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NON-COMPACT RIEMANNIAN MANIFOLDS WITH SPECIAL HOLONOMY

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Abstract

This thesis is dedicated to the study of the geometry and topology of non-compact Riemannian manifolds with special holonomy groups. After brief review of Riemannian holonomy theory we consider construction of resolution of standard cone over 3-Sasakian manifold. In particular, this gives a continuous family of explicit examples of non-compact Riemanian manifolds with Sin(7)-holonomy. We apply the idea of this construction to resolve cone G_2-holonomy metric over twistor space of 3-Sasakian manifold and to find continuous family of $SU(4)$-holonomy metrics connecting well-known Calabi metrics. The explicit construction of $SU(2)$-holonomy metric generalising Eguchi-Hanson metrics is studied and applications of these metrics to describing Calabi-Yau metrics on $K3$ surface are considered.

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Paper B: Bazaikin, Ya. V., *Noncompact Riemannian spaces with the holonomy group Spin(7) and 3-Sasakian manifolds.* Proceedings of the Steklov Institute of Mathematics. 2008. Vol. 263, Issue 1. P. 2–12.

Paper C: Bazaikin, Ya. V., Malkovich, E. G., *Spin(7)-structures on complex linear bundles and explicit Riemannian metrics with holonomy group SU(4).* Sbornik Mathematics. 2011. Vol. 202, Issue. 4. P. 467–493.

Paper D: Bazaikin, Ya. V., Malkovich, E. G., *G(2)-holonomy metrics connected with a 3-Sasakian manifold..* Siberian Mathematical Journal. 2008. Vol. 49, No. 1. P. 1–4.

Paper E: Bazaikin, Ya. V., *On some Ricci-flat metrics of cohomogeneity two on complex line bundles.* Siberian Mathematical Journal. 2004. Vol. 45, No. 3. P. 410–415.

Paper F: Bazaikin, Ya. V., *Special Kahler metrics on complex line bundles and the geometry of K3-surfaces.* Siberian Mathematical Journal. 2005. Vol. 46, No. 6. P. 995–1004.

1 Introduction

The first mention of holonomy (namely, the use of the term "holonomic" and "nonholonomic" constraints in classical mechanics) dates back to 1895 and belongs to Hertz [28, 38]. In mathematical works, the holonomy concept first appeared in 1923 in E. Cartan papers [15, 16, 18] and already had a modern meaning. Briefly speaking, the holonomy group $Hol(M^n)$ of the Riemannian manifold M^n is generated by the operators of parallel translations with respect to the Levi-Civita connection along paths starting from and ending at a fixed point $p \in M^n$. If we consider only contractible loops, we obtain restricted holonomy group $Hol_0(M^n)$, which is a connected component of identity map in the group $Hol(M^n)$. Further in the text we will assume M^n to be simply connected and therefore $Hol(M^n) = Hol_0(M^n)$. It is intuitively clear that if $Hol(M^n)$ does not coincide with the maximal possible group of isometries $SO(n)$ of the tangent space T_pM^n, then this should indicate the presence of restrictions on the geometry of the Riemannian manifold. Indeed, each special group of holonomy corresponds to one or another special geometry.

The global character of the holonomy group of a Riemannian manifold is emphasized by de Rum decomposition theorem [10]. Namely, it is obvious that if the Riemannian manifold M is a direct product of the Riemannian manifolds M_1 and M_2, then $Hol(M) = Hol(M_1) \times Hol(M_2)$ (together with the corresponding decomposition of the representation of the holonomy group). It turns out that if the Riemannian manifold is complete, then the converse is true.

The problem of classifying Riemannian holonomy groups naturally arises: which groups can be holonomy groups of a Riemannian manifold?

In solving this problem, we can immediately restrict ourselves to complete irreducible Riemannian manifolds, i.e. such whose holonomy representation does not have invariant subspaces in T_pM. By the de Rham decomposition theorem, such manifolds do not decompose into a direct product, and any complete Riemannian manifold decomposes into a product of irreducible ones.

Symmetric spaces gives an important example of Riemannian manifolds with special holonomy groups [18]:

Theorem 1. *Let M^n be a simply connected symmetric space and G be a Lie group of the isometries of M^n generated by all transvections. We assume that $H \subset G$ is the isotropy group of M, with respect to the chosen point. Then M is isometric to homogeneous* *space G/H, and the holonomy group $Hol(M)$ coincides with H, and the representation of holonomy coincides with the representation of isotropy.*

Cartan [17] reduced the problem of describing simply connected Riemannian symmetric spaces to the theory of Lie groups, and he obtained a list of all such spaces. Cartan's proof and a complete list of simply connected Riemannian symmetric spaces are discussed in [27, 8].

The next important advance in the classification problem was made by Berger:

Theorem 1.1. *Let M be a simply connected irreducible Riemannian manifold of dimension n that is not symmetric. Then one of the following cases takes place.*

1) $Hol(M) = SO(n)$, the general case,

2) $n = 2m$, where $m > 2$ and $Hol(M) = U(m) \subset SO(2m)$, Kähler manifolds,

3) $n = 2m$, where $m > 2$ and $Hol(M) = SU(m) \subset S(2m)$, special Kähler manifolds,

4) $n = 4m$, where $m > 2$ and $Ho1(M) = Sp(m) \subset SO(4m)$, hyperkähler manifolds,

5) $n = 4m$, where $m > 2$ and $Hol(M) = Sp(m)Sp(1) \subset SO(4m)$, quaternion-Kähler manifolds,

6) $n = 7$ and $Hol(M) = G_2 \subset SO(7)$,

7) $n = 8$ and $Ho1(M) = Spin(7) \subset SO(8)$.

In the original Berger list there was also the case $n = 16$ and $Hol(M) = Spin(9) \subset SO(16)$. However, in [7, 11] it was proved that in this case M is a symmetric and is isometrical to projective Cayley plane $\mathbb{C}aP^2 = F_4/Spin(9)$.

Thus, to solve the classification problem, one needs to understand which of the groups in the Berger list can be realized as holonomy groups of complete Riemannian manifolds. In this case, two aspects of the classification problem arise: the proof that the Berger group is realized as the holonomy group of the (incomplete) locally defined Riemannian metric; finding the complete Riemannian metric with a given holonomy group. The second problem, especially in the case of constructing a Riemannian metric on a closed manifold, is significantly more difficult. On the other hand, the construction of a complete metric seems to be a reasonable problem, due to the global nature of the holonomy group (we cannot potentially exclude the possibility that loops that can go "far enough" from a fixed point will have a decisive influence on the holonomy group). Furher we briefly go through the Berger list and comment each case.

- Kähler spaces are well studied, and many examples of Kähler spaces with the holonomy group $U(m)$ can be cited [26, 34].

- Riemannian manifolds whose holonomy group is contained in $SU(m)$ are called Calabi – Yau manifolds (the name is connected with the Calabi – Yau theorem, cited below), or special Kähler manifolds. It can be shown that special Kähler manifolds are Ricci-flat [34, 8]. Already from this fact it is clear that the construction of such manifolds is a difficult task. The first example of a complete Riemannian metric with the group $SU(m)$ was constructed by Calabi [14]:

$$d\tilde{s}^2 = \frac{d\rho^2}{1-\frac{1}{\rho^{2n}}} + \rho^2\left(1-\frac{1}{\rho^{2n}}\right)(d\tau - 2A)^2 + \rho^2 ds^2.$$

Here ds^2 is the Fubini – Study metric on the complex projective space $\mathbb{C}P^{n-1}$, A is the 1-form integrating the Kähler form on $\mathbb{C}P^{n-1}$, $\rho \geq 1$, and τ is the angular variable on the circle. It can be shown that metric $d\tilde{s}^2$ is a smooth globally defined special Kähler metric of the n-th tensor degree of the complex Hopf line bundle over $\mathbb{C}P^{n-1}$. Note the Calabi metric in such a form was found in [37], a similar approach to the construction of this metric was used in [9].

It became possible to show the existence of special Kähler metrics on compact manifolds after Yau proved the Calabi conjecture [39]: *a compact Kähler manifold with zero first Chern class admits a special Kähler metric whose Kähler form is cohomological to the original Kähler form.* The first and most famous example of such a manifold is an $K3$-surface, which, using the Kummer construction, can be represented as follows.

Consider the involution of the flat torus T^4 arising from the central symmetry of the Euclidean space $\mathbb{R}^4$. After factorisation, we obtain an orbifold with 16 singular points whose neighborhoods are locally isometric to $\mathbb{C}^2/\mathbb{Z}_2$. The blowup construction produced in every singular point gives two-dimensional complex manifold - $K3$ surface. Since its first Chern class is equal to zero, there is a special Kähler metric on $K3$ by the Calabi – Yau theorem. Moreover, the moduli space of such metrics has a dimension of 58. A geometric explanation of this dimension, as well as a "qualitative" description of special Kähler metrics on the $K3$, was given by Page [36]. The central role in the Page construction is played by the Eguchi – Hanson metric [23], which is obtained from above $d\tilde{s}^2$ for $n = 2$:

$$ds^2 = \frac{dr^2}{1-\frac{1}{r^4}} + r^2\left(1-\frac{1}{r^4}\right)(d\psi + \cos\theta d\varphi)^2 + r^2(d\theta^2 + sin^2\theta d\varphi^2)$$

(here φ, ψ are spherical coordinates). This metric is a metric with the holonomy group

$SU(2)$ on T^*S^2 and asymptotically looks like a flat metric on $\mathbb{C}^2/\mathbb{Z}_2$. Topologically, the construction of the blow-up of a singular point in $T^4/\mathbb{Z}_2$ is structured as follows: we need to cut off the singularity and identify its neighborhood with the total space of the disk subbundle in T^*S^2 without the zero section S^2.

Page suggested to consider on T^*S^2 a metric homothetic to the Eguchi – Hanson metric with a sufficiently small homothety coefficient, so that at the boundary of the glued disk bundle, the metric becomes arbitrarily close to flat. After this, it is necessary to slightly deform the metric on the torus so as to obtain a smooth metric on the $K3$-surface with the holonomy $SU(2)$. A simple calculation of the degrees of freedom during this operation shows that in this way a 58-dimensional family of metrics is obtained, which coincides with the known results on the dimension of the moduli space of such metrics [39].

- Hyperkähler manifolds are also Kalabi-Yau, but their holonomy group is smaller than $SU(2m)$ and coincides with $Sp(m)$. The Calabi-Yau theorem can also be used to construct them, and moreover, the construction of hyperkähler manifolds turned out to be easier than special Kähler manifolds. Details can be found in [31]. Note that the first complete Riemannian hyperkähler metric was found by Calabi [14].

- Quaternion-Kähler manifolds are interesting in that they are Einstein (not being generally Kähler). A classic example is the quaternionic projective spaces $\mathbb{H}P^n$, which are symmetric. There is a hypothesis (still not proved) that the only compact quaternion-Kähler manifolds are quaternionic projective spaces. In the non-compact case, there are many homogeneous quaternion-Kähler spaces classified in [7, 19].

- Finally, the last remaining cases on the Berger list are $Hol(M) = G_2$ and $Hol(M) = Spin(7)$. They are of particular interest for thesis. For a long time there were no Riemannian metric with exceptional holonomy groups was known. Only in 1987 examples of incomplete (locally defined) metrics with holonomy groups $Spin(7)$ and G_2 were constructed by Bryant in [12]. Then, in 1989, Bryant and Salamon [13] constructed the first examples of complete Riemannian metrics with exceptional holonomy on non-compact spaces. And only in 1996, Joyce [29, 30] with the help of a construction that goes back to Page and a rather delicate analysis was able to prove the existence of compact examples. A systematic presentation of Joyce's results can be found in [31]. Kovalev constructed new examples of compact manifolds with the holonomy group G_2, different from the Joyce examples, using the construction of a connected sum using Fano three-dimensional surfaces [35].

At the moment, the question of existing of Riemannian metrics with the holonomy groups $Spin(7)$ and G_2 on prescribed manifold (compact or noncompact) remains unclear, for example, the works of Joyce and Kovalev give a finite number of compact manifolds, admitting metrics with exceptional holonomy groups, and it is still unknown whether the number of topological types of such spaces can be infinite.

A new interest in noncompact manifolds with special holonomy has arisen relatively recently from mathematical physics. The use of noncompact metrics with $Spin(7)$-holonomy in the so-called M-theory has been suggested. A number of new complete examples were constructed in [20, 21, 22, 25, 32, 33], some of which are not manifolds, but orbifolds. All these metrics are automatically Ricci-flat and asymptotically behave either as cones or as products of cones on a circle. All the constructed examples are of homogeneity one, i.e. are stratifies on homogeneous 7-dimensional fibres.

Noncompact Riemannian manifolds with special holonomy groups (the thesis is devoted to this case) occupy their own position in the theory of holonomy groups of Riemannian spaces, and the importance of studying them is motivated by the following reasons. The Calabi-Yau theorem, although it provides a comprehensive answer to the question of the existence of special Kähler metrics, the question of the structure of such metrics remains unclear. There is no explicit construction of Calabi - Yau metrics on closed manifolds; and the Calabi – Yau theorem also does not clarify the "qualitative" structure of such metrics. Perhaps the only approach is related to the Page method described above for constructing Calabi - Yau metrics on a $K3$ surface: indeed, in this case, we can quite accurately understand how the metric is constructed with the holonomy group $SU(2)$ (at least near a singular plane metrics on $T^4/\mathbb{Z}_2$). Moreover, in the Page method, the explicit form of the Eguchi - Hanson metric on the noncompact manifold T^*S^2 is of fundamental importance. This example is model in a certain sense: Joyce, when constructing his metrics, used this idea.

Summarizing, we can say that for a qualitative understanding of metrics with special holonomy groups, metrics on noncompact manifolds are useful, because: firstly, the equations for them are much simpler and can be solved either explicitly, or there is a good qualitative description of the solutions; secondly, one can model using them metrics on compact manifolds (for example, in the spirit of Page's construction); thirdly, from the point of view of mathematical physics, metrics on noncompact manifolds (or orbifolds) are of a great interest.

Now describe briefly the results included in present thesis.

In [1, 2] the author suggested a general construction that allows one to construct metrics with the holonomy group $Spin(7)$ from a given 3-Sasakian 7-dimensional manifold M. The idea is as follows. If we choose a 3-Sasakian manifold M, then the cone over M is hyperkähler, i.e. will have the holonomy group $Sp(2) \subset Spin(7)$. We deform the cone metric so as to resolve the singularity at the vertex of the cone and obtain a metric whose holonomy group remains in $Spin(7)$. Moreover, the functions $A_1(t), A_2(t), A_3(t), B(t)$ are responsible for the deformation, depending on the radial variable t varying along the generator of the cone (Paper **A**, Theorem of Section 4; Paper **B**, Theorem 1).

In paper [3] author (jointly with E. Malkovich) give an explicit construction, in algebraic form, of a one- parameter family of complete Riemannian metrics 'connecting' two Calabi metrics: Calabi-Yau metric on m-th tensor power of canonical complex line bundle over $\mathbb{C}P^m$ and hyper Kaeler metric over $T^*\mathbb{C}P^m$ (Paper **C**, Theorems 1 and 2).

In [4] the author (jointly with E.G. Malkovich) used adea of this construction to smooth resolution of the cone metric over the twistor space Z associated with M.

Namely, for every 3-Sasakian manifold M, a twistor space Z defined as a quotient space of M by the action of a circle corresponding to the Reeb vector field is well known. Z is an orbifold with the Kähler – Einstein metric [34]. We consider metrics that are natural resolutions of the standard conical metric over Z (Paper **D**, Theorem in section 2).

The Riemannian metrics of homogeneity 2 on four-dimensional noncompact manifolds were investigated in [5]. This means that the desired metrics must be invariant under the action of the two-dimensional torus T^2, i.e. are invariant metrics on T^2-manifolds. As a result of studying the structure of the Ricci curvature of such metrics, Riemannian metrics generalising the Eguchi – Hanson metric were found (Paper **E**, Theorem in Section2).

As an application, in [6] we consider the representation of a $K3$ surface as a blow-up of the singularities of the $T^4/\mathbb{Z}_p$ orbifold with a prime $p > 2$. It turns out that the only possible case is $p = 3$, where 9 singular points of the form $\mathbb{C}^2/\mathbb{Z}_3$ must be cutted of and resolved with help of above mentioned generalised Eguchi-Hanson metrics. A simple calculation of the degrees of freedom of our construction shows that the dimension of the resulting family is 58 (paper **F**, Theorem 2 in Section 3).

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ON THE NEW EXAMPLES OF COMPLETE NONCOMPACT Spin(7)-HOLONOMY METRICS

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Abstract: We construct some complete Spin(7)-holonomy Riemannian metrics on the noncompact orbifolds that are $\mathbb{R}^4$-bundles with an arbitrary 3-Sasakian spherical fiber M. We prove the existence of the smooth metrics for $M = S^7$ and $M = SU(3)/U(1)$ which were found earlier only numerically.

Keywords: exceptional holonomy groups, 3-Sasakian manifold

§ 1. Introduction and Description of the Main Results

One of the interesting problems of modern differential geometry is the problem of constructing and studying the Riemannian metrics with the exotic holonomy groups G_2 and Spin(7). In this article we consider only the case of Spin(7).

The exotic holonomy groups occupy a special place in Berger's list since the question was open for a long time of existence of metrics with these holonomy groups. The first complete (noncompact) example was constructed explicitly in 1989 in [1]. Existence of compact spaces was proven by Joyce [2] in 1996. Joyce's construction gives no explicit description of the metrics, while their existence follows from a rather sophisticated analysis.

A new wave of interest in noncompact examples has been arisen just recently in mathematical physics. It was proposed to use the noncompact Spin(7)-holonomy metrics in the so-called M-theory. In [3–8] a series of new complete examples was constructed; some of them use orbifolds rather than manifolds. All these metrics are automatically Ricci-flat and behave asymptotically as either cones or the products of cones and circles (asymptotically locally conical metrics or ALC-metrics for short). All available examples represent the metrics of cohomogeneity one, i.e., fiber into 7-dimensional homogeneous fibers.

On the other hand, studying the question of existence of noncompact examples is also interesting for geometry itself, since we cannot exclude the possibility of constructing further compact examples from those noncompact by a construction similar to that of Kummer.

In this article we propose a general construction that enables us to construct the Spin(7)-holonomy metrics from a given 7-dimensional 3-Sasakian manifold M. The idea is as follows: If M is a 3-Sasakian manifold then the cone over M has the holonomy group $Sp(2) \subset \mathrm{Spin}(7)$. We deform the conical metric of the cone so as to resolve the singularity at the apex of the cone and obtain a metric with the holonomy group not broader than Spin(7). Deformation is governed here by the functions $A_1(t)$, $A_2(t)$, $A_3(t)$, and $B(t)$ depending on the radial variable t varying along the generator of the cone.

In more detail, consider a 3-Sasakian bundle $M \to \mathscr{O}$ with the common fiber diffeomorphic to either S^3 or $SO(3)$, over a quaternion Kähler orbifold $\mathscr{O}$. With this bundle we can associate the vector $\mathbb{H}$- and $\mathbb{C}$-bundles whose spaces are denoted by $\mathscr{M}_1$ and $\mathscr{M}_2$. The spaces $\mathscr{M}_1$ and $\mathscr{M}_2$ make it possible to resolve the conical singularity in two topologically different ways.

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Moreover, the metric on $\mathscr{M}_1$ and $\mathscr{M}_2$ looks as follows:

$$dt^2 + \sum_{i=1}^{3} A_i(t)^2\eta_i^2 + B(t)^2 g|_{\mathscr{H}}, \tag{$*$}$$

where g is the metric on the 3-Sasakian manifold M, while $\mathscr{H}$ is the distribution of horizontal vectors tangent to M, and η_i is the basis of 1-forms that annihilate $\mathscr{H}$. The condition that the holonomy group of $(*)$ be exotic reduces then to the following system of nonlinear differential equations:

$$\begin{gathered}\dot{A}_1 = \frac{2A_1^2}{B^2} + \frac{(A_2 - A_3)^2 - A_1^2}{A_2A_3}, \quad \dot{A}_2 = \frac{2A_2^2}{B^2} + \frac{(A_3 - A_1)^2 - A_2^2}{A_1A_3}, \\ \dot{A}_3 = \frac{2A_3^2}{B^2} + \frac{(A_1 - A_2)^2 - A_3^2}{A_1A_2}, \quad \dot{B} = -\frac{A_1 + A_2 + A_3}{B}.\end{gathered} \tag{$**$}$$

To obtain regular ALC-metrics, we need to pose some boundary value problem for the system $(**)$: a condition on one boundary must resolve the conical singularity, while the condition on the other must guarantee the required asymptotic behavior.

The examples of Spin(7)-holonomy metrics on $\mathscr{M}_1$ for some particular cases of M were constructed in [3, 4, 6–8]. The numerical analysis, reported in [5], suggests that $\mathscr{M}_2/\mathbb{Z}_4$ may carry metrics for $M = S^7$. Similarly, the results of numerical analysis in [8] indicate that $\mathscr{M}_2/\mathbb{Z}_2$ for $M = SU(3)/S^1$ might carry metrics. Among the other things, we rigorously prove the existence of these metrics. More exactly, the main result of this article is the following assertion:

Theorem. *Let M be a 7-dimensional compact 3-Sasakian manifold, and let $p = 2$ or $p = 4$ depending on whether the common fiber of the 3-Sasakian fibration M is $SO(3)$ or $Sp(1)$. Then the orbifold $\mathscr{M}_2/\mathbb{Z}_p$ carries a one-parameter family of complete regular Riemannian metrics of the form $(*)$ with* Spin(7)-*holonomy which tends at infinity to the product of the cone over the twistor space $\mathscr{Z}$ and the circle S^1.*

Moreover, any other complete regular metric of the form $(*)$ with Spin(7)-*holonomy on $\mathscr{M}_2/\mathbb{Z}_q$ is homothetic to one of the metrics of this family.*

There are many examples of 7-dimensional 3-Sasakian manifolds [9]. The spaces $\mathscr{M}_2/\mathbb{Z}_p$, which are orbifolds in general, are manifolds if M is a regular 3-Sasakian manifold. This is so only for $M = S^7$, $M = \mathbb{R}P^7$, and $M = SU(3)/S^1$. We should observe that the system (6) appeared in [5, 8] as a result of independent computations in various Lie algebras; of course, the coincidence of the equations is explained in this article from the viewpoint of the presence in both cases of the 3-Sasakian structure on homogeneous sections.

The article is organized as follows: In § 2 we give some facts on 3-Sasakian manifolds which we need below. In § 3 we consider deformation of the conical metric of the cone and derive the equations which reduce the holonomy to Spin(7). Here we give conditions which guarantee regularity of the metric which resolves the conical singularity of the cone. In § 4 we formulate and prove the main theorem and § 5 is devoted to justification of the regularity conditions.

§ 2. 3-Sasakian Manifolds

This section contains a survey of the basic necessary results on 3-Sasakian manifolds. The complete proofs and further references can be found in [9].

Let M be some smooth closed Riemannian manifold of dimension m and metric g. The *cone* $\overline{M}$ over M is the manifold $\mathbb{R}_+ \times M$ with metric $\bar{g} = dt^2 + t^2 g$.

The manifold M is called *Sasakian* if the holonomy group of $\overline{M}$ is contained in $U(\frac{m+1}{2})$ (in particular, m is odd). Hence, there is a parallel complex structure J on $\overline{M}$. Identify M with the embedded submanifold $M \times \{1\} \subset \overline{M}$ isometric to M and put $\xi = J(\partial_t)$. The vector field ξ is called the *characteristic* field of M. The characteristic 1-form η of M is defined by the relation $\eta(X) = g(X, \xi)$ for all fields X on M.

Lemma 1. *The field ξ is a unit Killing vector field on M.*

Proof. The fact that ξ is a unit field is immediate by definition. Let ∇ and $\overline{\nabla}$ be the Riemannian connections in M and $\overline{M}$. We see that

$$\overline{\nabla}_X Y = \nabla_X Y - t \cdot g(X,Y)\partial_t, \quad \overline{\nabla}_{\partial_t} X = \overline{\nabla}_X \partial_t = \frac{1}{t}X, \quad \overline{\nabla}_{\partial_t}\partial_t = 0$$

for arbitrary vector fields X and Y on M. Then

$$g(\nabla_X \xi, X) = \bar{g}(\nabla_X \xi, X) = \bar{g}(\overline{\nabla}_X \xi, X) = \bar{g}(J(\overline{\nabla}_X \partial_t), X) = \bar{g}(JX, X) = 0$$

for every vector field X on M. Consequently, ξ is a Killing field. The lemma is proven.

If three pairwise orthogonal Sasakian structures are given on M then M becomes 3-Sasakian. More exactly, a manifold M is called 3-*Sasakian* if the metric $\bar{g}$ on $\overline{M}$ is hyper-Kähler; i.e., its holonomy group is contained in $Sp\big(\frac{m+1}{4}\big)$ (in particular, $m = 4n+1$ and $n \geq 1$). The last means that three parallel complex structures J^1, J^2, and J^3 are defined on $\overline{M}$ and satisfy the relations $J^j J^i = -\delta^{ij} + \varepsilon_{ijk}J^k$. As in the Sasakian case we define the characteristic fields ξ^i and the 1-forms η_i:

$$\xi^i = J^i(\partial_t), \quad \eta_i(X) = g(X, \xi^i), \quad i = 1,2,3,$$

for all vector fields X on M.

Lemma 2. *The fields ξ^1, ξ^2, and ξ^3 are unit pairwise orthogonal Killing vector fields on M; moreover,*

$$\nabla_{\xi^i}\xi^j = \varepsilon_{ijk}\xi^k, \quad [\xi^i, \xi^j] = 2\varepsilon_{ijk}\xi^k.$$

Proof. The fact that ξ^i are unit pairwise orthogonal Killing fields is immediate from the definition and the previous lemma. Now,

$$\nabla_{\xi^i}\xi^j = \overline{\nabla}_{\xi^i}\xi^j + \delta^{ij}\partial_t = J^j\overline{\nabla}_{\xi^i}\partial_t + \delta^{ij}\partial_t = (J^jJ^i + \delta^{ij})\partial_t = \varepsilon_{ijk}\xi^k,$$

which implies immediately that

$$[\xi^i, \xi^j] = \nabla_{\xi^i}\xi^j - \nabla_{\xi^j}\xi^i = 2\varepsilon_{ijk}\xi^k.$$

The lemma is proven.

The fields ξ^1, ξ^2, and ξ^3 constitute the Lie subalgebra $sp(1)$ in the algebra of infinitesimal isometries. Hence, the group of all isometries contains either the subgroup $Sp(1)$ or $SO(3)$ whose orbits of action define a 3-dimensional fibration $\mathscr{F}$. The lemma implies that each fiber of $\mathscr{F}$ is a 3-dimensional totally geodesic submanifold of constant curvature 1.

We briefly recall the definition of an orbifold (V-manifold in Satake's terminology [10]). Let $\mathscr{S}$ be a Hausdorff space satisfying the second axiom of countability. A local uniformizing system for an open neighborhood $U \subset \mathscr{S}$ is a triple $(\widetilde{U}, \Gamma, \pi)$, where $\widetilde{U}$ is an open subset in $\mathbb{R}^n$, Γ is a finite group of diffeomorphisms of the neighborhood $\widetilde{U}$, and the projection $\pi : \widetilde{U} \to U$ is invariant under the group Γ and induces the homeomorphism $\tilde{\pi} : \widetilde{U}/\Gamma \to U$.

Suppose now that $\widetilde{U}_1$ and $\widetilde{U}_2$ are two open sets in $\mathbb{R}^n$ and finite groups Γ_1 and Γ_2 act by diffeomorphisms on $\widetilde{U}_1$ and $\widetilde{U}_2$. A continuous mapping $f : \widetilde{U}_1/\Gamma_1 \to \widetilde{U}_2/\Gamma_2$ is *smooth* if, for each point $p \in \widetilde{U}_1$, there exist neighborhoods V_1 and V_2 of the points p and $f(p)$ and local uniformizing systems $(\widetilde{V}_1, \Gamma_1, \pi_1)$ and $(\widetilde{V}_2, \Gamma_2, \pi_2)$ for V_1 and V_2 such that the mapping $f|_{V_1}$ has a smooth lifting $\tilde{f} : \widetilde{V}_1 \to \widetilde{V}_2$ invariant under the action of the groups Γ_1 and Γ_2. Similarly, we define the notion of submersion, immersion, diffeomorphism, and so on.

A *smooth V-atlas* for $\mathscr{S}$ is a covering of $\mathscr{S}$ by open sets U_i together with local uniformizing systems $(\widetilde{U}_i, \Gamma_i, \pi_i)$ such that the mapping $\mathrm{Id} : U_i \cap U_j \to U_i \cap U_j$ is a diffeomorphism (in the sense of the above definition). The space $\mathscr{S}$, together with a complete V-atlas, is called a *V-manifold* or *orbifold*. Obviously, we can define the notions of principal V-bundle, V-bundle associated with the principal bundle, differential form, Riemannian metric, Riemannian submersion, and so on.

A Riemann orbifold $\mathscr{O}$ is called a *quaternion Kähler orbifold* if the V-bundle of endomorphisms of the tangent space contains a parallel V-subbundle $\mathscr{I}$ of dimension 3 which is locally generated by the almost complex structures I^1, I^2, and I^3 satisfying the relations of the quaternion algebra and the bundle $\mathscr{I}$ is invariant under the action of the local uniformizing group $\mathscr{O}$.

Theorem 1. *Let M be a $(4n+3)$-dimensional closed 3-Sasakian manifold with the 3-dimensional fibration $\mathscr{F}$ defined above. Then the space of the sheets of the fibration $\mathscr{F}$ carries the structure of a $4n$-dimensional quaternion Kähler orbifold $\mathscr{O}$ such that the natural projection $\pi : M \to \mathscr{O}$ is a Riemannian submersion and the principal V-bundle having the structure group $Sp(1)$ or $SO(3)$. The common fiber π is isometric to either $Sp(1)$ or $SO(3)$.*

PROOF. Denote by $\mathscr{V}$ the 3-dimensional subbundle in TM generated by the characteristic fields ξ^1, ξ^2, and ξ^3. Let $TM = \mathscr{V} \oplus \mathscr{H}$ be the orthogonal decomposition with respect to the metric g. The subbundle $\mathscr{V}$ is called the *bundle of vertical vectors* and $\mathscr{H}$ is called the *bundle of horizontal vectors*.

Take $p \in M$. Suppose that the stabilizer of p under the action of $Sp(1)$ is a discrete subgroup Γ in $Sp(1)$; i.e., the sheet $\mathscr{F}_p$ passing through the point p is isometric to $Sp(1)/\Gamma$. Put

$$U = \{\exp_p(tX) \mid X \in \mathscr{H}_p,\ |X| = 1,\ 0 \le t \le \varepsilon\},$$

where $\varepsilon > 0$ is chosen so small that $\varepsilon < \operatorname{inj}(M)$, $\mathscr{F}_p$ intersects U once at the point p, and each sheet of the fibration $\mathscr{F}$ intersects U at most finitely many times. Then U is homeomorphic to $\mathbb{R}^{4n}$ and the group Γ acts on U by isometries according to the rule

$$\gamma \in \Gamma : \exp_p(tX) \mapsto \exp_p(t d_p\gamma(X)).$$

It is easy to see that the neighborhood $\mathscr{O}$ constituted by the sheets intersecting U is homeomorphic to U/Γ and the system of these neighborhoods constructed for all points p determines a uniformizing atlas on $\mathscr{O}$.

It is obvious that the metric g on M has the form

$$g = \sum_{i=1}^{3} \eta_i^2 + g|_{\mathscr{H}},$$

where $g|_{\mathscr{H}}$ is the restriction of g to the horizontal subbundle. Consider the projection $\pi : M \to \mathscr{O}$. Since the metric g is invariant under the action of $Sp(1)$, there is a Riemannian metric $g_{\mathscr{O}}$ on the orbifold $\mathscr{O}$ such that, for every point $p \in M$, the restriction $d\pi_p : \mathscr{H}_p M \to T_{\pi(p)}\mathscr{O}$ is an isometry; moreover, $d\pi^*(g_{\mathscr{O}}) = g|_{\mathscr{H}}$. Thus, the projection π becomes a Riemannian submersion and to each vector field Y on $\mathscr{O}$ there corresponds a unique horizontal $Sp(1)$-invariant vector field X on M such that $d\pi(X) = Y$. The Levi-Civita connection of the metric $g_{\mathscr{O}}$ is obtained by projection of the Levi-Civita connection of the metric g to $\mathscr{H}$ [11]. Now, if X is a horizontal vector field then

$$g(J^i(X), \xi^j) = g(X, \varepsilon_{ijk}\xi^k) = 0.$$

Thus, the operators J^1–J^3 on $\mathscr{H}$ take horizontal vectors into horizontal vectors and determine a quaternion structure on the orbifold $\mathscr{O}$.

Define the 2-form on M as follows:

$$\omega_i = d\eta_i + \sum_{j,k} \varepsilon_{ijk}\eta_j \wedge \eta_k, \quad i = 1, 2, 3.$$

We see immediately that

$$\begin{aligned} \omega_i(X,Y) &= \frac{1}{2}(X\eta_i(Y) - Y\eta_i(X) - \eta_i([X,Y])) = \frac{1}{2}\eta_i(-\nabla_X Y + \nabla_Y X) \\ &= \frac{1}{2}(g(\nabla_X \xi^i, Y) - g(\nabla_Y \xi^i, X)) = g(J^i(X), Y), \\ &\omega_i(X, \xi^j) = 0, \quad \omega_i(\xi^j, \xi^k) = 0 \end{aligned}$$

for arbitrary horizontal vector fields X and Y. Thus, the forms ω_i are obtained by lowering of the index from the restrictions of the operators J^i to $\mathscr{H}$.

Now,

$$L_{\xi^i}\eta_j(X) = \xi^i g(X,\xi^j) - g(\xi^j,[\xi^i,X]) = g(\nabla_{\xi^i}X,\xi^j) + g(X,\nabla_{\xi^i}\xi^j)$$
$$-g(\xi^j,[\xi^i,X]) = g(\nabla_X\xi^i,\xi^j) + g(X,\nabla_{\xi^i}\xi^j) = g(\overline{\nabla}_X J^i\partial_t,\xi^j)$$
$$+g(X,\overline{\nabla}_{\xi^i}J^j\partial_t) = -g(X,J^i\xi^j) + g(X,J^j\xi^i)$$
$$= 2g(X,J^jJ^i\partial_t) = 2g(X,\varepsilon_{ijk}\xi^k) = 2\varepsilon_{ijk}\eta_k(X).$$

Thus,

$$L_{\xi^i}\eta_j = 2\varepsilon_{ijk}\eta_k. \tag{1}$$

Differentiating (1), we obtain

$$L_{\xi^i}d\eta_j = 2\varepsilon_{ijk}d\eta_k. \tag{2}$$

Using (2) and the relation $L_{\xi^i}(\eta_j\wedge\eta_k) = L_{\xi^i}\eta_j\wedge\eta_k + \eta_j\wedge L_{\xi^i}\eta_k$, we infer $L_{\xi^i}\omega_j = 2\varepsilon_{ijk}\omega_k$. Therefore, the space of the forms generated by ω_i is invariant under the action of $Sp(1)$ which means that the space generated by the operators $J^i|_{\mathscr{H}}$ is $Sp(1)$-invariant, too, and can be descended to $\mathscr{O}$. Thus, in the bundle $\mathrm{End}(T\mathscr{O})$ we can define a 3-dimensional subspace locally generated by the almost complex structures J^1–J^3.

For horizontal fields X and Y we now obtain

$$\mathscr{H}(\nabla_X J^i)(Y) = \mathscr{H}(\nabla_X(J^iY) - J^i(\nabla_XY)) = \mathscr{H}\overline{\nabla}_X(J^iY) - \mathscr{H}J^i(\nabla_XY)$$
$$= \mathscr{H}J^i(\overline{\nabla}_XY)) - \mathscr{H}J^i(\nabla_XY) = 0.$$

Hence, we can conclude readily that the distribution of these subspaces is parallel along $\mathscr{O}$.

The proof of the assertion on the common fiber of the bundle π can be found in [9]. The theorem is proven.

The field ξ^1 corresponds to the subgroup S^1 either in $Sp(1)$ or $SO(3)$. Thus, we can consider the one-dimensional fibration $\mathscr{F}'$ on M generated by the field ξ^1. By analogy with the above theorem, we can prove that on the space of fibers of the fibration $\mathscr{F}'$ we can introduce the structure of the 6-dimensional Riemann orbifold $\mathscr{Z}$ which agrees with the Riemannian submersion $\pi' : M \to \mathscr{Z}$. It is well known that the metric on $\mathscr{Z}$ is a Kähler–Einstein metric [9]. The orbifold $\mathscr{Z}$ is called the *twistor space* of the manifold M.

With each 3-Sasakian manifold M we associate two orbifolds $\mathscr{M}_1$ and $\mathscr{M}_2$ which resolve the conical singularity of $\overline{M}$ in the following two ways.

1. Consider the standard action on $\mathbb{R}^4 = \mathbb{H}$ of the group $Sp(1)$ represented by unit quaternions and the corresponding action of $SO(3) = Sp(1)/\mathbb{Z}_2$ on $\mathbb{R}^4/\mathbb{Z}_2$:

$$q \in Sp(1) : x \in \mathbb{H} \mapsto qx \in \mathbb{H}.$$

Let $\mathscr{M}_1$ be the fiber space with fiber $\mathbb{R}^4$ or $\mathbb{R}^4/\mathbb{Z}_2$ associated with the principal bundle $M \to \mathscr{O}$ under the action considered above. Thus, the orbifold $\mathscr{O}$ is embedded in $\mathscr{M}_1$ as the zero fiber and $\mathscr{M}_1\backslash\mathscr{O}$ fibers into the spherical sections diffeomorphic to M and collapsing to the zero fiber $\mathscr{O}$.

2. Let $S \simeq S^1$ be a subgroup either in $Sp(1)$ or $SO(3)$ which integrates the Killing field ξ^1. Consider the action of S on $\mathbb{R}^2 = \mathbb{C}$:

$$e^{i\phi} \in S : z \in \mathbb{C} \to ze^{i\phi} \in \mathbb{C}.$$

The bundle $M \to \mathscr{Z}$ is the principal bundle with the structure group S. Let $\mathscr{M}_2$ be the fiber space with fiber $\mathbb{R}^2$ associated with $\pi' : M \to \mathscr{Z}$. Thus, the orbifold $\mathscr{Z}$ is embedded in $\mathscr{M}_2$ as the zero fiber and $\mathscr{M}_2\backslash\mathscr{Z}$ fibers into spherical sections diffeomorphic to M and collapsing to the zero fiber $\mathscr{Z}$. We need the following modification of this construction: For every natural number p, there is an obvious embedding $\mathbb{Z}_p \subset S$; moreover, $\mathbb{Z}_p$ acts on $\mathscr{M}_2$ by isometries. Consequently, we can correctly define the orbifold $\mathscr{M}_2/\mathbb{Z}_p$ which is a manifold if and only if so is $\mathscr{M}_2$. It is easy to see that $\mathscr{M}_2/\mathbb{Z}_p$ is the bundle with fiber $\mathbb{C}$ associated with the principal bundle $\pi' : M \to \mathscr{Z}$ by means of the action

$$e^{i\phi} \in S : z \in \mathbb{C} \to ze^{ip\phi} \in \mathbb{C}.$$

In the case when the bundle π is regular (i.e., the uniformizing group is trivial) we say that the 3-Sasakian manifold is *regular*. In this case the fiber π is either $S^3 = Sp(1)$ or $SO(3)$ and the orbifolds $\mathscr{O}$ and $\mathscr{Z}$ are smooth manifolds.

Theorem 2 [9]. *If M is a compact regular 3-Sasakian manifold of dimension 7 then M is isometric to one of the following homogeneous spaces: S^7, $\mathbb{R}P^7$, and $SU(3)/T_{1,1}$, where $T_{1,1}$ stands for the circle S^1 embedded in the maximal torus $T^2 \subset SU(3)$ with "weights" $(1,1,-2)$.*

In the case $M = S^7$ both spaces $\mathscr{M}_1$ and $\mathscr{M}_2$ are 8-dimensional smooth manifolds. If $M = \mathbb{R}P^7$ or $M = SU(3)/T_{1,1}$ then the common fiber is equal to $SO(3)$ and only the corresponding spaces $\mathscr{M}_2$ are manifolds.

