, not valid for complex numbers. Thus, some superfluous roots of the characteristic equation may appear if $0 < \alpha < 1$ (as illustrated via a counterexample in Remark 1). In this case, our stability and asymptotic formulae remain basically true, but we cannot confirm their strictness. In particular, we cannot claim that the above described rate of exponential growth of solutions is non-improvable. Nevertheless, we conjecture that a more thorough analysis of the corresponding branches of a complex power can overcome this problem, and thus achieve the strict asymptotic results for all $\alpha > 0$. Such an analysis provides another possible topic for the next research.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Appendix E

Paper on lower-order two-term FDDE [13] (AMC, 2017)

Fairly soon after entering the field of fractional delay equations, we began to deal with linear equations involving fractional derivative, delayed term and also an undelayed one. The addition of the undelayed term brings significant technical challenges, even in scalar case, which we first addressed in [13] (co-authors: J. Čermák, Z. Došlá; my author's share 45 %).

We derived explicit necessary and sufficient conditions for asymptotic stability, including asymptotic formulas for solutions (including algebraic decay rate towards zero). To achieve this, we further developed our inverse Laplace transform technique and introduced a broad family of functions containing exponentials, Mittag-Leffler functions and generalized delay exponential of Mittag-Leffler type as special cases.

Although the stability region in space of equation's coefficients appears simple, it comprises a region of delay-independent stability and region where the stability boundary depends on the delay. This work marked my first direct encounter with the phenomenon of stability switching, where the stability property changes with increasing delay. In this case, it meant just a one-time loss of stability, but in subsequent research, much richer situations awaited us.

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Fractional differential equations with a constant delay: Stability and asymptotics of solutions

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Asymptotic behaviour

ABSTRACT

The paper discusses stability and asymptotic properties of a fractional-order differential equation involving both delayed as well as non-delayed terms. As the main results, explicit necessary and sufficient conditions guaranteeing asymptotic stability of the zero solution are presented, including asymptotic formulae for all solutions. The studied equation represents a basic test equation for numerical analysis of delay differential equations of fractional type. Therefore, the knowledge of optimal stability conditions is crucial, among others, for numerical stability investigations of such equations. Theoretical conclusions are supported by comments and comparisons distinguishing behaviour of a fractional-order delay equation from its integer-order pattern.

1. Introduction

We investigate stability and asymptotic properties of the fractional delay differential equation

$$\mathrm{D}^{\alpha}y(t) = a\,y(t) + b\,y(t-\tau), \qquad t > 0 \tag{1}$$

with real coefficients a, b, a positive real lag τ and the fractional Caputo derivative operator D^{α} ($0 < \alpha < 1$ is assumed to be a real number).

Letting $\alpha \to 1$ from the left, $\mathrm{D}^{\alpha}y(t)$ becomes $y'(t)$ and (1) is reduced to the classical delay differential equation

$$y'(t) = a\,y(t) + b\,y(t-\tau), \qquad t > 0\,, \tag{2}$$

studied frequently due to its theoretical as well as practical importance (see, e.g. [12]). This equation serves, among others, as the basic test equation for stability analysis of various numerical discretizations of delay differential equations (see, e.g. [1,9]). In this connection, stability conditions for (2) are traditionally required in the optimal form, i.e. as the necessary and sufficient ones. There are known two types of such conditions for asymptotic stability of the zero solution of (2) that we recall in the following two assertions (see [12]). As it is customary, by asymptotic stability of the zero solution of (2) we understand the property that any solution y of (2) is eventually tending to the zero solution.

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Theorem 1. *Let a, b and $\tau > 0$ be real numbers. The zero solution of (2) is asymptotically stable if and only if the couple (a, b) is an interior point of the area bounded by the line $a + b = 0$ from above and by the parametric curve*

$$a = \varphi \cot(\tau\varphi), \quad b = -\frac{\varphi}{\sin(\tau\varphi)}, \qquad \varphi \in \left(0, \frac{\pi}{\tau}\right)$$

from below.

Theorem 2. *Let a, b and $\tau > 0$ be real numbers. The zero solution of (2) is asymptotically stable if and only if either*

$$a \le b < -a \qquad \text{and} \qquad \tau \text{ is arbitrary}, \tag{3}$$

or

$$|a| + b < 0 \qquad \text{and} \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}. \tag{4}$$

While Theorem 1 describes the stability boundary for (2) in the (a, b)-plane, the conditions of Theorem 2 seem to be more explicit. In particular, $(4)_2$ presents the value of the stability switch when (2) loses its stability property as the delay τ is monotonically increasing.

It is also well-known that asymptotic stability of the zero solution of (2), described via the conditions of Theorems 1 and 2, is of exponential type, i.e. there exists $\delta > 0$ such that $y(t) = \mathcal{O}(\exp[-\delta t])$ as $t \to \infty$ for any solution y of (2). The value δ depends on location of roots of the characteristic equation

$$s - a - b\exp[-s\tau] = 0 \tag{5}$$

with respect to the imaginary axis (its estimates are discussed, e.g. in [8]).

The involvement of fractional-order derivatives into delay differential equations represents a new type combining advantages of both delayed and non-integer derivative terms, especially hereditary properties, more degrees of freedom and other advantages of fractional modelling. Since application areas of fractional delay differential equations are especially control theory and robotics, the question of their stability (and asymptotics) is again of main interest. In general, stability and asymptotic analysis of fractional delay differential equations is just at the beginning. As it is evident from the literature (see, e.g. [6,16,17,21]), almost all the existing stability results on autonomous equations of this type are based on the root locus of appropriate characteristic equations, and they do not provide universally acceptable effective criteria for testing stability of a given fractional delay equation.

Therefore, the main goal of this paper is to extend the above stated properties of (2) to (1). Since (1) may serve as a basic prototype of fractional delay differential equations, formulation of non-improvable stability conditions and related asymptotic formulae is of a great importance in qualitative as well as numerical analysis of fractional delay differential equations.

Following the classical case, we say that the zero solution of (1) is asymptotically stable if any solution y of (1) is eventually tending to the zero solution. Similarly, the zero solution of (1) is called stable if any its solution is eventually bounded. To describe asymptotics of solutions of (1), we shall use the following asymptotic symbols:

$$f(t) \sim g(t) \qquad \text{if} \qquad \lim_{t\to\infty} \frac{|f(t)|}{|g(t)|} = K > 0,$$

$$f(t) \sim_{sup} g(t) \qquad \text{if} \qquad \limsup_{t\to\infty} \frac{|f(t)|}{|g(t)|} = K > 0.$$

We note that (1) has been studied in the purely delayed case (when $a = 0$) in [5,15]. While the first paper presents stability criterion based on a transcendent inequality involving the fractional Lambert function, the latter paper already contains explicit conditions (which are, as we have already mentioned, very rare for fractional delay differential equations). We recall here the explicit relevant results (reformulated to our notation) characterizing stability and asymptotics of (1) with $a = 0$ (see [15, Theorem 5.1]).

Theorem 3. *Let $0 < \alpha < 1$, $b \neq 0$ and $\tau > 0$ be real numbers.*

(i) *The zero solution of*

$$\mathrm{D}^\alpha y(t) = by(t - \tau), \qquad t > 0 \tag{6}$$

is asymptotically stable if and only if

$$-\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^\alpha < b < 0.$$

In this case, $y(t) \sim t^{-\alpha}$ as $t \to \infty$ for any solution y of (6).

(ii) *The zero solution of (6) is stable, but not asymptotically stable, if and only if*

$$b = -\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^\alpha.$$

A vector extension of Theorem 3 has been recently derived in [4]. As we have declared above, the goal of this paper is to provide its another extension, namely to (1) involving both the delayed as well as non-delayed term (for some preliminary stability results on (1) we refer to [2,11]). More precisely, we are going to derive direct fractional extensions of Theorems 1 and 2, including asymptotic descriptions of solutions of (1).

The paper is organized as follows. Section 2 recalls some basic necessary notions and properties of fractional calculus, especially those related to the Laplace transform. In Section 3, we discuss the characteristic equation associated with (1) and describe some of its root properties. Section 4 presents main results on stability and asymptotics of (1). In particular, we formulate here fractional analogues of Theorems 1 and 2 and present asymptotic formulae for solutions of (1). Some comments and comparisons related to these results are mentioned as well. Section 5 is devoted to the proof of a key auxiliary result describing asymptotics of (1) in terms of roots location of the associate characteristic equation. The final section summarizes the results and outlines perspectives of a future research.

2. Preliminaries

Throughout this paper, we use the following standard definitions of the fractional integral of a real function f

$$\mathrm{D}^{-\nu} f(t) = \int_0^t \frac{(t-\xi)^{\nu-1}}{\Gamma(\nu)} f(\xi)\mathrm{d}\xi, \qquad \nu > 0, \quad t > 0$$

and the Caputo fractional derivative of f

$$\mathrm{D}^{\alpha} f(t) = \mathrm{D}^{-(1-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right), \qquad 0 < \alpha < 1, \quad t > 0$$

where we put $\mathrm{D}^0 f(t) = f(t)$ (for more details on basics of fractional calculus theory we refer, e.g. to [14,20,25]).

The main analytical technique used in stability investigations of linear fractional differential equations is based on the well-known Laplace transform. For a given real function f, it is introduced via

$$\mathcal{L}(f(t))(s) = \int_0^{\infty} f(t)\exp[-st]\mathrm{d}t \ \big(\equiv F(s)\big), \qquad s \in \mathfrak{D}$$

where $\mathfrak{D} \subset \mathbb{C}$ contains all complex s such that the integral converges. The inverse Laplace transform of F can be expressed via the contour integral

$$\mathcal{L}^{-1}(F(s))(t) = \int_{\mathcal{C}} F(s)\exp[st]\mathrm{d}s$$

where the contour $\mathcal{C} \subset \mathfrak{D}$ is usually considered as a line $\Re(s) = c$ such that all singularities of F lie to the left of this line. However, in view of the fundamental theory of complex integration, this line can be changed into arbitrary contour $\mathcal{C}$ such that its orientation with respect to singularities of F remains preserved (we recall that contour is an oriented piecewise smooth curve). In our next analysis, we utilize the contour

$$\gamma(\mu, \vartheta) = \gamma_1 + \gamma_2 + \gamma_3 \tag{7}$$

such that its three oriented segments are given via

$$\gamma_1 = \{s \in \mathbb{C} : s = -u\exp[-\mathrm{i}\vartheta], u \in (-\infty, -\mu)\},$$
$$\gamma_2 = \{s \in \mathbb{C} : s = \mu\exp[\mathrm{i}u], u \in [-\vartheta, \vartheta]\},$$
$$\gamma_3 = \{s \in \mathbb{C} : s = u\exp[\mathrm{i}\vartheta], u \in (\mu, \infty)\}$$

where $\mu > 0$ and $\vartheta \in (-\pi, \pi]$. This contour is depicted on Fig. 1.

A key computational property, namely the Laplace transform of fractional derivative of f, is given by the formula

$$\mathcal{L}(D^{\alpha} f(t))(s) = s^{\alpha}\mathcal{L}(f(t))(s) - s^{\alpha-1} f(0), \qquad 0 < \alpha < 1 \tag{8}$$

extending the classical first-order case (see, e.g. [20]). To avoid a possible misunderstanding, we emphasize that the principal branch of the power functions occurring in (8) has to be considered. It is a consequence of the following property (for more details see, e.g. [7, pp. 8–10]).

Proposition 1. *Let $q > -1$. Then $\mathcal{L}(t^q)(s)$ is a single-valued function given by*

$$\mathcal{L}(t^q)(s) = s^{-q-1}\Gamma(q+1)$$

where the principal branch of the complex power function is considered.

As the last step of this section, we introduce the family of functions

$$\mathcal{R}_{\alpha,\beta}^{a,b,\tau}(t) = \mathcal{L}^{-1}\left(\frac{s^{\alpha-\beta}}{s^{\alpha} - a - b\exp[-s\tau]}\right)(t) \tag{9}$$

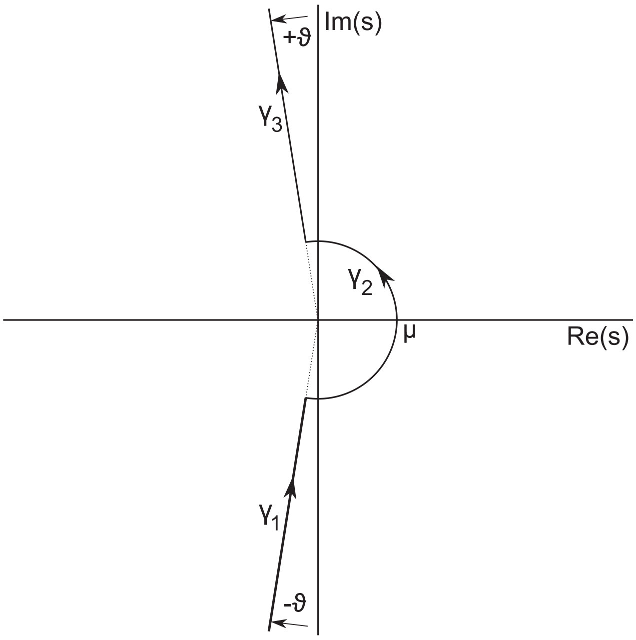

Fig. 1. The contour $\gamma(\mu, \vartheta) = \gamma_1 + \gamma_2 + \gamma_3$ in the complex plane.

where α, β, $\tau > 0$ and $a, b \in \mathbb{R}$. This family involves certain special functions appearing in fractional calculus such as

$$\mathcal{R}^{0,0,\tau}_{\alpha,\beta}(t) = t^{\beta-1}/\Gamma(\beta)\,,$$
$$\mathcal{R}^{a,0,\tau}_{\alpha,\beta}(t) = t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\,,$$
$$\mathcal{R}^{0,b,\tau}_{\alpha,\beta}(t) = G^{b,\tau,0}_{\alpha,\beta}(t)\,,$$
$$\mathcal{R}^{a,b,0}_{\alpha,\beta}(t) = t^{\beta-1}E_{\alpha,\beta}((a+b)t^{\alpha})\,.$$

Here, $E_{\alpha,\beta}$ is the two-parameter Mittag–Leffler function (see, e.g. [20]) and $G^{b,\tau,0}_{\alpha,\beta}$ is the generalized delay exponential function of Mittag–Leffler type (see [4]). We can see that for $\alpha = \beta = 1$ the functions coincide with the standard exponential functions or delayed exponential functions known from the theory of delay differential equations (see, e.g. [3]).

The functions $\mathcal{R}^{a,b,\tau}_{\alpha,\beta}$ generalize the notion of fundamental solution for classical delay differential equations (see, e.g. [10]); in particular, the representation formula

$$y(t) = \phi(0)\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) + b\int_{-\tau}^{0}\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-\tau-u)h(t-\tau-u)\phi(u)\mathrm{d}u \tag{10}$$

holds for all $t > 0$. Here, y is a solution of (1), h is the Heaviside step function and ϕ is the associated initial function (which is supposed to be piecewise continuous on $[-\tau, 0]$), i.e. it holds

$$y(t) = \phi(t)\,, \qquad -\tau < t \le 0\,. \tag{11}$$

The verification of (10) is analogous to that employed in [4, Theorem 3] and therefore it is omitted. Based on (9) and (10),

$$Q(s) \equiv s^{\alpha} - a - b\exp[-s\tau] = 0 \tag{12}$$

plays the role of a characteristic equation associated with (1). A more detailed explanation of this role and its connection to stability properties of (1) is discussed in the next sections.

3. Distribution of roots of the characteristic equation

Eq. (12) admits infinitely many complex roots. In this section, we state some of their basic properties and analyse their location in the complex plane.

Proposition 2. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers.*

(i) *If $a + b \ge 0$ then (12) has a non-negative real root.*
(ii) *If s is a root of (12) then its complex conjugate $\bar{s}$ is also a root of (12).*
(iii) *Let $0 < \omega < \pi$ be arbitrary. Then (12) has no more than a finite number of roots s such that $|\arg(s)| \le \omega$.*
(iv) *If $Q(s) = Q'(s) = 0$ for some complex s, then s is a positive real number.*

Proof. If $a+b \geq 0$ then $Q(0) = -a-b \leq 0$. At the same time, $Q(s) \to \infty$ as $s \to \infty$, s being real. This implies the property (i) (more precisely, (12) has the zero root if $a+b=0$ and a positive real root if $a+b>0$).

Now let

$$s = \varrho \exp(\mathrm{i}\psi)\,, \tag{13}$$

$\varrho = |s|$ and $\psi = \arg(s) \in (-\pi, \pi]$, be a root of (12). Then

$$\varrho^\alpha \exp(\mathrm{i}\alpha\psi) - a - b\exp[-\varrho\tau \exp(\mathrm{i}\psi)] = 0\,,$$

i.e. ϱ and ψ satisfy

$$\varrho^\alpha \cos(\alpha\psi) - a - b\exp[-\varrho\tau\cos(\psi)]\cos[\varrho\tau\sin(\psi)] = 0\,, \tag{14}$$

$$\varrho^\alpha \sin(\alpha\psi) + b\exp[-\varrho\tau\cos(\psi)]\sin[\varrho\tau\sin(\psi)] = 0\,. \tag{15}$$

This immediately yields the property (ii). To prove (iii), we first show the existence of a real $\varrho_1 > 0$ large enough such that (12) has no roots s with $|s| > \varrho_1$ and $|\arg(s)| \leq \omega$. Squaring and adding (14) and (15) we get the relation

$$\varrho^{2\alpha} - 2a\varrho^\alpha\cos(\alpha\psi) + a^2 = b^2\exp[-2\varrho\tau\cos(\psi)] \tag{16}$$

as the necessary condition for ϱ, ψ to satisfy (14) and (15). We distinguish two cases.

Let $\omega \leq \pi/2$. Then the property $|\arg(s)| \leq \omega$ implies $|\psi| \leq \pi/2$, i.e.

$$b^2\exp[-2\varrho\tau\cos(\psi)] \leq b^2\,.$$

On the other hand,

$$\varrho^{2\alpha} - 2a\varrho^\alpha\cos(\alpha\psi) + a^2 > (\varrho^\alpha - |a|)^2\,.$$

This particularly implies that (16) cannot be satisfied for any $\varrho \geq \varrho_1'$, where

$$\varrho_1' = (|a| + |b|)^{1/\alpha}\,,$$

hence (12) has no roots with $|s| \geq \varrho_1'$ and $|\arg(s)| \leq \pi/2$.

Now let $\omega > \pi/2$. Put $\kappa = -\tau\cos(\omega) > 0$. Since

$$\varrho^{2\alpha} - 2a\varrho^\alpha\cos(\alpha\psi) + a^2 < (\varrho^\alpha + |a|)^2\,,$$

it is enough to find $\varrho_1'' > 0$ such that

$$\varrho^\alpha + |a| < |b|\exp[\kappa\varrho]$$

for all $\varrho \geq \varrho_1''$. The existence of such a value is obvious and, using appropriate elementary calculations, we can give various specifications of ϱ_1''. To summarize it, for any given $0 < \omega < \pi$, we can find $\rho_1 > 0$ such that (12) has no roots s with $|s| > \varrho_1$ and $|\arg(s)| \leq \omega$.

Further, we can easily observe the existence of a real $\varrho_2 > 0$ sufficiently small such that (12) has no nonzero roots lying inside the circle $|s| < \varrho_2$. Indeed, the Taylor expansion enables to write (12) as

$$s^\alpha = a + b(1 - s\tau + s^2\tau^2/2! - \cdots)\,. \tag{17}$$

If $a+b \neq 0$ then (17) has no roots s with $|s| < \varrho_2$, $\varrho_2 > 0$ being sufficiently small. If $a+b=0$, then $s=0$ is a root of (17) and, assuming $s \neq 0$, (17) yields

$$s^{\alpha-1} = -b(\tau - s\tau^2/2! + \cdots)\,,$$

hence (17) has no nonzero roots s with $|s| < \varrho_2$, $\varrho_2 > 0$ being sufficiently small.

Finally, let Ω be the compact set of complex s such that $\varrho_1 \leq |s| \leq \varrho_2$ and $|\arg(s)| \leq \omega$. Then Q is analytic on Ω, hence it cannot have infinitely many roots in Ω. This completes the proof of (iii).

To prove the property (iv), we assume that the relations

$$s^\alpha - a - b\exp[-s\tau] = 0 \qquad \text{and} \qquad \alpha s^{\alpha-1} + \tau b\exp[-s\tau] = 0 \tag{18}$$

hold for a suitable nonzero complex s (obviously, $s=0$ cannot satisfy (18)). Elimination of the exponential term yields

$$a = s^{\alpha-1}\left(s + \frac{\alpha}{\tau}\right). \tag{19}$$

If we substitute (13) into (19) and compare imaginary parts, we obtain

$$\tau\varrho\sin(\alpha\psi) = \alpha\sin[(1-\alpha)\psi]\,. \tag{20}$$

Similarly, (18) implies

$$b = -\frac{\alpha}{\tau}s^{\alpha-1}\exp[s\tau] \tag{21}$$

and an analogous argumentation as used above leads to

$$\tau\varrho\sin(\psi)+(\alpha-1)\psi=k\pi \tag{22}$$

for a suitable integer k.

Now assume that $\psi\neq 0$. Then we can express $\tau\varrho$ from (20) and (22) to compare their corresponding right-hand sides. This yields

$$g_1(\psi)\equiv\frac{\alpha\sin(\psi)}{\sin(\alpha\psi)}=\frac{(1-\alpha)\psi+k\pi}{\sin[(1-\alpha)\psi]}\equiv g_2(\psi)\,. \tag{23}$$

We show that

$$g_1(\psi)<g_2(\psi)\qquad\text{for all } -\pi<\psi\le\pi,\ \psi\neq 0 \text{ and all integers } k. \tag{24}$$

Obviously, it is enough to check (24) for all $0<\psi\le\pi$ and $k=0$. In such a case, $g_1(\psi)=g_2(\psi)=1$ as $\psi\to 0$ and $g_1'(\psi)<0$, $g_2'(\psi)>0$ for all $0<\psi<\pi$ (verification of these sing derivative properties requires some routine and straightforward calculations which are omitted). This confirms (24), hence (23) cannot occur for any $-\pi<\psi\le\pi$, $\psi\neq 0$ and any integer k. In other words, (18) cannot be satisfied if s is a complex number with a nonzero argument ψ.

If $\psi=0$ then $s=\varrho$ and

$$a=\varrho^{\alpha-1}\left(\varrho+\frac{\alpha}{\tau}\right),\qquad b=-\frac{\alpha}{\tau}\varrho^{\alpha-1}\exp[\varrho\tau] \tag{25}$$

by use of (19) and (21). For all $\varrho>0$, (25) defines the set of all couples (a, b) such that there exists a positive real s satisfying (18). □

Remark 1.

(a) We can easily check that the condition $Q(s)=Q'(s)=Q''(s)=0$ does not hold for any complex s. In other words, if s is a multiple root of Q, then it is a (positive real) double root.
(b) The property Proposition 2 (iii) generalizes the well-known fact, namely that (5) has only finitely many roots with positive real parts (indeed, if we put $\omega=\pi/2$, we get such a conclusion also for (12)).

In classical analysis of (2), asymptotic stability of the zero solution occurs if and only if all roots of (5) have negative real parts (effective conditions ensuring this property are involved in Theorems 1 and 2). In the next section, we deduce an analogous requirement on distribution of roots of (12). Therefore, the next aim of this section is to derive effective conditions on parameters of (12) ensuring that all its roots have negative real parts.

In the case of standard (integer-order) delay differential equations and their associated characteristic equations, this matter is solved via the D-partition method (see, e.g. [12]). This method is applicable also in the fractional-order case and we perform its basic steps (routine calculations will be omitted).

Let $BL_\alpha^\tau(a,b)$ be the boundary locus for (12), i.e. the set of all real couples (a, b) such that (12) admits a purely imaginary root $s=\mathrm{i}\varphi=\varphi\exp[\mathrm{i}\pi/2]$ (note that it is enough to consider only the case $\varphi\ge 0$ due to the property Proposition 2 (ii)). By (14) and (15), such couples (a, b) satisfy

$$\varphi^\alpha\cos(\alpha\pi/2)-a-b\cos(\varphi\tau)=0\,, \tag{26}$$

$$\varphi^\alpha\sin(\alpha\pi/2)+b\sin(\varphi\tau)=0\,. \tag{27}$$

Now we distinguish two cases. First let $\varphi\tau=m\pi$ for a suitable integer m. Then (27) implies $\varphi=0$, i.e. $m=0$, and the relevant part of $BL_\alpha^\tau(a,b)$ corresponding to this case consists of the line $a+b=0$.

Now let $\varphi\tau\neq m\pi$ for all integers m. Then (26) and (27) yield the solution

$$a=\frac{\varphi^\alpha\sin(\varphi\tau+\alpha\pi/2)}{\sin(\varphi\tau)},\qquad b=-\frac{\varphi^\alpha\sin(\alpha\pi/2)}{\sin(\varphi\tau)}\,.$$

Hence, $BL_\alpha^\tau(a,b)$ corresponding to this case is formed by the system of curves

$$a_m(\varphi)=\frac{\varphi^\alpha\sin(\varphi\tau+\alpha\pi/2)}{\sin(\varphi\tau)},\qquad b_m(\varphi)=-\frac{\varphi^\alpha\sin(\alpha\pi/2)}{\sin(\varphi\tau)}\,, \tag{28}$$

$m\pi/\tau<\varphi<(m+1)\pi/\tau$, $m=0,1,\ldots$. The set $BL_\alpha^\tau(a,b)$ is depicted in the (a, b)-plane on Fig. 2. We note that the construction of this set for (12) involving multi-valued function s^α has been performed in [11]. As we have emphasized in the part preceding Proposition 1, it is enough to consider appropriate single-valued function in this problem.

Our next analysis of the area of couples (a, b) such that (12) has all roots with negative real parts originates from continuous dependance of roots of (12) on the coefficients a, b. This property particularly implies that the number of roots of (12) with positive real parts can be changed only when its coefficients a, b are crossing an element of $BL_\alpha^\tau(a,b)$. In other words, the number of roots of (12) having a positive real part remains unchanged in all open sets whose boundaries are formed by the line $a+b=0$ or by some curves (28) (or their parts). Then it is enough to choose representatives of these open sets to specify the number of roots of (12) with positive real parts within these sets.

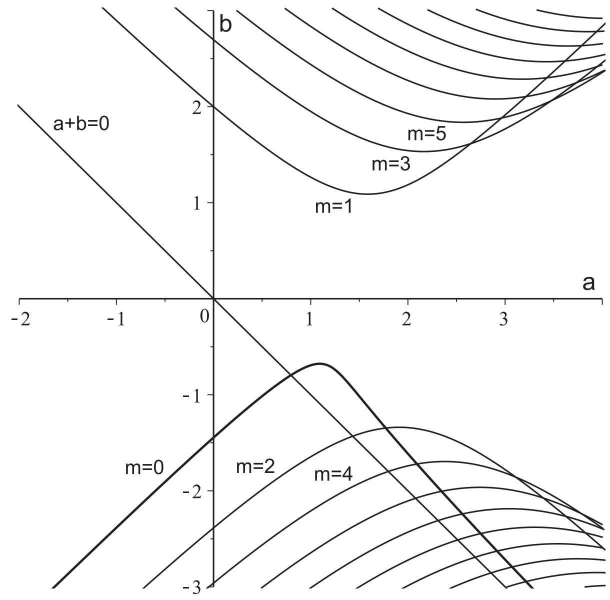

Fig. 2. The set $BL^{\tau}_{\alpha}(a, b)$ for $\alpha = 0.4,\ \tau = 1$.

Since we are looking for the couples (a, b) such that (12) has all roots with negative real parts, Proposition 2 (i) implies that it is enough to restrict on the half-plane $a + b < 0$. It is easy to check that the curves (28) with even m intersect the line $a + b = 0$ when $\varphi = (m + 1 - \alpha)\pi/\tau$, hence it remains to investigate their parts

$$(a_m(\varphi), b_m(\varphi)), \qquad (m+1-\alpha)\pi/\tau < \varphi < (m+1)\pi/\tau, \quad m = 0, 2, 4, \dots$$

lying to the left of this line. A more detailed computational analysis shows that these curves cross the b-axis at the points $b^*_m = -[(2m+2-\alpha)\pi/(2\tau)]^{\alpha}$, tend to the asymptotes $b = a - [(m+1)\pi/\tau]^{\alpha}\cos(\alpha\pi/2)$ as $\varphi \to (m+1)\pi/\tau$ from the left and, moreover, these curves do not intersect each other (see also Fig. 2). Then, choosing appropriate points on the b-axis as suitable representatives, previous considerations along with Theorem 3 yield that (12) has all roots with negative real parts if and only if the couple (a, b) is an interior point of the area bounded by the line $a + b = 0$ from above and by the parametric curve

$$a = \frac{\varphi^{\alpha}\sin(\tau\varphi + \alpha\pi/2)}{\sin(\tau\varphi)}, \quad b = -\frac{\varphi^{\alpha}\sin(\alpha\pi/2)}{\sin(\tau\varphi)}, \qquad \varphi \in \left(\frac{(1-\alpha)\pi}{\tau}, \frac{\pi}{\tau}\right) \tag{29}$$

from below.

