MASARYK UNIVERSITY
FACULTY OF SCIENCE

Wavelets on the Interval and Their Applications

Habilitation Thesis

Dana Černá

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Abstract

This habilitation thesis is concerned with constructions of new spline-wavelet bases on the interval and product domains, their adaptations to boundary conditions, and their applications. This thesis is a collection of eight previously published papers [10, 11, 13, 14, 15, 17, 19, 23]. First, we introduce the concept of a wavelet basis on a bounded interval and on tensor product domains. Then, we review the wavelet-Galerkin method and adaptive wavelet methods for the numerical solution of operator equations. Finally, we discuss constructions of quadratic and cubic spline wavelets and we comment on the collected papers. Papers [10, 11, 15] are focused on the construction of well-conditioned biorthogonal spline-wavelet bases on the interval where both primal and dual wavelets have compact support. In [13, 14, 19, 23], a local support of dual wavelets is not required which enables the construction of wavelets that have smaller supports and significantly smaller condition numbers than wavelets of the same type but with local duals. Another advantage is the simplicity of the construction. In [17], we constructed wavelets where the corresponding matrices, arising from discretization of second-order differential equations with coefficients that are piecewise polynomials of degree at most four on uniform grids, are sparse and not only quasi-sparse as for most wavelet bases. We used the constructed bases for solving various types of operator equations, e.g. Poisson's equation, the Helmholtz equation, a fourth-order boundary value problem, and the Black-Scholes equation with two state variables. We also applied the constructed bases for option pricing under Kou's double exponential jump-diffusion option pricing model which is represented by a partial integro-differential equation.

Contents

List of Symbols

$\mathbb{N}$	the set of all positive integers
$\mathbb{Z}$	the set of all integers
$\mathbb{R}$	the set of all real numbers
I	bounded interval $I \subset \mathbb{R}$
Ω	bounded domain
$\square$	hyperrectangle
$\lvert\cdot\rvert$	absolute value of a number or a level for an index λ
$\lfloor\cdot\rfloor$	floor function
$\delta_{i,j}$	Kronecker delta, $\delta_{i,i} = 1$, $\delta_{i,j} = 0$ for $i \neq j$
span	linear span of a set
supp	support of a function
diam	diameter of a bounded set
cond	spectral condition number of a matrix or condition number of a basis
$\oplus$	direct sum of two spaces
$V^{\perp}$	orthogonal complement of V
$\overline{V}$	closure of the subspace V
$L^2(\Omega)$	the space of all square integrable functions defined on Ω
$H^s(\Omega)$	Sobolev space of order $s \in \mathbb{R}$ on Ω
$H_0^m(\Omega)$	Sobolev space of H^m functions satisfying homogeneous Dirichlet boundary conditions of order m
$C^m(\mathbb{R})$	the space of m-times continuously differentiable functions
$l^2(\mathcal{J})$	the space of (infinite) vectors $\mathbf{v}$ with finite l^2-norm
$\Pi_m(\Omega)$	the space of all algebraic polynomials on Ω of degree at most m
$\lVert\cdot\rVert$	the l^2-norm of an (infinite) vector, the L^2-norm of a function, or the spectral norm of a matrix or an operator
$\lVert\cdot\rVert_H$	norm in the space H
$\lvert\cdot\rvert_{H^s(\Omega)}$	seminorm in $H^s(\Omega)$
$\langle\cdot,\cdot\rangle$	the L^2-inner product or a dual product
$\langle\cdot,\cdot\rangle_H$	inner product in H
j_0	the coarsest level in a multiresolution analysis in a given context
λ	wavelet index, usually $\lambda = (j, k)$

$\phi, \phi_{j,k}$	scaling functions
$\psi, \psi_{j,k}$	wavelets or more generally functions in a wavelet basis (scaling functions on the coarsest level and wavelets)
λ_{max}	eigenvalue that is maximal in the absolute value
λ_{min}	eigenvalue that is minimal in the absolute value
$\#$	cardinality of a set
Δ	Laplace operator
C	generic constant
$\forall$	for all
$\mathbf{M}^T$	transpose of a matrix $\mathbf{M}$
$\mathbf{I}$	identity matrix or identity operator
$\mathbf{0}$	zero matrix

Introduction

Wavelet bases and the fast wavelet transform are a powerful and useful tool for signal and image analysis, detection of singularities, data compression, and also for the numerical solution of partial differential equations, integral equations, and integro-differential equations. One of the most important properties of wavelets is that they have vanishing moments. Vanishing wavelet moments ensure the so-called compression property of wavelets. This means that integrals of a product of a function and a wavelet decay exponentially, dependent on the level of the wavelet if the function is smooth enough in the support of the wavelet. This enables the obtainment of sparse representations of functions as well as sparse representations of some operators, see e.g. [2, 28, 72].

There are two main classes of wavelet based methods for the numerical solution of operator equations. The first method is the wavelet-Galerkin method. Due to vanishing moments, the wavelet-Galerkin method leads to sparse matrices not only for differential equations but also for integral and integro-differential equations while the Galerkin method with the standard B-spline basis leads to full matrices if the equation contains an integral term. Another important property of wavelet bases is that they form Riesz bases in certain spaces, such as Lebesgue, Sobolev or Besov spaces. Due to this property, the diagonally preconditioned matrices arising from discretization using the Galerkin method with wavelet bases have uniformly bounded condition numbers for many types of operator equations.

The second class of methods are adaptive wavelet methods. We focus on adaptive wavelet methods that were originally designed in [29, 30] and later modified in many papers [43, 52, 67]. For a large class of operator equations, both linear and nonlinear, it was shown that these methods converge and are asymptotically optimal in the sense that the storage and the number of floating point operations, needed to resolve the problem with desired accuracy, depend linearly on the number of parameters representing the solution. Moreover, the method enables higher-order approximation if higher-order spline-wavelet bases are used. The solution and the right-hand side of the equation have sparse representations in a wavelet basis, i.e. they can be represented by a small number of numerically significant parameters. Similarly as in the case of the wavelet-Galerkin method, the differential and integral operators can be represented by sparse or quasi-sparse matrices. For a large class of problems, the matrices arising from a discretization using wavelet bases can be simply preconditioned by a diagonal preconditioner, and the condition numbers of these preconditioned matrices are uniformly bounded. For more details about adaptive wavelet methods, see [7, 29, 30, 43, 52, 67, 72].

The first wavelet methods used orthogonal wavelets, e.g. Daubechies wavelets or coiflets. Their disadvantages are that the most orthogonal wavelets are usually not known in an explicit form and their smoothness is typically dependent on the length of the support. In contrast, spline wavelets are known in a closed form, are smoother, and have shorter support than orthogonal wavelets with the same polynomial exactness and the same number of vanishing moments. Therefore, they are preferable in numerical methods for operator equations.

This habilitation thesis is concerned with constructions of new spline-wavelet bases on the interval and product domains, their adaptations to boundary conditions, and their applications. The thesis is conceived as a collection of the following eight previously published articles supplemented by commentary.

[10] Černá, D.; Finěk, V.: *Construction of optimally conditioned cubic spline wavelets on the interval*, Adv. Comput. Math. **34(2)**, (2011), pp. 219-252. My contribution to this paper was 60%.

[11] Černá, D.; Finěk, V.: *Cubic spline wavelets with complementary boundary conditions*, Appl. Math. Comput. **219(4)**, (2012), pp. 1853-1865. My contribution to this paper was 60%.

[13] Černá, D.; Finěk, V.: *Quadratic spline wavelets with short support for fourth-order problems*, Result. Math. **66(6)**, (2014), pp. 525-540. My contribution to this paper was 60%.

[14] Černá, D.; Finěk, V.: *Cubic spline wavelets with short support for fourth-order problems*, Appl. Math. Comput. **243**, (2014), pp. 44-56. My contribution to this paper was 60%.

[15] Černá, D.; Finěk, V.: *Wavelet basis of cubic splines on the hypercube satisfying homogeneous boundary conditions*, Int. J. Wavelets Multiresolut. Inf. Process. **13(3)**, (2015), article No. 1550014. My contribution to this paper was 60%.

[17] Černá, D.; Finěk, V.: *Sparse wavelet representation of differential operators with piecewise polynomial coefficients*, Axioms **6**, (2017), article No. 4. My contribution to this paper was 60%.

[19] Černá, D.; Finěk, V.: *Quadratic spline wavelets with short support satisfying homogeneous boundary conditions*, Electron. Trans. Numer. Anal. **48**, (2018), pp. 15-39. My contribution to this paper was 90%.

[23] Černá, D.: *Cubic spline wavelets with four vanishing moments on the interval and their applications to option pricing under Kou model*, Int. J. Wavelets Multiresolut. Inf. Process. **17(1)**, (2019), article No. 1850061.

Papers [10, 11, 15] are focused on constructions of well-conditioned biorthogonal spline wavelet bases on the interval where both primal and dual wavelets have compact support. In [13, 14, 19, 23] we do not require local support of dual wavelets, which enables us to construct wavelet bases that have smaller support and have significantly smaller condition number than wavelet bases with local duals. Moreover, their construction is significantly simpler than constructions of wavelets with local duals, which are typically quite long and technical. In [18], we constructed wavelets that are orthogonal to piecewise polynomials of degree at most seven on a uniform grid. Due to this property, matrices arising from discretization of second-order differential equations with coefficients that are piecewise polynomials of degree at most four on uniform grids are sparse. We use the constructed bases for solving various types of operator equations, e.g. Poisson's equation, the Helmholtz equation, fourth-order differential equations, and the Black-Scholes equation with two state variables. We also applied the constructed bases for option pricing under Kou's double exponential jump-diffusion option pricing model. Other applications are presented in Chapter 2.

This thesis is organized as follows. In Chapter 1 we briefly review a concept of a wavelet basis on a bounded interval, the fast wavelet transform on a bounded domain, and two constructions of wavelet bases on product domains that are based on tensorizing univariate wavelet bases. We also describe basic principles of the wavelet-Galerkin method and adaptive wavelet methods. Since all the papers collected in this thesis are concerned with constructions of quadratic or cubic spline wavelets on the interval, in Chapter 2 we present existing constructions of such types of wavelets and their applications.

Most of the papers presented in this thesis comes from a collaboration with my colleague Václav Finěk. I would like to thank him for this friendly and very helpful collaboration. Most of the presented work was done at Technical University of Liberec. I want to express my gratitude to all of my colleagues there for all the inspiration and help. In particular, I am grateful to Jirka Hozman and Prof. Jan Picek, who allowed me to collaborate with them on interesting projects.

Chapter 1

Wavelet Basis

In this chapter, we introduce the concept of a wavelet basis on the interval and product domains, the fast wavelet transform, the wavelet-Galerkin method, and adaptive wavelet methods.

Wavelet bases were originally constructed as orthonormal bases for the space $L^2(\mathbb{R})$ and later as Riesz bases for this space. One of the possibilities of how to use these bases and the fast wavelet transform on a bounded domain is the extension of a function or a signal near the boundary using for example zero padding, periodization, or symmetrization, see e.g. [7, 8]. However, this approach can lead to boundary effects, and it is not suitable for the numerical solution of operator equations in which a basis has to be adapted to boundary conditions. We present another approach, one where the wavelet bases on the real line are adapted to the interval with special boundary functions that have to be constructed, such that the resulting basis is a Riesz basis for a chosen space and the locality of the support, smoothness of basis functions, the number of vanishing wavelet moments, and polynomial exactness are preserved.

1.1 Wavelet Bases on Bounded Interval

In this section, we briefly recall the concept of a wavelet basis on a bounded interval $I \subset \mathbb{R}$; for more details refer to [7, 28, 31, 37, 41, 72]. Let $\mathcal{J}$ be at most countable index set such that each index $\lambda \in \mathcal{J}$ takes the form $\lambda = (j,k)$, where $|\lambda| = j \in \mathbb{Z}$ denotes a level. We define

$$\|\mathbf{v}\| = \sqrt{\sum_{\lambda \in \mathcal{J}} v_\lambda^2}, \quad \text{for} \quad \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \tag{1.1}$$

and

$$l^2(\mathcal{J}) = \left\{\mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \|\mathbf{v}\| < \infty\right\}. \tag{1.2}$$

We use the standard notation $L^2(I)$ for the space of all square-integrable functions defined on I, $\|\cdot\|$ for the L^2-norm, and $\langle \cdot, \cdot \rangle$ for the L^2-inner product. Let $H \subset L^2(I)$ be a real separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|\cdot\|_H$. For example, H can be the Sobolev space $H_0^1(I)$ of functions whose first weak derivatives are in $L^2(I)$ and that vanish at boundary points. First, a wavelet basis $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\}$ has to be a Riesz basis for H.

Definition 1. A family $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *Riesz basis* of H, if the span of Ψ is dense in H and there exist constants $c, C \in (0, \infty)$ such that

$$c\,\|\mathbf{b}\| \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \leq C\,\|\mathbf{b}\|, \quad \forall\ \mathbf{b} = \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{1.3}$$

We refer to the constants

$$c_\Psi = \sup\{c : c \text{ satisfies (1.3)}\} \quad \text{and} \quad C_\Psi = \inf\{C : C \text{ satisfies (1.3)}\} \tag{1.4}$$

as lower and upper Riesz bounds (with respect to the H-norm), respectively, and to the number $\operatorname{cond}\Psi = C_\Psi / c_\Psi$ as the *condition number* of Ψ. In some papers, the squares of the norms are used in (1.3) and the Riesz bounds are defined as c_Ψ^2 and C_Ψ^2. Riesz basis property is crucial for a uniform boundedness of condition numbers of the discretization matrices. The set of functions is called a *Riesz sequence* in H if there exist positive constants c and C that satisfy (1.3) but the closure of this set is not necessarily H.

We view countable sets of functions $\Gamma, \Theta \subset L^2(\Omega)$ also as the column vectors and we use the symbol $\langle \Gamma, \Theta \rangle_H$ to denote the matrix

$$\langle \Gamma, \Theta \rangle_H = \{\langle \gamma, \theta \rangle_H\}_{\gamma \in \Gamma, \theta \in \Theta}. \tag{1.5}$$

It is known that the constants c_Ψ and C_Ψ satisfy

$$c_\Psi = \sqrt{\lambda_{min}(\langle \Psi, \Psi \rangle)}, \quad C_\Psi = \sqrt{\lambda_{max}(\langle \Psi, \Psi \rangle)}, \tag{1.6}$$

where $\lambda_{min}(\langle \Psi, \Psi \rangle)$ and $\lambda_{max}(\langle \Psi, \Psi \rangle)$ are the smallest and the largest eigenvalues of the matrix $\langle \Psi, \Psi \rangle$, respectively. Furthermore, the functions ψ_λ have to be *local* in the sense that

$$\text{diam supp}\ \psi_\lambda \leq \mathrm{C}2^{-|\lambda|}, \quad \lambda \in \mathcal{J}, \tag{1.7}$$

where the constant C does not depend on λ, and at a given level j the supports of only finitely many wavelets overlap at any point $x \in I$.

Another desired property of a wavelet basis Ψ is its hierarchical structure, i.e. Ψ is of the form

$$\Psi = \Phi_{j_0} \cup \bigcup_{j=j_0}^{\infty} \Psi_j, \tag{1.8}$$

j_0 being the coarsest level. The functions from the set Φ_{j_0} are called *scaling functions*, and the functions from the set Ψ_j, $j \geq j_0$, are called *wavelets* on the level j. Wavelets in the inner part of the interval called *inner wavelets* are typically translations and dilations of one function ψ or several functions $\psi^1, \ldots, \psi^p$ also called *wavelets* (or *mother wavelet, wavelet generator*), i.e.

$$\psi_{j,k}(x) = 2^{j/2} \psi^l \left(2^j x - m\right), \tag{1.9}$$

for some $l \in \{1, \ldots, p\}$ and some $k, m \in \mathbb{Z}$, k dependent on m and l. Similarly the wavelets near the boundary are derived from functions called *boundary wavelets.*

Another desired property is a *polynomial exactness* of order $M \geq 1$. This means that the *multiscale basis*

$$\Psi^J = \Phi_{j_0} \cup \bigcup_{j=j_0}^{J} \Psi_j, \tag{1.10}$$

$j_0 \leq J$, is such that $span\Psi^J$ contains all polynomials of degree at most $M-1$. Polynomial exactness determines the convergence rate of methods for the numerical solution of operator equations.

Finally, we require that there exists $L \geq 1$, such that all functions $\psi_\lambda \in \Psi_j$, $j_0 \leq j$, have L *vanishing moments*, i.e.

$$\int_I x^k \, \psi_\lambda (x) \, dx = 0, \quad k = 0, \ldots, L-1. \tag{1.11}$$

Vanishing wavelet moments are important for sparse representation of functions and operators.

The concept of a wavelet basis is not unified in the mathematical literature and some of the above conditions can be omitted or generalized.

1.2 Construction of Wavelet Bases

The wavelet basis Ψ is typically constructed using a multiresolution analysis.

Definition 2. A sequence $\{V_j\}_{j=j_0}^{\infty}$ of closed linear subspaces $V_j \subset H$ is called a *multiresolution analysis*, if these subspaces are nested and their union is dense in H, i.e.,

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots \subset H, \quad \overline{\bigcup_{j=j_0}^{\infty} V_j} = H.$$

Let the set

$$\Phi_j = \{\phi_{j,k}, k \in \mathcal{I}_j\}$$

be a basis for V_j that is local and uniformly stable, i.e. the condition number of Φ_j is bounded and the bound is independent on j. The set Φ_j is called a *scaling basis* and similarly as functions from Φ_{j_0}, the functions $\phi_{j,k} \in \Phi_j$ are called *scaling functions.*

In the papers presented in this thesis the scaling functions are quadratic B-splines [10, 13, 19], cubic B-splines [10, 11, 14, 15, 23], or Hermite cubic splines [17].

Let W_j be complement spaces such that $V_j \oplus W_j = V_{j+1}$, where $\oplus$ denotes a direct sum, and let the sets Ψ_j be uniformly stable bases of W_j. Then, wavelets are constructed as the elements of a basis Ψ_j such that they have vanishing moments.

Now, using Φ_j and Ψ_j we define Ψ by (1.8). However, the fact that the spaces V_j form a multiresolution analysis, the scaling bases Φ_j are uniformly stable, and the one-level wavelet bases Ψ_j are uniformly stable, does not imply that Ψ is a Riesz basis for H. In the next section, we discuss several approaches to prove the Riesz basis property (1.3).

If Ψ is a Riesz basis of H, then there exists a unique Riesz basis $\tilde{\Psi}$ that is biorthogonal to Ψ, i.e.

$$\left\langle \Psi, \tilde{\Psi} \right\rangle_H = \mathbf{I}, \tag{1.12}$$

where $\mathbf{I}$ is the (infinite) identity matrix. The basis $\tilde{\Psi}$ is called the *dual* basis to Ψ. The dual basis generates a dual multiresolution spaces $\tilde{V}_j$, $j \geq j_0$.

In [10, 11], we constructed dual bases that are local. However, in some applications such as solving linear PDEs, the dual basis is not directly used. Therefore, in [13, 14, 15, 17, 19, 23] we were concerned with constructions of wavelet bases without requiring locality of duals, but with a shorter support or some special properties.

1.3 Proofs of Riesz Basis Property

While one can employ the Fourier transform to prove the Riesz basis property (1.3) for the space $L^2(\mathbb{R})$, the proof of the Riesz basis property for the space $H \subset L^2(I)$ is usually more complicated. We present here several possible approaches that we used in papers collected in this thesis.

In [10, 11, 15, 17], the proof of the Riesz basis property (1.3) for Ψ is based on the following theorem [28, 36, 45].

Theorem 3. *Let $j_0 \in \mathbb{N}$ and for $j \geq j_0$ let V_j and $\tilde{V}_j$ be subspaces of the space $H \subset L^2(I)$ such that $V_j \subset V_{j+1}$, $\tilde{V}_j \subset \tilde{V}_{j+1}$, and $\dim V_j = \dim \tilde{V}_j < \infty$. Let Φ_j be bases of V_j, $\tilde{\Phi}_j$ be bases of $\tilde{V}_j$, and Ψ_j be bases of $\tilde{V}_j^{\perp} \cap V_{j+1}$, where $\tilde{V}_j^{\perp}$ denotes the L^2-orthogonal complement of $\tilde{V}_j$ in H. Moreover, let the Riesz bounds with respect to the L^2-norm of Φ_j, $\tilde{\Phi}_j$, and Ψ_j, be uniformly bounded. Let Ψ be composed of Φ_{j_0} and Ψ_j, $j \geq j_0$, as in (1.8). Furthermore, we assume that*

$$\mathbf{\Gamma}_j = \left\langle \Phi_j, \tilde{\Phi}_j \right\rangle \tag{1.13}$$

is invertible and that the spectral norm of $\mathbf{\Gamma}_j^{-1}$ is bounded independently on j. In addition, for some positive constants C, γ and d, such that $\gamma < d$, let

$$\inf_{v_j \in V_j} \|v - v_j\|_{L^2(I)} \leq C 2^{-jt} \|v\|_{H^t(I)}, \; v \in H^t(I) \cap H, \; 0 \leq t \leq d, \tag{1.14}$$

and

$$\|v_j\|_{H^s(I)} \leq C 2^{js} \|v_j\|_{L^2(I)}, \; v_j \in V_j, \; 0 \leq s < \gamma, \tag{1.15}$$

and similarly let (1.14) and (1.15) hold for $\tilde{\gamma}$ and $\tilde{d}$ on the dual side. Then

$$\left\{ \psi_\lambda / \|\psi_\lambda\|_{H^s(I)}, \psi_\lambda \in \Psi \right\} \tag{1.16}$$

is a Riesz sequence in $H^s(I)$ for $s \in (-\tilde{\gamma}, \gamma)$.

Estimates of type (1.14) indicating the approximation properties of V_j are called *direct* or *Jackson estimates*. Estimates like (1.15) describe smoothness properties of V_j, and they are often referred to as *inverse* or *Bernstein estimates*. The parameters in these estimates depend on the polynomial exactness, the number of vanishing wavelet moments, and the smoothness of basis functions. For more details, refer to [28, 36, 41].

In [10, 11], we constructed several biorthogonal bases for the spaces $L^2(I)$, $H_0^1(I)$ and $H_0^2(I)$, such that functions from these bases are local. These constructions were quite technical and complicated, but due to biorthogonality and locality of the bases, the proof was relatively simple using Theorem 3, because the biorthogonality property implies that $\mathbf{\Gamma}_j$ is the identity matrix.

In [15, 17], we constructed spline-wavelet bases using dual multiresolution spaces $\tilde{V}_j$, but we did not construct corresponding biorthogonal bases because these do not have local supports. In these cases, the main part of the proof is finding appropriate bases $\tilde{\Phi}_j$ of the spaces $\tilde{V}_j$ and proving that the matrices $\mathbf{\Gamma}_j$ have desired properties.

In [13, 14, 19], we constructed wavelet bases without using dual spaces. Thus Theorem 3 can not be used for the proof of the Riesz basis property. In these papers, we employed the theory developed in [57], summarized in the following theorem.

Theorem 4. *Let Ω be a bounded domain and let the spaces V_j, $j \geq j_0$, form a multiresolution analysis for the space $L^2(\Omega)$. Let H_q for fixed $q > 0$ be a linear subspace of $L^2(\Omega)$ that is itself a normed linear space and assume that there exist positive constants A_1 and A_2 such that*

a) If $f \in H_q$ has decomposition $f = \sum_{j \geq j_0} f_j$, $f_j \in V_j$ then

$$\|f\|_{H_q}^2 \leq A_1 \sum_{j \geq j_0} 2^{qj} \|f_j\|^2 ; \tag{1.17}$$

b) For each $f \in H_q$ there exists a decomposition $f = \sum_{j \geq j_0} f_j$, $f_j \in V_j$, such that

$$\sum_{j \geq j_0} 2^{qj} \|f_j\|^2 \leq A_2 \|f\|_{H_q}^2 . \tag{1.18}$$

Furthermore, suppose that P_j is a linear projection from V_{j+1} onto V_j, W_j is the kernel space of P_j, $\Phi_j = \{\phi_{j,k}, k \in \mathcal{I}_j\}$ are Riesz bases of V_j with respect to the L_2–norm with uniformly bounded condition numbers, and $\Psi_j = \{\psi_{j,k}, k \in \mathcal{I}_j\}$ are Riesz bases of W_j with uniformly bounded condition numbers. If there exist constants C and p such that $0 < p < q$ and

$$\|P_m P_{m+1} \dots P_{n-1}\| \leq C \, 2^{p\,(n-m)}, \tag{1.19}$$

$\|\cdot\|$ being the spectral norm, then

$$\left\{2^{-j_0 q} \phi_{j_0,k}, k \in \mathcal{I}_{j_0}\right\} \cup \left\{2^{-jq} \psi_{j,k}, j \geq j_0, k \in \mathcal{I}_j\right\} \tag{1.20}$$

is a Riesz basis of H_q.

To employ Theorem 4, one has to find appropriate projectors P_j and prove the inequality (1.19). The advantage of this approach is that it enables proving the Riesz basis property in the Sobolev spaces H^s for values of s in some range (s_1, s_2), where s_1 can be positive. Therefore, it is possible to use this theorem to prove the Riesz basis property in H^s even if the Riesz basis property does not hold in L^2. For example in [19], we used this theorem to prove that the constructed set is a Riesz basis in the spaces $H_0^1(I)$ and $H_0^1(I^2)$ for $I = (0,1)$, but numerical experiments show that the L^2-condition numbers increase with the level and they seem to be unbounded. This suggests that the basis on I is not a Riesz basis in the space $L^2(I)$.

In [23], we used a completely different approach and derived a condition under which a union of Riesz sequences is also a Riesz sequence.

Theorem 5. *Let $\mathcal{I}$ and $\mathcal{J}$ be at most countable index sets, $\{f_k\}_{k\in\mathcal{I}}$ be a Riesz sequence with a Riesz lower bound c_f, and $\{g_l\}_{l\in\mathcal{J}}$ be a Riesz sequence with a Riesz lower bound c_g. Furthermore, let the matrix $\mathbf{G}$ with entries $\mathbf{G}_{k,l} = \langle f_k, g_l\rangle$, $k \in \mathcal{I}$, $l \in \mathcal{J}$, satisfy*

$$\|\mathbf{G}\| / (c_f c_g) < 1. \tag{1.21}$$

Then $\{f_k\}_{k\in\mathcal{I}} \cup \{g_l\}_{l\in\mathcal{J}}$ is a Riesz sequence with a Riesz lower bound c, and

$$c \geq \sqrt{1 - \frac{\|\mathbf{G}\|}{c_f c_g}} \cdot \min(c_f, c_g). \tag{1.22}$$

In [23], we proved the Riesz basis property separately for inner wavelets and for boundary wavelets, and then we verified the condition (1.21) to show that their union is also a Riesz basis.