§ 3. Construction of Spin(7)-Holonomy Metrics

Let $\{e^i\}$, $i = 0, 1, \dots, 7$, be an orthonormal basis of 1-forms on the standard Euclidean space $\mathbb{R}^8$. Put $e^{ijkl} = e^i \wedge e^j \wedge e^k \wedge e^l$ and define the 4-form Φ_0 on $\mathbb{R}^8$ as follows:

$$\Phi_0 = e^{0123} + e^{4567} + e^{0145} - e^{2345} - e^{0167} + e^{2367} + e^{0246}$$
$$+e^{1346} - e^{0275} + e^{1357} + e^{0347} - e^{1247} - e^{0356} + e^{1256}.$$

Let N be an 8-dimensional oriented Riemannian manifold. We say that a differential form $\Phi \in \Lambda^4 N$ *determines the* Spin(7)*-structure on* N if, in a neighborhood of each point $p \in N$, there is an orientation-preserving isometry $\phi_p : T_pN \to \mathbb{R}^8$ such that $\phi_p^*\Phi_0 = \Phi|_p$. If the form Φ is parallel then the holonomy group of the Riemannian manifold N reduces to the subgroup Spin(7) $\subset SO(8)$. It is well known [12] that Φ is parallel if and only if it is closed (coclosure follows automatically from the fact that the form Φ is selfadjoint with respect to the Hodge operator):

$$d\Phi = 0. \qquad (3)$$

As above, suppose that M is a 7-dimensional compact 3-Sasakian manifold. Consider the following metric on $(0, \infty) \times M$:

$$\bar{g} = dt^2 + \sum_{i=1}^{3} A_i(t)^2 \eta_i^2 + B(t)^2 g|_{\mathscr{H}}, \qquad (4)$$

where the functions $A_i(t)$ and $B(t)$ are defined on the interval $(0, \infty)$. Locally, we can choose an orthonormal system of 1-forms η_4, η_5, η_6, and η_7 which generate the annihilator of the horizontal subbundle $\mathscr{H}$ so that

$$\omega_1 = 2(\eta_4 \wedge \eta_5 - \eta_6 \wedge \eta_7), \quad \omega_2 = 2(\eta_4 \wedge \eta_6 - \eta_7 \wedge \eta_5), \quad \omega_3 = 2(\eta_4 \wedge \eta_7 - \eta_5 \wedge \eta_6).$$

Let $\Omega = \eta_4 \wedge \eta_5 \wedge \eta_6 \wedge \eta_7 = -\frac{1}{8}\omega_1 \wedge \omega_1 = -\frac{1}{8}\omega_2 \wedge \omega_2 = -\frac{1}{8}\omega_3 \wedge \omega_3$.

Consider the following 4-form:

$$\Phi = e^0 \wedge e^1 \wedge e^2 \wedge e^3 + B^4\Omega + \frac{1}{2}B^2(e^0 \wedge e^1 - e^2 \wedge e^3) \wedge \omega_1$$
$$+\frac{1}{2}B^2(e^0 \wedge e^2 - e^3 \wedge e^1) \wedge \omega_2 + \frac{1}{2}B^2(e^0 \wedge e^3 - e^1 \wedge e^2) \wedge \omega_3,$$

where

$$e^0 = dt, \quad e^i = A_i\eta_i, \ i = 1, 2, 3, \quad e^j = B\eta_j, \ j = 4, \dots, 7.$$

It is obvious that the form Φ is defined globally on $\overline{M}$ and coincides locally with Φ_0.

Using the obvious identities

$$d\eta_i = \omega_i - 2\eta_{i+1} \wedge \eta_{i+2},$$
$$d\omega_i = 2d(\eta_{i+1} \wedge \eta_{i+2}) = 2(\omega_{i+1} \wedge \eta_{i+2} - \eta_{i+1} \wedge \omega_{i+2}), \quad i = 1,2,3 \bmod 3,$$

we obtain the relations that close the exterior algebra of forms considered above:

$$\begin{aligned} de^0 &= 0, \\ de^i &= \frac{A_i'}{A_i} e^0 \wedge e^i + A_i \omega_i - \frac{2A_i}{A_{i+1}A_{i+2}} e^{i+1} \wedge e^{i+2}, \quad i = 1,2,3 \bmod 3, \\ d\omega_i &= \frac{2}{A_{i+2}} \omega_{i+1} \wedge e^{i+2} - \frac{2}{A_{i+1}} e^{i+1} \wedge \omega_{i+2}, \quad i = 1,2,3 \bmod 3. \end{aligned} \tag{5}$$

The following assertion is obtained by straightforward calculations with (5).

Lemma 3. *Condition (3) is equivalent to the system of ordinary differential equations*

$$\begin{aligned} A_1' &= \frac{2A_1^2}{B^2} + \frac{(A_2 - A_3)^2 - A_1^2}{A_2 A_3}, \quad A_2' = \frac{2A_2^2}{B^2} + \frac{(A_3 - A_1)^2 - A_2^2}{A_1 A_3}, \\ A_3' &= \frac{2A_3^2}{B^2} + \frac{(A_1 - A_2)^2 - A_3^2}{A_1 A_2}, \quad B' = -\frac{A_1 + A_2 + A_3}{B}. \end{aligned} \tag{6}$$

Under certain boundary conditions, the metric (4) gives a smooth Riemannian metric on $\mathscr{M}_1$ or $\mathscr{M}_2$. We now find out these conditions.

Lemma 4. *Let $A_i(t)$, $i = 1,2,3$, and $B(t)$ be a solution to (6) C^∞-smooth on $[0,\infty)$. Then (4) extends to a smooth metric on $\mathscr{M}_1$ if and only if the following conditions are met:*

(1) *$A_1(0) = A_2(0) = A_3(0) = 0$ and $|A_1'(0)| = |A_2'(0)| = |A_3'(0)| = 1$;*
(2) *$B(0) \neq 0$ and $B'(0) = 0$;*
(3) *the functions A_1, A_2, A_3, and B have definite sign on $(0,\infty)$.*

Lemma 5. *Suppose that the conditions of the above lemma are satisfied and $p = 4$ or $p = 2$ depending on whether the common fiber M is isometric to $Sp(1)$ or $SO(3)$. For (4) to admit extension to a smooth metric on $\mathscr{M}_2/\mathbb{Z}_p$, it is necessary and sufficient that the following be satisfied:*

(1) *$A_1(0) = 0$ and $|A_1'(0)| = 4$;*
(2) *$A_2(0) = -A_3(0) \neq 0$ and $A_2'(0) = A_3'(0)$;*
(3) *$B(0) \neq 0$ and $B'(0) = 0$;*
(4) *the functions A_1, A_2, A_3, and B have definite sign on $(0,\infty)$.*

Lemmas 4 and 5 are proven in § 5.

Before turning to studying (6), we list the exact solutions to this system available so far. If $A_1 = A_2 = A_3$ then (6) can be integrated by elementary methods and we obtain the following metric on $\mathscr{M}_1$ with holonomy Spin(7) [1]:

$$\bar{g} = \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{\frac{10}{3}}} + \frac{9}{25}\left(1 - \left(\frac{r_0}{r}\right)^{\frac{10}{3}}\right) \sum_{i=1}^{3} \eta_i^2 + \frac{9}{5} r^2 g|_{\mathscr{H}}. \tag{7}$$

Observe that (7) was the first example of a complete metric with holonomy Spin(7).

Now, if $A_2 = A_3$ then (6) can also be integrated explicitly, and in this case we arrive at the following metric on $\mathscr{M}_1$:

$$\begin{aligned} \bar{g} = &\frac{(r - r_0)^2}{(r + r_0)(r - 3r_0)} dr^2 + 4r_0^2 \frac{(r + r_0)(r - 3r_0)}{(r - r_0)^2} \eta_1^2 \\ &+ (r + r_0)(r - 3r_0)\left(\eta_2^2 + \eta_3^2\right) + 2\left(r^2 - r_0^2\right) g|_{\mathscr{H}}. \end{aligned} \tag{8}$$

The metric (8) was found in [3] for $M = S^4$. Observe that (6) was studied completely under the condition $A_2 = A_3$ in [3, 4]; moreover, explicit solutions were found in quadratures which led, in particular, to regular metrics on $\mathscr{M}_1$. Here we do not present the corresponding expressions in view of their bulkiness.

Finally, if $A_2 = -A_3$ then we obtain the following metric on $\mathscr{M}_2/\mathbb{Z}_p$ (where $p = 4$ or $p = 2$ depending on the common fiber M, as in Lemma 5) having the holonomy group $SU(4) \subset \mathrm{Spin}(7)$:

$$\bar{g} = \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^8} + r^2\left(1 - \left(\frac{r_0}{r}\right)^8\right)\eta_1^2 + r^2\left(\eta_2^2 + \eta_3^2\right) + r^2 g|_{\mathscr{H}}. \tag{9}$$

To the author's best knowledge, this metric was first described in [13, 14].

§ 4. Existence of Metrics on $\mathscr{M}_2$

The following definitions characterize the behavior of the metrics in question at infinity. The metric (4) is *conical* if the functions $A_i(t)$ and $B(t)$ are linear in t and none of them is a constant function. For example, the metric on $\overline{M}$ is obtained for $A_i = B = t$. Metric (4) is *locally conical* if the functions $A_i(t)$ and $B(t)$ are linear in t. Such metrics look locally like the direct product of a conical metric and a metric independent of t. Finally, the metric (4) determined by $A_i(t)$ and $B(t)$ is *asymptotically* (*locally*) *conical* (ALC or AC for short) if there exist functions $\widetilde{A}_i(t)$ and $\widetilde{B}(t)$ that determine a (locally) conical metric such that

$$\left|1 - \frac{A_i(t)}{\widetilde{A}_i(t)}\right| \to 0 \text{ as } t \to \infty, \ \ i = 1, 2, 3; \qquad \left|1 - \frac{B(t)}{\widetilde{B}(t)}\right| \to 0 \text{ as } t \to \infty.$$

It is easy to see that all metrics (7)–(9) listed above are ALC.

The goal of this article is to prove the following theorem:

Theorem. *Let M be a 7-dimensional compact 3-Sasakian manifold, and put $p = 2$ or $p = 4$ depending on whether the common fiber of the 3-Sasakian fibration M is $SO(3)$ or $Sp(1)$. Then the orbifold $\mathscr{M}_2/\mathbb{Z}_p$ carries the following complete regular Riemannian metrics $\bar{g}$ of the form (4) with the holonomy group $H \subset \mathrm{Spin}(7)$:*

(1) if $A_1(0) = 0$ and $-A_2(0) = A_3(0) = B(0) > 0$ then the metric $\bar{g}$ has the holonomy group $SU(4) \subset \mathrm{Spin}(7)$ and coincides with the AC-metric (9);

(2) for each collection of the initial values $A_1(0) = 0$ and $0 < -A_2(0) = A_3(0) < B(0)$, there is a regular ALC-metric $\bar{g}$ with holonomy $\mathrm{Spin}(7)$. At infinity these metrics tend to the product of a cone over the twistor space $\mathscr{Z}$ and the circle S^1.

Moreover, every complete regular metric of the form (4) on the space $\mathscr{M}_2/\mathbb{Z}_q$ with the holonomy group $H \subset \mathrm{Spin}(7)$ is isometric to one of those indicated above.

The remainder of the section is devoted to a proof of this theorem. We start with a sketch of the proof. We propose to pass from the system (6) on the metric $\bar{g}$ to the system on the conformal class of the metric $\bar{g}$. To this end, we normalize the vector-function $(A_i(t), B(t))$ and obtain the dynamical system (10) on S^3. Moreover, the metric $\bar{g}$ itself is reconstructed from its conformal class. It turns out that, for a metric $\bar{g}$ to be ALC, it is necessary that the trajectory of the normalized system tend to a stationary point (or a conditionally stationary point whose notion is introduced below). Now, we prove that for given initial data dictated by the regularity conditions, there is a outgoing trajectory of the normalized system. Finally, using some specially chosen directing functions of the normalized system, we find out the asymptotic behavior of the trajectories and prove convergence to (conditionally) stationary points.

Consider the standard space $\mathbb{R}^4$ and denote by $R(t) \in \mathbb{R}^4$ the vector with entries $A_1(t)$, $A_2(t)$, $A_3(t)$, and $B(t)$. Let $V : \mathbb{R}^4 \to \mathbb{R}^4$ be the function of R defined by the right-hand side of (6) (of course, the function V is defined only in the domain, where $A_i, B \neq 0$). Thus, (6) has the form $\frac{dR}{dt} = V(R)$. Using invariance of V under the homotheties of $\mathbb{R}^4$, we execute the change $R(t) = f(t)S(t)$, where

$$|S(t)| = 1, \quad f(t) = |R(t)|, \quad S(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t)).$$

Thus, we "normalize" the vector-function R and the system splits into the "radial" and "tangential" parts:

$$\frac{dS}{du} = V(S) - \langle V(S), S\rangle S = W(S), \tag{10}$$

$$\frac{1}{f}\frac{df}{du} = \langle V(S), S\rangle, \quad dt = f\,du. \tag{11}$$

Consequently, we first have to solve the autonomous system (10) on the 3-dimensional sphere $S^3 = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) | \sum_{i=1}^4 \alpha_i^2 = 1\}$ and we then find solutions to (6) from (11) by usual integration.

Lemma 6. *System* (10) *admits the discrete symmetry group* $S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *generated by the following transformations:*

$$\sigma \in S_3 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto (\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}, \alpha_4),$$
$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto (\alpha_1, \alpha_2, \alpha_3, -\alpha_4),$$
$$(\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha_4(u)) \mapsto (-\alpha_1(-u), -\alpha_2(-u), -\alpha_3(-u), \alpha_4(-u))$$

(*we denote by* S_3 *the symmetry group*).

Consider the 2-dimensional "equators" on S^3:

$$E_i = \{S \in S^3 \mid \alpha_i = 0\}, \quad i = 1, \dots, 4,$$
$$E_{ij}^+ = \{S \in S^3 \mid \alpha_i + \alpha_j = 0\}, \quad i, j = 1, 2, 3, \ i \neq j,$$
$$E_{ij}^- = \{S \in S^3 \mid \alpha_i - \alpha_j = 0\}, \quad i, j = 1, 2, 3, \ i \neq j.$$

Each of them is a standardly embedded 2-dimensional sphere $S^2 \subset S^3$. The following lemma is immediate from the symmetry of (10) under rearrangements of A_i.

Lemma 7. *The sphere* E_{ij}^- *is an invariant surface of the dynamical system determined by* (10).

Lemma 8. *The stationary solutions to* (10) *on* S^3 *are exhausted by the following list of zeros of the vector field* W*:*

$$\pm(\sqrt{2}/4, \sqrt{2}/4, \sqrt{2}/4, \pm\sqrt{10}/4), \ \pm(-1/2, -1/2, 1/2, \pm 1/2),$$
$$\pm(-1/2, 1/2, -1/2, \pm 1/2), \ \pm(1/2, -1/2, -1/2, \pm 1/2).$$

A point $S \in S^3$ at which the field W is undefined is called a *conditionally stationary point* if there is a smooth curve $\gamma(u)$ on S^3, $u \in (-\varepsilon, \varepsilon)$, $\gamma(0) = S$, such that the field W is defined at all points $\gamma(u)$, $u \in (-\varepsilon, \varepsilon)$, $u \neq 0$, and $\lim_{u\to 0} W(\gamma(u)) = 0$.

Lemma 9. *System* (10) *possesses the following conditionally stationary points on* S^3*:*

$$\pm(0, 1/2, 1/2, \pm 1/\sqrt{2}), \ \pm(1/2, 0, 1/2, \pm 1/\sqrt{2}), \ \pm(1/2, 1/2, 0, \pm 1/\sqrt{2}).$$

Proof. Suppose that a point $S = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\sum_{i=1}^4 \alpha_i^2 = 1$, is a conditionally stationary point; i.e., there is a curve $\gamma(u)$, $u \in (-\varepsilon, \varepsilon)$, with the above properties. It is obvious that the finite limit $\lim_{u\to 0} W(\gamma(u))$ exists only in the following three cases to within the symmetries described in Lemma 6: (1) $\alpha_1 = 0$ and $\alpha_2 = \alpha_3$; (2) $\alpha_1 = 0$ and $\alpha_2 = -\alpha_3$; (3) $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

1. Put $\lim_{u\to 0} \frac{\alpha_2(u) - \alpha_3(u)}{\alpha_1(u)} = h$. We immediately verify then that

$$\lim_{u\to 0} W_1(\gamma(u)) = 0, \quad \lim_{u\to 0} W_2(\gamma(u)) = \frac{2\alpha_2^2}{\alpha_4^2} - 2 - 2h - \alpha_2\left(\frac{4\alpha_2^3}{\alpha_4^2} - 6\alpha_2\right),$$
$$\lim_{u\to 0} W_3(\gamma(u)) = \frac{2\alpha_2^2}{\alpha_4^2} - 2 + 2h - \alpha_2\left(\frac{4\alpha_2^3}{\alpha_4^2} - 6\alpha_2\right),$$
$$\lim_{u\to 0} W_4(\gamma(u)) = -\frac{2\alpha_2}{\alpha_4} - \alpha_4\left(\frac{4\alpha_2^3}{\alpha_4^2} - 6\alpha_2\right).$$

Equating these limits to zero, we obtain the conditionally stationary points $\pm(0, 1/2, 1/2, \pm 1/\sqrt{2})$. The other points are obtained from those above by the symmetries of (10) described in Lemma 6.

2. Put $\lim_{u\to 0} \frac{\alpha_2(u)+\alpha_3(u)}{\alpha_1(u)} = h$. Then

$$\lim_{u\to 0} W_1(\gamma(u)) = -4, \quad \lim_{u\to 0} W_2(\gamma(u)) = \frac{2\alpha_2^2}{\alpha_4^2} - 2 + 2h,$$

$$\lim_{u\to 0} W_3(\gamma(u)) = \frac{2\alpha_2^2}{\alpha_4^2} - 2 + 2h, \quad \lim_{u\to 0} W_4(\gamma(u)) = 0.$$

It is obvious that there are no conditionally stationary points in this case either.

3. Put $\lim_{u\to 0} \frac{\alpha_1(u)}{\alpha_2(u)} = h$ and $\lim_{u\to 0} \frac{\alpha_1(u)}{\alpha_3(u)} = f$. Then

$$\lim_{u\to 0} W_1(\gamma(u)) = -2 + \frac{f}{h} + \frac{h}{f} - hf, \quad \lim_{u\to 0} W_2(\gamma(u)) = -2 + \frac{1}{f} + f - \frac{f}{h^2},$$

$$\lim_{u\to 0} W_3(\gamma(u)) = -2 + \frac{1}{h} + h - \frac{h}{f^2}, \quad \lim_{u\to 0} W_4(\gamma(u)) = 0.$$

We verify immediately that there are no conditionally stationary points in this case either. The lemma is proven.

Lemma 10. *To the stationary solutions to* (10) *there correspond some locally conical metrics on* $\overline{M}$, *whereas to the trajectories of* (10) *tending asymptotically to* (*conditionally*) *stationary solutions there correspond some asymptotically locally conical metrics on* $\overline{M}$.

PROOF. Let S_0 be a stationary solution to (10); i.e., $W(S_0) = 0$. Integrating (11), we obtain $f = e^{c_1 u + c_2}$, where c_1 and c_2 are constants and $c_1 = \langle V(S_0), S_0\rangle$. Then $R(t) = f(t)S_0 = (c_0 + c_1 t)S_0$ for some constant c_0. Thus, $R(t)$ determines a locally conical metric.

Suppose now that S_0 is a (conditionally) stationary solution to (10) and the trajectory $S(u)$ tends to S_0 as $u \to \infty$. It is clear that $W(S(u)) \to 0$ as $u \to \infty$. As above, consider the constant $c_1 = \langle V(S_0), S_0\rangle$. By smoothness of the field $V(S)$ along $S(u)$, we can conclude that $\langle V(S(u)), S(u)\rangle \to c_1$ as $u \to \infty$. Hence, $\frac{d}{du}(\log f(u)) \to c_1$ as $u \to \infty$, and we infer that f cannot vanish as $u \to \infty$. Consequently,

$$t = t_0 + \int_{u_0}^{u} f(\xi)\, d\xi \to \infty \quad \text{as} \quad u \to \infty$$

for some constants u_0 and t_0. Consider the quantity

$$\Delta = \left|1 - \frac{f(t)}{c_1 t}\right| = \frac{|c_1 t - f(t)|}{|c_1 t|}. \tag{12}$$

If the numerator of the right-hand side of (12) does not tend to ∞ then $\Delta \to 0$ as $u \to \infty$. Otherwise

$$\lim_{u\to\infty} \Delta = \left|\lim_{u\to\infty} \frac{c_1 \frac{dt}{du} - \frac{df}{du}}{c_1 \frac{dt}{du}}\right| = \left|\lim_{u\to\infty} \frac{c_1 - \frac{d}{du}(\log f)}{c_1}\right| = 0.$$

Thus, $\Delta \to 0$ as $u \to \infty$, i.e., as $t \to \infty$, too. It remains to note that $R(t) = f(t)S(t)$, where $S(t) \to S_0$ as $t \to \infty$. The lemma is proven.

REMARK. It is easy to see that the trajectory of (10) corresponding to (7) converges to the stationary point $(\sqrt{2}/4, \sqrt{2}/4, \sqrt{2}/4, \sqrt{10}/4)$; the trajectory corresponding to (9) converges to the stationary point $(1/2, 1/2, -1/2, 1/2)$; and the trajectory corresponding to (8) converges to the conditionally stationary point $(0, 1/2, 1/2, 1/\sqrt{2})$.

Consider the circles $C_i^{\pm}$, $i = 1, 2, 3$, standardly embedded in S^3: $C_1^{\pm} = E_1 \cap E_{23}^{\pm}$, $C_2^{\pm} = E_2 \cap E_{31}^{\pm}$, and $C_3^{\pm} = E_3 \cap E_{12}^{\pm}$. Let $Q_{\pm} = (0, 0, 0, \pm 1)$ be the poles of the sphere S^3 at which all these circles intersect.

By Lemma 5, to construct a regular metric on $\mathscr{M}_2/\mathbb{Z}_p$, we need a trajectory of (10) starting at an arbitrary point on the circles C_i^{+} different from the poles $Q_{\pm}$. By Lemma 6, without loss of generality we can take the initial point to be $S_0 = (0, -\lambda, \lambda, \mu)$, where $\lambda, \mu > 0$ and $2\lambda^2 + \mu^2 = 1$. The other solutions are obtained from the above-considered case by the symmetries of (10).

Lemma 11. *For every point* $S_0 = (0, -\lambda, \lambda, \mu) \in C_1^+$ *considered above, there is a unique smooth trajectory of* (10) *starting at* S_0 *in the domain* $\alpha_1 < 0$.

PROOF. Let $J = C_1^+ \cap \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mid \alpha_3 > 0,\ \alpha_4 > 0\}$ be the circle arc containing the point S_0. Denote by U an open ball in $\mathbb{R}^2$ with coordinates $x = \alpha_1$ and $y = \alpha_2 + \alpha_3$ of radius ε centered at the origin. Then in a neighborhood of the arc J we can consider the local coordinates $x, y, z = \alpha_4$. In these coordinates, the field W has the following components:

$$W_x = W_1, \quad W_y = W_2 + W_3, \quad W_z = W_4,$$

where $\widetilde{W}_j(S) = V_j(S) - \langle V(S), S\rangle \alpha_j$, $j = 1, 2, 3, 4$,

$$\begin{aligned} S &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \left(x, \frac{1}{2}(y - \sqrt{2 - 2z^2 - 2x^2 - y^2}), \frac{1}{2}(y + \sqrt{2 - 2z^2 - 2x^2 - y^2}), z\right), \end{aligned}$$

$$V_1(S) = -2 + 2\frac{x^2}{z^2} + 2\frac{1 - z^2 - 2x^2}{z^2 + x^2 + y^2 - 1},$$

$$\begin{aligned} V_2(S) &= \frac{(y - \sqrt{2 - 2z^2 - 2x^2 - y^2})^2}{2z^2} \\ -2 &+ \frac{2x}{y + \sqrt{2 - 2z^2 - 2x^2 - y^2}} + \frac{y}{x}\frac{2\sqrt{2 - 2z^2 - 2x^2 - y^2}}{y + \sqrt{2 - 2z^2 - 2x^2 - y^2}}, \\ V_3(S) &= \frac{(y + \sqrt{2 - 2z^2 - 2x^2 - y^2})^2}{2z^2} \\ -2 &+ \frac{2x}{y - \sqrt{2 - 2z^2 - 2x^2 - y^2}} - \frac{y}{x}\frac{2\sqrt{2 - 2z^2 - 2x^2 - y^2}}{y - \sqrt{2 - 2z^2 - 2x^2 - y^2}}, \\ V_4(S) &= -\frac{x + y}{z}, \end{aligned} \tag{13}$$

$$\begin{aligned} \langle V(S), S\rangle &= V_1(S)x + \frac{1}{2}V_2(S)(y - \sqrt{2 - 2z^2 - 2x^2 - y^2}) \\ &+ \frac{1}{2}V_3(S)(y + \sqrt{2 - 2z^2 - 2x^2 - y^2}) + V_4(S)z. \end{aligned}$$

In the neighborhood $J \times U$ consider the system

$$\frac{d}{dv}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xW_x \\ xW_y \\ xW_z \end{pmatrix}. \tag{14}$$

It is obvious that the trajectories of (14) coincide with the trajectories of (10) to within the change of variables $du = x\,dv$. The vector field xW is smooth in the neighborhood $J \times U$, and straightforward calculations demonstrate that, for a sufficiently small $\varepsilon > 0$, the stationary points of (14) in $J \times U$ are nothing but the points of the interval J. Consider the following linearization of (14) in a neighborhood of S_0:

$$\frac{dx}{dv} = -4x, \quad \frac{dy}{dv} = -\frac{2(3\mu^2 - 1)}{\mu^2}x + 4y, \quad \frac{dz}{dv} = 0.$$

The linearized system has three eigenvectors $e_1 = \left(8, \frac{2(3\mu^2 - 1)}{\mu^2}, 0\right)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$ with the eigenvalues -4, 4, and 0.

It follows from (13) that if $(x,y,z)\to S_0=(0,0,\mu)$ then $\langle (0,0,1),\frac{xW}{|xW|}\rangle\to 0$; i.e., the angle between the vector xW and the vector tangent to the arc J tends to $\pi/2$ as we approach the points of J. This enables us to reconstruct the "phase portrait" of (14) in the neighborhood $J\times U$ as in the classical case. Namely, consider the domain Γ in $J\times U$ bounded by the parabolic cylinders $-\frac{2(3\mu^2-1)}{\mu^2}x+8y-\alpha x^2=0$ and $-\frac{2(3\mu^2-1)}{\mu^2}x+8y+\alpha x^2=0$ and the plane $x=\delta$, where $\alpha,\delta>0$. It is easy to see that

$$\frac{d}{dv}\left(-\frac{2(3\mu^2-1)}{\mu^2}x+8y-\alpha x^2\right)=12\alpha x^2+O(x^2+y^2)\geq 0$$

at the points of the first parabolic cylinder if the constant α is chosen sufficiently large (moreover, the equality is attained only on J). Hence, the trajectories intersect the first parabolic cylinder from the inside of the domain Γ. Similarly, we can show that the trajectories of (14) intersect from the inside of the domain Γ the second parabolic cylinder that bounds Γ. Then, for each value $z=z_0$, there is a trajectory that starts on the flat wall of the domain at a point (δ,y,z_0) and arrives at a point on the axis J if δ is sufficiently small and α is sufficiently large (this follows from the fact that such trajectory cannot deviate strongly along J, since the angle between it and J tends to $\pi/2$). Consequently, if we fix a point $S_0=(0,0,\mu)$ on the arc J then, as δ decreases and α increases, we can find a trajectory arriving exponentially with order e^{-4v} at the point S_0 from the side of the domain $x>0$. Similarly, there is a unique trajectory arriving at S_0 from the opposite side, i.e., from the side of the domain $x<0$. Since x converges to zero with the convergence rate e^{-4v}, the trajectory "arrives" at the point S_0 with respect to the parameter u in finite time.

Note now that, as we pass from u to v, the orientation changes of the trajectories in the domain $x<0$. This means that, for each point S_0, there is a unique trajectory starting at S_0 in finite time and arriving at the domain $x<0$ in finite time. Moreover, starting at S_0 the trajectory touches the vector $e_1=\left(-8,-\frac{2(3\mu^2-1)}{\mu^2},0\right)$. The lemma is proven.

Thus, as follows from Lemma 11, there exist a trajectory $S(u)$ of (10) starting at $u=u_0$ at the point S_0 and a metric on $\mathscr{M}_2/\mathbb{Z}_p$ regular in some neighborhood of the zero section $\mathscr{O}$. Our further problem is to determine the behavior of the metric at large u.

The following lemma is immediate from analysis of (6) and (10).

Lemma 12. *If $S=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is a solution to (10) then the following hold:*

(1) $\frac{d}{du}\left(\log\left|\frac{\alpha_1}{\alpha_2}\right|\right)=(\alpha_1-\alpha_2)\left[\frac{2}{\alpha_4^2}-2\frac{\alpha_1+\alpha_2-\alpha_3}{\alpha_1\alpha_2\alpha_3}\right]$;

(2) $\frac{d}{du}(\alpha_2+\alpha_3)=\frac{4}{\alpha_4^2}(\alpha_2+\alpha_4)(\alpha_2-\alpha_4)$ *if* $\alpha_2+\alpha_3=0$;

(3) $\frac{d}{du}(\alpha_2+\alpha_4)=\frac{(\alpha_2+\alpha_3)(\alpha_1+\alpha_2)(\alpha_1-\alpha_2+\alpha_3)}{\alpha_1\alpha_2\alpha_3}$ *if* $\alpha_2+\alpha_4=0$;

(4) $\frac{d}{du}\left(\log\left|\frac{\alpha_2}{\alpha_3}\right|\right)=2\frac{(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_2+\alpha_4)}{\alpha_2^2\alpha_4^2}$ *if* $\alpha_1=\alpha_2$;

(5) $\frac{d}{du}(\alpha_1+\alpha_2)\to 16\left(\alpha_2-\frac{1}{2}\right)\left(\alpha_2+\frac{1}{2}\right)$ *as* $\alpha_1-\alpha_2\to 0$ *and* $\alpha_3\to 0$.

Lemma 13. *The trajectory of (10) defined by the initial point $S_0=(0,-\lambda,\lambda,\mu)$, $\lambda,\mu>0$, $2\lambda^2+\mu^2=1$, possesses one of the following asymptotic expansions depending on the parameter μ:*

(1) *if $\mu=1/\sqrt{3}$ then $S(u)$ tends to the stationary point $S_\infty=(-1/2,-1/2,1/2,1/2)$ as $u\to\infty$;*

(2) *if $\mu>1/\sqrt{3}$ then $S(u)$ tends to the conditionally stationary point $S_\infty=(-1/2,-1/2,0,1/\sqrt{2})$ as $u\to\infty$;*

(3) *if $\mu<1/\sqrt{3}$ then $S(u)$ tends to the point $S_1=(0,0,1,0)$ as $u\to u_1<\infty$.*

Proof. Introduce the notations for the following points in S^3:

$$O=Q_+=(0,0,0,1),\ A=(0,-1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}),\ B=(-1/2,-1/2,1/2,1/2),$$
$$C=(0,-1/\sqrt{2},0,1/\sqrt{2}),\quad D=(-1/\sqrt{3},-1/\sqrt{3},0,1/\sqrt{3}),$$
$$E=(0,-1/\sqrt{2},1/\sqrt{2},0),\quad F=(-1/\sqrt{3},-1/\sqrt{3},1/\sqrt{3},0),\quad G=(0,0,1,0).$$

Consider the two domains $\Pi, \Gamma \subset S^3$ defined by the inequalities

$$\Pi : \alpha_1 \le 0,\ \alpha_2 \le \alpha_1,\ \alpha_2 + \alpha_3 \le 0,\ \alpha_2 + \alpha_4 \ge 0,\ \alpha_3 \ge 0,$$
$$\Gamma : \alpha_1 \le 0,\ \alpha_2 \le \alpha_1,\ \alpha_2 + \alpha_3 \ge 0,\ \alpha_2 + \alpha_4 \le 0,\ \alpha_4 \ge 0.$$

It is easy to see that the domains Π and Γ are spherical pyramids $(OABCD)$ and $(GABEF)$. The boundaries of the pyramids are the following sets:

$\Pi_1 = (OAC) = \{\alpha_1 = 0, \alpha_2 + \alpha_3 \le 0, \alpha_2 + \alpha_4 \ge 0, \alpha_3 \ge 0\} \subset E_1$;
$\Pi_2 = (OBD) = \{\alpha_2 = \alpha_1, \alpha_2 + \alpha_3 \le 0, \alpha_2 + \alpha_4 \ge 0, \alpha_3 \ge 0\} \subset E_{12}^-$;
$\Pi_3 = (OCD) = \{\alpha_1 \le 0, \alpha_2 \le \alpha_1, \alpha_3 = 0, \alpha_2 + \alpha_4 \ge 0\} \subset E_3$;
$\Pi_4 = (OAB) = \{\alpha_1 \le 0, \alpha_2 \le \alpha_1, \alpha_2 + \alpha_3 = 0, \alpha_2 + \alpha_4 \ge 0\} \subset E_{23}^+$;
$\Pi_5 = (ABCD) = \{\alpha_1 \le 0, \alpha_2 \le \alpha_1, \alpha_2 + \alpha_3 \le 0, \alpha_2 + \alpha_4 = 0, \alpha_3 \ge 0\}$;
$\Gamma_1 = (GAE) = \{\alpha_1 = 0, \alpha_2 + \alpha_3 \ge 0, \alpha_2 + \alpha_4 \le 0, \alpha_4 \ge 0\} \subset E_1$;
$\Gamma_2 = (GBF) = \{\alpha_2 = \alpha_1, \alpha_2 + \alpha_3 \ge 0, \alpha_2 + \alpha_4 \le 0, \alpha_4 \ge 0\} \subset E_{12}^-$;
$\Gamma_3 = (GEF) = \{\alpha_1 \le 0, \alpha_2 \le \alpha_1, \alpha_2 + \alpha_3 \ge 0, \alpha_4 = 0\} \subset E_4$;
$\Gamma_4 = (ABFE) = \{\alpha_1 \le 0, \alpha_2 \le \alpha_1, \alpha_2 + \alpha_3 = 0, \alpha_2 + \alpha_4 \le 0, \alpha_4 \ge 0\} \subset E_{23}^+$;
$\Gamma_5 = (GAB) = \{\alpha_1 \le 0, \alpha_2 \le \alpha_1, \alpha_2 + \alpha_3 \ge 0, \alpha_2 + \alpha_4 = 0\}$.

The pyramids intersect over the common part of the boundary (AB). It is clear that the trajectory of (10) corresponding to metric (9) with the holonomy group $SU(4)$ passes along the arc (AB) from A to B. The initial point $S_0 = (0, -\lambda, \lambda, \mu)$ belongs to (OE).

1. Suppose that $S_0 \in (OA)$. Moreover, if $\mu = 1/\sqrt{3}$, i.e., $S_0 = A$; then the trajectory coincides with (AB). Let $S_0 \ne A$, i.e., $\mu > 1/\sqrt{3}$. Then the vector e_1 (see the proof of Lemma 11) is strictly inward with respect to the domain Π; i.e., for small u the trajectory of (10) arrives to Π. Suppose that the trajectory first attains the boundary at some point S_1 in a finite time $u = u_1$.

Define the function F_1 on S^3: $F_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \log \frac{-\alpha_1}{-\alpha_2}$. It follows from the relation (1) of Lemma 12 that the function F_1 is strictly increasing along the trajectories of (10) lying inside Π and Γ. Note that $F(S(u)) \to -\infty$ as $u \to u_0 + 0$ and the function $F(S(u))$ is strictly increasing for $u > u_0$. Hence, the trajectory cannot return to Π_1 (except for the point O) at least until it leaves Π. Thus, $S_1 \notin \Pi_1 \backslash \{O\}$. Now, the right-hand side of the relation (2) of Lemma 12 is negative for $S \in \Pi_4$, while the right-hand side of (3) of Lemma 12 is positive for $S \in \Pi_5$. Consequently, the vector field W is inward with respect to Π at the points of Π_4 and Π_5; i.e., $S_1 \notin \Pi_4, \Pi_5$.

Assume that $S_1 \in \Pi_3 \backslash \Pi_2$. Then we can note that the component W_3 of W is a smooth function at the points of Π_3. Consequently, we can pass to the new parameter α_3 on the trajectory $S(u)$ in some neighborhood of S_1; moreover, the point S_1 is attained at $\alpha_3 = 0$. Moreover, the tangential component of the field W has order $1/\alpha_3$ in a neighborhood of S_1. This means that the curve $S(u)$ cannot reach Π_3 in finite time. We are left with the only case $S_1 \in \Pi_2$ but $(\Pi_2 \backslash \Pi_3) \subset E_{12}^-$ is an invariant surface of (10) on which the system satisfies the uniqueness theorems. Therefore, the fact that the trajectory reaches the points of $\Pi_2 \backslash \Pi_3$ in finite time would contradict the uniqueness of the trajectories. Thus, we are left with the case $S_1 \in \Pi_2 \cap \Pi_3 = (OD)$.

Suppose that $S_1 = (\alpha, \alpha, 0, \sqrt{1 - 2\alpha^2})$, $0 \le -\alpha \le 1/\sqrt{3}$. Let $X = (x_1, x_2, x_3, x_4)$ be the tangent vector to the trajectory $S(u)$ at S_1. Then the following obvious relation holds:

$$\lim_{\varepsilon \to -0} W(S_1 + \varepsilon X) = \lim_{u \to u_1} W(S(u)) = X.$$

If $\alpha \ne 0$ then the limits of $W(S(u))$ as $u \to u_1$ are calculated in the item 1 of the proof of Lemma 9 (with the indices $1, 2, 3$ replaced with $3, 1, 2$). Hence, we obtain $x_3 = 0$, which is possible only for $X = 0$. Thus, $S_1 = (-1/2, -1/2, 0, 1/\sqrt{2})$ is a conditionally stationary point attained in infinite time. Now, if $\alpha = 0$, i.e., $S_1 = Q_+$, then $X = (x_1, x_2, x_3, 0)$, $x_i \ne 0$, $i = 1, 2, 3$, and the corresponding limit of $W(S(u))$ is calculated in the item 3 of the proof of Lemma 9. After simple calculations we see that the vector $X = (-1, -1, -1, 0)$ attached to the point Q_+ is not inward with respect to Π and so it does not fit.

Thus, the trajectory $S(u)$ cannot reach the boundary of Π in finite time; i.e., it lies entirely in Π, $u \in (u_0, \infty)$. We now find the limit point of $S(u)$ as $u \to \infty$.

First, as was observed above, the function F_1 is increasing along $S(u)$. Since there are no stationary points of W inside Π (Lemma 8), $S(u)$ tends to the maximal (in Π) level of F_1 as $u \to \infty$, i.e., to Π_2. Now, consider the function $F_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \log \frac{-\alpha_2}{\alpha_3}$.

It follows from the relation (4) of Lemma 12 that F_2 is increasing inside Π in a neighborhood of Π_2. Hence, the trajectory $S(u)$ tends either to the maximal level of F_2 on the wall Π_2, i.e., to $\Pi_2 \cap \Pi_3 = (OD)$, as $u \to \infty$ or to a stationary point lying on Π_2 that is to the point B. Introduce the coordinates $x = \alpha_1 + \frac{1}{2}$, $y = \alpha_2 + \frac{1}{2}$, and $z = \alpha_3 - \frac{1}{2}$ in a neighborhood of B and consider the linearization of (10):

$$\frac{dx}{du} = -19x - 3y + 6z, \quad \frac{dy}{du} = -3x - 19y + 6z, \quad \frac{dz}{du} = -13x - 13y + 10z. \tag{15}$$

We verify immediately that (15) has one positive eigenvalue 4 and two multiple eigenvalues equal to -16. To the multiple eigenvalues there corresponds the plane $x + y - 2z = 0$ constituted by the eigenvectors. Consequently, in some neighborhood of B there is a 2-dimensional invariant surface tangent to the plane $x + y - 2z = 0$ and constituted by the trajectories of (10) arriving asymptotically exponentially at B; moreover, no other trajectories arrive at B. It is easy to verify that this surface intersects the domains Π and Γ only by the arc (AB) and is transversal to the walls Π_4, Π_5, Γ_4, and Γ_5 adjacent to (AB). Consequently, except (AB), no other trajectory in question can arrive at B. Thus, the trajectory $S(u)$ tends to (OD) as $u \to \infty$.

Finally, consider the function $F_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 + \alpha_2$. It follows from the relation (5) of Lemma 12 that in a neighborhood of the arc (OD) the trajectory $S(u)$ tends to the point at which $\alpha_2 = -1/2$, i.e., $S(u) \to S_\infty = (-1/2, -1/2, 0, 1/\sqrt{2})$.

2. Suppose that $S_0 \in (AE)$ and $\mu < 1/\sqrt{3}$. We argue here by analogy with Section 1. We first find out successively at what points the trajectory can reach the boundary of Γ in finite time.

Suppose that $S_1 = S(u_1) \in \partial\Gamma$. On the walls Γ_4 and Γ_5 the field W is directed inside the domain which follows from the relations (2) and (3) of Lemma 12. Thus, $S_1 \notin \Gamma_4, \Gamma_5$; the wall Γ_1 cannot be reached in view of the increase of F_1. Now, if the trajectory $S(u)$ can reach the wall Γ_3 in finite time at some point $S_1 = S(u_1)$ then this is possible only in the case $S_1 = G$. Indeed, if $S_1 \neq G$ then we can pass to the parameter α_4^2 along the curve $S(u)$. We can easily establish that in this case the normal (with respect to Γ_3) component of W is bounded, while the tangential component tends to infinity; a contradiction. The same can be said about the wall Γ_2; namely, it is invariant with respect to the dynamical system and the uniqueness theorem for trajectories holds at all points but $\Gamma_2 \cap \Gamma_3 = (FG)$; therefore, none of the trajectories coming from inside intersects $\Gamma_2 \backslash \{G\}$. Thus, the only possible case is $S_1 = G$.

If the trajectory does not reach the boundary of Γ in finite time then it follows from the increase of F_1 that the trajectory tends to Γ_2 (here we also use the absence of stationary points inside Γ). The item (4) of Lemma 12 demonstrates that the function F_2 is decreasing along the trajectory. Hence, the trajectory tends either to a stationary point on Γ_2 (such a point is unique, B) or to the minimal level of F_2 which is the point G. The trajectory tends to B for $\mu = 1/\sqrt{3}$, and all other trajectories cannot converge to this point (here we argue as in the item 1 of the proof by using the linearization (15) at B).