Note also that an alternative way how to deduce such an area from a given boundary locus is based on differentiation of (12) with respect to some of its parameters. Following this way, it can be shown that $d(\Re(s))/da > 0$ at $s = \mathrm{i}\varphi$. This implies that all roots of (12) cross the imaginary axis at $s = \mathrm{i}\varphi$ from the left to the right as a increases, and thus we arrive at the same conclusion as above.

We can summarize the previous considerations in the following

Proposition 3. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers. Then all roots of (12) have negative real parts if and only if the couple (a, b) is an interior point of the area bounded by the line $a + b = 0$ from above and by the parametric curve (29) from below.*

Now we rewrite the parametric form (29) into the explicit one. Doing this, (29) implies that

$$\frac{a}{-b} = \frac{\sin(\varphi\tau + \alpha\pi/2)}{\sin(\alpha\pi/2)}.$$

Taking into account the restriction $\varphi \in ((1-\alpha)\pi/\tau, \pi/\tau)$, we can express φ and eliminate it via its substitution into $(29)_2$. This yields

$$b = -\frac{\left((1-\alpha)\pi/2 + \arccos[(-a/b)\sin(\alpha\pi/2)]\right)^{\alpha}\sin(\alpha\pi/2)}{\tau^{\alpha}\sin\left((1-\alpha)\pi/2 + \arccos[(-a/b)\sin(\alpha\pi/2)]\right)}.$$

Obviously, some additional calculations enable to determine the delay τ from this relation explicitly. Consequently, this argumentation implies that (29) can be written as $\tau = \tau^*$ where

$$\tau^* = \frac{(1-\alpha)\pi/2 + \arccos[(-a/b)\sin(\alpha\pi/2)]}{[a\cos(\alpha\pi/2) + (b^2 - a^2\sin^2(\alpha\pi/2))^{1/2}]^{1/\alpha}} \tag{30}$$

and $|a| + b < 0$. Using this notation we have

Proposition 4. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers. Then all roots of (12) have negative real parts if and only if it holds either*

$$a \le b < -a \qquad \text{and} \qquad \tau \text{ is arbitrary}, \tag{31}$$

or

$$|a| + b < 0 \qquad \text{and} \qquad \tau < \tau^*. \tag{32}$$

Remark 2. A similar issue has been discussed in [2] where (12) is analysed under the restrictive assumption $a < 0$ (in our notation). The computational technique used in [2] is different from ours and leads to a formally slightly more complicated evaluation of τ^*.

4. Stability and asymptotics of solutions

This section presents our main results on (1). First, we formulate the auxiliary assertion providing a key tool in stability and asymptotic analysis of (1).

Lemma 1. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers and let y be a solution of (1).*

(i) *Let all roots of (12) have negative real parts. Then*

$$y(t) \sim t^{-\alpha} \qquad \text{or} \qquad y(t) = \mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t \to \infty. \tag{33}$$

(ii) *Let all roots of (12) have non-positive real parts and let there exist a root with the zero real part. If (12) has no purely imaginary roots, then*

$$y(t) \sim 1 \qquad \text{or} \qquad y(t) = \mathcal{O}(t^{\alpha-1}) \qquad \text{as } t \to \infty. \tag{34}$$

If (12) has purely imaginary roots, then

$$y(t) \sim_{sup} 1 \qquad \text{or} \qquad y(t) = \mathcal{O}(1) \qquad \text{as } t \to \infty. \tag{35}$$

(iii) *Let there exist a root of (12) with a positive real part. Then $y(t) = \mathcal{O}(t \exp[Mt])$ as $t \to \infty$, where $M = \max_{s_i}(\Re(s_i))$, s_i being roots of (12). Moreover, there exists a solution y of (1) such that*

$$y(t) \sim_{sup} t \exp[Mt] \qquad \text{or} \qquad y(t) \sim_{sup} \exp[Mt] \qquad \text{as } t \to \infty. \tag{36}$$

Remark 3. The asymptotic property $(33)_1$ or $(33)_2$ occurs if $\phi(0) \neq 0$ or $\phi(0) = 0$, respectively, where ϕ is the initial function implying a particular solution of (1) via (11). The same comment holds also for (34) and (35) (a more detailed analysis is involved in Section 5). In particular, under the assumption of the part (ii), there always exist solutions of (1) not tending to the zero solution. Regarding (36), the type of asymptotic behaviour depends on multiplicity of the root of (12) with the maximal positive real part (see also Proposition 2 (iv) and Remark 1).

Now we are in a position to formulate our main stability and asymptotic criteria on (1). If we put through Propositions 3, 4 and Lemma 1, we arrive at the following two results.

Theorem 4. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers.*

(i) *The zero solution of (1) is asymptotically stable if and only if the couple (a, b) is an interior point of the area bounded by the line $a + b = 0$ from above and by the parametric curve (29) from below. In this case, (33) holds for any solution y of (1).*
(ii) *The zero solution of (1) is stable, but not asymptotically stable, if and only if either*

$$a + b = 0, \qquad a \le \frac{[\pi(1-\alpha)]^{\alpha}}{2\tau^{\alpha}\cos(\alpha\pi/2)}, \tag{37}$$

or a, b satisfy (29) for an admissible value φ. In this case, (34) or (35) holds for any solution y of (1).

Theorem 5. *Let $0 < \alpha < 1$, a, b and $\tau > 0$ be real numbers.*

(i) *The zero solution of (1) is asymptotically stable if and only if (31) or (32) is satisfied. In this case, (33) holds for any solution y of (1).*
(ii) *The zero solution of (1) is stable, but not asymptotically stable, if and only if either (37), or*

$$|a| + b < 0, \qquad \tau = \tau^*, \qquad \tau^* \text{ being given by (30)}$$

is satisfied. In this case, (34) or (35) holds for any solution y of (1).

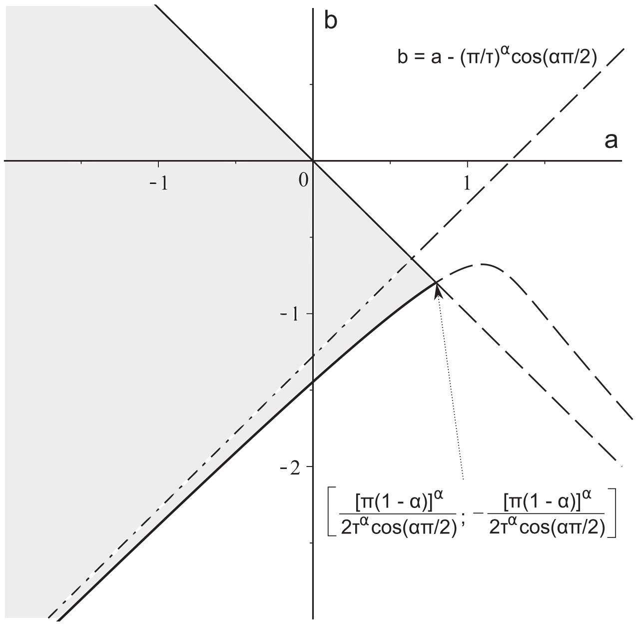

Fig. 3. The stability region for (1) with $\alpha = 0.4$ and $\tau = 1$.

The stability area for (1), described via the conditions of Theorems 4 and 5, is depicted in the (a, b)-plane on Fig. 3.

Remark 4. The assertions of Theorems 4 and 5 provide a direct extension of Theorems 1 and 2, respectively. Indeed, letting $\alpha \to 1$ from the left, one can observe a full agreement between appropriate formulae for the fractional and classical case. In particular, the value (30) of the fractional stability switch, when asymptotic stability of the zero solution of (1) is turning into instability as the delay τ is monotonically increasing, becomes the appropriate value from (4). Notice also, that if we put $a = 0$ in Theorem 5, we obtain just the assertion of Theorem 3 (where the condition $\phi(0) \neq 0$ is considered).

Now we make some comments on two qualitative dissimilarities between the fractional and classical delayed case. First, the decay rate of solutions in the asymptotically stable case is exponential when $\alpha = 1$, but only algebraic when $0 < \alpha < 1$. Second, the situation on the stability boundary differs at the cusp point

$$P_\alpha = \left[\frac{[\pi(1-\alpha)]^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)}; -\frac{[\pi(1-\alpha)]^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)}\right]$$

where the line $a + b = 0$ intersects the transcendental curve (29). The cusp point P_α corresponds to the zero (simple) root of (12) and Theorem 4 (ii) (as well as Theorem 5 (ii)) immediately yields

Corollary 1. *Let $0 < \alpha < 1$ and $\tau > 0$ be real numbers. The zero solution of*

$$D^\alpha y(t) = \frac{[\pi(1-\alpha)]^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)}[y(t) - y(t-\tau)], \qquad t > 0 \tag{38}$$

is stable, but not asymptotically stable.

Letting $\alpha \to 1$ from the left, (38) turns into

$$y'(t) = \frac{1}{\tau}[y(t) - y(t-\tau)], \qquad t > 0 \tag{39}$$

and the cusp point P_α becomes $P = [1/\tau; -1/\tau]$. A direct procedure shows that P corresponds to the double zero root of (5) with $a = -b = 1/\tau$, hence the zero solution of (39) cannot be stable. Indeed, we can easily check that (39) admits an unbounded solution, namely $y(t) = t$. Consequently, the stability property of (38) is not transferable to its limit case (39). Thus, this example confirms a positive impact of derivatives of real orders between 0 and 1 on stability properties of studied delay equation.

5. Proof of Lemma 1

First, we present several technical assertions where we use the notation

$$\sigma = \gamma(\varepsilon/t, \pi/2 + \delta), \qquad t > 0, \tag{40}$$

$$\sigma' = \gamma(\varepsilon^\alpha, \alpha\pi/2 + \alpha\delta) \tag{41}$$

with real constants ε, $\delta > 0$, $0 < \alpha < 1$ and the contour γ given by (7).

Proposition 5. *For given* $0 < \alpha < 1$ *and* $\tau > 0$, *we denote*

$$\omega_{x,n}(t) = \frac{1}{2\pi\alpha \mathrm{i}} \int_{\sigma'} \frac{u^x \exp[u^{1/\alpha}(1+n\tau/t)]}{(u - at^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha} \mathrm{d}u\,, \quad x > -1,\ n = 2, 3, \ldots .$$

(i) If there exists $\eta_0 > 0$ *such that*

$$|(u - at^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha| \geq \eta_0 |\exp[u^{1/\alpha}\tau/t]| \tag{42}$$

for all $u \in \sigma'$, *then*

$$\omega_{x,n}(t) = \mathcal{O}(1) \qquad \text{as } t \to \infty\,. \tag{43}$$

(ii) If there exists $\eta_1 > 0$ *such that*

$$|(u - at^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha| \geq \eta_1 t^\alpha |\exp[u^{1/\alpha}\tau/t]| \tag{44}$$

for all $u \in \sigma'$, *then*

$$\omega_{x,n}(t) = \mathcal{O}(t^{-\alpha}) \qquad \text{as } t \to \infty\,, \tag{45}$$

$$\omega_{x,n+m}(t) = \omega_{x,n}(t) + \mathcal{O}(t^{-1-\alpha}) \qquad \text{as } t \to \infty \tag{46}$$

where $m \in \mathbb{Z}^+$ *is arbitrary.*

Proof. First, we prove (43) and (45) simultaneously. Let $\kappa \in \{-\alpha, 0\}$. Utilizing (42) and (44), we obtain

$$\begin{aligned}|\omega_{x,n}(t)| &\leq \frac{t^\kappa}{2\pi\alpha\eta_0} \int_{\sigma'} |u^x||\exp[u^{1/\alpha}(1+(n-1)\tau/t)]||\mathrm{d}u| \\ &\leq \frac{t^\kappa}{2\pi\alpha\eta_0}\left(\int_{-\alpha\pi/2-\alpha\delta}^{\alpha\pi/2+\alpha\delta} \varepsilon^{\alpha(x+1)} \exp[\varepsilon(1+(n-1)\tau/t)\cos(\varphi/\alpha)]\mathrm{d}\varphi \right. \\ &\quad \left. + 2\int_{\varepsilon^\alpha}^{\infty} r^x \exp[r^{1/\alpha}(1+(n-1)\tau/t)\cos(\pi/2+\delta)]\mathrm{d}r \right) \\ &\leq \frac{t^\kappa}{2\pi\alpha\eta_0}\left(\alpha(\pi+2\delta) + \frac{\alpha\Gamma(\alpha(x+1))}{(\cos(\pi/2-\delta))^{\alpha(x+1)}} \right) = \mathcal{O}(t^\kappa) \qquad \text{as } t \to \infty\end{aligned}$$

where we have employed the integral definition of the Gamma function (computational details are omitted). This proves (43) and (45).

The formula (46) can be derived via a direct calculation:

$$\begin{aligned}\omega_{x,n+m}(t) &= \frac{1}{2\pi\alpha\mathrm{i}} \int_{\sigma'} \frac{u^x \exp[u^{1/\alpha}(1+n\tau/t)]}{(u-at^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha} \sum_{j=0}^{\infty} \frac{(m\tau)^j}{j!t^j} u^{j/\alpha}\mathrm{d}u \\ &= \omega_{x,n}(t) + \frac{1}{2\pi\alpha\mathrm{i}} \sum_{j=1}^{\infty} \frac{(m\tau)^j}{j!t^j} \int_{\sigma'} \frac{u^{x+j/\alpha}\exp[u^{1/\alpha}(1+n\tau/t)]}{(u-at^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha}\mathrm{d}u \\ &= \omega_{x,n}(t) + \sum_{j=1}^{\infty} \frac{(m\tau)^j}{j!t^j} \omega_{j/\alpha+x,n}(t) = \omega_{x,n}(t) + \mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t \to \infty\,.\end{aligned}$$

□

Proposition 6. *For given* $0 < \alpha < 1$ *and* $\tau > 0$, *we denote*

$$\theta_{x,n}(t) = \frac{1}{2\pi\alpha\mathrm{i}} \int_{\sigma'} u^x \exp[u^{1/\alpha}(1+n\tau/t)]\mathrm{d}u\,, \quad x > -1,\ n = 2, 3, \ldots .$$

Then

$$\theta_{x,n}(t) = \frac{t^{\alpha x+\alpha}(t+n\tau)^{-\alpha x-\alpha}}{\Gamma(1-\alpha x-\alpha)}\,.$$

Proof. We set $v = u(1+n\tau/t)^\alpha$ to get

$$\theta_{x,n}(t) = \frac{1}{2\pi\alpha\mathrm{i}}\left(\frac{t}{t+n\tau}\right)^{\alpha x+\alpha} \int_{\gamma((1+n\tau/t)^\alpha\varepsilon^\alpha, \alpha\pi/2+\alpha\delta)} v^x \exp[v^{1/\alpha}]\mathrm{d}v\,.$$

Applying the formula

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\alpha\mathrm{i}} \int_{\gamma(r,\psi)} \exp[\xi^{1/\alpha}]\xi^{1/\alpha - z/\alpha - 1}\mathrm{d}\xi\,, \quad r > 0,\ \pi\alpha/2 < \psi < \alpha\pi$$

(see, e.g. [20]), we obtain the result. □

Now we can formulate the key auxilliary assertion of this section dealing with asymptotic properties of the class of $\mathcal{R}^{a,b,\tau}_{\alpha,\beta}$ functions.

Lemma 2. *Let $0<\alpha<1$, $0<\beta\le 1$, a, b and $\tau>0$ be real numbers and let s_i be roots of (12).*

(i) If $\Re(s_i)<0$ for all s_i then

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)\sim t^{\beta-\alpha-1} \text{ for } \alpha\ne\beta \quad \text{and} \quad \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t)=\mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t\to\infty .$$

(ii) If there exists the zero root of (12) and $\Re(s_i)<0$ otherwise, then

$$\mathcal{R}^{a,b,\tau}_{\alpha,1}(t)\sim 1 \quad \text{and} \quad \mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)=\mathcal{O}(t^{\beta-1}) \text{ for } \beta<1 \qquad \text{as } t\to\infty .$$

(iii) If $\Re(s_i)\le 0$ for all s_i and some of the roots are purely imaginary, then

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)\sim_{sup} 1 \qquad \text{as } t\to\infty .$$

(iv) If $\Re(s_i)>0$ for some s_i then

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)\sim_{sup} (Bt+C)\exp[Mt] \qquad \text{as } t\to\infty$$

where $M=\max_{s_i}(\Re(s_i))$ and reals B, $C\ge 0$ are such that $B+C>0$.

Proof. It is enough to consider the case $b\ne 0$ (for the case $b=0$ we refer to [18]). Proposition 2 (iii) implies that there exists $\delta>0$ such that all nonzero roots s_i of (12) satisfy $|\arg(s_i)|\ne\pi/2+\delta$ and, moreover, there are only finitely many of them satisfying $|\arg(s_i)|<\pi/2+\delta$. Consequently, for every $t>1$, we can choose $R>\varepsilon>0$ such that all s_i lie to the left of $\gamma(R,\pi/2+\delta)$ and those satisfying $|\arg(s_i)|<\pi/2+\delta$ are located to the right of σ given by (40). Thus, we can split the inverse Laplace transform formula into

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma(R,\frac{\pi}{2}+\delta)}\frac{s^{\alpha-\beta}\exp[ts]}{s^\alpha-a-b\exp[-\tau s]}\mathrm{d}s=I_1(t)+I_2(t),$$

where I_1 and I_2 denote integrals over $\gamma(R,\pi/2+\delta)-\sigma$ and σ, respectively.

I. First, we analyse I_1. Clearly, the contour $\gamma(R,\frac{\pi}{2}+\delta)-\sigma$ is a simple positively oriented closed curve. Proposition 2 (iii) implies that the integrand has only finitely many poles s_i $(i=1,\dots,N)$ lying in the interior of $\gamma(R,\frac{\pi}{2}+\delta)-\sigma$. The residue theorem yields

$$I_1(t)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma(R,\frac{\pi}{2}+\delta)-\sigma}\frac{s^{\alpha-\beta}\exp[st]}{s^\alpha-a-b\exp[-\tau s]}\mathrm{d}s=\sum_{i=1}^{N}\mathrm{Res}_{s=s_i}\left(\frac{s^{\alpha-\beta}\exp[st]}{s^\alpha-a-b\exp[-\tau s]}\right).$$

Proposition 2 (iv) shows that all poles s_i with nonzero imaginary parts are simple, while real poles (more precisely, positive real poles) admit double multiplicity. Thus, for every pole s_i, we can utilize the Laurent expansions

$$\frac{s^{\alpha-\beta}}{s^\alpha-a-b\exp[-\tau s]}=a^i_{-2}(s-s_i)^{-2}+a^i_{-1}(s-s_i)^{-1}+a^i_0+a^i_1(s-s_i)+a^i_2(s-s_i)^2+\cdots,$$

$$\exp[st]=\exp[s_it]\left(1+t(s-s_i)+\frac{t^2}{2!}(s-s_i)^2+\frac{t^3}{3!}(s-s_i)^3+\cdots\right)$$

where a_j $(j=-2,-1,0\dots)$ are complex constants independent of t. Hence, a multiplication of both expansions enables us to write the sum of residues (i.e. the coefficients at $(s-s_i)^{-1}$) as

$$I_1(t)=\sum_{i=1}^{N}(a^i_{-1}+a^i_{-2}t)\exp[s_it],$$

where a^i_{-1}, a^i_{-2} $(i=1,\dots,N)$ are complex constants such that for any i at least one of the terms is nonzero.

II. Now, we analyse the term I_2. Employing the change of variables $s=u^{1/\alpha}/t$, which transforms the contour σ into σ' (given by (41)), we get

$$I_2(t)=\frac{1}{2\pi\mathrm{i}}\int_\sigma\frac{s^{\alpha-\beta}\exp[st]}{s^\alpha-a-b\exp[-\tau s]}\mathrm{d}s=t^{\beta-1}\omega_{(1-\beta)/\alpha,1}(t) \tag{47}$$

where $\omega_{(1-\beta)/\alpha,1}$ is introduced in Proposition 5.

II-1. Let (12) have the zero root, i.e. $a+b=0$. Although σ is chosen so that it does not contain any roots of (12), it approaches the zero root as $t\to\infty$. The closest points s of σ with respect to the root $s=0$ satisfy $s=\varepsilon/t\exp[\mathrm{i}\varphi]$, i.e. we can write

$$|s^\alpha-a-b\exp[-\tau s]|\ge|\varepsilon^\alpha t^{-\alpha}\exp[\mathrm{i}\alpha\varphi]-a-b\exp[-\varepsilon\tau/t]|\ge\eta_0t^{-\alpha}$$

for a suitable $\eta_0 > 0$ and sufficiently large t. Applying the change of variable $s = u^{1/\alpha}/t$, we can directly employ (43) to obtain the asymptotic estimate

$$I_2(t) = \mathcal{O}(t^{\beta-1}) \qquad \text{as } t \to \infty .$$

II-2. Let (12) have only nonzero roots, i.e. $a + b \neq 0$. Since the contour σ does not contain any root s_i of (12), there exists $\eta_1 > 0$ such that

$$|s^\alpha - a - b\exp[-\tau s]| \geq \eta_1$$

for all $s \in \sigma$. Similarly to II-1, as a direct consequence of (45) we get

$$I_2(t) = \mathcal{O}(t^{\beta-\alpha-1}) \qquad \text{as } t \to \infty .$$

In the sequel, we precise this estimate.

II-2-a. Let $a \neq b$. Substituting the above suggested change of variable, we get (along with some simple calculations) that (44) is satisfied. Now, we use the identity

$$\frac{1}{\xi - z} = -\frac{1}{z} - \frac{\xi}{z^2} + \frac{\xi^2}{z^2(\xi - z)} \tag{48}$$

(with $z = bt^\alpha$ and $\xi = (u - at^\alpha)\exp[u^{1/\alpha}\tau/t]$) and (45), (46) to get

$$I_2(t) = -\frac{t^{\beta-\alpha-1}}{b}\theta_{(1-\beta)/\alpha,1}(t) - \frac{t^{\beta-2\alpha-1}}{b^2}\theta_{(1-\beta)/\alpha+1,2}(t) + \frac{at^{\beta-\alpha-1}}{b^2}\theta_{(1-\beta)/\alpha,2}(t)$$
$$+ \frac{t^{\beta-2\alpha-1}}{b^2}\int_{\sigma'}\frac{u^{(1-\beta)/\alpha}(u - at^\alpha)^2\exp[u^{1/\alpha}(1+3\tau/t)]}{(u - at^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha}\mathrm{d}u .$$

Further decomposition of the last term yields

$$\begin{aligned} I_2(t) &= -\frac{\theta_{(1-\beta)/\alpha,1}(t)}{bt^{-\beta+\alpha+1}} - \frac{\theta_{(1-\beta)/\alpha+1,2}(t)}{b^2t^{-\beta+2\alpha+1}} + \frac{a\theta_{(1-\beta)/\alpha,2}(t)}{b^2t^{-\beta+\alpha+1}} \\ &+ \frac{\omega_{(1-\beta)/\alpha+2,3}(t)}{b^2t^{-\beta+2\alpha+1}} - \frac{2a\omega_{(1-\beta)/\alpha+1,3}(t)}{b^2t^{-\beta+\alpha+1}} + \frac{a^2\omega_{(1-\beta)/\alpha,3}(t)}{b^2t^{-\beta+1}} \\ &= -\frac{\theta_{(1-\beta)/\alpha,1}(t)}{bt^{-\beta+\alpha+1}} - \frac{\theta_{(1-\beta)/\alpha+1,2}(t)}{b^2t^{-\beta+2\alpha+1}} + \frac{a\theta_{(1-\beta)/\alpha,2}(t)}{b^2t^{-\beta+\alpha+1}} \\ &+ \frac{\omega_{(1-\beta)/\alpha+2,3}(t)}{b^2t^{-\beta+2\alpha+1}} - \frac{2a\omega_{(1-\beta)/\alpha+1,3}(t)}{b^2t^{-\beta+\alpha+1}} + \frac{a^2}{b^2}I_2(t) + \mathcal{O}(t^{\beta-\alpha-2}) \end{aligned}$$

as $t \to \infty$. Rearranging the expression, we arrive at

$$I_2(t) = \frac{b^2}{b^2 - a^2}\Bigg(-\frac{\theta_{(1-\beta)/\alpha,1}(t)}{bt^{-\beta+\alpha+1}} - \frac{\theta_{(1-\beta)/\alpha+1,2}(t)}{b^2t^{-\beta+2\alpha+1}} + \frac{a\theta_{(1-\beta)/\alpha,2}(t)}{b^2t^{-\beta+\alpha+1}}$$
$$+ \frac{\omega_{(1-\beta)/\alpha+2,3}(t)}{b^2t^{-\beta+2\alpha+1}} - \frac{2a\omega_{(1-\beta)/\alpha+1,3}(t)}{b^2t^{-\beta+\alpha+1}} + \mathcal{O}(t^{\beta-\alpha-2})\Bigg)$$

as $t \to \infty$. An analogous argumentation yields

$$\omega_{(1-\beta)/\alpha+1,3}(t) = \frac{b^2}{b^2 - a^2}\Bigg(-\frac{\theta_{(1-\beta)/\alpha+1,3}(t)}{bt^\alpha} - \frac{\theta_{(1-\beta)/\alpha+2,4}(t)}{b^2t^{2\alpha}}$$
$$+ \frac{a\theta_{(1-\beta)/\alpha+1,4}(t)}{b^2t^\alpha} + \frac{\omega_{(1-\beta)/\alpha+3,5}(t)}{b^2t^{2\alpha}} - \frac{2a\omega_{(1-\beta)/\alpha+2,5}(t)}{b^2t^\alpha} + \mathcal{O}(t^{-\alpha-1})\Bigg)$$

as $t \to \infty$. Combining these expressions with Propositions 5 (ii) and 6, we get

$$\begin{aligned} I_2(t) &= \frac{b^2}{b^2 - a^2}\Bigg(-\frac{(t+\tau)^{\beta-\alpha-1}}{b\Gamma(\beta-\alpha)} - \frac{(t+2\tau)^{\beta-2\alpha-1}}{b^2\Gamma(\beta-2\alpha)} + \frac{a(t+2\tau)^{\beta-\alpha-1}}{b^2\Gamma(\beta-\alpha)} \\ &+ \mathcal{O}(t^{\beta-3\alpha-1}) + \frac{2a(t+3\tau)^{\beta-2\alpha-1}}{b(b^2-a^2)\Gamma(\beta-2\alpha)} + \frac{2a(t+4\tau)^{\beta-3\alpha-1}}{b^2(b^2-a^2)\Gamma(\beta-3\alpha)} \\ &- \frac{2a^2(t+4\tau)^{\beta-2\alpha-1}}{b^2(b^2-a^2)\Gamma(\beta-2\alpha)} + \mathcal{O}(t^{\beta-4\alpha-1}) + \mathcal{O}(t^{\beta-3\alpha-1}) + \mathcal{O}(t^{\beta-2\alpha-2}) + \mathcal{O}(t^{\beta-\alpha-2})\Bigg) \\ &= \frac{b(t+\tau)^{\beta-\alpha-1} - a(t+2\tau)^{\beta-\alpha-1}}{-(b^2-a^2)\Gamma(\beta-\alpha)} \\ &+ \frac{(b^2-a^2)(t+2\tau)^{\beta-2\alpha-1} - 2ab(t+3\tau)^{\beta-2\alpha-1} + 2a^2(t+4\tau)^{\beta-2\alpha-1}}{-(b^2-a^2)^2\Gamma(\beta-2\alpha)} + \mathcal{O}(t^{\beta-\alpha-1-\min(2\alpha,1)}) \end{aligned}$$

as $t \to \infty$. Taking into account the property $1/\Gamma(0) = 0$, we can derive

$$\lim_{t\to\infty} \frac{|I_2(t)|}{t^{\beta-\alpha-1}} = \frac{-(a+b)^{-1}}{\Gamma(\beta-\alpha)} \quad \text{and} \quad \lim_{t\to\infty} \frac{|I_2(t)|}{t^{\beta-2\alpha-1}} = \frac{-(a+b)^{-2}}{\Gamma(\beta-2\alpha)} \tag{49}$$

for $\alpha \neq \beta$ and $\alpha = \beta$, respectively. Thus, the asymptotic behaviour of I_2 is described for $a \neq b$.