1.4 Fast Wavelet Transform

As we have already mentioned, we view the sets of functions such as Φ_j and Ψ_j also as columns vectors. The nestedness of the spaces V_j implies the existence of a *refinement matrix* $\mathbf{M}_{j,0}$ such that

$$\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}. \tag{1.23}$$

Since $W_j \subset V_{j+1}$, there exists a matrix $\mathbf{M}_{j,1}$ such that

$$\Psi_j = \mathbf{M}_{j,1}^T \Phi_{j+1}. \tag{1.24}$$

Applying (1.23) and (1.24) several times we find out that the multiscale basis Ψ^J defined by (1.10) and the scaling basis Φ_J are interrelated by the transform $\mathbf{T}_J$ such that

$$\Psi^J = \begin{pmatrix} \Phi_{j_0} \\ \Psi_{j_0} \\ \Psi_{j_0+1} \\ \vdots \\ \Psi_{J-1} \end{pmatrix} = \mathbf{T}_J^T \Phi_J, \tag{1.25}$$

and the transform $\mathbf{T}_J$ can be expressed by

$$\mathbf{T}_J = \mathbf{T}_{J,J-1} \dots \mathbf{T}_{J,j_0}, \quad \text{where} \quad \mathbf{T}_{J,j} = \begin{pmatrix} \mathbf{M}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad \mathbf{M}_j = (\mathbf{M}_{j,0}, \mathbf{M}_{j,1}), \tag{1.26}$$

where $\mathbf{0}$ and $\mathbf{I}$ are zero and identity matrices, respectively, of appropriate sizes. This transform is often called the *fast wavelet transform* (FWT).

Since $V_J = \text{span}\,\Phi_J = \text{span}\,\Psi^J$, any function $f \in V_J$ has a *single-scale representation*

$$f = \mathbf{c}_J^T \Phi_j = \sum_{k \in \mathcal{I}_J} c_{J,k} \phi_{J,k}, \quad \mathbf{c}_J = \{c_{J,k}\}_{k \in \mathcal{I}_J}, \tag{1.27}$$

and also a *multiscale representation*

$$f = \mathbf{c}_{j_0}^T \Phi_{j_0} + \mathbf{d}_{j_0}^T \Psi_{j_0} + \ldots + \mathbf{d}_{J-1}^T \Psi_{J-1} = \sum_{k \in \mathcal{I}_{j_0}} c_{j_0,k} \phi_{j_0,k} + \sum_{j=j_0}^{J-1} \sum_{k \in \mathcal{J}_j} d_{j,k} \psi_{j,k}, \tag{1.28}$$

where $\mathbf{c}_{j_0} = \{c_{j_0,k}\}_{k \in \mathcal{I}_{j_0}}$ and $\mathbf{d}_j = \{d_{j,k}\}_{k \in \mathcal{J}_j}$. Then, the vectors $\mathbf{c}_J$ and

$$\mathbf{d}^J = \left(\mathbf{c}_{j_0}^T, \mathbf{d}_{j_0}^T, \ldots \mathbf{d}_{J-1}^T\right)^T \tag{1.29}$$

are also interrelated by the fast wavelet transform $\mathbf{T}_J$, i.e.

$$\mathbf{c}_J = \mathbf{T}_J \mathbf{d}^J, \tag{1.30}$$

because from (1.23) and (1.24) we obtain

$$\mathbf{c}_j^T \Phi_j + \mathbf{d}_j^T \Psi_j = \left(\mathbf{M}_{j,0} \mathbf{c}_j + \mathbf{M}_{j,1} \mathbf{d}_j\right)^T \Phi_{j+1} = \mathbf{c}_{j+1}^T \Phi_{j+1}. \tag{1.31}$$

Schematically $\mathbf{T}_J$ applied on $\mathbf{d}^J$ can be visualized as a *pyramid scheme*,

$$\begin{array}{ccccccccc} \mathbf{c}_{j_0} & \xrightarrow{\mathbf{M}_{j_0,0}} & \mathbf{c}_{j_0+1} & \xrightarrow{\mathbf{M}_{j_0+1,0}} & \mathbf{c}_{j_0+2} & \xrightarrow{\mathbf{M}_{j_0+2,0}} & \ldots \ \mathbf{c}_{J-1} & \xrightarrow{\mathbf{M}_{J-1,0}} & \mathbf{c}_J \ . \\ & \mathbf{M}_{j_0,1} \nearrow & & \mathbf{M}_{j_0+1,1} \nearrow & & \mathbf{M}_{j_0+2,1} \nearrow & & \mathbf{M}_{J-1,1} \nearrow & \\ & \mathbf{d}_{j_0} & & \mathbf{d}_{j_0+1} & & \mathbf{d}_{j_0+2} & \ldots & \mathbf{d}_{J-1} & \end{array}$$

Due to the local support of basis functions, the matrices $\mathbf{M}_j$ are sparse and they can be applied in $\mathcal{O}(N_j)$ operations, where $N_j = \dim V_j$. Thus, the fast wavelet transform can be applied in $\mathcal{O}(N_J)$ operations when using a pyramid scheme.

Since the matrix $\mathbf{M}_j$ represents a basis transformation, its inverse exists. Let us define

$$\mathbf{G}_j = \left(\mathbf{G}_{j,0}, \mathbf{G}_{j,1}\right) = \mathbf{M}_j^{-1}, \tag{1.32}$$

where the matrix $\mathbf{G}_{j,0}$ is of the size $\#\mathcal{I}_{j+1} \times \#\mathcal{I}_j$ and the matrix $\mathbf{G}_{j,1}$ is of the size $\#\mathcal{I}_{j+1} \times \#\mathcal{J}_j$. Then

$$\mathbf{c}_j = \mathbf{G}_{j,0}^T \mathbf{c}_{j+1}, \quad \mathbf{d}_j = \mathbf{G}_{j,1}^T \mathbf{c}_{j+1}. \tag{1.33}$$

Thus, the *inverse fast wavelet transform* (IFWT) $\mathbf{T}_J^{-1}$ has the form

$$\mathbf{T}_J^{-1} = \mathbf{T}_{J,j_0}^{-1} \ldots \mathbf{T}_{J,J-1}^{-1}, \quad \text{where} \quad \mathbf{T}_{J,j}^{-1} = \begin{pmatrix} \mathbf{G}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{1.34}$$

The corresponding pyramid scheme is then

$$\begin{array}{ccccccccccc}
\mathbf{c}_J & \xrightarrow{\mathbf{G}_{J-1,0}} & \mathbf{c}_{J-1} & \xrightarrow{\mathbf{G}_{J-2,0}} & \mathbf{c}_{J-2} & \xrightarrow{\mathbf{G}_{J-3,0}} & \dots & \mathbf{c}_{j_0+1} & \xrightarrow{\mathbf{G}_{j_0,0}} & \mathbf{c}_{j_0} & . \\
& \searrow^{\mathbf{G}_{J-1,1}} & & \searrow^{\mathbf{G}_{J-2,1}} & & \searrow^{\mathbf{G}_{J-3,1}} & & & \searrow^{\mathbf{G}_{j_0,1}} & & \\
& \mathbf{d}_{J-1} & & \mathbf{d}_{J-2} & & \mathbf{d}_{J-3} & \dots & & \mathbf{d}_{j_0} & &
\end{array}$$

Clearly, different wavelet bases lead to different fast wavelet transforms. As mentioned above, FWT can be used to transform a scaling basis to a multiscale wavelet basis and vectors of multi-scale coefficients to vectors of single-scale coefficients. Furthermore, if $\mathcal{A}$ is a differential or an integral operator, then FWT can be used to transform the discretization matrix in a scaling basis $\langle \mathcal{A}\Phi_J, \Phi_J \rangle$ to a discretization matrix with respect to a multiscale basis $\langle \mathcal{A}\Psi_J, \Psi_J \rangle$ by

$$\langle \mathcal{A}\Psi_J, \Psi_J \rangle = \mathbf{T}_J^T \langle \mathcal{A}\Phi_J, \Phi_J \rangle \mathbf{T}_J.$$

In signal processing FWTs and IFWTs are widely used for signal analysis, signal compression and decompression. FWTs corresponding to a wavelet basis adapted to a bounded interval have an advantage that the boundary wavelets also have vanishing moments and thus the boundary effects that can occur when using the standard approach based on symmetrization of the signal are reduced. We studied this issue in [8, 9], where we used the fast wavelet transforms corresponding to wavelet bases that we constructed in [10] for image compression.

1.5 Wavelet Bases on Product Domains

There are several approaches for constructing a multi-dimensional wavelet basis on a tensor product domain, for example an isotropic approach [57, 72], an anisotropic approach [35, 45] or a sparse tensor product [44]. In this section, we recall an isotropic and an anisotropic approach. Both constructions are based on tensorizing univariate wavelet bases and they preserve their important properties.

We consider a product domain $\Box = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_d, b_d)$, where $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, $i = 1, \ldots, d$, $d \in \mathbb{N}$. The construction usually starts with a Riesz basis

$$\Psi = \{\phi_{j_0,k}, k \in \mathcal{I}_{j_0}\} \cup \{\psi_{j,k}, k \in \mathcal{J}_j, j \geq j_0\} \tag{1.35}$$

for the space $H \subset H^s(0,1)$ for s in some interval $(s1, s2)$. First, we use a simple linear transformation to obtain a wavelet basis for the space $H^s(a_i, b_i)$. Let us define

$$\phi^i_{j,k}(x) = \phi_{j,k}\left(\frac{x - a_i}{b_i - a_i}\right), \quad \psi^i_{j,k}(x) = \psi_{j,k}\left(\frac{x - a_i}{b_i - a_i}\right), \quad x \in (a_i, b_i), \tag{1.36}$$

then

$$\Psi^{(i)} = \left\{\phi^i_{j_0,k}, k \in \mathcal{I}_{j_0}\right\} \cup \left\{\psi^i_{j,k}, k \in \mathcal{J}_j, j \geq j_0\right\} \tag{1.37}$$

forms a Riesz basis in $H^s(a_i, b_i)$.

Isotropic wavelet bases. We define the multivariate scaling functions by

$$\phi_{j,k}(x) = \Pi_{l=1}^d \phi^l_{j,k_l}(x_l), \quad x = (x_1, \ldots, x_d) \in \Box, \tag{1.38}$$

with $k = (k_1, \ldots, k_d)$ now being a multi-index, $k \in \mathcal{I}_j^{\square} = \mathcal{I}_j \times \ldots \times \mathcal{I}_j$. We introduce the abbreviation

$$\mathcal{J}_{j,e} = \begin{cases} \mathcal{J}_j, & e = 1, \\ \mathcal{I}_j & e = 0, \end{cases} \tag{1.39}$$

i.e. the parameter e allows distinguishing between scaling functions and wavelets. Furthermore, we denote

$$E = \{e = (e_1, \ldots, e_d), e_i \in \{0, 1\}, e \neq (0, 0, \ldots, 0)\}, \tag{1.40}$$

and

$$\mathcal{J}_{j,e}^{\square} = \mathcal{J}_{j,e_1} \times \ldots \times \mathcal{J}_{j,e_d}, \quad \mathcal{J}_j^{\square} = \bigcup_{e \in E} \mathcal{J}_{j,e}. \tag{1.41}$$

For any $e = (e_1, \ldots, e_d) \in E$, $j \geq j_0$, and $k = (k_1, \ldots, k_d) \in \mathcal{J}_j^{\square}$, we define the multivariate wavelet

$$\psi_\lambda(x) = \Pi_{l=1}^d \psi_{j,e_l,k_l}^l(x_l), \quad x = (x_1, \ldots, x_d) \in \square, \quad \lambda = (j, e, k), \tag{1.42}$$

where

$$\psi_{j,e_l,k_l}^l = \begin{cases} \phi_{j,k_l}^l, & e_l = 1, \\ \psi_{j,k_l}^l, & e_l = 0. \end{cases} \tag{1.43}$$

The wavelet basis on the hyperrectangle $\square$ is then given by

$$\Psi = \left\{\psi_{j,e,k},\ e \in E,\ k \in \mathcal{J}_j^{\square}, j \geq j_0\right\} \cup \left\{\phi_{j_0,k},\ k \in \mathcal{I}_j^{\square}\right\}. \tag{1.44}$$

We denote the multiscale basis containing wavelets up to level J as

$$\Psi^J = \left\{\psi_{j,e,k},\ e \in E,\ k \in \mathcal{J}_j^{\square}, j_0 \leq j \leq J\right\} \cup \left\{\phi_{j_0,k},\ k \in \mathcal{I}_j^{\square}\right\}. \tag{1.45}$$

If we start with a Riesz basis in the space $L^2(0,1)$, then the resulting basis is a Riesz basis in the space $L^2(\Omega)$. The Riesz basis property in the space $H^s(\Omega)$ can be verified using e.g. Theorem 4. Furthermore, this approach preserves the regularity of basis functions, the full degree of polynomial exactness, vanishing wavelet moments, as well as locality of bases functions. For more details see e.g. [57, 72]. In this thesis, we constructed isotropic wavelet bases and used them for the numerical solution of differential equations in [10, 11, 13, 14, 15, 19].

Anisotropic wavelet basis. Let $\Psi^{(i)}$ be a wavelet basis on the interval (a_i, b_i) defined by (1.36) and (1.37). For notational simplicity, we denote $\mathcal{J}_{j_0-1} = \mathcal{I}_{j_0}$ and

$$\psi_{j_0-1,k}^i = \phi_{j_0,k}^i, \quad k \in \mathcal{J}_{j_0-1}, \quad \mathcal{J} = \{(j,k),\ j \geq j_0 - 1,\ k \in \mathcal{J}_j\}. \tag{1.46}$$

Then $\Psi^{(i)}$ can also be expressed as

$$\Psi^{(i)} = \left\{\psi_{j,k}^i,\ j \geq j_0 - 1,\ k \in \mathcal{J}_j\right\} = \left\{\psi_\lambda^i,\ \lambda \in \mathcal{J}\right\}. \tag{1.47}$$

Recall that for the index $\lambda = (j,k)$ we denote $|\lambda| = j$. We use $u \otimes v$ to denote the tensor product of functions u and v, i.e. $(u \otimes v)(x_1, x_2) = u(x_1)\, v(x_2)$. For $d \geq 1$ we generalize the definition of the index set $\mathcal{J}$:

$$\mathcal{J} = \{\lambda = (\lambda_1, \ldots, \lambda_d) : \lambda_i = (j_i, k_i), j_i \geq j_0 - 1, k_i \in \mathcal{J}_{j_i}\}. \tag{1.48}$$

We define multivariate basis functions as

$$\psi_\lambda = \otimes_{i=1}^d \psi_{\lambda_i}^i, \quad \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathcal{J}. \tag{1.49}$$

Then $|\lambda| = \max_{i=1,\ldots,d} |\lambda_i|$ represents a level. We also denote $[\lambda] = \min_{i=1,\ldots,d} |\lambda_i|$. Due to locality of the one-dimensional basis functions, i.e. $\operatorname{diam}\operatorname{supp} \psi_{\lambda_i}^i \leq C_i 2^{-|\lambda_i|}$, we have

$$\operatorname{diam}\operatorname{supp} \psi_\lambda \leq \sqrt{\sum_{i=1}^d C_i^2 2^{-2|\lambda_i|}} \leq C\sqrt{d}2^{-[\lambda]}, \quad C = \max_{i=1,\ldots,d} C_i. \tag{1.50}$$

In this case, basis functions are not local in the sense that $\operatorname{diam}\operatorname{supp} \psi_\lambda \leq C2^{-|\lambda|}$, but only in the sense that (1.50) holds. We define the set $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\}$, and the set

$$\Psi^J = \{\psi_\lambda : \lambda = (\lambda_1, \ldots, \lambda_d), \; |\lambda_i| \leq J\}. \tag{1.51}$$

If we start with a univariate Riesz basis in the space $L^2(0,1)$, then the set Ψ is a Riesz basis of the space $L^2(\Omega)$, see e.g. [48]. This approach also preserves the properties of the univariate basis, such as polynomial exactness, smoothness of basis functions, and vanishing moments, but as already mentioned the resulting functions are local only in the sense of (1.50). For more details see [35, 45, 48]. In this thesis, we used anisotropic wavelet bases in [15, 17, 19].

1.6 Wavelet-Galerkin Method

In this section, we recall the wavelet-Galerkin method for solving operator equations. Let Ω be a bounded domain, and let $H \subset L^2(\Omega)$ be a separable Hilbert space with the norm $\|\cdot\|_H$. We denote a dual space to H as H' and by $\langle \cdot, \cdot \rangle$ we denote a duality product. For an operator $\mathcal{A} : H \to H'$ and given $f \in H'$ we consider an operator equation

$$\mathcal{A}u = f. \tag{1.52}$$

We define a corresponding bilinear form $a : H \times H \to \mathbb{R}$ by

$$a(u,v) = \langle \mathcal{A}u, v \rangle \quad \forall\, u, v \in H. \tag{1.53}$$

The variational problem becomes: Given $f \in H'$, find $u \in H$ such that

$$a(u,v) = \langle f, v \rangle \quad \forall\, v \in H. \tag{1.54}$$

Let Ψ be a family of functions such that Ψ normalized in the H-norm is a wavelet basis of H. Let $\Psi^k \subset \Psi$ be a multiscale basis of the form (1.10) that contains scaling functions at a coarsest level j_0 and wavelets up to level k. Let us assume that the spaces $X_k = \operatorname{span} \Psi^k$ form a multiresolution analysis in H.

The Galerkin formulation of (1.54) reads as: Find $u_k \in X_k$ such that

$$a(u_k, v) = \langle f, v \rangle \quad \forall v \in X_k. \tag{1.55}$$

We focus on the case where the bilinear form a is continuous and coercive. Recall that a bilinear form $a : H \times H \to \mathbb{R}$ is called *continuous* if there exists a constant C such that

$$|a(u,v)| \leq C \|u\|_H \|v\|_H \;\; \forall\, u, v \in H, \tag{1.56}$$

and a is called *coercive* if there exists a constant $\alpha > 0$ such that

$$a(u,u) \geq \alpha \|u\|_H^2 \;\; \forall\, u \in H. \tag{1.57}$$

Under these assumptions, the existence and uniqueness of the solutions of equations (1.54) and (1.55) are a consequence of the Lax–Milgram theorem, see [27, 62].

Theorem 6. Lax–Milgram
Let H be a Hilbert space, let the bilinear form $a : H \times H \to \mathbb{R}$ be continuous and coercive with constants C and α as in (1.56) and (1.57), respectively, and let $f \in H'$. Then the solution u of the equation

$$a(u,v) = \langle f, v\rangle \quad \forall\, v \in H \tag{1.58}$$

exists and is unique, and the stability estimate

$$\|u\|_H \leq \frac{C}{\alpha} \|f\|_{H'} \tag{1.59}$$

holds.

The Lax–Milgramn theorem guaranties existence and uniqueness for solution u of the variational problem (1.54) as well as the existence and uniqueness of the approximate solution u_k by the Galerkin method.

Now, we study the convergence rate of the Galerkin method.

Theorem 7. Céa's lemma
If the bilinear form $a : H \times H \to \mathbb{R}$ is continuous and coercive with constants C and α as in (1.56) and (1.57), then

$$\|u - u_k\|_H \leq \frac{C}{\alpha} \inf_{v \in X_k} \|u - v\|_H \,. \tag{1.60}$$

Hence, Céa's lemma shows that the convergence rate of the Galerkin method depends on the approximation power of the spaces X_k. The term

$$E_k(u) = \inf_{v \in X_k} \|u - v\|_H \tag{1.61}$$

is known as the *error of the best approximation* in H. The study of this error is a subject of approximation theory. Nowadays, approximation order is known for several kinds of spaces X_k.

For instance, one starts with a univariate wavelet basis corresponding to a multiresolution analysis formed by spaces

$$V_k = \left\{ v \in C^m(0,1) : v|_{\left(\frac{l}{2^k}, \frac{l+1}{2^k}\right)} \in \Pi_r \left(\frac{l}{2^k}, \frac{l+1}{2^k} \right), l = 0, \ldots, 2^k - 1 \right\}, \tag{1.62}$$

where $0 < m < r$, $\Pi_r(a,b)$ is the space of all polynomials on (a,b) of degree less than r, and $C^m(0,1)$ is the space of m-times continuously differentiable functions on $(0,1)$. If then multiscale bases Ψ^k on Ω are constructed using an isotropic or anisotropic tensor product of bases of the spaces V_k, then the spaces $X_k = \text{span}\, \Psi^k$ satisfy

$$\inf_{v \in X_k} \|u - v\|_{H^s} \leq C 2^{-(r-s)k} |u|_{H^r}, \tag{1.63}$$

for any $u \in H^r(\Omega)$ provided that $0 \leq s < r$ and X_k is contained in $H^s(\Omega)$. Here, we view $H^s(\Omega)$ for $s = 0$ as the space $L^2(\Omega)$. Similar results hold for spaces of piecewise polynomial functions incorporating boundary conditions. Hence, $r = 3$ for the Galerkin method with quadratic spline wavelets from [13, 19], and $r = 4$ for the Galerkin method with cubic spline wavelets from [11, 14, 15, 17, 23].

From Theorem 6 and Theorem 7, the convergence rate depends on the chosen discretization spaces and not directly on the chosen bases of these spaces. Since a scaling basis Φ_k generates the same spaces as a multiscale basis Ψ^k, it can be expected that the error will be similar. However, the Galerkin method with a wavelet basis, called the wavelet-Galerkin method, has several advantages. This method seems to be superior to classical methods especially for operator equations with an integral term, because the discretization matrices can be approximated by sparse matrices while most other methods lead to full matrices, see [2, 23, 25]. The second advantage is that a simple diagonal preconditioner is optimal in the sense that diagonally rescaled discretization matrices have uniformly bounded condition numbers. This affects the number of iterations needed to resolve the problem with a desired accuracy. Finally, the solution has a sparse representation in a wavelet basis, which can be used for adaptive versions of the wavelet-Galerkin method that are based on analysis of the size of wavelet coefficients, see e.g. [72], or on a priori knowledge of singularity regions as we did in [20].

We write the function u_k as

$$u_k = \sum_{\psi_\lambda \in \Psi^k} c_\lambda^k \psi_\lambda. \tag{1.64}$$

Let the matrix $\mathbf{A}^k$ and the vector $\mathbf{f}^k$ have entries

$$\mathbf{A}_{\mu,\lambda}^k = a(\psi_\lambda, \psi_\mu), \quad f_\mu^k = \langle f, \psi_\mu \rangle, \quad \psi_\lambda, \psi_\mu \in \Psi^k, \tag{1.65}$$

and $\mathbf{c}^k$ be the column vector of coefficients c_λ^k. Substituting (1.64) into (1.55), we obtain the system

$$\mathbf{A}^k \mathbf{c}^k = \mathbf{f}^k. \tag{1.66}$$

Preconditioning. We apply the standard Jacobi diagonal preconditioning to the system (1.66). Let $\mathbf{D}^k$ be a diagonal matrix with diagonal elements

$$\mathbf{D}_{\lambda,\lambda}^k = \sqrt{\mathbf{A}_{\lambda,\lambda}^k} = \sqrt{a(\psi_\lambda, \psi_\lambda)}. \tag{1.67}$$

Then, we obtain the preconditioned system

$$\tilde{\mathbf{A}}^k \tilde{\mathbf{c}}^k = \tilde{\mathbf{f}}^k \tag{1.68}$$

with

$$\tilde{\mathbf{A}}^k = \left(\mathbf{D}^k\right)^{-1} \mathbf{A}^k \left(\mathbf{D}^k\right)^{-1}, \quad \tilde{\mathbf{f}}^k = \left(\mathbf{D}^k\right)^{-1} \mathbf{f}^k, \quad \tilde{\mathbf{c}}^k = \mathbf{D}^k \mathbf{c}^k. \tag{1.69}$$

The resulting system can be large, and therefore it is usually solved by an appropriate iterative method such as the method of generalized residuals or, in the case where the system matrix is symmetric and positive definite, one can use the conjugate gradient method.

Due to coercivity of the bilinear form a, the matrices $\tilde{\mathbf{A}}^k$ have uniformly bounded condition numbers (cond), i.e. there exists a constant C such that $\operatorname{cond} \tilde{\mathbf{A}}^k \leq C$ for all $k \geq j_0$, see [28, 35, 52].

Sparsity of discretization matrices. If $\mathcal{A}$ is a differential operator, then it may be convenient to compute the discretization matrix $\langle \mathcal{A}\Phi_J, \Phi_J \rangle$ in a scaling basis first, because this matrix is banded. However, the condition numbers of these matrices are not uniformly bounded. Therefore, we use the fast wavelet transform on its rows and columns to transform it to the matrix $\mathbf{A}^k$ for a wavelet basis. Using this approach, wavelets are not used directly in the computation, and the fast wavelet transform can be viewed together with diagonal rescaling (1.69) as optimal preconditioning of the system. For more details see e.g. [15, 72].

If $\mathcal{A}$ is an integral or integro-differential operator, then a discretization matrix in a scaling basis is typically full, and in this case, it is more convenient to compute entries of the matrix $\mathbf{A}^k$ directly rather than to use FWT. For a large class of integral operators this matrix can be approximated by a sparse or quasi-sparse matrix. Several estimates for decay estimates of these matrices are known [2, 23] that make it possible to compute only significant entries of these matrices.

We used the wavelet-Galerkin method in [15, 19, 23]. In [15, 19] we used a modification of the wavelet-Galerkin method called *multilevel Galerkin method* which first computates the solutions of (1.68) on some coarse scale and then use this solution to define an initial vector of the iterative method when solving the discrete problem on some finer scale. In [23] we used the Crank-Nicolson scheme for time discretization and the wavelet-Galerkin method for spatial discretization of the parabolic partial integro-differential equation representing Kou's model for option pricing. In [20], we proposed an adaptive version of the wavelet-Galerkin method for the numerical solution of differential equation with the Dirac measure on the right-hand side.