We have thus shown that the only possible case for $\mu < 1/\sqrt{3}$ is as follows: the trajectory $S(u)$ converges to G in finite or infinite time. However, note that if we take the parameter on the curve $S(u)$ to be the value α_4^2 then the change of parameter is smooth in a neighborhood of G. Consequently, the trajectory reaches the point G in finite or infinite time for both parameters u and α_4^2 simultaneously. On the other hand, $S(\alpha_4^2) \to G$ as $\alpha_4^2 \to 0$; i.e., $S(u)$ reaches G in finite time. The lemma is proven.

The following lemma completes the proof of our main theorem:

Lemma 14. *The holonomy group of the metric $\bar{g}$ on $\mathscr{M}_2$ defined by the initial point S_0 coincides with the whole group* Spin(7) *for $\mu > 1/\sqrt{3}$.*

Proof. Assume that $G \subset \mathrm{Spin}(7)$ is the holonomy group of the space $\mathscr{M}_2/\mathbb{Z}_p$ with metric (4). This metric (4) tends asymptotically to a metric locally isometric to the product $\mathbb{R} \times C(\mathscr{Z})$. Consequently, there is a subgroup $H \subset G$ presenting the holonomy group of the limit metric on $\mathbb{R} \times C(\mathscr{Z})$. By the de Rham theorem, the group H equals the product of the unit group (acting trivially on $\mathbb{R}$) and the holonomy group H_1 of the cone over $\mathscr{Z}$. Since the cone is 7-dimensional, we can say a priori that the three cases are only possible: $H_1 = SO(7)$, $H_1 = G_2$, or the cone over $\mathscr{Z}$ is a flat space. In the last case $\mathscr{Z}$ is a 6-dimensional sphere by necessity, which contradicts the fact that $\mathscr{Z}$ is a Kähler–Einstein manifold (or orbifold in the general case). The first case is impossible, since $H \subset G \subset \mathrm{Spin}(7)$. We are left with $G_2 \simeq H \subset G \subset \mathrm{Spin}(7)$. From the classification of simple Lie groups and the classification of holonomy groups we see that this is possible only for $G = \mathrm{Spin}(7)$. The lemma is proven.

§ 5. Justification of Regularity Conditions

We give proofs of Lemmas 4 and 5 to make exposition complete and rigorous, since the author failed in finding an exact reference. Nevertheless, these assertions undoubtedly are familiar to specialists (and very clear intuitively).

Proof of Lemma 4. To obtain a Riemannian metric on $\mathscr{M}_1$, it is necessary that the functions $A_i(t)$ and $B(t)$ have definite sign on $(0, \infty)$, while $A_i(0) = 0$ and $B(0) \neq 0$. Moreover, smoothness of the metric $\bar{g}$ outside $\mathscr{O} \subset \mathscr{M}_1$ is equivalent to smoothness of $A_i(t)$ and $B(t)$ for $t > 0$. Now, we find out what happens in a neighborhood of $\mathscr{O}$.

Let G be the common fiber of the 3-Sasakian fibration on M isometric to $Sp(1)$ or $SO(3)$. As above, denote by $\pi : M \to \mathscr{O}$ the principal G-bundle of M over $\mathscr{O}$. Let $F = \mathbb{H}$ for $G = Sp(1)$ and $F = \mathbb{H}/\mathbb{Z}_2$ for $G = SO(3)$ be the fiber of the bundle $\mathscr{M}_1$ over $\mathscr{O}$ associated with π. Consider an arbitrary point $q \in \mathscr{O}$. Then there exist an open set $\widetilde{U} \subset \mathbb{R}^4$ and a discrete group $\Gamma \subset G$ acting on $\widetilde{U}$ so that some neighborhood of $\pi^{-1}(q)$ is diffeomorphic to $(\widetilde{U} \times G)/\Gamma$ (see the proof of Theorem 1). The action of the group Γ on G by translations extends obviously to the action of Γ on F. Consequently, some neighborhood of q in $\mathscr{M}_1$ is homeomorphic to $(\widetilde{U} \times F)/\Gamma$.

It is obvious that in the case $G = Sp(1)$ the collection constituted by $\widetilde{U} \times \mathbb{H}$, the group Γ, and the corresponding homeomorphism gives a chart on the orbifold $\mathscr{M}_1$ in a neighborhood of q and all these charts agree pairwise. If $G = SO(3)$ then we should take the chart to be $\widetilde{U} \times \mathbb{H}$ (covering $\widetilde{U} \times \mathbb{H}/\mathbb{Z}_2$ by two sheets) and the group to be $\widetilde{\Gamma}$ which covers the group Γ under the standard $\mathbb{Z}_2$-covering $Sp(1) \to SO(3)$. Thus, to verify smoothness of (4) in a neighborhood $\mathscr{O}$, we should show smoothness of the metric lifted to each neighborhood $\widetilde{U} \times \mathbb{H}$ constructed above.

Given $p \in \widetilde{U}$, consider the restriction of the metric $\bar{g}$ to $\{p\} \times \mathbb{H}$:

$$\bar{g}_v = dt^2 + \sum_{i=1}^{3} A_i(t)^2 \eta_i^2. \tag{16}$$

Here t is the radial parameter on $\mathbb{H}$ and the metric $\bar{g}_v$ is independent of the choice of p. We need the following

Lemma 15 [11, 15]. *The metric $g = dr^2 + h^2(r)\, d\phi^2$ given in the polar coordinate system (r, ϕ) on the standard 2-dimensional disk $r \leq r_0$, $0 \leq \phi \leq 2\pi$, is a smooth Riemannian metric if and only if $|h(r)| > 0$ for $r \in (0, r_0]$ and the function $h(r)$ extends to a smooth odd function $h(r)$ on $(-r_0, r_0)$ such that $|h'(0)| = 1$.*

Suppose that metric (4) is smooth; in this case (16) is smooth as well. We assume that the vector fields ξ^i bounded on $S^3 \subset \mathbb{H}$ are represented as qi, qj, and qk, where $q = q_0 + q_1 i + q_2 j + q_3 k$. Consider the restriction of (16) to the plane generated by the vectors 1 and i in $\mathbb{H}$:

$$\bar{g}_v|_{1,i} = dt^2 + A_1(t)^2 \eta_1^2.$$

This metric is smooth if $\bar{g}_v$ is smooth; consequently, by Lemma 15, the function A_1 satisfies the condition (1) of Lemma 4. Similarly, smoothness of $\bar{g}_v$ implies validity of the condition (1) of Lemma 4 for all A_1–A_3. Now, (4) is the twisted product of (16) and the metric $g|_{\mathscr{H}}$ with the twisting function $B(t)$ considered as a function on $\mathbb{H}$. Hence, we see easily that $B'(0) = 0$.

Conversely, suppose that the conditions (1)–(3) of Lemma 4 are satisfied. The expression for dA_i/dt can be differentiated formally k times with respect to t. Let $V_i^{(k)} = d^k A_i/dt^k$ be a rational function of the variables A_1, A_2, A_3, and B, $i = 1, 2, 3$. Similarly, let $V_4^{(k)} = d^k B/dt^k$. It follows from the conditions on the functions A_i that there exist C^∞-smooth functions $a_i(t)$ defined for $t \geq 0$ such that $A_i(t) = ta_i(t)$ and $|a_i(0)| = 1$.

Put $\widetilde{A}_i(t) = -A_i(-t)$ and $\widetilde{B}(t) = B(-t)$ for $t \leq 0$. It is clear that the obtained functions $\widetilde{A}_i(t)$ and $\widetilde{B}(t)$ belong to the class C^∞ on the interval $t \leq 0$ and $\widetilde{A}_i(t) = ta_i(-t)$ for $t \leq 0$. Moreover, invariance of (6) under the transformation $(t, A_i, B) \mapsto (-t, -A_i, B)$ implies that $\widetilde{A}_i(t)$, $\widetilde{B}(t)$ is a solution to (6). Now,

$$\left.\frac{d^k}{dt^k}\right|_{t>0} (A_i(t)) = V_i^{(k)}(A_i(t), B(t)) = t^m \frac{P(a_i(t), B(t))}{Q(a_i(t), B(t))}, \tag{17}$$

where the polynomials P and Q have nonzero values at $t = 0$. Since the solutions $A_i(t)$ and $B(t)$ are infinitely smooth for $t \geq 0$ by condition, expression (17) has a limit as $t \to \infty$ and consequently $m \geq 0$. Inserting the curve $\widetilde{A}_i(t)$, $\widetilde{B}(t)$ in (17), we see that the right derivatives of all orders of the functions $A_i(t)$ at the point $t = 0$ coincide with the corresponding left derivatives of the functions $\widetilde{A}_i(t)$ at the point $t = 0$ (note that this implies in particular that the derivatives of even orders are trivial). Thus, the functions $A_i(t)$ and $\widetilde{A}_i(t)$ together constitute C^∞-smooth odd functions in a neighborhood of the point $t = 0$. Similarly, the functions $B(t)$ and $\widetilde{B}(t)$ constitute C^∞-smooth even functions in a neighborhood of the point $t = 0$.

At each point $q = q_0 + q_1 i + q_2 j + q_3 k$ we can expand the standard coordinate basis $\partial/\partial q_i$ over the basis $q/|q|$, qi, qj, qk dual to the forms dt, η_1, η_2, and η_3. This makes it possible to calculate immediately the components of the metric tensor (16) with respect to the standard coordinates q_0, q_1, q_2, and q_3 in $\mathbb{H}$:

$$g_{00}(q) = \frac{q_0^2|q|^2 + A_1^2(|q|)q_1^2 + A_2^2(|q|)q_2^2 + A_3^2(|q|)q_3^2}{|q|^4},$$

$$g_{11}(q) = \frac{q_1^2|q|^2 + A_1^2(|q|)q_0^2 + A_2^2(|q|)q_3^2 + A_3^2(|q|)q_2^2}{|q|^4},$$

$$g_{01} = \frac{q_0 q_1\big(|q|^2 - A_1^2(|q|)\big) + q_2 q_3\big(A_2^2(|q|) - A_3^2(|q|)\big)}{|q|^4},$$

$$g_{12} = \frac{q_1 q_2\big(|q|^2 - A_3^2(|q|)\big) + q_0 q_3\big(A_1^2(|q|) - A_2^2(|q|)\big)}{|q|^4}$$

(we give only some components; the others are obtained from those above by an appropriate rearrangement of the indices 1, 2, and 3). We use the following simple fact: If a smooth function $f(t)$ on a neighborhood of the point $t = 0$ is odd then $f^2(t)$ is a smooth function of $u = t^2$ in a neighborhood of the point $t = 0$. To prove this, we only need to note that the Taylor expansion of the function f to an arbitrary order contains only the even degrees of the variable t and $\frac{d}{du} = \frac{1}{t}\frac{d}{dt}$. Hence, we easily derive existence and continuity at the point $t = 0$ of the derivative of $f(u)$ of every order. Since it is proven that the functions $A_i(t)$ extend to odd functions, we have demonstrated that the components of the metric tensor and hence (15) are all smooth.

Now, recall that (4) is the twisted product of $\widetilde{U}$ and $\mathbb{H}$ with the twisting function $B(t)$ considered as a function on $\mathbb{H}$. Consequently, smoothness of the metric is equivalent to smoothness of $B(t)$. Using similar arguments (also see [15]) which we omit, we prove that smoothness of B on $\mathbb{H}$ is equivalent to the fact that $B(t)$ extends to an even function in a neighborhood of $t = 0$ which is guaranteed by the condition (2) of the lemma. Lemma 4 is proven.

Proof of Lemma 5. In general, the proof is carried out by analogy with that of Lemma 4; moreover, we preserve the former notations for G, π, and so on. First, to obtain a Riemannian metric on $\mathscr{M}_2/\mathbb{Z}_p$, it is necessary that the functions $A_i(t)$ and $B(t)$ have definite sign on $(0,\infty)$, while $A_1(0) = 0$, $A_2(0) \neq 0$, $A_3(0) \neq 0$, and $B(0) \neq 0$. Smoothness of the metric $\bar{g}$ outside $\mathscr{X} \subset \mathscr{M}_2$ is equivalent to smoothness of the functions $A_i(t)$ and $B(t)$ for $t > 0$. We now find out what happens in a neighborhood of $\mathscr{X}$.

Let $S \subset G$ be a circle that integrates the field ξ^1. Denote by $\pi' : \mathscr{X} \to \mathscr{O}$ the bundle with fiber $S^2 = G/S$. Take $q \in \mathscr{O}$. Consider a small neighborhood U of the point q and the corresponding neighborhood $\widetilde{U} \subset \mathbb{R}^4$ such that $\widetilde{U}/\Gamma = U$ for some discrete subgroup $\Gamma \subset G$ acting on $\widetilde{U}$ by diffeomorphisms. If the neighborhood $\widetilde{U}$ is sufficiently small then ${\pi'}^{-1}(U) = (\widetilde{U} \times (G/S))/\Gamma$. Then $(\pi' \circ \pi)^{-1}(U)$ is diffeomorphic to $(\widetilde{U} \times (G \times \mathbb{C})/S)/\Gamma$ and smoothness of metric (4) is equivalent to smoothness of the metric pulled-back to the each neighborhood $\widetilde{U} \times (G \times \mathbb{C})/S$.

Metric (4) is the twisted product of metric (16) on $(G \times \mathbb{C})/S$ and the metric $g|_{\mathscr{O}}$ on $\widetilde{U}$ with the twisting function $B(t)$. If (4) is a smooth metric then, obviously, the function B is smooth and extends to an even function of the radial parameter t on $\mathbb{C}$, i.e., satisfy the condition (3) of Lemma 5. Now, it follows from smoothness of the restriction of (16) on $\mathbb{C}$ and Lemma 15 that A_1 satisfies the condition (1) of Lemma 5. Finally, for A_2 and A_3 to have the derivatives at the point $t = 0$, it is necessary that either $A_2(0) = A_3(0)$ (which contradicts the condition (1) of the lemma) or $A_2(0) = -A_3(0)$, whence we easily derive the condition (2) of Lemma 5.

Suppose conversely that smooth functions A_i and B on $[0,\infty)$ satisfy all conditions of Lemma 5. As in the previous lemma, we show that the function B extends to an even smooth function of the argument t; therefore, to prove smoothness of the metric of the twisted product (4), it suffices to prove smoothness of (16).

Consider the projection $p : (G \times \mathbb{C})/S \to G/S$. It is clear that $(G \times \mathbb{C})/S$ is fiberwise diffeomorphic either to a one-dimensional canonical complex bundle for $G = Sp(1)$ or its double for $G = SO(3)$. Moreover, the bundle p is the canonical bundle over the 2-dimensional sphere G/S. Metric (16) makes p into a Riemannian submersion with a fiber diffeomorphic to $\mathbb{C}$. Let $\mathscr{V}$ and $\mathscr{H}$ be mutually orthogonal subbundles of vertical and horizontal vectors of the submersion p. The restriction of (16) to $\mathscr{V}$ looks as follows:

$$dt^2 + A_1^2(t)\eta_1^2. \tag{18}$$

As in the proof of Lemma 4, we show that $A_1(t)$ extends to a smooth odd function; hence, by Lemma 15, (18) is smooth on $\mathscr{V}$. Consider a small neighborhood $V \subset G/S$ and a point $gS \in V$. The inverse image of gS under the mapping p has the form $(g,z)S \in p^{-1}(gS)$, where $z \in \mathbb{C}$. The horizontal tangent vector at $(g,z)S$ can be identified with gX, where $X \in sp(1) = \mathrm{Im}(\mathbb{H})$, $X = x_2 j + x_3 k$, and $x_i \in \mathbb{R}$. Moreover, for $s \in S$ the vectors gX and gsY have the same projections tangent to G/S at gS if and only if $Y = s^{-1}Xs$. Consider the fields X_1 and X_2 on $(G \times \mathbb{C})/S$ defining them at each point $(g,z)s$ as $g(s^{-1}js)$ and $g(s^{-1}ks)$. Then the fields X_1 and X_2 project into some smooth fields $\widetilde{X}_1$ and $\widetilde{X}_2$ in the neighborhood V. It is clear that the fields $\widetilde{X}_1$ and $\widetilde{X}_2$ constitute a basis for the module of smooth vector fields in the neighborhood V and we have to verify smoothness of the components $g_{ij} = g(\widetilde{X}_i, \widetilde{X}_j) = g(X_i, X_j)$:

$$\begin{gathered} g_{11} = \frac{A_2^2(|w|) + A_3^2(|w|)}{2} + \frac{x}{2|w|}\big(A_2^2(|w|) - A_3^2(|w|)\big), \\ g_{22} = \frac{A_2^2(|w|) + A_3^2(|w|)}{2} + \frac{x}{2|w|}\big(A_3^2(|w|) - A_2^2(|w|)\big), \\ g_{12} = \frac{y}{2|w|}\big(A_2^2(|w|) - A_3^2(|w|)\big), \end{gathered} \tag{19}$$

where $s = e^{i\phi} \in S$, $t = |z|$, and $w = x + yi = te^{4i\phi}$.

Extend the function $A_2(t)$ to $t \leq 0$, by putting $A_2(t) = -A_3(-t)$ for $t \leq 0$, and similarly put $A_3(t) = -A_2(-t)$ for $t \leq 0$. As in the proof of Lemma 4, we show that the so-extended functions $A_2(t)$ and $A_3(t)$ are C^∞-smooth in a neighborhood of $t = 0$. Hence, we conclude in particular that the even coefficients in the Taylor expansions of the functions A_2 and A_3 have opposite signs, whereas the odd coefficients coincide. We can easily show now that $A_2^2 + A_3^2$ is a smooth function of the argument $|w|^2$. Similarly, $\frac{A_2^2 - A_3^2}{|w|}$ is a smooth function of the argument $|w|^2$. Consequently, the functions on the right-hand side of (19) are smooth, which implies smoothness of the restriction of (16) to $\mathscr{H}$ and smoothness of (4). Lemma 5 is proven.

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Noncompact Riemannian Spaces with the Holonomy Group Spin(7) and 3-Sasakian Manifolds

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Abstract—We complete the study of the existence of Riemannian metrics with Spin(7) holonomy that smoothly resolve standard cone metrics on noncompact manifolds and orbifolds related to 7-dimensional 3-Sasakian spaces.

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1. INTRODUCTION AND MAIN RESULTS

This paper is devoted to Riemannian metrics with special holonomy group Spin(7) and completes the study begun in [1].

The exceptional holonomy groups Spin(7) and G_2 deserve special attention: the question of existence of metrics with such holonomy was open for a long time. The first example of a Riemannian metric with Spin(7) holonomy was constructed in 1987 [2], but that metric was not complete. The first construction of a complete Riemannian metric with Spin(7) holonomy on a noncompact manifold appeared in 1989 [3]. The existence of compact spaces was proved by Joyce [4] in 1996. Joyce's construction gives no explicit description of the metrics; their existence follows from a fairly delicate analysis.

New interest in noncompact examples has recently arisen from mathematical physics. Applications of noncompact Spin(7)-holonomy metrics have been found in the so-called M-theory. In [5–10], a series of new examples of complete metrics were constructed; some of them were orbifolds rather than manifolds. All these metrics are automatically Ricci flat and asymptotically behave as cones or products of cones and circles (that is, are asymptotically locally conical (ALC)). All the examples constructed are metrics of cohomogeneity 1, i.e., are foliated by homogeneous 7-dimensional fibers.

In [1], we proposed a general scheme for constructing a Spin(7)-holonomy metric on the basis of a given 3-Sasakian 7-manifold M. The idea is as follows. For a 3-Sasakian manifold M, the cone over M has the holonomy group $\mathrm{Sp}(2) \subset \mathrm{Spin}(7)$. We deform the cone metric so as to resolve the singularity at the apex and obtain a metric whose holonomy group is not larger than Spin(7). The deformation is determined by functions $A_1(t)$, $A_2(t)$, $A_3(t)$, and $B(t)$ that depend on a radial variable t ranging over the generatrix of the cone.

In more detail, consider a 3-Sasakian bundle $M \to \mathcal{O}$ with the common fiber diffeomorphic either to S^3 or to SO(3) over a quaternion Kähler orbifold $\mathcal{O}$. With this bundle we can associate two vector bundles with fibers $\mathbb{H}$ and $\mathbb{C}$, whose spaces we denote by $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Using the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, we can resolve the conical singularity in two topologically different ways. The metric on $\mathcal{M}_1$ and $\mathcal{M}_2$ is

$$dt^2 + \sum_{i=1}^{3} A_i^2(t)\eta_i^2 + B^2(t)g\big|_{\mathcal{H}}, \qquad (*)$$

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where g is the metric on the 3-Sasakian manifold M, $\mathcal{H}$ is the distribution of horizontal vectors tangent to M, and η_i is a basis of 1-forms that annihilate $\mathcal{H}$. Requiring that the holonomy of metric $(*)$ should be exceptional, we arrive at the following system of nonlinear differential equations:

$$\begin{aligned} \dot{A}_1 &= \frac{2A_1^2}{B^2} + \frac{(A_2 - A_3)^2 - A_1^2}{A_2 A_3}, \\ \dot{A}_2 &= \frac{2A_2^2}{B^2} + \frac{(A_3 - A_1)^2 - A_2^2}{A_1 A_3}, \\ \dot{A}_3 &= \frac{2A_3^2}{B^2} + \frac{(A_1 - A_2)^2 - A_3^2}{A_1 A_2}, \\ \dot{B} &= -\frac{A_1 + A_2 + A_3}{B}. \end{aligned} \tag{$**$}$$

To obtain regular ALC metrics, we need to pose a boundary value problem for system $(**)$; the condition on one boundary must resolve the conical singularity, while that on the other boundary must ensure the required asymptotic behavior.

Examples of Spin(7)-holonomy metrics on $\mathcal{M}_1$ for particular manifolds M were constructed in [5–10]. We also mention paper [13], in which the local structure of noncompact cohomogeneity 1 spaces with holonomy $G \subset \mathrm{Spin}(7)$ was studied; the results of [13] overlap with those obtained in this paper.

In [1], we proved the existence of a one-parameter family of metrics on $\mathcal{M}_2/\mathbb{Z}_p$, but the problem of existence of solutions of the form $(*)$ to system $(**)$ on $\mathcal{M}_1$ remained open. In this paper, we solve this problem; namely, combining the results of this paper and those of [1], we obtain the following theorem.

Theorem 1. *Let M be a compact 3-Sasakian 7-manifold, and let $p = 2$ if the common fiber of the 3-Sasakian bundle M is* SO(3) *and $p = 4$ if the fiber is* Sp(1).

1. *There exists a two-parameter family of pairwise nonhomothetic* Spin(7)*-holonomy Riemannian metrics on $\mathcal{M}_1$ of the form $(*)$ that satisfy the initial conditions*

$$A_1(0) = A_2(0) = A_3(0) = 0, \qquad \dot{A}_1(0) = \dot{A}_2(0) = \dot{A}_3(0) = -1,$$
$$B(0) > 0, \qquad \dot{B}(0) = 0.$$

This family of metrics is parameterized by number triples $\lambda_3 \le \lambda_2 \le \lambda_3 < 0$ with $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \varepsilon^2$ for sufficiently small $\varepsilon > 0$; for each of these triples, there exists a value $t = t_0$ at which the trajectory (A_1, A_2, A_3) passes through this triple, i.e.,

$$A_1(t_0) = \lambda_1, \qquad A_2(t_0) = \lambda_2, \qquad A_3(t_0) = \lambda_3.$$

2. *There exists a one-parameter family of pairwise nonhomothetic Riemannian metrics on $\mathcal{M}_2/\mathbb{Z}_p$ of the form $(*)$ that satisfy the initial conditions*

$$A_1(0) = 0, \qquad -A_2(0) = A_3(0) > 0, \qquad \dot{A}_1(0) = -4, \qquad \dot{A}_2(0) = \dot{A}_3(0),$$
$$B(0) > 0, \qquad \dot{B}(0) = 0.$$

This family of metrics is parameterized by the ratio $\mu = A_3(0)/B(0)$. The metrics of family 2 *have the holonomy* Spin(7) *for $\mu \ne 1$ and* SU(4) *for $\mu = 1$.*

For $\lambda_1 = \lambda_2 = \lambda_3$, the metric of family 1 *is complete and behaves asymptotically as a cone over M; for $\lambda_1 \ne \lambda_2 = \lambda_3$, the metrics are also complete and behave asymptotically as the product* *of a cone over the twistor space of M and a circle of constant radius. The other metrics of family* 1 *are not complete.*

The metrics of family 2 *are complete for $\mu \leq 1$, behave asymptotically as the product of a cone over the twistor space of M and a circle for $\mu < 1$, and behave asymptotically as a cone over M for $\mu = 1$. For $\mu > 1$, the metrics of family* 2 *are not complete.*

Any other complete regular metric of the form $(*)$ *with holonomy* Spin(7) *on $\mathcal{M}_1$ and $\mathcal{M}_2/\mathbb{Z}_p$ is homothetic to one of the metrics specified above up to a permutation of indices of the variables.*

The complete metrics of family 1 and of family 2 with $\mu = 1$ can be obtained by explicitly integrating system $(**)$; the formulas are given at the end of the next section. So far, we have not been able to determine an explicit form of the metrics of family 2 for $\mu < 1$.

There exist many examples of 3-Sasakian 7-manifolds [12]. In the general case, the spaces $\mathcal{M}_1$ and $\mathcal{M}_2/\mathbb{Z}_p$ are orbifolds. The space $\mathcal{M}_1$ is a manifold for $M = S^7$, while the space $\mathcal{M}_2/\mathbb{Z}_p$ is a manifold for $M = S^7$, $M = \mathbb{R}\mathrm{P}^7$, and $M = \mathrm{SU}(3)/S^1$.

2. CONSTRUCTION OF Spin(7)-HOLONOMY METRICS

Let $\{e^i\}$, $i = 0, 1, \dots, 7$, be an orthonormal basis of 1-forms on the standard Euclidean space $\mathbb{R}^8$. We set $e^{ijkl} = e^i \wedge e^j \wedge e^k \wedge e^l$ and define a 4-form Φ_0 on $\mathbb{R}^8$ as

$$\Phi_0 = e^{0123} + e^{4567} + e^{0145} - e^{2345} - e^{0167} + e^{2367} + e^{0246} + e^{1346} - e^{0275} + e^{1357} + e^{0347} - e^{1247} - e^{0356} + e^{1256}.$$

Let N be an oriented Riemannian 8-manifold. We say that a differential form $\Phi \in \Lambda^4 N$ determines a Spin(7)-structure on N if, in a neighborhood of each point $p \in N$, there exists an orientation-preserving isometry $\phi_p\colon T_pN \to \mathbb{R}^8$ such that $\phi_p^*\Phi_0 = \Phi|_p$. If the form Φ is parallel, then the holonomy group of the Riemannian manifold N reduces to the subgroup $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. As is known [11], Φ is parallel if and only if it is closed (it is automatically coclosed because the form Φ is self-adjoint with respect to the Hodge operator):

$$d\Phi = 0. \tag{1}$$

Recall the definition of a 3-Sasakian manifold. A compact $(4k-1)$-manifold M with a Riemannian metric g is said to be 3-Sasakian if the standard metric

$$g_0 = dt^2 + t^2 g$$

on the cone $\overline{M} = \mathbb{R}_+ \times M$ is hyper-Kähler, i.e., if it has the holonomy $\mathrm{Sp}(k)$. Hereafter, we assume that M is a compact 3-Sasakian 7-manifold.

The manifold M possesses the following structure [12]: there exist three Killing fields ξ^1, ξ^2, and ξ^3 on M such that

$$[\xi^i, \xi^j] = 2\varepsilon_{ijk}\xi^k.$$

The fields ξ^i generate a locally free isometric action of the group SU(2) on M. A generic orbit of this action is isometric to either S^3 or SO(3) with a metric of constant curvature, and the orbit space $\mathcal{O}$ is, generally, a four-dimensional orbifold with quaternion Kähler structure. The natural projection $M \to \mathcal{O}$ is a Riemannian submersion; we denote the bundle of horizontal vectors with respect to this submersion by $\mathcal{H}$.

Let η_1, η_2, and η_3 denote the 1-forms dual to the fields ξ^1, ξ^2, and ξ^3, and let η_4, η_5, η_6, and η_7 be 1-forms generating the annihilator of the horizontal subbundle $\mathcal{H}$, so that

$$\omega_1 = 2(\eta_4 \wedge \eta_5 - \eta_6 \wedge \eta_7) = d\eta_1 + 2\eta_2 \wedge \eta_3,$$

$$\omega_2 = 2(\eta_4 \wedge \eta_6 - \eta_7 \wedge \eta_5) = d\eta_2 + 2\eta_3 \wedge \eta_1,$$

$$\omega_3 = 2(\eta_4 \wedge \eta_7 - \eta_5 \wedge \eta_6) = d\eta_3 + 2\eta_1 \wedge \eta_2.$$

We endow $\overline{M} = \mathbb{R}_+ \times M$ with the metric

$$\bar{g} = dt^2 + \sum_{i=1}^{3} A_i^2(t)\eta_i^2 + B^2(t)g\big|_{\mathcal{H}}, \tag{2}$$

where the functions $A_i(t)$ and $B(t)$ are defined on the interval $\mathbb{R}_+ = (0,\infty)$. Let

$$\Omega = \eta_4 \wedge \eta_5 \wedge \eta_6 \wedge \eta_7 = -\frac{1}{8}\omega_1 \wedge \omega_1 = -\frac{1}{8}\omega_2 \wedge \omega_2 = -\frac{1}{8}\omega_3 \wedge \omega_3.$$

Consider the 4-form

$$\Phi = e^0 \wedge e^1 \wedge e^2 \wedge e^3 + B^4\Omega + \frac{1}{2}B^2(e^0 \wedge e^1 - e^2 \wedge e^3) \wedge \omega_1$$
$$+ \frac{1}{2}B^2(e^0 \wedge e^2 - e^3 \wedge e^1) \wedge \omega_2 + \frac{1}{2}B^2(e^0 \wedge e^3 - e^1 \wedge e^2) \wedge \omega_3,$$

where

$$e^0 = dt, \qquad e^i = A_i\eta_i, \quad i = 1,2,3, \qquad \text{and} \qquad e^j = B\eta_j, \quad j = 4,\dots,7.$$

Obviously, the form Φ is defined globally on $\overline{M}$ and coincides locally with Φ_0.

Using the obvious identities

$$d\eta_i = \omega_i - 2\eta_{i+1} \wedge \eta_{i+2}, \qquad i = 1,2,3 \mod 3,$$

$$d\omega_i = 2d(\eta_{i+1} \wedge \eta_{i+2}) = 2(\omega_{i+1} \wedge \eta_{i+2} - \eta_{i+1} \wedge \omega_{i+2}), \qquad i = 1,2,3 \mod 3,$$

we obtain the following relations, which close the exterior algebra of the forms introduced above:

$$de^0 = 0,$$

$$de^i = \frac{\dot{A}_i}{A_i}e^0 \wedge e^i + A_i\omega_i - \frac{2A_i}{A_{i+1}A_{i+2}}e^{i+1} \wedge e^{i+2} \qquad \text{for} \quad i = 1,2,3 \mod 3, \tag{3}$$

$$d\omega_i = \frac{2}{A_{i+2}}\omega_{i+1} \wedge e^{i+2} - \frac{2}{A_{i+1}}e^{i+1} \wedge \omega_{i+2} \qquad \text{for} \quad i = 1,2,3 \mod 3.$$

The following lemma is proved by direct calculations based on relations (3).

Lemma 1. *Condition* (1) *is equivalent to the system of ordinary differential equations*

$$\dot{A}_1 = \frac{2A_1^2}{B^2} + \frac{(A_2 - A_3)^2 - A_1^2}{A_2A_3},$$

$$\dot{A}_2 = \frac{2A_2^2}{B^2} + \frac{(A_3 - A_1)^2 - A_2^2}{A_1A_3},$$

$$\dot{A}_3 = \frac{2A_3^2}{B^2} + \frac{(A_1 - A_2)^2 - A_3^2}{A_1A_2}, \tag{4}$$

$$\dot{B} = -\frac{A_1 + A_2 + A_3}{B}.$$

In [1], with each 3-Sasakian manifold M, we associated two orbifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ that resolve the conical singularity of $\overline{M}$ in the following two ways.

The space $\mathcal{M}_1$. Consider the standard action of the group $\mathrm{Sp}(1)$ represented by unit quaternions on $\mathbb{R}^4=\mathbb{H}$ and the corresponding action of $\mathrm{SO}(3)=\mathrm{Sp}(1)/\mathbb{Z}_2$ on $\mathbb{R}^4/\mathbb{Z}_2$:

$$q\in \mathrm{Sp}(1)\colon\; x\in\mathbb{H}\mapsto qx\in\mathbb{H}.$$

Let $\mathcal{M}_1$ be the fiber space with fiber $\mathbb{R}^4$ or $\mathbb{R}^4/\mathbb{Z}_2$ associated with the principal bundle $M\to\mathcal{O}$ with respect to this action. The orbifold $\mathcal{O}$ is embedded in $\mathcal{M}_1$ as the zero fiber, and $\mathcal{M}_1\setminus\mathcal{O}$ is foliated by spherical sections diffeomorphic to M and collapsing to the zero fiber $\mathcal{O}$.

The space $\mathcal{M}_2$. Let $S\simeq S^1$ be a subgroup in $\mathrm{Sp}(1)$ or $\mathrm{SO}(3)$ that integrates the Killing field ξ^1. Consider the action of S on $\mathbb{R}^2=\mathbb{C}$ defined by

$$e^{i\phi}\in S\colon z\in\mathbb{C}\to ze^{i\phi}\in\mathbb{C}.$$

Let $\mathcal{Z}$ denote the twistor space of the manifold M. The bundle $M\to\mathcal{Z}$ is principal with the structure group S. Let $\mathcal{M}_2$ be the fiber space with fiber $\mathbb{R}^2$ associated with $\pi'\colon M\to\mathcal{Z}$. Then the orbifold $\mathcal{Z}$ is embedded in $\mathcal{M}_2$ as the zero fiber, and $\mathcal{M}_2\setminus\mathcal{Z}$ is foliated by spherical sections diffeomorphic to M and collapsing to the zero fiber $\mathcal{Z}$. We will need the following modification of this construction. For any positive integer p, there exists an obvious embedding $\mathbb{Z}_p\subset S$, and $\mathbb{Z}_p$ acts on $\mathcal{M}_2$ by isometries. Therefore, the orbifold $\mathcal{M}_2/\mathbb{Z}_p$ is well defined, and it is a manifold if and only if $\mathcal{M}_2$ is a manifold. It is easy to see that $\mathcal{M}_2/\mathbb{Z}_p$ is a bundle with fiber $\mathbb{C}$ associated with the principal bundle $\pi'\colon M\to\mathcal{Z}$ by means of the action

$$e^{i\phi}\in S\colon\; z\in\mathbb{C}\to ze^{ip\phi}\in\mathbb{C}.$$

Under certain boundary conditions, metric (2) gives a smooth Riemannian metric on $\mathcal{M}_1$ or $\mathcal{M}_2$. The following lemmas, which were proved in [1], describe these conditions.

Lemma 2. *Let $A_i(t)$, $i=1,2,3$, and $B(t)$ be a C^∞-smooth solution of system* (4) *on the interval* $[0,\infty)$. *Then metric* (2) *can be extended to a smooth metric on $\mathcal{M}_1$ if and only if the following conditions hold*:

(i) $A_1(0)=A_2(0)=A_3(0)=0$ *and* $|\dot A_1(0)|=|\dot A_2(0)|=|\dot A_3(0)|=1$;

(ii) $B(0)\neq 0$ *and* $\dot B(0)=0$;

(iii) *the functions A_1, A_2, A_3, and B do not change sign on the interval* $(0,\infty)$.

Lemma 3. *Under the conditions of Lemma* 2, *let $p=4$ if the common fiber of M is isometric to* $\mathrm{Sp}(1)$ *and $p=2$ if the fiber is* $\mathrm{SO}(3)$. *Then metric* (2) *can be extended to a smooth metric on $\mathcal{M}_2/\mathbb{Z}_p$ if and only if the following conditions hold*:

(i) $A_1(0)=0$ *and* $|\dot A_1(0)|=4$;

(ii) $A_2(0)=-A_3(0)\neq 0$ *and* $\dot A_2(0)=\dot A_3(0)$;

(iii) $B(0)\neq 0$ *and* $\dot B(0)=0$;

(iv) *the functions A_1, A_2, A_3, and B do not change sign on the interval* $(0,\infty)$.

Before proceeding to study system (4), we list available exact solutions of this system. Setting $A_1=A_2=A_3$, we can integrate system (4) by elementary methods and obtain the following $\mathrm{Spin}(7)$-holonomy metric on $\mathcal{M}_1$ [3]:

$$\bar g=\frac{dr^2}{1-\left(\frac{r_0}{r}\right)^{10/3}}+\frac{9}{25}r^2\left(1-\left(\frac{r_0}{r}\right)^{10/3}\right)\sum_{i=1}^{3}\eta_i^2+\frac{9}{5}r^2 g|_{\mathcal{H}}. \tag{5}$$

Note that metric (5) was the first example of a complete metric with holonomy $\mathrm{Spin}(7)$.

For $A_2 = A_3$, system (4) can be explicitly integrated in terms of hypergeometric functions, which yields the following metric on $\mathcal{M}_1$:

$$\bar{g} = \frac{vf\,dz^2}{4z(1-z^2)(1-z)(v-2)} + \frac{16(v-2)zf}{(1+z)v^3}\eta_1^2 + \frac{4(v-2)zf}{(1+z)v}(\eta_2^2+\eta_3^2) + fg\big|_{\mathcal{H}}, \tag{6}$$

where

$$v(z) = \frac{2k\sqrt{z}}{(1-z^2)^{1/4}} - 2z\,{}_2F_1\left[1,\frac{1}{2};\frac{5}{4};1-z^2\right], \qquad f(z) = \left(\frac{1+z}{1-z}\right)^{1/2}\exp\left[\int^z \frac{dz'}{v(z')(1-z'^2)}\right],$$

and k is an integration constant.

For $k=0$, we obtain the following metric on $\mathcal{M}_1$, which can be expressed in terms of elementary functions:

$$\bar{g} = \frac{(r-r_0)^2}{(r+r_0)(r-3r_0)}\,dr^2 + 4r_0^2\frac{(r+r_0)(r-3r_0)}{(r-r_0)^2}\eta_1^2 + (r+r_0)(r-3r_0)(\eta_2^2+\eta_3^2) + 2(r^2-r_0^2)g\big|_{\mathcal{H}}. \tag{7}$$

Metrics (6) and (7) were found in [5] for $M = S^4$.

Finally, setting $A_2 = -A_3$, we come to the following metric on $\mathcal{M}_2/\mathbb{Z}_p$ (where, as in Lemma 5, $p=4$ or $p=2$ depending on the common fiber of M), which has the holonomy $\mathrm{SU}(4) \subset \mathrm{Spin}(7)$:

$$\bar{g} = \frac{dr^2}{1-\left(\frac{r_0}{r}\right)^8} + r^2\left(1-\left(\frac{r_0}{r}\right)^8\right)\eta_1^2 + r^2(\eta_2^2+\eta_3^2) + r^2 g\big|_{\mathcal{H}}. \tag{8}$$

As far as we know, this metric was first described in [14, 15].

3. Spin(7)-HOLONOMY METRICS ON $\mathcal{M}_1$

Consider the standard space $\mathbb{R}^4$ and let $R(t) \in \mathbb{R}^4$ denote the vector formed by the functions $A_1(t)$, $A_2(t)$, $A_3(t)$, and $B(t)$. Define a function $V\colon \mathbb{R}^4 \to \mathbb{R}^4$ as

$$V(a_1,a_2,a_3,a_4) = \left(\frac{2a_1^2}{a_4^2} + \frac{(a_2-a_3)^2-a_1^2}{a_2a_3}, \frac{2a_2^2}{a_4^2} + \frac{(a_3-a_1)^2-a_2^2}{a_1a_3}, \frac{2a_3^2}{a_4^2} + \frac{(a_1-a_2)^2-a_3^2}{a_1a_2}, -\frac{a_1+a_2+a_3}{a_4}\right)$$

(of course, V is defined only on the domain where $a_i \neq 0$). Then system (4) takes the form

$$\frac{dR}{dt} = V(R).$$

Since V is invariant with respect to the homotheties $(a_1,a_2,a_3,a_4) \mapsto (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4)$ in the space $\mathbb{R}^4$, we can make the change $R(t) = f(t)S(t)$, where

$$|S(t)| = 1, \qquad f(t) = |R(t)|, \qquad \text{and} \qquad S(t) = \big(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t)\big).$$

Thus, we "normalize" the vector function R, and the system under consideration decomposes into "radial" and "tangential" parts as

$$\frac{dS}{du} = V(S) - \langle V(S), S\rangle S = W(S), \tag{9}$$

$$\frac{1}{f}\frac{df}{du} = \langle V(S), S\rangle, \qquad dt = f\,du. \tag{10}$$

Thus, we need first to solve the autonomous system (9) on the 3-sphere $S^3 = \{(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \mid \sum_{i=1}^4 \alpha_i^2 = 1\}$, after which solutions of (4) can be found from equations (10) by ordinary integration. The following lemmas are obvious.