II-2-b. Now, let $a = b \neq 0$ and $\alpha \neq \beta$. We show that $(49)_1$ is transferable to this case through $a = b + \zeta$ as $\zeta \to 0$. Based on (47) and $(49)_1$, we denote

$$I_2^\zeta(t) = \frac{t^{\beta-1}}{2\pi\alpha\mathrm{i}} \int_{\sigma'} \frac{u^{(1-\beta)/\alpha} \exp[u^{1/\alpha}(1+\tau/t)]}{(u-(b+\zeta)t^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha} \mathrm{d}u,$$
$$L(\zeta) = \frac{-1}{(2b+\zeta)\Gamma(\beta-\alpha)}.$$

Some straightforward (but tedious) calculations enable us to write

$$\begin{aligned} I_2^\zeta(t) - I_2^0(t) = {} & \frac{\zeta t^{\beta-\alpha-1}}{b^2}\theta_{(1-\beta)/\alpha,2}(t) + \frac{t^{\beta-2\alpha-1}}{b^2} \\ & \times \int_{\sigma'} \Bigg(\frac{\zeta u^{(1-\beta)/\alpha} \exp[u^{1/\alpha}(1+4\tau/t)](-u^2t^\alpha + u(2b+\zeta)t^{2\alpha} - b(b+\zeta)t^{3\alpha})}{[(u-(b+\zeta)t^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha][(u-bt^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha]} \\ & + \frac{\zeta u^{(1-\beta)/\alpha} \exp[u^{1/\alpha}(1+3\tau/t)](2ubt^{2\alpha} - b(2b+\zeta)t^{3\alpha})}{[(u-(b+\zeta)t^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha][(u-bt^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha]} \Bigg) \mathrm{d}u. \end{aligned}$$

Utilizing similar estimates as in the proof of Proposition 5, we arrive at

$$|I_2^\zeta(t) - I_2^0(t)| \le |\zeta| K t^{\beta-\alpha-1}$$

where $K > 0$ is a suitable real constant (independent of t and ζ). Further, continuity of L at $\zeta = 0$ yields that for every $\varepsilon > 0$ there exists $\chi(\varepsilon) > 0$ such that if $|\zeta| < \chi(\varepsilon)$ then $|L(\zeta) - L(0)| < \varepsilon/3$. Similarly, (49) implies that for every $\varepsilon > 0$ and every ζ there exists $N(\varepsilon, \zeta) > 0$ such that if $t > N(\varepsilon, \zeta)$ then $|t^{1+\alpha-\beta}|I_2^\zeta(t)| - L(\zeta)| < \varepsilon/3$.

Employing the above stated properties, we can conclude that $(49)_1$ holds also for $a = b \neq 0$ and $\alpha \neq \beta$, i.e. for every $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that if $t > N(\varepsilon)$ then $|t^{1+\alpha-\beta}|I_2^0(t)| - L(0)| < \varepsilon$. Indeed, it holds

$$\begin{aligned} |t^{1+\alpha-\beta}|I_2^0(t)| - L(0)| \le {} & |t^{1+\alpha-\beta}|I_2^\zeta(t)| - L(\zeta)| \\ & + t^{1+\alpha-\beta}|I_2^\zeta(t) - I_2^0(t)| + |L(\zeta) - L(0)| \le \varepsilon/3 + \zeta K + \varepsilon/3 < \varepsilon \end{aligned}$$

where ζ is chosen such that $|\zeta| K < \varepsilon/3$.

II-2-c. Finally, we discuss the case $a = b \neq 0$ and $\alpha = \beta$. Applying (48) with $z = bt^\alpha(\exp[u^{1/\alpha}\tau/t] + 1)$ and $\xi = u\exp[u^{1/\alpha}\tau/t]$ to (47), we obtain

$$\begin{aligned} I_2(t) = {} & \frac{t^{\alpha-1}}{2\pi\alpha\mathrm{i}} \int_{\sigma'} \Bigg(\frac{-u^{1/\alpha-1}\exp[u^{1/\alpha}(1+\tau/t)]}{bt^\alpha(\exp[u^{1/\alpha}\tau/t]+1)} - \frac{u^{1/\alpha}\exp[u^{1/\alpha}(1+2\tau/t)]}{b^2t^{2\alpha}(\exp[u^{1/\alpha}\tau/t]+1)^2} \\ & + \frac{u^{1/\alpha+1}\exp[u^{1/\alpha}(1+3\tau/t)]}{b^2t^{2\alpha}(\exp[u^{1/\alpha}\tau/t]+1)^2((u-bt^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha)} \Bigg) \mathrm{d}u. \end{aligned}$$

Estimates employed in the proof of Proposition 5 yield

$$I_2(t) = \frac{-t^{-1}}{2\pi\alpha b\mathrm{i}} G(t) + \mathcal{O}(t^{-\alpha-1}) + \mathcal{O}(t^{-2\alpha-1}) \qquad \text{as } t \to \infty$$

where G, utilizing the change of variable $u^{1/\alpha} = p$, is given by

$$G(t) = \int_{\sigma'} \frac{u^{1/\alpha-1}\exp[u^{1/\alpha}(1+\tau/t)]}{\exp[u^{1/\alpha}\tau/t]+1} \mathrm{d}u = \alpha \int_{\gamma(\varepsilon,\pi/2+\delta)} \frac{\exp[p(1+\tau/t)]}{\exp[p\tau/t]+1} \mathrm{d}p.$$

We can see that the integrand of the second expression is a function analytical in the complex plane, hence integral of this function taken over any closed contour is zero. Utilizing this property, we can write

$$0 = \lim_{R\to\infty} \oint_{\Sigma(\varepsilon,\delta,R)} \frac{\exp[p(1+\tau/t)]}{\exp[p\tau/t]+1} \mathrm{d}p = \frac{G(t)}{\alpha} + \lim_{R\to\infty} G_R(t)$$

with the contour $\Sigma(\varepsilon, \delta, R)$ depicted on Fig. 4. G_R can be estimated as

$$|G_R(t)| = \left| \int_{\pi/2+\delta}^{3\pi/2-\delta} \frac{\exp[R(1+\tau/t)(\cos(\varphi) + \mathrm{i}\sin(\varphi))]}{\exp[R\tau/t(\cos(\varphi) + \mathrm{i}\sin(\varphi))] + 1} R\mathrm{i}\exp[\mathrm{i}\varphi]\mathrm{d}\varphi \right|$$

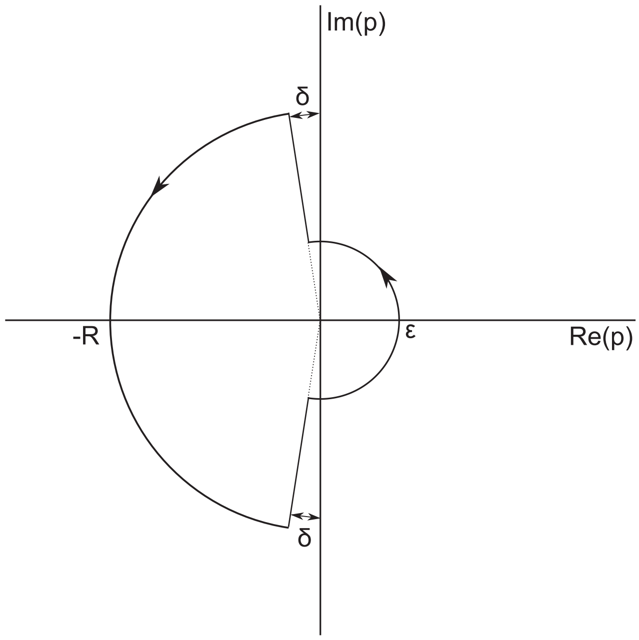

Fig. 4. The oriented closed contour $\Sigma(\varepsilon, \delta, R)$ in the complex plane.

$$\leq \frac{R}{2}\exp[-R(1+\tau/t)\sin(\delta)](\pi-2\delta)\,.$$

Since $G_R(t) \to 0$ as $R \to \infty$ for any $t > 0$, we arrive at $G(t) \equiv 0$. Thus, it holds

$$I_2(t) = \mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t \to \infty\,.$$

III. Combining all the obtained results, we get Lemma 2 (i), (iii) and (iv). In particular, if all the roots of (12) have negative real parts, then algebraic decay becomes dominant over the exponential with negative argument.

If (12) has the zero root, we have $\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) = \mathcal{O}(t^{\beta-1})$ as $t \to \infty$ (note that I_1 equals zero) which corresponds to Lemma 2 (ii) for $\beta < 1$. In the case $\beta = 1$, we can rewrite (9) as

$$\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{bs^{-1}(1-\exp[-\tau s])}{s^\alpha + b(1-\exp[-\tau s])}\right)(t) = 1 - b\int_{t-\tau}^{t}\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(\xi)\mathrm{d}\xi$$
$$= 1 - b\tau\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-c(t)) = 1 + \mathcal{O}(t^{\alpha-1}) \sim 1 \qquad \text{as } t \to \infty$$

$(0 \leq c(t) \leq \tau)$ which concludes the proof of Lemma 2 (ii). □

Remark 5. We note that our technique enables even a stronger formulation of Lemma 2 (i). In particular, if $a \neq b$ then $\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t) \sim t^{-\alpha-1}$ as $t \to \infty$.

To complete the proof of Lemma 1, we consider (10) implying that every solution y of (1) consists of terms

$$\phi(0)\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) \qquad \text{and} \qquad b\int_{-\tau}^{0}\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-\tau-u)h(t-\tau-u)\phi(u)\mathrm{d}u\,.$$

Let all the roots of (12) have negative real parts. By Lemma 2 (i), we have $\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) \sim t^{-\alpha}$ and $\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t) = \mathcal{O}(t^{-\alpha-1})$ as $t \to \infty$ as $t \to \infty$. For $t > \tau$, we get

$$\left|\int_{-\tau}^{0}\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-\tau-u)\phi(u)\mathrm{d}u\right| \leq \sup_{t-\tau\leq u\leq t}|\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(u)|\int_{-\tau}^{0}|\phi(u)|\mathrm{d}u = \mathcal{O}(t^{-\alpha-1})$$

as $t \to \infty$. Thus, we arrive at $(33)_1$ for $\phi(0) \neq 0$ and $(33)_2$ for $\phi(0) = 0$.

Analogously, Lemma 2 (ii), (iii) and (iv) implies (34), (35) and (36), respectively. Moreover, the existence of a constant solution of (1) in the case $a+b=0$ (when the zero root of (12) occurs) can be checked via a direct substitution into (1). Also, the property $(36)_1$ or $(36)_2$ occurs if the root of (12) with the maximal real part is double or single, respectively.

6. Concluding remarks

In this paper, we have discussed stability and asymptotic behaviour of a linear prototype of fractional delay differential equations. The main results (Theorems 4 and 5) have formulated two types of necessary and sufficient conditions for stability and asymptotic stability of the zero solution of (1), including asymptotic formulae for its solutions. The derived stability conditions confirm expectations on positive influence of fractional-order derivative $0<\alpha<1$ on stability properties of (1) compared with the classical case (2) when $\alpha=1$. The stability region for (1) is wider than that for (2) and, moreover, the zero solution of (1) is stable for all couples (a, b) lying on the stability boundary (this property does not hold in the case of (2) when the zero solution is not stable at the cusp point of the stability boundary). Another distinction consists in the form of decay rate of solutions of (1) in the asymptotically stable case, which is not exponential, but algebraic.

Following the topic of this paper, we outline some next possible research directions. The most preferable direction is an extension of our results to the vector case when (1) becomes

$$\mathrm{D}^\alpha y(t)=Ay(t)+By(t-\tau), \qquad t>0 \tag{50}$$

with $d\times d$ real matrices A, B. The knowledge od stability conditions for (50) is important from a theoretical as well as practical view ((50) describes, among others, the fractional-order time delay state space model of PD^α control of Newcastle robot, see [16]). It should be noted that the problem of necessary and sufficient conditions for (asymptotic) stability of the zero solution of (50) seems to be a very complicated matter which is not answered neither in the integer-order case $\alpha=1$ (for a particular result on this problem we refer to [13]). Also, these stability investigations can be extended to other types of fractional equations involving more general types of delays (see, e.g. [19,22,23]), to fractional evolution equations (see, e.g. [24,26]), or to basic numerical discretizations of (1). From this viewpoint, we hope that results of this paper might be a starting point for numerical stability investigations of fractional delay differential equations.

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Appendix F

Paper on lower-order complex two-term FDDE [11] (CNSNS, 2019)

The paper [11] (co-author: J. Čermák; my author's share 50 %) was a reaction on our growing interest in the phenomenon of stability switches. We decided to study a planar FDDS with three real entries, which can be transformed into a fractional generalization of the first-order delay differential equation with an imaginary coefficient by an undelayed term. This type of classical system is known to switch stability on and off with increasing time delay.

In this paper, we conducted a detailed analysis of stability switching, including conditions for its appearance, number, and exact calculations of stability switches. We discovered rich behaviour not present in the original first-order equation, such as different switching patterns starting from instability or stability for small delays and the presence of a delay-independent stability region.

The biggest challenge was the derivation of explicit form of the stability boundary, which is no longer given by a continuous parametric curve but by a union of segments from infinitely many parametric curves. Additionally, we explored how the stability boundary transitions with increasing order of the derivative, from a union of transcendental curves (for derivative orders less than one) to the known shape for first-order equations.

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Research paper

Delay-dependent stability switches in fractional differential equations

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ABSTRACT

This paper discusses stability properties of a linear fractional delay differential system involving both delayed as well as non-delayed terms. As a main result, the explicit stability dependence on a changing time delay is described, including conditions for the appearance, number and exact calculations of stability switches for this system when its stability property turns into instability and vice versa in view of a monotonically increasing lag. Some supporting asymptotic results are stated as well. The proof technique is based on analysis of the generalized delay exponential function of the Mittag-Leffler type combined with D-decomposition method. The obtained results are illustrated via a fractional Lotka-Volterra population model and applied to a stabilization problem of the control theory.

1. Introduction

The usefulness of delay and fractional differential equations in mathematical modeling of various phenomena is well known and described in many works. The presence of time delay parameters is connected especially with systems not capable to react promptly after receiving an information (typically in chemical processes, electric and high-speed communication networks, transportation phenomena and other related areas, see, e.g., [1]). Similarly, the involvement of non-integer derivative orders turned out to be useful especially in models having hereditary properties (typically in viscoelasticity, acoustics, diffusion, transmission lines and other problems, see, e.g., [2–5]).

Differential models comprising both a time delay as well as a non-integer derivative order found their applications especially in various problems of control theory, robotics and related disciplines (see, e.g., [6–9]). In particular, both these parameters can be viewed as control parameters in various stabilization and synchronization issues which makes stability analysis of these equations very important. Although stability theory for delay differential equations and fractional differential equations is well established, the corresponding unifying theory for fractional delay differential equations is still at the beginning.

Stability dependence on some of changing system parameters belongs among the key qualitative properties of studied systems. It is well known that the integer-order linear delay dynamical systems may repeatedly change their stability into

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instability and vice versa in view of a monotonically increasing time delay. This specific phenomenon was described, e.g., in the papers [10–12] where explicit values of these stability switches were described in terms of remaining system parameters (for switching laws designed to maintain stability of delayed switched nonlinear systems with stable and unstable modes see also [13]). The existence of stability switches indicates that the involvement of a delay parameter into appropriate dynamical system can result either into stabilizing or destabilizing effects depending on a concrete value of a delay. In the fractional-order case, the structure of stability dependence of linear dynamical systems on continuously changing derivative order is more simple: the corresponding stability area becomes larger in view of a decreasing derivative order (see [14]). All these facts play an important role in stabilization and synchronization problems connected with delayed feedback controls of dynamical systems (see, e.g., [15–17]). A general theoretical framework related to various dynamical models was presented, e.g., in [18,19].

The stability switches problem for fractional delay differential equations with delay-dependent coefficients was discussed in [20,21]. While the first paper primarily investigated the stability intervals for such equations, in the latter one, using general formulae based on the Jacobi determinant of some auxiliary functions, the occurrence of stability switches was observed for studied fractional equations with delay-dependent as well as delay-independent coefficients. Despite of interesting results involved in these papers, several important issues (concerning especially *a priori* conditions for the appearance, number and exact values of stability switches) remained open.

Our main goal is to follow the above commented research and perform a detailed analysis of this phenomenon for a basic fractional delay differential system admitting an arbitrary (finite) number of stability switches depending on values of entry coefficients and a derivative order. Doing this, the paper is structured as follows. In Sections 2 and 3, we precise a mathematical side of the problem, including related notions and tools. In particular, we derive the characteristic equation associated to the studied system. Section 4 analyzes the location of characteristic roots which is a crucial step in related stability investigations that are summarized in Section 5. We formulate here stability criteria for the studied fractional delay differential system, providing a simple algorithm how to reveal the occurrence of stability switches, their number and exact values. These values are derived explicitly in terms of system coefficients and a derivative order. In Section 6, we outline an applicability of our results in stability analysis of equilibria of fractional delay dynamical systems and control theory. Some final remarks conclude the paper.

2. A brief mathematical background

The fractional differential system

$$\mathrm{D}^\alpha x(t) = Ax(t) + Bx(t-\tau)\,, \tag{2.1}$$

where $0<\alpha<1$ is a real number, the symbol D^α means the Caputo derivative operator (its introduction is recalled below), A, B are real 2×2 matrices and τ is a positive real time delay, provides a mathematical model with large application potential, especially in control theory. In particular, (2.1) appears in stabilization of equilibria of fractional dynamical systems via delayed feedback controls (A usually serves as the Jacobi matrix of an uncontrolled system and B is the gain matrix). For its other use, we refer, e.g., to [8].

However, stability analysis of (2.1) represents a problem of a considerable difficulty. More precisely, if A, B are planar matrices with general real entries a_{ij}, b_{ij}, respectively, then easily applicable stability conditions are missing even in the integer-order case. Indeed, if $\alpha=1$, then the classical result says that the zero solution to (2.1) with $\alpha=1$ is asymptotically stable if and only if all roots of the characteristic quasi-polynomial

$$\begin{aligned} F(s;A,B,\tau) &\equiv \det[sI-A-B\exp(-s\tau)] \\ &= s^2-(\operatorname{tr}A)s+\det A+(-(\operatorname{tr}B)s+a_{11}b_{22}+a_{22}b_{11}-a_{12}b_{21}-a_{21}b_{12})\exp(-s\tau)+\det B\exp(-2s\tau) \end{aligned} \tag{2.2}$$

have negative real parts. Although there are several methods how to analyze this theoretical condition, necessary and sufficient stability conditions for (2.1) with $\alpha=1$, given explicitly in terms of entry coefficients, are known only in very particular cases. Perhaps, the best analyzed case is that when a factorization of $F(s;\ A,\ B,\ \tau)$ is possible, i.e. when A, B are commutative (or, slightly more generally, simultaneously triangularizable). For relevant results on this topic we refer to [22–25].

In the fractional-order case, an analogue of (2.2) is described and partially analyzed also in many special cases (for basic results in this direction, see, e.g., [26]). However, stability conditions expressed explicitly in terms of entry coefficients are still very rare. From a short history of searching for such stability conditions for (2.1), we refer to results on some of its basic particular cases derived in [27] (the case $A=0, B\in\mathbb{R}$), [28] (the case $A=0, B\in\mathbb{R}^d,\ d\in\mathbb{Z}^+$) and [29–31] (the case $A, B\in\mathbb{R}$).

In this paper, we follow the research and investigate stability properties of a planar fractional delay system

$$\begin{aligned} \mathrm{D}^\alpha x_1(t) &= ax_1(t-\tau)+bx_2(t) \\ \mathrm{D}^\alpha x_2(t) &= cx_1(t)+ax_2(t-\tau) \end{aligned} \tag{2.3}$$

with real entries a, b, c (b, $c\neq 0$) and a positive real delay τ. The conventional first-order pattern for (2.3) (with $\alpha=1$) was studied in [24] where necessary and sufficient stability conditions explicit with respect to delay τ were derived. In particular, the repeated switches between stability and instability with respect to changing time delay were observed and expressed explicitly in terms of parameters a, b, c. Thus, (2.3) with $\alpha = 1$ turned out to be the simplest conventional time delay system enabling the occurrence of this phenomenon.

Therefore, we wish to extend stability analysis from [24] to its fractional analogue (2.3). As it might be expected, this problem provides a considerably more difficult task. We show that stability switches with respect to changing time delay can occur also in (2.3) and describe conditions for their appearance and number, including their exact calculations in terms of a, b, c and α. Also, a supplemental information on algebraic decay rate of the solutions will be added.

3. Preliminary results

In this section, we recall some basic notions and formulate a theoretical basis related to stability analysis of (2.3). First, we recall the fractional derivative operator D^α, $0<\alpha<1$ that is understood here in the Caputo sense. For a given real function f, it is defined via the fractional integral

$$\mathrm{D}^{-\nu} f(t) = \int_0^t \frac{(t-\xi)^{\nu-1}}{\Gamma(\nu)} f(\xi)\mathrm{d}\xi, \qquad \nu > 0, \quad t > 0$$

as

$$\mathrm{D}^{\alpha} f(t) = \mathrm{D}^{-(1-\alpha)} f'(t), \qquad 0 < \alpha < 1, \quad t > 0\,.$$

Further, we put $\mathrm{D}^0 f(t) = f(t)$ (for more details on basics of fractional calculus theory we refer, e.g., to [4,32]).

The main analytical technique used in stability investigations of linear fractional differential equations is based on the Laplace transform. For a given real function f, it is introduced as

$$\mathcal{L}(f(t))(s) = \int_0^\infty f(t)\exp(-st)\mathrm{d}t\,, \qquad s \in \mathfrak{D}(f)$$

where $\mathfrak{D}(f) \subset \mathbb{C}$ contains all complex s such that the integral converges. Employing this approach, the Laplace transform of solution (x_1, x_2) to (2.3) can be expressed as

$$\mathcal{L}\left\{\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}\right\}(s) = M^{-1}(s)\cdot\left(s^{\alpha-1}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}(0) + a\exp(-s\tau)I\int_{-\tau}^0 \exp(-su)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}(u)\mathrm{d}u\right) \tag{3.1}$$

where

$$M(s) = \begin{pmatrix} s^\alpha - a\exp(-s\tau) & -b \\ -c & s^\alpha - a\exp(-s\tau)\end{pmatrix},$$

I is the planar identity matrix and ϕ_1, ϕ_2 are the associated initial functions that are piecewise continuous on $[-\tau, 0]$. Obviously, (3.1) is well posed if and only if $M(s)$ is regular, i.e. it holds

$$\det M(s) = (s^\alpha - a\exp(-s\tau))^2 - bc = \left(s^\alpha - \sqrt{bc} - a\exp(-s\tau)\right)\left(s^\alpha + \sqrt{bc} - a\exp(-s\tau)\right) \neq 0\,. \tag{3.2}$$

Then the inverse of $M(s)$ can be rewritten via the Jordan canonical form as

$$M^{-1}(s) = \frac{1}{2}\begin{pmatrix}\sqrt{\frac{b}{c}} & -\sqrt{\frac{b}{c}} \\ 1 & 1\end{pmatrix}\begin{pmatrix}(s^\alpha - \sqrt{bc} - a\exp(-s\tau))^{-1} & 0 \\ 0 & (s^\alpha + \sqrt{bc} - a\exp(-s\tau))^{-1}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{c}{b}} & 1 \\ -\sqrt{\frac{c}{b}} & 1\end{pmatrix}.$$

Utilizing methods of the inverse Laplace transform, we can write the components x_1, x_2 of the solution to (2.3) in the form

$$\begin{aligned} x_1(t) = {} & \frac{\phi_1(0) - \sqrt{\frac{b}{c}}\phi_2(0)}{2}\mathcal{R}_{\alpha,1}^{-\sqrt{bc},a,\tau}(t) + \frac{\phi_1(0) + \sqrt{\frac{b}{c}}\phi_2(0)}{2}\mathcal{R}_{\alpha,1}^{\sqrt{bc},a,\tau}(t) \\ & + a\int_{-\tau}^0\left[\frac{\phi_1(u) - \sqrt{\frac{b}{c}}\phi_2(u)}{2}\mathcal{R}_{\alpha,\alpha}^{-\sqrt{bc},a,\tau}(t-\tau-u) + \frac{\phi_1(u) + \sqrt{\frac{b}{c}}\phi_2(u)}{2}\mathcal{R}_{\alpha,\alpha}^{\sqrt{bc},a,\tau}(t-\tau-u)\right]h(t-\tau-u)\mathrm{d}u\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned} x_2(t) = {} & \frac{\phi_2(0) - \sqrt{\frac{c}{b}}\phi_1(0)}{2}\mathcal{R}_{\alpha,1}^{-\sqrt{bc},a,\tau}(t) + \frac{\phi_2(0) + \sqrt{\frac{c}{b}}\phi_1(0)}{2}\mathcal{R}_{\alpha,1}^{\sqrt{bc},a,\tau}(t) \\ & + a\int_{-\tau}^0\left[\frac{\phi_2(u) - \sqrt{\frac{c}{b}}\phi_1(u)}{2}\mathcal{R}_{\alpha,\alpha}^{-\sqrt{bc},a,\tau}(t-\tau-u) - \frac{\phi_2(u) + \sqrt{\frac{c}{b}}\phi_1(u)}{2}\mathcal{R}_{\alpha,\alpha}^{\sqrt{bc},a,\tau}(t-\tau-u)\right]h(t-\tau-u)\mathrm{d}u \end{aligned} \tag{3.4}$$

for $t > 0$, where h is the Heaviside step function and $\mathcal{R}^{v,w,\tau}_{\alpha,\beta}$ is a generalized delay exponential function of the Mittag-Leffler type introduced in [31] as

$$\mathcal{R}^{v,w,\tau}_{\alpha,\beta}(t) = \mathcal{L}^{-1}\left(\frac{s^{\alpha-\beta}}{s^{\alpha} - v - w\exp[-s\tau]}\right)(t), \qquad \alpha, \beta, \tau \in \mathbb{R}^+ \text{ and } v, w \in \mathbb{C}. \tag{3.5}$$

Thus, the relations (3.3) and (3.4) enable us to study asymptotic properties of x_1, x_2 through asymptotics of $\mathcal{R}^{v,w,\tau}_{\alpha,\beta}$ with $v = \pm\sqrt{bc}$, $w = a$ and $\beta \in \{\alpha, 1\}$.

Now we discuss next procedures that depend on the sign of bc. If $bc > 0$, then both v, w are real numbers, hence the behaviour of $\mathcal{R}^{v,w,\tau}_{\alpha,\beta}$ is covered by some parts of [31]. In particular, the results relevant for our purposes can be summarized in the following

Lemma 1. *Let $0 < \alpha < 1$, $0 < \beta \le 1$, $\tau > 0$, v and w be real numbers. We denote*

$$\tau^*(v, w) = \frac{(2-\alpha)\pi/2 + \arcsin(v/w\sin(\alpha\pi/2))}{\left(\sqrt{w^2 - v^2\sin^2(\alpha\pi/2)} + v\cos(\alpha\pi/2)\right)^{1/\alpha}}. \tag{3.6}$$

Then $\mathcal{R}^{v,w,\tau}_{\alpha,\beta}$ tends to zero as $t \to \infty$ if and only if

$$v \le w < -v \quad \text{and} \quad \tau > 0, \qquad \text{or} \qquad |v| + w < 0 \quad \text{and} \quad \tau < \tau^*(v, w).$$

Using this conclusion, we can describe asymptotic properties of (3.3) and (3.4) when $bc > 0$.

If $bc < 0$, then v remains a real number, but w is purely imaginary, hence the above mentioned result cannot cover this case. Thus, the next section is devoted solely to study of the case $bc < 0$. In particular, we focus on root analysis of characteristic equations implied by (3.2) (and consequently (3.5)) in the form

$$Q_1(s) \equiv s^{\alpha} - \mathrm{i}p - q\exp(-s\tau) = 0 \qquad \text{and} \qquad Q_2(s) \equiv s^{\alpha} + \mathrm{i}p - q\exp(-s\tau) = 0 \tag{3.7}$$

where $p = \sqrt{-bc} \neq 0$, $q = a$ are real numbers.

4. Analysis of characteristic roots

As recalled above, the question of stability of linear autonomous differential systems, both integer-order as well as fractional-order, is closely related to the location of all characteristic roots to the left of the imaginary axis. As we shall confirm later, (2.3) is not an exception to this rule. Thus, this section is devoted to a study of conditions guaranteeing that all roots of (3.7) have negative real parts.