1.7 Adaptive Wavelet Methods

We briefly review a class of adaptive methods that were originally designed by A. Cohen, W. Dahmen, and R. DeVore in [29, 30] and later modified in many papers [7, 12, 33, 34, 41, 43, 53, 67, 72]. The results presented in this section are known and fuller details can be found in these papers.

While the classical adaptive methods use refining a mesh according to a posteriori local error estimates, the wavelet approach is different and comprises the following steps:

- We start with a variational formulation but instead of its finite dimensional approximation as in the case of the wavelet-Galerkin method, we expand the solution in a wavelet basis and transform the continuous problem into an infinite-dimensional l^2–problem.
- We propose an iteration scheme for the infinite-dimensional problem.
- We replace all infinite-dimensional quantities by finitely supported ones, and we design the routine for an approximate multiplication of an infinite matrix and a finite vector.

As in the previous section, we consider the problem (1.52) and the corresponding variational formulation (1.54). We focus on the case that the bilinear form $a : H \times H \to \mathbb{R}$ is symmetric, continuous, and coercive. Then, by the Lax-Milgram theorem, the problem (1.54) has a unique solution.

Let Ψ, when normalized with respect to the H-norm, be a wavelet basis in H. Let $\mathbf{D}$ be a bi-infinite diagonal matrix with diagonal elements

$$\mathbf{D}_{\lambda,\lambda} = \sqrt{a\left(\psi_\lambda, \psi_\lambda\right)}, \quad \psi_\lambda \in \Psi. \tag{1.70}$$

Then the original equation (1.52) can be reformulated as an equivalent bi-infinite matrix equation

$$\mathbf{Ac} = \mathbf{f}, \tag{1.71}$$

where $\mathbf{A} = \mathbf{D}^{-1} \langle A\Psi, \Psi \rangle \mathbf{D}^{-1}$ is a diagonally preconditioned discretization matrix, $u = \mathbf{c}^T \mathbf{D}^{-1} \Psi$, and $\mathbf{f} = \mathbf{D}^{-1} \langle f, \Psi \rangle$. Then u solves (1.52) if and only if $\mathbf{c}$ solves the matrix equation (1.71). Moreover, the condition number of the matrix $\mathbf{A}$ is finite.

The simplest convergent iteration for the l^2-problem (1.71) is a *Richardson iteration* which has the following form

$$\mathbf{c}_0 = 0, \quad \mathbf{c}_{n+1} = \mathbf{c}_n + \omega\left(\mathbf{f} - \mathbf{A}\mathbf{c}_n\right), \quad n = 0, 1, \ldots. \tag{1.72}$$

The method is convergent if $0 < \omega < 2/\lambda_{max}(\mathbf{A})$, where $\lambda_{max}(\mathbf{A})$ is the largest eigenvalue of $\mathbf{A}$. It is known that the optimal relaxation parameter ω and the corresponding estimate of the error reduction are given by

$$\omega = \frac{2}{\lambda_{min}(\mathbf{A}) + \lambda_{max}(\mathbf{A})}, \quad \rho = \frac{\text{cond }(\mathbf{A}) - 1}{\text{cond }(\mathbf{A}) + 1}, \tag{1.73}$$

where $\lambda_{min}(\mathbf{A})$ is the smallest eigenvalue of $\mathbf{A}$. Then,

$$\left\|\mathbf{c}_{n+1} - \mathbf{c}\right\| \le \rho \left\|\mathbf{c}_n - \mathbf{c}\right\|. \tag{1.74}$$

Hence, the small condition number of $\mathbf{A}$, which depends on the chosen wavelet basis, guaranties the small value of a reduction parameter ρ.

Structure of the discretization matrix. Since the matrices $\tilde{\mathbf{A}}^k$ defined by (1.69) arising from discretization using the wavelet-Galerkin method are submatrices of biinfinite matrix $\mathbf{A}$, the matrices $\tilde{\mathbf{A}}^k$ and $\mathbf{A}$ have similar structure. For differential equations the discretization matrices typically have a so-called finger pattern, see e.g. [17]. Therefore these matrices are quasi-sparse, i.e. they have $\mathcal{O}(N \log N)$ nonzero entries, where $N \times N$ is the size of the matrix. For equations containing the integral term the discretization matrices can be approximated by sparse or quasi-sparse matrices. In some papers, e.g. [17, 45], a construction of a wavelet basis was proposed which leads to discretization matrices that are truly sparse, i.e. they have $\mathcal{O}(N)$ nonzero entries.

Coarsening of vectors. To control the number of degrees of freedom in the algorithm, one needs a routine for approximation of a vector $\mathbf{v}$ by its N-term approximation, i.e. $\mathbf{v}_N$ is obtained by retaining the N largest components of $\mathbf{v}$. It can be done simply by sorting and thresholding as in the following algorithm.

COARSE[**v**, N] $\rightarrow \mathbf{v}_N$

1. Sort $|v_\lambda|$, $\lambda \in \mathcal{J}$, in descending order and denote the resulting vector as $\tilde{\mathbf{v}}$.
2. Denote the N-th element of $\tilde{\mathbf{v}}$ as P.
3. If $|v_\lambda| \geq P$ then set $(\mathbf{v}_N)_\lambda = v_\lambda$, else set $(\mathbf{v}_N)_\lambda = 0$.

The sorting of all nonzero elements of $\mathbf{v}$ requires $N_\mathbf{v} \log N_\mathbf{v}$ arithmetic operations, where $N_\mathbf{v} = \#\text{supp}\, \mathbf{v}$. However, it is possible to avoid sorting to obtain the algorithm with linear complexity. Such algorithm uses so-called *binning* and can be found in Stevenson [67].

Approximation of the right-hand side. We assume that it is possible to compute the vector $\mathbf{f} = \mathbf{D}^{-1} \langle f, \Psi \rangle$ of wavelet coefficients of the right-hand side $f \in H'$ with a desired accuracy. More precisely, we require that for any $\epsilon > 0$, there exists a finitely supported vector $\mathbf{f}_\epsilon \in l^2(\mathcal{J})$, such that

$$\|\mathbf{f} - \mathbf{f}_\epsilon\| \leq \epsilon. \tag{1.75}$$

In the following, the computation of $\mathbf{f}_\epsilon$ will be referred to as the routine **RHS** $[\mathbf{f}, \epsilon] \rightarrow \mathbf{f}_\epsilon$. This can be realized by computing a highly accurate approximation to $\mathbf{f}$ as a pre-processing step and then applying the routine **COARSE** to this finitely supported array of coefficients.

Matrix vector multiplication. Solution of the equation (1.71) by some iterative method requires a multiplication of the infinite-dimensional matrix $\mathbf{A}$ with a finitely supported vector $\mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}$. There are several routines available. Here, we present the routine **APPLY** that we proposed in [12]. The idea is the following: We truncate $\mathbf{A}$ in scale by zeroing its entries $\mathbf{A}_{\lambda,\mu}$ whenever $||\lambda| - |\mu|| > k$, $k \in \mathbb{N} \cup \{0\}$, and denote the resulting matrix by $\mathbf{A}_k$. Let us denote $S_{\mathbf{A}_k} = \max\{|\mathbf{A}_{\lambda,\mu}|, ||\lambda| - |\mu|| = k\}$. Then we multiply the matrix $\mathbf{A}_0$ with vector entries that are greater than given tolerance ϵ, the matrix $\mathbf{A}_1 - \mathbf{A}_0$ with vector entries that are greater than $\epsilon / S_{\mathbf{A}_1}$, ..., and the matrix $\mathbf{A}_K - \mathbf{A}_{K-1}$ with vector entries that are greater than $\epsilon / S_{\mathbf{A}_K}$. In the case that $S_{\mathbf{A}_k} = 0$ for some k, we can formally define $\epsilon / S_{\mathbf{A}_k} = \infty$ and no multiplications with matrix $\mathbf{A}_k - \mathbf{A}_{k-1}$ are necessary. More precisely, let

$$\mathbf{z}_k^\epsilon = \left\{ v_\lambda : |v_\lambda| > \frac{\epsilon}{S_{\mathbf{A}_k}} \right\}, \quad \mathbf{z}_{-1}^\epsilon = \mathbf{0}, \quad \mathbf{v}_k^\epsilon = \mathbf{z}_k^\epsilon - \mathbf{z}_{k-1}^\epsilon, \quad \tilde{\mathbf{w}}^\epsilon = \sum_{k=0}^{K} \mathbf{A}_{K-k} \mathbf{v}_k^\epsilon. \quad (1.76)$$

The parameter K is the smallest number such that $\mathbf{v}_k^\epsilon$ is an empty set for all $k > K$. In [12] we proposed two algorithms based on these ideas. We present one of them below.

APPLY $[\mathbf{A}, \mathbf{v}, \epsilon] \to \mathbf{w}_\epsilon$
For $j \in \mathbb{N} \cup \{0\}$, let e_j be such that $\|\mathbf{A} - \mathbf{A}_j\| \le e_j$.
1. Set $S_{\mathbf{A}_k} := \max\{|\mathbf{A}_{\lambda,\mu}|, \, ||\lambda| - |\mu|| = k\}$.
2. Set $G = 1.1$ and $\delta = \lfloor \log_G \epsilon \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function. Compute $\mathbf{w}_1 := \tilde{\mathbf{w}}^{G^\delta}$ and $\mathbf{w}_2 := \tilde{\mathbf{w}}^{2G^\delta}$ according to (1.76).
3. While $\|\mathbf{w}_1 - \mathbf{w}_2\| > \epsilon$
   $\delta := \delta - 1$
   Compute $\mathbf{w}_1 := \tilde{\mathbf{w}}^{G^\delta}$ and $\mathbf{w}_2 := \tilde{\mathbf{w}}^{2G^\delta}$ using (1.76).
   end while.
4. $\mathbf{w}^\epsilon := \mathbf{w}_1$.

Algorithm SOLVE. Since we consider here a class of adaptive methods, there are many algorithms representing these methods. We present one example of such an algorithm that we used in [19].

The method insists in solving the infinite preconditioned system (1.71) with Richardson iterations. We compute the relaxation parameter ω and the error reduction factor ρ by (1.73). Then we set $\theta = 0.3$ and $K \in \mathbb{N}$ such that $2\rho^K/\theta < 0.6$.

The resulting algorithm is of the form:

SOLVE $[\mathbf{A}, \mathbf{f}, \epsilon] \to \mathbf{c}_\epsilon$
1. Set $j := 0$, $\mathbf{u}_0 := 0$, and $\epsilon_0 \ge \|\mathbf{c}\|_2$.
2. While $\epsilon_j > \epsilon$ do
   $\mathbf{z}_0 := \mathbf{c}_j$,
   For $l = 1, \ldots, K$ do
   $\mathbf{z}_l := \mathbf{z}_{l-1} + \omega \left(\mathbf{RHS}[\mathbf{f}, \frac{\epsilon_j \rho^l}{2\omega K}] - \mathbf{APPLY}[\mathbf{A}, \mathbf{z}_{l-1}, \frac{\epsilon_j \rho^l}{2\omega K}] \right)$,
   end for,
   $j := j + 1$
   $\epsilon_j := \frac{2\rho^K \epsilon_{j-1}}{\theta}$,
   $\mathbf{c}_j := \mathbf{COARSE}[\mathbf{z}_K, (1-\theta)\,\epsilon_j]$,
   end while,
   $\mathbf{c}_\epsilon := \mathbf{c}_j$.

It is known that the coefficients of a function in the wavelet basis are small in regions where the function is smooth and large in regions where the function has some singularity or a large gradient. Since we work with a sparse representation of the right-hand side and a sparse representation of the vector representing the solution, the method is adaptive.

For analysis of the method we refer to e.g. [30, 41, 72]. Roughly speaking, the methods converge with the same rate as the wavelet-Galerkin method, but for a wider class of functions, because the error estimates are derived in Besov spaces and not only in Sobolev spaces. Other advantages are a small number of parameters representing the solution with desired accuracy, asymptotical optimality in the sense that the number of floating point operations depend linearly on the number of degrees of freedom, optimality of diagonal preconditioner, a sparse structure of matrices also for equations containing an integral term, and a higher-order convergence if higher-order basis functions are used.

In this thesis, we used adaptive wavelet methods in [10, 11, 13, 14, 17, 19].

Chapter 2

Constructions of Quadratic and Cubic Spline-Wavelet Bases

Since this thesis is a collection of eight articles that are all concerned with constructions of quadratic or cubic spline-wavelet bases on the interval, we review here existing constructions of such types of bases, discuss their advantages and disadvantages, and comment on the papers presented in this thesis. We also review applications of these bases.

We focus on concrete quadratic and cubic spline wavelet bases for which the Riesz basis property was proven. There also exist general methods for construction of wavelet bases on the interval, e.g. [51], spline wavelets without adaptation to boundary conditions and without the proof of the Riesz basis property, e.g. [59, 66], and quadratic and cubic finite element wavelets [39].

2.1 Quadratic Spline-Wavelet Bases

In [38, 40], W. Dahmen, A. Kunoth and K. Urban proposed a construction of a spline-wavelet biorthogonal wavelet basis on the interval. The inner wavelets were the same as wavelets from [32], where wavelet bases were constructed on the whole real line. The order of spline is any $N \geq 1$, and the number of vanishing moments is $L \geq N$ such that $N + L$ is even. Both the primal and dual wavelets are local. A disadvantage of these bases is their relatively large condition number. Therefore many modifications of this construction were proposed, see e.g. [1, 3, 5, 70]. The construction by M. Primbs [63] outperforms previous constructions for the linear and quadratic spline-wavelet bases with respect to their conditioning. In [8, 10, 43] the construction was significantly improved in the case of cubic spline wavelet basis, but the condition numbers of quadratic spline-wavelet bases was comparable to those constructed by M. Primbs. In the case of quadratic spline wavelet basis adapted to homogeneous Dirichlet boundary conditions, these bases are even the same up to a normalization, see also the comparison in [19]. In [46], a method for a construction of the L^2-orthogonal wavelet basis on the real line was proposed starting from a non-orthogonal wavelet basis. In [64], the L^2-orthogonal spline-wavelet bases on the unit interval were constructed using this method.

Quadratic spline wavelet bases with nonlocal duals have also been constructed and adapted to some types of boundary conditions [26, 56]. The main advantages of these types of bases in comparison to bases with local duals are usually the shorter support of wavelets, the lower condition numbers of the bases and the corresponding stiffness matrices, and also the simplicity of the construction.

In [13, 19], we also constructed quadratic spline-wavelet bases on the interval. Here, we comment on these constructions.

[13] **Černá, D.; Finěk, V.: *Quadratic spline wavelets with short support for fourth-order problems*, Result. Math. 66(6), (2014), pp. 525–540.**

In [13], we proposed two constructions of quadratic spline-wavelet bases for the space $H_0^2(0,1)$. The inner wavelets have one vanishing moment and boundary wavelets are of two types, wavelets with one vanishing moment and wavelet with shorter support but without vanishing moments. Since we did not require local support of dual wavelets, we were able to construct wavelets with a short support of length 2, which is the shortest possible support for wavelets with one vanishing moment corresponding to the quadratic B-spline multiresolution analysis. We used the isotropic tensor product to obtain a wavelet basis for the space $H_0^2\left((0,1)^2\right)$. We studied the quantitative behaviour of the adaptive wavelet method for the numerical solution of the fourth-order differential equation $\Delta^2 u = f$ on the unit square, Δ being the Laplace operator. Due to the short support, the discretization matrices are sparser than for other quadratic spline wavelets of the same type. The condition numbers of discretization matrices are uniformly bounded and small, e.g. for the stiffness matrix of the size 64516×64516 the condition number is 11.1.

[19] **Černá, D.; Finěk, V.: *Quadratic spline wavelets with short support satisfying homogeneous boundary conditions*, Electron. Trans. Numer. Anal. 48, (2018), pp. 15–39.**

In [19], we constructed wavelets of the similar type as in the previous paper, but adapted to the first-order homogeneous Dirichlet boundary conditions, i.e. quadratic spline wavelets on the interval and on a unit square with one vanishing moment and the shortest possible support. The matrices arising from discretization of the second-order elliptic problems using the constructed wavelet basis have uniformly bounded condition numbers and the condition numbers are small, e.g. the condition number was 2.84 for a matrix of the size 1024×1024 corresponding to one-dimensional Poisson's equation, and it was 18.3 for the matrix of the size 1048576×1048576 corresponding to a two-dimensional Poisson's equation. We also provided numerical examples to show that the Galerkin method and the adaptive wavelet method using our wavelet basis require a smaller number of iterations than methods with other quadratic spline wavelet bases of the same type, i.e. bases from [3, 26, 43, 63]. Moreover, due to the short support of our wavelets, one iteration requires a smaller number of floating-point operations than for these bases.

In [20], we propose post-processing for the Galerkin method with this basis, such that the resulting method has a convergence rate the same as the rate of convergence for the Galerkin method with cubic spline wavelets under the assumption that the solution is smooth enough. We show theoretically as well as numerically that the presented method outperforms the Galerkin method with many other quadratic or cubic spline wavelets with respect to the number of floating point operations needed to compute a sufficiently accurate solution. Furthermore, we proposed local post-processing for example with an equation with Dirac measure on the right-hand side.

In Table 2.1, we list parameters and properties for several constructions of quadratic spline-wavelet bases such as the order of homogeneous Dirichlet boundary conditions (bound. cond.), the number of vanishing moments (vanish. moments), the maximal length of the support of generators of inner scaling functions (supp scal.), the maximal length of the support of generators of inner wavelets (supp wav.), locality of duals (loc. duals), the number of generators of inner scaling functions (scal. gen.), the number of generators of inner wavelets (wav. gen.), and special properties. The property *short. sup.* means that a wavelet basis is such that wavelets have the shortest possible support among all wavelets with the same number of vanishing moments corresponding to the same scaling basis. The other very important parameters that characterize wavelet bases are the condition number of the basis and the condition numbers of discretization matrices. These numbers are problem dependent and can be found in the attached papers.

Table 2.1: Parameters characterizing quadratic spline-wavelet bases.

wavelet basis	bound. cond.	vanish. moments	supp scal.	supp wav.	loc. duals	scal. gen.	wav. gen.	special property
DKU [38]	0	$L \geq 3$ odd	3	$L+2$	loc.	1	1	
D [43]	≥ 0	$L \geq 3$ odd	3	$L+2$	loc.	1	1	
B [3]	0	$L \geq 3$ odd	3	$L+2$	loc.	1	1	
P [63]	0-1	$L \geq 3$ odd	3	$L+2$	loc.	1	1	
CF [10]	0-1	$L \geq 3$ odd	3	$L+2$	loc.	1	1	
R [64]	1	3	2	2	loc.	6	6	L^2-orth.
CQ [26]	0	3	3	5	glob.	1	1	semiorth.
J [56]	2	1	3	3	glob.	1	1	
J [56]	2	3	3	5	glob.	1	1	
CF [13]	2	1	3	2	glob.	1	1	short. sup.
CF [19]	1	1	3	2	glob.	1	1	short. sup.

2.2 Cubic Spline-Wavelet Bases

As already mentioned in the previous section, biorthogonal cubic spline-wavelet bases with local support of primal and dual wavelets were constructed in [38, 40], and this construction was modified in several papers. In [10, 43] the construction was significantly improved with regard to conditioning of the bases. Biorthogonal cubic Hermite spline multiwavelet bases on the interval with local duals were designed in

[40, 65]. The L^2-orthogonal piecewise cubic basis was constructed in [64] using the method from [46].

Several cubic spline wavelet and multiwavelet bases with nonlocal duals have been constructed and adapted to various types of boundary conditions in [26, 45, 55, 56, 57]. Their properties are summarized in Table 2.2. Similarly to the case of quadratic spline wavelets the main advantages of these types of bases in comparison with bases with local duals are usually shorter supports of wavelets, lower condition numbers of the bases and corresponding discretization matrices, and also simplicity of the construction.

Below we comment on the constructions of cubic spline wavelets that are presented in the papers [10, 11, 14, 15, 17, 23] collected in this thesis.

[10] **Černá, D.; Finěk, V.: *Construction of optimally conditioned cubic spline wavelets on the interval*, Adv. Comput. Math. 34(2), (2011), pp. 219–252.**

In [10], we constructed biorthogonal spline-wavelet bases such that both primal and dual wavelets are local and they have the desired number of vanishing wavelet moments. Inner wavelets are translated and dilated versions of the well-known wavelets designed by A. Cohen, I. Daubechies, and J.-C. Feauveau in [32]. Our objective was to construct interval spline-wavelet bases with condition numbers close to the condition numbers of spline wavelet bases on the real line, especially in the case of cubic spline wavelets. We showed that the constructed set of functions is indeed a Riesz basis for the space $L^2(0,1)$ and for the Sobolev space $H^s(0,1)$ for a certain range of s. Then we adapted the primal bases to the homogeneous Dirichlet boundary conditions of the first order and the dual bases to complementary boundary conditions. We compared the efficiency of an adaptive wavelet scheme for our wavelets and cubic spline wavelets constructed in [63] by M. Primbs and we showed the superiority of our construction. Numerical examples are presented for the one-dimensional and two-dimensional Poisson's equations where the solution has steep gradients.

[11] **Černá, D.; Finěk, V.: *Cubic spline wavelets with complementary boundary conditions*, Appl. Math. Comput. 219(4), (2012), pp. 1853–1865.**

In [11], we focused on a construction of a cubic spline-wavelet basis on the interval with local duals satisfying complementary boundary conditions of the second order. This means that a primal wavelet basis is adapted to homogeneous Dirichlet boundary conditions of the second order, while the dual wavelet basis preserves the full degree of polynomial exactness. We showed superiority of our construction in comparison to spline wavelet bases of the same type, i.e. those from [63, 65], with respect to conditioning of wavelet bases and the number of iterations in an adaptive wavelet method for the numerical solution of the partial differential equation $\Delta^2 u = f$ in two dimensions. For example, the discretization matrix for this problem in one dimension when a basis with seven levels of wavelets is used has the condition number 66.7 for our basis, 693.0 for the basis from [65], and 1117.0 for the basis from [63]. Hence, this basis can be recommended for problems where local duals are needed and second-order Dirichlet boundary condition are prescribed.

[14] **Černá, D.; Finěk, V.:** ***Cubic spline wavelets with short support for fourth-order problems*****, Appl. Math. Comput. 243, (2014), pp. 44–56.**

In [14], we proposed a construction of new cubic spline-wavelet bases on the unit cube satisfying homogeneous Dirichlet boundary conditions of the second order. Wavelets have short supports of the length 3 and two vanishing moments. In this paper, we were inspired by the construction of cubic spline-wavelet basis satisfying similar type of boundary conditions proposed by R.Q. Jia and W. Zhao in [57], where the wavelets have no vanishing moments. They used their basis for solving fourth-order problems, and they showed that the Galerkin method with this basis has superb convergence and it outperforms the Galerkin method with cubic splines preconditioned using BPX preconditioner or multigrid method. The discretization matrices for the equation $\Delta^2 u = f$ on a unit square have very small and uniformly bounded condition numbers. In our paper [14], we designed wavelet bases with the same scaling functions, but with different wavelets. We showed that our basis has an even smaller condition number than the basis in [57] and additionally the wavelets have vanishing moments. For example, the condition number of the discretization matrix of the size 65025×65025 was 18.6. Vanishing moments enable the use of this wavelet basis in adaptive wavelet methods and wavelet-based methods for equations with an integral term.

[15] **Černá, D.; Finěk, V.:** ***Wavelet basis of cubic splines on the hypercube satisfying homogeneous boundary conditions*****, Int. J. Wavelets Multiresolut. Inf. Process. 13(3), (2015), article No. 1550014.**

In [15], we proposed a construction of new cubic spline wavelets on the hypercube that have two vanishing moments and satisfy first-order homogeneous Dirichlet boundary conditions. In comparison with [14] where the duals are not discussed, here we defined dual spaces as linear spline spaces. We defined bases of dual scaling spaces that have compact support and used them for the proof of the Riesz basis property. The biorthogonal wavelet basis contains functions with global support. The matrices arising from discretization of second-order elliptic problems using a constructed wavelet basis have uniformly bounded condition numbers and these condition numbers are relatively small. We constructed wavelet bases on the hypercube using both isotropic and anisotropic tensor product, studied condition numbers of discretization matrices corresponding to the Helmholtz equation with various parameters, and we provided a numerical example to show the efficiency of the multilevel-Galerkin method using the constructed basis. This basis was studied by L. Calderón, M.T. Martín, and V. Vampa in [6]. They used our basis in numerical experiments and showed that the additional advantage is that the stiffness matrix corresponding to the one-dimensional Poisson's equation is banded. This also affects the structure of discretization matrices for the Helmholtz equation in higher dimensions, because they are computed using tensor products of the stiffness matrix and the mass matrix.

[17] **Černá, D.; Finěk, V.: *Sparse wavelet representation of differential operators with piecewise polynomial coefficients*, Axioms 6, (2017), article No. 4.**

In [17], we proposed a construction of a Hermite cubic spline-wavelet basis on the interval and hypercube. The basis is adapted to homogeneous Dirichlet boundary conditions. We focused on the structure of discretization matrices rather than on the length of the support and the condition number of the basis as in the previous papers. Here, the wavelets are orthogonal to piecewise polynomials of degree at most seven on a uniform grid. Therefore the wavelets have eight vanishing moments and the matrices arising from discretization of differential equations with coefficients that are piecewise polynomials of degree at most four on uniform grids are sparse and not only quasi-sparse as for most wavelet bases. This greatly simplifies the routine APPLY needed for computation of the multiplication of a biinfinite matrix with a finitely supported vector in adaptive wavelet methods. Numerical examples showed the efficiency of an adaptive wavelet method with the constructed wavelet basis for solving a second-order elliptic equation and the Black–Scholes equation with two state variables and quadratic volatility.