Lemma 4. *System* (9) *admits a discrete group of symmetries* $S_3\times\mathbb{Z}_2\times\mathbb{Z}_2$ *generated by the transformations*

$$\sigma\in S_3\colon\ (\alpha_1,\alpha_2,\alpha_3,\alpha_4)\mapsto\big(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\alpha_{\sigma(3)},\alpha_4\big),$$

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\mapsto(\alpha_1,\alpha_2,\alpha_3,-\alpha_4),$$

$$\big(\alpha_1(u),\alpha_2(u),\alpha_3(u),\alpha_4(u)\big)\mapsto\big(-\alpha_1(-u),-\alpha_2(-u),-\alpha_3(-u),\alpha_4(-u)\big)$$

(*here,* S_3 *denotes the symmetric group*).

Lemma 5. *The stationary solutions of system* (9) *on* S^3 *are exhausted by the following zeros of the vector field* W:

$$\pm\left(\frac{\sqrt2}{4},\frac{\sqrt2}{4},\frac{\sqrt2}{4},\pm\frac{\sqrt{10}}{4}\right),\quad \pm\left(-\frac12,-\frac12,\frac12,\pm\frac12\right),\quad \pm\left(-\frac12,\frac12,-\frac12,\pm\frac12\right),\quad \pm\left(\frac12,-\frac12,-\frac12,\pm\frac12\right).$$

We say that a point $S\in S^3$ at which the field W is not defined is *conditionally stationary* if there exists a smooth curve $\gamma(u)$ on S^3 with $u\in(-\varepsilon,\varepsilon)$ and $\gamma(0)=S$ such that the field W is defined at all points of $\gamma(u)$ with nonzero $u\in(-\varepsilon,\varepsilon)$ and $\lim_{u\to0}W(\gamma(u))=0$. The following two lemmas were proved in [1].

Lemma 6. *System* (9) *has the following conditionally stationary points in* S^3:

$$\pm\left(0,\frac12,\frac12,\pm\frac{1}{\sqrt2}\right),\quad \pm\left(\frac12,0,\frac12,\pm\frac{1}{\sqrt2}\right),\quad \pm\left(\frac12,\frac12,0,\pm\frac{1}{\sqrt2}\right).$$

Lemma 7. *The stationary solutions of system* (9) *give rise to locally conical metrics on* $\overline{M}$, *and the trajectories of system* (9) *that tend asymptotically to* (*conditionally*) *stationary solutions give rise to asymptotically locally conical metrics on* $\overline{M}$.

Remark. It is easy to see that the trajectory of system (9) corresponding to solution (5) converges to the stationary point $\big(\frac{\sqrt2}{4},\frac{\sqrt2}{4},\frac{\sqrt2}{4},\frac{\sqrt{10}}{4}\big)$, the trajectory corresponding to solution (8) converges to the stationary point $\big(\frac12,\frac12,-\frac12,\frac12\big)$, and the trajectories corresponding to solutions (6) and (7) converge to the conditionally stationary point $\big(0,\frac12,\frac12,\frac{1}{\sqrt2}\big)$.

To construct a smooth metric on $\mathcal{M}_1$, we blow up the sphere S^3 at the points $Q_\pm=(0,0,0,\pm1)$. Recall what the blow-up operation looks like. In a neighborhood of the point Q_+, consider local coordinates $(\alpha_1,\alpha_2,\alpha_3)$ and the ball $U=\{(\alpha_1,\alpha_2,\alpha_3)\mid\sum_{i=1}^3\alpha_i^2\le\varepsilon^2\}$ of radius ε.

In the neighborhood U, we introduce a geodesic coordinate system; i.e., we consider two coordinates, a radial one $-\varepsilon<r<\varepsilon$ and a tangential one $s\in S^2$, where $S^2=\{(\alpha_1,\alpha_2,\alpha_3)\mid\sum_{i=1}^3\alpha_i^2=1\}$. Thus, $(\alpha_1,\alpha_2,\alpha_3)=rs$. Now, consider the product $(-\varepsilon,\varepsilon)\times S^2$ and the action of the group $\mathbb{Z}_2$ on it defined by

$$(r,s)\mapsto(-r,-s).$$

Clearly, this action is free, and we obtain the quotient space $\widetilde U=(-\varepsilon,\varepsilon)\times S^2/\mathbb{Z}_2$. The correspondence

$$\pm(r,s)\mapsto rs$$

defines a smooth mapping $\widetilde U\to U$, which is obviously a diffeomorphism $\widetilde U\setminus P\to U\setminus Q_+$, where $P=\{(r,s)\mid r=0\}$ is the projective plane embedded in $\widetilde U$.

We remove the point Q_+ from the neighborhood U and attach $\widetilde{U}$ using the diffeomorphism constructed above. The manifold thus obtained is said to be the blow-up of S^3 at the point Q_+.

Let $\widetilde{S}$ denote the sphere S^3 blown up at the points $Q_\pm$. By symmetry, it suffices to consider only a neighborhood of Q_+. We need local coordinates in a neighborhood of P. Consider $U_i = \{\pm(r,s) \mid \alpha_i \neq 0\}$, where $i = 1,2,3$. In each neighborhood U_i, we set

$$\alpha_i^i = \alpha_i \qquad \text{and} \qquad \alpha_j^i = \frac{\alpha_j}{\alpha_i} \quad \text{for } i \neq j.$$

We have thereby defined local coordinates $\alpha_1^i, \alpha_2^i, \alpha_3^i$ on $\widetilde{U}$ in the neighborhoods U_i for $i = 1,2,3$.

Lemma 8. *A smooth metric of the form* (2) *on* $\mathcal{M}_1$ *satisfies, in addition to the conditions of Lemma* 2*, the following boundary conditions for* $t = 0$:

$$\dot{A}_1(0) = \dot{A}_2(0) = \dot{A}_3(0) = -1.$$

Proof. Suppose that $A_i(t) = c_i t + o(t)$, where $|c_i| = 1$ by Lemma 2. It follows immediately from (4) that

$$c_1 = \frac{(c_2 - c_3)^2 - c_1^2}{c_2 c_3}, \qquad c_2 = \frac{(c_1 - c_3)^2 - c_2^2}{c_1 c_3}, \qquad \text{and} \qquad c_3 = \frac{(c_1 - c_2)^2 - c_3^2}{c_1 c_2}.$$

Subtracting the second equation multiplied by $c_1 c_3$ from the first multiplied by $c_2 c_3$, we obtain either $c_1 = c_2$ or $c_1 + c_2 = 2c_3$. Since the equations are symmetric with respect to permutations, we have $c_1 = c_2 = c_3 = -1$, which proves the lemma.

Lemma 9. *There exists a two-parameter family of solutions to system* (4) *in a neighborhood of* $t = 0$ *that satisfy the boundary conditions of Lemma* 8 *and thereby deliver* Spin(7)*-holonomy metrics on* $\mathcal{M}_1$ *that are smooth in a neighborhood of* $t = 0$.

This family of solutions is parameterized by the number triples $\lambda_1, \lambda_2, \lambda_3 < 0$ *such that* $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \varepsilon^2$ *for a sufficiently small* $\varepsilon > 0$; *for every such triple, there exists a value* $t = t_0$ *at which the trajectory* (A_1, A_2, A_3) *passes through this triple, i.e.,*

$$A_1(t_0) = \lambda_1, \qquad A_2(t_0) = \lambda_2, \qquad \textit{and} \qquad A_3(t_0) = \lambda_3.$$

Proof. We carry over system (9) to $\widetilde{S}$, after which the projections of the trajectories on S^3 give the required solutions. According to the above considerations, we need to study the trajectories of system (9) on $\widetilde{S}$ that start at the point $\alpha_1^1 = 0$, $\alpha_2^1 = \alpha_3^1 = 1$. Let us rewrite the field W in the neighborhood U_1 in the new coordinates. For simplicity, we set $x = \alpha_1^1$, $y = \alpha_2^1$, and $z = \alpha_3^1$. Then system (9) is equivalent to

$$\begin{aligned} \frac{dx}{dv} &= xW_1(x, xy, xz) = \widetilde{W}_1(x,y,z), \\ \frac{dy}{dv} &= W_2(x, xy, xz) - yW_1(x, xy, xz) = \widetilde{W}_2(x,y,z), \\ \frac{dz}{dv} &= W_3(x, xy, xz) - zW_3(x, xy, xz) = \widetilde{W}_3(x,y,z), \end{aligned} \tag{11}$$

where $du = x\,dv$.

It can be verified directly that the vector field $\widetilde{W}$ vanishes at $p = (0,1,1)$. Consider the linearization of system (11) in a neighborhood of this point:

$$\frac{dx}{dv} = -x, \qquad \frac{dy}{dv} = -2y, \qquad \frac{dz}{dv} = -2z.$$

Thus, there exists a three-parameter family of trajectories of system (9) that reach the point p exponentially with respect to the variable v. Changing the parameter v for u, we obtain a two-parameter family of trajectories that reach p from the side of positive x in finite time and a two-parameter family of trajectories that emanate from p in the direction of negative x in finite time u and, therefore, in finite time t. The required parameterization of the family of trajectories is quite obvious. This completes the proof of the lemma.

The next lemma is proved by a direct analysis of systems (4) and (9).

Lemma 10. *If $S = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a solution of system* (10), *then*

$$\frac{d}{du}\left(\ln\left|\frac{\alpha_2}{\alpha_1}\right|\right) = 2\frac{(\alpha_2-\alpha_1)(\alpha_2-\alpha_4)(\alpha_2+\alpha_4)}{\alpha_2^2\alpha_4^2} \qquad \text{if} \quad \alpha_2 = \alpha_3, \tag{12}$$

$$\frac{d}{du}(\alpha_2+\alpha_4) = \frac{(\alpha_2+\alpha_3)(\alpha_1+\alpha_2)(\alpha_1-\alpha_2+\alpha_3)}{\alpha_1\alpha_2\alpha_3} \qquad \text{if} \quad \alpha_2+\alpha_4 = 0, \tag{13}$$

$$\frac{d}{du}(\alpha_2+\alpha_3) \to 16\left(\alpha_2-\frac{1}{2}\right)\left(\alpha_2+\frac{1}{2}\right) \qquad \text{as} \quad \alpha_2-\alpha_3 \to 0 \quad \text{and} \quad \alpha_1 \to 0, \tag{14}$$

$$\frac{d}{du}\left(\ln\left|\frac{\alpha_1(\alpha_2-\alpha_3)}{\alpha_2(\alpha_1-\alpha_3)}\right|\right) = \frac{4}{\alpha_2} - \frac{4}{\alpha_1}. \tag{15}$$

Lemma 11. *Consider a trajectory of system* (9). *According to Lemma* 9, *this trajectory is parameterized by a number triple $\lambda_1, \lambda_2, \lambda_3 < 0$, where $\lambda_1^2+\lambda_2^2+\lambda_3^2 = \varepsilon^2$. From symmetry considerations, we can assume without loss of generality that $\lambda_3 \le \lambda_2 \le \lambda_1$. Then*

(1) *for $\lambda_1 = \lambda_2 = \lambda_3$, the trajectory converges to the stationary point $\left(-\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, \frac{\sqrt{10}}{4}\right)$ as $u \to \infty$;*

(2) *for $\lambda_1 \ne \lambda_2 = \lambda_3$, the trajectory converges to the conditionally stationary point $\left(0, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ as $u \to \infty$;*

(3) *in the remaining cases, the trajectory converges to the point* $(0,0,1,0)$ *in finite time, i.e., as $u \to u_0 < \infty$.*

Proof. We use the notation

$$O = Q_+ = (0,0,0,1), \qquad A = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right), \qquad B = (0,0,-1,0),$$

$$C = \left(0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \qquad D = \left(-\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, \frac{\sqrt{10}}{4}\right), \qquad E = \left(0, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right).$$

Since system (9) is symmetric with respect to permutations of the variables α_1, α_2, and α_3, we can assume without loss of generality that the trajectory $S(u)$ starts from the point O at $u = u_0$, enters the spherical tetrahedron $\Pi = OABC$, and is tangent to OA at O. Note that the tetrahedron Π is determined by the relations $\alpha_3 \le \alpha_2 \le \alpha_1 \le 0$ and $\alpha_4 \ge 0$.

Consider possible situations.

1. The trajectory $S(u)$ goes along the segment OA. In this case, $A_1 = A_2 = A_3$, and we obtain solution (5), which converges to D.

2. The trajectory $S(u)$ goes along the invariant wall OAC but not along the segment OA, i.e., $\alpha_2 = \alpha_3$. On the wall OAC, consider the function

$$F_1 = \frac{\alpha_2}{\alpha_1}.$$

It follows from relation (12) in Lemma 10 that the function F_1 strictly increases along the trajectories of system (9) while $\alpha_2 + \alpha_4 > 0$. This condition surely holds in a neighborhood of $u = u_0$, i.e., at the beginning of the trajectory. Consider the function

$$F_2 = \alpha_2 + \alpha_4.$$

It follows from relation (13) in Lemma 10 that the function F_2 strictly increases along the trajectories of system (9) in a neighborhood of those points at which $F_2 = 0$. Therefore, the trajectory can approach the set $F_2 = 0$ only from the side of negative values of the function F_2, which implies that $F_2 > 0$ along the whole trajectory. Thus, the function F_2 is positive for all u, and so F_1 strictly increases along the trajectory $S(u)$.

Finally, relation (14) in Lemma 10 shows that the trajectory $S(u)$ converges to the conditionally stationary point E.

3. The trajectory $S(u)$ goes along the invariant wall OAB, i.e., $\alpha_1 = \alpha_2$. Consider the function

$$F_3 = \frac{\alpha_1}{\alpha_3}$$

defined on OBA. Relation (12) in Lemma 10 (after a suitable permutation of indices) shows that this function strictly decreases at all points where $\alpha_1 + \alpha_4 > 0$. By analogy with the preceding case, consider the function

$$F_4 = \alpha_1 + \alpha_4.$$

It follows from (13) that F_4 increases at all points where $F_4 = 0$. At the initial point $t = t_0$, we have $F_4 > 0$; therefore, F_4 is positive along the whole trajectory, i.e., $S(u)$ tends to the segment OB. A direct verification shows that, inside the interval OB, the α_3-component of the vector field W is strictly negative, i.e., the trajectory $S(u)$ converges to the point B.

4. After $t = t_0$, the trajectory $S(u)$ goes strictly inside the domain Π. Consider the function

$$F_5 = \frac{\alpha_1(\alpha_2 - \alpha_3)}{\alpha_2(\alpha_1 - \alpha_3)}$$

defined on the interior of Π. Relation (15) in Lemma 10 shows that F_5 strictly increases along the trajectories of system (9) in the domain Π. The function F_5 has no extremal points inside Π; therefore, the trajectory $S(u)$ approaches one of the boundary walls of Π. Moreover, $F_5 \geq 0$, and $F_5 = 0$ on the wall $\alpha_1 = 0$. Thus, the wall OAC is unattainable for the trajectory $S(u)$. Since the wall OAB is an invariant subset for system (9), the trajectory cannot cross it. If $S(u)$ tends to OAB, then the trajectories converge to B, because the trajectories in OAB converge to B.

Next, since the function F_5 increases, the trajectory cannot approach the wall OBC (at least, outside the interval OB). Consider the remaining case where the trajectory reaches the wall ABC. If the trajectory $S(u)$ can reach ABC in finite time at some point $S_1 = S(u_1)$, then $S_1 = B$. Indeed, if $S_1 \neq B$, then we can consider the parameter α_4^2 varying along the curve $S(u)$. It is easy to show that the component of W normal to ABC is then bounded, while the tangential component tends to infinity, which is a contradiction.

Thus, only one case is possible, in which the trajectory reaches the point B in finite time. This completes the proof of the lemma. □

The following lemma, together with Lemmas 9 and 11 and the results of [1], immediately implies the main theorem.

Lemma 12. *The metrics whose existence is claimed in Lemmas* 9 *and* 11 *have the holonomy* Spin(7).

Proof. It is sufficient to consider only the case of an incomplete metric. In this case, the trajectory of system (9) converges to the point $(0, 0, -1, 0)$ in finite time (as $t \to t_1$). This means that the limit tangent cone to $\mathcal{M}_1$ at $t = t_1$ is locally isometric to the product of a circle (corresponding to the variable A_3) and a Riemannian space that is topologically a cone over $\mathcal{Z}$. Repeating an argument from [1, Lemma 14] almost without changes, we see that the holonomy group of metric (2) must coincide with the whole group Spin(7). This proves the lemma.

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Spin(7)-structures on complex linear bundles and explicit Riemannian metrics with holonomy group SU(4)

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Abstract. A system of differential equations with 5 unknowns is fully investigated; this system is equivalent to the existence of a parallel Spin(7)-structure on a cone over a 3-Sasakian manifold. A continuous one-parameter family of solutions to this system is explicitly constructed; it corresponds to metrics with a special holonomy group, SU(4), which generalize Calabi's metrics.

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Keywords: holonomy group, 3-Sasakian manifold.

§ 1. Introduction

1.1. The first example of a complete Riemannian metric with holonomy group SU(n) was the Calabi metric, which was described explicitly in terms of algebraic functions in [1]. The Calabi metric is constructed on the total space of an appropriate complex linear bundle on an arbitrary Kähler-Einstein manifold F. If we take F to be the complex projective space $\mathbb{C}P^{n-1}$, the resulting Calabi metric is asymptotically locally Euclidean (ALE), otherwise it is asymptotically conical (AC). In [1] Calabi also considered hyper-Kähler metrics and constructed a complete Riemannian metric on $T^*\mathbb{C}P^m$ with holonomy group Sp(m) explicitly; this was the first explicit example of a hyper-Kähler metric.

In this paper we make an explicit construction, in algebraic form, of a one-parameter family of complete Riemannian metrics 'connecting' these two Calabi metrics in the space of special Kähler metrics in 8-dimensional spaces when F is the complex 3-flag manifold of $\mathbb{C}^3$; we also carry out a full investigation of the existence problem for metrics with holonomy group Spin(7) on an appropriate bundle on F. In this case the tangent space of the 4-dimensional quaternionic Kähler manifold $\mathscr{O}$

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associated with F can be 'split', which allows us to introduce an additional parameter describing deformations of the metric and to obtain a well-defined system of differential equations.

There is quite a lot of interest in explicit metrics with special holonomy groups (and, in particular, in special Kähler metrics), because only a few examples of this type are known. For instance, Joyce [2], 8.2.5, conjectured that all the other ALE-metrics with holonomy group $\mathrm{SU}(n)$ for $n \geqslant 3$ (apart from the Calabi metric for $F = \mathbb{C}P^{n-1}$) are 'transcendental', that is, they cannot be represented in algebraic form. We stress that the metrics we construct here are asymptotically conical (AC), but not ALE, so our example does not refute Joyce's conjecture.

1.2. More precisely, let

$$M = \mathrm{SU}(3)/\mathrm{U}(1)_{1,1,-2}$$

be the Aloff-Wallach space, which carries the structure of a 7-dimensional 3-Sasakian manifold. We consider the Riemannian metric of the following form on $\overline{M} = M \times \mathbb{R}_+$:

$$dt^2 + A_1(t)^2\eta_1^2 + A_2(t)^2\eta_2^2 + A_3(t)^2\eta_3^2 + B(t)^2(\eta_4^2 + \eta_5^2) + C(t)^2(\eta_6^2 + \eta_7^2), \qquad (1.1)$$

where t is the variable on $\mathbb{R}_+$ and $\{\eta_i\}$ is an orthonormal coframe on M agreeing with the 3-Sasakian structure (see § 2 for details). We resolve the conical singularity of $\overline{M}$ (for $t = 0$) as follows: on the level $\{t = 0\}$ we contract each circle corresponding to the covector η_1 to a point. This gives us a manifold whose quotient by $\mathbb{Z}_2$ is diffeomorphic to $H/\mathbb{Z}_2$, the square of the canonical complex linear bundle over the flag space of $\mathbb{C}^3$.

Theorem 1. *For $0 \leqslant \alpha < 1$ each Riemannian metric in the family*

$$\begin{aligned}\overline{g}_\alpha = \frac{r^4(r^2-\alpha^2)(r^2+\alpha^2)}{r^8 - 2\alpha^4(r^4-1) - 1}dr^2 + \frac{r^8 - 2\alpha^4(r^4-1) - 1}{r^2(r^2-\alpha^2)(r^2+\alpha^2)}\eta_1^2 + r^2(\eta_2^2+\eta_3^2) \\ + (r^2+\alpha^2)(\eta_4^2+\eta_5^2) + (r^2-\alpha^2)(\eta_6^2+\eta_7^2)\end{aligned} \qquad (1.2)$$

is a complete smooth Riemannian metric on $H/\mathbb{Z}_2$ with holonomy group $\mathrm{SU}(4)$. *For $\alpha = 0$ the metric* (1.2) *is isometric to the Calabi metric with holonomy group* $\mathrm{SU}(4)$; *for $\alpha = 1$,* (1.2) *is isometric to the Calabi metric on $T^*\mathbb{C}P^2$ with holonomy group* $\mathrm{Sp}(2) \subset \mathrm{SU}(4)$ (*see* [1]).

Note that the metrics (1.2) in Theorem 1 corresponding to $\alpha = 0$ and $\alpha = 1$ have a different form from the metrics in [1]; Calabi metrics in this form were investigated in [3] and [4]. The metric (1.2) for $M = \mathrm{SU}(3)/\mathrm{U}(1)_{1,1,-2}$ was also obtained in [5] as a particular solution of a system of equations for metrics with holonomy group $\mathrm{Spin}(7)$.

The above result was obtained when we were making a systematic investigation of metrics of the form (1.1) with holonomy group $\mathrm{Spin}(7)$ by a method developed in [6] and then used in [7] and [8]: a metric (1.1) is constructed for an arbitrary 7-dimensional 3-Sasakian manifold M and carries a natural $\mathrm{Spin}(7)$-structure. The condition that this structure be parallel reduces to the following nonlinear system of ordinary differential equations:

$$
\begin{aligned}
A_1' &= \frac{(A_2-A_3)^2-A_1^2}{A_2A_3}+\frac{A_1^2(B^2+C^2)}{B^2C^2},\\
A_2' &= \frac{A_1^2-A_2^2+A_3^2}{A_1A_3}-\frac{B^2+C^2-2A_2^2}{BC},\\
A_3' &= \frac{A_1^2+A_2^2-A_3^2}{A_1A_2}-\frac{B^2+C^2-2A_3^2}{BC},\\
B' &= -\frac{CA_1+BA_2+BA_3}{BC}-\frac{(C^2-B^2)(A_2+A_3)}{2A_2A_3C},\\
C' &= -\frac{BA_1+CA_2+CA_3}{BC}-\frac{(B^2-C^2)(A_2+A_3)}{2A_2A_3B}.
\end{aligned}
\tag{1.3}
$$

Note that for $B=C$ the system (1.3) was investigated fully in [6] and [8]. To obtain a smooth metric (1.1) we must resolve the conical singularity of $\overline{M}$ using one of two methods, which gives a space $\mathscr{M}_1$ or $\mathscr{M}_2$. We shall describe this scheme in § 2. Then the family of metrics (1.2) on $\mathscr{M}_2/\mathbb{Z}_2$ is obtained by integrating the system (1.3) for $A_2=-A_3$ (this is the subject of § 3).

1.3. In § 4 we prove the following result, which completes our analysis of system (1.2) in the case of the space $\mathscr{M}_2$.

Theorem 2. *Let M be a 3-Sasakian 7-manifold. Let $p=2$ or $p=4$, depending on whether the general leaf of the 3-Sasakian foliation on M is* SO(3) *or* SU(2). *Then the orbifold $\mathscr{M}_2/\mathbb{Z}_p$ carries the following complete regular Riemannian metrics $\overline{g}$ of the form* (1.2), *with holonomy group $H\subset$* Spin(7):

1) *if $A_1(0)=0$, $-A_2(0)=A_3(0)>0$ and $2A_2^2(0)=B^2(0)+C^2(0)$, then the metric $\overline{g}$ in* (1.1) *has holonomy group* SU(4) $\subset$ Spin(7) *and is homothetical to some metric in* (1.2);
2) *if $A_1(0)=0$ and $-A_2(0)=A_3(0)<B(0)=C(0)$, then there exists a regular ALC-metric $\overline{g}$ of the form* (1.1) *with holonomy group* Spin(7); *this was found in* [6]. *At infinity such metrics converge to the product of a cone over the twistor space $\mathscr{Z}$ and the circle S^1.*

Moreover, each complete regular metric of the form (1.1) *on $\mathscr{M}_2/\mathbb{Z}_p$, which has the* Spin(7)*-structure mentioned above and a holonomy group $H\subset$* Spin(7) *is isometric to one of the above metrics.*

§ 2. The description of a Spin(7)-structure on a cone over a 3-Sasakian manifold

2.1. In this section we describe briefly how to construct the spaces on which we find metrics with holonomy Spin(7). In our notation and definitions relating to a 3-Sasakian manifold we follow [6]. For more detail we direct the reader to [9].

By a *cone $\overline{M}$ over a smooth closed Riemannian manifold* (M,g) we mean a Riemannian manifold $(\mathbb{R}_+\times M, dt^2+t^2g)$, $t\in\mathbb{R}_+=(0,\infty)$. The manifold M is said to be 3-*Sasakian* if the metric on $\overline{M}$ is hyper-Kähler, that is, its holonomy group lies in $\mathrm{Sp}\bigl(\frac{m+1}{4}\bigr)$. Then there exist three parallel complex structures J^1, J^2 and J^3 on $\overline{M}$, which satisfy $J^jJ^i=-\delta^{ij}+\varepsilon_{ijk}J^k$. Identifying M with the 'base'

$\overline{M}\cap\{t=1\}$ of the cone, we consider the vector fields $\xi^i = J^i(\partial_t)$, $i=1,2,3$, on M. The fields ξ^i are called *characteristic fields of the 3-Sasakian manifold* M, and the dual 1-forms η_i are called *characteristic forms*. We can show that the ξ^i form the Lie algebra su(2) with respect to the Lie bracket of vector fields, so we have a fibration $\pi\colon M\to\mathscr{O}$ with general fibre $\mathrm{SU}(2)=S^3$ (or $\mathrm{SO}(3)=\mathbb{R}P^3$) over some 4-dimensional quaternionic Kähler orbifold $\mathscr{O}$. Let $\mathscr{H}$ be the bundle of horizontal vectors (with respect to π) on M.

We consider the following 2-forms on M:

$$\omega_i = d\eta_i + \sum_{j,k}\varepsilon_{ijk}\eta_j\wedge\eta_k, \qquad i=1,2,3.$$

We see immediately (see [6]) that the ω_i span a subspace of $\Lambda^2\mathscr{H}^*$, so we can pick an orthonormal system of 1-forms $\eta_4,\eta_5,\eta_6,\eta_7$ in $\mathscr{H}$ such that

$$\omega_1 = 2(\eta_4\wedge\eta_5-\eta_6\wedge\eta_7), \qquad \omega_2 = 2(\eta_4\wedge\eta_6-\eta_7\wedge\eta_5),$$
$$\omega_3 = 2(\eta_4\wedge\eta_7-\eta_5\wedge\eta_6).$$

Consider the standard Euclidean space $\mathbb{R}^8$ with coordinates $x^0,\dots,x^7$. Setting $e^{ijkl}=dx^i\wedge dx^j\wedge dx^k\wedge dx^l$, we define the following self-dual 4-form on $\mathbb{R}^8$:

$$\Phi_0 = e^{0123}+e^{4567}+e^{0145}-e^{2345}-e^{0167}+e^{2367}+e^{0246} + e^{1346}-e^{0275}+e^{1357}+e^{0347}-e^{1247}-e^{0356}+e^{1256}.$$

We know that the group of linear transformations of $\mathbb{R}^8$ which preserve Φ_0 is isomorphic to Spin(7), and this group Spin(7) also preserves the orientation and the metric $g_0=\sum_{i=0}^7(e^i)^2$. Let N be an oriented Riemannian 8-manifold. We say that a differential form $\Phi\in\Lambda^4N$ defines a Spin(7)-structure on N if there exists an orientation-preserving isometry $\varphi_p\colon T_pN\to\mathbb{R}^8$ in a neighbourhood of each point $p\in N$ such that $\varphi_p^*\Phi_0=\Phi|_p$. If Φ is a parallel form, then the holonomy group of the Riemannian manifold N reduces to the subgroup $\mathrm{Spin}(7)\subset\mathrm{SO}(8)$, that is, $\mathrm{Hol}(N)\subset\mathrm{Spin}(7)$. It is well-known that Φ is parallel if and only if it is closed:

$$\nabla\Phi=0 \iff d\Phi=0$$

(see [10]). We shall construct a Spin(7)-structure on $\overline{M}$. We take the following form for Φ:

$$\Phi = e^{0123}+C^2B^2\eta_4\wedge\eta_5\wedge\eta_6\wedge\eta_7+\frac{B^2+C^2}{4}(e^{01}-e^{23})\wedge\omega_1 + \frac{B^2-C^2}{4}(e^{01}-e^{23})\wedge\omega+\frac{BC}{2}(e^{02}-e^{31})\wedge\omega_2+\frac{BC}{2}(e^{03}-e^{12})\wedge\omega_3,$$

where

$$e^0=dt, \qquad e^i=A_i\eta_i, \quad i=1,2,3, \qquad e^j=B\eta_j, \quad j=4,5,$$
$$e^k=C\eta_k, \quad k=6,7,$$

and $A_1(t)$, $A_2(t)$, $A_3(t)$, $B(t)$ and $C(t)$ are some smooth functions. It is easy to see that Φ corresponds to a Riemannian metric of the form (1.1) on $\overline{M}$.

2.2. We shall assume that the quaternionic Kähler orbifold $\mathscr{O}$ is Kähler, so we can pick a basis η_i, $i = 4, 5, 6, 7$, such that the form $\omega = 2(\eta_4 \wedge \eta_5 + \eta_6 \wedge \eta_7)$ defines a Kähler structure on $\mathscr{O}$ and, in particular, is closed. This assumption closes the exterior algebra of forms under consideration and allows us to derive a well-defined system of equations with respect to the functions A_i, B, C. Note that if $\mathscr{O}$ is not assumed to be Kähler, then generally speaking we must assume that $B = C$ to close the algebra of forms.

Lemma 1. *The fact that the form* Φ *is parallel is equivalent to the following system of ordinary differential equations*:

$$\begin{aligned}
A_1' &= \frac{(A_2 - A_3)^2 - A_1^2}{A_2A_3} + \frac{A_1^2(B^2 + C^2)}{B^2C^2},\\
A_2' &= \frac{A_1^2 - A_2^2 + A_3^2}{A_1A_3} - \frac{B^2 + C^2 - 2A_2^2}{BC},\\
A_3' &= \frac{A_1^2 + A_2^2 - A_3^2}{A_1A_2} - \frac{B^2 + C^2 - 2A_3^2}{BC},\\
B' &= -\frac{CA_1 + BA_2 + BA_3}{BC} - \frac{(C^2 - B^2)(A_2 + A_3)}{2A_2A_3C},\\
C' &= -\frac{BA_1 + CA_2 + CA_3}{BC} - \frac{(B^2 - C^2)(A_2 + A_3)}{2A_2A_3B}.
\end{aligned} \tag{2.1}$$

Proof. Using the relations imposed on the exterior algebra of forms in [6],

$$\begin{aligned}
de^0 &= 0,\\
de^i &= \frac{A_i'}{A_i}e^0 \wedge e^i + A_i\omega_i - \frac{2A_i}{A_{i+1}A_{i+2}}e^{i+1} \wedge e^{i+2}, \qquad i = 1, 2, 3 \ (\mathrm{mod}\, 3),\\
d\omega_i &= \frac{2}{A_{i+2}}\omega_{i+1} \wedge e^{i+2} - \frac{2}{A_{i+1}}e^{i+1} \wedge \omega_{i+2}, \qquad i = 1, 2, 3 \ (\mathrm{mod}\, 3),
\end{aligned}$$

and also the relations $d\omega = 0$ and $\omega_1 \wedge \omega_1 = \omega_2 \wedge \omega_2 = \omega_3 \wedge \omega_3$, after straightforward calculations we obtain

$$\begin{aligned}
d\Phi = {} & \left[\frac{B^2 - C^2}{2A_2A_3}A_1 - \frac{BB' - CC'}{2} - \frac{B^2 - C^2}{4A_2}A_2' - \frac{B^2 - C^2}{4A_3}A_3'\right]e^{023} \wedge \omega\\
& + \left[-A_1 - \frac{BC}{A_3} - \frac{BC}{A_2} + \frac{B^2 + C^2}{2A_2A_3}A_1 - \frac{BB' + CC'}{2}\right.\\
& \qquad \left. - \frac{B^2 + C^2}{4A_2}A_2' - \frac{B^2 + C^2}{4A_3}A_3'\right]e^{023} \wedge \omega_1\\
& + \left[A_2 + \frac{BC}{A_1} - \frac{BCA_2}{A_1A_3} + \frac{B^2 + C^2}{2A_3} + \frac{B'C + BC'}{2} + \frac{BCA_1'}{2A_1} + \frac{BCA_3'}{2A_3}\right]e^{013} \wedge \omega_2\\
& - \left[A_3 + \frac{BC}{A_1} - \frac{BCA_3}{A_1A_2} + \frac{B^2 + C^2}{2A_2} + \frac{B'C + BC'}{2} + \frac{BCA_1'}{2A_1} + \frac{BCA_2'}{2A_2}\right]e^{012} \wedge \omega_3\\
& - \frac{1}{4}\left[2BCA_2 + BCA_3 + (B^2 + C^2)A_1 + C^2BB' + B^2CC'\right]e^0 \wedge \omega_1 \wedge \omega_1.
\end{aligned}$$

Solving a system of 5 linear equations with respect to the 'unknowns' A_1', A_2', A_3', B' and C' we obtain (2.1). The proof is complete.

For $B = C$ we obtain the following system, which was investigated in [6]:

$$A_1' = \frac{2A_1^2}{B^2} + \frac{(A_2 - A_3)^2 - A_1^2}{A_2 A_3}, \qquad A_2' = \frac{2A_2^2}{B^2} + \frac{(A_3 - A_1)^2 - A_2^2}{A_1 A_3},$$
$$A_3' = \frac{2A_3^2}{B^2} + \frac{(A_1 - A_2)^2 - A_3^2}{A_1 A_2}, \qquad B' = -\frac{A_1 + A_2 + A_3}{B}. \tag{2.2}$$

To get a smooth Riemannian metric on a manifold (orbifold) we must prescribe boundary conditions for (2.1). In [6] the spaces $\mathscr{M}_1$ and $\mathscr{M}_2$ corresponding to the two different methods of resolving the conical singularity of $\overline{M}$ were described. Below we describe the space $\mathscr{M}_2$, on which we shall seek a metric with holonomy group $H \subset \mathrm{Spin}(7)$.

Let $S \simeq S^1$ be the subgroup of SU(2) (or SO(3)) integrating one of the Killing fields, for instance ξ^1. Then we have a principal bundle $\pi' \colon M \to \mathscr{Z}$ with structure group S, where $\mathscr{Z} = M/S$ is the twistor space. We consider the natural action of S on $\mathbb{R}^2 = \mathbb{C}$: $e^{i\varphi} \in S \colon z \to ze^{i\varphi}$ and associate the fibred space $\mathscr{M}_2$ with fibre $\mathbb{C}$ on which we have the above action with π'. Thus the orbifold $\mathscr{Z}$ is embedded in $\mathscr{M}_2$ as the zero section, and $\mathscr{M}_2 \setminus \mathscr{Z}$ is foliated by 'spherical' sections diffeomorphic to M and contracting to the zero section $\mathscr{Z}$ as $t \to 0$.

Now let $p \in \mathbb{N}$ and $\mathbb{Z}_p \subset S$. The group $\mathbb{Z}_p$ acts by isometries on $\mathscr{M}_2$, so we have a well-defined orbifold $\mathscr{M}_2/\mathbb{Z}_p$, which is a manifold if and only if $\mathscr{M}_2$ is a manifold. It is easy to see that $\mathscr{M}_2/\mathbb{Z}_p$ is a bundle with fibre $\mathbb{C}$, which is associated with the principal bundle $\pi' \colon M \to \mathscr{Z}$ by means of the action $e^{i\varphi} \in S \colon z \to ze^{ip\varphi}$.

Note that if M is a regular 3-Sasakian manifold (that is, the foliation by 3-dimensional 3-Sasakian leaves is regular), then all the fibres of π are isometric to $S^3 = \mathrm{SU}(2)$ or SO(3) and the orbifolds $\mathscr{O}$, $\mathscr{Z}$ and $\mathscr{M}_2$ are smooth manifolds. We know that this is possible only when M is isometric to S^7, $\mathbb{R}P^7$ or $N_{1,1} = \mathrm{SU}(3)/T_{1,1}$ (see [9]). However, among these examples only the Aloff-Wallach space $N_{1,1}$ has a Kähler base, so we can only obtain new metrics on a smooth manifold in that case.

2.3. The following lemma presents conditions on the functions A_i, B and C which ensure that a solution of system (2.1) defines a smooth metric (1.1) on $\mathscr{M}_2$.

Lemma 2. *Let* $(A_1(t), A_2(t), A_3(t), B(t), C(t))$ *be a* C^∞*-smooth solution of* (2.1), $t \in [0, \infty)$. *Let* $p = 4$ *or* $p = 2$ *depending on whether the general fibre in* M *is isometric to* Sp(1) *or* SO(3). *The metric* (1.1) *extends to a smooth metric on* $\mathscr{M}_2/\mathbb{Z}_p$ *if and only if the following conditions are satisfied*:

1) $A_1(0) = 0$, $|A_1'(0)| = 4$;
2) $A_2(0) = -A_3(0) \neq 0$, $A_2'(0) = A_3'(0)$;
3) $B(0) \neq 0$, $B'(0) = 0$;
4) $C(0) \neq 0$, $C'(0) = 0$;
5) *the functions* A_1, A_2, A_3, B, C *have constant sign on the interval* $(0, \infty)$.

Lemma 2 was proved in [6] for $B = C$. The proof can be carried over to the general case with no modifications apart from the following observation: in the construction of $\mathscr{M}_2$ in [6] it is not important how we choose the field ξ^i along which the circle S is 'collapsed', because the system (2.2) has extra symmetries. However, a simple analysis of (2.1) shows that we must take ξ^1 as a generator for S, so that only the function A_1 can vanish at the initial moment of time.

§ 3. Constructing explicit solutions on $\mathscr{M}_2$

3.1. In (2.1) we make the substitution $A_2 = -A_3$. Then adding together the second and third equations we obtain $B^2 + C^2 = 2A_2^2$, and subtracting the fifth equation form the fourth we conclude that $(B^2 - C^2)' = 0$. Thus we may assume that $B^2 = A_2^2 + \alpha^2$ and $C^2 = A_2^2 - \alpha^2$ for some nonnegative constant α, and (2.1) reduces to the system

$$A_1' = -4 + \frac{A_1^2}{A_2^2} + 2\frac{A_1^2 A_2^2}{A_2^4 - \alpha^4}, \qquad (A_2^2)' = -A_1.$$

This is easy to integrate. Namely, we introduce a new variable ρ by setting $d\rho = -2A_1 dt$. By shifting in ρ we can always ensure that $A_2^2 = \rho$. Setting $A_1^2 = F$ we obtain

$$\frac{dF}{d\rho} + FG = 4,$$

where

$$G(\rho) = \frac{1}{\rho} + \frac{1}{\rho - \alpha^2} + \frac{1}{\rho + \alpha^2}.$$

This system is solved in the standard way (by introducing an integrating factor). Setting $r^2 = \rho$ we obtain

$$F = \frac{r^8 - 2\alpha^4 r^4 + \beta}{r^2(r^4 - \alpha^4)},$$

where β is the integration constant. So the metric (1.1) takes the following form:

$$\begin{aligned}\overline{g} = \frac{r^4(r^2-\alpha^2)(r^2+\alpha^2)}{r^8 - 2\alpha^4 r^4 + \beta} dr^2 + \frac{r^8 - 2\alpha^4 r^4 + \beta}{r^2(r^2-\alpha^2)(r^2+\alpha^2)}\eta_1^2 + r^2(\eta_2^2 + \eta_3^2) \\ + (r^2+\alpha^2)(\eta_4^2 + \eta_5^2) + (r^2 - \alpha^2)(\eta_6^2 + \eta_7^2).\end{aligned}$$

To have a regular metric on $\mathscr{M}_2/\mathbb{Z}_p$ we need the polynomial $r^8 - 2\alpha^4 r^4 + \beta$ to have real roots and its largest root r_0 to be greater than α. In this case the metric will be defined for $r \geqslant r_0$. Obviously, taking a metric homothetical to the original one, we can normalize the largest root by the condition $r_0 = 1$. Thus we can readily calculate that $0 \leqslant \alpha < 1$ and $\beta = 2\alpha^4 - 1$. Thus the metric (1.1) takes the following form:

$$\begin{aligned}\overline{g}_\alpha = \frac{r^4(r^2-\alpha^2)(r^2+\alpha^2)}{r^8 - 2\alpha^4(r^4-1) - 1} dr^2 + \frac{r^8 - 2\alpha^4(r^4-1) - 1}{r^2(r^2-\alpha^2)(r^2+\alpha^2)}\eta_1^2 + r^2(\eta_2^2 + \eta_3^2) \\ + (r^2+\alpha^2)(\eta_4^2 + \eta_5^2) + (r^2 - \alpha^2)(\eta_6^2 + \eta_7^2),\end{aligned} \qquad (3.1)$$

where $0 \leqslant \alpha < 1$ and $r \geqslant 1$. An immediate verification of the assumptions of Lemma 2 demonstrates that for $r \geqslant 1$, (3.1) represents a family of smooth metrics on $\mathscr{M}/\mathbb{Z}_p$ for $0 \leqslant \alpha < 1$, and $\overline{g}_0$ coincides with the Calabi metric with holonomy group SU(4) constructed in [1].