We start with a survey of several generic properties of roots of (3.7).

Proposition 1. *Let $0 < \alpha < 1$ and p, q, τ be real numbers such that $p \neq 0$ and $\tau > 0$.*

(i) *If s is a root of Q_1, then its complex conjugate $\bar{s}$ is a root of Q_2.*
(ii) *Q_1 and Q_2 have no non-negative real roots.*
(iii) *Let $0 < \omega < \pi$ be arbitrary. Then Q_1 and Q_2 have only a finite number of roots s such that $|\arg(s)| \le \omega$.*
(iv) *All roots of Q_1 and Q_2 have multiplicity one or two.*

Proof. The property (i) follows immediately by splitting Q_1 and Q_2 into real and imaginary parts. Its validity enables us to prove the remaining properties only with respect to Q_1.

Let s_0 be a non-negative real root of Q_1. Then the expression $s_0^{\alpha} - q\exp(-s_0\tau)$ has to be purely imaginary (and non-zero) which is a contradiction. Hence, the property (ii) is valid.

The technique how to verify (iii) is the same as that used in the proof of [31, Proposition 2 (iii)].

Finally, let s_0 be a root of Q_1 with multiplicity larger than two. Then

$$Q_1'(s_0) = \alpha s_0^{\alpha-1} + q\tau\exp(-s_0\tau) = 0 \qquad \text{and} \qquad Q_1''(s_0) = \alpha(\alpha-1)s_0^{\alpha-2} - q\tau^2\exp(-s_0\tau) = 0.$$

This implies that s_0 is a positive real number, namely $s_0 = (1-\alpha)/\tau$, which contradicts (ii). Thus, the property (iv) holds as well. □

Due to Proposition 1 (i), real parts of roots of Q_1 and Q_2 are identical, hence it is enough to consider Q_1 only. To (formally) simplify later discussions, we introduce a substitution $z = s\tau$ enabling us to study $\tilde{Q}_1$ instead of Q_1 where

$$\tilde{Q}_1(z) \equiv z^{\alpha} - \mathrm{i}\tau^{\alpha}p - \tau^{\alpha}q\exp(-z) = 0, \tag{4.1}$$

$0<\alpha<1$, $\tau>0$, $p\neq 0$ and q still being real numbers. We wish to find an explicit characterization of a property guaranteeing that all roots of $\tilde{Q}_1$ are located in the left half-part of the complex plane. For this purpose, we first use the D-decomposition method. Let $z=\rho\exp(\mathrm{i}\varphi)$ for suitable reals $\rho>0$ and $-\pi<\varphi\leq\pi$. Then z is the root of $\tilde{Q}_1$ if and only if

$$\begin{aligned}&\rho^\alpha\cos\alpha\varphi-\tau^\alpha q\exp\left(-\rho\cos\varphi\right)\cos(\rho\sin\varphi)=0\\&\rho^\alpha\sin\alpha\varphi-\tau^\alpha p+\tau^\alpha q\exp\left(-\rho\cos\varphi\right)\sin(\rho\sin\varphi)=0\,.\end{aligned}$$

For $\varphi=\pm\pi/2$, we get the boundary locus curves in a form

$$\left.\begin{aligned}\tau^\alpha p&=\pm\frac{\rho^\alpha\sin(\rho+\alpha\pi/2)}{\cos\rho}\\ \tau^\alpha q&=\frac{\rho^\alpha\cos\alpha\pi/2}{\cos\rho}\end{aligned}\right.,\qquad \rho>0\,,\quad \rho\neq\frac{\pi}{2}+j\pi\,,\quad \rho+\alpha\pi/2\neq j\pi\,,\quad j=0,1,2,\ldots\,. \tag{4.2}$$

Consequently, for any couple $(\tau^\alpha p,\ \tau^\alpha q)$ given by (4.2), (4.1) has a purely imaginary root. Moreover, it holds

Proposition 2. *Let $0<\alpha<1$, $\tau>0$ and $p\neq 0$ be fixed real numbers and let q be a free real number such that $(\tau^\alpha p,\ \tau^\alpha q)$ is given by (4.2) for a suitable $\rho>0$ and a suitable sign combination. If $|q|$ increases, then a new root of (4.1) having a positive real part appears.*

Proof. For any purely imaginary z, there exists its neighborhood where $\tilde{Q}_1$ is an analytic function and its roots are continuously depending on parameters p, q, τ, α (in fact, $\tilde{Q}_1$ is analytic for all points apart from non-positive reals). We show that if a couple $(\tau^\alpha p,\ \tau^\alpha q)$ from (4.2) is considered and $|q|$ increases, then the real part of the corresponding purely imaginary root becomes positive. Indeed, if we consider z as a function of p, q given implicitely by (4.1) and differentiate z with respect to q, we get

$$\frac{\mathrm{d}z}{\mathrm{d}q}=\frac{\tau^\alpha\exp(-z)}{\alpha z^{\alpha-1}+\tau^\alpha q\exp(-z)}=\frac{z^\alpha-\mathrm{i}\tau^\alpha p}{q(\alpha z^{\alpha-1}+z^\alpha-\mathrm{i}\tau^\alpha p)}\,,$$

which, in the case of purely imaginary root $z=\mathrm{i}\rho$ ($\rho>0$), yields

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}q}\Big|_{z=+\mathrm{i}\rho}&=\frac{(\mathrm{i}\rho)^\alpha-\mathrm{i}\tau^\alpha p}{q(\alpha(\mathrm{i}\rho)^{\alpha-1}+(\mathrm{i}\rho)^\alpha-\mathrm{i}\tau^\alpha p)}\\&=\frac{1}{q}\frac{\rho^\alpha\cos(\alpha\pi/2)+\mathrm{i}(\rho^\alpha\sin(\alpha\pi/2)-\tau^\alpha p)}{\alpha\rho^{\alpha-1}\cos((\alpha-1)\pi/2)+\rho^\alpha\cos(\alpha\pi/2)+\mathrm{i}(\alpha\rho^{\alpha-1}\sin((\alpha-1)\pi/2)+\rho^\alpha\sin(\alpha\pi/2)-\tau^\alpha p)}\\&=\frac{1}{q}\frac{a_1+\mathrm{i}a_2}{a_3+\mathrm{i}a_4}=\frac{1}{q}\frac{(a_1a_3+a_2a_4)+\mathrm{i}(a_2a_3-a_1a_4)}{a_3^2+a_4^2}\end{aligned}$$

where $a_1=\rho^\alpha\cos(\alpha\pi/2)$, $a_2=\rho^\alpha\sin(\alpha\pi/2)-\tau^\alpha p$, $a_3=\alpha\rho^{\alpha-1}\cos((\alpha-1)\pi/2)+\rho^\alpha\cos(\alpha\pi/2)$ and $a_4=\alpha\rho^{\alpha-1}\sin((\alpha-1)\pi/2)+\rho^\alpha\sin(\alpha\pi/2)-\tau^\alpha p$. Considering the derivative of a real part of the root with respect to q, we obtain

$$\frac{\mathrm{d}\Re(z)}{\mathrm{d}q}\Big|_{z=+\mathrm{i}\rho}=\Re\left(\frac{\mathrm{d}z}{\mathrm{d}q}\right)\Big|_{z=+\mathrm{i}\rho}=\frac{1}{q}\frac{a_1a_3+a_2a_4}{a_3^2+a_4^2}$$

where

$$a_1a_3+a_2a_4=(\rho^\alpha-\tau^\alpha p)^2+2\rho^\alpha\tau^\alpha p(1-\sin(\alpha\pi/2))+\alpha\rho^{\alpha-1}\tau^\alpha p\cos(\alpha\pi/2)>0\qquad\text{for }\ p>0\,.$$

Due to Proposition 1 (i), the real parts of roots of (4.1) are not depending on the sign of p, so we can extend this result for all real p to get

$$\operatorname{sgn}\left(\frac{\mathrm{d}\Re(z)}{\mathrm{d}q}\Big|_{z=+\mathrm{i}\rho}\right)=\operatorname{sgn}(q)\,. \tag{4.3}$$

Analogously, we can extend (4.3) also for $z=-\mathrm{i}\rho$ ($\rho>0$). Thus, whenever we are crossing any curve from the system (4.2) with α, τ, p fixed and $|q|$ increasing, a new root of (4.1) with a positive real part actually appears. □

The next assertion is useful in calculations of cusp points where relevant parts of the boundary locus system intersect each other.

Proposition 3. *Let $0<\alpha<1$. Then, for any $j=0,1,2,\ldots,$ the equation*

$$-\frac{\sin(\theta+\alpha\pi/2)}{\sin(\theta-\alpha\pi/2)}=\left(\frac{(2j+3-\alpha)\pi}{\theta+(j+1/2-\alpha/2)\pi}-1\right)^\alpha \tag{4.4}$$

has a unique root θ_j^* such that $\theta_j^*\in(0,\alpha\pi/2)$. Moreover, it holds $\theta_j^*>\theta_{j+1}^*$ for all $j=0,1,2,\ldots$ and $\theta_j^*\to 0^+$ as $j\to\infty$.

Proof. Put

$$g_j(\theta)=\frac{\sin(\theta+\alpha\pi/2)}{\sin(\theta-\alpha\pi/2)}+\left(\frac{(2j+3-\alpha)\pi}{\theta+(j+1/2-\alpha/2)\pi}-1\right)^\alpha\,.$$

We show that $g_j(\theta)$ has a unique root θ_j^* in $(0, \alpha\pi/2)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\sin(\theta+\alpha\pi/2)}{\sin(\theta-\alpha\pi/2)}\right] = -\frac{\sin(\alpha\pi/2)}{\sin^2(\theta-\alpha\pi/2)} < 0\,,$$

$g_j(\theta)$ is decreasing in $(0, \alpha\pi/2)$. Moreover,

$$g_j(\theta) \to \left(1+\frac{4}{2j+1-\alpha}\right)^{\alpha} - 1 > 0 \quad \text{as } \theta \to 0^+$$

and

$$g_j(\theta) \to -\infty \quad \text{as } \theta \to \alpha\pi/2^-\,.$$

The uniqueness of θ_j^* is now ensured by continuity of $g_j(\theta)$ in $(0, \alpha\pi/2)$.

Moreover, a direct calculation shows that $g_j(\theta) > g_{j+1}(\theta)$ for all $\theta \in (0, \alpha\pi/2)$ and $j = 0, 1, 2, \ldots$. This fact, along with continuity and decrease of $g_j(\theta)$ with respect to θ, implies the property $\theta_j^* > \theta_{j+1}^*$. Letting $j \to \infty$, $g_j(\theta)$ is reduced to $g_\infty(\theta) = \sin(\theta+\alpha\pi/2)/\sin(\theta-\alpha\pi/2)+1$ with a root $\theta_\infty^* = 0$ which concludes the proof. □

To describe the boundary of the set of all $(\tau^\alpha p,\ \tau^\alpha q)$ such that all roots of (4.1) have negative real parts, we consider two piecewise smooth curves Γ^+ and Γ^- defined in the $(\tau^\alpha p,\ \tau^\alpha q)$-plane as follows: Let $\theta_j^* \in (0, \alpha\pi/2)$ be the unique root of (4.4) for $j = 0, 1, 2, \ldots$. We put

$$\rho_j^* = \frac{(2j+1-\alpha)\pi}{2} + \theta_j^*, \qquad \rho_j^{**} = \frac{(2j+1-\alpha)\pi}{2} - \theta_{j-2}^*, \qquad j = 0, 1, 2, \ldots$$

where we define

$$\theta_{-2}^* = \frac{(\alpha-1)\pi}{2}, \qquad \theta_{-1}^* = \frac{\pi}{2}$$

(this leads to $\rho_0^{**} = 0$ and $\rho_1^{**} = (2-\alpha)\pi/2$). Further, for any $j = 0, 1, 2, \ldots$, we introduce the system of curves Γ_j in the $(\tau^\alpha p,\ \tau^\alpha q)$-plane given via a parametric form

$$\begin{aligned} \tau^\alpha p &= \pm\frac{\rho^\alpha \sin(\rho+\alpha\pi/2)}{\cos(\rho)} \\ \tau^\alpha q &= \frac{\rho^\alpha \cos(\alpha\pi/2)}{\cos(\rho)} \end{aligned}\,, \qquad \rho_j^{**} \le \rho \le \rho_j^* \tag{4.5}$$

(in the first relation, both the sign cases have to be considered, and thus any curve Γ_j has two branches symmetric with respect to $\tau^\alpha q$-axis). Finally, we define $\Gamma^+ = \cup_{j=0}^\infty \Gamma_{2j}$ and $\Gamma^- = \cup_{j=0}^\infty \Gamma_{2j+1}$.

Using this notation, we have

Lemma 2. *Let $0<\alpha<1$, $\tau>0$, $p \neq 0$ and q be real numbers. All roots of (4.1) have negative real parts if and only if the couple $(\tau^\alpha p,\ \tau^\alpha q)$ is located inside the area bounded by Γ^+ from above and by Γ^- from below.*

Proof. For $q = 0$ and $p \neq 0$, (4.1) is reduced to the equation $z^\alpha - \mathrm{i}\tau^\alpha p = 0$ having roots with a negative real part only (we consider the principal branches). Then Proposition 2 implies that all roots of (4.1) have negative real parts if and only if the couple $(\tau^\alpha p,\ \tau^\alpha q)$ lies in a region containing the points $\tau^\alpha q = 0$ and bounded by appropriate boundary locus curves. We show that these curves are just Γ^+ and Γ^-.

As already noted above, the boundary locus consists of countably many continuous branches of (4.2). For any $j = 0, 1, 2, \ldots$, we put

$$f_j(\rho) = \frac{\cos(\alpha\pi/2)}{\sin(\rho+\alpha\pi/2)}\,, \quad (j-1)\pi+\pi/2 < \rho < j\pi+\pi/2\,, \quad \rho+\alpha\pi/2 \neq j\pi\,. \tag{4.6}$$

Using this notation we can describe the ratio $q/|p|$, where p, q are given by (4.2), as

$$\frac{q}{|p|} = f_j(\rho)\,, \quad (j-1)\pi+\pi/2 < \rho < j\pi+\pi/2\,, \quad \rho+\alpha\pi/2 \neq j\pi\,.$$

Obviously, f_j has an extremal value $(-1)^j\cos(\alpha\pi/2)$ occurring when $\rho = (j+(1-\alpha)/2)\pi \in [\rho_j^{**}, \rho_j^*]$ (indeed, $\mathrm{d}f_j/\mathrm{d}\rho = -q/|p|\cot(\rho+\alpha\pi/2)$). In particular, the minimal value of the ratio $q/|p|$ related to Γ^+ equals $\cos(\alpha\pi/2)$ for all its branches (the case when j is even), while the maximal value of $q/|p|$ related to Γ^- equals $-\cos(\alpha\pi/2)$ for all its branches (the case when j is odd). By Proposition 2, it is enough, for each of the branches, to find intersections with boundary locus curves closest to the corresponding minimum or maximum.

We consider an arbitrary even $j \ge 2$ and calculate the intersection between two adjacent branches of boundary locus curves from the system (4.2). Let

$$\sin(\rho_1+\alpha\pi/2) = \sin(\rho_2+\alpha\pi/2) \tag{4.7}$$

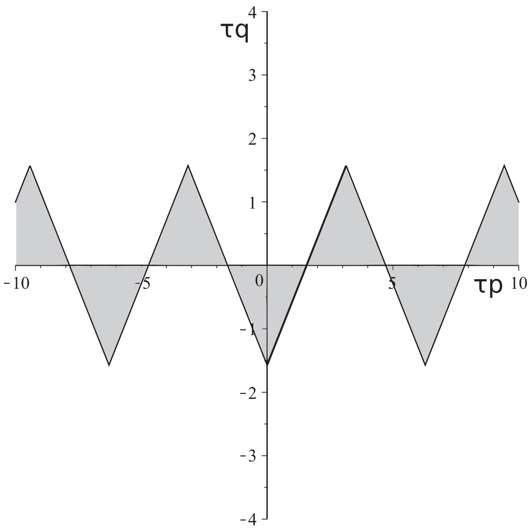

Fig. 1. A classical result for the stability region ($\alpha = 1$).

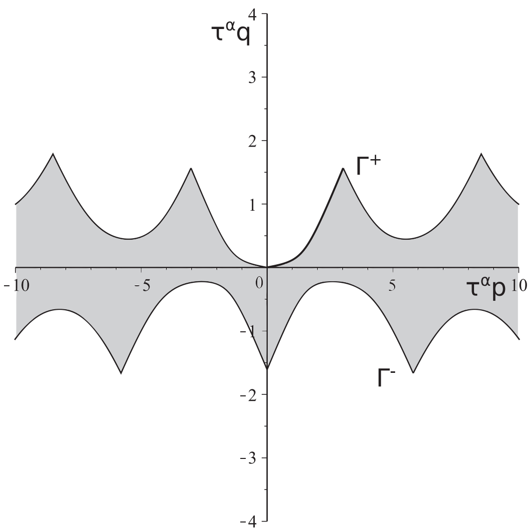

Fig. 2. The stability region for $\alpha = 0.95$.

and

$$\frac{\rho_1^\alpha}{\cos(\rho_1)} = \frac{\rho_2^\alpha}{\cos(\rho_2)} \tag{4.8}$$

for a suitable $(j-1)\pi + \pi/2 < \rho_1 < j\pi + \pi/2$ and $(j+1)\pi + \pi/2 < \rho_2 < (j+2)\pi + \pi/2$ (note that for $j = 0$ this interval is replaced by $0 < \rho_1 < \pi/2$). Then (4.7) implies either $\rho_1 + 2\pi = \rho_2$ (that, however, contradicts (4.8)), or $\rho_1 + \rho_2 = (2j + 3 - \alpha)\pi$. Substituting this into (4.8) we have

$$\frac{\rho_1^\alpha}{\cos(\rho_1)} = \frac{((2j+3-\alpha)\pi - \rho_1)^\alpha}{\cos((2j+3-\alpha)\pi - \rho_1)}, \qquad \text{i.e.} \quad -\frac{\cos(\rho_1 + \alpha\pi)}{\cos(\rho_1)} = \left(\frac{(2j+3-\alpha)\pi}{\rho_1} - 1\right)^\alpha.$$

Denoting $\theta = \rho_1 - (j + 1/2 - \alpha/2)\pi$, Proposition 3 directly implies that this equation has a unique root in a given interval. This root corresponds to the (unique) intersection of appropriate boundary locus curves. It is easy to check that relevant parts of these curves form just the system of curves Γ_j. Employing the same technique, we can show that this intersection is actually the closest one to the extreme (in the given direction), i.e. intersections with any non-adjacent branch occur for $\rho > \rho_j^*$ or $\rho < \rho_j^{**}$. The same argumentation is true also when j is odd. □

Relevant parts of $(\tau^\alpha p, \tau^\alpha q)$-plane, where all roots of (4.1) have negative real parts, are depicted in Figs. 1–4 for various values of α, including the first-order case discussed in [24] (these parts are referred to as stability regions in comments to figures and are depicted in a grey color). Contrary to the first-order case (Fig. 1), any stability region for $0 < \alpha < 1$ (Figs. 2–4) forms a set connected in the $(\tau^\alpha p, \tau^\alpha q)$-plane. Also, we can observe larger stability regions for decreasing α (more detailed comments will be given in Remark 2).

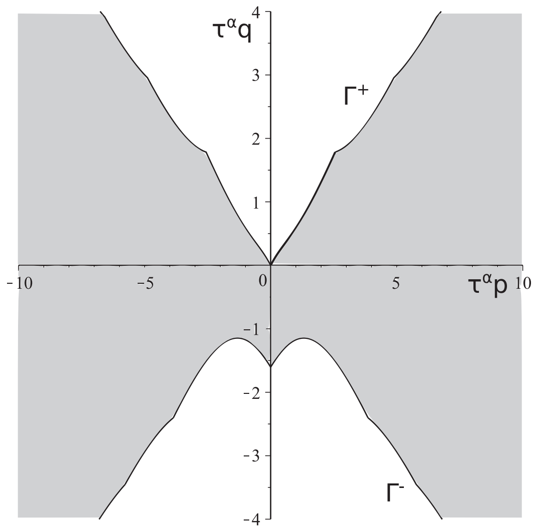

Fig. 3. The stability region for $\alpha = 0.6$.

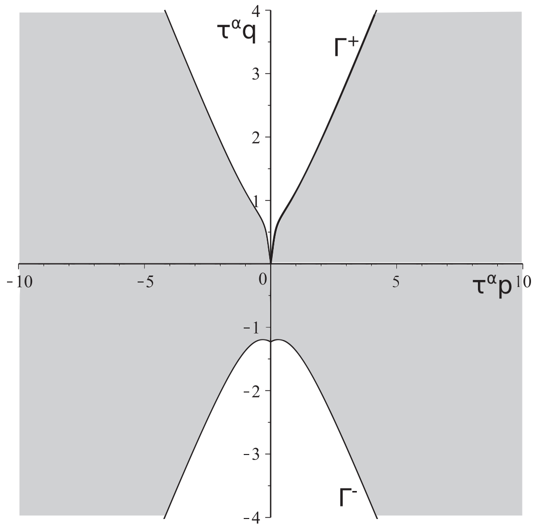

Fig. 4. The stability region for $\alpha = 0.2$.

For a given ratio $|q/p|$, we can give an analytical description of delays τ such that $(\tau^\alpha p, \tau^\alpha q) \in \Gamma^+ \cup \Gamma^-$. On this account, we put

$$\tau^*_{i,\kappa}(p,q) = \frac{(i + (1-\alpha)/2)\pi - \kappa \arccos(|p/q|\cos(\alpha\pi/2))}{\left(\kappa\sqrt{q^2 - p^2\cos^2(\alpha\pi/2)} + |p|\sin(\alpha\pi/2)\right)^{1/\alpha}}, \qquad i = 0, 1, 2, \dots, \quad \kappa = \pm 1. \tag{4.9}$$

If $|q/p| \geq \cot(\alpha\pi/2)$, then the symbol $\tau^*_{0,1}$ is not defined.

Using this notation we have

Lemma 3. *Let $0 < \alpha < 1$, $\tau > 0$, $p \neq 0$ and q be real numbers such that $|q/p| > \cos(\alpha\pi/2)$. Further, let $\theta^*_j \in (0, \alpha\pi/2)$ be the unique root of (4.4) for $j = 0, 1, \dots$. Then it holds:*

(i) *There exists a unique couple of consecutive non-negative integers m such that*

$$\frac{\cos(\alpha\pi/2)}{\cos(\theta^*_m)} \leq \frac{|q|}{|p|} < \frac{\cos(\alpha\pi/2)}{\cos(\theta^*_{m-2})}, \quad \text{if } m \geq 2, \qquad \text{or} \qquad \frac{\cos(\alpha\pi/2)}{\cos(\theta^*_m)} \leq \frac{|q|}{|p|}, \quad \text{if } m \in \{0, 1\}. \tag{4.10}$$

(ii) *If $(\tau^\alpha p, \tau^\alpha q) \in \Gamma^+ \cup \Gamma^-$, then $\tau = \tau^*_{i,\kappa}(p,q)$ for a suitable $i = 0, 1, \dots, m$ and $\kappa = \pm 1$ (see (4.9)). Here, i and m given by (4.10) are even (odd) when $q \geq 0$ ($q < 0$), respectively.*

(iii) *$\tau^*_{i,1} \leq \tau^*_{i,-1} < \tau^*_{i+2,1}$ for all $i = 0, 1, \dots, m-2$.*

Proof. (i) Proposition 3 guarantees that $\theta^*_m > \theta^*_{m+2}$ are both lying in $(0, \alpha\pi/2)$ for all $m = 0, 1, 2 \dots$, which ensures the well posedness of (4.10). Moreover, Proposition 3 implies that $\cos(\alpha\pi/2)/\cos(\theta^*_m) \to \cos(\alpha\pi/2)$ as $m \to \infty$ which, along with the condition $|q/p| > \cos(\alpha\pi/2)$, proves (i).

(ii) If we substitute the endpoints of Γ_j into (4.6), we get

$$f_j(\rho_j^*) = \frac{\cos(\alpha\pi/2)}{(-1)^j\cos(\theta_j^*)}\,, \qquad f_j(\rho_j^{**}) = \frac{\cos(\alpha\pi/2)}{(-1)^j\cos(\theta_{j-2}^*)}\,.$$

We note that the endpoints of Γ_j (corresponding to ρ_j^* and ρ_j^{**}) are actually cusp points of Γ^+ or Γ^- (in the sequel, we call them the j-th and the $(j-2)$-nd cusp point, respectively; obviously, the cusp points are not defined when $j \in \{0, 1\}$). Even values of j relate to Γ^+ and odd j correspond to Γ^-. We point out that $f_j(\rho)$ is convex or concave for ρ lying in appropriate intervals corresponding to Γ^+ or Γ^-, respectively. The property (i) implies that (4.10) determines two couples of adjacent cusp points (the m-th and the $(m-2)$-nd) of Γ^+ and Γ^-, respectively. Each of couples is characterized by the property that tangents of two lines ℓ_m, ℓ_{m-2} $(m \ge 2)$ starting from the origin and going through the corresponding cusp points define an interval containing $|q/p|$.

Let $(4.10)_1$ be valid. The above arguments yield that Γ_m contains a unique point of the given ratio $|q/p|$ and every Γ_{m-2j} $(j = 1, \dots (m-1)/2)$ has two such points with a possible exception of Γ_0. Indeed, since Γ_0 starts from the origin with the tangent equal to $\cot(\alpha\pi/2)$, it has two such points only when $|q/p| < \cot(\alpha\pi/2)$, otherwise it has only one. In the case of $(4.10)_2$, there is only one point of the given ratio $|q/p|$ lying on Γ_0 (provided $|q/p| < \cot(\alpha\pi/2)$) or Γ_1 depending on the sign of q. Thus, for $|q/p|$ being fixed, all points $(\tau^\alpha p, \tau^\alpha q) \in \Gamma^+ \cup \Gamma^-$ lie on branches Γ_j such that $j \le m$ (j being even or odd for Γ^+ or Γ^-, respectively). Now we are going to derive the precise expressions for the values of τ characterizing these points.

First, we consider a part of Γ_j ($j \ge 2$ being an even integer) corresponding to the parameter range $\rho \in (\rho_j^{**}, (j + 1/2 - \alpha/2)\pi)$, i.e. the part of Γ_j corresponding to decreasing $f_j(\rho)$ (see the proof of Lemma 2). Then (4.6) yields

$$\rho + \alpha\pi/2 = j\pi + \arcsin\left(|p|/q\cos(\alpha\pi/2)\right), \qquad \text{i.e.} \quad \rho = (j + (1-\alpha)/2)\pi - \arccos\left(|p|/q\cos(\alpha\pi/2)\right).$$

A substitution into $(4.5)_2$ leads to

$$q = \frac{[(j + (1-\alpha)/2)\pi - \arccos(|p|/q\cos(\alpha\pi/2))]^\alpha \cos(\alpha\pi/2)}{\tau^\alpha \cos[(j + (1-\alpha)/2)\pi - \arccos(|p|/q\cos(\alpha\pi/2))]}\,.$$

Since

$$q\cos[(j + (1-\alpha)/2)\pi - \arccos(|p|/q\cos(\alpha\pi/2))] = \left[\sqrt{q^2 - p^2\cos^2(\alpha\pi/2)} + |p|\sin(\alpha\pi/2)\right]\cos(\alpha\pi/2))\,,$$

we have

$$\tau^\alpha = \frac{[(j + (1-\alpha)/2)\pi - \arccos\left(|p|/q\cos(\alpha\pi/2)\right)]^\alpha}{\sqrt{q^2 - p^2\cos^2(\alpha\pi/2)} + |p|\sin(\alpha\pi/2)}\,.$$

Similarly, for a part of Γ_j (where $j \ge 2$ is still even) with $\rho \in ((j + 1/2 - \alpha/2)\pi, \rho_j^*)$, i.e. the part corresponding to increasing $f_j(\rho)$, one gets

$$\tau^\alpha = \frac{[(j + (1-\alpha)/2)\pi + \arccos(|p|/q\cos(\alpha\pi/2))]^\alpha}{-\sqrt{q^2 - p^2\cos^2(\alpha\pi/2)} + |p|\sin(\alpha\pi/2)}.$$

That yields (4.9) for $j \ge 2$ being even and $\kappa = \pm 1$, where positive or negative κ corresponds to decreasing or increasing $f_j(\rho)$, respectively. If $j = 0$, then all the calculations are analogous up to the case $|q/p| \ge \cot(\alpha\pi/2)$ when computations for $\rho \in (0, (1-\alpha)\pi/2)$ leads to $\rho < 0$ (or $\tau < 0$) which is a contradiction. Therefore, (4.9) is not defined for $j = 0$ and $\kappa = 1$ in this case.