[23] **Černá, D.: *Cubic spline wavelets with four vanishing moments on the interval and their applications to option pricing under Kou model*, Int. J. Wavelets Multiresolut. Inf. Process. 17(1), (2019), article No. 1850061.**

As in our paper [14], our aim was to construct cubic spline wavelets with the shortest possible support corresponding to B-spline multiresolution analysis, but with a larger number of vanishing moments, namely four vanishing moments. We constructed bases that satisfies no boundary conditions and bases that satisfy first-order homogeneous Dirichlet boundary conditions. Inner wavelets are the same as inner wavelets for a wavelet basis on the real line constructed in [24, 50]. To illustrate the applicability of the constructed bases we used the wavelet-Galerkin method with our bases to option pricing under the double exponential jump-diffusion model represented by a partial integro-differential equation. We used the Crank-Nicolson scheme for time discretization and the wavelet-Galerkin method for spatial discretization. We compared the results with B-spline bases and cubic spline wavelet bases from [10], because they are adapted to the same type of boundary conditions. Since the equation contains an integral term, most classical methods lead to full matrices. Hence the advantage of the proposed method is the quasi-sparse structure of the discretization matrices. In comparison with methods from [49, 54, 58, 61, 71], the presented method required significantly smaller number of degrees of freedom needed to compute the solution with desired accuracy.

In Table 2.2, we list parameters and properties for cubic spline-wavelet bases. The property *sparse Lapl.* means that matrices arising from discretization of Laplacian are truly sparse and the property *sparse diff.* means that the discretization matrices are sparse for some class of differential operators with piecewise polynomial coefficients.

Table 2.2: Parameters characterizing cubic spline-wavelet bases.

wavelet basis	bound. cond.	vanish. moments	supp scal.	supp wav.	loc. duals	scal. gen.	wav. gen.	special property
DKU [38]	0	$L \geq 4$ even	4	$L+3$	loc.	1	1	
D [43]	$m \geq 0$	$L \geq 4$ even	4	$L+3$	loc.	1	1	
B [3]	0	$L \geq 4$ even	4	$L+3$	loc.	1	1	
P [63]	0-1	$L \geq 4$ even	4	$L+3$	loc.	1	1	
CF [10]	0-1	$L \geq 4$ even	4	$L+3$	loc.	1	1	
CF [11]	2	6	4	9	loc.	1	1	
S [65]	2	2	2	3	loc.	2	2	
R [64]	1	4	2	2	loc.	6	6	L^2-orth.
CQ [26]	0	4	4	7	glob.	1	1	semiorth.
J [56]	3	2	4	5	glob.	1	1	
J [56]	3	4	4	7	glob.	1	1	
JZ [57]	2	0	4	3	glob.	1	1	
JL [55]	1	2	2	2	glob.	2	2	
CF [14]	2	2	4	3	glob.	1	1	short. sup.
CF [15]	1	2	4	5	glob.	1	1	
CF [16]	1	4	2	4	glob.	2	4	sparse Lapl.
CF [17]	1	8	2	8	glob.	2	8	sparse diff.
CF [23]	0-1	4	4	4	glob.	1	1	short. sup.
D [45]	1	4	2	4	glob.	2	4	sparse Lapl.

2.3 Applications of Constructed Bases

Wavelet bases on the interval and product domains are useful in a wide range of applications including signal and image analysis, data compression, and numerical solution of various types of operator equations. In this section, we mention several concrete examples where wavelets constructed in the enclosed papers were used. First, we mention applications from these papers.

Second-order linear elliptic equations

These equations represent a wide range of applications, typically governing equilibrium problems in physics such as displacement of a membrane, electric potential, gravity fields, or pressure fields. We focused on the Poisson and Helmholtz equations, which we solved by the wavelet-Galerkin method in [15, 17] and by the adaptive wavelet method in [10].

Fourth-order linear elliptic problems

These equations arise for example in linear elasticity theory, mechanics of elastic plates, or slow flows of viscous fluids. We solved these equations in [11, 13, 14]. As already mentioned, we were motivated by the results in [57], where a cubic spline-wavelet basis adapted to second-order boundary conditions was constructed. The wavelet-Galerkin method has superb convergence and outperformed BPX preconditioner and multigrid methods, but wavelets did not have vanishing moments. We improved their results in [14], where we constructed wavelets of similar type, but even better conditioned and with vanishing moments. This enabled us to reduce the number of iterations needed to find a sufficiently accurate solution and to apply these wavelets in adaptive wavelet methods.

Option pricing under the Black-Scholes model

In [17], we solved the Black-Scholes equation with two state variables and quadratic volatility by an adaptive wavelet method. The advantages of an adaptive wavelet approach were not only the small number of degrees of freedom needed to find a solution with desired accuracy, but also that the routine APPLY was greatly simplified due to the fact that the discretization matrices have uniformly bounded number of nonzero entries in each row. This is not the case for most other wavelet bases, which have a so-called finger pattern.

Option pricing under a double exponential jump diffusion model

In [23], we studied option pricing under a double exponential jump diffusion model proposed by Kou in [60]. Since this model is represented by partial integro-differential equation, most classical methods suffer from the fact that discretization matrices are full. We used the wavelet-Galerkin method combined with the Crank-Nicolson scheme and showed that the discretization matrices can be approximated by quasi-sparse matrices. Furthermore, we showed that our method enables a solution to the problem with desired accuracy and smaller number of degrees of freedom than methods from [49, 54, 58, 61, 71]. Hence, smaller matrices are involved in computation.

We also used the wavelets from this thesis in applications in our other works. Several of them are mentioned below.

Option pricing under stochastic volatility models

Wavelet methods are a very promising tool for option pricing, for a survey see [52, 64]. We used an adaptive wavelet method with bases constructed in this thesis for option pricing under stochastic volatility models that are improvement of the famous Black and Scholes model, where volatility is a constant or deterministic function. For instance in [21], we used wavelets from [19] for option pricing under the Heston model.

Valuation of Asian options

In [18], we used the adaptive wavelet method with a linear spline-wavelet basis from [10] for valuation of two-asset Asian options with a floating strike. We compared this method with the wavelet-Galerkin method with the same basis, and we found that the adaptive method required a significantly smaller number of degrees of freedom to compute the solution with a desired accuracy. Moreover, the optimal convergence rate with respect to the L^2-norm was achieved for the adaptive wavelet method, while this was not the case of the wavelet-Galerkin method.

Sensitivity analysis of options

In the book [53], we were concerned with option pricing and the numerical computation of the Greeks, i.e. derivatives of the option price with respect to underlying parameters such as the underlying asset price, time to expiration, volatility, and interest rates. The Greeks measure the sensitivity of the option price to these parameters and their computation is important for hedging. Since Greeks are defined as derivatives of option price with respect to these parameters, the convergence rate for the methods for their computation is typically smaller than the convergence rate for the methods for computation of the price of the option. Therefore, it is beneficial to use a higher-order method such as the adaptive method with quadratic and cubic spline-wavelet basis. In [53], comparison with other methods such as the finite difference method, a discontinuous Galerkin method and a fuzzy method was provided. Our method was superior in the sense that it enabled achieving a significantly smaller error for the same number of degrees of freedom as these methods.

Differential equations with Dirac right-hand side

In [20], we proposed an adaptive method that uses wavelets constructed in [19] for the numerical solution of the partial differential equation with the right-hand side that contains the Dirac delta function.

Sparse representation of images and image compression

As already mentioned in Section 1.4, the wavelet bases on the interval lead to the fast wavelet transforms that use special boundary filters. In [8, 9], we used the FWT corresponding to spline wavelets from [10] to sparse representation of images and image compression. We compared our method with methods based on the signal extension such as zero padding, symmetrization, periodization, etc. and we showed that the error near the boundary is significantly smaller for our method and that the method enables to reduce boundary artefacts.

Singularly perturbed boundary value problems

We also used the adaptive wavelet method for the solution of singularly perturbed boundary value problems in [4].

In summary, wavelets on the interval can be used directly in methods for the numerical solution of operator equations and in signal and image processing for decomposition, analysis, and compression. In addition, constructions of wavelets on the interval can be used as the first step of constructions of wavelets on more general domains and constructions of wavelets satisfying some special conditions. For example as mentioned in [69], a construction of divergence-free wavelets starts with the pair of two biorthogonal wavelet bases such as those from [11, 45, 63]. Divergence free wavelets then can be used for the numerical solution of the Navier–Stokes equations representing the flow of viscous fluid. Moreover, wavelets on the interval can be used in many engineering applications, e.g. the method from [13] was used as the part of the algorithm for building venting system on complex surfaces of injection molds in [73]. For other applications of wavelets on the interval we refer to [25, 28, 37, 41, 72].

Conclusions and Further Research

The previous text summarizes the results presented in the attached papers. In these papers, we constructed several quadratic and cubic spline wavelets on the interval and product domains and compared them to existing constructions. All constructed bases were well-conditioned. Bases from [13, 14, 23] have the shortest possible support, therefore discretization matrices are sparse and one iteration of the method for the solution of the resulting discrete system requires less floating point operations than for other wavelets of the same type applied to the same equations. Bases from [17] lead to truly sparse matrices for a class of differential operators with polynomial coefficients, while other wavelet bases lead to only quasi-sparse matrices. For applications, where global support of dual functions is needed we can recommend bases from [10, 11]. We used the constructed basis for the numerical solution of many types of equations, and we presented other possible applications of the bases.

In terms of further research, we would like to extend our previous results to higher-dimensional problems (dimension $d \geq 4$), especially for solving partial integro-differential equations representing pricing multi-asset options under jump diffusion models. Furthermore, we recently constructed wavelet dictionaries for ECG signal modelling in [22]. Here, the dictionaries were constructed from wavelets on the real line simply by restriction. We would also like to focus on a construction of dictionaries that are boundary adapted and compare them with dictionaries based on restriction for ECG signal modelling and also use them in other applications.

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Selected papers

This chapter contains articles [10, 11, 13, 14, 15, 17, 19, 23]. Due to copyright restrictions, we present postprints of the papers [10, 11, 13, 14, 17, 19] and preprints of the papers [15, 23].

Construction of Optimally Conditioned Cubic Spline Wavelets on the Interval

Dana Černá · **Václav Finěk**

Abstract The paper is concerned with a construction of new spline-wavelet bases on the interval. The resulting bases generate multiresolution analyses on the unit interval with the desired number of vanishing wavelet moments for primal and dual wavelets. Both primal and dual wavelets have compact support. Inner wavelets are translated and dilated versions of well-known wavelets designed by Cohen, Daubechies, and Feauveau. Our objective is to construct interval spline-wavelet bases with the condition number which is close to the condition number of the spline wavelet bases on the real line, especially in the case of the cubic spline wavelets. We show that the constructed set of functions is indeed a Riesz basis for the space $L^2([0,1])$ and for the Sobolev space $H^s([0,1])$ for a certain range of s. Then we adapt the primal bases to the homogeneous Dirichlet boundary conditions of the first order and the dual bases to the complementary boundary conditions. Quantitative properties of the constructed bases are presented. Finally, we compare the efficiency of an adaptive wavelet scheme for several spline-wavelet bases and we show a superiority of our construction. Numerical examples are presented for the one-dimensional and two-dimensional Poisson equations where the solution has steep gradients.

Keywords Biorthogonal wavelets · Interval · Spline · Condition number

Mathematics Subject Classification (2000) 65T60 · 65N99

1 Introduction

Wavelets are by now a widely accepted tool in signal and image processing as well as in numerical simulation. In the field of numerical analysis, methods based on wavelets are successfully used especially for preconditioning of large systems arising from discretization of elliptic partial differential equations, sparse representations of some types of operators and adaptive solving of operator equations. The quantitative performance of such methods strongly depends on a choice of a wavelet basis, in particular on its condition number.

Wavelet bases on a bounded domain are usually constructed in the following way: Wavelets on the real line are adapted to the interval and then by a tensor product technique to the n-dimensional cube. Finally by splitting the domain into subdomains which are images of $(0,1)^n$ under appropriate parametric mappings one can obtain wavelet bases on a fairly general domain. Thus, the properties of the employed wavelet basis on the interval are crucial for the properties of the resulting bases on general a domain.

Biorthogonal spline-wavelet bases on the unit interval were constructed in [16]. The disadvantage of them is their bad condition which causes problems in practical applications. Some

D. Černá
Department of Mathematics and Didactics of Mathematics, Technical University of Liberec,
Studentská 2, 461 17 Liberec, Czech Republic
Tel.: +420-48-535-2407
Fax: +420-48-535-2332
E-mail: dana.cerna@tul.cz

V. Finěk
Department of Mathematics and Didactics of Mathematics, Technical University of Liberec,
Studentská 2, 461 17 Liberec, Czech Republic

modifications which lead to better conditioned bases were proposed in [2], [17], [24], and [33]. The recent construction by M. Primbs, see [12], [24], or [25], seems to outperform the previous constructions with respect to the Riesz bounds as well as spectral properties of the corresponding stiffness matrices in the case of linear and quadratic spline-wavelets. In this paper, we focus on cubic spline wavelets and we construct interval spline-wavelet bases with the condition number which is close to the condition number of the spline wavelet bases on the real line. It is known that the condition number of the wavelet basis on the real line is less than or equal to the condition number of the interval wavelet basis, where the inner functions are restrictions of scaling functions and wavelets on the real line.

First of all, we summarize the desired properties:

- *Riesz basis property.* The functions form a Riesz basis of the space $L^2([0,1])$.
- *Locality.* The basis functions are local. Then the corresponding decomposition and reconstruction algorithms are simple and fast.
- *Biorthogonality.* The primal and dual wavelet bases form a biorthogonal pair.
- *Polymial exactness.* The primal MRA has polynomial exactness of order N and the dual MRA has polynomial exactness of order $\tilde{N}$. As in [9], $N+\tilde{N}$ has to be even and $\tilde{N} \geq N$.
- *Smoothness.* The smoothness of primal and dual wavelet bases is another desired property. It ensures the validity of norm equivalences, for details see below.
- *Closed form.* The primal scaling functions and wavelets are known in the closed form. It is a desirable property for the fast computation of integrals involving primal scaling functions and wavelets.
- *Well-conditioned bases.* Our objective is to construct wavelet bases with an improved condition number, especially for larger values of N and $\tilde{N}$.

From the viewpoint of numerical stability, ideal wavelet bases are orthogonal wavelet bases. However, they are usually avoided in the numerical treatment of partial differential and integral equations, because they are not accessible analytically, the complementary boundary conditions can not be satisfied and it is not possible to increase the number of vanishing wavelet moments independent from the order of accuracy. Moreover, sufficiently smooth orthogonal wavelets typically have a large support.

Biorthogonal wavelet bases on the unit interval derived from B-splines were constructed also in [8] and [19] and they were adapted to homogeneous Dirichlet boundary conditions in [20]. These bases are well-conditioned, but have globally supported dual basis functions. Another construction of spline-wavelets was proposed in [4], but the corresponding dual bases are unknown so far. We should also mention the construction of spline multiwavelets [15], [22], and [28], though the dual wavelets have a low Sobolev regularity.

The paper is organized as follows. Section 2 provides a short introduction to the concept of wavelet bases. Section 3 is concerned with the construction of primal multiresolution analysis on the interval. The primal scaling functions are B-splines defined on the Schoenberg sequence of knots, which have been used also in [4], [8], and [24]. In Section 4 we construct dual multiresolution analysis. There are two types of boundary scaling functions. The functions of the first type are defined in order to preserve the full degree of polynomial exactness as in [1] and [10]. The construction of the scaling functions of the second type is a delicate task, because the low condition number and nestedness of the multiresolution spaces have to be preserved. Section 5 is concerned with the computation of refinement matrices. In Section 6 wavelets are constructed by the method of stable completion proposed in [18]. The construction of initial stable completion is along the lines of [16]. In Section 7 we show that the constructed set of functions is indeed a Riesz basis for the space $L^2([0,1])$ and for the Sobolev space $H^s([0,1])$ for a certain range of s. In Section 8 we adapt the primal bases to the homogeneous Dirichlet boundary conditions of the first order and the dual bases to the complementary boundary conditions. Quantitative properties of the constructed bases are presented in Section 9. Finally, in Section 10, we compare the efficiency of an adaptive wavelet scheme for several spline-wavelet bases and we show a superiority of our construction. Numerical examples are presented for one-dimensional and two-dimensional Poisson equations where the solution has steep gradients.

2 Wavelet bases

This section provides a short introduction to the concept of wavelet bases. Let us introduce some notation. We use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ to denote the set of positive integers, integers, rational numbers, and real numbers, respectively. Let $\mathbb{N}_{j_0}$ denote the set of integers which are greater than or equal to j_0.

We consider a domain $\Omega \subset \mathbb{R}^d$ and the space $L^2(\Omega)$ with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Let $\mathscr{J}$ be some index set and let each index $\lambda \in \mathscr{J}$ takes the form $\lambda = (j,k)$, where $|\lambda| = j \in \mathbb{Z}$ is a *scale* or a *level*. Let $l^2(\mathscr{J})$ be a space of all sequences $b = \{b_\lambda\}_{\lambda \in \mathscr{J}}$ such that

$$\|b\|_{l_2(\mathscr{J})} := \left(\sum_{\lambda \in \mathscr{J}} |b_\lambda|^2 \right)^{\frac{1}{2}} < \infty. \tag{1}$$

Definition 1. A family $\Psi := \{\psi_\lambda \in \mathscr{J}\} \subset L^2(\Omega)$ is called a *wavelet basis* of $L^2(\Omega)$, if

i) Ψ is a *Riesz basis* for $L^2(\Omega)$, it means that the linear span of Ψ is dense in $L^2(\Omega)$ and there exist constants $c, C \in (0, \infty)$ such that

$$c\,\|\mathrm{b}\|_{l_2(\mathscr{J})} \le \left\| \sum_{\lambda \in \mathscr{J}} b_\lambda \psi_\lambda \right\| \le C\,\|\mathrm{b}\|_{l_2(\mathscr{J})} \quad \text{for all} \quad \mathrm{b} = \{b_\lambda\}_{\lambda \in \mathscr{J}} \in l^2(\mathscr{J}). \tag{2}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies } (2)\}, C_\psi := \inf\{C : C \text{ satisfies } (2)\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi / c_\psi$ is called the *condition number* of Ψ.

ii) The functions are *local* in the sense that

$$\operatorname{diam}(\Omega_\lambda) \le C2^{-|\lambda|} \quad \text{for all} \quad \lambda \in \mathscr{J}, \tag{3}$$

where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap in any point $x \in \Omega$.

By the Riesz representation theorem, there exists a unique family $\tilde{\Psi} = \left\{ \tilde{\psi}_\lambda, \lambda \in \tilde{\mathscr{J}} \right\} \subset L^2(\Omega)$ biorthogonal to Ψ, i.e.

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle = \delta_{i,j}\delta_{k,l}, \quad \text{for all} \quad (i,k) \in \mathscr{J}, \quad (j,l) \in \tilde{\mathscr{J}}. \tag{4}$$

Here, $\delta_{i,j}$ denotes the Kronecker delta, i.e. $\delta_{i,i} := 1$, $\delta_{i,j} := 0$ for $i \ne j$. This family is also a Riesz basis for $L^2(\Omega)$. The basis Ψ is called a *primal* wavelet basis, $\tilde{\Psi}$ is called a *dual* wavelet basis.

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis.

Definition 2. A sequence $S = \left\{S_j\right\}_{j \in \mathbb{N}_{j_0}}$ of closed linear subspaces $S_j \subset L^2(\Omega)$ is called a *multiresolution* or *multiscale analysis*, if

$$S_{j_0} \subset S_{j_0+1} \subset \ldots \subset S_j \subset S_{j+1} \subset \ldots L^2(\Omega) \quad \text{and} \quad \overline{\left(\cup_{j \in \mathbb{N}_{j_0}} S_j\right)} = L^2(\Omega). \tag{5}$$

The nestedness and the closedness of the multiresolution analysis implies the existence of the *complement spaces* W_j such that

$$S_{j+1} = S_j \oplus W_j, \tag{6}$$

where $\oplus$ denotes the direct sum.

We now assume that S_j and W_j are spanned by sets of basis functions

$$\Phi_j := \left\{\phi_{j,k}, k \in \mathscr{I}_j\right\}, \quad \Psi_j := \left\{\psi_{j,k}, k \in \mathscr{J}_j\right\}, \tag{7}$$

where $\mathscr{I}_j$, $\mathscr{J}_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis is given by

$$\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j \tag{8}$$

and the overall wavelet basis of $L^2(\Omega)$ is obtained by

$$\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j. \tag{9}$$

The single-scale and the multiscale bases are interrelated by the *wavelet transform* $\mathbf{T}_{j,s} : l^2(I_{j+s}) \to l^2(I_{j+s})$,

$$\Psi_{j,s} = \mathbf{T}_{j,s} \Phi_{j+s}. \tag{10}$$

The dual wavelet system $\tilde{\Psi}$ generates a dual multiresolution analysis $\tilde{S}$ with a dual scaling basis $\tilde{\Phi}$.

Polynomial exactness of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that

$$\Pi_{N-1}(\Omega) \subset S_j, \quad \Pi_{\tilde{N}-1}(\Omega) \subset \tilde{S}_j, \quad j \geq j_0, \tag{11}$$

where $\Pi_m(\Omega)$ is the space of all algebraic polynomials on Ω of a degree at most m.

3 Primal Scaling Basis

The primal scaling bases will be the same as bases designed by Chui and Quak in [8], because they are known to be well-conditioned. A big advantage of this approach is that it readily adapts to the bounded interval by introducing multiple knots at the endpoints. Let N be the desired order of the polynomial exactness of the primal scaling basis and let $\mathbf{t}^j = \left(t_k^j\right)_{k=-N+1}^{2^j+N-1}$ be a *Schoenberg sequence of knots* defined by

$$\begin{aligned} t_k^j &:= 0, \qquad k = -N+1, \ldots, 0, \\ t_k^j &:= \frac{k}{2^j}, \quad k = 1, \ldots, 2^j - 1, \\ t_k^j &:= 1, \qquad k = 2^j, \ldots, 2^j + N - 1. \end{aligned} \tag{12}$$

The corresponding *B-splines of order N* are defined by

$$B_{k,N}^j(x) := \left(t_{k+N}^j - t_k^j\right) \left[t_k^j, \ldots, t_{k+N}^j\right] (t-x)_+^{N-1}, \quad x \in \langle 0, 1 \rangle, \tag{13}$$

where $(x)_+ := \max\{0, x\}$. The symbol $[t_k, \ldots t_{k+N}] f$ is the N-th divided difference of f which is recursively defined as

$$[t_k, \ldots, t_{k+N}] f = \begin{cases} \frac{[t_{k+1}, \ldots, t_{k+N}] f - [t_k, \ldots, t_{k+N-1}] f}{t_{k+N} - t_k} & \text{if} \quad t_k \neq t_{k+N}, \\ \frac{f^{(N)}(t_k)}{N!} & \text{if} \quad t_k = t_{k+N}, \end{cases} \tag{14}$$

with $[t_k] f = f(t_k)$.

The set $\Phi_j = \left\{\phi_{j,k}, k = -N+1, \ldots, 2^j - 1\right\}$ of primal scaling functions is then simply defined by

$$\phi_{j,k} = 2^{j/2} B_{k,N}^j, \quad k = -N+1, \ldots, 2^j - 1, \quad j \geq 0. \tag{15}$$

Thus there are $2^j - N + 1$ inner scaling functions and $N - 1$ functions at each boundary. Figure 1 shows the primal scaling functions for $N = 4$ and $j = 3$. The inner scaling functions are translations and dilations of one function ϕ which corresponds to the primal scaling function constructed by Cohen, Daubechies, and Feauveau in [9]. In the following, we consider ϕ from [9] which is shifted so that its support is $[0, N]$.

We define the primal multiresolution spaces by

$$S_j := \text{span}\, \Phi_j. \tag{16}$$

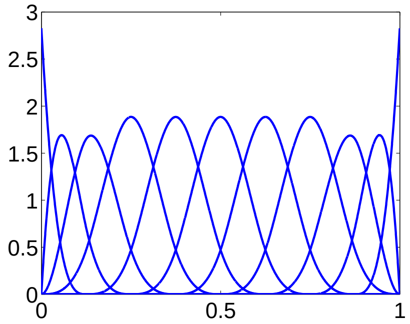

Fig. 1 Primal scaling functions for $N = 4$ and $j = 3$ without boundary conditions.

Lemma 3. *Under the above assumptions, the following holds:*
i) For any $j_0 \in \mathbb{N}$ the sequence $\mathscr{S} = \{S_j\}_{j \geq j_0}$ forms a multiresolution analysis of $L^2([0,1])$.
ii) The spaces S_j are exact of order N, i.e.

$$\Pi_{N-1}([0,1]) \subset S_j, \quad j \geq 1. \tag{17}$$

The proof can be found in [8], [24], [29].

4 Dual Scaling Basis

The desired property of the dual scaling basis $\tilde{\Phi}$ is the biorthogonality to Φ and the polynomial exactness of order $\tilde{N}$. Let $\tilde{\phi}$ be the dual scaling function which was designed by Cohen, Daubechies, and Feauveau in [9] and which is shifted so that $\langle \phi, \tilde{\phi} \rangle = 0$, i.e. its support is $[-\tilde{N}+1, N+\tilde{N}-1]$. In this case $\tilde{N} \geq N$ and $\tilde{N}+N$ has to be an even number. It is known that there exist sequences $\{h_k\}_{k\in\mathbb{Z}}$ and $\{\tilde{h}_k\}_{k\in\mathbb{Z}}$ such that the functions ϕ and $\tilde{\phi}$ satisfy the *refinement equations*

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k \phi(2x-k), \quad \tilde{\phi}(x) = \sum_{k\in\mathbb{Z}} \tilde{h}_k \tilde{\phi}(2x-k), \quad x \in \mathbb{R}. \tag{18}$$

The parameters h_k and $\tilde{h}_k$ are called *scaling coefficients*. By biorthogonality of ϕ and $\tilde{\phi}$, we have

$$2\sum_{k\in\mathbb{Z}} h_{2m+k}\tilde{h}_k = \delta_{0,m}, \quad m \in \mathbb{Z}. \tag{19}$$

Note that only coefficients $h_0, \ldots, h_N$ and $\tilde{h}_{-\tilde{N}+1}, \ldots, \tilde{h}_{N+\tilde{N}-1}$ may be nonzero.