It follows from Lemma 1 that the holonomy group $\mathrm{Hol}(\overline{g}_\alpha)$ of the metric (3.1) lies in Spin(7). Now consider the 2-form

$$\overline{\Omega}_1 = -e^0 \wedge e^1 + e^2 \wedge e^3 + e^4 \wedge e^5 - e^6 \wedge e^7.$$

Obviously, it is compatible with the metric (1.1). A direct calculation shows that it is closed precisely when $A_2 = -A_3$, so that it is the Kähler form of the metric (3.1). Thus $\mathrm{Hol}(\overline{g}_\alpha) \subset \mathrm{SU}(4)$.

3.2. Now we make a detailed investigation of the case when the metrics $\overline{g}_\alpha$ are defined on a smooth manifold, that is, when $M = N_{1,1}$. We start by introducing our notation for the following subgroups of SU(3):

$$S_{1,1} = \{\mathrm{diag}(z, z, \overline{z}^2) \mid z \in S^1 \subset \mathbb{C}\}, \qquad T = \{\mathrm{diag}(z_1, z_2, \overline{z}_1\overline{z}_2) \mid z_1, z_2 \in S^1 \subset \mathbb{C}\},$$
$$K_1 = \left\{ \begin{pmatrix} \det(\overline{A}) & 0 \\ 0 & A \end{pmatrix} \,\middle|\, A \in U(2) \right\}, \qquad K_3 = \left\{ \begin{pmatrix} A & 0 \\ 0 & \det(\overline{A}) \end{pmatrix} \,\middle|\, A \in U(2) \right\}.$$

Consider the 3-dimensional complex space $\mathbb{C}^3$ and the unit sphere $S^5 \subset \mathbb{C}^3$ in it. Assume that the unit circle S^1 acts diagonally on $\mathbb{C}^3$ and associated spaces. We use square brackets for equivalence classes defined by such an action: $[u, v]$, $[u]$ etc.

Let
$$\widetilde{E} = \bigl\{(u_1, u_2) \bigm| |u_1| = 1,\ \langle u_1, u_2\rangle_{\mathbb{C}} = 0\bigr\} \subset S^5 \times \mathbb{C}^3.$$

Consider the diagonal action of the circle S^1 on the space $\widetilde{E}$ and the projection $\widetilde{\pi}_1\colon (u_1, u_2) \mapsto u_1$ of $\widetilde{E}$ onto S^5, which is a fibration with fibre $\mathbb{C}^2$. The space of the spherical subbundle of $\widetilde{\pi}_1$ is

$$\widetilde{E}^1 = \bigl\{(u_1, u_2) \in E \bigm| |u_1| = |u_2| = 1,\ \langle u_1, u_2\rangle_{\mathbb{C}} = 0\bigr\};$$

it is diffeomorphic to the group SU(3). The bundle $\widetilde{\pi}_1$ gives rise (by means of the action of S^1) to the vector bundle $\pi_1\colon E = \widetilde{E}/S^1 \to \mathbb{C}P^2$ with fibre $\mathbb{C}^2$ and spherical subbundle $E^1 = \widetilde{E}^1/S^1 = \mathrm{SU}(3)/S_{1,1} = N_{1,1} \to \mathbb{C}P^2 = \mathrm{SU}(3)/K_1$. It is easy to see that π_1 can be identified with the cotangent bundle $T^*\mathbb{C}P^2 \to \mathbb{C}P^2$.

In a similar way we consider the space

$$\widetilde{H} = \bigl\{(u_1, u_2, [u_3]) \bigm| |u_1| = |u_3| = 1,\ \langle u_i, u_j\rangle_{\mathbb{C}} = 0,\ i, j = 1, 2, 3\bigr\} \subset S^5 \times \mathbb{C}^3 \times \mathbb{C}P^2$$

and the projection $\widetilde{\pi}_2\colon (u_1, u_2, [u_3]) \mapsto (u_1, [u_3])$ of the space $\widetilde{H}$ onto

$$\widetilde{F} = \bigl\{(u_1, [u_3]) \bigm| |u_1| = |u_3| = 1,\ \langle u_1, u_3\rangle_{\mathbb{C}} = 0\bigr\},$$

with fibre $\mathbb{C}$. The total space of the spherical subbundle of $\widetilde{\pi}_2$ coincides with

$$\widetilde{H}^1 = \bigl\{(u_1, u_2, [u_3]) \bigm| \langle u_i, u_j\rangle_{\mathbb{C}} = 0,\ |u_i| = 1,\ i, j = 1, 2, 3\bigr\}$$

and can be identified with $\mathrm{SU}(3) = \widetilde{E}^1$ in the obvious way. Through the same action of S^1 the bundle $\widetilde{\pi}_2$ gives rise to a bundle $\pi_2\colon H = \widetilde{H}/S^1 \to F = \widetilde{F}/S^1$ with fibre $\mathbb{C}$, whose spherical subbundle coincides with the map $E^1 = \widetilde{E}^1 = N_{1,1} \to \mathrm{SU}(3)/T$. The base of π_2 is the complex flag manifold $F = \mathrm{SU}(3)/T$, which can be represented as follows:

$$F = \bigl\{([u_1], [u_3]) \bigm| u_i \in \mathbb{C}^3,\ |u_i| = 1,\ \langle u_1, u_3\rangle_{\mathbb{C}} = 0,\ i = 1, 3\bigr\}.$$

Definition 1. We call the complex linear bundle π_2: $H \to F$ the *canonical bundle on the complex flag manifold* F *of* $\mathbb{C}^3$.

Thus the canonical bundle on F and the cotangent bundle of $\mathbb{C}P^2$ have the same total space $N_{1,1}$ of the spherical subbundle, which can be fibred in two different ways. It is known that $M = N_{1,1}$ carries the structure of a 3-Sasakian manifold, whose twistor bundle coincides with $\pi_2\colon N_{1,1} \to F = \mathscr{X}$ and whose 3-Sasakian foliation is given by the projection $\pi_2'\colon N_{1,1} \to \mathrm{SU}(3)/K_3 = \mathbb{C}P_2 = \mathscr{O}$ with fibre SO(3) (see [9]). Obviously, in this case $\mathscr{M}_2$ coincides with the space H of the fibration π_2 on the complex flag manifold F which was considered above. For $0 \leqslant \alpha < 1$ the metric (3.1) we have constructed is a smooth metric on $H/\mathbb{Z}_2$, the space of the complex linear bundle $\pi_2 \otimes \pi_2$. For $\alpha = 1$, (3.1) reduces to a metric on $E = T^*\mathbb{C}P^2$ coinciding with the Calabi metric (see [1]).

3.3. The proof of Theorem 1. Here we finish the proof and present a fuller statement of Theorem 1 given in the introduction.

Theorem 3. *For $M = N_{1,1}$ the Riemannian metrics $\overline{g}_\alpha$ constructed explicitly in* (3.1) *are pairwise nonhomothetical smooth complete metrics. They have the following properties*:

1) *for $0 \leqslant \alpha < 1$, $\overline{g}_\alpha$ is a smooth metric on the space $H/\mathbb{Z}_2$ of the tensor square of the canonical bundle $\pi_2\colon H \to F$ on the complex flag manifold F of $\mathbb{C}^3$ and has the holonomy group* SU(4); *$\overline{g}_0$ coincides with the Calabi metric* (*see* [1]);
2) *the metric $\overline{g}_1$ has holonomy* $\mathrm{Sp}(2) \subset \mathrm{SU}(4)$ *and coincides with Calabi's hyper-Kähler metric* (*see* [1]) *on $T^*\mathbb{C}P^2$.*

Proof. To see that $\overline{g}_1$ is hyper-Kähler, it is sufficient to consider an additional pair of Kähler forms, which together with $\overline{\Omega}_1$ form a hyper-Kähler structure:

$$\overline{\Omega}_2 = e^0 \wedge e^2 + e^1 \wedge e^3 - e^4 \wedge e^6 + e^7 \wedge e^5 = e^0 \wedge e^2 + e^1 \wedge e^3 - \frac{BC}{2}\omega_2,$$

$$\overline{\Omega}_3 - e^0 \wedge e^3 + e^1 \wedge e^2 - e^4 \wedge e^7 + e^5 \wedge e^6 = -e^0 \wedge e^3 + e^1 \wedge e^2 - \frac{BC}{2}\omega_3.$$

Direct calculation shows that the forms $\overline{\Omega}_2$ and $\overline{\Omega}_3$ are closed precisely for $\alpha = 1$, which reduces the holonomy group to $\mathrm{Sp}(2) \subset \mathrm{SU}(4)$ in the case of the metric $\overline{g}_1$.

To complete the proof it remains to show that $\overline{g}_\alpha$ is not hyper-Kähler for $0 \leqslant \alpha < 1$. In fact, if

$$\mathrm{Hol}(\overline{g}_\alpha) = \mathrm{Hol}(\mathscr{M}_2/\mathbb{Z}_2) \subset \mathrm{Sp}(2), \qquad 0 \leqslant \alpha < 1,$$

then the limiting metric has the same property: $\mathrm{Hol}(\overline{M}/\mathbb{Z}_2) \subset \mathrm{Sp}(2)$. However, it is clear that after taking the quotient of the cone $\overline{M}$ by $\mathbb{Z}_2$, the generator of $\mathbb{Z}_2$ must be added to the holonomy group of $\overline{M}$. This generator corresponds to the transformation $\mathbb{H}^2 \to \mathbb{H}^2\colon (q_1, q_2) \mapsto (q_1', q_2')$, where $q_l = u_l + v_l j$ and $q_l' = -u_l + v_l j$, $u_l, v_l \in \mathbb{C}$, $l = 1, 2$. It is clear that although this transformation belongs to SU(4), it stays outside Sp(2). Hence $\mathrm{Hol}(\overline{M}/\mathbb{Z}_2)$ does not lie in Sp(2) and $\mathrm{Hol}(\overline{g}_\alpha) = \mathrm{SU}(4)$, which completes the proof.

§ 4. Analysing the general problem of the existence of solutions on $\mathscr{M}_2$

4.1. Recall that a metric (1.1) is called *locally conical* if the functions (A_i, B, C) are linear in t. If moreover, none of (A_i, B, C) is a constant function, the metric (1.1) is called *conical*. If there exists a (locally) conical metric defined by functions $(\widetilde{A_i}, \widetilde{B}, \widetilde{C})$ such that

$$\lim_{t\to\infty}\left|1 - \frac{A_i(t)}{\widetilde{A}_i(t)}\right| = 0, \qquad \lim_{t\to\infty}\left|1 - \frac{B(t)}{\widetilde{B}(t)}\right| = 0, \qquad \lim_{t\to\infty}\left|1 - \frac{C(t)}{\widetilde{C}(t)}\right| = 0,$$

then (1.1) is called an *asymptotically* (*locally*) *conical metric* (which we abbreviate to AC- or ALC-metric).

This section is mainly devoted to the proof of Theorem 2.

The central idea of the proof is to use the fact that system (2.1) has a homogeneous right-hand side and to come over to a dynamical system on the sphere $S^4 \subset \mathbb{R}^5$. Consider a vector $R(t) = (A_1(t), A_2(t), A_3(t), B(t), C(t)) \in \mathbb{R}^5$ and the map $V\colon \mathbb{R}^5 \to \mathbb{R}^5$ defined by the right-hand side of (2.1) (strictly speaking, V is only partially defined, for $A_i, B, C \neq 0$). Thus we can write system (2.1) in the following form:

$$\frac{dR}{dt} = V(R).$$

Now we consider the substitution $R(t) = f(t)S(t)$, where $f(t) = |R(t)|$ and

$$S(t) = (\alpha_1(t), \dots, \alpha_5(t)) \in S^4 = \biggl\{(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \biggm| \sum_{i=1}^{5} \alpha_i^2 = 1\biggr\}.$$

Since $V(fR) = V(R)$, the original system splits into its tangential and radial parts:

$$\frac{dS}{du} = V(S) - \langle V(S), S\rangle S = W(S), \tag{4.1}$$

$$\frac{1}{f}\,\frac{df}{du} = \langle V(S), S\rangle, \qquad dt = f\,du. \tag{4.2}$$

We see that to solve (2.1) it is sufficient to solve the autonomous system (4.1) on S^4, after which we can find a solution to (2.1) by simply integrating equations (4.2).

The remaining part of the proof of Theorem 2 is structured as follows. First we find all the stationary and conditionally stationary points of system (4.1) (Lemmas 4 and 5); they determine the asymptotic behaviour of the corresponding metrics (Lemma 6). Next we describe the initial points S_0 corresponding to the necessary conditions for the smoothness of the metric in Lemma 2; we prove that there is a unique trajectory of system (4.1) going out of any such point (Lemma 7). After that it remains to understand the limiting behaviour of these trajectories. To do this we define invariant domains Π and Γ of system (4.1) and establish some differential relations, which hold along trajectories of the system and are useful for what follows (Lemma 8); these relations demonstrate that certain specially selected functions are monotonic along trajectories, so that their asymptotic behaviour can be described precisely (Proposition 1).

4.2. Symmetries, stationary and conditionally stationary points of system (4.1). The following lemma is obvious.

Lemma 3. *System* (4.1) *has the discrete symmetry group* $G = D_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *generated by the following transformations*:

$$\begin{aligned}
&(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_4),\\
&(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_1, -\alpha_2, -\alpha_3, \alpha_5, -\alpha_4),\\
&(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_1, \alpha_3, \alpha_2, \alpha_4, \alpha_5),\\
&(\alpha_1(u), \dots, \alpha_5(u)) \mapsto (-\alpha_1(-u), \dots, -\alpha_5(-u)),
\end{aligned}$$

where D_4 *is the dihedral group.*

Recall that a point S is said to be *stationary* for (4.1) if $W(S)=0$. Obviously, a vector field W is defined precisely at the points $S\in S^4$ at which the vector field V is defined. We say that a point $S\in S^4$ at which W is not defined is *conditionally stationary* if there exists a real analytic curve $\gamma(u)\colon(-\varepsilon,\varepsilon)\to S^4$ on the sphere such that $\gamma(0)=S$ and $\lim_{u\to 0}W(\gamma(u))=0$.

Lemma 4. *All the stationary points of* (4.1) *can be obtained from the points*

$$\left(\frac{1}{\sqrt{13}},\frac{1}{\sqrt{13}},\frac{1}{\sqrt{13}},\frac{\sqrt5}{\sqrt{13}},\frac{\sqrt5}{\sqrt{13}}\right)\quad\text{and}\quad\left(-\frac{1}{\sqrt5},\frac{1}{\sqrt5},\frac{1}{\sqrt5},\frac{1}{\sqrt5},\frac{1}{\sqrt5}\right)$$

by the action of G.

Proof. Let T denote the expression $\langle V(S),S\rangle$ in (4.1). We are looking for stationary solutions of the equation $W(S)=0$, that is, for $S=(\alpha_1,\dots,\alpha_5)$ such that no α_i vanishes.

We start with the case $\alpha_4=\pm\alpha_5$. Note that system (2.1) is invariant under the substitution $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\to(\alpha_1,-\alpha_2,-\alpha_3,\alpha_4,-\alpha_5)$, so we can limit ourselves to the subsystem $\alpha_4=\alpha_5$, which coincides with (2.2). Note that this case was considered in [6], but there the corresponding argument was left out for reasons of space, so we present the proof in full here. It is also easy to see that (2.2) is invariant under permutations of the variables $\alpha_1,\alpha_2,\alpha_3$.

Suppose $\alpha_1\neq\alpha_2$. Setting $\alpha_2W_1-\alpha_1W_2$ equal to zero we obtain the relation

$$\alpha_1+\alpha_2=\alpha_3+\frac{\alpha_1\alpha_2\alpha_3}{\alpha_4^2}.$$

Note that if all the quantities α_1, α_2, α_3 are different, then we also obtain the relation

$$\alpha_2+\alpha_3=\alpha_1+\frac{\alpha_1\alpha_2\alpha_3}{\alpha_4^2}$$

so that taking the difference of the last two equations we arrive at a contradiction: $\alpha_1=\alpha_3$.

Thus in view of the symmetry of the system, we shall assume that $\alpha_1\neq\alpha_2$ and $\alpha_2=\alpha_3$. Then from the penultimate equation we obtain $\alpha_2^2=\alpha_4^2$, and since $W_2=0$, it follows that $T=\alpha_1/\alpha_2^2$. Substituting all these relations in the equation $W_4=0$ we see that $\alpha_1+\alpha_2=0$. In combination with $\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2+\alpha_5^2=1$, up to the symmetries in Lemma 3, all these relations give us the stationary point

$$\left(-\frac{1}{\sqrt5},\frac{1}{\sqrt5},\frac{1}{\sqrt5},\frac{1}{\sqrt5},\frac{1}{\sqrt5}\right).$$

The case $\alpha_1=\alpha_2=\alpha_3$ remains: here (2.2) degenerates into $\alpha_4=\sqrt5\alpha_1$, which in combination with the equations $\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2+\alpha_5^2=1$ and $\alpha_4=\alpha_5$, up to the symmetries in Lemma 3 immediately gives us the stationary point

$$\left(\frac{1}{\sqrt{13}},\frac{1}{\sqrt{13}},\frac{1}{\sqrt{13}},\frac{\sqrt5}{\sqrt{13}},\frac{\sqrt5}{\sqrt{13}}\right).$$

Now we return to the general case. Getting an expression for T from the equation $W_1(S)=0$ we obtain

$$T=\frac{\alpha_2}{\alpha_1\alpha_3}+\frac{\alpha_3}{\alpha_1\alpha_2}-\frac{\alpha_1}{\alpha_2\alpha_3}-\frac{2}{\alpha_1}+\frac{\alpha_1}{\alpha_4^2}+\frac{\alpha_1}{\alpha_5^2}.$$

Since $0 = W_4\alpha_5 - W_5\alpha_4 = V_4\alpha_5 - V_5\alpha_4$, we arrive at the following equation

$$\frac{(\alpha_4^2 - \alpha_5^2)}{\alpha_2\alpha_3\alpha_4\alpha_5}[\alpha_1\alpha_2\alpha_3 + \alpha_4\alpha_5(\alpha_2 + \alpha_3)] = 0.$$

As we explained earlier, we can assume that $\alpha_4 \neq \pm\alpha_5$. Hence substituting

$$\alpha_1 = -\frac{\alpha_4\alpha_5(\alpha_2 + \alpha_3)}{\alpha_2\alpha_3}$$

into the equation $V_2\alpha_3 - V_3\alpha_2 = 0$ yields

$$\frac{(\alpha_2 - \alpha_3)(4\alpha_2\alpha_3 + \alpha_4^2 + \alpha_5^2)}{\alpha_4\alpha_5} = 0.$$

We shall consider two cases: 1) $\alpha_2 - \alpha_3 = 0$ and 2) $4\alpha_2\alpha_3 + \alpha_4^2 + \alpha_5^2 = 0$.

1) In this case the equation $W_4 = 0$ takes the following form:

$$\frac{\alpha_2^2(2\alpha_2^2 - 3\alpha_5^2) + \alpha_4^2(2\alpha_5^2 - 3\alpha_2^2)}{\alpha_5\alpha_2^3} = 0.$$

For convenience we set $\alpha_5 = 1$. If we find a nontrivial solution, we can normalize it, but from this point on, the equations will no longer be homogeneous. Setting

$$\alpha_4^2 = \frac{\alpha_2^2(3 - 2\alpha_2^2)}{2 - 3\alpha_2^2}$$

in $W_2 = 0$ we obtain the biquadratic equation $4\alpha_2^4 - 6\alpha_2^2 + 5 = 0$, which has no real roots.

2) Setting

$$\alpha_2 = -\frac{\alpha_4^2 + \alpha_5^2}{4\alpha_3}$$

in the equation $W_2 = 0$, we obtain an expression in the numerator which is biquadratic with respect to α_3:

$$2\alpha_4^6\alpha_5^2 + 4\alpha_4^4\alpha_5^4 + 2\alpha_4^2\alpha_5^6 + 32\alpha_3^4\alpha_4^2\alpha_5^2 - 19\alpha_3^2\alpha_4^4\alpha_5^2 - 19\alpha_3^2\alpha_4^2\alpha_5^4 - \alpha_3^2\alpha_4^6 - \alpha_3^2\alpha_5^6,$$

which must be equal to zero. Obtaining an expression for α_3 from this equation and substituting the result in $W_4 = 0$, after some calculations we obtain an equation for α_4 and α_5:

$$(3\alpha_4^4 - 2\alpha_4^2\alpha_5^2 + 3\alpha_5^4)(\alpha_4^2 + \alpha_5^2)^4 \times \left[\alpha_4^4 + 18\alpha_4^2\alpha_5^2 + \alpha_5^4 + (\alpha_4^2 + \alpha_5^2)\sqrt{\alpha_4^4 + 34\alpha_4^2\alpha_5^2 + \alpha_5^4}\right] = 0,$$

which has no real roots either. The proof of Lemma 4 is complete.

The next lemma demonstrates that, by comparison with a similar system for equation (2.2), system (4.1) has no essentially new conditionally stationary points either.

Lemma 5. *All the conditionally stationary points of* (4.1) *are obtained from the point*

$$\left(0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

by the action of G.

Proof. [1] Let $S = (\alpha_1, \alpha_2, \alpha_3, a_4, \alpha_5) \in S^4$ be a conditionally stationary point of (4.1), so that there exists a curve

$$\gamma(u) = \bigl(\alpha_1(u), \alpha_2(u), \alpha_3(u), \alpha(u), \alpha_5(u)\bigr), \qquad u \in (-\varepsilon, \varepsilon), \quad S = \gamma(0),$$

with the properties detailed above. Note that here u is a smooth parameter, which does not necessarily coincide with the independent variable of system (4.1) with respect to which derivatives are taken.

First we note that we can assume that $\alpha_4(0) \neq \alpha_5(0)$ (although formally system (2.1) can have new conditionally stationary points by comparison with (2.2) for $\alpha_4 = \alpha_5$, it is easy to see that the corresponding argument in [6] rules this out). Hence, taking account of the symmetry group G, we can conclude that the field W can only have singularities for $\alpha_1(0) = 0$, $\alpha_2(0) = 0$, $\alpha_3(0) = 0$ or $\alpha_5(0) = 0$.

1) We start with the case when all the four relations hold: $\alpha_1(0) = \alpha_2(0) = \alpha_3(0) = \alpha_5(0) = 0$, $\alpha_4(0) = 1$ (the case $\alpha_4(0) = -1$ reduces to this by a symmetry in the group G). Then we set

$$\alpha_i = c_i u^{k_i}(1 + o(1)), \qquad u \to 0,$$

where $c_i \neq 0$, $k_i \geqslant 1$, $i = 1, 2, 3, 5$. Consider the following function:

$$\begin{aligned} \alpha_5 W_4 - \alpha_4 W_5 &= \frac{\alpha_4^2 - \alpha_5^2}{\alpha_4\alpha_5}\alpha_1 + (\alpha_4^2 - \alpha_5^2)\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_3}\right) \\ &= \frac{c_1}{c_5}u^{k_1 - k_5}(1 + o(1)) + \frac{1}{c_2}u^{-k_2}(1 + o(1)) + \frac{1}{c_3}u^{-k_3}(1 + o(1)) = o(1). \end{aligned}$$

It is obvious that if $k_2 \neq k_3$, then the last summands contain terms of maximal growth having distinct growth orders, so the first summand cannot compensate for both of them, which leads to a contradiction. So we must have $k_2 = k_3$. Hence we are either in case 1, a): $k_1 \geqslant k_5$, when $c_2 = -c_3$, or in case 1, b): all the three summands display the same maximal order of growth to infinity. In this case $k_1 - k_5 = -k_2 = -k_3$, so $k_5 = k_1 + k_2$, and since the three leading terms must compensate for one another we have

$$\frac{c_1}{c_5} + \frac{1}{c_2} + \frac{1}{c_3} = 0. \tag{4.3}$$

[1] In preparing the English edition of the paper some inconsistencies were noticed in the proof of Lemma 5: in cases 1), 2,b) and 3,c) we hastily (and generally speaking, incorrectly) concluded that if a sum of three monomials is $o(1)$, then two of the monomials must have equal degrees. To complete the proof it is sufficient to consider the following additional relations on the curve γ:

$$W_2\alpha_3 - W_3\alpha_2 = W_4\alpha_5 - W_5\alpha_4 = W_1\alpha_2 - W_2\alpha_1 = W_1 = o(1).$$

Then we obtain relations for the coefficients that either lead to a contradiction or yield the required solutions. The authors thank N. Kruzhilin for pointing out these inconsistencies. — *The authors' note to the English edition.*

In case 1, b) consider the function

$$\begin{aligned}\alpha_2W_1-\alpha_1W_2&=2\frac{\alpha_2^2}{\alpha_3}-2\frac{\alpha_1^2}{\alpha_3}-2\alpha_2+\alpha_1^2\alpha_2\frac{\alpha_4^2+\alpha_5^2}{\alpha_4^2\alpha_5^2}+\alpha_1\frac{\alpha_4^2+\alpha_5^2-2\alpha_2^2}{\alpha_4\alpha_5}\\&=-\frac{2c_1^2}{c_3}u^{2k_1-k_2}(1+o(1))+\frac{c_1^2c_2}{c_5^2}u^{2k_1+k_2-2k_5}(1+o(1))+\frac{c_1}{c_5}u^{k_1-k_5}(1+o(1))\\&=-\frac{2c_1^2}{c_3}u^{2k_1-k_2}(1+o(1))+\frac{c_1^2c_2}{c_5^2}u^{-k_2}(1+o(1))+\frac{c_1}{c_5}u^{-k_2}(1+o(1))=o(1).\end{aligned}$$

Since the order of the leading term in the first summand is definitely greater that those in the second and third summands, for equality we require that

$$\frac{c_1^2c_2}{c_5^2}+\frac{c_1}{c_5}=0,$$

and in view of (4.3), this yields $1/c_3=0$, which is a contradiction.

In case 1, a) (that is, when $k_2=k_3$, $k_1\geqslant k_5$ and $c_2=-c_3$) we consider W_1 and W_2 (we absorb in $o(1)$ the part that we know does not contain terms with greatest growth order in this case):

$$W_1=\frac{c_1^2}{c_2^2}u^{2k_1-2k_2}(1+o(1))=o(1),\qquad W_2=-\frac{c_1}{c_2}u^{k_1-k_2}-\frac{1}{c_5}u^{-k_5}=o(1).$$

It follows from the first relation that $k_1\geqslant k_2+1$. Then in the second relation we have a contradiction.

We see that the four variables cannot vanish simultaneously for $u=0$.

2) Next we consider the cases when precisely three variables vanish. First assume that we have case 2, a): $\alpha_1(0)=\alpha_2(0)=\alpha_3(0)=0$, $\alpha_4(0)=a_4\neq0$ and $\alpha_5(0)=a_5\neq0$. Then

$$\alpha_5W_4-\alpha_4W_5=(a_4^2-a_5^2)\biggl(\frac{1}{c_2}u^{-k_2}(1+o(1))+\frac{1}{c_3}u^{-k_3}(1+o(1))\biggr)=o(1).$$

Hence necessarily $k_2=k_3$ and $c_2=-c_3$. Again, we consider the functions W_1 and W_2:

$$W_1=-4(1+o(1))+\frac{c_1^2}{c_2^2}u^{2k_1-2k_2}(1+o(1))=o(1),$$

$$W_2=-\frac{a_4^2+a_5^2}{a_4a_5}(1+o(1))-\frac{c_1}{c_2}u^{k_1-k_2}(1+o(1))=o(1).$$

It follows from the first relation that $c_1^2/c_2^2=4$, and then the second relation shows that $a_4=a_5$, but we have ruled this possibility out. Therefore, case a) is impossible.

We consider case 2, b): $\alpha_1(0)=a_1\neq0$, $\alpha_2(0)=\alpha_3(0)=\alpha_5(0)=0$ and $\alpha_4(0)=a_4\neq0$. Then

$$\begin{aligned}&\alpha_5W_4-\alpha_4W_5\\&\qquad=\frac{a_4a_1}{c_5}u^{-k_5}(1+o(1))+\frac{a_4}{c_2}u^{-k_2}(1+o(1))+\frac{a_4}{c_3}u^{-k_3}(1+o(1))=o(1).\end{aligned}$$

Thus $k_2 = k_3 = k_5 = k$ and

$$\frac{a_1}{c_5} + \frac{1}{c_2} + \frac{1}{c_3} = 0.$$

Now we have

$$W_4 = \frac{a_4}{c_5}\left(\frac{c_5 a_1^3}{c_2 c_3} - \frac{a_1^3}{c_5} - \frac{a_4^3}{2c_3} - \frac{a_4^3}{2c_2} + \frac{a_4}{2c_3} + \frac{a_4}{2c_2}\right)u^{-2k}(1+o(1)) + O(1) = o(1),$$
$$W_5 = \left(\frac{c_5 a_1^3}{c_2 c_3} - \frac{a_1}{c_5} - \frac{a_1^3}{c_5} - \frac{a_4^3}{2c_3} - \frac{a_4^3}{2c_2} - \frac{a_4}{2c_3} - \frac{a_4}{2c_2}\right)u^{-k}(1+o(1)) = o(1).$$

Hence the corresponding coefficients of u^{-2k} and u^{-k} vanish. Taking their difference (after scaling the first coefficient) we obtain

$$\frac{a_4}{c_3} + \frac{a_4}{c_2} + \frac{a_1}{c_5} = 0.$$

In combination with the previous equality of a similar form this allows us to conclude that $a_4 = 1$, so that $a_1 = 0$, contradicting our assumptions.

Consider case 2, c): $\alpha_1 = \alpha_2 = \alpha_5 = 0$, $\alpha_3(0) = a_3 \neq 0$ and $\alpha_4(0) = a_4 \neq 0$. Then

$$\alpha_5 W_4 - \alpha_4 W_5 = \frac{a_4 c_1}{c_5} u^{k_1 - k_5}(1+o(1)) + \frac{a_4^2}{c_2} u^{-k_2}(1+o(1)) = o(1),$$

which shows that $k_1 - k_5 = -k_2$ and $c_1 c_2 + a_4 c_5 = 0$. Hence

$$W_5 = \frac{a_4(1-a_4^2)}{2c_2} u^{-k_2}(1+o(1)) + O(1) = o(1),$$

that is, $a_4 = 1$, contradicting the assumptions made in this case.

Taking account of the symmetries in the group G we see that we have fully investigated case 2): there are no conditionally stationary points in this case.

3) Assume that precisely two variable vanish for $u = 0$. We shall consider all the possible cases (modulo the action of G).

Suppose we have case 3, a): $\alpha_1(0) = \alpha_2(0) = 0$, $\alpha_3(0) = a_3 \neq 0$, $\alpha_4(0) = a_4 \neq 0$ and $\alpha_5(0) = a_5 \neq 0$. Then

$$W_4 = \frac{a_3^3 a_4}{c_1 c_2} u^{-k_1 - k_2}(1+o(1)) = o(1),$$

which is a contradiction.

Consider case 3, b): $\alpha_1(0) = \alpha_5(0) = 0$, $\alpha_2(0) = a_2 \neq 0$, $\alpha_3(0) = a_3 \neq 0$ and $\alpha_4(0) = a_4 \neq 0$. Then

$$\alpha_5 W_4 - \alpha_4 W_5 = \frac{a_4 c_1}{c_5} u^{k_1 - k_2}(1+o(1)) = o(1),$$

so that $k_1 \geqslant k_5$. Suppose $k_1 > k_5$; then

$$W_4 = \left(\frac{a_2^3 a_4}{a_3 c_1} + \frac{a_3^3 a_4}{a_2 c_1} - \frac{2a_2 a_3 a_4}{c_1}\right)u^{-k_1}(1+o(1)) = o(1),$$

which yields $a_2^2 = a_3^2$. If $a_2 = a_3$, then

$$W_4 = \frac{a_4}{c_5 a_2}\left(-\frac{2a_2^2}{a_4} + a_4 + 4a_4 a_2^2 - 4\frac{a_2^4}{a_4} - a_4^3\right)u^{-k_5}(1+o(1)) = o(1),$$
$$W_3 = \frac{1}{c_5}\left(-a_4 + \frac{2a_2^2}{a_4} + 4a_2^2 a_4 - \frac{4a_2^4}{a_4} - a_4^3\right)u^{-k_5}(1+o(1)) = o(1).$$

The expressions in brackets must vanish, so subtracting one expression from the other we obtain $a_4^2 = 2a_2^2$. Without loss of generality we can set $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{2}$ and $a_4 = \frac{1}{\sqrt{2}}$. Then we arrive at a contradiction as follows:

$$W_5 = -2\sqrt{2} + o(1) = o(1).$$

On the other hand if $a_3 = -a_2$, then we again obtain a contradiction:

$$W_1 = -4 + o(1) = o(1).$$

Now let $k_1 = k_5$. Then

$$\alpha_5 W_4 - \alpha_4 W_5 = a_4\left(\frac{a_4}{a_2} + \frac{a_4}{a_3} + \frac{c_1}{c_5}\right)(1+o(1)) = o(1).$$

We express c_1 in terms of the other parameters and obtain

$$\alpha_2\alpha_3 W_1 - \alpha_1\alpha_3 W_2 = \left(\frac{a_2 a_4^2}{a_3} + 4a_2^2 + a_4^2\right)(1+o(1)) = o(1),$$
$$\alpha_2\alpha_3 W_1 - \alpha_1\alpha_2 W_3 = \left(\frac{a_3 a_4^2}{a_2} + 4a_3^2 + a_4^2\right)(1+o(1)) = o(1).$$

From these two equations it necessarily follows that

$$(a_2^2 - a_3^2)\left[4 + \frac{a_4^2}{a_2 a_3}\right] = 0.$$

The first factor cannot vanish for otherwise

$$\text{either}\quad \frac{-a_2 a_4^2}{a_2} + 4a_2^2 + a_4^2 = 4a_4^2 = 0 \quad\text{or}\quad \frac{a_4}{a_2} + \frac{a_4}{a_3} + \frac{c_1}{c_5} = \frac{c_1}{c_5} = 0.$$

The second factor cannot vanish since otherwise we have

$$\frac{-4a_3 a_2 a_3}{a_2} + 4a_3^2 - 4a_2 a_3 = 0.$$

Consider case 3, c): $\alpha_2(0) = \alpha_3(0) = 0$, $\alpha_1(0) = a_1 \neq 0$, $\alpha_4(0) = a_4 \neq 0$ and $\alpha_5(0) = a_5 \neq 0$. Then

$$\alpha_5 W_4 - \alpha_4 W_5 = (a_4^2 - a_5^2)\left(\frac{a_1}{a_4 a_5}(1+o(1)) + \frac{1}{c_2 u^{k_2}}(1+o(1)) + \frac{1}{c_3 u^{k_3}}(1+o(1))\right)$$
$$= o(1).$$

The last relation gives us a contradiction because $k_2, k_3 \geqslant 1$, and the other parameters do not vanish.

Consider case 3, d): $\alpha_2(0) = \alpha_5(0) = 0$, $\alpha_1(0) = a_1 \neq 0$, $\alpha_3(0) = a_3 \neq 0$ and $\alpha_4(0) = a_4 \neq 0$. Then

$$\alpha_2 W_3 - \alpha_3 W_2 = \frac{a_3 a_4}{c_5} u^{-k_5}(1 + o(1)) = o(1),$$

which is a contradiction.

4) It remains to analyse the case when any one variable takes the value zero.

Assume that we have case 4, a): $\alpha_1(0) = 0$ and $\alpha_i(0) = a_i \neq 0$, $i = 2, \dots, 5$. Then

$$\alpha_5 W_4 - \alpha_4 W_5 = (a_4^2 - a_5^2)\frac{a_2 + a_3}{a_2 a_3}(1 + o(1)),$$

which means that $a_2 = -a_3$. Then

$$W_1 = -4 + o(1),$$

which leads to a contradiction.

Consider case 4, b): $\alpha_2(0) = 0$, while the other $\alpha_i(0) = a_i \neq 0$. Then

$$\alpha_5 W_4 - \alpha_4 W_5 = \frac{a_4^2 - a_5^2}{c_2} u^{-k_2}(1 + o(1)),$$

and we obtain a contradiction.

Consider case 4, c): $\alpha_5(0) = 0$, while the other $\alpha_i(0) = a_i \neq 0$. Then we also arrive at contradiction:

$$\alpha_5 W_4 - \alpha_4 W_5 = \frac{a_1 a_4}{c_4} u^{-k_5}(1 + o(1)).$$

The proof of Lemma 5 is complete.

4.3. The behaviour of trajectories in a neighbourhood of the initial point. The following lemma was proved in [6].

Lemma 6. *Stationary solutions of system* (4.1) *are associated with locally conical metrics on* $\overline{M}$*, and trajectories of* (4.1) *asymptotically tending to* (*conditionally*) *stationary solutions correspond to asymptotically* (*locally*) *conical metrics on* $\overline{M}$.

We set

$$J = \bigl\{0, -\alpha_2, \alpha_2, \alpha_4, \alpha_5) \in S^4 \bigm| \alpha_2 > 0,\ \alpha_4 \geqslant \alpha_5 > 0\bigr\}.$$

By Lemma 2, to construct a regular metric on $\mathscr{M}_2/\mathbb{Z}_p$ we must find a trajectory of system (4.1) going out of a point of the form $(0, -\lambda, \lambda, \mu, \nu)$, where $2\lambda^2 + \mu^2 + \nu^2 = 1$. The symmetries in Lemma 3 let us limit ourselves to the case when the initial point $S_0 = (0, -\lambda, \lambda, \mu, \nu)$ lies in the region J, which is a geodesic triangle in the 2-sphere $\{\alpha_1 = \alpha_2 + \alpha_3 = 0\}$.

Lemma 7. *A unique trajectory of system* (4.1) *goes out of the above-mentioned point* $S_0 = (0, -\lambda, \lambda, \mu, \nu)$ *into the domain* $\alpha_1 < 0$.

Proof. Consider an open ball $U \subset \mathbb{R}^2$ of small radius $\varepsilon < 1-\mu^2-\nu^2$ in the system of coordinates $x = \alpha_1$, $y = \alpha_2+\alpha_3$. Then in the neighbourhood $J\times U$ of the domain J the variables $x, y, z = \alpha_4 > 0$, $w = \alpha_5 > 0$ form a local system of coordinates on S^4,

$$S = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \biggl(x, \frac{1}{2}\biggl(y - \sqrt{2}\sqrt{1 - z^2 - x^2 - w^2 - \frac{y^2}{2}}\biggr), \frac{1}{2}\biggl(y + \sqrt{2}\sqrt{1 - z^2 - x^2 - w^2 - \frac{y^2}{2}}\biggr), z, w\biggr).$$

In these new coordinates the vector field $V(S)$ looks as follows:

$$V_1(S) = -4 + \frac{x^2}{z^2} + \frac{x^2}{w^2} + 2\frac{y^2 - x^2}{x^2 + y^2 + z^2 + w^2 - 1},$$

$$V_2(S) = \frac{-\sqrt{2 - 2x^2 - 2z^2 - 2w^2 - y^2}(-2zwy + 2xz^2 + 2xw^2 - x - x^3)}{xzw(y + \sqrt{2 - 2z^2 - 2x^2 - 2w^2 - y^2})} + \frac{2zwx + y^3 - y + x^2y}{zw(y + \sqrt{2 - 2z^2 - 2x^2 - 2w^2 - y^2})},$$

$$V_3(S) = \frac{\sqrt{2 - 2x^2 - 2z^2 - 2w^2 - y^2}(-2zwy + 2xz^2 + 2xw^2 - x - x^3)}{xzw(y - \sqrt{2 - 2z^2 - 2x^2 - 2w^2 - y^2})} + \frac{2zwx + y^3 - y + x^2y}{zw(y - \sqrt{2 - 2z^2 - 2x^2 - 2w^2 - y^2})},$$

$$V_4(S) = -\frac{x}{z} - \frac{y}{w} - \frac{y(w^2 - z^2)}{w(x^2 + y^2 + z^2 + w^2 - 1)},$$

$$V_5(S) = -\frac{x}{w} - \frac{y}{z} - \frac{y(z^2 - w^2)}{z(x^2 + y^2 + z^2 + w^2 - 1)}.$$

In this system of coordinates the field W, which is tangent to S^4, has the components

$$W_x = W_1, \qquad W_y = W_2 + W_3, \qquad W_z = W_4, \qquad W_w = W_5,$$

with the W_i defined by (4.1).

In $J \times U$ we consider the system

$$\frac{d}{dv}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} xW_x \\ xW_y \\ xW_z \\ xW_w \end{pmatrix}, \tag{4.4}$$

where $du = x\,dv$. Obviously, (4.4) has the same trajectories as (4.1). Since $z, w > 0$, xW is a smooth field and its stationary points in $J \times U$ are described by $x = y = 0$, so that all of them lie on J. Now consider the linearization of system (4.4) in a neighbourhood of the point S_0:

$$\frac{dx}{dv} = -4x, \qquad \frac{dy}{dv} = -\frac{2(2\mu^2 + 2\nu^2 - 1)}{\mu\nu}x + 4y, \qquad \frac{dz}{dv} = 0, \qquad \frac{dw}{dv} = 0.$$

The linearized system is degenerate, with eigenvalues -4, 4, 0, 0 and eigenvectors

$$e_1 = \left(8, \frac{2(2\mu^2+2\nu^2-1)}{\mu\nu}, 0, 0\right),$$
$$e_2 = (0,1,0,0), \qquad e_3 = (0,0,1,0), \qquad e_4 = (0,0,0,1),$$

respectively.