The technique can be directly extended for odd integers $j \ge 1$, where $\kappa = 1$ and $\kappa = -1$ correspond to decreasing and increasing $|f_j(\rho)|$, respectively.

(iii) The last assertion of Lemma 3 follows from the above argumentation. E.g., if j is even, then $\tau = \tau_{j,1}^*$ $(\tau = \tau_{j,-1}^*)$ corresponds to the point $(\tau^\alpha p,\ \tau^\alpha q)$ lying on the decreasing (increasing) segment of $\Gamma_j \subset \Gamma^+$, respectively. The validity of the inequalities now follows from a simple computational analysis of (4.5). □

Remark 1. Lemma 3 plays a key role when finding the relation between the ratio $|q/p|$ and cusp points of Γ^+, Γ^-. Another significant value for $|q/p|$ is the tangent $\cot(\alpha\pi/2)$ of Γ^+ at the origin. As it was already pointed out, lines ℓ_m ($m \ge 0$ being even) connecting the origin and cusp points of Γ^+ have decreasing tangents with respect to increasing index of given cusp points (a similar comment is true also for Γ^-). It might be useful to discuss a relationship between the cusp points and the above mentioned value $\cot(\alpha\pi/2)$. On this account, we consider the first cusp point of Γ^+ (i.e. the endpoint of Γ_0). If we equate tangents of Γ^+ at the origin and a line ℓ_0, and supplied with the condition for the first cusp point (see the proof of Lemma 2), we arrive at the system of two equations

$$\begin{aligned} \cot(\alpha\pi/2) &= \frac{\cos(\alpha\pi/2)}{\cos(\theta)} \\ -\frac{\sin(\theta + \alpha\pi/2)}{\sin(\theta - \alpha\pi/2)} &= \left(\frac{(3-\alpha)\pi}{\theta + (1-\alpha)\pi/2} - 1\right)^\alpha \end{aligned} \tag{4.11}$$

for two unkowns $0<\alpha<1$ and $0<\theta<\alpha\pi/2$. It can be shown that (4.11) has a unique solution that can be numerically calculated as

$$\alpha^* \approx 0.6150768144 \qquad \text{and} \qquad \theta_0^* \approx 0.562788670\,. \tag{4.12}$$

Consequently, if $\alpha>\alpha^*$, then the tangent of Γ^+ at the origin is larger than the tangent of ℓ_0 passing through the origin and the first cusp point of Γ^+.

5. Main results

In this section, we summarize previous considerations to obtain stability criteria for (2.3) that are given in terms of entry coefficients with respect to the sign of bc.

Theorem 1. *Let $0<\alpha<1$, $\tau>0$ and a, b, c be real numbers such that $bc>0$ and let τ^* be defined by (3.6). The zero solution to (2.3) is asymptotically stable if and only if*

$$a+\sqrt{bc}<0 \qquad \text{and} \qquad \tau<\tau^*(\sqrt{bc},a)\,.$$

Proof. By (3.3), (3.4), the solution to (2.3) can be expressed in terms of the functions $\mathcal{R}_{\alpha,\alpha}^{\pm\sqrt{bc},a,\tau}$, $\mathcal{R}_{\alpha,1}^{\pm\sqrt{bc},a,\tau}$. More precisely, the zero solution to (2.3) is asymptotically stable if and only if both $\mathcal{R}_{\alpha,\alpha}^{\sqrt{bc},a,\tau}(t)$ and $\mathcal{R}_{\alpha,\alpha}^{-\sqrt{bc},a,\tau}(t)$ tend to zero as $t\to\infty$. Although Lemma 1 admits two families of conditions guaranteeing this property, we can see that the first of them (corresponding to a delay-independent part) cannot be satisfied simultaneously for both $v=\sqrt{bc}$ and $v=-\sqrt{bc}$. A direct sign analysis of (3.6) yields that $\tau^*(\sqrt{bc},a)<\tau^*(-\sqrt{bc},a)$ for all $a<0$ (positive values of a are excluded due to the condition $a+\sqrt{bc}<0$). The assertion now follows from Lemma 1. □

Theorem 2. *Let $0<\alpha<1$, $\tau>0$ and a, b, c be real numbers such that $bc<0$, let α^* be given by (4.12) and let $\theta_j^*\in(0,\alpha\pi/2)$ be the unique root of (4.4) for $j=0,1,2,\dots$. Further, assuming $|a|/\sqrt{-bc}\ge\cos(\alpha\pi/2)$, let $m\ge0$ be an even integer (if $a\ge0$), or an odd integer (if $a<0$), uniquely determined by*

$$\frac{\cos(\alpha\pi/2)}{\cos(\theta_m^*)}\le\frac{|a|}{\sqrt{-bc}}<\frac{\cos(\alpha\pi/2)}{\cos(\theta_{m-2}^*)}\,,\quad m\ge2\,,\qquad \text{or}\qquad \frac{\cos(\alpha\pi/2)}{\cos(\theta_m^*)}\le\frac{|a|}{\sqrt{-bc}}\,,\quad m\in\{0,1\}\,. \tag{5.1}$$

The zero solution to (2.3) is asymptotically stable if and only if any of the following conditions holds:

$$\frac{|a|}{\sqrt{-bc}}<\cos(\alpha\pi/2)\,; \tag{5.2}$$

$$\cos(\alpha\pi/2)\le\frac{a}{\sqrt{-bc}}<\cot(\alpha\pi/2)\quad\text{and}\quad \tau\in\bigcup_{j=-1}^{m/2-1}\left(\tau_{2j,-1}^*,\tau_{2j+2,1}^*\right); \tag{5.3}$$

$$\alpha>\alpha^*,\quad \cot(\alpha\pi/2)\le\frac{a}{\sqrt{-bc}}<\frac{\cos(\alpha\pi/2)}{\cos(\theta_0^*)}\quad\text{and}\quad \tau\in\bigcup_{j=0}^{m/2-1}\left(\tau_{2j,-1}^*,\tau_{2j+2,1}^*\right); \tag{5.4}$$

$$\frac{a}{\sqrt{-bc}}\le-\cos(\alpha\pi/2)\quad\text{and}\quad \tau\in\bigcup_{j=-1}^{(m-1)/2-1}\left(\tau_{2j+1,-1}^*,\tau_{2j+3,1}^*\right), \tag{5.5}$$

where $\tau_{i,\kappa}^*=0$ for negative integers i and $\tau_{i,\kappa}^*=\tau_{i,\kappa}^*(\sqrt{-bc},a)$ are given by (4.9) for non-negative integers i.

Remark 2. (i) The condition (5.2) describes the delay-independent asymptotic stability area for (2.3). While this area is empty when $\alpha=1$ (see also [24]), it becomes non-empty for any $0<\alpha<1$. Moreover, an explicit dependence of this area on a changing derivative order α is described via (5.2). The remaining conditions (5.3)–(5.5) capture delay-dependent asymptotic stability areas for (2.3). It might be interesting to discuss their form with respect to α approaching 1. Similarly to (5.2), the condition (5.3) becomes empty when $\alpha\to1$. Further, in this limit case, (5.4) turns into the stability condition (1.6) of [24], and (5.5) can be split into (1.4)–(1.5) of [24]. The same comment is true for the stability condition of Theorem 1 that can be converted into the condition (1.3) of [24]. Thus, stability conclusions of Theorems 1 and 2 with α approaching 1 fully agree with existing stability conditions for the corresponding integer-order system.

(ii) As observed in Figs. 1–4, stability regions become larger with decreasing α which can be utilized, e.g., in stabilization or synchronization issues, or in design of fractional PID controllers which were studied, along with influence of α as a control parameter, by many authors (see, e.g., [4,17]). If we formally put $\alpha=0$ in (2.3) and assume $bc\neq1$, then we obtain, after some technical rearrangements, a difference system with the continuous time in a form

$$x_1(t)=\frac{a}{1-bc}(x_1(t-\tau)+bx_2(t-\tau))$$
$$x_2(t)=\frac{a}{1-bc}(cx_1(t-\tau)+x_2(t-\tau))$$

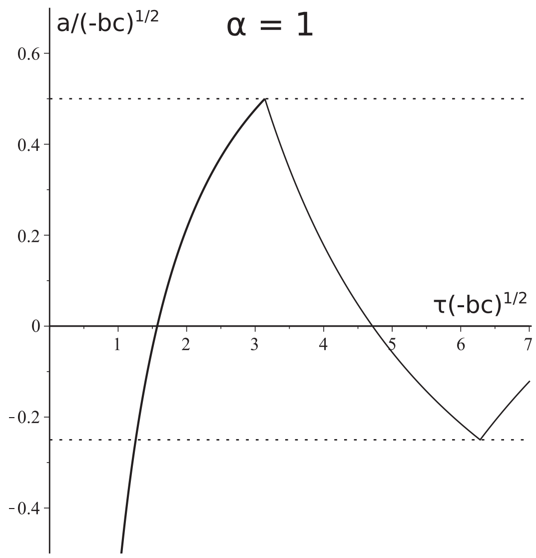

Fig. 5. A classical result for the stability region ($\alpha = 1$), see [24]; delay-independent instability above the upper dotted line, stability switching between dotted lines, one-time stability loss below the lower dotted line.

whose stability properties depend on location of the eigenvalues of the system matrix inside the unit circle of the complex plane. The use of the Schur–Cohn test (see, e.g., [33]) implies that this occurs if and only if $a^2 + bc < 1$ (hence, the stability boundaries depicted in Figs. 1–4 are simplified to the hyperbola when $\alpha = 0$). This agrees with the conclusion of Theorem 2, whose stability condition actually become $a^2 + bc < 1$ when α is approaching zero.

(iii) The conditions (5.3)–(5.5) of Theorem 2 reveal that (2.3) actually provides (probably the simplest) fractional differential system admitting the stability switches phenomenon. If $a \geq 0$, then repeated stability switching (with more than one change of stability properties) occurs when $\cos(\alpha\pi/2) \leq a/\sqrt{-bc} < \cos(\alpha\pi/2)/\cos(\theta_0^*)$, while for $a < 0$ when $\cos(\alpha\pi/2)/\cos(\theta_1^*) > |a|/\sqrt{-bc} \geq \cos(\alpha\pi/2)$. To determine an exact number of stability switches, nonlinear conditions (4.4) and (5.1) have to be solved. We note that the uniqueness of appropriate solutions to (4.4) and (5.1) is guaranteed by Proposition 3 and Lemma 3 (i) (with $p = \sqrt{-bc}$, $q = a$), respectively.

(iv) If $|a|/\sqrt{-bc} = \cos(\alpha\pi/2)$, then $\tau_{i,1}^* = \tau_{i,-1}^*$ for all $i \geq 0$. Hence, the zero solution to (2.3) is asymptotically stable for all $\tau > 0$ except for infinitely (but countably) many values

$$\tau = \tau_{i,1}^* = \tau_{i,-1}^* = \frac{(i + (1-\alpha)/2)\pi}{(-bc - a^2)^{1/(2\alpha)}}$$

with i being even for $a \geq 0$ and odd for $a < 0$. At these delay values, a non-asymptotic stability appears.

(v) In the asymptotically stable case, any solution (x_1, x_2) to (2.3) is tending to the zero solution with an algebraic decay rate

$$|x_j(t)| \sim t^{-\alpha} \qquad \text{as } t \to \infty, \quad j = 1, 2$$

(the symbol $\sim$ means an asymptotic equivalence). This asymptotic property is commented at the end of the following proof.

(vi) A rich stability behaviour of (2.3) with $bc < 0$ (described in Theorem 2) is graphically illustrated in Figs. 5–8.

Proof. Throughout the proof, we keep the notation $p = \sqrt{-bc}$, $q = a$, hence $p > 0$ and $q \in \mathbb{R}$. First, we prove that conditions (5.2)–(5.5) are necessary and sufficient for all roots of Q_1 and Q_2 (defined by (3.7)) to have negative real parts. By Proposition 1 (i) and a discussion following its proof, it is enough to consider $\tilde{Q}_1$ given by (4.1). By Lemma 2, it is sufficient to prove that the conditions (5.2)–(5.5) capture just couples $(\tau^\alpha p, \tau^\alpha q)$ located between Γ^+ and Γ^-.

As it was shown in the proof of Lemma 3, all the couples $(\tau^\alpha p, \tau^\alpha q)$ with $|q|/p < \cos(\alpha\pi/2)$ lie between Γ^+ and Γ^-, which proves (5.2).

Now, let $q \geq 0$. Then a position of $(\tau^\alpha p, \tau^\alpha q)$ has to be evaluated only with respect to Γ^+ (for $q/p \geq \cos(\alpha\pi/2)$). Consider a line ℓ starting from the origin with a tangent q/p. Lemma 3 implies that ℓ has either m or $m+1$ intersections with Γ^+ where m is determined by (4.10) (and therefore by (5.1)). These intersections $\tau_{i,\kappa}^*$ are characterized by (4.9) for $i = 0, 2, \ldots, m$ (i, m being even) and $\tau_{i,1}^* \leq \tau_{i,-1}^* < \tau_{i+2,1}^*$.

If $\cos(\alpha\pi/2) \leq q/p < \cot(\alpha\pi/2)$, then the line ℓ starts below Γ_0, i.e. below Γ^+. Indeed, as indicated in the proof of Lemma 3, Γ^+ starts from the origin with a tangent equal to $\cot(\alpha\pi/2)$ (i.e. it is lying in the first quadrant of $(\tau^\alpha p, \tau^\alpha q)$-plane). Each of intersections $\tau_{i,\kappa}^*$ ($i = 0, 2, \ldots, m$) then describes a point where ℓ crosses (or in the case $q/p = \cos(\alpha\pi/2)$ only touches) Γ^+ which implies (5.3). If $q/p \geq \cot(\alpha\pi/2)$, the line ℓ starts above Γ_0, i.e. above Γ^+. Then Remark 1 implies that ℓ intersects Γ^+ only when $\alpha > \alpha^*$, and a similar argumentation as used above leads to (5.4). Finally, if $q/p \geq \cot(\alpha\pi/2)$ and $\alpha \leq \alpha^*$, or $q/p \geq \cos(\alpha\pi/2)/\cos(\theta_0^*)$ and $\alpha > \alpha^*$, then $(\tau^\alpha p, \tau^\alpha q)$ is located above Γ^+.

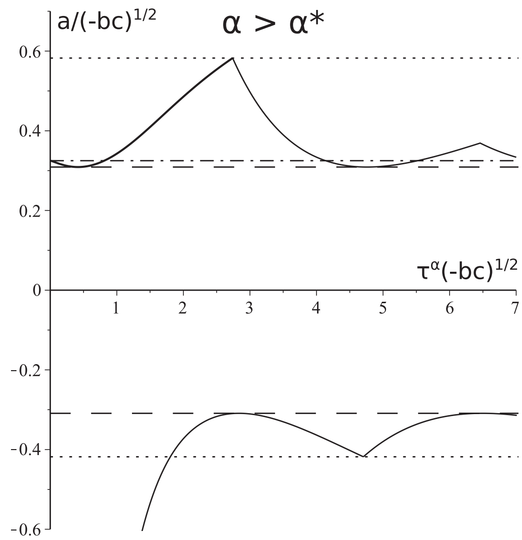

Fig. 6. The stability region for $\alpha = 0.8 > \alpha^*$; delay-independent instability above the upper dotted line, stability switching starting from instability between upper dotted and dash-dotted lines, stability switching starting from stability between dash-dotted and upper dashed lines and between lower dashed and dotted lines, delay-independent stability between dashed lines, one-time stability loss below the lower dotted line.

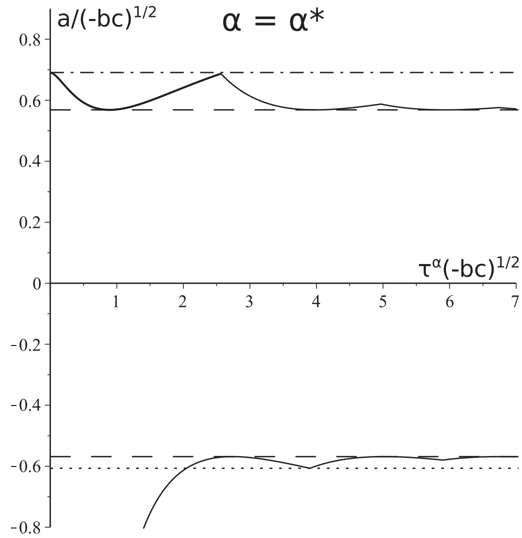

Fig. 7. The stability region for $\alpha = \alpha^*$; delay-independent instability above the dash-dotted line, stability switching starting from stability between dash-dotted and upper dashed lines and between lower dashed and dotted lines, delay-independent stability between dashed lines, one-time stability loss below the lower dotted line.

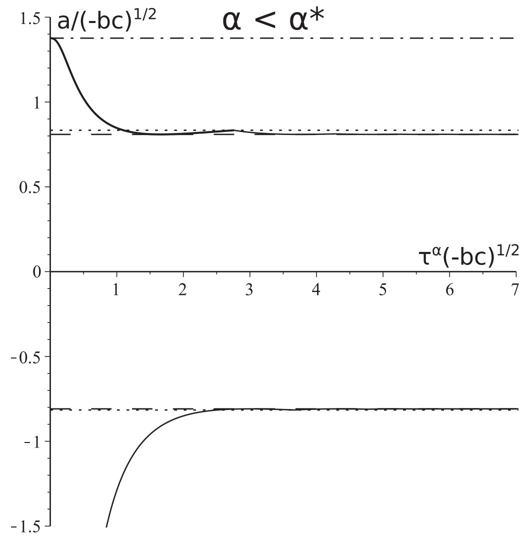

Fig. 8. The stability region for $\alpha = 0.4 < \alpha^*$; delay-independent instability above the dash-dotted line, one-time stability loss between the dash-dotted and upper dotted lines and below the lower dotted line, stability switching starting from stability between the upper dotted and dashed lines and between lower dashed and dotted lines, delay-independent stability between dashed lines.

Let $q<0$. Then a position of $(\tau^\alpha p,\ \tau^\alpha q)$ has to be evaluated only with respect to Γ^- (for $q/p \le -\cos(\alpha\pi/2)$). We again introduce ℓ as a line starting from the origin with a tangent q/p (i.e. lying in the fourth quadrant of $(\tau^\alpha p,\ \tau^\alpha q)$-plane). (5.5) can be now proved by the same line of arguments as used for (5.3).

Thus we have shown that the conditions (5.2)–(5.5) are valid if and only if Q_1 and Q_2 have all roots with negative real parts.

Analogously to the proof of Theorem 1, stability properties of (2.3) with $bc<0$ are determined by appropriate asymptotic properties of the functions $\mathcal{R}_{\alpha,\alpha}^{\pm i\sqrt{-bc},a,\tau}$, $\mathcal{R}_{\alpha,1}^{\pm i\sqrt{-bc},a,\tau}$. Based on characteristic root properties stated in Proposition 1 (ii)-(iv), we can utilize the proof technique from [31] (supplied with some straightforward modifications) to show that $\mathcal{R}_{\alpha,\alpha}^{\pm i\sqrt{-bc},a,\tau}(t)$ and $\mathcal{R}_{\alpha,1}^{\pm i\sqrt{-bc},a,\tau}(t)$ tend to zero as $t\to\infty$ if and only if (3.7) has all roots with negative real parts. This proves the assertion of Theorem 2. A note on algebraic decay of the solutions (see Remark 2 (iv)) follows from related considerations originating from the technique used in [31]. It implies that, in the asymptotically stable case, $\mathcal{R}_{\alpha,1}^{\pm i\sqrt{-bc},a,\tau}(t) \sim t^{-\alpha}$ and $\mathcal{R}_{\alpha,\alpha}^{\pm i\sqrt{-bc},a,\tau}(t) = \mathcal{O}(t^{-\alpha-1})$ as $t\to\infty$. □

Remark 3. We note that the stability conditions of Theorems 1 and 2 can be reformulated for (2.1), where A is a planar matrix having either two real eigenvalues of the form $\pm v$ $(v>0)$, or a couple of (complex conjugate) purely imaginary eigenvalues $\pm iv$ $(v>0)$. In such a case, the assertions of Theorems 1 and 2 are expressed in terms of eigenvalues of matrices A, B and parameters α, τ. In particular, this enables us to reformulate Theorems 1 and 2 in the case when the matrices A, B are given in the corresponding Jordan canonical forms. Moreover, it seems that our results could be generalized, with some technical adjustments, also for the matrix A having two distinct real eigenvalues. Under the above stated restrictions on eigenvalues of the system matrices, a generalization into higher dimensions is possible as well.

6. Some applications

This section illustrates a simple use and applicability of Theorems 1 and 2. We start with

Example 1. We consider a fractional delay predator-prey system

$$\begin{aligned} D^\alpha x_1(t) &= x_1(t)[1+x_2(t)-x_1(t-\tau)-x_2(t)/u] \\ D^\alpha x_2(t) &= x_2(t)[-1+x_1(t)-x_2(t-\tau)+x_1(t)/u] \end{aligned} \tag{6.1}$$

where $0<\alpha,\ u<1$ and $\tau>0$ are real numbers, whose linearization (for linearization theorem on fractional differential equations, we refer to [34]) along the unique positive equilibrium $(u,\ u)$ leads to

$$\begin{aligned} D^\alpha x_1(t) &= -ux_1(t-\tau)+(u-1)x_2(t) \\ D^\alpha x_2(t) &= (u+1)x_1(t)-ux_2(t-\tau) \end{aligned} \tag{6.2}$$

(we note that this population model was studied in the first-order case $\alpha=1$ in [24,35]). To discuss its stability properties with respect to the entries α, u and τ, we apply Theorem 2 with $a=-u$, $b=u-1$ and $c=u+1$. Thus, we get that the zero solution to (6.2) is asymptotically stable if and only if

$$\frac{u}{\sqrt{1-u^2}} < \cos(\alpha\pi/2)\,, \tag{6.3}$$

$$\text{or}\quad \frac{u}{\sqrt{1-u^2}} \ge \frac{\cos(\alpha\pi/2)}{\cos(\theta_1^*)} \quad\text{and}\quad \tau\in(0,\tau_{1,1}^*)\,, \tag{6.4}$$

$$\text{or}\quad \frac{\cos(\alpha\pi/2)}{\cos(\theta_m^*)} \le \frac{u}{\sqrt{1-u^2}} < \frac{\cos(\alpha\pi/2)}{\cos(\theta_{m-2}^*)} \quad\text{and}\quad \tau\in\bigcup_{j=-1}^{(m-1)/2-1}\left(\tau_{2j+1,-1}^*,\ \tau_{2j+3,1}^*\right) \tag{6.5}$$

where $m\ge 3$ is an odd positive integer, $\tau_{-1,-1}^*=0$ and $\tau_{2j+1,\kappa}^*=\tau_{2j+1,\kappa}^*(\sqrt{1-u^2},-u)$ are given by (4.9) for $j\ge 0$. The stability boundary is depicted in Fig. 9 for $\alpha=0.8$ (appropriate stability region lies below this boundary) along with some significant values of u. Some graphical comparisons of stability boundaries for various values of α are illustrated in Fig. 10. We can observe here a delay-independent asymptotic stability case occurring when $u<u_\infty$ which corresponds to (6.3) (note that this case cannot occur provided $\alpha=1$). For $u\ge u_1$ (i.e. when (6.4) occurs), we have a one-time stability loss with increasing τ, while for $u_\infty\le u<u_1$ (when (6.5) occurs), a repeated stability switches appear. We point out that if $u=u_\infty$, then the zero solution to (6.2) is asymptotically stable for all τ up to infinitely (countably) many values $\tau=\tau_{2j+1,1}^*$ $(j=0,1,\dots)$ when it is stable, but not asymptotically (note again that such a phenomenon cannot be obtained for the classical first-order model). As it can be confirmed by Fig. 10, the delay-independent asymptotic stability is not actually present for $\alpha=1$, while the conditions (6.4) and (6.5) agree with expectations based on [24, Corollary 3.2].

Finally, we choose the value $u=0.5$ (i.e. $u(1-u^2)^{-1/2}=3^{-1/2}$), which was studied for $\alpha=1$ in [24], and discuss stability properties of (6.2) with respect to changing τ and α. We can find three qualitatively different situations depending on α with respect to two special values calculated numerically as $\alpha_1\approx 0.6432790899$ (in fact, $\cos(\alpha_1\pi/2)=3^{-1/2}\cos(\theta_1^*)$, see (6.4) with respect to (4.4)) and $\alpha_\infty\approx 0.6081734480$ (in fact, $\cos(\alpha_\infty\pi/2)=3^{-1/2}$, see (6.3)):

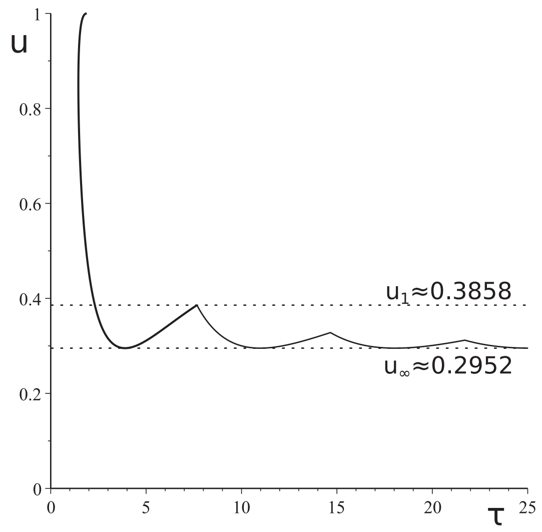

Fig. 9. The stability boundary for $\alpha = 0.8$.

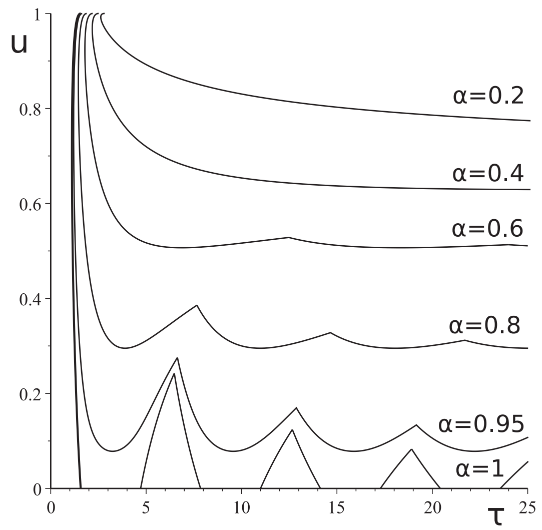

Fig. 10. The stability boundaries for various α.

- If $\alpha_1 < \alpha < 1$, then the zero solution to (6.2) is asymptotically stable if and only if $0 < \tau < \tau^*_{1,1}$ (it agrees with [24, Remark 3.2] for $\alpha = 1$);
- If $\alpha_\infty \le \alpha \le \alpha_1$, then the zero solution to (6.2) is asymptotically stable if and only if $\tau \in \bigcup_{j=-1}^{(m-1)/2-1} (\tau^*_{2j+1,-1}, \tau^*_{2j+3,1})$ where a positive odd integer $m \ge 3$ depends on α and is uniquely determined by (6.5);
- If $0 < \alpha < \alpha_\infty$, then the zero solution to (6.2) is asymptotically stable for all values of $\tau > 0$.

To illustrate these results, we calculate the corresponding stability switches for particular values of α. The generic procedure is as follows: For each specified α, we check if (6.3) or (6.4) holds (the latter one requires the knowledge of the root $\theta^*_1 \in (0, \alpha\pi/2)$ to (4.4)). If none of them is valid, we sequentially calculate the roots $\theta^*_m \in (0, \alpha\pi/2)$ to (4.4) for $m = 3, 5, \ldots$ until we find m satisfying (6.5). Then we evaluate the corresponding $\tau^*_{i,\kappa}$ given by (4.9) for $p = (0.75)^{1/2}$, $q = -0.5$, $\kappa = \pm 1$ and $i = 1, 3, \ldots, m$. Thus we get:

- If $\alpha = 0.8$, then the zero solution to (6.2) is asymptotically stable if and only if

$$\tau \in (0, 1.781364151); \tag{6.6}$$

- If $\alpha = 0.64$, then $m = 3$ and the zero solution to (6.2) is asymptotically stable if and only if

$$\tau \in (0, 3.589360507) \cup (10.06793272, 10.77857157);$$

- If $\alpha = 0.615$, then $m = 5$ and the zero solution to (6.2) is asymptotically stable if and only if

$$\tau \in (0, 4.850664442) \cup (7.976214463, 13.89430971) \cup (21.45019407, 22.93795499).$$

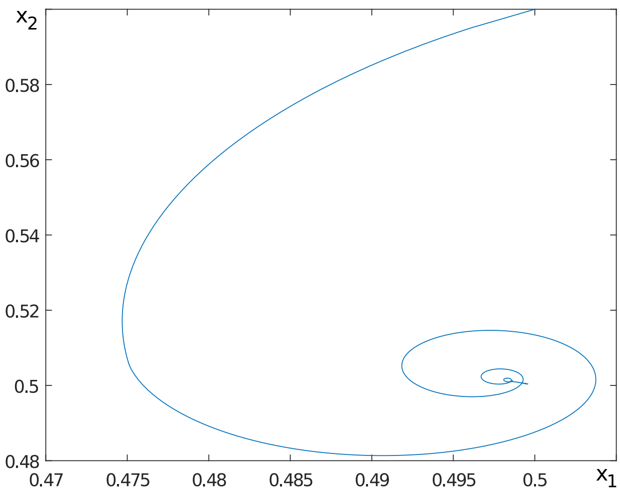

Fig. 11. The phase trajectory of the solution to (6.1) with $\alpha = 0.8$, $u = 0.5$, $\tau = 1.0$ and initial conditions $x_1(t) \equiv 0.5$ and $x_2(t) \equiv 0.6$ on $(-1, 0]$; the stepsize $h = 0.03$.