In the sequel, we assume that

$$j \geq j_0 := \lceil \log_2(N+2\tilde{N}-3) \rceil \tag{20}$$

so that the supports of the boundary functions are contained in $[0,1]$. We define inner scaling functions as translations and dilations of $\tilde{\phi}$:

$$\theta_{j,k} := 2^{j/2}\tilde{\phi}(2^j \cdot -k), \quad k = \tilde{N}-1, \ldots, 2^j - N - \tilde{N}+1. \tag{21}$$

There will be two types of basis functions at each boundary. In the following, it will be convenient to abbreviate the boundary and inner index sets by

$$\mathscr{I}_j^{L,1} = \{-N+1, \ldots, -N+\tilde{N}\}, \tag{22}$$
$$\mathscr{I}_j^{L,2} = \{-N+\tilde{N}+1, \ldots, \tilde{N}-2\}, \tag{23}$$
$$\mathscr{I}_j^{0} = \{\tilde{N}-1, \ldots, 2^j-N-\tilde{N}+1\}, \tag{24}$$
$$\mathscr{I}_j^{R,2} = \{2^j-N-\tilde{N}+2, \ldots 2^j-\tilde{N}-1\}, \tag{25}$$
$$\mathscr{I}_j^{R,1} = \{2^j-\tilde{N}, \ldots, 2^j-1\}, \tag{26}$$

and

$$\mathscr{I}_j^L = \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{L,2} = \left\{-N+1,\ldots,\tilde{N}-2\right\}, \tag{27}$$

$$\mathscr{I}_j^R = \mathscr{I}_j^{R,2} \cup \mathscr{I}_j^{R,1} = \left\{2^j - N - \tilde{N} + 2,\ldots,2^j - 1\right\}, \tag{28}$$

$$\mathscr{I}_j = \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^0 \cup \mathscr{I}_j^{R,2} \cup \mathscr{I}_j^{R,1} = \left\{-N+1,\ldots,2^j-1\right\}. \tag{29}$$

Basis functions of the first type are defined to preserve polynomial exactness by the same way as in [1], [10]:

$$\theta_{j,k} = 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle p_{k+N-1}, \phi\left(\cdot - l\right)\right\rangle \tilde{\phi}\left(2^j \cdot - l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,1}, \tag{30}$$

where $\left\{p_0,\ldots,p_{\tilde{N}-1}\right\}$ is a basis of $\Pi_{\tilde{N}-1}([0,1])$. In Lemma 6 we show that the resulting dual scaling functions do not depend on the choice of the polynomial basis. In our case, p_k are the Bernstein polynomials defined by

$$p_k(x) := b^{-\tilde{N}+1} \binom{\tilde{N}-1}{k} x^k (b-x)^{\tilde{N}-1-k}, \quad k = 0,\ldots,\tilde{N}-1, \quad x \in \mathbb{R}. \tag{31}$$

The Bernstein polynomials were used also in [16]. On the contrary to [16], in our case the choice of polynomials does not affect the resulting dual scaling basis $\tilde{\Psi}$, but it has only the effect of stabilization of the computation, for details see Lemma 6 and the discussion below.

The definition of basis functions of the second type is a delicate task, because the low condition number and the nestedness of the multiresolution spaces have to be preserved. This means that the relation $\theta_{j,k} \subset \tilde{V}_j \subset \tilde{V}_{j+1}$, $k \in \mathscr{I}_j^{L,2}$, should hold. Therefore we define $\theta_{j,k}$, $k \in \mathscr{I}_j^{L,2}$, as linear combinations of functions which are already in $\tilde{V}_{j+1}$. To obtain well-conditioned bases, we define functions $\theta_{j,k}$, $k \in \mathscr{I}_j^{L,2}$, which are close to $\tilde{\phi}_{j,k}^{\mathbb{R}} := 2^{j/2}\tilde{\phi}\left(2^j \cdot - k\right)$, because $\tilde{\phi}_{j,k}^{\mathbb{R}}$, $k \in \mathscr{I}_j^{L,2}$, are biorthogonal to the inner primal scaling functions and the condition of $\left\{\tilde{\phi}_{j,k}^{\mathbb{R}}, k \in \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^0\right\}$ is the same as the condition of the set of inner dual basis functions.

For this reason, we define the basis functions of the second type by

$$\theta_{j,k} = 2^{\frac{j}{2}} \sum_{l=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}\left(2^{j+1} \cdot - 2k - l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,2}, \tag{32}$$

where $\tilde{h}_i$ are the scaling coefficients corresponding to the scaling function $\tilde{\phi}$. Then $\theta_{j,k}$ is close to $\tilde{\phi}_{j,k}^{\mathbb{R}}|_{[0,1]}$, because by (18) we have

$$\tilde{\phi}_{j,k}^{\mathbb{R}}|_{[0,1]} = 2^{\frac{j}{2}} \sum_{k=-\tilde{N}+1}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}\left(2^{j+1} \cdot - 2k - l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,2}. \tag{33}$$

Figure 2 shows the functions $\theta_{j,k}$ and $\tilde{\phi}_{j,k}^{\mathbb{R}}$ for $N = 4$, $\tilde{N} = 6$, and $j = 4$.

The boundary functions at the right boundary are defined to be symmetric with the left boundary functions:

$$\theta_{j,k} = \theta_{j,2^j-k}(1-\cdot), \quad k \in \mathscr{I}_j^R. \tag{34}$$

It is easy to see that

$$\theta_{j+1,k} = 2^{1/2}\theta_{j,k}(2\cdot), \quad k \in \mathscr{I}_j^L \tag{35}$$

for the left boundary functions and

$$\theta_{j+1,k}(1-\cdot) = 2^{1/2}\theta_{j,k}(1-2\cdot), \quad k \in \mathscr{I}_j^R \tag{36}$$

for the right boundary functions.

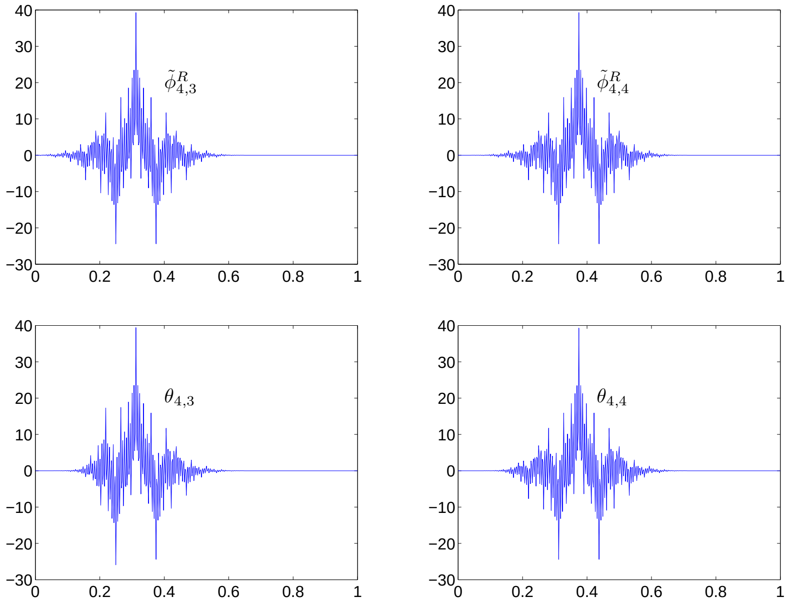

Fig. 2 The functions $\tilde{\phi}^{\mathbb{R}}_{4,k}$ and $\theta_{4,k}$ for $N=4$ and $\tilde{N}=6$.

Since the set $\Theta_j := \left\{\theta_{j,k}, k \in \mathscr{I}_j\right\}$ is not biorthogonal to Φ_j, we derive a new set

$$\tilde{\Phi}_j := \left\{\tilde{\phi}_{j,k}, k \in \mathscr{I}_j\right\} \tag{37}$$

from Θ_j by biorthogonalization. Let

$$\mathbf{Q}_j = \left(\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle\right)_{k,l \in \mathscr{I}_j}. \tag{38}$$

Then viewing $\tilde{\Phi}_j$ and Θ_j as column vectors we define

$$\tilde{\Phi}_j := \mathbf{Q}_j^{-T} \Theta_j, \tag{39}$$

assuming that $\mathbf{Q}_j$ is invertible, which is the case of all choices of N and $\tilde{N}$ considered in our numerical examples below.

Then $\tilde{\Phi}_j$ is biorthogonal to Φ_j, because

$$\left\langle \Phi_j, \tilde{\Phi}_j \right\rangle = \left\langle \Phi_j, \mathbf{Q}_j^{-T} \Theta_j \right\rangle = \mathbf{Q}_j \mathbf{Q}_j^{-1} = \mathbf{I}_{\#\mathscr{I}_j}, \tag{40}$$

where the symbol # denotes the cardinality of the set and $\mathbf{I}_m$ denotes the identity matrix of the size $m \times m$.

Lemma 4. *i) Let Φ_j, Θ_j be defined as above. Then the matrices*

$$\mathbf{Q}_{j,L} = \left(\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle\right)_{k,l \in \mathscr{I}_j^L} \quad \text{and} \quad \mathbf{Q}_{j,R} = \left(\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle\right)_{k,l \in \mathscr{I}_j^R} \tag{41}$$

are independent of j, i.e. there are matrices $\mathbf{Q}_L$, $\mathbf{Q}_R$ such that

$$\mathbf{Q}_{j,L} = \mathbf{Q}_L, \quad \mathbf{Q}_{j,R} = \mathbf{Q}_R. \tag{42}$$

Moreover, the matrix $\mathbf{Q}_R$ results from the matrix $\mathbf{Q}_L$ by reversing the ordering of rows and columns, which means that

$$(\mathbf{Q}_R)_{k,l} = (\mathbf{Q}_L)_{2^j-N-k,2^j-N-l}, \quad k,l \in \mathscr{I}_j^R. \tag{43}$$

ii) The following holds:

$$(\mathbf{Q}_j)_{k,l} = \delta_{k,l}, \quad k \in \mathscr{I}_j,\ l \in \mathscr{I}_j^0. \tag{44}$$

iii) The following holds:

$$(\mathbf{Q}_j)_{k,l} = 0, \quad k \in \mathscr{I}_j^0,\ l \in \mathscr{I}_j^L \cup \mathscr{I}_j^R. \tag{45}$$

Proof Due to (35) and by substitution we have for $k,l \in \mathscr{I}_j^L$

$$\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle = \left\langle 2^{\frac{j-j_0}{2}} \phi_{j_0,k}\left(2^{j-j_0}\cdot\right), 2^{\frac{j-j_0}{2}} \theta_{j_0,l}\left(2^{j-j_0}\cdot\right) \right\rangle = \left\langle \phi_{j_0,k}, \theta_{j_0,l} \right\rangle. \tag{46}$$

Therefore, $\mathbf{Q}_{j,L} = \mathbf{Q}_{j_0,L} = \mathbf{Q}_L$, i.e. the matrix $\mathbf{Q}_{j,L}$ is independent of j. Due to (36) $\mathbf{Q}_{j,R}$ is independent of j too. The property (43) is a direct consequence of the reflection invariance (34).

The property *ii*) follows from the biorthogonality of $\{\phi(\cdot-k)\}_{k\in\mathbb{Z}}$ and $\left\{\tilde{\phi}(\cdot-l)\right\}_{l\in\mathbb{Z}}$. It also implies (45) for $k \in \mathscr{I}_j^0$, $l \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{R,1}$. It remains to prove (45) for $k \in \mathscr{I}_j^0$, $l \in \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^{R,2}$. By the definition of the dual scaling functions of the second type (32), the refinement relation (18) for the dual scaling function $\tilde{\phi}$, and (19), we have for $k \in \mathscr{I}_j^0$, $l \in \mathscr{I}_j^{L,2}$,

$$\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle = \left\langle \phi(\cdot-k), \sqrt{2} \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}(2\cdot-2l-m)|_{[0,1]} \right\rangle \tag{47}$$

$$= 2\left\langle \sum_{n=0}^{N} h_n \phi(2\cdot-2k-n), \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_m \tilde{\phi}(2\cdot-2l-m)|_{[0,1]} \right\rangle \tag{48}$$

$$= 2\sum_{n=0}^{N} \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} h_n \tilde{h}_m \delta_{2k+n,2l+m} = 2 \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} h_{2l-2k+m} \tilde{h}_m \tag{49}$$

$$= 2 \sum_{m\in\mathbb{Z}} h_{2l-2k+m} \tilde{h}_m = 0. \tag{50}$$

By (34), the relation (45) holds also for $k \in \mathscr{I}_j^0$, $l \in \mathscr{I}_j^{R,2}$.

Thus, we can write

$$\tilde{\Phi}_j := \mathbf{Q}_j^{-T} \Theta_j = \begin{pmatrix} \mathbf{Q}_L & & \\ & \mathbf{I}_{\#\mathscr{I}_j^0} & \\ & & \mathbf{Q}_R \end{pmatrix}^{-T} \Theta_j = \begin{pmatrix} \mathbf{Q}_L^{-T} & & \\ & \mathbf{I}_{\#\mathscr{I}_j^0} & \\ & & \mathbf{Q}_R^{-T} \end{pmatrix} \Theta_j, \tag{51}$$

Since the matrix $\mathbf{Q}_j$ is symmetric in the sense of (43), the properties (34), (35), and (36) hold for $\tilde{\phi}_{j,k}$ as well.

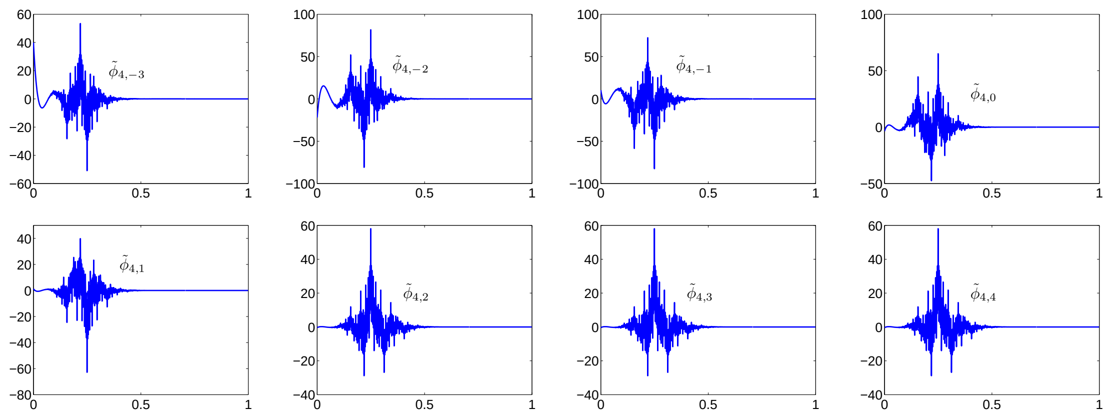

Fig. 3 Boundary dual scaling functions for $N = 4$ and $\tilde{N} = 6$ without boundary conditions.

Remark 5. *It is known that the scaling function $\tilde{\phi}$ has typically a low Sobolev regularity for smaller values of $\tilde{N}$. Thus the functions $\theta_{j,k}$ have a low Sobolev regularity for smaller values of $\tilde{N}$, too. Therefore, we do not obtain the sufficiently accurate entries of the matrix $\mathbf{Q}_j$ directly by* *classical quadratures. Fortunately, we are able to compute the matrix $\mathbf{Q}_j$ precisely up to the round off errors. For $k \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{L,2}$, $l \in \mathscr{I}_j^{L,1}$ we have*

$$\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle = \sum_{m=-N-\tilde{N}+2}^{\tilde{N}-2} \sum_{n=0}^{\tilde{N}-1} c_{l,n} \left\langle (\cdot)^n, \phi(\cdot - m) \right\rangle \left\langle \phi(\cdot - k), \tilde{\phi}(\cdot - m) \right\rangle_{L^2(\langle 0,1 \rangle)}, \tag{52}$$

with $c_{l,n}$ given by (64). Since ϕ is a piecewise polynomial function and $\tilde{\phi}$ is refinable, for $k \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{L,2}$, $l \in \mathscr{I}_j^{L,1}$ we can compute the entries of $\mathbf{Q}_j$ by the method from [11]. By the refinement relation we easily obtain the following relations for the computation of the remaining entries of $\mathbf{Q}^L$:

$$\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle = \begin{cases} \sum_{m=\tilde{N}-1-2l}^{N+\tilde{N}-1} \tilde{h}_m \left\langle \phi_{0,k}, \tilde{\phi}(\cdot - 2k - m) \right\rangle, & k = -N+1, \ldots, -1,\ l \in \mathscr{I}_j^{L,2}, \\ 2^{-1} \sum_{m=\tilde{N}-1-2l}^{N+\tilde{N}-1} h_{2k-2l+m} \tilde{h}_m, & k = 0, \ldots, \tilde{N}-2,\ l \in \mathscr{I}_j^{L,2}. \end{cases} \tag{53}$$

Since the submatrix $\mathbf{Q}_R$ is obtained from a matrix $\mathbf{Q}_L$ by reversing the ordering of rows and columns, the matrix $\mathbf{Q}_j$ can be indeed computed precisely up to the round off errors.

Now we show that the resulting dual scaling basis $\tilde{\Phi}$ does not depend on a choice of a polynomial basis of the space $\Pi_{\tilde{N}}([0,1])$ in the formula (30).

Lemma 6. *We suppose that $P^1 = \left\{ p_0^1, \ldots, p_{\tilde{N}-1}^1 \right\}$, $P^2 = \left\{ p_0^2, \ldots, p_{\tilde{N}-1}^2 \right\}$ are two different bases of the space $\Pi_{\tilde{N}}([0,1])$ and for $i = 1,2$ we define the sets $\Theta_j^i = \left\{ \theta_{j,k}^i \right\}_{k=-N+1}^{2^j-1}$ by*

$$\theta_{j,k}^i = \begin{cases} 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle p_{k+N-1}^i, \phi(\cdot - l) \right\rangle \tilde{\phi}\left(2^j \cdot - l\right)|_{[0,1]}, & k \in \mathscr{I}_j^{L,1}, \\ \theta_{j,2^j-N-k}^i, & k \in \mathscr{I}_j^{R,1}, \\ \theta_{j,k}, & k \in \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^0 \cup \mathscr{I}_j^{R,2}. \end{cases} \tag{54}$$

Furthermore, we define

$$\mathbf{Q}_j^i = \left\langle \Phi_j, \Theta_j^i \right\rangle, \quad \tilde{\Phi}_j^i = \left(\mathbf{Q}_j^i\right)^{-T} \Theta_j^i, \quad i = 1,2, \tag{55}$$

and we assume that $\mathbf{Q}_j^i$ is nonsingular. Then $\tilde{\Phi}_j^1 = \tilde{\Phi}_j^2$.

Proof Since P^1 and P^2 are both bases of the space $\Pi_{\tilde{N}}([0,1])$, there exists a regular matrix $\mathbf{B}_L$ such that $P^2 = \mathbf{B}_L P^1$. The consequence is that

$$\Theta^2 = \mathbf{B}_j \Theta^1, \tag{56}$$

with

$$\mathbf{B}_j = \begin{pmatrix} \mathbf{B}_L & & \\ & \mathbf{I}_{\#\mathscr{I}_j^0} & \\ & & \mathbf{B}_R \end{pmatrix}, \tag{57}$$

where $\mathbf{B}_R$ results from a matrix $\mathbf{B}_L$ by reversing the ordering of rows and columns, which means that

$$(\mathbf{B}_R)_{k,l} = (\mathbf{B}_L)_{2^j-N-k,2^j-N-l}, \quad k,l \in \mathscr{I}_j^{L,1}. \tag{58}$$

Therefore, we have

$$\tilde{\Phi}_j^2 = \left(\mathbf{Q}_j^2\right)^{-T} \Theta_j^2 = \left(\mathbf{Q}_j^1\right)^{-T} \mathbf{B}_j^{-1} \mathbf{B}_j \Theta_j^1 = \tilde{\Phi}_j^1. \tag{59}$$

Although a choice of a polynomial basis does not influence the resulting dual scaling basis, it has an influence on the stability of the computation and the preciseness of the results, because some choices of the polynomial bases lead to the critical values of the condition number of the biorthogonalization matrix. We present the condition numbers of the matrix $\mathbf{Q}_L$ for the monomial basis $\left\{ 1, x, x^2, \ldots x^{\tilde{N}-1} \right\}$ and Bernstein polynomials (31) with the parameters $b = 4$ and $b = 10$ in Table 4. In our numerical experiments in Section 9 we choose $b = 10$.

Remark 7. *In the case of linear primal basis, i.e. $N = 2$, there are no boundary dual functions of the second type. In [24] the primal scaling functions and the inner dual scaling functions are the same as ours. The boundary dual functions before biorthogonalization are defined by (30) with the same choice of polynomials $p_0, \ldots, p_{\tilde{N}-1}$ as in [10]. Due to the Lemma 6, for $N = 2$ the wavelet basis in [24] is identical to the wavelet basis constructed in this section.*

The main difference of the construction by M. Primbs [24] in comparison with our construction is the definition of dual basis functions of the second type $\theta_{j,k}^{Primbs}$, $k = -N+1, \ldots, -2$. Note that they correspond to different indexes than ours. These functions are defined as linear combination of functions $\theta_{j+1,n}^{Primbs}$, $n > k$, in order to be already biorthogonal to the primal scaling functions. The refinement coefficients for them are obtained by solving certain system of linear algebraic equations. In case $N > 3$, the functions $\theta_{j,k}^{Primbs}$, $k = -N+1, \ldots, -2$, take much larger values than primal scaling functions and than the inner dual scaling functions. Then some of the boundary wavelets take much larger values than inner wavelets which probably causes bad conditioning of wavelet bases. Furthermore, the dual boundary functions of the first type which are defined to preserve the polynomial exactness correspond to the first N scaling functions in our case and they correspond to the primal scaling functions indexed by $-1, \ldots, \tilde{N}-2$ in case of the construction from [24]. It leads to better matching of the supports and values of the primal and dual functions in our construction. This better localization and 'almost biorthogonality' of the dual functions of the second type to the primal scaling functions lead to optimally conditioned wavelet bases for $N \leq 4$ and to an improvement of the condition number also for $N = 5$, see Section 9.

The constructions of primal and dual boundary scaling functions in [16] and [17] is based on the relation (30) with various choices of polynomials. There are no boundary generators of the second type. This construction also leads to some boundary functions which take larger values than the inner functions and the condition number of wavelet bases is bad for $N > 3$, see figures in [16], [17], and [35].

For the proof of Theorem 9 below and also for deriving of refinement matrices we will need the following lemma.

Lemma 8. *For the left boundary functions of the first type there exist refinement coefficients $m_{n,k}$, $k \in \mathscr{I}_j^{L,1}$, $n \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^3$, $\mathscr{I}_j^3 := \left\{\tilde{N}-1, \ldots, 3\tilde{N}+N-5\right\}$ such that*

$$\theta_{j,k} = \sum_{n=-N+1}^{-N+\tilde{N}} m_{n,k} \theta_{j+1,n} + \sum_{n=\tilde{N}-1}^{3\tilde{N}+N-5} m_{n,k} \theta_{j+1,n}, \quad k \in \mathscr{I}_j^{L,1}. \tag{60}$$

Proof Let $\Theta_j^0 = \left\{\theta_{j,k}, k \in \mathscr{I}_j^3\right\}$ and $\Theta_j^{L,1,mon} = \left\{\theta_{j,k}^{mon}, k \in \mathscr{I}_j^{L,1}\right\}$ be defined by

$$\theta_{j,k}^{mon} = 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle (\cdot)^i, \phi(\cdot - l) \right\rangle \tilde{\phi}\left(2^j \cdot -l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,1}. \tag{61}$$

Then

$$\Theta_j^{L,1,mon} = \left(\mathbf{M}^{mon}\right)^T \begin{pmatrix} \Theta_{j+1}^{L,1,mon} \\ \Theta_{j+1}^0 \end{pmatrix}, \tag{62}$$

Table 1 Condition numbers of the matrices $\mathbf{Q}_L$

N	$\tilde{N}$	mon.	$b=4$	$b=10$	N	$\tilde{N}$	mon.	$b=4$	$b=10$
2	2	6.68e+00	9.94e+00	3.16e+01	4	4	2.46e+04	6.75e+02	1.33e+04
2	4	4.66e+02	1.94e+01	9.48e+02	4	6	1.30e+07	2.94e+04	7.34e+04
2	6	1.40e+05	1.00e+02	4.47e+03	4	8	1.24e+10	6.24e+06	9.42e+04
2	8	1.03e+08	8.52e+03	5.81e+03	4	10	1.92e+13	2.26e+09	5.24e+04
2	10	1.48e+11	1.67e+06	1.58e+03	5	5	5.34e+06	3.29e+04	1.26e+05
3	3	2.18e+02	1.07e+02	1.00e+03	5	7	5.62e+09	6.91e+06	3.73e+05
3	5	3.73e+04	1.88e+02	1.05e+04	5	9	9.39e+12	2.57e+09	3.47e+05
3	7	1.64e+07	1.20e+04	2.26e+04	6	6	1.20e+09	3.68e+06	6.81e+05
3	9	1.54e+10	2.90e+06	1.33e+04	6	8	2.97e+12	1.92e+09	1.81e+06

where the refinement matrix $\mathbf{M}^{mon} = \left\{m_{n,k}^{mon}\right\}_{n\in\mathscr{I}_j^{L,1}\cup\mathscr{I}_j^3, k\in\mathscr{I}_j^{L,1}}$ is given by

$$m_{n,k}^{mon} = \begin{cases} \frac{1}{\sqrt{2}}2^{-k}, & k=n,\ n\in\mathscr{I}_j^{L,1}, \\ \frac{1}{\sqrt{2}}\sum_{q=\left\lceil\frac{n-N-\tilde{N}+1}{2}\right\rceil}^{\tilde{N}-2}\left\langle(\cdot)^{k+N-1},\phi(\cdot-q)\right\rangle\tilde{h}_{n-2q}, & k\in\mathscr{I}_j^{L,1},\ n\in\mathscr{I}_j^3, \\ 0, & \text{otherwise.}\end{cases} \tag{63}$$

For deriving of $\mathbf{M}^{mon}$ see [16]. It is known that the coefficients of Bernstein polynomials in a monomial basis are given by

$$c_{l,n} = \begin{cases}(-1)^{l-n}\binom{\tilde{N}-1}{n}\binom{n}{l}b^{-n}, & \text{if} \quad n\geq l, \\ 0, & \text{otherwise.}\end{cases} \tag{64}$$

Hence, the matrix $\mathbf{C} = \{c_{l,n}\}_{l,n=-N+1}^{-N+\tilde{N}}$ is an upper triangular matrix with nonzero entries on the diagonal which implies that $\mathbf{C}$ is invertible. We denote $\Theta_j^{L,1} = \left\{\theta_{j,k}, k\in\mathscr{I}_j^{L,1}\right\}$ and we obtain

$$\Theta_j^{L,1} = \mathbf{C}\Theta_j^{L,1,mon} = \mathbf{C}\left(\mathbf{M}^{mon}\right)^T\begin{pmatrix}\Theta_{j+1}^{L,1,mon}\\ \Theta_{j+1}^0\end{pmatrix} = \mathbf{C}\left(\mathbf{M}^{mon}\right)^T\begin{pmatrix}\mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\Theta_{j+1}^{L,1}\\ \Theta_{j+1}^0\end{pmatrix}. \tag{65}$$

Therefore, the refinement matrix $\mathbf{M} = \{m_{n,k}\}_{n\in\mathscr{I}_j^{L,1}\cup\mathscr{I}_j^3, k\in\mathscr{I}_j^{L,1}}$ is given by

$$\mathbf{M} = \begin{pmatrix}\mathbf{C}^{-T} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}\mathbf{M}^{mon}\mathbf{C}^T. \tag{66}$$

We define the dual multiresolution spaces by

$$\tilde{S}_j := \text{span } \tilde{\boldsymbol{\Phi}}_j. \tag{67}$$

Theorem 9. *Under the above assumptions, the following holds*

i) The sequence $\tilde{\mathscr{S}} = \left\{\tilde{S}_j\right\}_{j\geq j_0}$ *forms a multiresolution analysis of* $L^2([0,1])$.

ii) The spaces $\tilde{S}_j$ *are exact of order* $\tilde{N}$, *i.e.*

$$\Pi_{\tilde{N}-1}([0,1]) \subset \tilde{S}_j, \quad j > j_0. \tag{68}$$

Proof To prove $i)$ we have to show the nestedness of the spaces $\tilde{S}_j$, i.e. $\tilde{S}_j \subset \tilde{S}_{j+1}$. Note that

$$\tilde{S}_j = \text{span } \tilde{\boldsymbol{\Phi}}_j = \text{span } \Theta_j. \tag{69}$$

Therefore, it is sufficient to prove that any function from Θ_j can be written as a linear combination of the functions from Θ_{j+1}. For the left boundary functions of the first type it is a consequence of Lemma 8. By definition (32) it holds also for the left boundary functions of the second type. Since the inner basis functions are just translated and dilated scaling function $\tilde{\phi}$, they obviously satisfy the refinement relation. Finally, the right boundary scaling functions are derived by the reflection from the left boundary scaling functions and therefore, they satisfy the refinement relation, too. It remains to prove that

$$\overline{\bigcup_{j\geq j_0}\tilde{S}_j} = L^2([0,1]), \tag{70}$$

where $\overline{M}$ denotes the closure of the set M in $L^2([0,1])$. It is known [26] that for the spaces generated by inner functions

$$\tilde{S}_j^0 := \left\{\theta_{j,k}, k\in\mathscr{I}_j^0\right\} \tag{71}$$

we have

$$\overline{\bigcup_{j\geq j_0}\tilde{S}_j^0} = L^2([0,1]). \tag{72}$$

Hence, (70) holds independently of the choice of boundary functions.