The tangent space to J is spanned by e_3 and e_4. Calculations show that $\left\langle (0,0,a,b), \frac{xW}{|xW|}\right\rangle \to 0$ as $(x,y,z,w) \to S_0$. This means that trajectories of (4.1) meet J at right angles. This allows us to consider the behaviour of the system in the (x,y)- and (z,w)-planes separately. In the (x,y)-plane we consider the parabolas

$$F_1(x,y) = -\frac{2(2\mu^2+2\nu^2-1)}{\mu\nu}x + 8y - \alpha x^2 = 0,$$
$$F_2(x,y) = -\frac{2(2\mu^2+2\nu^2-1)}{\mu\nu}x + 8y + \alpha x^2 = 0$$

and the line $x = -\delta$, where $\alpha, \delta > 0$. They form a bounded region $\Gamma \subset U$. At points of the first parabola we have

$$\begin{aligned}\langle \nabla F_1, (xW_x, xW_y)\rangle &= \frac{d}{dv}\left(-\frac{2(2\mu^2+2\nu^2-1)}{\mu\nu}x + 8y - \alpha x^2\right)\\ &= 12\alpha x^2 + O(x^2+y^2).\end{aligned}$$

The resulting expression vanishes only on J. Obviously, for sufficiently large values of the parameter α the angle between the outward normal ∇F_1 and (xW_x, xW_w) is acute, that is, the projections of trajectories onto (x,y) cross the first parabola going out of Γ. Hence these trajectories leave the region $\Gamma \times J$. In a similar way, for large α the angle between the inward normal ∇F_2 and (xW_x, xW_y) is obtuse, so again the trajectories intersecting the second cylindrical surface $\{F_2 = 0\} \times J$ go outwards. The parabolas meet at $(0,0)$ and the projections of trajectories intersect the parabolas, therefore there exists a trajectory whose projection comes into the point $(0,0)$ for sufficiently small δ. Since trajectories reach J at a right angle, for sufficiently large α and sufficiently small δ there exists a trajectory of the system coming in S_0. Obviously, the projection of its tangent vector at $(x,y) = (0,0)$ coincides with the tangent vector $\left(8, \frac{2(2\mu^2+2\nu^2-1)}{\mu\nu}\right)$ to the parabolas and the tangent vector itself is equal to e_1.

The x-coordinate converges to zero as e^{-4v}. Hence with respect to the parameter u the trajectory of (4.1) comes into S_0 in finite time u_0, because in a neighbourhood of S_0 we have $-4u = c_1 e^{-4v} + c_2$ asymptotically. Note that in the domain $x < 0$ the parameters u and v are 'inversely proportional' since $c_1 < 0$. We see that for each $S_0 \in J$ there exists a unique trajectory leaving S_0 in finite time and going in the domain $\alpha_1 = x < 0$ with tangent vector at S_0 equal to $-e_1$. The proof of Lemma 7 is complete.

We have shown that for each point S_0 system (4.1) has a trajectory going out of this point. Hence by Lemma 2 there exists a metric on $\mathscr{M}_2/\mathbb{Z}_p$ which is regular in a neighbourhood of the orbifold $\mathscr{O}$.

4.4. Now we describe the behaviour of trajectories of the system at infinity. Recall that we are looking for asymptotically locally conical metrics.

Lemma 8. *If $S=(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ is a solution of* (4.1), *then the following relations hold*:

1) $\dfrac{d}{du}\ln\biggl(\dfrac{\alpha_2}{\alpha_1}\biggr)=2\dfrac{\alpha_1^2-\alpha_2^2}{\alpha_1\alpha_2\alpha_3}+2\biggl(\alpha_2-\dfrac{\alpha_4^2+\alpha_5^2}{2\alpha_4\alpha_5}\alpha_1\biggr)\biggl(\dfrac{1}{\alpha_1\alpha_2}+\dfrac{1}{\alpha_4\alpha_5}\biggr)$;

2) $\dfrac{d}{du}\ln\biggl(\dfrac{\alpha_2}{\alpha_3}\biggr)=2(\alpha_2-\alpha_3)\biggl(\dfrac{1}{\alpha_4\alpha_5}+\dfrac{1}{\alpha_1\alpha_2\alpha_3}\biggl(\dfrac{\alpha_4^2+\alpha_5^2}{2\alpha_4\alpha_5}\alpha_1-\alpha_2-\alpha_3\biggr)\biggr)$;

3) $\dfrac{d}{du}\ln\biggl(\dfrac{\alpha_4}{\alpha_5}\biggr)=\dfrac{(\alpha_4-\alpha_5)^2}{\alpha_4\alpha_5}\biggl(\dfrac{\alpha_1}{\alpha_4\alpha_5}+\dfrac{\alpha_2+\alpha_3}{\alpha_2\alpha_3}\biggr)$;

4) $\dfrac{d}{du}\ln(\alpha_5^2)=-2\alpha_1(1+\alpha_1^2)$ *for* $\alpha_5=0$;

5) $\dfrac{d}{du}\alpha_1=\dfrac{(\alpha_2-\alpha_3)^2}{\alpha_2\alpha_3}\bigl(1-(\alpha_2+\alpha_3)^2\bigr)$ *for* $\alpha_1=0$;

6) $\dfrac{d}{du}(\alpha_1-\alpha_2)=\dfrac{(\alpha_4-\alpha_5)^2}{\alpha_4\alpha_5}\biggl(\dfrac{\alpha_1^2}{\alpha_4\alpha_5}+1\biggr)$ *for* $\alpha_1=\alpha_2$;

7) $\dfrac{d}{du}(\alpha_2+\alpha_3)=\dfrac{2}{\alpha_4\alpha_5}(2\alpha_2^2-\alpha_4^2-\alpha_5^2)$ *for* $\alpha_2+\alpha_3=0$;

8) $$\begin{aligned}\frac{d}{du}(2\alpha_2^2-\alpha_4^2-\alpha_5^2)=(\alpha_2+\alpha_3)\biggl(&\frac{4}{\alpha_1\alpha_3}\bigl(\alpha_2\alpha_3+(\alpha_1-\alpha_2)(\alpha_1+\alpha_2)\bigr)\\&-2\frac{\alpha_2}{\alpha_3}\,\frac{(\alpha_4^2-\alpha_5^2)^2}{\alpha_4\alpha_5(\alpha_4^2+\alpha_5^2)}+2\frac{\alpha_4^2+\alpha_5^2}{\alpha_4\alpha_5}\biggr)\ \textit{for}\ 2\alpha_2^2=\alpha_4^2+\alpha_5^2;\end{aligned}$$

9) $\dfrac{d}{du}\ln(\alpha_4-\alpha_5)\sim\dfrac{2}{\alpha_3}$ *for* $\alpha_1=\alpha_2$, $\alpha_4-\alpha_5\to 0$, $\alpha_3\to 0$;

10) $$\begin{aligned}\frac{d}{du}\ln\biggl(\alpha_2-\frac{\alpha_4^2+\alpha_5^2}{2\alpha_4\alpha_5}\alpha_1\biggr)=-\biggl(\frac{\alpha_4^2-\alpha_5^2}{\alpha_4^2+\alpha_5^2}\biggr)^2\frac{1}{\alpha_3\alpha_4\alpha_5}\bigl(&(\alpha_4^2+\alpha_5^2)(\alpha_2+\alpha_3)\\&+\alpha_2(\alpha_4^2+\alpha_5^2+2\alpha_2\alpha_3)\bigr)\ \textit{for}\ \alpha_2=\frac{\alpha_4^2+\alpha_5^2}{2\alpha_4\alpha_5}\alpha_1.\end{aligned}$$

Lemma 8 is proved by direct calculation. However, certain observations simplify the calculations significantly. Let

$$F(S)=F(\alpha_1,\dots,\alpha_5)=\frac{P_1(S)}{P_2(S)}$$

be a homogeneous rational system of degree zero. Obviously,

$$\frac{d}{dt}\ln F(R)=\frac{1}{F(R)}\biggl\langle\frac{\partial F(R)}{\partial R},\frac{dR}{dt}\biggr\rangle=\frac{1}{F(R)}\biggl\langle\frac{\partial F(R)}{\partial R},V(R)\biggr\rangle$$

is a homogeneous rational function of R of degree -1, therefore

$$\frac{d}{du}\ln F(S)=f\frac{d}{dt}\ln F(fS)=f\frac{d}{dt}\ln F(R)=\frac{d}{dt}\ln F(S)$$

(here and below we use the notation from (4.1) and (4.2)). Thus, to find $\frac{d}{du}\ln F(S)$ it is sufficient to calculate $\frac{d}{dt}\ln F(S)$, that is, to do the calculations in parts 1)–3) of Lemma 8 we can use system (2.1), which is pretty manageable, instead of the very cumbersome (4.1).

Now let $F(S)$ be a rational function of degree $k > 0$. We will calculate its derivative at a point where $F(S) = 0$ (this is what we have in parts 4)–10) of Lemma 8). Obviously,

$$\frac{d}{dt}F(R) = \left\langle \frac{\partial F(R)}{\partial R}, \frac{dR}{dt} \right\rangle = \left\langle \frac{\partial F(R)}{\partial R}, V(R) \right\rangle$$

is a homogeneous rational function of degree $k-1$ of R. Then

$$\begin{aligned}\frac{d}{du}F(S) &= f\frac{d}{dt}(F(fS)f^{-k}) = f^{1-k}\frac{d}{dt}F(R) - kF(R)f^{-k-1}\frac{df}{du} \\ &= \frac{d}{dt}F(S) - kF(R)f^{-k}\langle V(S), S\rangle = \frac{d}{dt}F(S) - kF(S)\langle V(S), S\rangle = \frac{d}{dt}F(S).\end{aligned}$$

So as before, in calculating the derivative of $F(S)$ we use (2.1) in place of the intractable relations (4.1).

4.5. The behaviour of trajectories at infinity. The next assertion is the basis of the proof of Theorem 2.

Proposition 1. *The trajectory of* (4.1), *which is specified by an initial point* $S_0 = (0, -\lambda, \lambda, \mu, \nu)$, $\lambda > 0$, $\mu \geqslant \nu > 0$, $2\lambda^2 + \mu^2 + \nu^2 = 1$, *displays one of the following patterns of asymptotic behaviour, depending on the parameter* μ:

1) *if* $\lambda = \frac{1}{2}$, *then* $S(u)$ *converges to the stationary point*

$$S_\infty = \left(-\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

as $u \to \infty$; *the trajectory* $S(u)$ *corresponds to the metric* $\overline{g}_\alpha$ *with* $\alpha = \sqrt{\mu^2 - \nu^2}$ *in the family* (3.1);

2) *if* $\lambda < \frac{1}{2}$ *and* $\mu = \nu$, *then* $S(u)$ *converges to the conditionally stationary points*

$$S'_\infty = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

as $u \to \infty$;

3) *if* $\lambda < \frac{1}{2}$ *and* $\mu > \nu$, *then* $S(u)$ *converges to the point* $S_1 = (0, 0, 0, 1, 0)$ *as* $u \to u_1 < \infty$;

4) *if* $\lambda > \frac{1}{2}$, *then* $S(u)$ *converges to the point* $S_2 = (0, 0, 1, 0, 0)$ *as* $u \to u_2 < \infty$.

Proof. Consider the regions Π and Γ in S^4 defined by

$$\begin{aligned}\Pi &= \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_3 \geqslant 0,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2 + \alpha_3 \leqslant 0,\ 2\alpha_2^2 \leqslant \alpha_4^2 + \alpha_5^2\}, \\ \Gamma &= \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2 + \alpha_3 \geqslant 0,\ 2\alpha_2^2 \geqslant \alpha_4^2 + \alpha_5^2\}.\end{aligned}$$

The boundaries of Π and Γ are formed by the following subsets of S^4 (which we call 'walls' in what follows):

$$\Pi_1 = \{\alpha_1 = 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_3 \geqslant 0,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \leqslant 0,\ 2\alpha_2^2 \leqslant \alpha_4^2+\alpha_5^2\},$$
$$\Pi_2 = \{\alpha_1 \leqslant 0,\ \alpha_2 = \alpha_1,\ \alpha_3 \geqslant 0,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \leqslant 0,\ 2\alpha_2^2 \leqslant \alpha_4^2+\alpha_5^2\},$$
$$\Pi_3 = \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_3 = 0,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \leqslant 0,\ 2\alpha_2^2 \leqslant \alpha_4^2+\alpha_5^2\},$$
$$\Pi_4 = \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_3 \geqslant 0,\ \alpha_4 = \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \leqslant 0,\ 2\alpha_2^2 \leqslant \alpha_4^2+\alpha_5^2\},$$
$$\Pi_5 = \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_3 \geqslant 0,\ \alpha_4 \geqslant \alpha_5 = 0,\ \alpha_2+\alpha_3 \leqslant 0,\ 2\alpha_2^2 \leqslant \alpha_4^2+\alpha_5^2\},$$
$$\Pi_6 = \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_3 \geqslant 0,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 = 0,\ 2\alpha_2^2 \leqslant \alpha_4^2+\alpha_5^2\},$$
$$\Pi_7 = \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_3 \geqslant 0,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \leqslant 0,\ 2\alpha_2^2 = \alpha_4^2+\alpha_5^2\},$$
$$\Gamma_1 = \{\alpha_1 = 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \geqslant 0,\ 2\alpha_2^2 \geqslant \alpha_4^2+\alpha_5^2\},$$
$$\Gamma_2 = \{\alpha_1 \leqslant 0,\ \alpha_2 = \alpha_1,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \geqslant 0,\ 2\alpha_2^2 \geqslant \alpha_4^2+\alpha_5^2\},$$
$$\Gamma_3 = \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_4 = \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \geqslant 0,\ 2\alpha_2^2 \geqslant \alpha_4^2+\alpha_5^2\},$$
$$\Gamma_4 = \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_4 \geqslant \alpha_5 = 0,\ \alpha_2+\alpha_3 \geqslant 0,\ 2\alpha_2^2 \geqslant \alpha_4^2+\alpha_5^2\},$$
$$\Gamma_5 = \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 = 0,\ 2\alpha_2^2 \geqslant \alpha_4^2+\alpha_5^2\},$$
$$\Gamma_6 = \{\alpha_1 \leqslant 0,\ \alpha_2 \leqslant \alpha_1,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \geqslant 0,\ 2\alpha_2^2 = \alpha_4^2+\alpha_5^2\}.$$

In addition, we partition Π by the wall

$$\Pi_8 = \left\{\alpha_1 \leqslant 0,\ \alpha_2 = \frac{\alpha_4^2+\alpha_5^2}{2\alpha_4\alpha_5}\alpha_1,\ \alpha_3 \geqslant 0,\ \alpha_4 \geqslant \alpha_5 \geqslant 0,\ \alpha_2+\alpha_3 \leqslant 0,\ 2\alpha_2^2 \leqslant \alpha_4^2+\alpha_5^2\right\}$$

into the subdomains

$$\Pi' = \Pi \cap \left\{\alpha_2 \leqslant \frac{\alpha_4^2+\alpha_5^2}{2\alpha_4\alpha_5}\alpha_1\right\} \quad \text{and} \quad \Pi'' = \Pi \cap \left\{\frac{\alpha_4^2+\alpha_5^2}{2\alpha_4\alpha_5}\alpha_1 \leqslant \alpha_2 \leqslant \alpha_1\right\}.$$

In accordance with Lemma 7, the trajectory $S(u)$ goes out of the point

$$S_0 = (0, -\lambda, \lambda, \mu, \nu), \qquad \lambda > 0, \quad \mu \geqslant \nu > 0, \quad 2\lambda^2+\mu^2+\nu^2 = 1,$$

with tangent vector $e_1 = \bigl(-8, -\frac{2(2\mu^2+2\nu^2-1)}{\mu\nu}, 0, 0\bigr)$ (expressed in the system of coordinates $(\alpha_1, \alpha_2+\alpha_3, \alpha_4, \alpha_5)$). If $\mu = \nu$, then (2.1) reduces to system (2.2), which was investigated in [6], and in the case $\mu = \nu$ assertions 2) and 4) follow from the results of [6]. For this reason we shall assume that $\mu > \nu$. If $\lambda < \frac{1}{2}$, then $\mu^2+\nu^2 > \frac{1}{2}$. Now, $\alpha_2+\alpha_3 < 0$ for the coordinates of e_1, therefore at the initial instant the trajectory $S(u)$ enters the domain $\Pi' \subset \Pi$. In a similar way, if $\lambda > \frac{1}{2}$, then $\mu^2+\nu^2 < \frac{1}{2}$ and for u close to u_0 this curve enters the domain Γ. Finally, if $\lambda = \frac{1}{2}$, then $\mu^2+\nu^2 = \frac{1}{2}$, and the family of solutions explicitly described in (3.1) satisfies this condition; the trajectories of the solutions in (3.1) fill the intersection of Π and Γ. We shall analyse the behaviour of trajectories in the regions Π and Γ thoroughly; in each of them we must consider two significantly different cases: when the trajectory attains the boundary of the region in finite time and when it remains in the interior of the region for all values of u.

We split the rest of the proof of Proposition 1 into Lemmas 9–12.

Lemma 9. *Assume that* $\lambda < \frac{1}{2}$, *so that the trajectory* $S(u)$ *enters the domain* Π' *at the initial instant. Then* $S(u)$ *intersects the boundary of* Π *for some* $u = u_1 < \infty$.

Proof. Assume the contrary: $S(u)$ remains in Π for all values of u. The region Π contains one stationary point

$$S_2 = \left(-\frac{1}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

and one conditionally stationary point

$$S_3 = \left(-\frac{\sqrt{2}}{2\sqrt{3}}, -\frac{\sqrt{2}}{2\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

Both of them lie in $\Pi_2 \cap \Pi_4 \subset \partial\Pi$. Let $S(u) \to S_2$ as $u \to \infty$. In a neighbourhood of S_2 we introduce the system of coordinates $x = \alpha_1 + \frac{1}{\sqrt{5}}$, $y = \alpha_2 + \frac{1}{\sqrt{5}}$, $z = \alpha_3 - \frac{1}{\sqrt{5}}$, $w = \alpha_4 - \alpha_5$ and consider the linearization of (4.1) in this neighbourhood:

$$\begin{aligned} \frac{dx}{du} &= \frac{2}{\sqrt{5}}(-21x - y + 3z), \qquad \frac{dy}{du} = \frac{2}{\sqrt{5}}(-x - 21y + 3z), \\ \frac{dz}{du} &= \frac{2}{\sqrt{5}}(-9x - 9y + 7z), \qquad \frac{dw}{du} = -\frac{10}{\sqrt{5}}w. \end{aligned} \tag{4.5}$$

We can immediately verify that the system (4.5) has eigenvalues $-8\sqrt{5}$, $-8\sqrt{5}$, $-2\sqrt{5}$ and $2\sqrt{5}$. The eigenvectors corresponding to the negative eigenvalues span the hyperplane $x + y - 3z = 0$. Hence there exists a hypersurface tangent to the hyperplane $x + y - 3z = 0$ which is formed by trajectories of (4.1). These enter S_2 exponentially, and no other trajectory of (4.1) comes into S_2. Clearly, this hypersurface contains the intersection $\Pi \cap \Gamma$, which consists of the trajectories of solutions in the family (3.1). A direct calculation shows that in a neighbourhood of S_2 this hypersurface is transversal to the other walls of the regions Π and Γ, but is disjoint from the regions proper. Thus no trajectory of system (4.1) can approach S_2 save the trajectories corresponding to (3.1).

Now let $S(u) \to S_3$ as $u \to \infty$, so that in particular, $\alpha_3 \to 0$ and $\alpha_4 - \alpha_5 \to 0$. It follows from part 9) of Lemma 8 that $\ln(\alpha_4 - \alpha_5)$ increases for $u \to \infty$, which is a contradiction. Thus we see that S_2 and S_3 are not limit points of $S(u)$.

Note that as follows from part 10) of Lemma 8, either a trajectory lies entirely in Π' or after traversing a wall of Π' it goes over to the domain Π'' and cannot then return. We consider these two cases separately.

First assume that $S(u) \in \Pi'$ for all u. The relation in part 1) of Lemma 8 demonstrates that $F_1 = \ln\frac{\alpha_2}{\alpha_1}$ is a decreasing function along the trajectory $S(u)$ in Π'. Hence as $u \to \infty$, $S(u)$ approaches the minimum level of the function F_1 on Π', which is $\Pi_2 \cap \Pi' = \Pi_2 \cap \Pi_4 \cap \Pi'$. The relation in part 2) of Lemma 8 demonstrates that $F_2 = \ln\frac{\alpha_2}{\alpha_3}$ is an increasing function along the trajectories in a neighbourhood of $\Pi_2 \cap \Pi_4$. Hence $S(u)$ approaches the maximum level of F_2 in $\Pi_2 \cap \Pi_4$, so that $\alpha_3 \to 0$. Then however, it follows from part 9) of Lemma 8 that $\alpha_4 - \alpha_5$ is increasing, which contradicts the trajectory approaching the wall Π_4. This contradiction shows that the trajectory must go over to the domain Π''.

From part 2) of Lemma 8 we easily see that the function F_2 is increasing along trajectories in Π''. This means that $S(u)$ approaches the maximum level of the function F_2, that is, $\alpha_3 \to 0$. Next, in a neighbourhood of $\{\alpha_3 = 0\}$ F_1 is decreasing on trajectories, so $\alpha_1 - \alpha_2 \to 0$. An argument similar to the proof of the previous lemma (the part which deals with a neighbourhood of the wall Π_3) demonstrates that the trajectory either converges to the conditionally stationary point S_2 (which is impossible as we have just shown) or to the point $(0,0,0,1,0)$.

Thus we have shown that the trajectory $S(u)$ 'attains' the point $(0,0,0,1,0)$ in infinite time. However, it is easy to see that α_5^2 is a smooth function in a neighbourhood of $(0,0,0,1,0)$, so we can take α_5^2 for a new smooth parameter on $S(u)$. Now, we can attain the point $(0,0,0,1,0)$ only for some finite value of α_5^2, and therefore only for a finite value of u. The proof of Lemma 9 is complete.

Lemma 10. *Suppose* $\lambda < \frac{1}{2}$ *and assume that a trajectory* $S(u)$ *enters* Π' *at the initial instant and intersects the boundary of the domain* Π *at a point* $S_1 = S(u_1)$, $u_1 < \infty$, *for the first time. Then* $S_1 = (0,0,0,1,0)$.

Proof. In fact, since $\Pi_4 \setminus (\Pi_1 \cup \Pi_3 \cup \Pi_5)$ lies in an invariant subspace of the system (4.1), the wall Π_4 can be attained in finite time only at points in which Π_4 intersects the walls Π_1, Π_3 and Π_5.

Let $S_1 = (0,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \in \Pi_1$. If $\alpha_3 > 0$, then $\alpha_2 < 0$ and $\alpha_2 \neq \alpha_3$. Hence $(\alpha_2+\alpha_3)^2 < \alpha_2^2+\alpha_3^2 \leqslant 1$ and the relation in part 5) of Lemma 8 demonstrates that the α_1-coordinate is decreasing in a neighbourhood of the wall Π_1, so Π_1 cannot be attained in finite time from inside Π. Let $\alpha_3 = 0$ and $\alpha_2 \neq 0$. Then again α_1 is decreasing, except perhaps for the case $\alpha_2 = -1$, when $S_1 = (0,-1,0,0,0)$. However, in this case too the relation in part 5) of Lemma 8 demonstrates that in a neighbourhood of S_1, for $u < u_1$ the derivative of the variable α_1 is equal to $-2\frac{1+\alpha_2}{\alpha_3} < 0$, so α_1 is decreasing, which is a contradiction. Hence the only possibility is when $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $S_1 = (0,0,0,\alpha_4,\alpha_5)$. In this case assume that $\alpha_4 \geqslant \alpha_5 > 0$. Let $X = (x_1,x_2,x_3,x_4,x_5)$ be the tangent vector to $S(u)$ at S_1. Then we set

$$\overline{\lim_{u\to u_1}}\,\frac{\alpha_2(u)}{\alpha_1(u)} = h \quad\text{and}\quad \overline{\lim_{u\to u_1}}\,\frac{\alpha_3(u)}{\alpha_1(u)} = f.$$

Calculating the limits of $W(S(u))$ as $u \to u_1$ we obtain

$$\begin{aligned} x_1 &= \frac{h}{f} + \frac{f}{h} - 2 - \frac{1}{fh} - \frac{(\alpha_4^2-\alpha_5^2)^2}{2\alpha_4\alpha_5}\biggl(\frac{f}{h}+\frac{h}{f}\biggr), \\ x_2 &= \frac{1}{f} - \frac{h^2}{f} + f - \frac{\alpha_4^2+\alpha_5^2}{\alpha_4\alpha_5} - \frac{(\alpha_4^2-\alpha_5^2)^2}{2\alpha_4\alpha_5}\biggl(1+\frac{h}{f}\biggr), \\ x_3 &= \frac{1}{h} - \frac{f^2}{h} + h - \frac{\alpha_4^2+\alpha_5^2}{\alpha_4\alpha_5} - \frac{(\alpha_4^2-\alpha_5^2)^2}{2\alpha_4\alpha_5}\biggl(1+\frac{f}{h}\biggr). \end{aligned}$$

Now taking account of the equalities $x_2/x_1 = h$ and $x_3/x_1 = f$ we obtain

$$h = f = 2\frac{\alpha_4\alpha_5}{\alpha_4^2+\alpha_5^2}.$$

However, this means that the x_1-, x_2- and x_3-components of X have the same sign, so that the trajectory cannot attain S_1 from the domain Π, which is a contradiction. Hence the wall Π_1 is attainable in finite time only if $\alpha_5 = 0$, that is, $S_1 = (0, 0, 0, 1, 0)$.

Now let $S_1 \in \Pi_5$. The relation in part 4) of Lemma 8 implies that α_5^2 is increasing in a neighbourhood of $\Pi_5 \setminus \Pi_1$. Hence $S_1 \in \Pi_5 \cap \Pi_1$, so bearing in mind the above description of the trajectory in a neighbourhood of Π_1 we see that $S_1 = (0, 0, 0, 1, 0)$.

Let $S_1 \in \Pi_3$. Based of the above arguments we can assume that $\alpha_5 \neq 0$ and $\alpha_1 \neq 0$, so the component W_3 is smooth in a neighbourhood of Π_3. We take α_3 for a smooth parameter in a neighbourhood of $u = u_1$. If $\alpha_1 \neq \alpha_2$ or $\alpha_4 \neq \alpha_5$, then the component of W tangential to Π_3 in a neighbourhood of S_1 has order $1/\alpha_3$, so the trajectory cannot attain Π_3 in finite time. We see that $\alpha_1 = \alpha_2$, $\alpha_4 = \alpha_5$ and $S_1 = \left(\alpha, \alpha, 0, \sqrt{\frac{1-2\alpha^2}{2}}, \sqrt{\frac{1-2\alpha^2}{2}}\right)$. Let $X = (x_1, x_2, x_3, x_4, x_5)$ be the tangent vector to $S(u)$ at S_1. Obviously,

$$\lim_{\varepsilon \to -0} W(S_1 + \varepsilon X) = \lim_{u \to u_1} W(S(u)) = X.$$

It is clear that $\lim_{u \to u_1} W_3(S(u)) = 0$, so $x_3 = 0$. Hence $X = 0$ and therefore S_1 is a conditionally stationary point, which cannot be attained in finite time.

Let $S_1 \in \Pi_2$. In view of the above, we can assume that $S_1 \notin \Pi_1 \cup \Pi_3 \cup \Pi_5$ and therefore $S_1 \notin \Pi_4$ and $\alpha_4 > \alpha_5 > 0$. Then relation 6) in Lemma 8 demonstrates that the function $\alpha_1 - \alpha_2$ increases in a neighbourhood of Π_2, so Π_2 cannot be attained in finite time.

Finally, relations 7) and 8) in Lemma 8 demonstrate that at points in the walls $\Pi_6 \setminus \Pi_7$ and $\Pi_7 \setminus \Pi_6$ the vector W points inwards the domain, so for Π_6 and Π_7, only points in $\Pi_6 \cap \Pi_7$ can be attained in finite time. On the other hand the intersection $\Pi_6 \cap \Pi_7$ consists of the trajectories corresponding to the family of solutions (3.1), so this part also cannot be attained by trajectories $S(u)$ for $u = u_1$.

From the above we conclude that only the case $S_1 = (0, 0, 0, 1, 0)$ is possible under the assumptions of Lemma 10, which completes the proof.

Lemma 11. *Assume that* $\lambda > \frac{1}{2}$, *so that the trajectory going out of the initial point lies in the domain* Γ. *Then* $S(u)$ *intersects the boundary of the domain* Γ *for* $u = u_1 < \infty$.

Proof. Assume the converse: $S(u)$ remains in Γ for all u. First, by Lemma 4 there are no stationary points in the interior of Γ. Second, as already noted, the function F_3 is decreasing on trajectories of (4.1) in Γ, hence as $u \to \infty$, $S(u)$ approaches the set of points at which F_3 takes the minimum value in Γ, namely Γ_3. Now we observe that for $\alpha_4 = \alpha_5$ system (4.1) reduces to a system investigated in [6] for the same boundary values. Here the whole of our domain Γ_3 corresponds to the domain Γ considered in the proof of Lemma 13 in [6]. The functions increasing or decreasing on trajectories that were used in the proof in [6] are well defined in the whole of Γ and are increasing or decreasing, respectively, in some neighbourhood of Γ_3. Hence our argument in [6] shows that the trajectory $S(u)$ converges to the stationary point S_2 or to $(0, 0, 1, 0, 0)$, and in the second case the convergence takes finite time. Since the linearization (4.5) of system (4.1) in a neighbourhood of S_2, which we obtained in the first part of the proof of Lemma 9, demonstrates that $S(u)$

cannot tend to S_2, we have arrived at a contradiction and the proof of Lemma 11 is complete.

Lemma 12. *Suppose $\lambda > \frac{1}{2}$, and assume that the trajectory from the initial point lies in the domain Γ and intersects the boundary of Γ for the first time in $S_1 = S(u_1)$, $u_1 < \infty$. Then $S_1 = (0,0,1,0,0)$.*

Proof. Since $\Gamma_3 \setminus (\Gamma_1 \cup \Gamma_4)$ lies in a subspace invariant under the system (4.1), the wall Γ_3 can be attained in finite time only where it intersects the walls Γ_1 and Γ_4.

Consider the function $F_3 = \ln \frac{\alpha_4}{\alpha_5}$ on S^4. Formula 3) in Lemma 9 shows that F_3 is decreasing on the trajectories of system (4.1) in the domain Γ.

Let $S_1 = (0, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \Gamma_1$. If $\alpha_2 < 0$, then $\alpha_3 > 0$ and $\alpha_2 \neq \alpha_3$. Hence the α_1-coordinate decreases in a neighbourhood of Γ_1, which is a contradiction. If $\alpha_2 = 0$ and $\alpha_3 \neq 0$, then α_1 is not decreasing only when $\alpha_3 = 1$, that is, $S_1 = (0,0,1,0,0)$. Finally assume that $\alpha_2 = \alpha_3 = 0$, that is, $S_1 = (0,0,0,\alpha_4,\alpha_5)$. Repeating the argument used in the proof of Lemma 10 under the assumption that $\alpha_5 > 0$ word for word we arrive at a contradiction. Hence the remaining case is $\alpha_5 = 0$, so that $S_1 = (0,0,0,1,0)$. However, this means that in the interior of the region Γ the function F_3 increases without limit on the trajectory, which contradicts it being decreasing.

Let $S_1 \in \Gamma_4$. Relation 4) in Lemma 8 means that α_5^2 is increasing along trajectories in a neighbourhood of $\Gamma_4 \setminus \Gamma_1$, which leads to a contradiction (we have already shown that Γ_1 cannot be attained).

Finally, repeating the proof of Lemma 10 word for word we can show that the cases $S_1 \in \Gamma_2$, $S_1 \in \Gamma_5$ and $S_1 \in \Gamma_6$ are also impossible.

Assume that $S(u)$ remains in Γ for all u. First, it follows from Lemma 4 that there are no stationary points in the interior of Γ. Second, as we pointed out above, F_3 is a decreasing function on trajectories of (4.1) in the domain Γ. Hence as $u \to \infty$, the trajectory $S(u)$ converges to the set of points at which F_3 takes the minimum value in Γ, that is, to Γ_3. Now we observe that for $\alpha_4 = \alpha_5$ system (4.1) reduces to a system investigated in [6] for the same boundary data, and the whole of our Γ_3 corresponds to the domain Γ considered in the proof of Lemma 13 in [6].

The functions increasing or decreasing on trajectories that were used in the proof in [6], are well defined in the whole of our domain Γ and are increasing or decreasing, respectively, in some neighbourhood of Γ_3. Hence our argument in [6] shows that the trajectory $S(u)$ tends to the stationary point S_2 or to $(0,0,1,0,0)$, and in the second case the convergence takes finite time. Since the linearization of system (4.1) in a neighbourhood of S_2 which we performed in the first part of the proof of the lemma demonstrates that $S(u)$ cannot tend to S_2, we have thus proved that in the case under consideration the trajectory $S(u)$ attains the point $(0,0,1,0,0)$ in finite time. The proof of Lemma 12 is complete.

Lemmas 9–12 complete the proof of Proposition 1.

Obviously, Proposition 1 yields Theorem 2. Indeed, trajectories in part 1) of Proposition 1 correspond to metrics in family 1) from Theorem 2, trajectories in part 2) correspond to metrics in family 2) from Theorem 2, and trajectories in parts 3) and 4) of Proposition 1 converge in finite time to singular points, so they correspond to incomplete metrics on $\mathscr{M}_2$. The fact that the metrics from family 2) in Theorem 2 have the holonomy group Spin(7) was proved in [6]. The proof of Theorem 2 is now complete.

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G_2-HOLONOMY METRICS CONNECTED WITH A 3-SASAKIAN MANIFOLD

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UDC 514.763.3

Abstract: We construct complete noncompact Riemannian metrics with G_2-holonomy on noncompact orbifolds that are $\mathbb{R}^3$-bundles with the twistor space $\mathscr{Z}$ as a spherical fiber.

Keywords: exceptional holonomy group, 3-Sasakian manifold, twistor space

1. Introduction

This article addressing G_2-holonomy metrics is a natural continuation of the study of Spin(7)-holonomy metrics which was started in [1]. We consider an arbitrary 7-dimensional compact 3-Sasakian manifold M and discuss the existence of a smooth resolution of the conic metric over the twistor space $\mathscr{Z}$ associated with M.

Briefly speaking, a manifold M is 3-Sasakian if and only if the standard metric on the cone over M is hyper-Kähler. Each manifold of this kind M is closely related to the twistor space $\mathscr{Z}$ which is an orbifold with a Kähler–Einstein metric. We consider the metrics that are natural resolutions of the standard conic metric over $\mathscr{Z}$:

$$\bar{g} = dt^2 + A(t)^2\big(\eta_2^2 + \eta_3^2\big) + B(t)^2\big(\eta_4^2 + \eta_5^2\big) + C(t)^2\big(\eta_6^2 + \eta_7^2\big), \tag{$*$}$$

where η_2 and η_3 are the characteristic 1-forms of M, η_4, η_5, η_6, and η_7 are the forms that annul the 3-Sasakian foliation on M, and A, B, and C are real functions.

One of the main results of the article is the construction (in the case when $M/SU(2)$ is Kähler) of a G_2-structure which is parallel with respect to $(*)$ if and only if the following system of ordinary differential equations is satisfied:

$$A' = \frac{2A^2 - B^2 - C^2}{BC}, \quad B' = \frac{B^2 - C^2 - 2A^2}{CA}, \quad C' = \frac{C^2 - 2A^2 - B^2}{AB}. \tag{$**$}$$

In case $(**)$ we thus see that $(*)$ has holonomy G_2; hence, $(*)$ is Ricci-flat. The system of equations $(**)$ was previously obtained in [2] in the particular case $M = SU(3)/S^1$.

For a solution to $(**)$ to be defined on some orbifold or manifold, some additional boundary conditions are required at t_0 that we will state them later. These conditions cannot be satisfied unless $B = C$, which leads us to the functions that give rise to the solutions found originally in [3] when $M = S^7$ and $M = SU(3)/S^1$. If $B = C$ then $(*)$ is defined on the total space of an $\mathbb{R}^3$-bundle $\mathscr{N}$ over a quaternionic-Kähler orbifold $\mathscr{O}$. In general, $\mathscr{N}$ is not an orbifold except in the event that $M = S^7$ and $M = SU(3)/S^1$. Note that it is unnecessary for $\mathscr{O}$ to be Kähler in case $B = C$.

Finally, we consider the well-known examples of the 3-Sasakian manifolds constructed in [4] and describe the topology of the corresponding orbifolds $\mathscr{N}$.

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2. Construction of a Parallel G_2-Structure

The definition of 3-Sasakian manifolds, their basic properties, and further references can be found in [1]. We mainly take our notation from [1].

Let M be a 7-dimensional compact 3-Sasakian manifold with characteristic fields ξ^1, ξ^2, and ξ^3 and characteristic 1-forms η_1, η_2, and η_3. Consider the principal bundle $\pi : M \to \mathscr{O}$ with the structure group $Sp(1)$ or $SO(3)$ over the quaternionic-Kähler orbifold $\mathscr{O}$ associated with M. We are interested in the special case when $\mathscr{O}$ additionally possesses a Kähler structure.

The field ξ^1 generates a locally free action of the circle S^1 on M, and the metric on the twistor space $\mathscr{Z} = M/S^1$ is a Kähler–Einstein metric. It is obvious that $\mathscr{Z}$ is topologically a bundle over $\mathscr{O}$ with fiber $S^2 = Sp(1)/S^1$ (or $S^2 = SO(3)/S^1$) associated with π. Consider the obvious action of $SO(3)$ on $\mathbb{R}^3$. The two-fold cover $Sp(1) \to SO(3)$ determines the action of $Sp(1)$ on $\mathbb{R}^3$, too. Now, let $\mathscr{N}$ be a bundle over $\mathscr{O}$ with fiber $\mathbb{R}^3$ associated with π. It is easy to see that $\mathscr{O}$ is embedded in $\mathscr{N}$ as the zero section, and $\mathscr{Z}$ is embedded in $\mathscr{N}$ as a spherical section. The space $\mathscr{N}\backslash\mathscr{O}$ is diffeomorphic to the product $\mathscr{Z} \times (0, \infty)$. Note that $\mathscr{N}$ can be assumed to be the projectivization of the bundle $\mathscr{M}_1 \to \mathscr{O}$ of [1]. In general, $\mathscr{N}$ is a 7-dimensional orbifold; however, if M is a regular 3-Sasakian space then $\mathscr{N}$ is a 7-dimensional manifold.

Let $\{e^i\}, i = 0, 2, 3, \dots, 7$, be an orthonormal basis of 1-forms on the standard Euclidean space $\mathbb{R}^7$ (the numeration here is chosen so as to emphasize the connection with the constructions of [1] and to keep the original notation wherever possible). Putting $e^{ijk} = e^i \wedge e^j \wedge e^k$, consider the following 3-form Ψ_0 on $\mathbb{R}^7$:

$$\Psi_0 = -e^{023} - e^{045} + e^{067} + e^{346} - e^{375} - e^{247} + e^{256}.$$

A differential 3-form Ψ on an oriented 7-dimensional Riemannian manifold N defines a G_2-structure if, for each $p \in N$, there exists an orientation-preserving isometry $\phi_p : T_pN \to \mathbb{R}^7$ defined in a neighborhood of p such that $\phi_p^*\Psi_0 = \Psi|_p$. In this case the form Ψ defines the unique metric g_Ψ such that $g_\Psi(v, w) = \langle \phi_p v, \phi_p w \rangle$ for $v, w \in T_pN$ [3]. If the form Ψ is parallel ($\nabla\Psi = 0$) then the holonomy group of the Riemannian manifold N lies in G_2. The parallelness of the form Ψ is equivalent to its closeness and cocloseness [5]:

$$d\Psi = 0, \quad d * \Psi = 0. \tag{1}$$

Note that the form $\Phi_0 = e^1 \wedge \Psi_0 - *\Psi_0$, where $*$ is the Hodge operator in $\mathbb{R}^7$, determines a Spin(7)-structure on $\mathbb{R}^8$ with the orthonormal basis $\{e^i\}_{i=0,1,2,\dots,7}$.

Locally choose an orthonormal system $\eta_4, \eta_5, \eta_6, \eta_7$ that generates the annihilator of the vertical subbundle $\mathscr{V}$ so that

$$\omega_1 = 2(\eta_4 \wedge \eta_5 - \eta_6 \wedge \eta_7), \quad \omega_2 = 2(\eta_4 \wedge \eta_6 - \eta_7 \wedge \eta_5), \quad \omega_3 = 2(\eta_4 \wedge \eta_7 - \eta_5 \wedge \eta_6),$$

where the forms ω_i correspond to the quaternionic-Kähler structure on $\mathscr{O}$. It is clear that $\eta_2, \eta_3, \dots, \eta_7$ is an orthonormal basis for M annulling the one-dimensional foliation generated by ξ^1; therefore, we can consider the metric of the following form on $(0, \infty) \times \mathscr{Z}$:

$$\bar{g} = dt^2 + A(t)^2\big(\eta_2^2 + \eta_3^2\big) + B(t)^2\big(\eta_4^2 + \eta_5^2\big) + C(t)^2\big(\eta_6^2 + \eta_7^2\big). \tag{2}$$

Here $A(t)$, $B(t)$, and $C(t)$ are defined on the interval $(0, \infty)$.