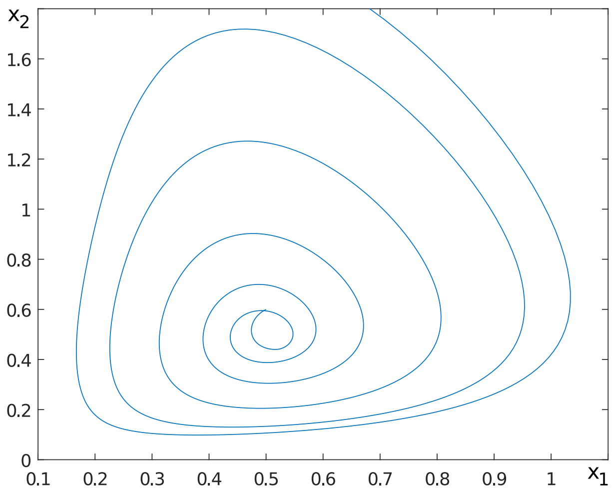

Fig. 12. The phase trajectory of the solution to (6.1) with $\alpha = 0.8$, $u = 0.5$, $\tau = 3.0$ and initial conditions $x_1(t) \equiv 0.5$ and $x_2(t) \equiv 0.6$ on $(-3, 0]$; the stepsize $h = 0.03$.

A further decrease of α below the critical value $\alpha_\infty \approx 0.6081734480$ already results into a delay-independent asymptotic stability (i.e., the zero solution to (6.2) is asymptotically stable for all $\tau > 0$).

Remark 4. We can supply the previous results also with respect to the original nonlinear problem (6.1). Figs. 11–14 depict phase portraits of the solutions to (6.1) with $\alpha = 0.8$, $u = 0.5$ and constant initial conditions $x_1(t) \equiv c_1$, $x_2(t) \equiv c_2$ $(c_1, c_2 \in \mathbb{R})$ prescribed on $(-\tau, 0]$ (for the used numerical algorithm based on the Adams method extended for fractional equations with a delay see [36]).

In particular, Figs. 11 and 12 show how the choice of τ inside and outside the interval (6.6) results into stability and instability of the positive equilibrium (0.5, 0.5), respectively. Further, Figs. 11, 13 and 14 illustrate the effect of initial conditions gradually moving away from the equilibrium (0.5, 0.5) with respect to the asymptotic convergence of trajectories.

Now we show an application potential of Theorems 1 and 2 in stabilization problems utilizing a delayed feedback control.

Example 2. We consider the fractional differential system

$$\begin{aligned} D^\alpha x_1(t) &= x_2(t) \\ D^\alpha x_2(t) &= \omega x_1(t) \end{aligned} \tag{6.7}$$

$0 < \alpha < 1$ and $\omega \neq 0$ being real numbers, whose characteristic equation $\lambda^{2\alpha} - \omega = 0$ is identical with that for a basic differential model

$$D^{2\alpha} x(t) - \omega x(t) = 0$$

of the linear fractional oscillator. In the standard case $\omega < 0$, the spectrum of the matrix system (6.7) is formed by a couple of purely imaginary numbers $\mu_{1,2} = \pm i\sqrt{-\omega}$ lying inside the Matignon stability sector $|\arg(\mu)| > \pi/2$ for any $0 < \alpha < 1$. In the

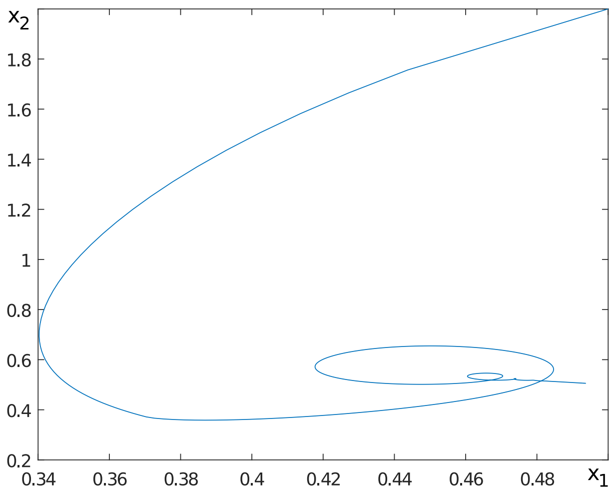

Fig. 13. The phase trajectory of the solution to (6.1) with $\alpha = 0.8$, $u = 0.5$, $\tau = 1.0$ and initial conditions $x_1(t) \equiv 0.5$ and $x_2(t) \equiv 2.0$ on $(-1, 0]$; the stepsize $h = 0.03$.

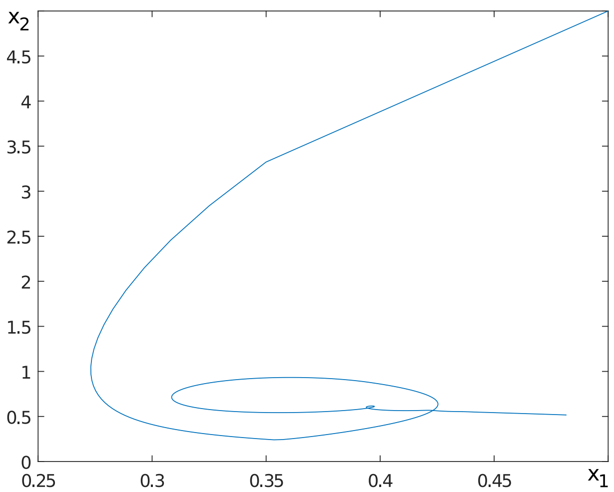

Fig. 14. The phase trajectory of the solution to (6.1) with $\alpha = 0.8$, $u = 0.5$, $\tau = 1.0$ and initial conditions $x_1(t) \equiv 0.5$ and $x_2(t) \equiv 5.0$ on $(-1, 0]$; the stepsize $h = 0.03$.

inverted case $\omega > 0$, a couple of real eigenvalues $\mu_{1,2} = \pm\sqrt{\omega}$ appears, hence the zero solution to (6.7) is not asymptotically stable for any $0 < \alpha < 1$. Therefore, we investigate the stabilization of this zero equilibrium via appropriate diagonal delay feedback control, i.e. we consider the system

$$\begin{aligned} D^{\alpha} x_1(t) &= x_2(t) + u_1(t-\tau) \\ D^{\alpha} x_2(t) &= \omega x_1(t) + u_2(t-\tau) \end{aligned}$$

under the control law $u_i(t) = kx_i(t)$, $i = 1, 2$, where k is a real gain parameter and τ is a (positive) real time delay. Thus we obtain the closed-loop system

$$\begin{aligned} D^{\alpha} x_1(t) &= kx_1(t-\tau) + x_2(t) \\ D^{\alpha} x_2(t) &= \omega x_1(t) + kx_2(t-\tau) \end{aligned} \tag{6.8}$$

and wish to find conditions on stabilizing control parameters k, τ. Doing this, we apply Theorem 1 with $a = k$, $b = 1$ and $c = \omega > 0$ and get the conditions $k + \sqrt{\omega} < 0$ and $\tau < \tau^*(\mathrm{sgn}(k)\sqrt{\omega}, k)$ for asymptotic stability of the zero solution to (6.8). In other words, the zero solution to the controlled system (6.8) is stabilized if and only if

$$k < -\sqrt{\omega} \qquad \text{and} \qquad \tau < \frac{(2-\alpha)\pi/2 + \arcsin(\sqrt{\omega}/k \sin(\alpha\pi/2))}{\left(\sqrt{k^2 - \omega\sin^2(\alpha\pi/2)} + \sqrt{\omega}\cos(\alpha\pi/2)\right)^{1/\alpha}}.$$

Thus, the stabilization is possible for all $\omega > 0$ provided there are no additional restrictions on τ or k.

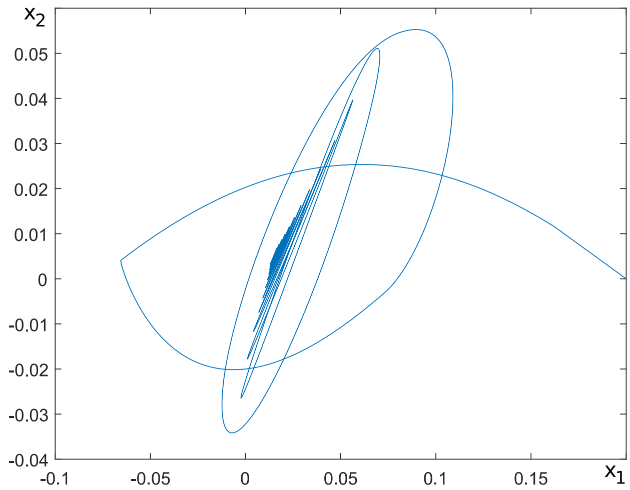

Fig. 15. The phase trajectory of the solution to (6.8) with $\alpha = 0.5$, $k = -3$, $\omega = 1$ and $\tau = 0.16$; the stepsize $h = 0.0016$.

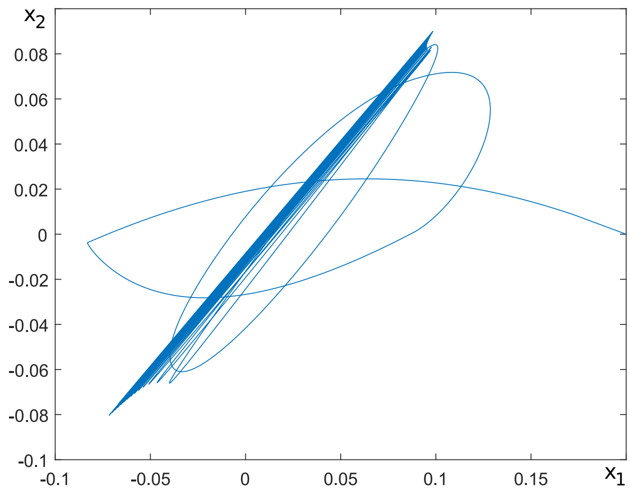

Fig. 16. The phase trajectory of the solution to (6.8) with $\alpha = 0.5$, $k = -3$, $\omega = 1$ and $\tau = 0.18$; the stepsize $h = 0.0018$.

To illustrate our results, we present phase portraits of the solutions to (6.8) with $\alpha = 0.5$, $k = -3$, $\omega = 1$ (for the numerical algorithm see again [36]). For this combination of parameters, the zero solution is stabilized when $\tau < 0.161$. Thus, Fig. 15 represents the stabilized case, while Fig. 16 shows the situation where the zero solution to (6.8) is not stabilized due to an unsuitable choice of the control τ.

More generally, we can study this stabilization problem in a more complex form. Instead of (6.7), we consider a general fractional differential system with an unstable equilibrium that we wish to stabilize via a diagonal delayed feedback control. Thus we obtain the closed-loop system (2.1) where A is the Jacobi matrix of the uncontrolled system (evaluated at a given equilibrium) and $B = kI$ is the gain matrix (with a gain parameter k, I being the identity matrix). Stability analysis of this controlled system leads to root analysis of quasi-polynomials of the type (3.7) where instead of the purely imaginary coefficient we consider appropriate imaginary coefficient (with arbitrary real part). Such an analysis is a qualitatively more difficult problem requiring some additional tools and can be viewed as an inspiration for the next research.

7. Concluding remarks

In this paper, the problem of delay-dependent stability switches for a linear planar autonomous fractional differential system has been discussed. We have formulated stability criteria describing a nature of this phenomenon, including conditions for its appearance as well as a number and exact calculations of corresponding stability switches. Contrary to the conventional first-order case, a much more rich stability behaviour of a studied system has been observed, commented and depicted. In particular, a delay-independent stability area has appeared for any derivative order between 0 and 1 (in the first-order model, this area is empty). A border between delay-independent and delay-dependent stability areas has been described explicitly in terms of entry parameters. In this specific case, the asymptotic stability property has been guaranteed for almost all positive real time lags up to a uniform grid of isolated lags when non-asymptotic stability has occurred.

Also, we have explored dependence of stability areas on changing derivative order, especially with respect to the limit values 1 and 0. Our analysis of these limit cases has shown that the derived stability criteria turn into the existing conditions known for the appropriate first-order differential system and non-differential system, respectively. Besides these stability investigations, an asymptotic result on algebraic decay rate of the solutions in the asymptotically stable case has been presented.

As illustrated in the previous section, these results can be applied either in local stability analysis of nonlinear fractional delay dynamical models, where the effect of stability switches appears, or in stabilization and synchronization problems of control theory for fractional systems. In particular, a formulation of stabilizing conditions explicit with respect to a time delay control provides a significant motivation for further research in this direction. The results presented in this paper enable to formulate this type of stabilizing conditions for diagonal delayed feedback controls of unstable equilibria of some fractional oscillators and other related models (where the eigenvalues of the appropriate Jacobi matrix are real or purely imaginary numbers). The above indicated possible extension of such results to general fractional dynamical systems (with arbitrary complex eigenvalues of the Jacobi matrix) could provide a very efficient tool for delayed feedback controls of chaos (appearing, e.g., in the fractional Lorenz model of convective fluid) and other stabilization and synchronization issues leading to the fractional delay system (2.1). We note that, under specific choices of the system matrices, (2.1) appears also in PD^{α} control of Newcastle robot (see [37], for outlines concerning more advanced models see also [38,39]). Our stability procedures seem to be extendable also to this problem and thus may contribute to a more complex answer to stability questions connected with this model.

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Appendix G

Paper on higher-order two-term FDDE [12] (CNSNS, 2023)

Previously in [13], we studied two-term FDDE of orders less than one. It is known that the stability regions for first and second-order equations are qualitatively very different, in this case a connected unbounded set with a boundary formed by a line and a transcendental curve, in contrast to infinitely many touching triangles. Thus, in [12] (co-author: J. Čermák; my author's share 50 %), we aimed to describe the transition between first-order and second-order two-term equations and perform a comprehensive analysis of the expected stability switches.

Building on our previous experience and the proving techniques developed in earlier works, we described the region of delay-independent stability, the stability boundary, and provided a detailed analysis of stability switches. All of that in the explicit form. We included all necessary details for precise calculations of their number and values, which were then applied on the stabilization and destabilization of fractional oscillators. The equation studied in this paper represents the simplest case of real-valued problem where such a rich stability properties occur.

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Research paper

Stabilization and destabilization of fractional oscillators via a delayed feedback control

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ABSTRACT

This paper discusses the problem of stabilization and destabilization of fractional oscillators by use of a delayed feedback control. A mathematical part of the problem consists in stability analysis of appropriate fractional delay differential equations with the derivative order varying between 1 and 2. Derived stability criteria are efficient and easy to apply when stabilizing or destabilizing fractional oscillators in the standard as well as inverted form. As a by-product of our results, we explicitly describe critical values of a delay control parameter when stability property turns into instability and vice versa. Evaluations of these stability switches are possible also in the limit harmonic case which brings new insights into classical stability results on this topic.

1. Introduction

The model of a fractional oscillator originates from the equation of motion for the classical harmonic oscillator where the second-order derivative (appearing in the acceleration term) is replaced by a fractional-order derivative between 1 and 2. Existing analysis of this fractional model revealed some similarities compared to a damped harmonic oscillator. However, this damping property does not follow from friction (or other external sources) as in the classical harmonic case, but from the internal structure of the fractional oscillator itself. For more details, including the cause of this intrinsic damping, physical interpretations of fractional oscillators, and their response to some external forces, we refer to [1–4].

The decay of amplitudes of fractional oscillators is supported by another specific property, namely a weak oscillation. While in the classical subcritically damped case, an oscillation around the equilibrium state occurs, the fractional oscillator approaches the equilibrium from one side only (after a few overshoots) with oscillating velocity. Thus, from a certain moment, its behavior begins to resemble rather critical or supercritical damping whose decay rate is not exponential, but algebraic [5].

The classical damped harmonic oscillator has also its fractional extension. When studying the problem of the motion of a rigid plate in a Newtonian fluid, the usefulness of a fractional damping term compared to classical damping was justified theoretically as well as empirically in [6], and later analyzed in [7,8]. From the mathematical point of view, the first derivative term (representing a damping proportional to velocity) was replaced by terms with rational derivative orders (mostly 1/2 or 3/2). As it was shown in these and several following papers [9,10], this fractional model keeps

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the same stability properties compared to the classical one, but has a different asymptotics. In particular, the fractional damping term implies an algebraic decay of amplitudes to the equilibrium position.

Coming back to the fractional oscillator model, its fractional-order derivative guarantees asymptotic stability of the system (as recalled above, this is not true for the oscillator with fractional damping). On the other hand, considering the inverted oscillator (when the internal force linearly depending on the deflection is acting towards this deflection), the asymptotic stability property does not hold for any derivative order between 1 and 2.

These observations can be extended via adding a suitable feedback control. The most simple feedback control is that proportional to a current state of the system. However, it is easy to check that such a control cannot change stability properties of neither harmonic nor fractional oscillator (including their inverted forms). Moreover, in the practical implementation of feedback controls, it is very likely that time delays will occur. Therefore, it is important to understand the sensitivity of the control system with respect to a time delay in the feedback loop. Depending on the values of a delay, (asymptotic) stabilization or destabilization of many integer-order dynamical systems were reported. While general reasons for a feedback stabilization consist in bringing the unstable (or even chaotic) system into a stable position, the importance of a feedback destabilization appears in situations when we need to destabilize the stable, but in a certain sense undesirable (e.g. pathological) state [11–13].

A mathematical platform for such feedback (de)stabilization of fractional models is provided by fractional delay differential equations (FDDEs); for basics of the corresponding theory we refer, e.g. to [14–16]. Their stability analysis currently belongs among rapidly developing research topics due to its practical as well as theoretical importance. However, effective stability criteria for FDDEs are still rare (for some basic results we refer to [17–22]). From this viewpoint, fractional and harmonic oscillators (whose position is at the border between stability and instability) represent very suitable test models for stability analysis of their extended delayed feedback loops.

Following such outlines, we wish to discuss these problems: As a preliminary motivation, we consider the (undamped) harmonic oscillator controlled via a time-delayed feedback loop that is linearly depending on the state of the system (but not on its velocity). We wish to describe the structure of all delay and gain control parameters that enable either damping of the oscillator (more precisely, its asymptotic stabilization around the equilibrium), or obversely its destabilization. As the main problem, we consider the fractional oscillator (including its inverted form) with the same type of a control and put similar questions on critical values of the control parameters, including a derivative order.

The paper is organized as follows. Section 2 introduces a short survey of some existing results, related notions and methods. These methods involve, among others, techniques originating from root analysis of the appropriate characteristic equation. A detailed description of locations of characteristic roots including their basic properties is provided in Section 3. Section 4 presents stability criteria for studied linear FDDEs with the derivative order between 1 and 2. These results indicate that when considering the derivative order as a varying bifurcation parameter, the first-order derivative can be taken for the critical bifurcation value. More precisely, if the derivative order is changing between 0 and 1, the stability areas display similar topological properties. However, exceeding the value 1, quite new features accompanying stability investigations appear. Section 5 contains applications of these results to the problems of stabilization and destabilization of fractional oscillators (including their inverted forms) via delayed feedback controls. As a by-product of these investigations, we present appropriate results for the classical harmonic oscillator as well. The final section concludes the paper with a survey of the presented results and possible perspectives.

2. A brief mathematical background

The dynamics of a fractional oscillator is given by the fractional differential equation

$$D^{\alpha}y(t)+\omega^{\alpha}y(t)=0, \qquad t>0 \tag{2.1}$$

where $\alpha\in(1,2)$ and $\omega>0$ is a parameter corresponding to frequency [3]. The symbol $D^{\alpha}f(t)$ used here stands for the Caputo fractional derivative which is, for any positive real α and a given function f, introduced as a composition of the standard $\lceil\alpha\rceil$th order derivative ($\lceil\cdot\rceil$ means a ceiling function) and the fractional $(\lceil\alpha\rceil-\alpha)$th order integral

$$I^{\lceil\alpha\rceil-\alpha}f(t)=\int_0^t\frac{(t-\xi)^{\lceil\alpha\rceil-\alpha-1}}{\Gamma(\lceil\alpha\rceil-\alpha)}f(\xi)\mathrm{d}\xi\,, \qquad t>0,$$

i.e.

$$D^{\alpha}f(t)=I^{\lceil\alpha\rceil-\alpha}\frac{\mathrm{d}^{\lceil\alpha\rceil}f(t)}{\mathrm{d}t^{\lceil\alpha\rceil}}\,, \qquad t>0.$$

If we put $I^0f(t)\equiv f(t)$ and α is a positive integer, then the Caputo derivative coincides with the standard (integer-order) derivative (for more details on fractional operators we refer, e.g. to [23,24]). Thus, if particularly $\alpha=2$, then (2.1) becomes the model for the classical harmonic oscillator

$$y''(t)+\omega^2y(t)=0, \qquad t>0. \tag{2.2}$$

It is well-known [2] that while (2.2) is non-asymptotically stable, its fractional analogue (2.1) is asymptotically stable for any $\alpha\in(1,2)$. In the inverted case, both the classical model

$$y''(t)-\omega^2y(t)=0, \qquad t>0$$

as well as its fractional analogue

$$D^{\alpha}y(t) - \omega^{\alpha}y(t) = 0, \qquad t > 0$$

are unstable.

To change stability or instability properties of these models, we introduce the control term u into their right-hand sides. If we employ the basic feedback control $u(t) = Ky(t)$, where K is a real gain parameter, then it is known (and easy to verify) that its impact on stability properties of these models is limited. In particular, we are not able to achieve asymptotical stabilization of (2.2) for any real K. We show that this situation changes if we consider the delay feedback control of the form $u(t) = Ky(t-\tau)$ where, in addition to a gain parameter K, we employ also the real time lag $\tau > 0$ as the second control parameter.

From the mathematical point of view, we investigate stability properties of the FDDE

$$D^{\alpha}y(t) = ay(t) + by(t-\tau), \qquad t > 0 \tag{2.3}$$

where $\alpha \in (1, 2)$, $\tau > 0$ and $a,\ b$ are real parameters. Stability properties of (2.3) are known in both the limit integer-order cases. It might be useful to recapitulate them [25,26].

Theorem 1. *(i) Let $\alpha = 1$. Then (2.3) is asymptotically stable if and only if either*

$$a \le b < -a \qquad \text{and} \qquad \tau \text{ is arbitrary},$$

or

$$|a| + b < 0 \qquad \text{and} \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}.$$

(ii) Let $\alpha = 2$ and $b > 0$. Then (2.3) is asymptotically stable if and only if $a < 0$ and there exists a non-negative even integer ℓ such that

$$2\ell\pi < \tau\sqrt{-a} < (2\ell + 1)\pi \tag{2.4}$$

and

$$\tau^2 b < \min(-(2\ell)^2\pi^2 - \tau^2 a, (2\ell + 1)^2\pi^2 + \tau^2 a). \tag{2.5}$$

Remark 1 (*a*)**.** Both the parts (i) and (ii) were derived by use of root analysis of the corresponding characteristic quasi-polynomials

$$P_1(z) \equiv z - a - b\exp(-z\tau) \qquad \text{and} \qquad P_2(z) \equiv z^2 - a - b\exp(-z\tau), \tag{2.6}$$

respectively. More precisely, the well-known stability condition says that (2.3) with $\alpha = 1$ or $\alpha = 2$ is asymptotically stable if and only if all roots of $(2.6)_1$ or $(2.6)_2$, respectively, have negative real parts (see, e.g. [27]). A crucial problem, namely how to convert this theoretical condition into an efficient form, was solved by use of the D-partition method [26] (the case $\alpha = 1$) and the Pontryagin criterion for location of quasi-polynomial roots in the open left complex half-plane [25] (the case $\alpha = 2$).

(b) The case $b < 0$ was not explicitly discussed in [25]. However, using similar arguments as those used for $b > 0$, the extension of the part (ii) to $b < 0$ can be provided as well. We add that these results, even in a more efficient form, can be also obtained as particular (limit) cases of our next considerations.

The above stated stability conditions for both the integer-order cases are of a quite different nature as illustrated by Figs. 1 and 2 depicting appropriate stability regions in the (a, b)–plane.

Following both the integer–order cases, conditions for asymptotic stability of (2.3) originate from the requirement on location of all characteristic roots left to the imaginary axis. More precisely, when considering (2.3), the generalized characteristic quasi-polynomial takes the form

$$P_{\alpha}(z) = z^{\alpha} - a - b\exp(-z\tau)$$

and the corresponding theoretical stability condition can be stated as follows:

Lemma 1. *Let $\alpha > 0$, $\tau > 0$ and a, b be real numbers.*

(i) If all the roots of $P_{\alpha}(z)$ have negative real parts, then (2.3) is asymptotically stable.

(ii) If there exists a root of $P_{\alpha}(z)$ with a positive real part, then (2.3) is not stable.

The proof of this assertion originates from the Laplace transform method and can be found in several earlier papers for $\alpha \in (0, 1)$ (see, e.g. [21, Lemma 1]). Its extension to arbitrary positive real values α uses an analogous argumentation.

Recently, results on effective stability conditions for (2.3) with $\alpha \in (0, 1)$ have appeared in [17,18,21]. The stability area in the (a, b)-plane was described either via boundary parametric curves, or via implicit formulae involving the (unique)

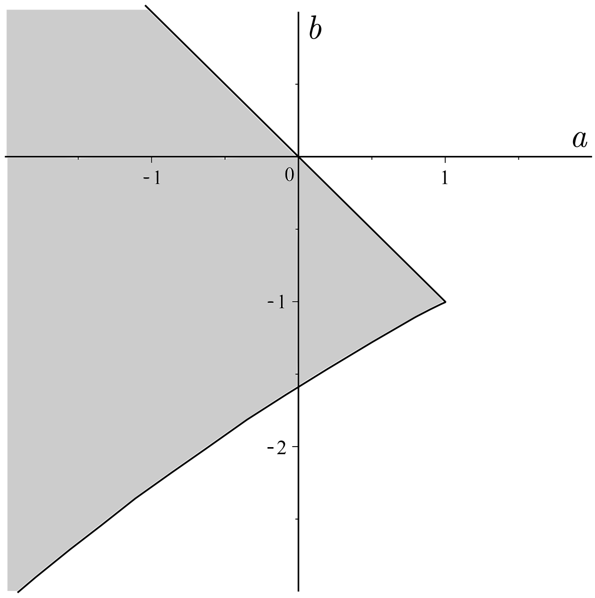

Fig. 1. Stability region of (2.3) for $\alpha = 1$ and $\tau = 1$.

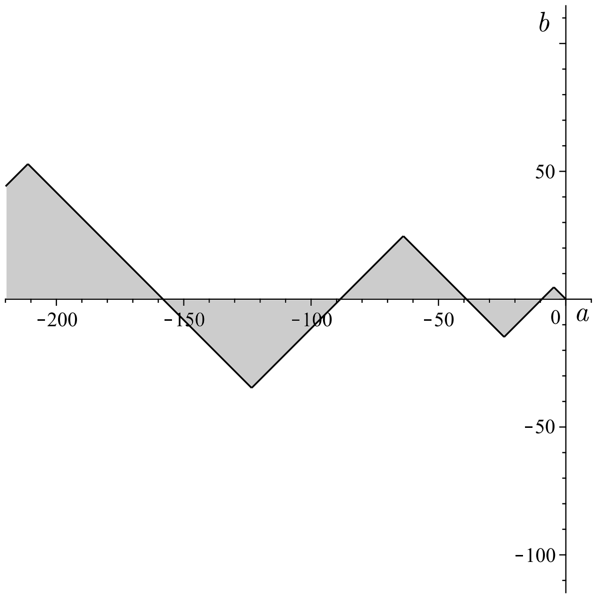

Fig. 2. Stability region of (2.3) for $\alpha = 2$ and $\tau = 1$.

switch between stability and instability with respect to increasing τ (this type of description is used in Theorem 1(i) for $\alpha = 1$). In both these descriptions, stability properties of (2.3) with $\alpha \in (0, 1)$ essentially share the same structure as in the case $\alpha = 1$. As Figs. 1 and 2 indicate, if $\alpha \in (1, 2)$, then the structure of stability properties of (2.3) become qualitatively different. From this point of view, $\alpha = 1$ can be viewed as an important bifurcation value with respect to a changing derivative order. This fact brings another reason why to study (2.3): we wish to clarify a mathematical background of this phenomenon and provide an interpolation between both the systems of stability conditions for (2.3) (recalled in Theorem 1) with α continuously varying between 1 and 2.