To prove ii) we recall that the scaling function $\tilde{\phi}$ is exact of order $\tilde{N}$, i.e.

$$2^{j(r+1/2)}x^r = \sum_{k\in\mathbb{Z}} \alpha_{k,r} 2^{j/2} \tilde{\phi}\left(2^j x - k\right), \quad x \in \mathbb{R}\ a.e.,\ r = 0,\dots,\tilde{N}-1, \tag{73}$$

where

$$\alpha_{k,r} = \left\langle (\cdot)^k, \phi\left(\cdot - r\right)\right\rangle. \tag{74}$$

It implies that for $r = 0,\dots,\tilde{N}-1$, $x \in \langle 0,1\rangle$, the following holds

$$\begin{aligned} 2^{j(r+1/2)}x^r|_{\langle 0,1\rangle} = & \sum_{k=-N-\tilde{N}+2}^{\tilde{N}-2} \alpha_{k,r} 2^{j/2} \tilde{\phi}\left(2^j x - k\right)|_{\langle 0,1\rangle} + \sum_{k=\tilde{N}-1}^{2^j-N-\tilde{N}+1} \alpha_{k,r} 2^{j/2} \tilde{\phi}\left(2^j x - k\right)|_{\langle 0,1\rangle} \\ & + \sum_{k=2^j-N-\tilde{N}+2}^{2^j+\tilde{N}-2} \alpha_{k,r} 2^{j/2} \tilde{\phi}\left(2^j x - k\right)|_{\langle 0,1\rangle}. \end{aligned}$$

By (30), (34), and (69), we immediately have

$$\Pi_{\tilde{N}-1}\left([0,1]\right) \subset \operatorname{span}\left\{\tilde{\phi}_{j,k}, k \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{0} \cup \mathscr{I}_j^{R,1}\right\} \subset \tilde{S}_j. \tag{75}$$

5 Refinement Matrices

Due to the length of the support of the primal scaling functions, the refinement matrix $M_{j,0}$ corresponding to $\boldsymbol{\Phi}$ has the following structure:

$$\mathbf{M}_{j,0} = \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{A}_j & \\ & & \mathbf{M}_R \end{pmatrix}. \tag{76}$$

where $\mathbf{M}_L$, $\mathbf{M}_R$ are blocks of the size $(2N-2)\times(N-1)$ and $\mathbf{A}_j$ is a $\left(2^{j+1}-N+2\right)\times\left(2^j-N+2\right)$ matrix given by

$$\left(\mathbf{A}_j\right)_{m,n} = \frac{1}{\sqrt{2}} h_{m-2n}, \quad 0 \le m-2n \le N. \tag{77}$$

Since the matrix $\mathbf{M}_L$ is given by

$$\begin{pmatrix} \phi_{j,-N+1} \\ \phi_{j,-N+2} \\ \vdots \\ \phi_{j,-1} \end{pmatrix} = \mathbf{M}_L^T \begin{pmatrix} \phi_{j+1,-N+1} \\ \phi_{j+1,-N+2} \\ \vdots \\ \phi_{j+1,N-1} \end{pmatrix}, \tag{78}$$

it could be computed by solving the system

$$\mathbf{P}_1 = \mathbf{M}_L^T \mathbf{P}_2, \tag{79}$$

where

$$\mathbf{P}_1 = \begin{pmatrix} \phi_{0,-N+1}(0) & \phi_{0,-N+1}(1) & \dots & \phi_{0,-N+1}(2N-3) \\ \phi_{0,-N+2}(0) & \phi_{0,-N+2}(1) & \dots & \phi_{0,-N+2}(2N-3) \\ \vdots & & & \vdots \\ \phi_{0,-1}(0) & \phi_{0,-1}(1) & \dots & \phi_{0,-1}(2N-3) \end{pmatrix} \tag{80}$$

and

$$\mathbf{P}_2 = \begin{pmatrix} \phi_{1,-N+1}(0) & \phi_{1,-N+1}(1) & \dots & \phi_{1,-N+1}(2N-3) \\ \phi_{1,-N+2}(0) & \phi_{1,-N+2}(1) & \dots & \phi_{1,-N+2}(2N-3) \\ \vdots & & & \vdots \\ \phi_{1,N-1}(0) & \phi_{1,N-1}(1) & \dots & \phi_{1,N-1}(2N-3) \end{pmatrix}. \tag{81}$$

The solution of the system (79) exists and is unique if and only if the matrix $\mathbf{P}_2$ is nonsingular. The proof of a nonsingularity of $\mathbf{P}_2$ can be found in [36].

To compute the refinement matrix corresponding to the dual scaling functions, we need to identify first the structure of the refinement matrices $\mathbf{M}^{\Theta}_{j,0}$ corresponding to Θ.

$$\mathbf{M}^{\Theta}_{j,0} = \begin{pmatrix} \mathbf{M}^{\Theta}_{L} & & \\ & \tilde{\mathbf{A}}_j & \\ & & \mathbf{M}^{\Theta}_{R} \end{pmatrix}, \tag{82}$$

where $\mathbf{M}^{\Theta}_{L}$ and $\mathbf{M}^{\Theta}_{R}$ are blocks of the size $\left(2N+3\tilde{N}-5\right) \times \left(N+\tilde{N}-2\right)$ and $\tilde{\mathbf{A}}_j$ is a matrix of the size $\left(2^{j+1}-N-2\tilde{N}+3\right) \times \left(2^{j}-N-2\tilde{N}+3\right)$ given by

$$\left(\tilde{\mathbf{A}}_j\right)_{m,n} = \frac{1}{\sqrt{2}} \tilde{h}_{m-2n}, \quad 0 \le m-2n \le N+\tilde{N}-2. \tag{83}$$

The recipe for the computation of the refinement coefficients for the left boundary functions of the first type is the proof of Lemma 8. The refinement coefficients for the left boundary functions of the second type are given by the definition (32). The matrix $\mathbf{M}^{\Theta}_{R}$ can be computed by the similar way.

Since we have

$$\tilde{\Phi}_j = \mathbf{Q}_j^{-T} \Theta_j = \mathbf{Q}_j^{-T} \left(\mathbf{M}^{\Theta}_{j,0}\right)^T \Theta_{j+1} = \mathbf{Q}_j^{-T} \left(\mathbf{M}^{\Theta}_{j,0}\right)^T \mathbf{Q}^T_{j+1} \tilde{\Phi}_{j+1}, \tag{84}$$

the refinement matrix $\tilde{\mathbf{M}}_{j,0}$ corresponding to the dual scaling basis $\tilde{\Phi}_j$ is given by

$$\tilde{\mathbf{M}}_{j,0} = \mathbf{Q}_{j+1} \mathbf{M}^{\Theta}_{j,0} \mathbf{Q}_j^{-1}. \tag{85}$$

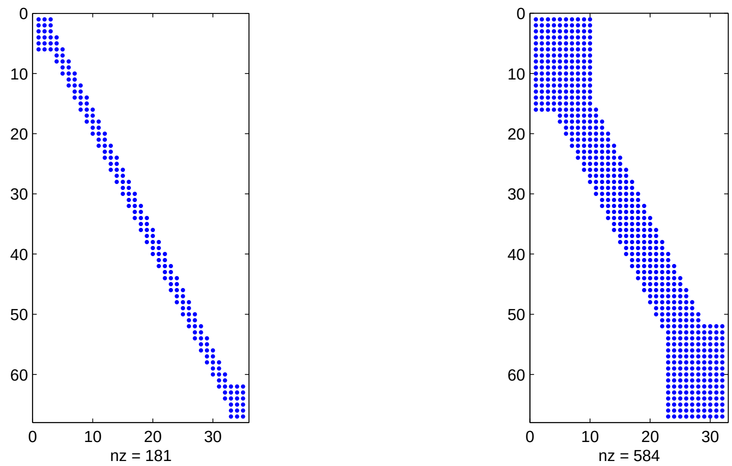

Fig. 4 Nonzero pattern of the matrices $\mathbf{M}_{5,0}$ and $\tilde{\mathbf{M}}_{5,0}$ for $N = 4$ and $\tilde{N} = 6$, *nz* is the number of nonzero entries.

6 Wavelets

Our next goal is to determine the corresponding wavelet bases. This is directly connected to the task of determining an appropriate matrices $\mathbf{M}_{j,1}$ and $\tilde{\mathbf{M}}_{j,1}$. Thus, the problem has been transferred from functional analysis to linear algebra. We follow a general principle called a *stable completion* which was proposed in [6].

Definition 10. Any $\mathbf{M}_{j,1} : l_2(J_j) \to l_2(I_{j+1})$ is called a *stable completion* of $\mathbf{M}_{j,0}$, if

$$\left\|\mathbf{M}_j\right\|, \left\|\mathbf{M}_j^{-1}\right\| = O(1), \quad j \to \infty, \tag{86}$$

where $\mathbf{M}_j := (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$.

The idea is to determine first an initial stable completion and then to project it to the desired complement space W_j determined by $\left\{\tilde{V}_j\right\}_{j \geq j_0}$. This is summarized in the following theorem [6].

Theorem 11. *Let Φ_j and $\tilde{\Phi}_j$ be a primal and dual scaling basis, respectively. Let $\mathbf{M}_{j,0}$ and $\tilde{\mathbf{M}}_{j,0}$ be the refinement matrices corresponding to these bases. Suppose that $\check{\mathbf{M}}_{j,1}$ is some stable completion of $\mathbf{M}_{j,0}$ and $\check{\mathbf{G}}_j = \check{\mathbf{M}}_j^{-1}$. Then*

$$\mathbf{M}_{j,1} := \left(\mathbf{I} - \mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^T\right)\check{\mathbf{M}}_{j,1} \tag{87}$$

is also a stable completion and $\mathbf{G}_j = \mathbf{M}_j^{-1}$ has the form

$$\mathbf{G}_j = \begin{pmatrix} \tilde{\mathbf{M}}_{j,0}^T \\ \check{\mathbf{G}}_{j,1} \end{pmatrix}. \tag{88}$$

Moreover, the collections

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j := \check{\mathbf{G}}_{j,1}^T \tilde{\Phi}_{j+1} \tag{89}$$

form biorthogonal systems

$$\left\langle \Psi_j, \tilde{\Psi}_j \right\rangle = \mathbf{I}, \quad \left\langle \Phi_j, \tilde{\Psi}_j \right\rangle = \left\langle \Psi_j, \tilde{\Phi}_j \right\rangle = \mathbf{0}. \tag{90}$$

We found the initial stable completion by the method from [16], [18] with some small changes. The difference is only in the dimensions of the involved matrices and in the definion of the matrix $\mathbf{F}_j$. Recall that $\mathbf{A}_j$ is the interior block in the matrix $\mathbf{M}_{j,0}$ of the form

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} h_0 & 0 & \dots & 0 \\ h_1 & 0 & & \vdots \\ h_3 & h_0 & & \\ \vdots & \vdots & & \vdots \\ h_N & h_{N-2} & & \vdots \\ 0 & h_{N-1} & & 0 \\ 0 & h_N & & h_0 \\ \vdots & & & \vdots \\ 0 & & & h_N \end{pmatrix}, \tag{91}$$

where $h_0,\dots,h_N$ are the scaling coefficients corresponding to ϕ. By a suitable elimination we will successively reduce the upper and lower bands from $\mathbf{A}_j$ such that after i steps we obtain

$$\mathbf{A}_j^{(i)} = \left(\begin{array}{cccc} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \\ h^{(i)}_{\lceil \frac{i}{2}\rceil} & 0 & & \\ h^{(i)}_{\lceil \frac{i}{2}\rceil+1} & 0 & & \\ \vdots & h^{(i)}_{\lceil \frac{i}{2}\rceil} & & \\ \vdots & \vdots & & \\ h^{(i)}_{N-\lfloor \frac{i}{2}\rfloor} & & & \\ 0 & & & \\ \vdots & & & \\ & & & h^{(i)}_{N-\lfloor \frac{i}{2}\rfloor} \\ & & & 0 \\ & & & \vdots \\ 0 & & & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \lceil \frac{i}{2}\rceil \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \lfloor \frac{i}{2}\rfloor \end{array}, \quad \mathbf{A}_j^{(0)} := \mathbf{A}_j. \tag{92}$$

In [16], it was proved for B-spline scaling functions that

$$h^{(i)}_{\lceil i/2\rceil},\dots,h_{N-\lfloor i/2\rfloor} \neq 0, \quad i = 1,\dots,N. \tag{93}$$

Therefore, the ellimination is always possible. The elimination matrices are of the form

$$H_j^{(2i-1)} := \operatorname{diag}\left(\mathbf{I}_{i-1},\mathbf{U}_{2i-1},\dots,\mathbf{U}_{2i-1},\mathbf{I}_{N-1}\right), \tag{94}$$

$$H_j^{(2i)} := \operatorname{diag}\left(\mathbf{I}_{N-i},\mathbf{L}_{2i},\dots,\mathbf{L}_{2i},\mathbf{I}_{i-1}\right), \tag{95}$$

where

$$\mathbf{U}_{i+1} := \begin{pmatrix} 1 & -\frac{h^{(i)}_{\lceil i/2\rceil}}{h^{(i)}_{\lceil i/2\rceil+1}} \\ 0 & 1 \end{pmatrix}, \quad \mathbf{L}_{i+1} := \begin{pmatrix} 1 & 0 \\ -\frac{h^{(i)}_{N-\lfloor i/2\rfloor}}{h^{(i)}_{N-\lfloor i/2\rfloor-1}} & 1 \end{pmatrix}. \tag{96}$$

It is easy to see that indeed

$$\mathbf{A}_j^{(i)} = \mathbf{H}_j^{(i)}\mathbf{A}_j^{(i-1)}. \tag{97}$$

After N elimination steps we obtain the matrix $\mathbf{A}_j^{(N)}$ which looks as follows

$$\mathbf{A}_j^{(N)} = \mathbf{H}_j\mathbf{A}_j = \left(\begin{array}{cccc} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \\ b & 0 & & \\ 0 & 0 & & \\ 0 & b & & \\ \vdots & 0 & \ddots & \\ & & & b \\ & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \lceil \frac{N}{2}\rceil \\ \\ \\ \\ \\ \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \lfloor \frac{N}{2}\rfloor \end{array}, \quad \text{where} \quad \mathbf{H}_j := \mathbf{H}_j^{(N)}\dots\mathbf{H}_j^{(1)}, \tag{98}$$

with $b \neq 0$. We define

$$\mathbf{B}_j := \left(\mathbf{A}_j^{(N)}\right)^{-1} = \begin{pmatrix} \underbrace{0\ \dots\ 0}_{\lceil \frac{N}{2}\rceil} & b^{-1} & 0 & 0 & 0 & \dots & & 0 \\ 0\ \dots\ 0 & 0 & 0 & b^{-1} & 0 & \dots & & 0 \\ & & & & & \ddots & & \\ & & & & & & b^{-1} & \underbrace{0\ \dots\ 0}_{\lfloor \frac{N}{2}\rfloor} \end{pmatrix} \tag{99}$$

and

$$
\mathbf{F}_j := \begin{pmatrix}
0 & 0 & & \\
\vdots & \vdots & & \\
0 & 0 & & \\
1 & 0 & & \\
0 & 0 & & \\
0 & 1 & & \\
\vdots & 0 & \ddots & \\
 & & & 1 \\
 & & & 0 \\
 & & & \vdots \\
 & & & 0
\end{pmatrix}
\begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} \lceil \frac{N}{2} \rceil - 1 \\ \\ \\ \\ \\ \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} \lfloor \frac{N}{2} \rfloor + 1 \end{matrix}
. \tag{100}
$$

Then, we have

$$
\mathbf{B}_j \mathbf{F}_j = \mathbf{0}. \tag{101}
$$

After these preparations we define extended versions of the matrices $\mathbf{H}_j$, $\mathbf{A}_j$, $\mathbf{A}_j^{(N)}$, and $\mathbf{B}_j$ by

$$
\hat{\mathbf{H}}_j := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{H}_j & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \quad \hat{\mathbf{A}}_j^{(N)} := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{A}_j^{(N)} & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \tag{102}
$$

$$
\hat{\mathbf{A}}_j := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{A}_j & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \quad \hat{\mathbf{B}}_j^T := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{B}_j^T & \\ & & \mathbf{I}_{N-1} \end{pmatrix}. \tag{103}
$$

Note that $\hat{\mathbf{H}}_j$, $\hat{\mathbf{A}}_j$, $\hat{\mathbf{A}}_j^{(N)}$, and $\hat{\mathbf{B}}_j$ are all matrices of the size $(\#\mathscr{I}_{j+1}) \times (\#\mathscr{I}_j)$. Hence, the completion of $\hat{\mathbf{A}}_j^{(N)}$ has to be a $(\#\mathscr{I}_{j+1}) \times 2^j$. On the contrary to the construction in [16], we define an expanded version of $\mathbf{F}_j$ as in [5], because it leads to a more natural formulation, when the entries of both the refinement matrices belong to $\sqrt{2}\mathbb{Q}$. The difference is in multiplication by $\sqrt{2}$,

$$
\hat{\mathbf{F}}_j := \sqrt{2} \begin{pmatrix}
 & \mathbf{O} & \\
\mathbf{I}_{\lceil \frac{N}{2} \rceil - 1} & & \\
 & \mathbf{F}_j & \\
 & & \mathbf{I}_{\lfloor \frac{N}{2} \rfloor} \\
 & \mathbf{O} &
\end{pmatrix}
\begin{matrix} \}\, N-1 \\ \\ \\ \\ \}\, N-1 \end{matrix}
. \tag{104}
$$

The above findings can be summarized as follows.

Lemma 12. *The following relations hold:*

$$
\hat{\mathbf{B}}_j \hat{\mathbf{A}}_j^{(N)} = \mathbf{I}_{\#\mathscr{I}_j}, \quad \frac{1}{2} \hat{\mathbf{F}}_j^T \hat{\mathbf{F}}_j = \mathbf{I}_{2^j} \tag{105}
$$

and

$$
\hat{\mathbf{B}}_j \hat{\mathbf{F}}_j = \mathbf{0}, \quad \hat{\mathbf{F}}_j^T \hat{\mathbf{A}}_j^{(N)} = \mathbf{0}. \tag{106}
$$

The proof of this lemma is similar to the proof in [16]. Note the refinement matrix $\mathbf{M}_{j,0}$ can be factorized as

$$
\mathbf{M}_{j,0} = \mathbf{P}_j \hat{\mathbf{A}}_j = \mathbf{P}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{A}}_j^{(N)} \tag{107}
$$

with

$$\mathbf{P}_j := \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{I}_{\#\mathscr{I}_{j+1-2N}} & \\ & & \mathbf{M}_R \end{pmatrix}. \tag{108}$$

Now we are able to define the initial stable completions of the refinement matrices $\mathbf{M}_{j,0}$.

Lemma 13. *Under the above assumptions, the matrices*

$$\check{\mathbf{M}}_{j,1} := \mathbf{P}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{F}}_j, \quad j \geq j_0, \tag{109}$$

are uniformly stable completions of the matrices $\mathbf{M}_{j,0}$. Moreover, the inverse

$$\check{\mathbf{G}}_j = \begin{pmatrix} \check{\mathbf{G}}_{j,0} \\ \check{\mathbf{G}}_{j,1} \end{pmatrix} \tag{110}$$

of $\check{\mathbf{M}}_j = \left(\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1}\right)$ is given by

$$\check{\mathbf{G}}_{j,0} = \hat{\mathbf{B}}_j \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}, \quad \check{\mathbf{G}}_{j,1} = \frac{1}{2} \hat{\mathbf{F}}_j^T \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}. \tag{111}$$

The proof of this lemma is straightforward and similar to the proof in [16]. Then using the initial stable completion $\check{\mathbf{M}}_{j,1}$ we are already able to contruct wavelets according to the Theorem 11.

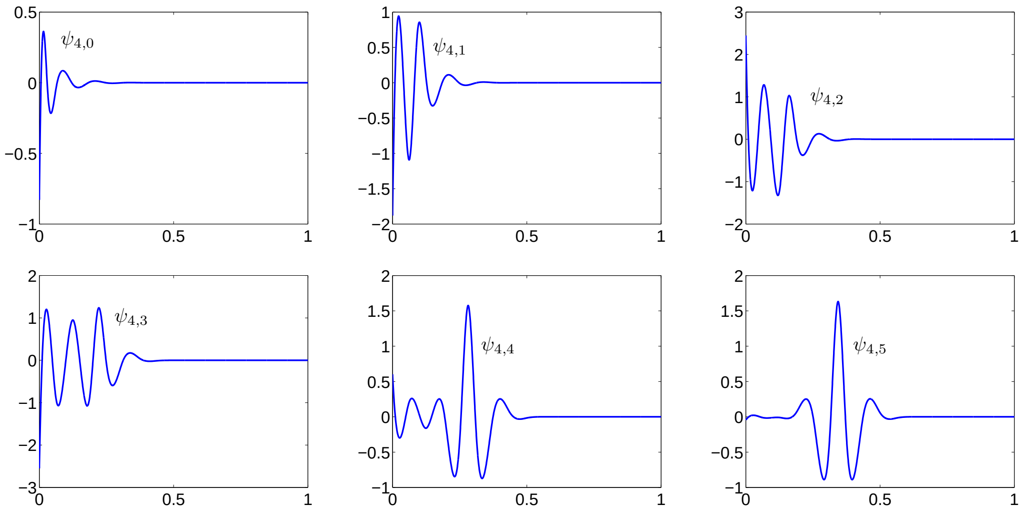

Fig. 5 Some primal wavelets for $N = 4$ and $\tilde{N} = 6$ without boundary conditions.

7 Norm equivalences

In this section, we prove norm equivalences and we show that Ψ and $\tilde{\Psi}$ are Riesz bases for the space $L^2([0,1])$. Furthermore, we show that $\left\{2^{-s|\lambda|}\psi_\lambda, \lambda \in \mathscr{J}\right\}$ is a Riesz basis for Sobolev space $H^s([0,1])$ for some s specified below. The proofs are based on the theory developed in [13] and [16].

Let us define

$$\gamma := \sup\left\{s : \phi \in H^s(\mathbb{R})\right\}, \quad \tilde{\gamma} := \sup\left\{s : \tilde{\phi} \in H^s(\mathbb{R})\right\}. \tag{112}$$

It is known that $\gamma = N - \frac{1}{2}$. The Sobolev exponent of smoothness $\tilde{\gamma}$ can be computed by the method from [21]. The functions in $\boldsymbol{\Phi}_j$ and $\boldsymbol{\Psi}_j$, $j \geq j_0$, have the Sobolev regularity at least γ, because the primal scaling functions are B-splines and the primal wavelets are finite linear combinations of the primal scaling functions. Similarly, the functions in $\tilde{\boldsymbol{\Phi}}_j$ and $\tilde{\boldsymbol{\Psi}}_j$, $j \geq j_0$, have the Sobolev regularity at least $\tilde{\gamma}$.