We suppose that $\mathscr{O}$ is a Kähler orbifold; therefore, $\mathscr{O}$ has the closed Kähler form that can be lifted to the horizontal subbundle $\mathscr{H}$ as a closed form ω. Without loss of generality we can assume that we locally have

$$\omega = 2(\eta_4 \wedge \eta_5 + \eta_6 \wedge \eta_7).$$

If we now put

$$e^0 = dt, \quad e^i = A\eta_i,\ i = 2, 3, \quad e^j = B\eta_j,\ j = 4, 5, \quad e^k = C\eta_k,\ k = 6, 7,$$

then the forms Ψ_0 and $*\Psi_0$ become

$$\Psi_1 = -e^{023} - \frac{B^2+C^2}{4} e^0 \wedge \omega_1 - \frac{B^2-C^2}{4} e^0 \wedge \omega + \frac{BC}{2} e^3 \wedge \omega_2 - \frac{BC}{2} e^2 \wedge \omega_3,$$

$$\Psi_2 = C^2 B^2 \Omega - \frac{B^2+C^2}{4} e^{23} \wedge \omega_1 - \frac{B^2-C^2}{4} e^{23} \wedge \omega + \frac{BC}{2} e^{02} \wedge \omega_2 + \frac{BC}{2} e^{03} \wedge \omega_3,$$

where $\Omega = \eta_4 \wedge \eta_5 \wedge \eta_6 \wedge \eta_7 = -\frac{1}{8}\omega_1 \wedge \omega_1 = -\frac{1}{8}\omega_2 \wedge \omega_2 = -\frac{1}{8}\omega_3 \wedge \omega_3$.

It is now obvious that Ψ_1 and Ψ_2 are defined globally and independently of the local choice of η_i; consequently, they uniquely define the metric $\bar{g}$ given locally by (2). Then the condition (1) that the holonomy group lies in G_2 is equivalent to the equation

$$d\Psi_1 = d\Psi_2 = 0. \tag{3}$$

Theorem. *If $\mathscr{O}$ possesses a Kähler structure then (2) on $\mathscr{N}$ is a smooth metric with holonomy G_2 given by the form Ψ_1 if and only if the functions A, B, and C defined on the interval $[t_0, \infty)$ satisfy the system of ordinary differential equations*

$$A' = \frac{2A^2 - B^2 - C^2}{BC}, \quad B' = \frac{B^2 - C^2 - 2A^2}{CA}, \quad C' = \frac{C^2 - 2A^2 - B^2}{AB} \tag{4}$$

with the initial conditions

(1) *$A(0) = 0$ and $|A_1'(0)| = 2$;*
(2) *$B(0), C(0) \neq 0$, and $B'(0) = C'(0) = 0$;*
(3) *the functions A, B, and C have fixed sign on the interval (t_0, ∞).*

Proof. In [1] the following relations were obtained, closing the algebra of forms:

$$de^0 = 0,$$
$$de^i = \frac{A_i'}{A_i} e^0 \wedge e^i + A_i \omega_i - \frac{2A_i}{A_{i+1}A_{i+2}} e^{i+1} \wedge e^{i+2}, \quad i = 1,2,3 \bmod 3,$$
$$d\omega_i = \frac{2}{A_{i+2}} \omega_{i+1} \wedge e^{i+2} - \frac{2}{A_{i+1}} e^{i+1} \wedge \omega_{i+2}, \quad i = 1,2,3 \bmod 3.$$

By adding the relation $d\omega = 0$ and carrying out some calculations to be omitted here, we obtain the sought system.

The smoothness conditions for the metric at t_0 are proven by analogy with the case of holonomy Spin(7) which was elaborated in [1]. We only note that, taking the quotient of the unit sphere S^3 by the Hopf action of the circle, we obtain the sphere of radius 1/2, which explains the condition $|A'(0)| = 2$.

In case $B = C$ the system reduces to the pair of equations

$$A' = 2\left(\frac{A^2}{B^2} - 1\right), \quad B' = -2\frac{A}{B}$$

whose solution gives the metric

$$\bar{g} = \frac{dr^2}{1 - r_0^4/r^4} + r^2\left(1 - \frac{r_0^4}{r^4}\right)(\eta_2^2 + \eta_3^2) + 2r^2(\eta_4^2 + \eta_5^2 + \eta_6^2 + \eta_7^2).$$

The regularity conditions hold. This smooth metric was originally found in [3] in the event that $M = SU(3)/S^1$ and $M = S^7$ (observe that we need not require $\mathscr{O}$ to be Kähler when $B = C$).

In the general case $B \neq C$ system (4) can also be integrated [2]. However, the resulting solutions do not enjoy the regularity conditions.

3. Examples

Some interesting family of examples arises when we consider the 7-dimensional biquotients of the Lie group $SU(3)$ as 3-Sasakian manifolds. Namely, let p_1, p_2, and p_3 be pairwise coprime positive integers. Consider the following action of S^1 on the Lie group $SU(3)$:

$$z \in S^1 : A \mapsto \operatorname{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \cdot A \cdot \operatorname{diag}(1, 1, z^{-p_1-p_2-p_3}).$$

This action is free; moreover, it was demonstrated in [4] that there is a 3-Sasakian structure on the orbit space $\mathscr{S} = \mathscr{S}_{p_1,p_2,p_3}$. Moreover, the action of $SU(2)$ on $SU(3)$ by right translations

$$B \in SU(2) : A \mapsto A \cdot \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

commutes with the action of S^1 and can be pushed forward to the orbit space $\mathscr{S}$. The corresponding Killing fields will be the characteristic fields ξ_i on $\mathscr{S}$. Therefore, the corresponding twistor space $\mathscr{Z} = \mathscr{Z}_{p_1,p_2,p_3}$ is the orbit space of the following action of the torus T^2 on $SU(3)$:

$$(z, u) \in T^2 : A \mapsto \operatorname{diag}(z^{p_1}, z^{p_2}, z^{p_3}) \cdot A \cdot \operatorname{diag}(u, u^{-1}, z^{-p_1-p_2-p_3}). \tag{5}$$

Lemma. *The space $\mathscr{Z}_{p_1,p_2,p_3}$ is diffeomorphic to the orbit space of $U(3)$ with respect to the following action of T^3:*

$$(z, u, v) \in T^3 : A \mapsto \operatorname{diag}(z^{-p_2-p_3}, z^{-p_1-p_3}, z^{-p_1-p_2}) \cdot A \cdot \operatorname{diag}(u, v, 1). \tag{6}$$

It suffices to verify that each T^3-orbit in $U(3)$ exactly cuts out an orbit of the T^2-action (5) in $SU(3) \subset U(3)$.

Action (6) makes it possible to describe the topology of $\mathscr{Z}$ and, consequently, the topology of $\mathscr{N}$ clearly. Here we use the construction of [6]. Consider the submanifold $E = \{(u, [v]) \mid u \perp v\} \subset S^5 \times \mathbb{C}P^2$. It is obvious that E is diffeomorphic to $U(3)/S^1 \times S^1$ (the "right" part of (6)) and is the projectivization of the $\mathbb{C}^2$-bundle $\widetilde{E} = \{(u, v) \mid u \perp v\} \subset S^5 \times \mathbb{C}^3$ over S^5. By adding the trivial one-dimensional complex bundle over S^5 to $\widetilde{E}$, we obtain the trivial bundle $S^5 \times \mathbb{C}^3$ over S^5.

The group S^1 acts from the left by the automorphisms of the vector bundle $\widetilde{E}$, and $\mathscr{Z} = S^1 \backslash E$ is the projectivization of the $\mathbb{C}^2$-bundle $S^1 \backslash \widetilde{E}$ over the weighted complex projective space $\mathscr{O} = \mathbb{C}P^2(q_1, q_2, q_3) = S^1 \backslash S^5$, where $q_i = (p_{i+1} + p_{i+2})/2$ for p_i all odd and $q_i = (p_{i+1} + p_{i+2})$ otherwise.

The above implies that the bundle $S^1 \backslash \widetilde{E}$ is stably equivalent to the bundle $S^1 \backslash (S^5 \times \mathbb{C}^3)$ over $\mathscr{O}$. The last bundle splits obviously into the Whitney sum $\sum_{i=1}^{3} \xi^{q_i}$, where ξ is an analog of the one-dimensional universal bundle of $\mathscr{O}$.

Corollary. *The twistor space $\mathscr{Z}$ is diffeomorphic to the projectivization of a two-dimensional complex bundle over $\mathbb{C}P^2(q_1, q_2, q_3)$ which is stably equivalent to $\xi^{q_1} \oplus \xi^{q_2} \oplus \xi^{q_3}$.*

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ON SOME RICCI-FLAT METRICS OF COHOMOGENEITY TWO ON COMPLEX LINE BUNDLES

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UDC 514.764.226

Abstract: We construct a family of four-dimensional smooth Ricci-flat Riemann orbifolds of cohomogeneity two which possess the structure of complex line bundles.

Keywords: Einstein manifold, Einstein equation

1. Introduction

In this article we study some Ricci-flat Riemannian metrics that are interesting because of their application in theoretical physics. An *Einstein metric* is a metric g_{ij} satisfying the Einstein equation

$$R_{ij} = \lambda g_{ij}, \tag{1}$$

where R_{ij} is the Ricci tensor of g_{ij}. If $\lambda = 0$ then we arrive at the case of the zero Ricci curvature which this article is devoted to. The Einstein equation has an extremely complicated structure and there is no general approach to its solution at present. Therefore, it is natural to try and search solutions under the additional assumption that the metric is symmetric. Some results were obtained in this direction, and now much is known about the homogeneous metrics and the metrics of cohomogeneity one (see the survey in [1, 2]). However, of great importance is the problem of constructing the solutions to the Einstein equation with possibly smaller groups of isometries. For example, one of the unsolved problems of general relativity is the problem of construction of the Lorentz metric describing the gravitational field between two bodies. It is clear from physical considerations that the corresponding solution to (1) can have an at most one-dimensional group of isometries (i.e., of cohomogeneity three). At present, in the Riemannian case there are several exact solutions to the Einstein equation with small groups of isometries. For example, using the principal torus bundles over products of Kähler–Einstein manifolds, it is possible to construct solutions to (1) with arbitrarily high cohomogeneity; however, these solutions reduce analytically to metrics of cohomogeneity one. It would be interesting to study solutions with more complicated analytic structure (an example of a “more complicated” metric is the Kerr metric of general relativity). It seems like this question has been studied better in the four-dimensional case in which there are exact solutions called “multi-instantons” and constructed in [3]. One of the characteristic features of these solutions is their topological “opacity.” The metrics we construct are intermediate in a sense: their topological structure is well controlled; but instead we obtain two singular points whose neighborhoods are diffeomorphic to cones over lens spaces.

Some of the well-known Ricci-flat metrics of cohomogeneity one are naturally defined on spaces of vector bundles. All these bundles with Kählerian base and the corresponding metrics were classified in [4] and studied in [5] from somewhat different standpoints. In particular, complete Ricci-flat metrics exist on part of complex line bundles over $\mathbb{C}P^n$, namely on all such bundles with the Chern class not exceeding n in magnitude (we take the first class to be the Chern class of the universal line bundle). We are particularly interested in one of these metrics in [6] called the Eguchi–Hanson metric:

$$d\tilde{s}^2 = \frac{dr^2}{1 - \frac{1}{r^4}} + r^2 \left(1 - \frac{1}{r^4}\right)(d\tau - A)^2 + r^2 ds^2. \tag{2}$$

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This metric is a smooth and complete Riemannian metric on the cotangent bundle T^*S^2 of the sphere S^2; here ds^2 is the standard metric on S^2, dA is the Kählerian form of this metric, and τ and r are the angular and radial coordinates on the complex fibers of T^*S^2.

In this article we construct a family of Ricci-flat metrics (6) depending on a rational parameter a ($|a| < 1$) and defined on the space M_a of the complex line bundle over the two-dimensional sphere S^2 lying in M_a as the "zero fiber." Moreover, S^2 bears the structure of an orbifold with two conic points (the "angles" at these points depend on a and differ from one another in general) and the bundle itself is not locally trivial. The space M_a has the structure of an orbifold; moreover, it is a manifold everywhere but two singular points on $S^2 \subset M_a$ whose neighborhoods are diffeomorphic to cones over lens spaces. The metric (6) is smooth and complete on M_a. For $a = 0$ we obtain (2) (which is generalized by our solution in the same way as the Kerr solution generalizes the Schwarzschild solution), and for $a \neq 0$ the constructed metric has cohomogeneity two.

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2. Construction of Solutions

Given some domain U with coordinates $(\rho, \theta, \phi, \psi) = (x^0, x^1, x^2, x^3)$, consider the Riemannian metric

$$ds^2 = f(d\rho^2 + d\theta^2) + g_{ij}dx^i dx^j, \tag{3}$$

where $i, j = 2, 3$ and the function f and the matrix $g = (g_{ij})$ depend only on ρ and θ. A metric of a similar form (the Lorentz version) was considered in [7]; in the same article the Ricci tensor was calculated for this metric. The condition $R_{ij} = 0$ takes the form

$$\frac{\partial}{\partial\rho}\left(\sqrt{\det g}\frac{\partial g}{\partial\rho}g^{-1}\right) + \frac{\partial}{\partial\theta}\left(\sqrt{\det g}\frac{\partial g}{\partial\theta}g^{-1}\right) = 0, \tag{4}$$

$$\begin{gathered}\frac{\partial(\log f)}{\partial\rho}\frac{\partial(\log\det g)}{\partial\theta} + \frac{\partial(\log f)}{\partial\theta}\frac{\partial(\log\det g)}{\partial\rho} = 2\frac{\partial^2(\log\det g)}{\partial\rho\partial\theta} + \mathrm{Tr}\left(\frac{\partial g}{\partial\rho}g^{-1}\frac{\partial g}{\partial\theta}g^{-1}\right),\\ \frac{\partial(\log f)}{\partial\rho}\frac{\partial(\log\det g)}{\partial\rho} - \frac{\partial(\log f)}{\partial\theta}\frac{\partial(\log\det g)}{\partial\theta} = \left(\frac{\partial^2}{\partial\rho^2} - \frac{\partial^2}{\partial\theta^2}\right)(\log\det g)\\ +\frac{1}{2}\,\mathrm{Tr}\left(\frac{\partial g}{\partial\rho}g^{-1}\frac{\partial g}{\partial\rho}g^{-1} - \frac{\partial g}{\partial\theta}g^{-1}\frac{\partial g}{\partial\theta}g^{-1}\right).\end{gathered} \tag{5}$$

We see that the main of these is the equation (4) in the matrix g; having found g we can determine the function f from (5) just by integration. Equation (4) in the Lorentz case was also studied in [8, 9], where an L–A pair for (4) was found and the soliton approach was developed. We assume that the variables ϕ and ψ are periodic and defined on some two-dimensional torus; obviously, this torus acts by isometries on the space endowed with the metric ds^2.

Studying the system of (4) and (5), we have managed to find the following solution:

$$f = b\cosh\rho - a\cos\theta,$$
$$g = \frac{1}{f}\left[\sinh^2\rho\begin{pmatrix}\cos^2\theta & b\cos\theta\\ b\cos\theta & b^2\end{pmatrix} + \sin^2\theta\begin{pmatrix}\cosh^2\rho & a\cosh\rho\\ a\cosh\rho & a^2\end{pmatrix}\right],$$

where a and b are arbitrary real parameters. Thus, we obtain the Ricci-flat metric of the form

$$\begin{gathered}ds^2 = (\cosh\rho - a\cos\theta)\left(d\rho^2 + d\theta^2\right) + \frac{\sinh^2\rho}{\cosh\rho - a\cos\theta}\left(d\psi + \cos\theta d\phi\right)^2\\ +\frac{\sin^2\theta}{\cosh\rho - a\cos\theta}\left(ad\psi + \cosh\rho d\phi\right)^2\end{gathered} \tag{6}$$

(we put $b = 1$ for simplicity; the case of an arbitrary b reduces to the general case by homothety). In analysis of the so-obtained metric, it is sometimes convenient to pass to a "geometric" coordinate systems. Namely, introduce the coordinate r by the relation

$$r^2 = \cosh \rho.$$

With respect to the coordinates (r, θ, ϕ, ψ), the metric (6) takes the form

$$ds^2 = (r^2 - a\cos\theta)\left(\frac{4r^2dr^2}{r^4-1} + d\theta^2\right) + \frac{r^4-1}{r^2 - a\cos\theta}(d\psi + \cos\theta d\phi)^2 + \frac{r^4\sin^2\theta}{r^2 - a\cos\theta}\left(\frac{a}{r^2}d\psi + d\phi\right)^2. \quad (6')$$

To study the topology of the space endowed with the metric (6), we need some preliminaries. Let

$$S^3 = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 = 1\}$$

be the three-dimensional sphere in $\mathbb{C}^2$. Take an arbitrary pair k, l of coprime numbers and denote by $\omega = e^{2\pi i \frac{1}{k+l}}$ the generator of the group $\mathbb{Z}_{k+l}$ of roots of unity of degree $k + l$. Consider the action of the group $\mathbb{Z}_{k+l}$ on S^3:

$$\omega \cdot (z_1, z_2) = (\omega^k z_1, \omega^l z_2).$$

This action is free and its factor-space is the lens space $L(-1, k + l)$. Now, consider the following action of the circle S^1 on S^3:

$$z \in S^1 : (z_1, z_2) \mapsto (z^k z_1, z^l z_2).$$

This action is not free, and its factor-space is the two-dimensional orbifold $S^2(k, l)$ which is topologically homeomorphic to the sphere but has two singular conic points at the poles with "angles" $2\pi/k$ and $2\pi/l$. It is obvious that the orbits of the action of S^1 contain the orbits of the action of $\mathbb{Z}_{k+l}$; therefore, we obtain the natural projection

$$p : L(-1, k + l) \to S^2(k, l).$$

By analogy with the construction of the universal complex line bundle over $\mathbb{C}P^1 = S^2$ for the Hopf bundle S^3 over S^2, we define the space $M_{k,l}$ as the cylinder of the mapping p; i.e., we have to consider the cylinder over $L(-1, k + l)$ and shrink to a point the orbits of the action of S^1 on one base. The space $M_{k,l}$ has the structure of a smooth four-dimensional orbifold and fibers naturally over $S^2(k, l)$ with one-dimensional complex fibers, although this bundle is no longer locally trivial in general. It is obvious that $M_{k,l}$ is a smooth manifold everywhere but possibly two poles in $S^2(k, l) \subset M_{k,l}$.

Theorem. (i) *The metric* (6) *is a smooth complete Ricci-flat Riemannian metric of cohomogeneity two on* $M_a = M_{k,l}$ *for* $-1 < a = \frac{p}{q} < 1$, $a \neq 0$, *where* p, q *is a pair of coprime integers,* $q > 0$, *and*

$$k = \begin{cases} q - p, \ l = q + p & \text{if } q \pm p \text{ is odd,} \\ \frac{q-p}{2}, \ l = \frac{q+p}{2} & \text{if } q \pm p \text{ is even.} \end{cases}$$

(ii) *If* $a = 0$ *then the metric* (6) *is a complete Ricci-flat Riemannian metric of cohomogeneity one on* $M_{1,1}$ *and coincides with the Eguchi–Hanson metric* (2).

Proof. By hypothesis, the pair k, l is such that $a = \frac{l-k}{l+k}$ and the numbers k and l are coprime. Consider the coordinate system (θ, α, β) on S^3:

$$z_1 = \cos\frac{\theta}{2}e^{2\pi i\alpha}, \quad z_2 = \sin\frac{\theta}{2}e^{2\pi i\beta},$$

where $0 \le \theta \le \pi$ and $(\alpha, \beta) \in \mathbb{R}^2/(\mathbb{Z} \oplus \mathbb{Z})$. Factorizing S^3 under the action of $\mathbb{Z}_{k+l}$, we supplement the lattice $\mathbb{Z} \oplus \mathbb{Z}$ that is generated by the vectors $e_1 = \frac{\partial}{\partial\beta}$ and $e_2 = \frac{\partial}{\partial\alpha}$ with the element

$$e_3 = \frac{k}{k+l}\frac{\partial}{\partial\alpha} + \frac{l}{k+l}\frac{\partial}{\partial\beta}.$$

Thereby the coordinates (θ, α, β) determine the lens space $L(-1, k+l)$, if $(\alpha, \beta) \in \mathbb{R}^2/\Gamma$, where

$$\Gamma = \langle e_1, e_2, e_3 \rangle.$$

On $M_{k,l}$, we add the radial coordinate r which is the "distance to the bottom" of the cylinder along its element plus one. Thus, $r \geq 1$; moreover, we have $r = 1$ exactly for the points of $S^2(k,l)$. We obtain the coordinate system $(r, \theta, \alpha, \beta)$ on $M_{k,l}$. Now, consider the coordinates (r, θ, ψ, ϕ) connected with the previous coordinates by the relations

$$\alpha = \frac{\psi + \phi}{2}, \quad \beta = \frac{\psi - \phi}{2}.$$

With respect to the new coordinates, the generators of the lattice Γ take the form

$$e_1 = \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \phi}, \quad e_2 = \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \phi}, \quad e_3 = \frac{\partial}{\partial \psi} + \frac{k-l}{k+l}\frac{\partial}{\partial \phi}.$$

Now, return to metric (6). We assume that the coordinates (r, θ, ϕ, ψ) are defined in the domain $U \times \mathbb{R}^2/\Gamma$, where

$$U = \{(r, \theta) \mid 1 < r < \infty,\ 0 < \theta < \pi\}.$$

It is obvious that the metric is Riemannian (i.e., has signature $++++$) and regular in $U \times \mathbb{R}^2/\Gamma$, if

$$-1 < a < 1.$$

Degeneration of the metric on the boundary of the domain implies "gluing" of points of the boundary to one another and determines the topology of the resulting Riemannian space which has singularities a priori. Obviously, the set of degeneration points of the metric splits into the three components defined by the respective equations $\theta = 0$, $\theta = \pi$, and $r = 1$. On each of these components, a one-dimensional tangent distribution appears along which (6) is zero. Namely, we obtain the one-dimensional distributions

$$V_1 = \{dr = d\theta = d\psi + d\phi = 0\},$$

$$V_2 = \{dr = d\theta = d\psi - d\phi = 0\},$$

$$V_3 = \{dr = d\theta = a d\psi + d\phi = 0\}$$

for the respective components $\theta = 0$, $\theta = \pi$, and $r = 1$. Observe that on the intersection of the components of degeneration points ($\{\theta = 0, r = 1\}$ and $\{\theta = \pi, r = 1\}$) two pairs V_1, V_3 and V_2, V_3 of linearly independent distributions are defined. The integral curves of the distributions V_i which are linear windings of the torus $\mathbb{R}^2/\Gamma$ have zero arclength; consequently, to extend smoothly the metric to the boundary $\partial U \times \mathbb{R}^2/\Gamma$, we have to shrink each integral curve of the distributions V_i to a point.

Fix $r > 1$. Then, for $\theta = 0$ ($\theta = \pi$), the distribution V_1 (V_2) is generated by the vector e_1 (e_2). It means that, in the product $(0, \pi) \times \mathbb{R}^2/\langle e_1, e_2 \rangle$, we have to shrink the parallels on the left boundary and the meridians of the torus on the right boundary. Thus, we obtain the sphere S^3. Now, taking the element e_3 of the lattice Γ into account, we find that, for a fixed $r > 1$, the space defined by metric (6) is diffeomorphic to the lens $L(-1, k+l)$.

Now, let us see what is happening on the lower base $r = 1$. Here we have to shrink to a point each integral curve of the distribution V_3 on S^3. It is easy to see that the distribution V_3 is generated by the vector e_3; therefore, the integral curve of V_3 is the winding $\mathbb{R}e_3$; i.e., shrinking the curves on the lower base, we obtain $S^2(k,l)$. Thus, metric (6) is defined on $M_{k,l}$. Now, find out how smooth the metric is and what the structure of $M_{k,l}$ is at two "singular" points $\theta = 0$, $r = 1$ and $\theta = \pi$, $r = 1$.

First, studying smoothness of metric (6), we can divide it by a smooth factor f and consider the metric

$$d\tilde{s}^2 = \frac{4r^2 dr^2}{r^4 - 1} + d\theta^2 + \frac{r^4 - 1}{(r^2 - a\cos\theta)^2}(d\psi + \cos\theta d\phi)^2 + \frac{r^4 \sin^2\theta}{(r^2 - a\cos\theta)^2}\left(\frac{a}{r^2}d\psi + d\phi\right)^2.$$

On $M_{k,l}$, consider two two-dimensional mutually orthogonal distributions D_1 and D_2:

$$D_1 = \{dr = 0,\ d\psi + \cos\theta d\phi = 0\}, \quad D_2 = \left\{d\theta = 0,\ \frac{a}{r^2}d\psi + d\phi = 0\right\}.$$

These distributions are smooth on $M_{k,l}$ and nondegenerate. Hence, it suffices to prove smoothness of the restrictions of the metric $d\tilde{s}^2$ to D_1 and D_2 in neighborhoods of the sets $\theta = 0, \pi; r = 1$. We can easily calculate

$$d\tilde{s}^2|_{D_1} = d\theta^2 + \sin^2\theta d\phi^2, \quad d\tilde{s}^2|_{D_2} = d\rho^2 + \tanh^2\rho d\psi^2.$$

Consider the mapping

$$u_1 : (r, \theta, \psi, \phi) \mapsto (\theta, \phi)$$

in a neighborhood of the points $\theta = 0$ and $\theta = \pi$ and the mapping

$$u_2 : (r, \theta, \psi, \phi) \mapsto (\rho, \psi)$$

in a neighborhood of the points $r = 1$. We see that the restrictions of the differentials du_1 and du_2 to D_1 and D_2 are linear isomorphisms. Thus, the metric $d\tilde{s}^2|_{D_1}$ is the pull-back under u_1 of the metric $d\theta^2 + \sin^2\theta d\phi^2$ on the plane with polar coordinates θ and ϕ. Similarly, the metric $d\tilde{s}^2|_{D_2}$ is the pull-back under u_2 of the metric $d\rho^2 + \tanh^2\rho d\psi^2$. These two metrics are smooth if and only if the total periods of the variables ϕ and ψ are equal to 2π. If $r > 1$ and $\theta = 0$ (or $\theta = \pi$) then the torus curves $\mathbb{R}e_1$ (or $\mathbb{R}e_2$) shrink to a point. We can easily verify that periodicity of these curves is determined by the element e_1 (or e_2) of the lattice Γ which corresponds to the total period 2π of the variable ϕ. Similarly, for $r = 1$ and $\theta \neq 0, \pi$ the torus curves $\mathbb{R}e_3$ shrink to a point; and periodicity on $\mathbb{R}e_3$ is determined by the vector e_3 and the variable ψ has the total period 2π. Now, the curves $\mathbb{R}e_1$ and $\mathbb{R}e_3$ shrink simultaneously at the point $\theta = 0$, $r = 1$. After shrinking a neighborhood of this point becomes diffeomorphic to the product of two two-dimensional disks (for example, with polar coordinate systems (ρ, ψ) and (θ, ϕ)) with a smooth metric pulled back by u_1 and u_2 as above. However, we have to factorize this neighborhood under the action of the element $e_2 \in \Gamma$. Since

$$e_2 \equiv -\frac{1}{k}e_1 + \frac{1}{k}e_3 (\operatorname{mod} e_1, e_3),$$

we finally find that a neighborhood of the point $\theta = 0$, $r = 1$ in $M_{k,l}$ is diffeomorphic to a cone over the lens space $L(-1, k)$; moreover, the metric on the cone is smooth in the sense of smoothness of a metric on an orbifold, i.e., is obtained by factorization of a smooth metric on $\mathbb{R}^4$ under the discrete group of isometries generated by the element e_2. Similarly, at the point $\theta = \pi$, $r = 1$ we obtain

$$e_1 \equiv -\frac{1}{l}e_2 + \frac{1}{l}e_3 (\operatorname{mod} e_2, e_3);$$

consequently, a neighborhood of this point in $M_{k,l}$ is diffeomorphic to a cone over the lens space $L(-1, l)$ with a smooth metric.

Prove now that the cohomogeneity of (6) indeed equals two for $a \neq 0$. Since this question is of local character, consider the metric ds^2 on the domain $U \times \mathbb{R}^2$ with coordinates $(\rho, \theta, \psi, \phi)$. For arbitrary values ψ_0 and ϕ_0 of ψ and ϕ, the surface M_{ψ_0,ϕ_0} in M_a defined by the equations $\psi = \psi_0$ and $\phi = \phi_0$ is a totally geodesic surface with coordinates (ρ, θ) as the set of fixed points of the isometry

$$\sigma_{\psi_0,\phi_0} : (\rho, \theta, \psi, \phi) \mapsto (\rho, \theta, 2\psi_0 - \psi, 2\phi_0 - \phi).$$

Assume given a Killing field K on $U \times \mathbb{R}^2$. Then $\operatorname{Ad}\sigma_{\psi_0,\phi_0}(K)$ is a Killing field as well. Consider the average of the field K over σ:

$$\overline{K} = \frac{1}{2}(K + \operatorname{Ad}\sigma_{\psi_0,\phi_0}(K)).$$

The field $\overline{K}$ is tangent to M_{ψ_0,ϕ_0}, is a Killing field on M_a, and consequently is a Killing field on M_{ψ_0,ϕ_0}. However, the metric on the surface M_{ψ_0,ϕ_0} is independent of ψ_0 and ϕ_0 and has the form

$$d\bar{s}^2 = (\cosh\rho - a\cos\theta)(d\rho^2 + d\theta^2).$$

We verify immediately that the metric $d\bar{s}^2$ has no nontrivial Killing vector fields for $a \neq 0$; therefore, $\overline{K} = 0$ on M_{ψ_0,ϕ_0}. But it means that the field K is orthogonal to the surface M_{ψ_0,ϕ_0}. Since ψ_0 and ϕ_0 are arbitrary, we find that the field K is orthogonal to the fields $\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial\theta}$ everywhere. But it means that every Killing field K is tangent to the orbits of the action of the torus T^2. Thus, the cohomogeneity of M_a equals $\dim M_a/T^2 = 2$. The theorem is proven.

3. Some Remarks

1. We have shown that, in neighborhoods of two singular points, the space $M_{k,l}$ has the structure of a cone over the respective lens spaces $L(-1,k)$ and $L(-1,l)$. Hence, for $q = p+1$ a neighborhood of the first point becomes a manifold and for $q = -p+1$ the same happens to a neighborhood of the second point. However, $M_{k,l}$ is a manifold at both singular points only for $a = 0$.

2. If the parameter a is irrational then the distribution V_3 is a dense winding of the torus $\mathbb{R}^2/\langle e_1, e_2\rangle$. Hence, for $r = 1$ we have to shrink to a point each such torus; i.e., instead of $S^2(k,l)$ in M_a we obtain a segment such that a neighborhood of each point of the segment in M_a is homeomorphic to the product of a cone over the 2-torus and a segment. Thus, M_a is not an orbifold in this case, and we cannot speak of smoothness or completeness of the metric on M_a. In the limit case $a = \pm 1$ we can establish straightforwardly that $R_{ijkl} = 0$; i.e., ds^2 is a planar metric.

3. The form (3) of the metric guarantees decomposition of the Einstein equations into two groups of equations, (4) and (5); moreover, (4) is an equation in the matrix g only and (5) can be elementary integrated, once g is known. This suggests a possible way of generalization to the case of higher dimensions: consider the following metric with the variables $(x^1,\dots,x^n,y^1,\dots,y^m)$: $ds^2 = h_{\alpha\beta}dx^\alpha dx^\beta + g_{ij}dy^i dy^j$, where the matrices $h_{\alpha\beta}$ and g_{ij} depend only on $(x^1,\dots,x^n)$. Then the problem is to find a class of metrics $h_{\alpha\beta}$ such that the Einstein equation (1) splits into two groups: a matrix nonlinear equation containing only the unknowns g_{ij} and an elementary integrable equation with g_{ij} known. Moreover, we can assume that the variables $(y_1,\dots,y_n)$ are cyclic and defined on some m-torus T^m acting by isometries on the space with the metric ds^2. Thereby we find that ds^2 is defined on some torus manifold (or orbifold) and has cohomogeneity m. Perhaps, the most interesting examples will come from the case $m = n$.

4. Undoubtedly, it is interesting to construct solutions of the form (3) to (1) for $\lambda \neq 0$, especially for $\lambda > 0$. In the latter case, by the Myers theorem, a solution (provided that it is smooth and complete) is defined on a compact manifold (orbifold); therefore, the domain of the variables (ρ,θ) acquires an extra "piece" of the boundary, which imposes more stringent regularity requirements on the boundary.

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SPECIAL KÄHLER METRICS ON COMPLEX LINE BUNDLES AND THE GEOMETRY OF $K3$-SURFACES

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UDC 514.764.22

Abstract: We construct metrics with the holonomy group $SU(2)$ on the tangent bundles of weighted complex projective lines. We give a geometric description of a neighborhood of the moduli space of special Kähler metrics on a $K3$-surface.

Keywords: special Kähler manifold, $K3$-surface

§ 1. Introduction

In this article we continue studying the Ricci-flat Riemannian metrics that were constructed in [1]. On closer examination it turned out that they possess a number of remarkable properties; in particular, they have the holonomy group $SU(2)$, so presenting special Kähler metrics.

The metrics of holonomy $SU(2)$ are interesting because of their applications in mathematical physics. In superstring theory and M-theory there appear compact manifolds with special holonomy groups. Moreover, if we admit the presence of physically isolated singularities then it suffices to study asymptotically flat metrics on the normal bundles of these singularities. Thus, we arrive at the problem of studying asymptotically locally Euclidean metrics with special holonomies on bundles over orbifolds.

One of the most topical examples of special Kähler metrics is the Eguchi–Hanson metric [2] on the cotangent bundle T^*S^2 of the standard two-dimensional sphere (without singularities). The Eguchi–Hanson metric has played an important role in studying special holonomy groups. Namely, Page [3] proposed a description of the space of special Kähler metrics on a $K3$-surface in which the Eguchi–Hanson metric plays the role of an "elementary brick." More exactly, represent a $K3$-surface using Kummer's construction; i.e., consider the involution of the flat torus T^4 which arises from the central symmetry of the Euclidean space $\mathbb{R}^4$. Factorizing, we obtain an orbifold with 16 singular points whose neighborhoods look like $\mathbb{C}^2/\mathbb{Z}_2$. Blowing up the resulting orbifold in a neighborhood of each singular point, we obtain a $K3$-surface. Topologically, the construction of blowing up a singular point of the form $\mathbb{C}^2/\mathbb{Z}_2$ is carried out as follows: We have to delete the singularity and identify its neighborhood with the space of the spherical bundle in T^*S^2 without the zero fiber S^2. Page proposed to consider a metric on T^*S^2 which is homothetic to the Eguchi–Hanson metric with a sufficiently small homothety coefficient so that the metric on the boundary of the glued spherical bundle becomes arbitrarily close to a flat metric. After that we need to deform slightly the metric on the torus so as to obtain a smooth metric on a $K3$-surface with holonomy $SU(2)$. A simple evaluation of the degrees of freedom in the process of this operation demonstrates that we obtain a 58-dimensional family of metrics which agrees with the well-known results on the dimension of the moduli space of such metrics [4]. Later, Page's idea was used by Joyce [5, 6] for constructing the first compact examples of manifolds with the exotic holonomy groups G_2 and Spin(7).

We can identify the two-dimensional sphere S^2 with the complex projective line $\mathbb{C}P^1$. Consider its natural generalization $\mathbb{C}P^1(k, l)$, the weighted complex projective line, which is a complex orbifold with two singular points. In this article we obtain the following

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Theorem. *On the cotangent bundle $M_{k,l} = T^*\mathbb{C}P^1(k,l)$ of the weighted complex projective line, there is a metric with the holonomy group $SU(2)$.*

This metric was found in [1] in a special coordinate system as a solution to the equation of the zero Ricci curvature on torus bundles over two-dimensional surfaces. Also, in [1] it was proven that the metric has the isometry group $U(1) \times U(1)$ and cohomogeneity 2 except for the case $k = l = 1$. For $k = l = 1$ our metric coincides with the Eguchi–Hanson metric.

Asymptotically, the constructed metrics behave as follows: At infinity the metric tends to the Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_{k+l}$, while in a neighborhood of each of the two singular points it tends to the respective Euclidean metrics on $\mathbb{C}^2/\mathbb{Z}_k$ and $\mathbb{C}^2/\mathbb{Z}_l$. Therefore, with Page's idea in mind, we propose to use the metrics on $M_{k,l}$ for blowing up the singularities of the form $\mathbb{C}^2/\mathbb{Z}_p$ on the orbifolds with holonomy $SU(2)$ in several steps: successively replace each singularity with two singularities of less order gluing the space $M_{k,l}$ with the constructed metric; hopefully, we eventually "remove" all singular points.

As an application, we consider a representation of a $K3$-surface as the blow-up of the singularities of the orbifold $T^4/\mathbb{Z}_p$ for a prime $p \neq 2$. It turns out that the only possible case is $p = 3$ in which we have to blow up 9 singular points of the form $\mathbb{C}^2/\mathbb{Z}_3$. This is done in two steps: first, using $M_{1,2}$, we obtain 9 singular points of the form $\mathbb{C}^2/\mathbb{Z}_2$ and then remove them by means of $M_{1,1} = T^*S^2$. Each time we slightly deform the metric on the "glued" space and eventually obtain a $K3$-surface with a family of metrics with holonomy $SU(2)$. Simple calculations demonstrate that the dimension of the so-obtained family equals 58, as expected. However, the so-described metrics on a $K3$-surface differ essentially from the family constructed by Page: actually, we give an asymptotic description of the moduli space of the metrics with holonomy $SU(2)$ in a neighborhood of a flat metric on $T^4/\mathbb{Z}_3$, while Page gave an asymptotic description of the same space in a neighborhood of a flat metric on $T^4/\mathbb{Z}_2$. To justify our construction rigorously, we establish a connection between the constructed metrics and multi-instantons [7, 8]; our metrics are the limit case of a multi-instanton corresponding to two sources with different "masses."

In the next section we consider weighted complex projective spaces and describe the structure of $M_{k,l}$ and the metric on it. In § 3 and § 4 we discuss applications to the geometry of $K3$-surfaces.

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§ 2. A Special Kählerian Structure on $M_{k,l}$

Let $\mathbf{k} = (k_0, k_1, \dots, k_n)$ be a set of coprime positive integers. Consider the action of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ on $\mathbb{C}^{n+1}\setminus\{0\}$ with weights $k_0, \dots, k_n$:

$$\lambda \in \mathbb{C}^* : (z_0, z_1, \dots, z_n) \mapsto (z_0\lambda^{k_0}, z_1\lambda^{k_1}, \dots, z_n\lambda^{k_n}).$$

The orbit space of this action of $\mathbb{C}P^n(\mathbf{k})$ possesses the structure of a complex orbifold and is called the *weighted complex projective space*. We denote the orbit of a point $(z_0, \dots, z_n)$ by $[z_0 : \dots : z_n]$. The structure of singularities of the orbifold $\mathbb{C}P^n(\mathbf{k})$ can be rather complicated and depends in general on the mutual divisibility of different sets of the numbers $k_0, \dots, k_n$. In the case when each pair k_i, k_j is coprime (this is the case we are interested in), the situation becomes somewhat simpler and $\mathbb{C}P^n(\mathbf{k})$ possesses only a discrete collection of isolated singularities $[1 : 0 : \dots : 0], [0 : 1 : \dots : 0], \dots, [0 : 0 : \dots : 1]$ beyond which it is a complex analytic manifold. As a uniformizing atlas we have to consider the collection of charts:

$$\phi_i(z_1, \dots, z_n) = [z_1 : \dots : z_i : 1 : z_{i+1} : \dots : z_n], \quad i = 0, \dots, n.$$

Moreover, for each chart, the uniformizing group is the group $\Gamma_i = \mathbb{Z}_{k_i}$ generated by the element $\omega_i = e^{\frac{2\pi i}{k_i}}$ whose action is given as follows:

$$\omega_i(z_1, \dots, z_n) = \left(z_1\omega_i^{k_0}, \dots, z_i\omega_i^{k_{i-1}}, z_{i+1}\omega_i^{k_{i+1}}, \dots, z_n\omega_i^{k_n}\right).$$

We need a generalization of the above construction. Suppose that we have collections $\mathbf{k} = (k_0, \dots, k_n)$ and $\mathbf{l} = (l_0, \dots, l_m)$ of pairwise coprime integers such that $k_i > 0$ and $l_j < 0$. As above, we can formally consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+m+2}\backslash\{0\}$ with weights $k_0, \ldots, k_n, l_0, \ldots, l_m$. Moreover, as the orbit space $\mathbb{C}P^{n+m+1}(\mathbf{k}, \mathbf{l}) = (\mathbb{C}^{n+m+2}\backslash\{0\})/\mathbb{C}^*$ we obtain a topological space possessing a uniformizing atlas which is given by the collection of charts ϕ_i and ψ_j, $i = 0, \ldots, n$, $j = 0, \ldots, m$, defined as above. However, the so-obtained orbit space does not possess the Hausdorff property: the singular points corresponding to positive weights are not separated from those corresponding to negative weights. More exactly, consider the obvious embeddings of $\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \{0\}$ and $\mathbb{C}^{m+1} = \{0\} \times \mathbb{C}^{m+1}$ into $\mathbb{C}^{n+m+2} = \mathbb{C}^{n+1} \times \mathbb{C}^{m+1}$. Passing to the orbit space, we obtain two orbifolds $\mathbb{C}P^n(\mathbf{k})$ and $\mathbb{C}P^m(\mathbf{l})$ that are naturally embedded into $\mathbb{C}P^{n+m+1}(\mathbf{k}, \mathbf{l})$ and cover together all singular points. It is obvious that $\mathbb{C}P^n(\mathbf{k})$ cannot be separated from $\mathbb{C}P^m(\mathbf{l})$ in the quotient topology of the orbit space. Therefore, define the two weighted complex projective spaces:

$$\mathbb{C}P_+^{n+m+1}(\mathbf{k}, \mathbf{l}) = \mathbb{C}P^{n+m+1}(\mathbf{k}, \mathbf{l})\backslash\mathbb{C}P^m(\mathbf{l}),$$
$$\mathbb{C}P_-^{n+m+1}(\mathbf{k}, \mathbf{l}) = \mathbb{C}P^{n+m+1}(\mathbf{k}, \mathbf{l})\backslash\mathbb{C}P^n(\mathbf{k}).$$

It is clear that $\mathbb{C}P_+^{n+m+1}(\mathbf{k}, \mathbf{l})$ and $\mathbb{C}P_-^{n+m+1}(\mathbf{k}, \mathbf{l})$ are noncompact orbifolds with uniformizing atlases given by the respective collections of charts ϕ_i, $i = 0, \ldots, n$, and ψ_j, $j = 0, \ldots, m$. Note that there is an obvious isomorphism between the complex manifolds $\mathbb{C}P_+^{n+m+1}(\mathbf{k}, \mathbf{l})\backslash\mathbb{C}P^n(\mathbf{k})$ and $\mathbb{C}P_-^{n+m+1}(\mathbf{k}, \mathbf{l})\backslash\mathbb{C}P^m(\mathbf{l})$ induced by the identical transformation of the space $\mathbb{C}^{n+m+2}$.