3. Distribution of characteristic roots

In this section, we analyze location of characteristic roots of $P_\alpha(z)$ with respect to the imaginary axis (and discuss also some related root properties). Thus we prepare a necessary mathematical apparatus for conversion of Lemma 1 into an effective from.

In addition to some elementary and well-known root properties of $P_\alpha(z)$ (such as complex conjugacy), the series of the following properties holds.

Proposition 1. *Let $\alpha \in (1,2)$, $\tau > 0$ and a, b be real numbers. Then $z = r\exp(\mathrm{i}\varphi)$ ($r \geq 0$, $\varphi \in (-\pi, \pi]$) is a root of $P_\alpha(z)$ with multiplicity greater than one if and only if either $z = 0$ or there exists an integer k such that $\alpha\varphi - \varphi + \tau r\sin(\varphi) = k\pi$ and $\tau r\sin(\alpha\varphi) + \alpha\sin(\alpha\varphi - \varphi) = 0$. Moreover, any root of $P_\alpha(z)$ has multiplicity at most three.*

Proof. Let z be a root of $P_\alpha(z)$ with multiplicity two. Then it also must hold

$$\alpha z^{\alpha-1} + b\tau\exp(-z\tau) = 0, \tag{3.1}$$

and we get $a = z^\alpha + \frac{\alpha}{\tau}z^{\alpha-1}$, $b = -\frac{\alpha}{\tau}z^{\alpha-1}\exp(z\tau)$. Since a, b are real numbers, the first part of the assertion follows by elaborating these two expressions for zero imaginary parts with respect to $z = r\exp(\mathrm{i}\varphi)$ ($r \geq 0$, $\varphi \in (-\pi, \pi]$).

Further, a triple root of $P_\alpha(z)$ satisfies, in addition to (3.1), also $\alpha(\alpha-1)z^{\alpha-2} - b\tau^2\exp(-z\tau) = 0$. Such a system admits a unique solution $z = (1-\alpha)/\tau$ occurring if $a = (1-\alpha)^{\alpha-1}/\tau^\alpha$, $b = -\alpha(1-\alpha)^{\alpha-1}\exp(1-\alpha)/\tau^\alpha$ (provided these values are feasible). Similarly we can show that there are no roots of multiplicity greater than three. □

Proposition 2. *Let $\alpha \in (1,2)$, $\tau > 0$, $a < 0$ and b be real numbers. Then there exists $\delta = \delta(\alpha, a) > 0$ such that all the roots z of $P_\alpha(z)$ satisfy $|\arg(z)| > \pi/2$ whenever $|b| < \delta$.*

Proof. Let $z = r\exp(\mathrm{i}\varphi)$ where $r \geq 0$ and $|\varphi| < \pi/2$. We substitute it into $P_\alpha(z) = 0$ and rewrite the real and imaginary parts as

$$r^\alpha\cos(\alpha\varphi) + |a| - b\exp(-r\tau\cos(\varphi))\cos(r\tau\sin(\varphi)) = 0,$$
$$r^\alpha\sin(\alpha\varphi) + b\exp(-r\tau\cos(\varphi))\sin(r\tau\sin(\varphi)) = 0,$$

which implies

$$(r^\alpha - |a|)^2 + 2r^\alpha|a|(1+\cos(\alpha\varphi)) = b^2\exp(-2r\tau\cos(\varphi)). \tag{3.2}$$

Since $r \geq 0$ and $|\varphi| < \pi/2$, the left-hand side of (3.2) has a minimum equal either to a^2 or $a^2\sin^2(\alpha\varphi)$ for $|\alpha\varphi| \leq \pi/2$ or $|\alpha\varphi| > \pi/2$, respectively. The right-hand side of (3.2) reaches its maximum for $r = 0$ and its value is b^2. Thus, (3.2) has no solution for any b such that $|b| < \delta = |a|\sin(\alpha\pi/2)$ which implies the assertion. □

Lemma 2. *Let $\alpha \in (1,2)$, $\tau > 0$ and a, b be real numbers. Then all functions $z = z(a,b)$ defined implicitly by $P_\alpha(z) = 0$ are continuous whenever $z(a,b)$ is not zero.*

Proof. Denote $z = r\exp(\mathrm{i}\varphi)$ and let F and G be real and imaginary parts of P_α, respectively. For the sake of clarity, we will use the notation $P_\alpha(a,b;z)$ to point out the dependence on parameters a, b. Then we can expand $P_\alpha(a,b;z) = 0$ as

$$F(a,b;r,\varphi) \equiv r^\alpha\cos(\alpha\varphi) - a - b\exp(-r\tau\cos(\varphi))\cos(r\tau\sin(\varphi)) = 0\,,$$
$$G(a,b;r,\varphi) \equiv r^\alpha\sin(\alpha\varphi) + b\exp(-r\tau\cos(\varphi))\sin(r\tau\sin(\varphi)) = 0\,.$$

The implicit function theorem states that if the fixed values $\hat{a}, \hat{b}, \hat{r}, \hat{\varphi}$ satisfy $F(\hat{a},\hat{b};\hat{r},\hat{\varphi}) = 0$, $G(\hat{a},\hat{b};\hat{r},\hat{\varphi}) = 0$ and $F_r(\hat{a},\hat{b};\hat{r},\hat{\varphi})G_\varphi(\hat{a},\hat{b};\hat{r},\hat{\varphi}) - F_\varphi(\hat{a},\hat{b};\hat{r},\hat{\varphi})G_r(\hat{a},\hat{b};\hat{r},\hat{\varphi}) \neq 0$, then there exists an open set $\mathcal{O} \in \mathbb{R}^2$ containing $(\hat{a},\hat{b})$ such that there exist unique continuously differentiable functions r, φ such that $r(\hat{a},\hat{b}) = \hat{r}$, $\varphi(\hat{a},\hat{b}) = \hat{\varphi}$ and $F(a,b;r(a,b),\varphi(a,b)) = 0$, $G(a,b;r(a,b),\varphi(a,b)) = 0$ for all $(a,b) \in \mathcal{O}$.

In our case, $F_rG_\varphi - F_\varphi G_r$ equals

$$r\cdot\left[\left(\alpha r^{\alpha-1} - \tau|b|\exp(-r\tau\cos(\varphi))\right)^2 + 2\alpha r^{\alpha-1}\tau|b|\exp(-r\tau\cos(\varphi))\,(1+\operatorname{sgn}(b)\cos(r\tau\sin\varphi + \alpha\varphi - \varphi))\right]. \tag{3.3}$$

Since z (then also r) is nonzero, (3.3) can be equal to zero provided the term in the square bracket is zero. Assume that this does occur. Since both the terms in the square brackets are nonnegative, we obtain

$$|b| = \frac{\alpha}{\tau}r^{\alpha-1}\exp(r\tau\cos(\varphi)) \qquad \text{and} \qquad r\tau\sin(\varphi) + \alpha\varphi - \varphi = \left(2k + \frac{1+\operatorname{sgn}(b)}{2}\right)\pi \quad \text{for a suitable } k \in \mathbb{Z}.$$

After some technical rearrangements, we can see that these conditions are identical to those for multiple roots described in Proposition 1.

Thus we get that if $\hat{a}, \hat{b}$ are fixed and $\hat{z}$ is a nonzero root of $P_\alpha(\hat{a},\hat{b};z)$, i.e. $\hat{z}^\alpha - \hat{a} - \hat{b}\exp(-\hat{z}\tau) = 0$, then there exists a complex-valued function $z(a,b)$ such that $z(\hat{a},\hat{b}) = \hat{z}$ and $P_\alpha(a,b;z(a,b)) = 0$. By a continuous extension of the corresponding open set $\mathcal{O}$, we obtain that this function is unique and continuous for all points $(a,b,z(a,b))$ provided $z(a,b)$ is not a multiple root of $P_\alpha(a,b;z)$.

$P_\alpha(a,b;z)$ has countably many nonzero roots. It defines countably many branches of $z(a,b)$ which do not intersect each other unless a, b imply a multiple root z of $P_\alpha(a,b;z)$; in this case, the corresponding branches cross each other. Since all the functions involved in $P_\alpha(a,b;z)$ are continuous at these points, we conclude that the branches $z(a,b)$ do not lose their continuity property there. □

Remark 2. Lemma 2 admits that the zero root occurring for $a+b=0$ might not depend continuously on a, b. Indeed, for values of α not admitting negative real roots of $P_\alpha(z)$, we can observe that the zero root "disappears" when crossing the line $a+b=0$.

Now we proceed to analysis of conditions ensuring that all the roots of $P_\alpha(z)$ have negative real parts. Doing it, we employ the D-partition method. This method, when applied to $P_\alpha(z)$, originates from analytical descriptions of curves in the (a,b)-plane such that $P_\alpha(z)$ admits either zero or purely imaginary roots just when its coefficients a, b are lying on these curves. Such descriptions are independent of values of the parameter α and appeared in several earlier papers [18,21]. It holds that $P_\alpha(z)$ has the zero root if and only if $a+b=0$, and $P_\alpha(z)$ has a purely imaginary root $z=\pm \mathrm{i}s/\tau$ $(s>0)$ if and only if $s \neq j\pi$ for any integer j and

$$a = \frac{s^\alpha \sin(s+\alpha\pi/2)}{\tau^\alpha \sin(s)} \qquad \text{and} \qquad b = -\frac{s^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(s)}. \tag{3.4}$$

From the geometrical point of view, $P_\alpha(z)$ admits the zero or purely imaginary roots if and only if the couple (a,b) is lying either on the line $a+b=0$, or on some of the curves Γ_j $(j=0,1\dots)$ given in the parametric forms

$$\begin{aligned} \Gamma_j:\ a_j(s) &= \frac{s^\alpha \sin(s+\alpha\pi/2)}{\tau^\alpha \sin(s)}, \\ b_j(s) &= -\frac{s^\alpha \sin(\alpha\pi/2)}{\tau^\alpha \sin(s)}, \end{aligned} \qquad s \in (j\pi, j\pi+\pi). \tag{3.5}$$

This system of curves plays a crucial role in stability investigations of (2.3). Properties of these curves significantly differ with respect to $\alpha \in (0,1]$ or $\alpha \in (1,2)$. In the sequel, we explore the latter case in a more detail.

Lemma 3. *Let $\alpha \in (1,2)$, $\tau > 0$ be real numbers and let Γ_j $(j=0,1\dots)$ be the curves defined by (3.5). Then it holds:*

(i) The line $a+b=0$ is tangent to the curve Γ_0 at the origin, and the line

$$p_0^-: b = a - \left(\frac{\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)$$

is the asymptote to Γ_0 as $s \to \pi^-$. Moreover, $b_0(s) < 0$, $b_0(s) < a_0(s) - (\pi/\tau)^\alpha \cos(\alpha\pi/2)$ and $b_0(s) < -a_0(s)$ for all $s \in (0,\pi)$.

(ii) If j is a positive odd integer, then Γ_j has asymptotes p_j^+ (as $s \to j\pi^+$) and p_j^- (as $s \to (j+1)\pi^-$) given by

$$p_j^+: b = a - \left(\frac{j\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \qquad \text{and} \qquad p_j^-: b = -a + \left(\frac{j\pi+\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Moreover, $b_j(s) > 0$, $b_j(s) > a_j(s) - (j\pi/\tau)^\alpha \cos(\alpha\pi/2)$ and $b_j(s) > -a_j(s) + ((j\pi+\pi)/\tau)^\alpha \cos(\alpha\pi/2)$ for all $s \in (j\pi, j\pi+\pi)$.

(iii) If j is a positive even integer, then Γ_j has asymptotes p_j^+ (as $s \to j\pi^+$) and p_j^- (as $s \to (j+1)\pi^-$) given by

$$p_j^+: b = -a + \left(\frac{j\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \qquad \text{and} \qquad p_j^-: b = a - \left(\frac{j\pi+\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Moreover, $b_j(s) < 0$, $b_j(s) < -a_j(s) + (j\pi/\tau)^\alpha \cos(\alpha\pi/2)$ and $b_j(s) < a_j(s) - ((j\pi+\pi)/\tau)^\alpha \cos(\alpha\pi/2)$ for all $s \in (j\pi, j\pi+\pi)$.

Proof. Verifications of all the three parts are of a technical nature. We derive the part (i), the remaining parts can be proven analogously.

Let $j=0$. Then $\lim_{s\to 0^+}(a_0(s), b_0(s)) = (0,0)$ due to $\alpha > 1$, and the form of the tangent line $a+b=0$ easily follows from the property $\lim_{s\to 0^+} b_0(s)/a_0(s) = -1$. Similarly, the value of slope k and b-intercept q of the asymptote p_0^- to Γ_0 (as $s \to \pi^-$) follow from the standard formulae

$$k = \lim_{s\to\pi^-} \frac{b_0(s)}{a_0(s)} = 1 \qquad \text{and} \qquad q = \lim_{s\to\pi^-} (b_0(s) - a_0(s)) = -\left(\frac{\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right).$$

Further, the property $b_0(s) < 0$ is evident. The second inequality appearing in the part (i) is equivalent to

$$\frac{s^\alpha}{\tau^\alpha \sin(s)}\left(\sin\left(\frac{\alpha\pi}{2}\right) + \sin(s)\cos\left(\frac{\alpha\pi}{2}\right) + \cos(s)\sin\left(\frac{\alpha\pi}{2}\right)\right) > \left(\frac{\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right),$$

and also to

$$s^\alpha \left(1 + \tan\left(\frac{\alpha\pi}{2}\right) \frac{1+\cos(s)}{\sin(s)}\right) < \pi^\alpha$$

due to $s \in (0,\pi)$ and $\alpha \in (1,2)$. The validity of the last inequality is easy to verify.

Similar calculations also imply $b_0(s) < -a_0(s)$ for all $s \in (0,\pi)$. □

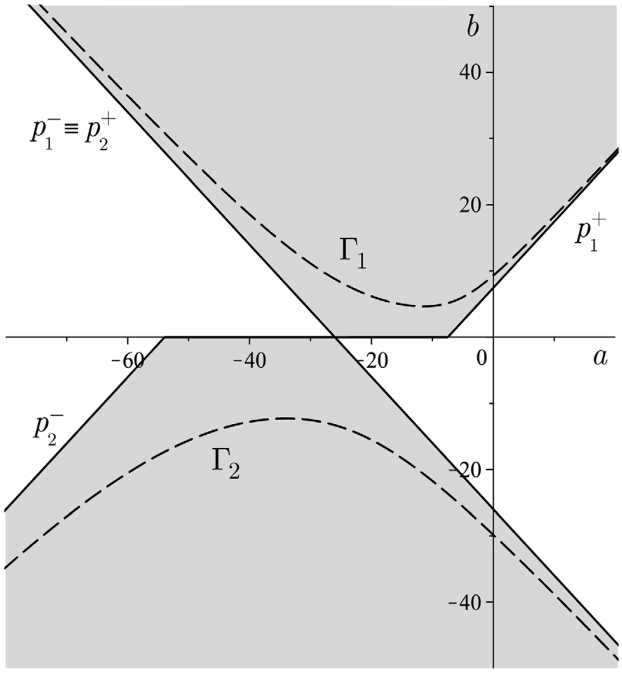

Fig. 3. The common asymptote to Γ_1 and Γ_2 and the corresponding trapezoids for $\alpha = 1.8$ and $\tau = 1$.

Remark 3. Lemma 3 implies that each of the curves Γ_j $(j = 0, 1 \dots)$ is contained in an infinite trapezoid bounded by the a-axis and two straight lines (namely its asymptotes). These asymptotes intersect the a-axis at the values $a = ((j\pi + \pi)/\tau)^\alpha \cos(\alpha\pi/2)$. Also, we can see that each pair Γ_j, Γ_{j+1} shares a common asymptote. The situation is depicted in Fig. 3.

Lemma 4. *Let $\alpha \in (1, 2)$, $\tau > 0$ be real numbers, and let Γ_j $(j = 0, 1 \dots)$ be the curves defined by (3.5). Further, let $X_{m,n} = (a_{m,n}, b_{m,n})$ be intersections of Γ_m and Γ_n (if they exist). Then it holds:*
(i) The intersection $(a_{m,n}, b_{m,n})$ exists (and it is unique) if and only if m, n have the same parity.
(ii) $a_{m,m+2k} < 0$ for all $k \in \mathbb{Z}$ such that $k > -m/2$.
(iii) $a_{m,m+2k} > a_{m,m+2(k+1)}$ for all $k \in \mathbb{Z}$ such that $k > -m/2$.
(iv) $a_{m,m+2k} > a_{m+2\ell,m+2k+2\ell}$ for all $k \in \mathbb{Z}$ such that $k > -m/2$ and $\ell = 1, 2 \dots$.

Proof. (i) By Lemma 3, the sign of functions $b_j(s)$ from (3.5) depends on parity of j. Hence, to find the intersections between Γ_m and Γ_n, we put $n = m + 2k$ for a suitable $k \in \mathbb{Z}$ and search for a couple $(s_{m,m+2k}, s_{m+2k,m})$ such that $s_{m,m+2k} \in (m\pi, m\pi + \pi)$, $s_{m+2k,m} \in ((m + 2k)\pi, (m + 2k + 1)\pi)$ and

$$a_m(s_{m,m+2k}) = a_{m+2k}(s_{m+2k,m}), \qquad b_m(s_{m,m+2k}) = b_{m+2k}(s_{m+2k,m}).$$

Substituting (3.5) into these relations one gets

$$\frac{(s_{m,m+2k})^\alpha \sin(s_{m,m+2k} + \alpha\pi/2)}{\sin(s_{m,m+2k})} = \frac{(s_{m+2k,m})^\alpha \sin(s_{m+2k,m} + \alpha\pi/2)}{\sin(s_{m+2k,m})} \quad \text{and} \quad \frac{(s_{m,m+2k})^\alpha}{\sin(s_{m,m+2k})} = \frac{(s_{m+2k,m})^\alpha}{\sin(s_{m+2k,m})} \tag{3.6}$$

which leads to $\sin(s_{m,m+2k} + \alpha\pi/2) = \sin(s_{m+2k,m} + \alpha\pi/2)$, hence

$$s_{m+2k,m} = (2m + 2k + 3 - \alpha)\pi - s_{m,m+2k}.$$

Let $s_{m,m+2k} = m\pi + \xi$ for a suitable $\xi \in (0, \pi)$. Then using (3.6), intersections between Γ_m and Γ_{m+2k} can be expressed via roots of the function

$$g_{m,m+2k}(\xi) = \frac{\sin(\alpha\pi + \xi)}{\sin(\xi)} - \left(\frac{(2m + 2k + 3 - \alpha)\pi}{m\pi + \xi} - 1\right)^\alpha. \tag{3.7}$$

Since $\lim_{\xi\to 0^+} g_{m,m+2k}(\xi) = \infty$, $\lim_{\xi\to\pi^-} g_{m,m+2k}(\xi) = -\infty$ and $g'_{m,m+2k}(\xi) < 0$ for all $\xi \in (0, \pi)$, the root $\xi_{m,m+2k}$ of $g_{m,m+2k}$ exists and it is determined uniquely.

(ii) Let s_j be a root of $a_j(s)$, $s \in (j\pi, j\pi + \pi)$. Obviously, this root is determined uniquely as $s_j = (j + 1 - \alpha/2)\pi$ for all $j = 0, 1, \dots$. Moreover, $a_j(s) > 0$ for $s < s_j$ and $a_j(s) < 0$ for $s > s_j$. We put $\xi_0 = s_j - j\pi = (1 - \alpha/2)\pi$. The property (ii) now follows from $g_{m,m+2k}(\xi_0) > 0$ for any m, k.

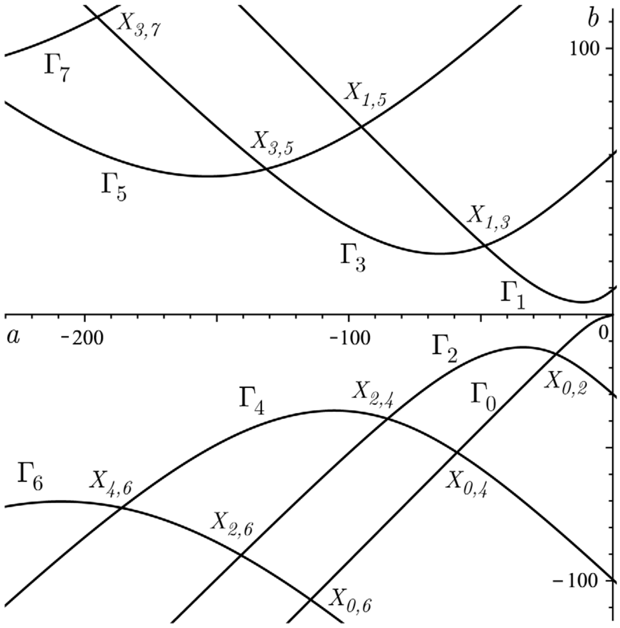

Fig. 4. Some intersections $X_{m,n} = (a_{m,n}, b_{m,n})$ for $\alpha = 1.8$, $\tau = 1$ and $m, n \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

(iii) Since $g_{m,m+2k}(\xi) < g_{m,m+2(k+1)}(\xi)$ for all $\xi \in (0, \pi)$, the roots of these functions satisfy $\xi_{m,m+2k} > \xi_{m,m+2(k+1)}$. Moreover, $a_j(s)$ is monotonically decreasing whenever $a_j(s) < 0$, which implies $a_{m,m+2k} > a_{m,m+2(k+1)}$.

(iv) The statement is a consequence of the property (iii). Indeed, $a_{m,m+2k} > a_{m,m+2k+2\ell} = a_{m+2k+2\ell,m} > a_{m+2k+2\ell,m+2\ell} = a_{m+2\ell,m+2k+2\ell}$. □

Remark 4. For the sake of lucidity, the location and ordering of intersections $X_{m,n} = (a_{m,n}, b_{m,n})$ between Γ_m and Γ_n is depicted in Fig. 4.

Lemma 5. *Let $\alpha \in (1, 2)$, $\tau > 0$, $a < 0$ and b be real numbers, and let a couple $(a, b) \in \Gamma_j$ for a unique j. If $|b|$ increases, then a new root of $P_\alpha(z)$ with a positive real part appears.*

Proof. Let $z = z(a, b)$ be the corresponding purely imaginary root of $P_\alpha(z)$. By Lemma 2, there exists a neighborhood of (a, b) such that $z(a, b)$ is a continuous function on this neighborhood. Moreover, we can calculate

$$\frac{\partial z(a,b)}{\partial b} = \frac{\exp(-z\tau)}{\alpha z^{\alpha-1} + b\tau \exp(-z\tau)} = \frac{z^\alpha - a}{b(\alpha z^{\alpha-1} + \tau z^\alpha - \tau a)},$$

which is the well-defined expression because the root $z = z(a, b)$ is simple. Substituting $z = \mathrm{i}s$ ($s > 0$) and evaluating the real part we get

$$\mathcal{R}\left(\frac{\partial z(a,b)}{\partial b}\right)\Big|_{z=\mathrm{i}s} = \frac{1}{b}\frac{x_1x_3 + x_2x_4}{x_3^2 + x_4^2}$$

where $x_1 = s^\alpha \cos(\alpha\pi/2) - a$, $x_2 = s^\alpha \sin(\alpha\pi/2)$, $x_3 = \alpha s^{\alpha-1}\cos((\alpha-1)\pi/2) + \tau s^\alpha \cos(\alpha\pi/2) - \tau a$ and $x_4 = \alpha s^{\alpha-1}\sin((\alpha-1)\pi/2) + \tau s^\alpha \sin(\alpha\pi/2)$. If $a < 0$ then

$$x_1x_3 + x_2x_4 = \tau(s^\alpha + a)^2 - 2as^\alpha\tau(1 + \cos(\alpha\pi/2)) - \alpha a s^{\alpha-1}\sin(\alpha\pi/2) > 0.$$

Since analogously we can dispose with the complex conjugate case $z = -\mathrm{i}s$ ($s > 0$), unifying both the cases we get

$$\operatorname{sgn}\left(\frac{\partial \mathcal{R}(z(a,b))}{\partial b}\Big|_{z=\pm\mathrm{i}s}\right) = \operatorname{sgn}(b)$$

which proves the assertion. □

4. Effective stability criteria for (2.3) with $\alpha \in (1, 2)$

Lemma 1 supported by the results of Section 3 shows that the line $a+b=0$ and the curves Γ_j given by (3.5) will play a crucial role in the formulation of efficient stability conditions for (2.3). Before we state these conditions, some auxiliary notation might be useful.

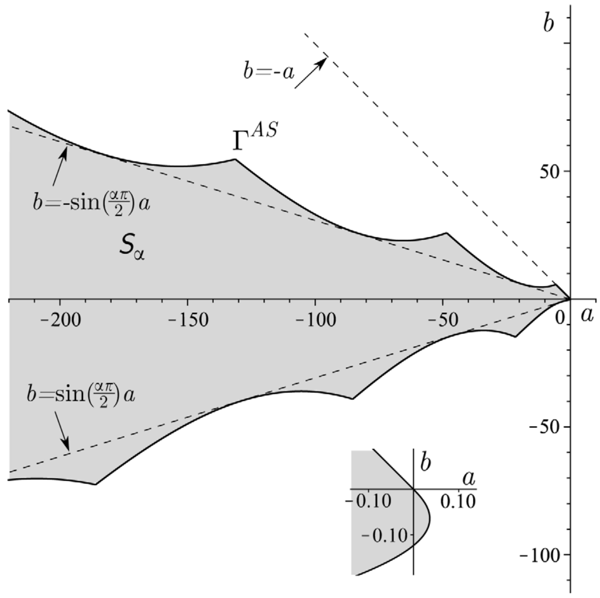

Fig. 5. Stability boundary $\varGamma^{AS}$ and stability region $\mathcal{S}_\alpha$ of (2.3) for $\alpha = 1.8$ and $\tau = 1$.

Let P be the line segment

$$a = -s, \quad b = s, \qquad s \in (0, T)$$

where $T = (3\pi - \alpha\pi)^\alpha/(2\tau^\alpha|\cos(\alpha\pi/2)|)$, and let $\tilde{\varGamma}_j$ $(j = 0, 1, \dots)$ be the parts of $\varGamma_j$ with the endpoints $X_{j,j-2}$ and $X_{j,j+2}$ given by its intersections with the neighboring curves $\varGamma_{j-2}$ and $\varGamma_{j+2}$ (see Fig. 4). If $j = 0$ or $j = 1$, then the curves $\varGamma_{-2}$ and $\varGamma_{-1}$ are not defined; in such a case, we introduce the corresponding endpoints as $(0, 0)$ and $(-T, T)$, respectively (for analytical descriptions of all the other endpoints we refer to Lemma 4 and its proof). Further, we put

$$\varGamma^{AS} = \bigcup_{j=0}^{\infty} \tilde{\varGamma}_j \cup P$$

and denote by $\mathcal{S}_\alpha$ the open area in the (a, b)-plane containing the negative part of a-axis and bounded by $\varGamma^{AS}$. Also, we set $\mathcal{U}_\alpha = \mathbb{R}^2 \setminus \mathrm{cl}(\mathcal{S}_\alpha)$ where $\mathrm{cl}(\cdot)$ means the closure of a given set. The area $\mathcal{S}_\alpha$ (including some angular bounds of the curve $\varGamma^{AS}$ and a detail of the situation near the origin) is depicted in Figs. 5 and 6.

Then an effective reformulation of Lemma 1 is provided by

Theorem 2. *Let $\alpha \in (1, 2)$, $\tau > 0$ and a, b be real numbers.*
(i) If $(a, b) \in \mathcal{S}_\alpha$, then (2.3) is asymptotically stable.
(ii) If $(a, b) \in \mathcal{U}_\alpha$, then (2.3) is not stable.