Theorem 14. *i) The sets* $\{\boldsymbol{\Phi}_j\} := \{\boldsymbol{\Phi}_j\}_{j \geq j_0}$ *and* $\{\tilde{\boldsymbol{\Phi}}_j\} := \{\tilde{\boldsymbol{\Phi}}_j\}_{j \geq j_0}$ *are uniformly stable, i.e.*

$$c \, \|b\|_{l_2(\mathscr{I}_j)} \leq \left\| \sum_{k \in \mathscr{I}_j} b_k \phi_{j,k} \right\| \leq C \, \|b\|_{l_2(\mathscr{I}_j)} \quad \text{for all } b = \{b_k\}_{k \in \mathscr{I}_j} \in l^2(\mathscr{I}_j), \ j \geq j_0. \tag{113}$$

ii) For all $j \geq j_0$*, the Jackson inequalities hold, i.e.*

$$\inf_{v_j \in S_j} \|v - v_j\| \lesssim 2^{-sj} \|v\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < N, \tag{114}$$

and

$$\inf_{v_j \in \tilde{S}_j} \|v - v_j\| \lesssim 2^{-sj} \|v\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < \tilde{N}. \tag{115}$$

iii) For all $j \geq j_0$*, the Bernstein inequalities hold, i.e.*

$$\|v_j\|_{H^s([0,1])} \lesssim 2^{sj} \|v_j\| \quad \text{for all } v_j \in S_j \text{ and } s < \gamma, \tag{116}$$

and

$$\|v_j\|_{H^s([0,1])} \lesssim 2^{sj} \|v_j\| \quad \text{for all } v_j \in \tilde{S}_j \text{ and } s < \tilde{\gamma}. \tag{117}$$

Proof i) Due to Lemma 2.1 in [16], the collections $\{\boldsymbol{\Phi}_j\} := \{\boldsymbol{\Phi}_j\}_{j \geq j_0}$ and $\{\tilde{\boldsymbol{\Phi}}_j\} := \{\tilde{\boldsymbol{\Phi}}_j\}_{j \geq j_0}$ are uniformly stable, if $\boldsymbol{\Phi}_j$ and $\tilde{\boldsymbol{\Phi}}_j$ are biorthogonal,

$$\|\phi_{j,k}\| \lesssim 1, \|\tilde{\phi}_{j,k}\| \lesssim 1, \quad k \in \mathscr{I}_j, \ j \geq j_0, \tag{118}$$

and $\boldsymbol{\Phi}_j$ and $\tilde{\boldsymbol{\Phi}}_j$ are locally finite, i.e.

$$\#\{k' \in \mathscr{I}_j : \Omega_{j,k'} \cap \Omega_{j,k} \neq \emptyset\} \lesssim 1, \quad \text{for all } k \in \mathscr{I}_j, \ j \geq j_0, \tag{119}$$

and

$$\#\{k' \in \mathscr{I}_j : \tilde{\Omega}_{j,k'} \cap \tilde{\Omega}_{j,k} \neq \emptyset\} \lesssim 1, \quad \text{for all } k \in \mathscr{I}_j, \ j \geq j_0, \tag{120}$$

where $\Omega_{j,k} := \text{supp } \phi_{j,k}$ and $\tilde{\Omega}_{j,k} := \text{supp } \tilde{\phi}_{j,k}$. By (40) the sets $\boldsymbol{\Phi}_j$ and $\tilde{\boldsymbol{\Phi}}_j$ are biorthogonal. The properties (118), (119), and (120) follow from (15), (21), and (35).

ii) By Lemma 2.1 in [16], the Jackson inequalities are the consequences of i) and the polynomial exactness (17) and (68).

iii) The Bernstein inequalities follow from i) and the regularity of basis functions, for details see [14].

The following fact follows from [13].

Corollary 1. *We have the norm equivalences*

$$\|v\|_{H^s}^2 \sim 2^{2sj_0} \left\| \sum_{k \in \mathscr{I}_{j_0}} \langle v, \tilde{\phi}_{j_0,k} \rangle \phi_{j_0,k} \right\|^2 + \sum_{j=j_0}^{\infty} 2^{2sj} \left\| \sum_{k \in \mathscr{J}_j} \langle v, \tilde{\psi}_{j,k} \rangle \psi_{j,k} \right\|^2, \tag{121}$$

where $v \in H^s([0,1])$ *and* $s \in (-\tilde{\gamma}, \gamma)$.

The norm equivalence for $s = 0$, Theorem 11, and Lemma 13, imply that

$$\boldsymbol{\Psi} := \boldsymbol{\Phi}_{j_0} \cup \bigcup_{j=j_0}^{\infty} \boldsymbol{\Psi}_j \quad \text{and} \quad \tilde{\boldsymbol{\Psi}} := \tilde{\boldsymbol{\Phi}}_{j_0} \cup \bigcup_{j=j_0}^{\infty} \tilde{\boldsymbol{\Psi}}_j \tag{122}$$

are biorthogonal Riesz bases of the space $L^2([0,1])$. Let us define

$$\mathbf{D} = \left(\mathbf{D}_{\lambda,\tilde{\lambda}}\right)_{\lambda,\tilde{\lambda} \in \mathscr{J}}, \quad \mathbf{D}_{\lambda,\tilde{\lambda}} := \delta_{\lambda,\tilde{\lambda}} 2^{|\lambda|}, \quad \lambda, \tilde{\lambda} \in \mathscr{J}. \tag{123}$$

The relation (121) implies that $\mathbf{D}^{-s}\boldsymbol{\Psi}$ is a Riesz basis of the Sobolev space $H^s([0,1])$ for $s \in (-\tilde{\gamma}, \gamma)$.

8 Adaptation to Complementary Boundary Conditions

In this section, we introduce a construction of well-conditioned spline-wavelet bases on the interval satisfying complementary boundary conditions of the first order. This means that the primal wavelet basis is adapted to homogeneous Dirichlet boundary conditions of the first order, whereas the dual wavelet basis preserves the full degree of polynomial exactness. This construction is based on the spline-wavelet bases constructed above. As already mentioned in Remark 7, in the linear case, i.e. $N = 2$, our bases are identical to the bases constructed in [24]. The adaptation of these bases to complementary boundary conditions can be found in [24]. Thus, we consider only the case $N \geq 3$.

Let $\Phi_j = \left\{\phi_{j,k}, k = -N+1, \ldots, 2^j - 1\right\}$ be defined as above. Note that the functions $\phi_{j,-N+1}$, $\phi_{j,2^j-1}$ are the only two functions which do not vanish at zero. Therefore, defining

$$\Phi_j^{comp} = \left\{\phi_{j,k}, k = -N+2, \ldots, 2^j - 2\right\} \tag{124}$$

we obtain the primal scaling bases satisfying complementary boundary conditions of the first order.

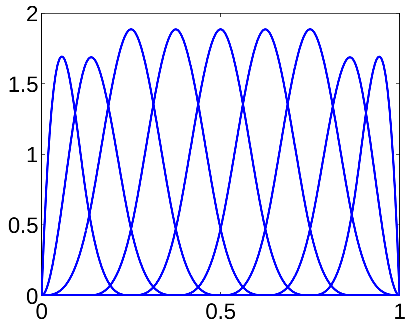

Fig. 6 Primal scaling functions for $N = 4$ and $j = 3$ satisfying complementary boundary conditions of the first order.

On the dual side, we also need to omit one scaling function at each boundary, because the number of the primal scaling functions must be the same as the number of the dual scaling functions. Let $\Theta_j = \left\{\theta_{j,k}, k \in \mathscr{I}_j\right\}$ be the dual scaling basis on the level j before biorthogonalization from Section 4. There are the boundary functions of two types. Recall that the functions $\theta_{j,-N+1}$, ..., $\theta_{j,-N+\tilde{N}}$ are the left boundary functions of the first type which are defined to preserve polynomial exactness of the order $\tilde{N}$. The functions $\theta_{j,-N+\tilde{N}+1}$, ..., $\theta_{j,\tilde{N}-2}$ are the left boundary functions of the second type. The right boundary scaling functions are then derived by the reflection of the left boundary functions. Since we want to preserve the full degree of polynomial exactness, we omit one function of the second type at each boundary. Thus, we define

$$\theta_{j,k}^{comp} = \begin{cases} \theta_{j,k-1}, & k = -N+2, \ldots, -N+\tilde{N}+1, \\ \theta_{j,k}, & k = -N+\tilde{N}+2, \ldots, 2^j - \tilde{N} - 2, \\ \theta_{j,k+1}, & k = 2^j - \tilde{N} - 1, \ldots, 2^j - 2. \end{cases} \tag{125}$$

Since the set $\Theta_j^{comp} := \left\{\theta_{j,k}^{comp} : k = -N+2, \ldots, 2^j - 2\right\}$ is not biorthogonal to Φ_j, we derive a new set $\tilde{\Phi}_j^{comp}$ from Θ_j^{comp} by biorthogonalization. Let $\mathbf{Q}_j^{comp} = \left(\left\langle \phi_{j,k}, \theta_{j,l}^{comp} \right\rangle\right)_{k,l=-N+2}^{2^j-2}$, then viewing $\tilde{\Phi}_j^{comp}$ and Θ_j^{comp} as column vectors we define

$$\tilde{\Phi}_j^{comp} := \left(\mathbf{Q}_j^{comp}\right)^{-T} \Theta_j^{comp}. \tag{126}$$

Our next goal is to determine the corresponding wavelets

$$\Psi_j^{comp} := \left\{\psi_{j,k}^{comp}, k = 0, \ldots, 2^j - 1\right\}, \quad \tilde{\Psi}_j^{comp} := \left\{\tilde{\psi}_{j,k}^{comp}, k = 0, \ldots, 2^j - 1\right\}. \tag{127}$$

It can be done by the method of a stable completion as in Section 6.

9 Quantitative Properties of Constructed Bases

In this section the condition numbers of the scaling bases, the single-scale wavelet bases and the multiscale wavelet bases are computed. As in [24] it can be improved by the L^2-normalization on the primal side. It will be shown that in the case of cubic spline wavelets bases the construction presented in this paper yields optimal L^2-stability, which is not the case of constructions in [16] and [24]. The condition numbers of the scaling bases and the wavelet bases satisfying the complementary boundary conditions of the first order are presented as well. The other criteria for the effectiveness of the wavelet bases is the condition number of the corresponding preconditioned stiffness matrix. To improve it further we apply an orthogonal transformation to the scaling basis on the coarsest level and then we use a diagonal matrix for preconditioning.

It is known that Riesz bounds (2) of the basis Φ_j can be computed by

$$c = \sqrt{\lambda_{min}\left(\mathbf{G}_j\right)}, \quad C = \sqrt{\lambda_{max}\left(\mathbf{G}_j\right)}, \tag{128}$$

where $\mathbf{G}_j$ is the Gram matrix, i.e. $\mathbf{G}_j = \left(\left\langle \phi_{j,k}, \phi_{j,l} \right\rangle\right)_{k,l\in\mathscr{I}_j}$, and $\lambda_{min}\left(\mathbf{G}_j\right)$, $\lambda_{max}\left(\mathbf{G}_j\right)$ denote the smallest and the largest eigenvalue of $\mathbf{G}_j$, respectively. The Riesz bounds of $\tilde{\Phi}_j$, Ψ_j and $\tilde{\Psi}_j$ are computed in a similar way.

The condition numbers of the constructed bases are presented in Table 2. To improve it further we provide a diagonal rescaling in the following way:

$$\phi_{j,k}^N = \frac{\phi_{j,k}}{\sqrt{\left\langle \phi_{j,k}, \phi_{j,k} \right\rangle}}, \quad \tilde{\phi}_{j,k}^N = \tilde{\phi}_{j,k} * \sqrt{\left\langle \phi_{j,k}, \phi_{j,k} \right\rangle}, \quad k \in \mathscr{I}_j, \quad j \geq j_0, \tag{129}$$

$$\psi_{j,k}^N = \frac{\psi_{j,k}}{\sqrt{\left\langle \psi_{j,k}, \psi_{j,k} \right\rangle}}, \quad \tilde{\psi}_{j,k}^N = \tilde{\psi}_{j,k} * \sqrt{\left\langle \psi_{j,k}, \psi_{j,k} \right\rangle}, \quad k \in \mathscr{J}_j, \quad j \geq j_0. \tag{130}$$

Then the new primal scaling and wavelet bases are normalized with respect to the L^2-norm. As already mentioned in Remark 7, the resulting bases for $N = 2$ are the same as those designed in [24] and [25]. For the quadratic spline-wavelet bases, i.e. $N = 3$, the condition of our bases is comparable to the condition of the bases from [24] and [25]. In [3], it was shown that for any spline wavelet basis of order N on the real line, the condition is bounded below by 2^{N-1}. This result readily carries over to the case of spline wavelet bases on the interval. Now, the constructions from [24], [25] yields the wavelet bases whose Riesz bounds are nearly optimal, i.e. cond $\Psi_j^N \approx 2^{N-1}$ for $N = 2$ and $N = 3$. Unfortunately, the L^2-stability gets considerably worse for $N \geq 4$. As can

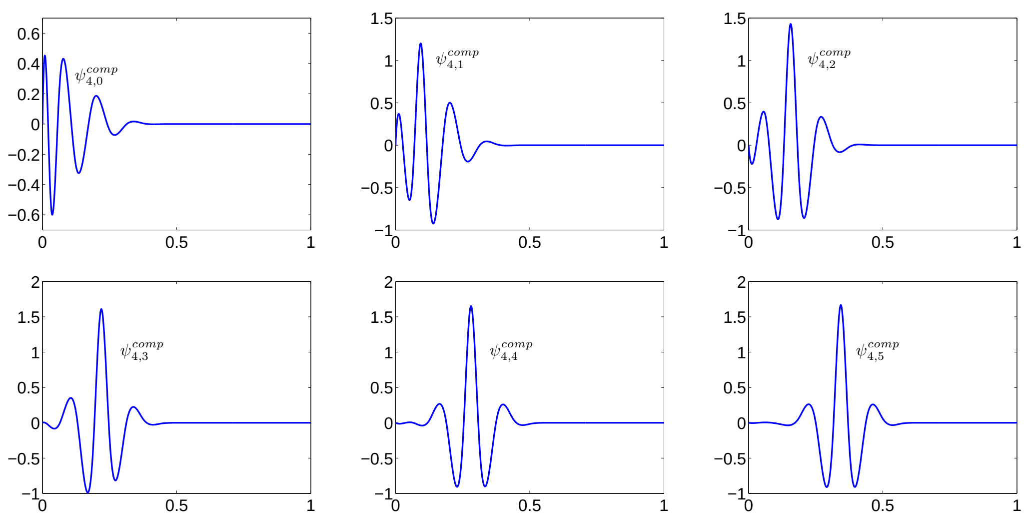

Fig. 7 Some primal wavelets for $N = 4$ and $\tilde{N} = 6$ satisfying the complementary boundary conditions of the first order.

be seen in Table 2, the column "Ψ_j^N", the presented construction seems to yield the optimal L^2-stability also for $N = 4$. Note that the cases $N = 4$, $\tilde{N} = 4$ and $N = 5$, $\tilde{N} < 9$ are not included in Table 2. It was shown in [9] that the corresponding scaling functions and wavelets do not belong to the space L^2.

Table 2 The condition of single-scale scaling and wavelet bases

N	$\tilde{N}$	j	Φ_j	Φ_j^N	$\tilde{\Phi}_j$	$\tilde{\Phi}_j^N$	Ψ_j	Ψ_j^N	$\tilde{\Psi}_j$	$\tilde{\Psi}_j^N$
2	2	10	2.00	1.73	2.30	1.97	2.00	2.00	2.02	2.00
2	4	10	2.00	1.73	2.09	1.80	2.00	2.00	2.04	2.00
2	6	10	2.00	1.73	2.26	2.03	2.00	2.00	2.30	2.26
2	8	10	2.00	1.73	2.90	2.78	2.34	2.22	3.14	3.81
3	3	10	3.25	2.76	7.58	6.37	4.49	4.00	7.07	4.27
3	5	10	3.25	2.76	3.93	3.49	4.63	4.00	5.55	4.05
3	7	10	3.25	2.76	3.53	3.11	4.55	4.00	5.13	4.01
3	9	10	3.25	2.76	3.75	3.32	4.44	4.00	5.51	4.23
4	6	10	5.18	4.42	10.88	9.07	14.02	8.00	24.36	9.23
4	8	10	5.18	4.42	6.69	5.88	13.96	8.00	16.98	8.20
4	10	10	5.18	4.42	5.83	5.16	13.82	8.00	15.27	8.00
5	9	10	8.32	7.13	29.87	25.23	67.74	27.44	169.76	68.90
5	11	10	8.32	7.13	12.10	11.74	16.00	16.00	45.12	21.65
5	13	10	8.32	7.13	28.49	45.60	16.00	16.00	22.64	22.23

In Table 3 the condition of the multiscale wavelet bases $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ is presented.

It is known that the condition number of the original basis on the real line from [9] is less than or equal to the condition number of the interval wavelet basis where the inner functions are identical to the basis functions from [9]. Therefore, we use the condition number of the wavelet bases from [9] as a benchmark. In Table 4, we compare the condition number of our wavelet bases and the wavelet bases from [9], [24].

In case $N = 5$, the condition numbers of the scaling bases and the single-scale wavelet bases seem to be optimal, but the condition numbers of the multiscale wavelet bases are not close to the condition numbers of the corresponding wavelet bases on the real line. However, in comparison with [24] the condition number is significantly improved for $N = 5$ and $\tilde{N} = 9$. Therefore the construction of well-conditioned high-order biorthogonal spline wavelets is still an open problem.

Table 3 The condition of the multiscale wavelet bases

N	$\tilde{N}$	j_0	$\Psi_{j_0,1}^N$	$\Psi_{j_0,2}^N$	$\Psi_{j_0,3}^N$	$\Psi_{j_0,4}^N$	$\Psi_{j_0,5}^N$	$\tilde{\Psi}_{j_0,1}^N$	$\tilde{\Psi}_{j_0,2}^N$	$\tilde{\Psi}_{j_0,3}^N$	$\tilde{\Psi}_{j_0,4}^N$	$\tilde{\Psi}_{j_0,5}^N$
2	2	2	1.98	2.27	2.52	2.65	2.76	2.20	2.42	2.65	2.78	2.87
2	4	3	2.13	2.25	2.30	2.33	2.34	2.15	2.26	2.31	2.33	2.35
2	6	4	2.47	2.71	2.84	2.92	2.99	2.60	2.78	2.88	2.94	3.00
2	8	4	3.71	4.77	5.35	5.68	5.89	4.44	5.17	5.57	5.82	5.98
3	3	3	4.92	6.01	7.15	7.87	8.50	7.25	8.54	9.50	10.08	10.48
3	5	4	4.51	4.82	5.01	5.10	5.14	4.63	4.98	5.11	5.15	5.16
3	7	4	4.19	4.38	4.44	4.46	4.48	4.24	4.39	4.45	4.48	4.49
3	9	5	4.44	4.55	4.61	4.64	4.65	4.48	4.58	4.62	4.64	4.66
4	6	4	9.55	10.90	11.88	12.50	12.90	10.88	12.90	13.35	13.48	13.58
4	8	5	8.01	8.31	8.54	8.68	8.76	8.23	8.60	8.73	8.79	8.81
4	10	5	7.89	8.02	8.09	8.12	8.13	7.93	8.05	8.11	8.13	8.14
5	9	5	30.22	64.60	75.17	81.03	84.81	72.34	83.19	87.93	90.11	91.27
5	11	5	84.40	631.61	3004.08	$> 10^4$	$> 10^4$	54.08	401.23	3004.08	$> 10^4$	$> 10^4$

The condition of the single-scale bases adapted to complementary boundary condition of the first order are listed in Table 5. We improve the condition of the constructed bases by the L^2-normalization. For $N = 4$ the condition number of the bases constructed in this paper is again significantly better than the condition of the bases from [24].

The other criteria for the effectiveness of a wavelet basis is the condition number of the corresponding stiffness matrix. Here, let us consider the stiffness matrix for the Poisson equation:

$$\mathbf{A}_{j_0,s} = \left(\left\langle \left(\psi_{j,k}^{comp}\right)', \left(\psi_{l,m}^{comp}\right)' \right\rangle \right)_{\psi_{j,k}^{comp}, \psi_{l,m}^{comp} \in \Psi_{j_0,s}^{comp}}, \tag{131}$$

where $\Psi_{j_0,s}^{comp} = \Phi_{j_0}^{comp} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j^{comp}$ denotes the multiscale basis adapted to complementary boundary conditions. It is well-known that the condition number of $\mathbf{A}_{j_0,s}$ increases quadratically with the matrix size. To remedy this, we use the diagonal matrix for preconditioning

$$\mathbf{A}_{j_0,s}^{prec} = \mathbf{D}_{j_0,s}^{-1} \mathbf{A}_{j_0,s} \mathbf{D}_{j_0,s}^{-1}, \quad \mathbf{D}_{j_0,s} = \operatorname{diag} \left(\left\langle \left(\psi_{j,k}^{comp}\right)', \left(\psi_{j,k}^{comp}\right)' \right\rangle^{1/2} \right)_{\psi_{j,k}^{comp} \in \Psi_{j_0,s}^{comp}}. \tag{132}$$

To improve further the condition number of $\mathbf{A}_{j_0,s}^{prec}$ we apply the orthogonal transformation to the scaling basis on the coarsest level as in [7] and then we use the diagonal matrix for preconditioning. We denote the obtained matrix by $\mathbf{A}_{j_0,s}^{ort}$. The condition numbers of the resulting matrices are listed in Table 6.

10 Adaptive wavelet methods

In recent years adaptive wavelet methods have been successfully used for solving partial differential as well as integral equations, both linear and nonlinear. It has been shown that these methods converge and that they are asymptotically optimal in the sense that a storage and a number of floating point operations, needed to resolve the problem with desired accuracy, remain proportional to the problem size when the resolution of the discretization is refined. Thus, the computational complexity for all steps of the algorithm is controlled.

The effectiveness of adaptive wavelet methods is strongly influenced by the choice of a wavelet basis, in particular by the condition of the basis. In this section, our intention is to compare the quantitative behaviour of the adaptive wavelet method for the cubic spline wavelet bases constructed in this paper and the cubic spline wavelet bases from [24].

Table 4 The condition number of our multiscale wavelet bases $\Psi_{j_0,5}^N$ and $\tilde{\Psi}_{j_0,5}^N$ and multiscale wavelet bases from [9] and [24]

N	$\tilde{N}$	j_0	s	$\Psi_{j_0,5}^{CDF}$	$\Psi_{j_0,5}^{Primbs}$	$\Psi_{j_0,5}^{N}$	$\tilde{\Psi}_{j_0,5}^{CDF}$	$\tilde{\Psi}_{j_0,5}^{Primbs}$	$\tilde{\Psi}_{j_0,5}^{N}$
3	3	3	5	6.68	6.25	8.50	8.52	8.17	10.48
3	5	4	5	4.36	5.31	5.14	4.37	5.36	5.16
3	7	4	5	4.04	8.57	4.48	4.04	8.63	4.49
3	9	5	5	4.00	25.40	4.65	4.00	25.76	4.66
4	6	4	5	9.89	141.95	12.90	10.43	160.54	13.58
4	8	5	5	8.27	257.41	8.76	8.27	258.56	8.81
4	10	5	5	8.04	917.10	8.13	8.04	935.38	8.14
4	12	5	5	8.01	3971.65	8.44	8.01	3992.29	8.45
5	9	5	5	17.64	$> 10^4$	84.81	18.01	$> 10^4$	91.27

Table 5 The condition of scaling bases and single-scale wavelet bases satisfying complementary boundary conditions of the first order

N	$\tilde{N}$	j	Φ_j	Φ_j^N	$\tilde{\Phi}_j$	$\tilde{\Phi}_j^N$	Ψ_j	Ψ_j^N	$\tilde{\Psi}_j$	$\tilde{\Psi}_j^N$
3	3	10	2.74	2.74	4.49	4.34	4.00	4.00	4.13	4.00
3	5	10	2.74	2.74	4.94	4.58	4.00	4.00	6.68	6.27
3	7	10	2.74	2.74	8.61	8.33	4.84	4.27	12.11	16.05
3	9	10	2.74	2.74	17.94	17.78	8.16	6.25	25.17	46.10
4	6	10	4.53	4.31	7.90	6.83	9.47	8.00	16.46	8.00
4	8	10	4.53	4.31	11.16	10.04	8.46	8.03	25.40	15.32
4	10	10	4.53	4.31	17.90	16.97	8.39	8.42	37.78	35.93
5	9	10	7.58	6.89	15.81	13.85	35.01	16.02	80.84	33.60
5	11	10	7.58	6.89	29.00	26.39	16.00	16.00	132.90	74.70
5	13	10	7.58	6.89	289.13	440.54	118.19	89.12	720.32	5884.77

Example 15. We consider the one-dimensional Poisson equation with homogeneous Dirichlet boundary conditions

$$-u'' = f, \quad \text{in} \quad \Omega = (0,1), \quad u(0) = u(1) = 0, \tag{133}$$

whose solution u is given by

$$u(x) = 4\frac{e^{50x}-1}{e^{50}-1}\left(1 - \frac{e^{50x}-1}{e^{50}-1}\right) + x(1-x), \quad x \in \Omega. \tag{134}$$

The solution exhibits steep a gradient near the boundary, see Figure 8. Let us define the diagonal

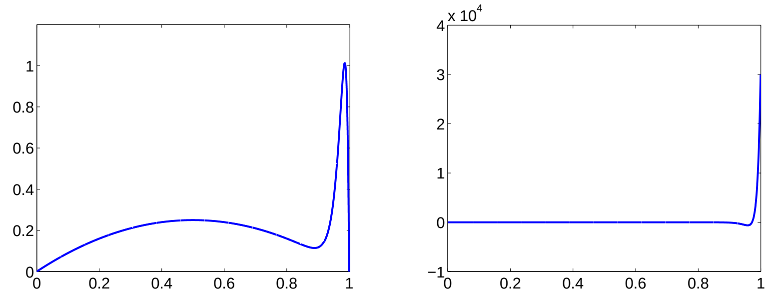

Fig. 8 The exact solution and the right hand side of (133).

matrix

$$\mathbf{D} = \operatorname{diag}\left(\left\langle \psi'_{j,k}, \psi'_{j,k}\right\rangle^{1/2}\right)_{\psi_{j,k}\in\Psi} \tag{135}$$

and operators

$$\mathbf{A} = \mathbf{D}^{-1}\left\langle \Psi', \Psi'\right\rangle \mathbf{D}^{-1}, \quad \mathbf{f} = \mathbf{D}^{-1}\left\langle f, \Psi\right\rangle. \tag{136}$$

Then the variational formulation of (133) is equivalent to

$$\mathbf{AU} = \mathbf{f} \tag{137}$$

and the solution u is given by $u = \mathbf{U}\mathbf{D}^{-1}\Psi$. We solve the infinite dimensional problem (137) by the inexact damped Richardson iterations. This algorithm was originally proposed by Cohen, Dahmen and DeVore in [10]. Here, we use a modified version from [30].