We turn to description of some necessary spaces. Consider a pair k, l of coprime positive numbers. We obtain the weighted complex projective line $S^2(k, l) = \mathbb{C}P^1(k, l)$. Let the uniformizing atlas on $S^2(k, l)$ consist of the two charts ϕ_0 and ϕ_1 defining the coordinates $z \in \mathbb{C}$ and $w \in \mathbb{C}$:

$$\phi_0(z) = [1 : z], \quad \phi_1(w) = [w : 1].$$

The coordinates are connected by the relation $z^k w^l = 1$. Thus, $S^2(k, l)$ has two singular points $z = 0$ and $w = 0$ with the respective uniformizing groups $\mathbb{Z}_k$ and $\mathbb{Z}_l$. Beyond the singular points, $S^2(k, l)$ possesses the structure of a complex manifold and, topologically, is a two-dimensional sphere.

Now, consider the cotangent bundle $M_{k,l} = T^*S^2(k, l)$ and study its structure. In the tangent bundle $T(\mathbb{C}^2\backslash\{0\})$, we naturally distinguish the subbundle constituted by the vertical tangent vectors with respect to the action of $\mathbb{C}^*$ (the vertical vectors are those tangent to the orbits of the action). In the cotangent bundle $T^*(\mathbb{C}^2\backslash\{0\}) = \Lambda^{1,0}\mathbb{C}^2 \times (\mathbb{C}^2\backslash\{0\})$ consider the subbundle E constituted by the covectors vanishing on the vertical vectors. It is easy to see that

$$E = \{(z_0(lz_2dz_1 - kz_1dz_2), z_1, z_2) \mid z_0 \in \mathbb{C}, (z_1, z_2) \in \mathbb{C}^2\backslash\{0\}\}.$$

Then the action of $\mathbb{C}^*$ on $\mathbb{C}^2\backslash\{0\}$ induces the action of $\mathbb{C}^*$ on E which has the following structure:

$$\lambda \in \mathbb{C}^* : (z_0, z_1, z_2) \mapsto (z_0\lambda^{k+l}, z_1\lambda^{-k}, z_2\lambda^{-l}).$$

Since the quotient space of E by the action of $\mathbb{C}^*$ coincides obviously with $T^*S^2(k, l)$, we thereby find that $M_{k,l}$ can be identified with $\mathbb{C}P_-^2(k+l, -k, -l)$. In this event, the projective line $S^2(k, l)$ is embedded into $M_{k,l}$ as a complex submanifold with singularities $\{z_0 = 0\}$.

Thus, $M_{k,l}$ is a complex orbifold with two singular points and the uniformizing groups $\mathbb{Z}_k$ and $\mathbb{Z}_l$ at these points. In particular, if one of the numbers k and l equals 1 then there is only one singularity, and if $k = l = 1$ then we obtain the cotangent bundle over the standard two-dimensional sphere without singularities. Consider the two charts with local coordinates $(z, \alpha) \in \mathbb{C}^2$ and $(w, \beta) \in \mathbb{C}^2$ which determine an atlas on $M_{k,l}$:

$$\psi_1(\alpha, z) = [\alpha : 1 : z], \quad \psi_2(\beta, w) = [\beta : w : 1].$$

The coordinate systems are connected by the relations $z^k w^l = 1$ and $\alpha z = \beta w$. Moreover, the uniformizing groups $\mathbb{Z}_k$ and $\mathbb{Z}_l$ acting in each coordinate system are presented in the group $SU(2)$ acting standardly on $\mathbb{C}^2$:

$$\mathbb{Z}_k = \left\{ \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \mid \omega \in \mathbb{C},\ \omega^k = 1 \right\},$$
$$\mathbb{Z}_l = \left\{ \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \mid \omega \in \mathbb{C},\ \omega^l = 1 \right\}.$$

The projective line $S^2(k,l)$ is embedded into $M_{k,l}$ as a complex submanifold (with singularities) $\{\alpha=\beta=0\}$.

Below we need the analytic structure of $M_{k,l}$ at infinity. Therefore, consider the isomorphism of complex manifolds

$$\begin{gathered}\tau_{k,l}:(\mathbb{C}^2/\mathbb{Z}_{k+l})\backslash\{0\}=\mathbb{C}P^2_+(k+l,-k,-l)\backslash\{[1:0:0]\}\to M_{k,l}\backslash S^2(k,l),\\ \tau_{k,l}[z_0:z_1:z_2]=[z_0:z_1:z_2].\end{gathered}\tag{1}$$

It is easy to see that $\tau_{k,l}$ "glues in" the projective line $S^2(k,l)$ instead of the origin in $\mathbb{C}^2/\mathbb{Z}_{k+l}$. Moreover, different orbits of the action of $\mathbb{C}^*$ on $\mathbb{C}^2\backslash\{0\}$ "intersecting" at the point $0\in\mathbb{C}^2$ do not intersect any longer in $M_{k,l}$; that is, we have an analog of the blow-up operation.

In [1] the author constructed Ricci-flat metrics on $M_{k,l}$ by means of special coordinates ρ,θ,ϕ,ψ. The orbifold $M_{k,l}$ appeared as a cylinder of the bundle of the lens space $L(-1,k+l)$ over $S^2(k,l)$. Moreover, (θ,ϕ,ψ) were the coordinates in $L(-1,k+l)$, while ρ was the coordinate along the element of the cylinder. The metrics looked as follows:

$$\begin{aligned}ds^2=(\cosh\rho-a\cos\theta)(d\rho^2+d\theta^2)+\frac{\sinh^2\rho}{\cosh\rho-a\cos\theta}(d\psi+\cos\theta d\phi)^2&\\ +\frac{\sin^2\theta}{\cosh\rho-a\cos\theta}(ad\psi+\cosh\rho d\phi)^2,&\end{aligned}\tag{2}$$

where $a=\frac{l-k}{l+k}$. For $k=l=1$ (i.e., for $a=0$) the metric (2) coincides with the Eguchi–Hanson metric of cohomogeneity 1. For $a\neq 0$ we obtain a new metric of cohomogeneity 2.

It is easy to note that the space $L(-1,k+l)$ is a spherical subbundle in $T^*S^2(k,l)=M_{k,l}$ over $S^2(k,l)$. This enables us to establish a connection between the coordinates (θ,ρ,ψ,ϕ) and (z,α) or (w,β). Namely, we consider the following change of coordinates:

$$z=\frac{\sin\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}\right)^{l/k}}e^{-il(a\psi+\phi)},\quad \alpha=kl\sinh\frac{\rho}{2}\left(\cos\frac{\theta}{2}\right)^{1+\frac{l}{k}}e^{il(\psi+\phi)},$$

$$w=\frac{\cos\frac{\theta}{2}}{\left(\sin\frac{\theta}{2}\right)^{k/l}}e^{ik(a\psi+\phi)},\quad \beta=kl\sinh\frac{\rho}{2}\left(\sin\frac{\theta}{2}\right)^{1+\frac{k}{l}}e^{ik(\psi-\phi)}.$$

The metric ds^2 becomes smooth upon this change in each of two charts (the exact expressions for the metric in the variables (z,α) and (w,β) are rather bulky and not given here).

Recall that a Riemannian metric on a two-dimensional complex manifold (orbifold) is called a *special Kähler metric* if its holonomy group lies in the group $SU(2)$ presented standardly in the tangent space. One of the main results of the article is the following assertion:

Theorem 1. *The space $M_{k,l}$ with metric* (2) *is a special Kähler orbifold.*

PROOF. Here is a Kählerian form ω which agrees with metric (2):

$$\omega=\sinh\rho\,d\rho\wedge d\psi-a\sin\theta\,d\theta\wedge d\psi+\sinh\rho\cos\theta\,d\rho\wedge d\phi-\cosh\rho\sin\theta\,d\theta\wedge d\phi.$$

Straightforward calculations show that this form is smooth outside the singular points and is closed (and consequently parallel). Hence, we can immediately conclude that $M_{k,l}$ is a Kähler manifold and the holonomy group lies in $U(2)$. One well-known result of differential geometry (for example, see [9]) states that a smooth Ricci-flat Kähler metric on a simply connected manifold is a special Kähler metric; i.e., its holonomy group lies in $SU(2)$. This result is of local character; therefore, applying it to our case, we can conclude that parallel translation along small loops away from the pair of singular points lies in $SU(2)$. Among the other things, the uniformizing group of the orbifold has a contribution into translation "around" a singular point. However, the elements of the uniformizing groups as well lie in $SU(2)$; therefore, the holonomy group $M_{k,l}$ coincides with $SU(2)$.

The theorem is proven.

§ 3. Applications to the Geometry of $K3$-Surfaces

Kummer's construction. Recall the construction of a $K3$-surface by means of Kummer's construction. Consider a complex two-dimensional torus $T^4 = \mathbb{C}^2/\Lambda$, where $\Lambda = \{(a+ib, c+id) \mid a,b,c,d \in \mathbb{Z}\}$ is a lattice in $\mathbb{C}^2$. Define the involution $\sigma : T^4 \to T^4$ as follows:

$$\sigma : (z_1, z_2) + \Lambda \mapsto (-z_1, -z_2) + \Lambda.$$

The involution σ has 16 fixed points; namely, the points (z_1, z_2) for $z_i \in \left\{0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2} + \frac{1}{2}i\right\}$. If we consider a flat metric on T^4 then σ becomes an isometry and $X = T^4/\sigma$ represents a special Kähler orbifold with 16 singular points. In this event, a neighborhood of each point looks like $\mathbb{C}^2/\{\pm 1\}$. Now, consider the complex surface Y obtained by blowing up each of the 16 singular points; this is a $K3$-surface.

The blow-up construction can be carried out as follows: Let $u = (u_1, u_2)$ be the Euclidean coordinates in $\mathbb{C}^2$; consider a spherical neighborhood $\Delta = \{|u|^2 \le \varepsilon\}/\mathbb{Z}_2 \subset \mathbb{C}^2/\mathbb{Z}_2$ in which the blow-up will happen. Now, consider $M_{1,1} = T^*S^2$ and the mapping $\tau_{1,1} : \mathbb{C}^2 \backslash \{0\} \to M_{1,1} \backslash S^2$ constructed in (1). The mapping $\tau_{1,1}$ induces a complex isomorphism between the open submanifold $\Delta \backslash \{0\}$ and an open submanifold in $M_{1,1} \backslash S^2$. Deleting the point $0 \in \Delta$ and carrying out the identification by means of the indicated isomorphism, we obtain a complex manifold without a singular point.

Page's construction. It is well known [4] that Y possesses a 58-dimensional family $\mathscr{S}$ of metrics with holonomy $SU(2)$. In [3] Page proposed a geometric description of the moduli space of the metrics with holonomy $SU(2)$. We briefly describe his approach. Considering all flat metrics on T^4, we obtain the family $\mathscr{S}_2$ of metrics of holonomy $SU(2)$ on the orbifold X. Neighborhoods of 16 singular points of X are isometric to neighborhoods of the origin in $\mathbb{C}/\{\pm 1\}$. For each singular point we cut its neighborhood $(0, \varepsilon_1) \times S^3/\{\pm 1\}$. Then the "collar" of the boundary $(\varepsilon_1, \varepsilon_2) \times S^3/\{\pm 1\}$ is "almost" isometric to the open spherical layer $M_{1,1}$ with a metric homothetic to the Eguchi–Hanson metric with the homothety coefficient t. Decreasing t, we can make the metrics arbitrarily close on the collar; moreover, both metrics have holonomy $SU(2)$. Now, using the analytic tools, we can show that if a neighborhood is sufficiently small then deformation of both metrics gives a smooth metric on Y with holonomy $SU(2)$. A rigorous justification of this construction was given later in [10, 11, 5].

This approach to the moduli space of a $K3$-surface gives a geometrically clear explanation of the dimension 58. Indeed, the space $\mathscr{S}_2$ is ten-dimensional. Now, in a neighborhood of each singular point, the metrics in $\mathscr{S}_2$ possess the group of isometries $SO(4)$ on $(\varepsilon_1, \varepsilon_2) \times S^3/\{\pm 1\}$. The subgroup $U(2) \subset SO(4)$ leaves the Eguchi–Hanson metric unchanged. This yields a family of metrics of dimension $\dim(SO(4)/U(2)) = 2$. If we use the parameter t, we obtain a three-dimensional family of different metrics in a neighborhood of each singular point. Thus, taking all singular points into account, we conclude that the dimension of the whole family equals $10 + 16 \cdot 3 = 58$. Actually, Page proposed a geometric description of the moduli space $\mathscr{S}$ of the metrics of holonomy $SU(2)$ on Y in a neighborhood of the limit family $\mathscr{S}_2$.

Resolution of singularities of type $\mathbb{C}^2/\mathbb{Z}_{k+l}$ by means of $M_{k,l}$. We propose to consider $M_{k,l}$ as a space which enables us to resolve singularities of higher order by reducing them to singularities of less order. More exactly, consider a singular point of some complex manifold such that the manifold in a neighborhood of this point looks like $\mathbb{C}^2/\mathbb{Z}_{k+l}$. Let $u = (u_1, u_2)$ be the Euclidean coordinates in $\mathbb{C}^2$ and let $\Delta = \{|u|^2 \le \varepsilon\}/\mathbb{Z}_{k+l}$ be a neighborhood of the manifold in which the blow-up happens. Now, consider $M_{k,l}$ and the mapping $\tau_{k,l} : (\mathbb{C}^2/\mathbb{Z}_{k+l}) \backslash \{0\} \to M_{k,l} \backslash S^2(k,l)$ defined in (1). This mapping induces a complex isomorphism between the manifold $\Delta \backslash \{0\}$ and an open submanifold in $M_{k,l} \backslash S^2(k,l)$. Removing the singular point $0 \in \Delta$ and carrying out the identification by means of the indicated isomorphism, we obtain a complex manifold which has two singular points of types $\mathbb{C}^2/\mathbb{Z}_k$ and $\mathbb{C}^2/\mathbb{Z}_l$ instead of one singular point of the form $\mathbb{C}^2/\mathbb{Z}_{k+l}$. Repeating this procedure, we can now resolve all resulting singular points.

Pursuing the goal to generalize Page's construction, consider the following question: What groups $\mathbb{Z}_p \subset SU(2)$ for $p > 2$ can act on $T^4 = \mathbb{C}^2/\Lambda$ by isometries? It is clear that after an appropriate choice of a unitary basis we can assume that

$$\mathbb{Z}_p = \left\{ \begin{pmatrix} \omega^q & 0 \\ 0 & \overline{\omega}^q \end{pmatrix} \mid q \in \mathbb{Z} \right\},$$

where $\omega = e^{\frac{2\pi}{p} i}$ is the primitive root of degree p of unity. Consider a nonzero element $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ of the lattice. Let $\Pi = \{(\lambda_1 z, \lambda_2 \bar{z}) \mid z \in \mathbb{C}\}$ be a real two-dimensional plane invariant under $\mathbb{Z}_p$. Since Λ is invariant under the action of $\mathbb{Z}_p$, $\Pi \cap \Lambda$ is a lattice containing the vectors λ, $\lambda\omega$, and $\lambda\omega^2$. Consequently, there is a polynomial of the second degree with integer coefficients whose root is the primitive root ω. Hence, this is a polynomial of division of the disk which leaves only three possibilities: $p = 3, 4, 6$. However, the cases $p = 4, 6$ mean the presence of singular points in $T^4/\mathbb{Z}_p$ whose neighborhoods are not modeled with the cone $\mathbb{C}^2/\mathbb{Z}_p$, and we cannot resolve them using our construction. We are left with the only case $p = 3$ which will be considered in detail.

Suppose that the flat metric on $\mathbb{C}^2$ is determined by the real part of the standard Hermitian product and $\Lambda_0 = \{ae_1 + be_2 \mid a, b \in \mathbb{Z}\}$, where $e_1 = 1$ and $e_2 = e^{\frac{\pi}{3} i}$. The arguments above demonstrate that the general lattice Λ invariant under the action of $\mathbb{Z}_3$ has the form

$$\Lambda = \{(\lambda_1 z_1 + \lambda_2 z_2, \mu_1 z_1 + \mu_2 z_2) \mid z_1, z_2 \in \Lambda_0\} \cong \Lambda_0 \oplus \Lambda_0,$$

where λ_1, λ_2, μ_1, and μ_2 are complex parameters which Λ depends on, and $\lambda_1\mu_2 - \lambda_2\mu_1 \neq 0$. The action of the group $\mathbb{Z}_3$ on $T^4 = \mathbb{C}^2/\Lambda$ is generated by the transformation

$$\gamma : (z_1, z_2) + \Lambda \mapsto (z_1 e^{\frac{2\pi}{3} i}, z_2 e^{-\frac{2\pi}{3} i}) + \Lambda.$$

The quotient space $X = T^4/\mathbb{Z}_3$ is a Kähler orbifold with nine singular points which correspond to the fixed (with respect to γ) points in T^4. These points have the form $(\lambda_1 z_1 + \lambda_2 z_2, \mu_1 z_1 + \mu_2 z_2)$, where $z_i \in \left\{0, \frac{1}{3}e_1 + \frac{1}{3}e_2, \frac{2}{3}e_1 + \frac{2}{3}e_2\right\}$. We denote by $\{s_1, \dots, s_9\}$ the singular points in X. Let $\mathscr{S}_3$ be the moduli space of the flat metrics on X with holonomy $SU(2)$ corresponding to all possible values of the parameters λ_i and μ_i. The action of the group $U(2) \subset GL_{\mathbb{C}}(2)$ leaves the metric on $\mathbb{C}^2$ unchanged and consequently induces the trivial action on $\mathscr{S}_3$. Therefore, $\dim(\mathscr{S}_3) = \dim(GL_{\mathbb{C}}(2)/U(2)) = 4$.

Let X' be the complex surface obtained by successive resolution of X at the singular points by means of $M_{1,2}$ and $M_{1,1}$. Namely, suppose that $B_i \subset T^4/\mathbb{Z}_3$, $i = 1, \dots, 9$, are open balls of radius ε centered at the singular points of X and $B_i' \subset B_i$ are the closed balls of radius $\varepsilon' < \varepsilon$ centered at the same points. Thus, we obtain a system of concentric neighborhoods of the singular points in X: $s_i \in B_i' \subset B_i$. Choose a sufficiently small ε such that $B_i \cap B_j = \varnothing$ for $i \neq j$. It is clear that $B_i \backslash B_i'$ is diffeomorphic to $(\varepsilon', \varepsilon) \times S^3/\mathbb{Z}_3$. Consider the space $M_{1,2}$ with the metric $t^2 ds^2$, where ds^2 is the metric (2) for $a = 1/3$. In the space $M_{1,2}$ consider the collar $\tau_{1,2}((\varepsilon', \varepsilon) \times S^3/\mathbb{Z}_3)$, where the mapping $\tau_{k,l}$ is defined in (1). Then the mapping $\tau_{1,2}$ defined on the collar tends to an isometry as $t \to 0$. Consider the orbifold Y obtained by identification of $X \backslash (\bigcup_{i=1}^9 B_i')$ with nine copies of $\tau_{1,2}((0, \varepsilon) \times S^3/\mathbb{Z}_3)$ by means of the mapping $\tau_{1,2}$ bounded on the collar $(\varepsilon', \varepsilon) \times S^3/\mathbb{Z}_3$. On the glued domains the metric $t^2 ds^2$ is arbitrarily close to a locally flat metric on X as $t \to 0$. The orbifold Y has nine singularities $s_1', \dots, s_9'$ and looks locally like $\mathbb{C}^2/\mathbb{Z}_2$ in their neighborhoods. Similarly, we consider a sufficiently small $\delta' < \delta < \varepsilon'$ and a system of concentric neighborhoods $s_i' \in C_i' \subset C_i$ in Y of radii δ' and δ. Using the mapping $\tau_{1,1}$, we identify $Y \backslash (\bigcup_{i=1}^9 C_i')$ with nine copies of $\tau_{1,1}((0, \delta) \times S^3/\mathbb{Z}_2) \subset M_{1,1}$ over the collar $(\delta', \delta) \times S^3/\mathbb{Z}_2$ and obtain a complex surface X' without singularities. Moreover, in the glued domains the metric on Y is arbitrarily close to the metric $u^2 ds'^2$ as $u, \delta \to 0$, where ds'^2 is the metric (2) on $M_{1,1}$ for $a = 0$. Let $d\tilde{s}^2$ be the metric on Y obtained by smoothing the metrics on X, $M_{1,2}$, and $M_{1,1}$ in the domains of the described identification.

Recall that $\mathscr{S}$ is the moduli space of the metrics with the holonomy group $SU(2)$ on a $K3$-surface, $\mathscr{S}_3$ is the moduli space of the flat metrics on X with the holonomy group $SU(2)$.

Theorem 2. *The surface X' is a $K3$-surface and consequently $\mathscr{S}_3$ is a limit space for $\mathscr{S}$. A sufficiently small neighborhood of $\mathscr{S}_3$ in $\mathscr{S}$ consists of the* 58*-dimensional family of metrics obtained by small deformation of the family of metrics $d\tilde{s}^2$ constructed as described above as $\delta, t, u \to 0$.*

We give a proof of this theorem in the next section.

§ 4. Connection with Multi-Instantons and the Proof of Theorem 2

The easiest way to justify the above construction rigorously is to use the connection of the constructed metrics with multi-instantons. Namely, multi-instantons are metrics with the holonomy group $SU(2)$ of the following form [7]:

$$ds^2 = \frac{1}{U}(d\tau + \boldsymbol{\omega} \cdot d\mathbf{x})^2 + U d\mathbf{x} \cdot d\mathbf{x}, \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^3$, the variable τ is periodic,

$$U = \sum_{i=1}^{s} \frac{1}{|\mathbf{x} - \mathbf{x}_i|}, \quad \operatorname{rot} \boldsymbol{\omega} = \operatorname{grad} U,$$

and $\mathbf{x}_i$ is a set of s different points in the Euclidean space $\mathbb{R}^3$. A multi-instanton (3) is a smooth Riemannian metric on some four-dimensional manifold.

Let $d\sigma$ be the area form of the level surface of U in $\mathbb{R}^3$. It is easy to verify that the form

$$\omega_1' = dU \wedge (d\tau + \boldsymbol{\omega} \cdot d\mathbf{x}) + |\operatorname{grad} U| U d\sigma \tag{4}$$

is a closed Kählerian form which agrees with metric (3).

Consider the limit case when there are only two points $\mathbf{x}_1 = (-1, 0, 0)$ and $\mathbf{x}_2 = (1, 0, 0)$; moreover, the first has multiplicity l and the second has multiplicity k. Take the coordinates (r, θ, ϕ, ψ) as follows:

$$x_1 = \cosh\rho \cos\theta, \quad x_2 = \sinh\rho \sin\theta \cos\psi, \quad x_3 = \sinh\rho \sin\theta \sin\psi, \quad \tau = (l + k)\phi.$$

Straightforward calculations show that with these coordinates (3) coincides with (2) to within multiplication by a constant; i.e., the constructed metric on $T^*\mathbb{C}P^1(k, l)$ is a limit case of a multi-instanton.

Now, we can prove Theorem 2. Our proof is similar to the arguments of [5]; therefore, as far as it is possible we will try to keep the corresponding notations. Let T^4 be the flat torus with the above-defined action of the group $\mathbb{Z}_3$. Consider the torus $T^7 = T^3 \times T^4$ with a flat metric and extend the action of $\mathbb{Z}_3$ to T^7 by making it trivial on T^3. Then the set S of singular points in $T^7/\mathbb{Z}_3$ is the disjoint union of nine tori T^3; moreover, a neighborhood T of the set S is isometric to the disjoint union of nine copies of $T^3 \times B^4_\zeta/\mathbb{Z}_3$, where B^4_ζ are open balls of radius ζ in $\mathbb{R}^4 = \mathbb{C}^2$ for an appropriate constant $\zeta > 0$.

Now, choose an arbitrary $\varepsilon > 0$ and consider a multi-instanton $ds^2(t)$ on the four-dimensional manifold M_t given by the following three points: $\mathbf{x}_1 = (-4t^2/3, 0, 0)$, $\mathbf{x}_2 = (2t^2/3, t^2\varepsilon, 0)$, and $\mathbf{x}_3 = (2t^2/3, -t^2\varepsilon, 0)$. Denote by U_t the corresponding potential. Let $\overline{U}(\mathbf{x}) = 3/|\mathbf{x}|$ be the potential of the center of gravity with multiplicity 3. It is easy to verify that

$$|\nabla^i(U_t(\mathbf{x}) - \overline{U}(\mathbf{x}))| = O(t^4) \tag{5}$$

for $|\mathbf{x}| > \zeta^2/16$ and all $i \geq 0$ as $t \to 0$, where ∇^i is the set of partial derivatives of order i. The metric $d\bar{s}^2$ constructed for the potential $\overline{U}$ is isometric to the flat metric on $\mathbb{C}^2/\mathbb{Z}_3$; moreover, each domain $\{|\mathbf{x}| \leq r^2\}$ is isometric to the ball B^4_r. It follows from (5) that

$$|\nabla^i(ds^2(t) - d\bar{s}^2)| = O(t^4) \tag{6}$$

for $|\mathbf{x}| > \zeta^2/16$ as $t \to 0$.

Now, cut each of the nine neighborhoods $B^4_\zeta/\mathbb{Z}_3$ of the singular points in $X = T^4/\mathbb{Z}_3$ and, instead of each of them, glue the domain in M_t defined by the condition $|\mathbf{x}| \leq \zeta^2$. We obtain a smooth four-dimensional manifold X'. In X' we define three domains A, B, and C as follows: the domain A is the union of the neighborhoods given by the condition $|\mathbf{x}| \leq \zeta^2/9$ in each of the nine glued copies of M_t; the domain C is the complement in X' of the glued neighborhoods and is isometric to $X \backslash (\bigcup_{i=1}^{9} B^4_\zeta)$; and finally $B = X' \backslash (A \cup C)$.

Consider the Kählerian forms $\omega_1'(t)$ and $\overline{\omega}_1$ of the metrics $ds^2(t)$ and $d\bar{s}^2$ described in (4). Since the metric $ds^2(t)$ has the holonomy group $SU(2)$, we have a parallel complex volume form $\mu(t)$ on the Kähler manifold $(M_t, \omega_1'(t))$. Put

$$\omega_2'(t) = \mathrm{Re}(\mu(t)), \quad \omega_3'(t) = \mathrm{Im}(\mu(t)).$$

Then the triple of the parallel Kählerian forms $\omega_1'(t)$, $\omega_2'(t)$, and $\omega_3'(t)$ determines the hypercomplex structure on M_t with the metric $ds^2(t)$. Similarly, we construct a triple of parallel constant forms $\overline{\omega}_1$, $\overline{\omega}_2$, and $\overline{\omega}_3$ which determine the flat hypercomplex structure on $\mathbb{C}^2/\mathbb{Z}_3$ with the metric $d\bar{s}^2$.

It follows from (4) and (5) that

$$|\nabla^k(\omega_i'(t) - \overline{\omega}_i)| = O(t^4), \quad i = 1, 2, 3,$$

for $|\mathbf{x}| > \zeta^2/16$ as $t \to 0$.

Consider the union B of the spherical layers. There exist 1-forms $\eta_i'(t)$ and $\bar{\eta}_i$ such that $\omega_i'(t) = d\eta_i'(t)$ and $\overline{\omega}_i = d\bar{\eta}_i$; moreover,

$$|\nabla^k(\eta_i'(t) - \bar{\eta}_i)| = O(t^4) \quad \text{as } t \to 0.$$

Consider a real smooth increasing function $u(r)$ defined on the interval $[0, \zeta]$ and possessing the following properties:

$$0 \le u(r) \le 1; \quad u(r) = 0 \text{ for } 0 \le r \le \zeta/3; \quad u(r) = 1 \text{ for } \zeta/2 \le r \le \zeta.$$

Put $\eta_i(t) = u\bar{\eta}_i + (1 - u)\eta_i'(t)$ and $\omega_i(t) = d\eta_i(t)$, $i = 1, 2, 3$. We have thus constructed a triple of closed 2-forms on X' which coincide with the forms $\overline{\omega}_i$ in the domain C and with the forms $\omega_i'(t)$ in the domain A. Moreover, it is easy to see that

$$|\nabla^k(\omega_i(t) - \overline{\omega}_i)| = O(t^4) \tag{7}$$

in the domain $B \cup C$ as $t \to 0$.

Recall the definition of the G_2-structure. Define a 3-form ϕ_0 on the space $\mathbb{R}^7$ with the standard Euclidean metric and orientation as follows:

$$\phi_0 = y_1 \wedge y_2 \wedge y_7 + y_1 \wedge y_3 \wedge y_6 + y_1 \wedge y_4 \wedge y_5 + y_2 \wedge y_3 \wedge y_5$$
$$-y_2 \wedge y_4 \wedge y_6 + y_3 \wedge y_4 \wedge y_7 + y_5 \wedge y_6 \wedge y_7,$$

where $y_1, \dots, y_7$ is the standard orthonormal positively oriented basis for $(\mathbb{R}^7)^*$. Moreover, the dual 4-form with respect to the Hodge operator looks like:

$$*\phi_0 = y_1 \wedge y_2 \wedge y_3 \wedge y_4 + y_1 \wedge y_2 \wedge y_5 \wedge y_6 - y_1 \wedge y_3 \wedge y_5 \wedge y_7 + y_1 \wedge y_4 \wedge y_6 \wedge y_7$$
$$+y_2 \wedge y_3 \wedge y_6 \wedge y_7 + y_2 \wedge y_4 \wedge y_5 \wedge y_7 + y_3 \wedge y_4 \wedge y_5 \wedge y_6.$$

The subgroup in $GL_+(\mathbb{R}^7)$, preserving the form ϕ_0 or $*\phi_0$, coincides with the group G_2.

Now, if M is a seven-dimensional oriented manifold then let $\Lambda^3_+ M$ and $\Lambda^4_+ M$ be the subbundles in $\Lambda^3 T^*M$ and $\Lambda^4 T^*M$ constituted by the forms that have the above-mentioned form ϕ_0 and $*\phi_0$ at each point $p \in M$ in an appropriate oriented basis $T^*_p M$. It is easily seen that $\Lambda^3_+ M$ and $\Lambda^4_+ M$ are open subbundles in $\Lambda^3 T^*M$ and $\Lambda^4 T^*M$, and we obtain the natural identification mapping $\Theta : \Lambda^3_+ M \to \Lambda^4_+ M$ which takes each form looking locally like ϕ_0 into a form looking locally like $*\phi_0$. We say that a section ϕ of the bundle $\Lambda^3_+ M$ *determines a G_2-structure on M*. This form ϕ determines uniquely the Riemannian metric with respect to which the identification operator Θ becomes the Hodge operator $*$. Moreover, if the forms ϕ and $*\phi$ are closed then the G_2-structure is torsion-free and the holonomy group of the Riemannian manifold M is contained in $G_2 \subset SO(7)$.

Defined a flat G_2-structure $\bar{\phi}$ on $T^3 \times X = T^7/\mathbb{Z}_3$ and its dual $*\bar{\phi}$ as follows:

$$\bar{\phi} = \overline{\omega}_1 \wedge \delta_1 + \overline{\omega}_2 \wedge \delta_2 + \overline{\omega}_3 \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3,$$

$$*\bar{\phi} = \overline{\omega}_1 \wedge \delta_2 \wedge \delta_3 + \overline{\omega}_2 \wedge \delta_3 \wedge \delta_1 + \overline{\omega}_3 \wedge \delta_1 \wedge \delta_2 + \frac{1}{2}\overline{\omega}_1 \wedge \overline{\omega}_1,$$

where δ_1, δ_2, and δ_3 are constant orthonormal 1-forms on T^3 extended to the whole $T^7/\mathbb{Z}_3 = T^3 \times X$.

Define the following 3- and 4-forms on $M = T^3 \times X'$:

$$\phi_t = \omega_1(t) \wedge \delta_1 + \omega_2(t) \wedge \delta_2 + \omega_3(t) \wedge \delta_3 + \delta_1 \wedge \delta_2 \wedge \delta_3,$$

$$v_t = \omega_1(t) \wedge \delta_2 \wedge \delta_3 + \omega_2(t) \wedge \delta_3 \wedge \delta_1 + \omega_3(t) \wedge \delta_1 \wedge \delta_2 + \frac{1}{2}\omega_1(t) \wedge \omega_1(t).$$

It is clear that these forms coincide with $\bar{\phi}$ and $*\bar{\phi}$ respectively in the domain $T^3 \times C$. Since all forms $\omega_i(t)$ and δ_i are closed, the forms ϕ_t and v_t are closed on M too.

In the domains $T^3 \times A$ and $T^3 \times C$, the triple of the forms $\omega_i(t)$ is a triple of Kählerian forms determining a hypercomplex structure; therefore, the form ϕ_t determines a torsion-free G_2-structure and $v_t = \Theta(\phi_t)$. In the domain $T^3 \times B$, the triple $\omega_i(t)$ does not determine a hypercomplex structure in general, thereby we cannot even guarantee a priori that ϕ_t determines a G_2-structure. However, $\Lambda^3_+(M)$ is open in $\Lambda^3 T^*(M)$; therefore, it follows from (7) that for a sufficiently small t we have $\phi_t \in C^\infty(\Lambda^3_+(M))$. Moreover, the form v_t differs from $\Theta(\phi_t)$ in general. Define the 3-form ψ_t on M by the relation $*\psi_t = \Theta(\phi_t) - v_t$, where the Hodge operator is defined with respect to the Riemannian metric g given by the G_2-structure ϕ_t. It is obvious that $d^*\psi_t = d^*\phi_t$.

The following theorems are proven in [5]:

Theorem A. *Let $E_1, \dots, E_5$ be positive constants. Then there are positive constants κ and K depending on $E_1, \dots, E_5$ and such that the following property holds for every $0 < t < \kappa$.*

Let M be a compact seven-dimensional manifold and let ϕ be a smooth closed form in $C^\infty\big(\Lambda^3_+ M\big)$. Suppose that ψ is a smooth 3-form on M such that $d^*\psi = d^*\phi$ and the following are fulfilled:

(i) *$\|\psi\|_2 \le E_1 t^4$ and $\|\psi\|_{C^{1,1/2}} \le E_1 t^4$;*

(ii) *if $\chi \in C^{1,1/2}(\Lambda^3 T^* M)$ and $d\chi = 0$ then*

$$\|\chi\|_{C^0} \le E_2(t\|\nabla\chi\|_{C^0} + t^{-7/2}\|\chi\|_2),$$

$$\|\nabla\chi\|_{C^0} + t^{1/2}[\nabla\chi]_{1/2} \le E_3(\|d^*\chi\|_{C^0} + t^{1/2}[d^*\chi]_{1/2} + t^{-9/2}\|\chi\|_2);$$

(iii) *$1 \le E_4 \operatorname{vol}(M)$;*

(iv) *if f is a smooth real function and $\int_M f\,d\mu = 0$ then $\|f\|_2 \le E_5\|df\|_2$.*

Then there is $\eta \in C^\infty(\Lambda^2 T^* M)$ such that $\|d\eta\|_{C^0} \le K t^{1/2}$ and $\tilde{\phi} = \phi + d\eta$ is a smooth torsion-free G_2-structure on M.

Theorem B. *Let $D_1, \dots, D_5$ be positive constants. Then there are positive constants $E_1, \dots, E_5$ and λ depending only on $D_1, \dots, D_5$ such that the following property holds for each $t \in (0, \lambda]$.*

Let M be a compact seven-dimensional manifold and let ϕ be a closed form in $C^\infty\big(\Lambda^3_+ M\big)$. Let g be the metric associated with ϕ. Suppose that ψ is a smooth 3-form on M such that $d^*\phi = d^*\psi$ and the following are fulfilled:

(i) *$\|\psi\|_2 \le D_1 t^4$ and $\|\psi\|_{C^2} \le D_1 t^4$;*

(ii) *the injectivity radius $\delta(g)$ satisfies the inequality $\delta(g) \ge D_2 t$;*

(iii) *the Riemannian tensor $R(g)$ of the metric g satisfies the inequality $\|R(g)\|_{C^0} \le D_3 t^{-2}$;*

(iv) *the volume $\operatorname{vol}(M)$ satisfies the inequality $\operatorname{vol}(M) \ge D_4$;*

(v) *the diameter $\operatorname{diam}(M)$ satisfies the inequality $\operatorname{diam}(M) \le D_5$.*

Then the conditions (i)–(iv) *of Theorem* A *are satisfied for (M, ϕ).*

We want to apply these theorems to the form ϕ_t on M with the associated metric g. Indeed, (7) implies validity of condition (i), while condition (iv) follows trivially from the construction. Now, note that $ds^2(t) = t^2\,ds^2(1)$. Hence, we find that the injectivity radius grows linearly with the increase of t which proves (ii) and (iii). The same arguments imply boundedness of the diameter, i.e., (v). Hence, there is a torsion-free G_2-structure $\tilde{\phi}$ close to ϕ_t. Let g' be the associated metric with the holonomy group G_2 on $M = T^3 \times X'$.

Since X is simply connected, X' is simply connected too. Therefore, $\pi_1(T^3 \times X') = \mathbb{Z}^3$ and from [5] we can conclude that the holonomy group (M, g') is equal to $SU(2) \subset G_2$. Hence, the metric on M is the direct product of a flat metric on T^3 and the metric ds'^2 with the holonomy group $SU(2)$ on X'. In particular, X' is a $K3$-surface.

Now, we find out how the metric ds'^2 looks like near the remote singular points, i.e., in the domain A. The metric g on $T^3 \times X'$ close to g' in the domain A is the direct product of the flat metric on T^3 and the multi-instanton $ds^2 = ds^2(t)$. The multi-instanton ds^2 tends to a flat metric on $\mathbb{C}^2/\mathbb{Z}_3$ as $t \to 0$ and to the metric (2) on $M_{1,2}$ as $\varepsilon \to 0$. Hence, as $\varepsilon \to 0$, the metric ds^2 is obtained from X by the resolution of the singular points by means of $M_{1,2}$ described in Theorem 2. Now, if we consider a small neighborhood of the points $\mathbf{x}_2$ and $\mathbf{x}_3$ and choose ε so small that the contribution of $\mathbf{x}_1$ in the potential U_t is small as compared with the contribution of $\mathbf{x}_2$ and $\mathbf{x}_3$ then the metric ds^2 in this neighborhood is close to the metric (2) on $M_{1,1}$. Thus, the metric ds^2 is obtained by the double resolution of singular points in X indicated in Theorem 2, while the metric ds'^2 is its small deformation.

Estimate the dimension of the family of metrics constructed above. In the process of resolution of the singularities s_i the freedom of gluing of $M_{1,2}$ is determined by the group $U(2)$ which does not change the complex structure on T^4. However, the metric on $M_{1,2}$ has the group of isometries $U(1) \times U(1) \subset U(2)$; therefore, if we use the presence of the parameter t responsible for homothety then we obtain a family of different metrics with holonomy $SU(2)$ of dimension 3 in a neighborhood of each point s_i. In the processes of resolution of the singularities s'_i, as in Page's method, we also obtain a family of dimension 3. The dimension of $\mathscr{S}_3$ is equal to 4; therefore, summing up the dimensions, we conclude that the dimension of $\mathscr{S}$ in a neighborhood of $\mathscr{S}_3$ equals 58; this is exactly the dimension of the moduli space of the metrics with holonomy $SU(2)$ on a $K3$-surface.

The proof of the theorem is complete.

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