Proof. By Lemma 1, it is enough to show that if $(a, b) \in \mathcal{S}_\alpha$, then all the roots of $P_\alpha(z)$ have negative real parts, and if $(a, b) \in \mathcal{U}_\alpha$, then there exists a root of $P_\alpha(z)$ with a positive real part.

Proposition 2 implies that there exists a neighborhood $\mathcal{O}$ of the negative a-axis where all the roots of $P_\alpha(z)$ have negative real parts. In addition, all the roots of $P_\alpha(z)$ depend continuously on its parameters a, b due to Lemma 2. Hence, the neighborhood $\mathcal{O}$ can be expanded until it is bounded by the appropriate parts of the line $a + b = 0$ and the curves $\varGamma_j$ $(j = 0, 1 \dots)$. The properties described in Lemmas 3 and 4 (supported by some simple calculations) yield that such an expansion of $\mathcal{O}$ is bounded just by the curve $\varGamma^{AS}$. Finally, by Lemma 5, when crossing $\varGamma^{AS}$ through any curve $\tilde{\varGamma}_j$ (while expanding the neighborhood $\mathcal{O}$), a root of $P_\alpha(z)$ with a positive real part appears. To complete the proof, it is enough to employ an obvious property claiming that $P_\alpha(z)$ admits a root with the positive real part whenever $a + b > 0$. □

Remark 5. Geometry of the stability boundary $\varGamma^{AS}$ can be described as follows: If $a < 0$ and $b > 0$, then this curve is formed by a part of the line $a + b = 0$ connecting the origin and the curve $\varGamma_1$. Then $\varGamma^{AS}$ follows $\varGamma_1$ until it is crossed by $\varGamma_3$, etc. If $b < 0$, then $\varGamma^{AS}$ follows the curve $\varGamma_0$ (that crosses the b-axis at $b = -(\pi/\tau - \alpha\pi/(2\tau))^\alpha$) until it is crossed by $\varGamma_2$, then follows appropriate part of $\varGamma_2$ until reaching $\varGamma_4$, etc.

Theorem 2 yields a geometric description of the stability area $\mathcal{S}_\alpha$ of (2.3) in the (a, b)-plane. In the sequel, we provide an alternative stability criterion for the case $a < 0$ that better agrees with the form of the conditions of Theorem 1 (the

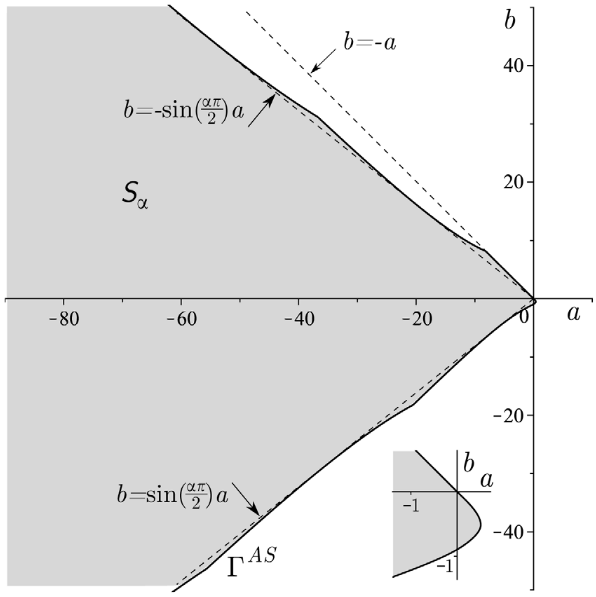

Fig. 6. Stability boundary Γ^{AS} and stability region $\mathcal{S}_\alpha$ of (2.3) for $\alpha = 1.4$ and $\tau = 1$.

case $a > 0$ is not considered here because the corresponding stability conditions are quite straightforward as illustrated in Figs. 5 and 6). On this account, for any $\alpha \in (1, 2)$, $\tau > 0$, $a < 0$ and b being real numbers, we introduce the symbols

$$\tau_\ell^+ = \frac{\ell\pi + \frac{(2-\alpha)\pi}{2} + \arcsin\left(\left|\frac{a}{b}\right|\sin(\frac{\alpha\pi}{2})\right)}{\left(a\cos(\frac{\alpha\pi}{2}) + \sqrt{b^2 - a^2\sin^2(\frac{\alpha\pi}{2})}\right)^{1/\alpha}} \quad \text{and} \quad \tau_\ell^- = \frac{\ell\pi + \pi + \frac{(2-\alpha)\pi}{2} - \arcsin\left(\left|\frac{a}{b}\right|\sin(\frac{\alpha\pi}{2})\right)}{\left(a\cos(\frac{\alpha\pi}{2}) - \sqrt{b^2 - a^2\sin^2(\frac{\alpha\pi}{2})}\right)^{1/\alpha}},$$

$\ell = 0, 1, 2 \dots$. Then it holds

Theorem 3. *Let $\alpha \in (1, 2)$, $\tau > 0$, $a < 0$ and b be real numbers.*

(i) If $-\sin(\alpha\pi/2) < b/a < \sin(\alpha\pi/2)$, then (2.3) is asymptotically stable.

(ii) If $b/a > \sin(\alpha\pi/2)$, then there exists an integer $N_1 \geq 0$ such that (2.3) is asymptotically stable for any $\tau \in (\tau_{2k-2}^-, \tau_{2k}^+)$, and it is not stable for any $\tau \in (\tau_{2k}^+, \tau_{2k+2}^-)$ where $k = 0, \dots, N_1$ (here we set $\tau_{-2}^- = 0$, $\tau_{2N_1+2}^- = \infty$).

(iii) If $-1 < b/a < -\sin(\alpha\pi/2)$, then there exists an integer $N_2 \geq 0$ such that (2.3) is asymptotically stable for any $\tau \in (\tau_{2k-1}^-, \tau_{2k+1}^+)$, and it is not stable for any $\tau \in (\tau_{2k+1}^+, \tau_{2k+3}^-)$ where $k = 0, \dots, N_2$ (here we set $\tau_{-1}^- = 0$, $\tau_{2N_2+3}^- = \infty$).

(iv) If $b/a < -1$, then (2.3) is not stable.

Proof. By (3.4),

$$\frac{b}{a} = -\frac{\sin(\alpha\pi/2)}{\sin(s + \alpha\pi/2)}. \tag{4.1}$$

This immediately implies the property (i) because all the couples (a, b) lying on Γ^{AS} have to satisfy $|b/a| \geq \sin(\alpha\pi/2)$.

The proofs of the parts (ii), (iii) are technical. The formulae for τ_ℓ^+, τ_ℓ^- can be derived via expressing s from (4.1) in the form

$$s = (-1)^{\ell+1}\arcsin\left(\frac{a}{b}\sin\left(\frac{\alpha\pi}{2}\right)\right) + \ell\pi - \frac{\alpha\pi}{2},$$

using its proper sign analysis, substitution into (3.4) and some straightforward rearrangements. The existence of the values N_1, N_2 follows from the property of the intersections $X_{j,j+2} = (a_{j,j+2}, b_{j,j+2})$ between the curves Γ_j, Γ_{j+2} discussed in Lemma 4. Indeed, the ratio

$$\left|\frac{b_{j,j+2}}{a_{j,j+2}}\right| = \frac{\sin(\alpha\pi/2)}{|\sin(s_{j,j+2} + \alpha\pi/2)|}$$

is monotonically decreasing and tending to $\sin(\alpha\pi/2)$ as $j \to \infty$ because the sequence $\{s_{j,j+2} - j\pi\}_{j=0}^\infty$ is decreasing and its limit equals $(3-\alpha)\pi/2$ due to (3.7) (see the proof of Lemma 4).

The property (iv) is a direct consequence of the fact that the line $a + b = 0$ forms a part of Γ^{AS}. □

Remark 6. It might be interesting to discuss the conclusions of Theorem 2 with $\alpha \to 2^-$ and compare them with the stability conditions of Theorem 1 (ii) derived for $\alpha = 2$. To emphasize dependence on the derivative order α, we denote the curves from (3.5) as $\Gamma_{\alpha,j}$ and their asymptotes (described in Lemma 3) as $p^+_{\alpha,j}$, $p^-_{\alpha,j}$ throughout this remark.

We describe the calculation with j being odd. First, we determine the distance between the curve $\Gamma_{\alpha,j}$ and its asymptotes $p^+_{\alpha,j}$, $p^-_{\alpha,j}$ in the sense of the supremum metric. On this account, we introduce the symbol $d_I(\Gamma_{\alpha,j}, p)$ as the difference between b–coordinates of $\Gamma_{\alpha,j}$ and a line p maximized over all $a \in I$. Further, we denote $a_j^+ = (j\pi)^\alpha/\tau^\alpha \cos(\alpha\pi/2)$, $a_j^- = (j\pi+\pi)^\alpha/\tau^\alpha \cos(\alpha\pi/2)$, and let $s_j^+ \in (j\pi, j\pi+\pi)$ and $s_j^- \in (j\pi, j\pi+\pi)$ be such that $a(s_j^+) = a_j^+$ and $a(s_j^-) = a_j^-$, respectively. Using this notation, we wish to show that $d_{[a_j^+,\infty)}(\Gamma_{\alpha,j}, p^+_{\alpha,j}) \to 0$ as $\alpha \to 2^-$. Indeed, it holds

$$\begin{aligned} d_{[a_j^+,\infty)}(\Gamma_{\alpha,j}, p^+_{\alpha,j}) &= \sup_{t\in(j\pi,s_j^+]} \frac{s^\alpha}{\tau^\alpha \sin(s)}\left(\sin\left(t+\frac{\alpha\pi}{2}\right)+\sin\left(\frac{\alpha\pi}{2}\right)\right) - \left(\frac{j\pi}{\tau}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right) \\ &= \sup_{t\in(j\pi,s_j^+]} \frac{s^\alpha-(j\pi)^\alpha}{\tau^\alpha}\left|\cos\left(\frac{\alpha\pi}{2}\right)\right| - \frac{s^\alpha}{\tau^\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\frac{1+\cos(s)}{\sin(s)} \\ &= \frac{(s_j^+)^\alpha-(j\pi)^\alpha}{\tau^\alpha}\left|\cos\left(\frac{\alpha\pi}{2}\right)\right| - \frac{(s_j^+)^\alpha}{\tau^\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\frac{1+\cos(s_j^+)}{\sin(s_j^+)} \end{aligned}$$

due to the monotony property of the maximized expression on the interval $(j\pi, s_j^+]$. Moreover, by use of the identity $((s_j^+)^\alpha - (j\pi)^\alpha)|\cos(\alpha\pi/2)|/\cos(s_j^+) = (s_j^+)^\alpha \sin(\alpha\pi/2)/\sin(s_j^+)$, we obtain

$$d_{[a_j^+,\infty)}(\Gamma_{\alpha,j}, p^+_{\alpha,j}) = \frac{((s_j^+)^\alpha-(j\pi)^\alpha)\left|\cos\left(\frac{\alpha\pi}{2}\right)\right|}{\tau^\alpha|\cos(s_j^+)|}.$$

Since $s_j^+ \to j\pi$ as $\alpha \to 2^-$, we actually get $d_{[a_j^+,\infty)}(\Gamma_{\alpha,j}, p^+_{\alpha,j}) \to 0$ as $\alpha \to 2^-$. Similarly, we can show that $d_{(-\infty,a_j^-]}(\Gamma_{\alpha,j}, p^-_{\alpha,j}) \to 0$ as $\alpha \to 2^-$ and, if we denote by p_* the a-axis, then also $d_{[a_j^+,a_j^-]}(\Gamma_{\alpha,j}, p_*) \to 0$ as $\alpha \to 2^-$. A similar line of arguments leads to analogous results also for j being even.

Note that the asymptotes $p^+_{\alpha,j}$, $p^-_{\alpha,j}$ themselves tend to the lines $b = -a - (j\pi)^2/\tau^2$ and $b = a + (j\pi)^2/\tau^2$ as $\alpha \to 2^-$. Hence, one can observe that the stability area described in Theorem 2 actually approaches (when $\alpha \to 2^-$) the system of triangles bounded by these lines and the negative a-axis (see Fig. 2). Thus the stability conditions of Theorem 2 are converting to those of Theorem 1 (ii) when $\alpha \to 2^-$.

This fact provides another interesting consequence. If we make this limit transition also in Theorem 3 (serving as an alternative description of $\mathcal{S}_\alpha$ using τ as a driving parameter), then we can easily deduce that (2.3) with $\alpha = 2$ is asymptotically stable if and only if $0 < |b| < -a$ and

$$\frac{\ell\pi}{\sqrt{-a-|b|}} < \tau < \frac{(\ell+1)\pi}{\sqrt{-a+|b|}} \tag{4.2}$$

where ℓ is a non-negative integer that is even for $b > 0$ and odd for $b < 0$. This form of conditions seems to be more effective compared to that of Theorem 1 (ii), especially with respect to explicit evaluations of stability switches for a varying delay parameter.

5. Stabilization and destabilization of harmonic and fractional oscillator

Now we apply conclusions of the previous section to the problem stated in the introductory part. First, we consider the classical harmonic oscillator and show how it can be asymptotically stabilized or destabilized via a feedback control depending only on a position of the controlled object. Although these results can be partially derived from Theorem 1 (ii), we deduce them more straightforwardly from (4.2). Then we study the linear fractional oscillator (which is, in the uncontrolled case, asymptotically stable for any derivative order between 1 and 2) and describe conditions for its destabilization. Also, we consider both these types of oscillators in their inverted forms and discuss conditions for their stabilizations.

5.1. Delayed feedback control of a harmonic oscillator

We consider a controlled harmonic oscillator in the form

$$y''(t) + \omega^2 y(t) = u(t), \qquad t > 0, \tag{5.1}$$

$\omega > 0$ is a real number representing frequency. Based on an elementary analysis, the uncontrolled linear oscillator (when u is identically zero) is non-asymptotically stable and cannot be asymptotically stabilized via a feedback control of the

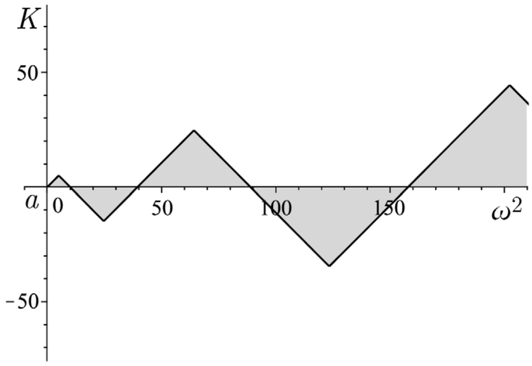

Fig. 7. The stability region for harmonic oscillator (5.1) with control (5.2) depicted in the (ω^2, K)-plane for $\tau = 1$.

form $u(t) = Ky(t)$ for any real gain parameter K (obviously, it can be destabilized whenever $K \geq \omega^2$). When implementing a time lag into the controller, i.e. when

$$u(t) = Ky(t - \tau), \tag{5.2}$$

the situation becomes different. There exists a nonempty set of couples (K, τ) such that (5.1)–(5.2) is either asymptotically stable or unstable. To describe these sets, we can set $a = -\omega^2$ and $b = K$ in (2.4), (2.5) and obtain their analytical descriptions explicitly depending on the gain parameter K. More precisely, (5.1)–(5.2) is asymptotically stable if and only if there exists a non-negative integer ℓ such that

$$2\ell\pi/\tau < \omega < (2\ell + 1)\pi/\tau \qquad \text{and} \qquad |K| < \min\left(\omega^2 - (\ell\pi/\tau)^2, ((\ell + 1)\pi/\tau)^2 - \omega^2\right)$$

where ℓ is even or odd for K positive or negative, respectively (see Fig. 7 for the corresponding stabilization region). If we need to evaluate an explicit dependence of stability of (5.1)–(5.2) on a time lag parameter, then (4.2) provides a more suitable platform. Indeed, (4.2) with $a = -\omega^2$ and $b = K$ immediately implies that if the gain K is fixed such that $0 < |K| < \omega^2$, then (5.1)–(5.2) is asymptotically stable if and only if

$$\frac{\ell\pi}{\sqrt{\omega^2 - |K|}} < \tau < \frac{(\ell + 1)\pi}{\sqrt{\omega^2 + |K|}} \tag{5.3}$$

where ℓ is a non-negative integer that is even for $K > 0$ and odd for $K < 0$. In addition, we can notice that the inequality

$$\frac{\ell\pi}{\sqrt{\omega^2 - |K|}} < \frac{(\ell + 1)\pi}{\sqrt{\omega^2 + |K|}}$$

can actually occur provided ℓ is less than the value

$$\ell^* = \frac{\omega^2 - |K| + \sqrt{\omega^4 - K^2}}{2|K|}$$

that enables to determine the number of such stability switches.

Quite analogously, as a counterpart of the condition (5.3), we can formulate the conditions for destabilization of the harmonic oscillator via the control (5.2).

Finally, it may be useful to describe this asymptotic stabilization or destabilization area in the domain (K, τ) of both the control parameters. Appropriate boundary curves $\tau_\ell = \tau_\ell(K)$ can be obtained directly from (5.3) in the form

$$\tau_\ell = \frac{\ell\pi}{\sqrt{\omega^2 + (-1)^{\ell+1}K}}, \qquad -\omega^2 < K < \omega^2, \quad \ell = 0, 1, \ldots$$

and they are depicted in Fig. 8.

5.2. Delayed feedback control of an inverted linear oscillator

We consider the controlled inverted linear oscillator

$$y''(t) - \omega^2 y(t) = u(t), \qquad t > 0 \tag{5.4}$$

that is for $u = 0$ unstable. Obviously, using the control $u(t) = Ky(t)$, it can be stabilized (only non-asymptotically) provided $K < -\omega^2$ (in fact, thus we convert (5.4) into the standard harmonic case). If we employ the control (5.2), then our previous analysis easily shows that asymptotic stabilization of (5.4) is not possible for any couple (K, τ).

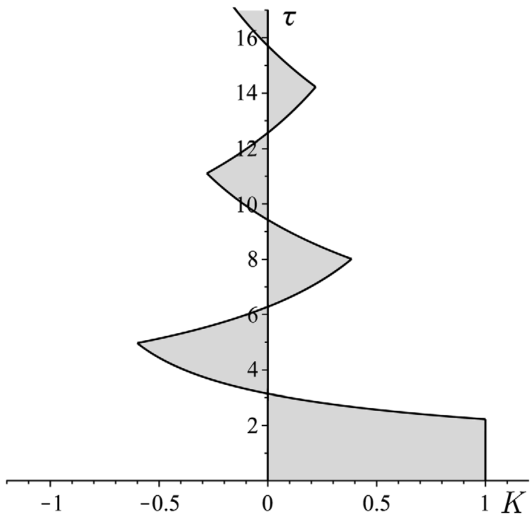

Fig. 8. The stability region for harmonic oscillator (5.1) with control (5.2) depicted in the (K, τ)-plane for $\omega = 1$.

5.3. Delayed feedback control of a fractional oscillator

Now we consider the controlled fractional oscillator

$$D^{\alpha}y(t) + \omega^{\alpha}y(t) = u(t), \qquad t > 0 \tag{5.5}$$

where $\alpha \in (1, 2)$ and $\omega > 0$ are real numbers. It is well-known [2] that this fractional model considered without a control term is asymptotically stable for any $\alpha \in (1, 2)$. Thus, using our results from the previous section, we can discuss its possible destabilization via (5.2).

By Theorem 3, if $|K| < \omega^{\alpha}\sin(\alpha\pi/2)$, then (5.5) remains asymptotically stable for any non-negative value of τ, hence its destabilization is possible only when $|K| \geq \omega^{\alpha}\sin(\alpha\pi/2)$. If, particularly, $K > \omega^{\alpha}$, then (5.5) is destabilized via (5.2) unconditionally, i.e. for any $\tau > 0$. For other values of K, the stability behavior depends on τ. If $\omega^{\alpha}\sin(\alpha\pi/2) < K < \omega^{\alpha}$, then (5.5) becomes destabilized via (5.2) provided

$$K > \frac{s_{\omega,\ell}^{\alpha}\sin(\alpha\pi/2)}{\tau^{\alpha}\sin(s_{\omega,\ell})} \qquad \text{for a non-negative odd integer } \ell, \tag{5.6}$$

$s_{\omega,\ell}$ being determined uniquely by $\omega^{\alpha} = -s_{\omega,\ell}^{\alpha}\sin(s_{\omega,\ell} + \alpha\pi/2)/(\tau^{\alpha}\sin(s_{\omega,\ell}))$, $s_{\omega,\ell} \in (\ell\pi, \ell\pi + \pi)$. If $K < -\omega^{\alpha}\sin(\alpha\pi/2)$, then (5.5) becomes destabilized via (5.2) provided

$$K < \frac{-s_{\omega,\ell}^{\alpha}\sin(\alpha\pi/2)}{\tau^{\alpha}\sin(s_{\omega,\ell})} \qquad \text{for a non-negative even integer } \ell, \tag{5.7}$$

$s_{\omega,\ell}$ being introduced as above. The corresponding stability region in the (ω^{α}, K)-plane is depicted in Fig. 9.

The use of criteria (5.6) and (5.7) requires a numerical solving of nonlinear equations which negatively affects their practical usability. Therefore, we describe the destabilization boundary in the (K, τ)–plane since it can provide explicit formulae. Indeed, using Theorem 3 (parts (ii) and (iii)), we can evaluate stability switches with respect to a time lag. First, we determine the number ℓ^* of stability switches via roots $\xi_{j,j+2}$ of $g_{j,j+2}(\xi) = 0$ (see (3.7)) for j odd (if $K > 0$) and even (if $K < 0$). The value ℓ^* is given uniquely by

$$|\sin(\xi_{\ell^*,\ell^*+2} + \alpha\pi/2)| \geq \frac{\omega^{\alpha}}{|K|}\sin(\alpha\pi/2) \quad \text{and} \quad |\sin(\xi_{\ell^*-2,\ell^*} + \alpha\pi/2)| < \frac{\omega^{\alpha}}{|K|}\sin(\alpha\pi/2).$$

Then (5.5) is destabilized via (5.2) when

$$\tau_{\ell}^{+} < \tau < \tau_{\ell}^{-}, \qquad \ell < \ell^*,$$
$$\tau > \tau_{\ell^*}^{+}$$

where ℓ is odd (if $K > 0$) or even ($K < 0$), and τ_{ℓ}^{+}, τ_{ℓ}^{-} are defined by Theorem 3. Similarly as in the harmonic case $\alpha = 2$, formulae for τ_{ℓ}^{+}, τ_{ℓ}^{-} can be interpreted as parts of the stability boundary curve $\tau = \tau(K)$ in the (K, τ)-plane (see Fig. 10).

5.4. Delayed feedback control of an inverted fractional oscillator

Finally, we consider the controlled inverted fractional oscillator

$$D^{\alpha}y(t) - \omega^{\alpha}y(t) = u(t), \qquad t > 0 \tag{5.8}$$

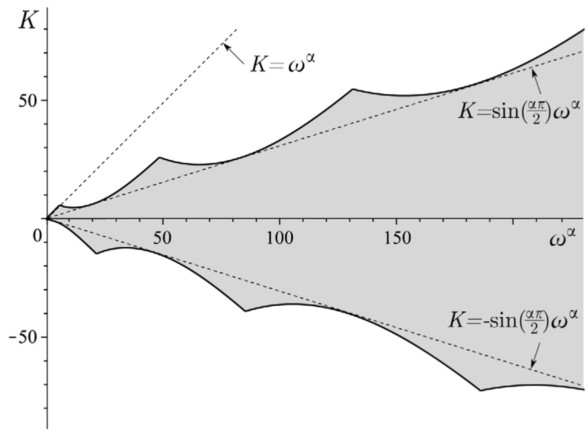

Fig. 9. The stability region for fractional oscillator (5.5) with control (5.2) depicted in the (ω^α, K)-plane for the values $\alpha = 1.8$ and $\tau = 1$.

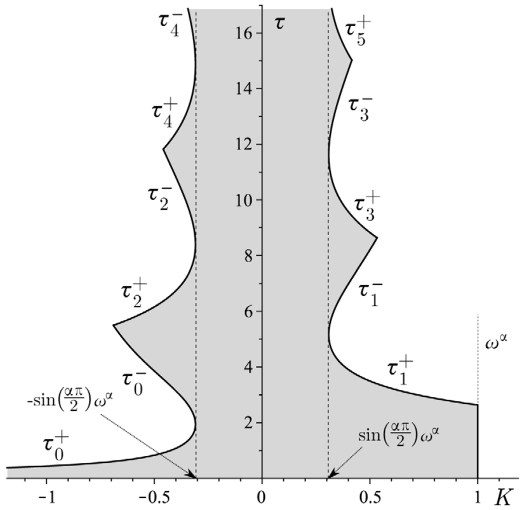

Fig. 10. The stability region for fractional oscillator (5.5) with control (5.2) depicted in the (K, τ)-plane for the values $\alpha = 1.8$ and $\omega = 1$.

that is unstable if $u = 0$. Introducing the control term (5.2) one can stabilize this system as demonstrated by the stability region in Fig. 11 (note that stabilization is not possible for $\alpha = 2$). Unlike the classical fractional oscillator case, there are no stability switches. The boundary curve for the asymptotic stability region in the (K, τ)-plane can be described as

$$\tau(K) = \frac{\frac{(2-\alpha)\pi}{2} + \arcsin\left(\frac{\omega^\alpha}{K}\sin(\frac{\alpha\pi}{2})\right)}{\left(\omega^\alpha\cos(\frac{\alpha\pi}{2}) + \sqrt{K^2 - \omega^{2\alpha}\sin^2(\frac{\alpha\pi}{2})}\right)^{1/\alpha}}, \qquad K \le -\omega^\alpha.$$

Asymptotic stabilization of (5.8) via (5.2) then occurs whenever $\tau < \tau(K)$ as depicted in Fig. 12.

6. Concluding remarks

We have discussed the problems connected with (asymptotic) stabilization and destabilization of fractional oscillators via a delayed feedback loop. The mathematical background of the problem consisted in analysis of appropriate FDDEs. Stability properties of these equations with the derivative order between 0 and 1 were described previously. Our analysis showed that the topological structure of stability regions (considered in the parameter space) significantly changes when the derivative order exceeds the value 1. In particular, a repeated occurrence of finitely many stability switches with respect to a changing delay parameter is a product of this analysis.

The derived stability criteria are easy to apply and result into effective conditions on control parameters ensuring required asymptotic stabilization or destabilization of the studied models. In the limit case, when the fractional oscillator becomes the harmonic one, stability switches can be evaluated fully explicitly. Thus, our stability criteria offer a new computational view on the existing results also in this classical case.

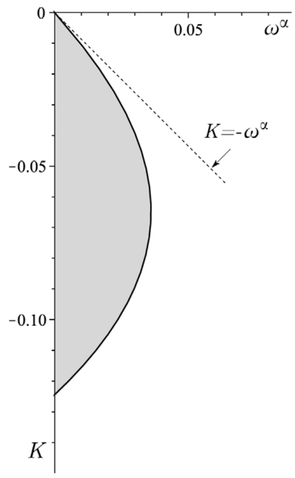

Fig. 11. The stability region for inverted fractional oscillator (5.8) with control (5.2) depicted in the (ω^α, K)-plane for the values $\alpha = 1.8$ and $\tau = 1$.

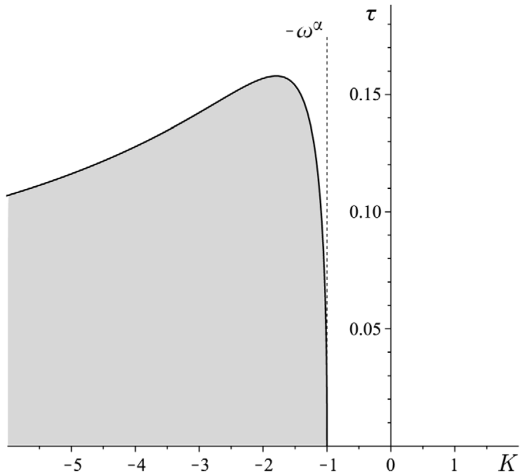

Fig. 12. The stability region for inverted fractional oscillator (5.8) with control (5.2) depicted in the (K, τ)-plane for the values $\alpha = 1.8$ and $\omega = 1$.

This research can be continued towards more advanced types of (nonlinear) fractional oscillators and controls of their stability or oscillatory properties. It seems to be unavoidable that the development of techniques for analysis of appropriate (nonlinear) FDDEs will require new approaches.

CRediT authorship contribution statement

Jan Čermák: Conceptualization, Methodology, Calculations, Investigation, Validation, Writing – review & editing. **Tomáš Kisela:** Conceptualization, Visualisation, Methodology, Calculations, Investigation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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