Figure 9 shows a convergence history for the spline-wavelet bases designed in this contribution with $N = 4$ and $\tilde{N} = 6$ denoted by CF and the spline-wavelet bases with the same polynomial exactness from [24]. We use also the algorithm with the stiffness matrix $\mathbf{A}^{ort}$ which has lower condition number, see Table 6. Its convergence history is denoted by CFort. Note that the relative error in the energy norm for an adaptive scheme with our bases is significantly smaller even though the number of the involved basis functions is about half compared with the bases in [24].

Table 6 The condition number of the stiffness matrices $\mathbf{A}^{prec}_{j,s}$, $\mathbf{A}^{ort}_{j,s}$ of the size $M \times M$

N	$\tilde{N}$	j	s	M	$\mathbf{A}^{prec}_{j,s}$	$\mathbf{A}^{ort}_{j,s}$	N	$\tilde{N}$	j	s	M	$\mathbf{A}^{prec}_{j,s}$	$\mathbf{A}^{ort}_{j,s}$
3	3	3	1	16	12.24	3.78	4	6	4	1	33	48.98	15.25
			4	128	12.82	5.05				4	259	51.61	16.15
			7	1024	12.86	5.37				7	2049	50.28	16.31
3	5	4	1	32	52.97	4.20	4	8	5	1	65	205.56	15.92
			4	256	55.09	8.41				4	513	208.88	26.80
			7	2048	55.24	9.47				7	4097	209.31	27.69
3	7	4	1	32	71.07	10.74	5	7	5	1	66	183.57	159.08
			4	256	71.90	33.52				4	514	214.27	214.40
			7	2048	71.91	38.66				7	4098	222.57	222.62
4	4	4	1	33	47.02	15.38	5	9	5	1	66	191.19	171.91
			4	259	50.01	18.13				4	514	225.92	225.69
			7	2049	50.28	18.91				7	4098	233.24	233.24

Example 16. We consider the two-dimensional Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega = (0,1)^2, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{138}$$

with the solution u given by

$$u(x,y) = u(x)\,u(y), \quad (x,y) \in \Omega, \tag{139}$$

where $u(x)$, $u(y)$ are given by (134). We use the adaptive wavelet scheme with the cubic wavelet basis adapted to homogeneous Dirichlet boundary conditions of the first order. The convergence history for our wavelet bases with and without orthogonalization and wavelet bases from [24] is shown in Figure 10.

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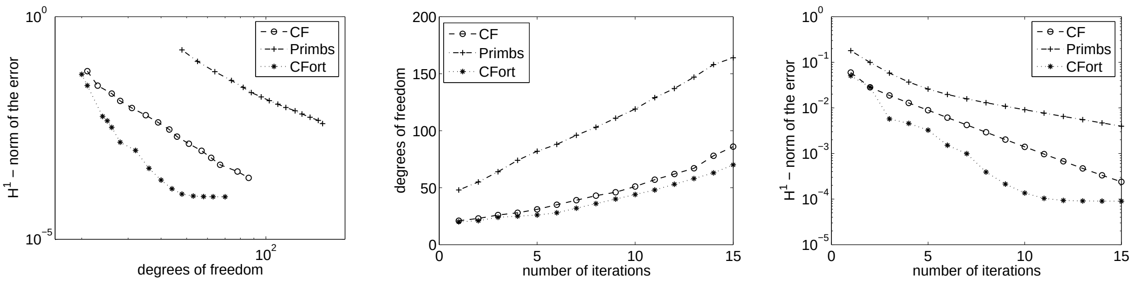

Fig. 9 Convergence history for 1d example, comparison of our wavelet bases with and without orthogonalization and wavelet bases from [24].

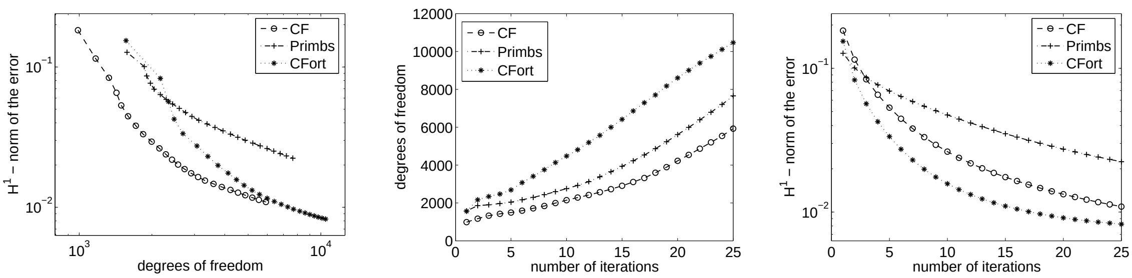

Fig. 10 Convergence history for 2d example, comparison of our wavelet bases with and without orthogonalization and wavelet bases from [24].

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Cubic spline wavelets with complementary boundary conditions

Dana Černá[a], Václav Finěk[a]

[a]*Department of Mathematics and Didactics of Mathematics, Technical University in Liberec, Studentská 2, 461 17 Liberec, Czech Republic*

Abstract

We propose a new construction of a stable cubic spline-wavelet basis on the interval satisfying complementary boundary conditions of the second order. It means that the primal wavelet basis is adapted to homogeneous Dirichlet boundary conditions of the second order, while the dual wavelet basis preserves the full degree of polynomial exactness. We present quantitative properties of the constructed bases and we show superiority of our construction in comparison to some other known spline wavelet bases in an adaptive wavelet method for the partial differential equation with the biharmonic operator.

Keywords: wavelet, cubic spline, complementary boundary conditions, homogeneous Dirichlet boundary conditions, condition number
2000 MSC: 46B15, 65N12, 65T60

1. Introduction

In recent years wavelets have been successfully used for solving partial differential equations [2, 11, 12, 16, 27] as well as integral equations [22, 24, 25], both linear and nonlinear. Wavelet bases are useful in the numerical treatment of operator equations, because they are stable, enable high order-approximation, functions from Besov spaces have sparse representation in wavelet bases, condition numbers of stiffness matrices are uniformly bounded and matrices representing operators are typically sparse or quasi-sparse. The quantitative properties of wavelet methods strongly depend on the choice of a wavelet basis, in particular on its condition number. Therefore, a construction of a wavelet basis is always an important issue.

Wavelet bases on a bounded domain are usually constructed in the following way: Wavelets on the real line are adapted to the interval and then by tensor product technique to the n-dimensional cube. Finally by splitting the domain into overlapping or non-overlapping subdomains which are images of a unit cube under appropriate parametric mappings one can obtain a wavelet basis or a wavelet frame on a fairly general domain. Thus, the properties of the employed wavelet basis on the interval are crucial for the properties of the resulting bases or frames on a general domain.

In this paper, we propose a construction of cubic spline wavelet basis on the interval that is adapted to homogeneous Dirichlet boundary conditions of the second order on

Email addresses: dana.cerna@tul.cz (Dana Černá), vaclav.finek@tul.cz (Václav Finěk)

the primal side and preserves the full degree of polynomial exactness on the dual side. Such boundary conditions are called complementary boundary conditions [18]. We compare properties of wavelet bases such as the condition number of the basis and the condition number of the corresponding stiffness matrix. Finally, quantitative behaviour of adaptive wavelet method for several boundary-adapted cubic spline wavelet bases is studied.

First of all, we summarize the desired properties of a constructed basis:

- *Polymial exactness.* Since the primal basis functions are cubic B-splines, the primal multiresolution analysis has polynomial exactness of order four. The dual multiresolution analysis has polynomial exactness of order six. As a consequence, the primal wavelets have six vanishing moments.
- *Riesz basis property.* The functions form a Riesz basis of the space $L^2([0,1])$ and if scaled properly they form a Riesz basis of the space $H_0^2([0,1])$.
- *Locality.* The primal and dual basis functions are local, see definition of locality below. Then the corresponding decomposition and reconstruction algorithms are simple and fast.
- *Biorthogonality.* The primal and dual wavelet bases form a biorthogonal pair.
- *Smoothness.* The smoothness of primal and dual wavelet bases is another desired property. It ensures the validity of norm equivalences.
- *Closed form.* The primal scaling functions and wavelets are known in the closed form. It is a desirable property for the fast computation of integrals involving primal scaling functions and wavelets.
- *Complementary boundary conditions.* Our wavelet basis satisfy complementary boundary conditions of the second order.
- *Well-conditioned bases.* Our objective is to construct a well conditioned wavelet basis.

Many constructions of cubic spline wavelet or multiwavelet bases on the interval have been proposed in recent years. In [5, 17, 26] cubic spline wavelets on the interval were constructed. In [14] cubic spline multiwavelet bases were designed and they were adapted to complementary boundary conditions of the second order in [28]. In this case dual functions are known and are local. Cubic spline wavelet bases were also constructed in [1, 9, 20, 21]. A construction of cubic spline multiwavelet basis was proposed in [19] and this basis was already used for solving differential equations in [8, 23]. However, in these cases duals are not known or are not local. Locality of duals are important in some methods and theory, let us mention construction of wavelet bases on general domain [18], adaptive wavelet methods especially for nonlinear equations, data analysis, signal and image processing. A general method of adaptation of biorthogonal wavelet bases to complementary boundary conditions was presented in [18], but this method often leads to very badly conditioned bases.

This paper is organized as follows: In Section 2 we briefly review the concept of wavelet bases. In Section 3 we propose a construction of primal and dual scaling bases. The refinement matrices are computed in Section 4 and in Section 5 primal and dual wavelets are constructed. Quantitative properties of constructed bases and other known cubic spline wavelet and multiwavelet bases are studied in Section 6. In Section 7 we compare the number of basis functions and the number of iterations needed to resolve the problem with desired accuracy for our bases and bases from [28]. A numerical example is presented for an equation with the biharmonic operator in two dimensions.

2. Wavelet bases

This section provides a short introduction to the concept of wavelet bases in Sobolev spaces. We consider the domain $\Omega \subset \mathbb{R}^d$ and the Sobolev space or its subspace $H \subset H^s(\Omega)$ for nonnegative integer s with an inner product $\langle \cdot, \cdot \rangle_H$, a norm $\|\cdot\|_H$ and a seminorm $|\cdot|_H$. In case $s = 0$ we consider the space $L^2(\Omega)$ and we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the L^2-inner product and the L^2-norm, respectively. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale* or a *level*. Let

$$l^2(\mathcal{J}) := \left\{ \mathbf{v} : \mathcal{J} \to \mathbb{R}, \sum_{\lambda \in \mathcal{J}} |\mathbf{v}_\lambda|^2 < \infty \right\}. \tag{1}$$

A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *wavelet basis* of H, if

i) Ψ is a *Riesz basis* for H, i.e. the closure of the span of Ψ is H and there exist constants $c, C \in (0, \infty)$ such that

$$c \|\mathrm{b}\|_{l^2(\mathcal{J})} \le \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \le C \|\mathrm{b}\|_{l^2(\mathcal{J})}, \quad \mathrm{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{2}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies } (2)\}$, $C_\psi := \inf\{C : C \text{ satisfies } (2)\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi / c_\psi$ is called the *condition number* of Ψ.

ii) The functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \le \mathrm{C} 2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap at any point $x \in \Omega$.

By the Riesz representation theorem, there exists a unique family

$$\tilde{\Psi} = \left\{ \tilde{\psi}_\lambda, \lambda \in \tilde{\mathcal{J}} \right\} \subset H \tag{3}$$

biorthogonal to Ψ, i.e.

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle_H = \delta_{i,j} \delta_{k,l}, \quad \text{for all} \quad (i,k) \in \mathcal{J}, \quad (j,l) \in \tilde{\mathcal{J}}. \tag{4}$$

This family is also a Riesz basis for H. The basis Ψ is called a *primal* wavelet basis, while $\tilde{\Psi}$ is called a *dual* wavelet basis.

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis. A sequence $\mathcal{V} = \{V_j\}_{j \ge j_0}$, of closed linear subspaces $V_j \subset H$ is called a *multiresolution* or *multiscale analysis*, if

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots H \tag{5}$$

and $\cup_{j \ge j_0} V_j$ is complete in H.

The nestedness and the closedness of the multiresolution analysis implies the existence of the *complement spaces* W_j such that $V_{j+1} = V_j \oplus W_j$.

We now assume that V_j and W_j are spanned by sets of basis functions

$$\Phi_j := \{\phi_{j,k}, k \in \mathcal{I}_j\}, \quad \Psi_j := \{\psi_{j,k}, k \in \mathcal{J}_j\}, \tag{6}$$

where $\mathcal{I}_j$ and $\mathcal{J}_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis is given by $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ and the wavelet basis of H is obtained by $\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j$. The dual wavelet system $\tilde{\Psi}$ generates a dual multiresolution analysis $\tilde{\mathcal{V}}$ with a dual scaling basis $\tilde{\Phi}_{j_0}$.

Polynomial exactness of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that

$$\mathbb{P}_{N-1}(\Omega) \subset V_j, \quad \mathbb{P}_{\tilde{N}-1}(\Omega) \subset \tilde{V}_j, \quad j \geq j_0, \tag{7}$$

where $\mathbb{P}_m(\Omega)$ is the space of all algebraic polynomials on Ω of degree less or equal to m.

By Taylor theorem, the polynomial exactness of order $\tilde{N}$ on the dual side is equivalent to $\tilde{N}$ vanishing wavelet moments on the primal side, i.e.

$$\int_\Omega P(x)\, \psi_\lambda(x)\, dx = 0, \quad P \in \mathbb{P}_{\tilde{N}-1}, \ \psi_\lambda \in \bigcup_{j \geq j_0} \Psi_j. \tag{8}$$

3. Construction of Scaling Functions

We propose a new cubic spline wavelet basis with six vanishing wavelet moments satisfying homogeneous Dirichlet boundary conditions of order two. Six vanishing wavelet moments on the primal side is equivalent to the polynomial exactness of order six on the dual side. We choose polynomial exactness of this order, because the dual scaling function of order four does not belong to $L^2(\mathbb{R})$ and the polynomial exactness of order greater than six leads to a larger support of primal wavelets which makes the computation more expensive.

The first step is the construction of primal scaling functions on the unit interval. Primal scaling basis is formed by cubic B-splines on the knots t_k^j defined by

$$t_{-2}^j = t_{-1}^j := 0, \quad t_0^j := \frac{1}{2^{j+1}}, \quad t_k^j := \frac{k}{2^j}, \quad k = 1, \ldots 2^j - 1, \tag{9}$$

$$t_{2^j}^j := \frac{2^{j+1}-1}{2^{j+1}}, \quad t_{2^j+1}^j = t_{2^j+2}^j := 1.$$

The corresponding cubic B-splines are defined by

$$B_k^j(x) := \left(t_{k+4}^j - t_k^j\right) \left[t_k^j, \ldots, t_{k+4}^j\right]_t (t-x)_+^3, \quad x \in [0,1],$$

where $(x)_+ := \max\{0, x\}$ and $[t_1, \ldots t_N]_t f$ is the N-th divided difference of f. The set $\Phi_j := \{\phi_{j,k}, k = -2, \ldots, 2^j - 2\}$ of primal scaling functions is simply given by

$$\phi_{j,k} := 2^{j/2} B_k^j, \quad k = -2, \ldots, 2^j - 2, \quad j \geq 0. \tag{10}$$

Thus there are $2^j - 5$ inner scaling functions and 3 boundary functions at each edge. The inner functions are translations and dilations of a function ϕ which corresponds to

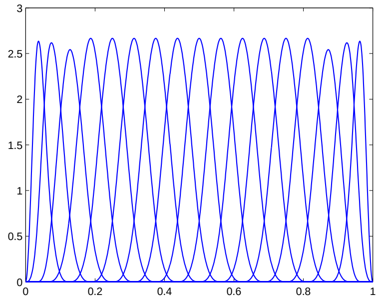

Figure 1: Primal scaling functions for the scale $j = 4$.

the primal scaling function constructed by Cohen, Daubechies, and Feauveau in [10]. Note that the primal scaling basis differs from the primal scaling basis constructed in [4, 5, 17, 26], because there are additional knots $\frac{1}{2^{j+1}}$ and $\frac{2^{j+1}-1}{2^{j+1}}$.

The desired property of a dual scaling basis $\tilde{\Phi}$ is the biorthogonality to Φ and the polynomial exactness of order six. Let $\tilde{\phi}$ be the dual scaling function which was designed by Cohen, Daubechies, and Feauveau in [10] and which is shifted so that $\tilde{\phi}$ is orthogonal to ϕ, i.e. its support is $[-5, 9]$. It is known that there exist sequences $\{h_k\}_{k=0}^{4}$ and $\{\tilde{h}_k\}_{k=-5}^{9}$ such that the functions ϕ and $\tilde{\phi}$ satisfy the *refinement equations*

$$\phi\left(x\right) = \sum_{k=0}^{4} h_k \phi\left(2x-k\right), \quad \tilde{\phi}\left(x\right) = \sum_{k=-5}^{9} \tilde{h}_k \tilde{\phi}\left(2x-k\right), \quad x \in \mathbb{R}. \tag{11}$$

The parameters h_k and $\tilde{h}_k$ are called *scaling coefficients*.

In the sequel, we assume that $j \geq j_0 := 4$. We define inner scaling functions as translations and dilations of $\tilde{\phi}$:

$$\theta_{j,k} = 2^{j/2} \tilde{\phi}\left(2^j \cdot -k\right), \quad k = 5, \ldots, 2^j - 9. \tag{12}$$

There will be two types of basis functions at each boundary. In the following, it will be convenient to abbreviate the boundary and inner index sets by

$$\begin{aligned} \mathcal{I}_j^{L,1} &= \{-2, \ldots, 3\}, \quad \mathcal{I}_j^{L,2} = \{4\}, \quad \mathcal{I}_j^{0} = \left\{5, \ldots, 2^j - 9\right\}, \\ \mathcal{I}_j^{R,2} &= \left\{2^j - 8\right\}, \quad \mathcal{I}_j^{R,1} = \left\{2^j - 7, \ldots, 2^j - 2\right\}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \mathcal{I}_j^{L} &= \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} = \{-2, \ldots, 4\}, \\ \mathcal{I}_j^{R} &= \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{2^j - 8, \ldots, 2^j - 2\right\}, \\ \mathcal{I}_j &= \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} \cup \mathcal{I}_j^{0} \cup \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{-2, \ldots, 2^j - 2\right\}. \end{aligned} \tag{14}$$

Basis functions of the first type are defined to preserve polynomial exactness and the nestedness of multiresolution spaces by the same way as in [17]:

$$\theta_{j,k}\left(x\right) = 2^{j/2} \sum_{l=-8}^{4} \left\langle p_{k+2}, \phi\left(\cdot - l\right)\right\rangle \tilde{\phi}\left(2^j x - l\right), \quad k \in \mathcal{I}_j^{L,1}, \quad x \in [0,1], \tag{15}$$

where $\{p_0, \ldots, p_5\}$ is a monomial basis of $\mathbb{P}_5([0,1])$, i.e. $p_i(x) = x^i$, $x \in [0,1]$, $i = 0, \ldots, 5$.

The definition of basis functions of the second type is a delicate task, because the low condition number and the nestedness of the multiresolution spaces have to be preserved. This means that the relation $\theta_{j,4} \in \tilde{V}_j \subset \tilde{V}_{j+1}$ should hold. Therefore we define $\theta_{j,4}$ as linear combinations of functions that are already in $\tilde{V}_{j+1}$. To obtain well-conditioned basis, we define a function $\theta_{j,4}$ which is close to $\tilde{\phi}^{\mathbb{R}}_{j,4} := 2^{j/2}\tilde{\phi}(2^j \cdot -4)$, because $\tilde{\phi}^{\mathbb{R}}_{j,4}$ is biorthogonal to the inner primal scaling functions and the condition of $\left\{\tilde{\phi}^{\mathbb{R}}_{j,4}, k \in \mathcal{I}^{L,2}_j \cup \mathcal{I}^0_j\right\}$ is close to the condition of the set of inner dual basis functions.

For this reason, we define the basis function of the second type by

$$\theta_{j,4}(x) = 2^{j/2} \sum_{l=-3}^{9} \tilde{h}_l \tilde{\phi}\left(2^{j+1}x - 8 - l\right), \quad x \in [0,1], \tag{16}$$

where $\tilde{h}_i$ are the scaling coefficients corresponding to the scaling function $\tilde{\phi}$. Then $\theta_{j,4}$ is close to $\tilde{\phi}^{\mathbb{R}}_{j,4}$ restricted to the interval $[0,1]$, because by (11) we have

$$\tilde{\phi}^{\mathbb{R}}_{j,4}(x) = 2^{j/2} \sum_{l=-5}^{9} \tilde{h}_l \tilde{\phi}\left(2^{j+1}x - 8 - l\right), \quad x \in [0,1]. \tag{17}$$

Figure 2 shows the functions $\theta_{4,4}$ and $\tilde{\phi}^{\mathbb{R}}_{4,4}$.

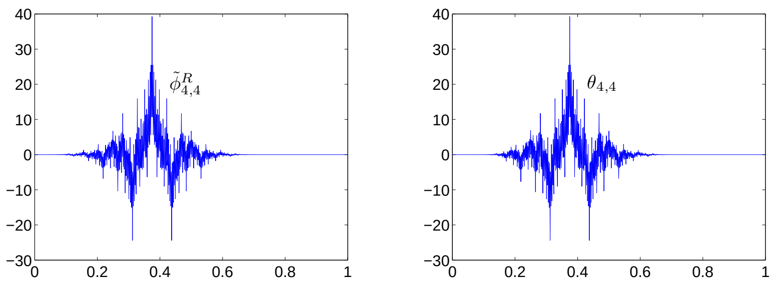

Figure 2: The functions $\tilde{\phi}^{\mathbb{R}}_{4,4}$ and $\theta_{4,4}$.

The boundary functions at the right boundary are defined to be symmetric with the left boundary functions:

$$\theta_{j,k}(x) = \theta_{j,2^j-4-k}(1-x), \quad x \in [0,1], \quad k \in \mathcal{I}^R_j. \tag{18}$$

It is easy to see that

$$\theta_{j+1,k}(x) = \sqrt{2}\,\theta_{j,k}(2x), \quad x \in [0,1], \quad k \in \mathcal{I}^L_j, \tag{19}$$

for left boundary functions and

$$\theta_{j+1,k}(1-x) = \sqrt{2}\,\theta_{j,k}(1-2x), \quad x \in [0,1], \quad k \in \mathcal{I}^R_j, \tag{20}$$

for right boundary functions.

Since the set $\Theta_j := \{\theta_{j,k}, k \in \mathcal{I}_j\}$ is not biorthogonal to Φ_j, we derive a new set

$$\tilde{\Phi}_j := \left\{\tilde{\phi}_{j,k}, k \in \mathcal{I}_j\right\} \tag{21}$$

from Θ_j by biorthogonalization. Let

$$\mathbf{Q}_j = \left(\langle \phi_{j,k}, \theta_{j,l} \rangle\right)_{k,l \in \mathcal{I}_j}. \tag{22}$$

We verify numerically that $\mathbf{Q}_j$ is invertible. Viewing $\tilde{\Phi}_j$ and Θ_j as column vectors we define

$$\tilde{\Phi}_j := \mathbf{Q}_j^{-T} \Theta_j. \tag{23}$$

Then $\tilde{\Phi}_j$ is biorthogonal to Φ_j, because

$$\left\langle \Phi_j, \tilde{\Phi}_j \right\rangle = \left\langle \Phi_j, \mathbf{Q}_j^{-T} \Theta_j \right\rangle = \mathbf{Q}_j \mathbf{Q}_j^{-1} = \mathbf{I}_{\#\mathcal{I}_j}, \tag{24}$$

where the symbol # denotes the cardinality of the set and $\mathbf{I}_m$ denotes the identity matrix of the size $m \times m$.

Remark 1. General approach of adapting wavelet bases to the unit interval was proposed in [18]. The idea is to remove certain boundary scaling functions to achieve homogeneous boundary conditions on the primal side. Then it is necessary to have the same number of basis functions on the dual side. Therefore an appropriate number of inner dual functions is used for the definition of boundary dual generators in formula (15). Applying this approach to cubic spline basis constructed in [5] and basis constructed in [26] we obtain the same resulting basis, because these constructions differs in the definition of some functions which are discarded during adaptation to complementary boundary conditions of the second order. Unfortunately, this basis has large condition number, although the starting basis in [5] is well conditioned. Its quantitative properties are presented in Section 6.

4. Refinement matrices

From the nestedness and the closedness of multiresolution spaces it follows that there exist *refinement matrices* $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that

$$\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}, \quad \tilde{\Phi}_j = \mathbf{M}_{j,1}^T \tilde{\Phi}_{j+1}. \tag{25}$$

Due to the length of support of primal scaling functions, the refinement matrix $\mathbf{M}_{j,0}$ has the following structure:

$$\mathbf{M}_{j,0} = \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{A}_j & \\ & & \mathbf{M}_R \end{pmatrix}. \tag{26}$$

where $\mathbf{A}_j$ is a $(2^{j+1}-5)\times(2^j-5)$ matrix given by

$$\begin{aligned} (\mathbf{A}_j)_{m,n} &= \frac{h_{m+1-2n}}{\sqrt{2}}, \quad n = 1,\dots,2^j-5,\ 0 \le m+1-2n \le 4, \\ &= 0, \qquad\qquad \text{otherwise,} \end{aligned} \tag{27}$$

where h_m are primal scaling coefficients (11), and $\mathbf{M}_L$, $\mathbf{M}_R$ are given by

$$\mathbf{M}_L = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{7}{8} & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & \frac{3}{20} & \frac{29}{40} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{8} \end{pmatrix}, \quad \mathbf{M}_R = \mathbf{M}_L^{\updownarrow}. \tag{28}$$

The symbol $\mathbf{M}^{\updownarrow}$ denotes a matrix that results from a matrix $\mathbf{M}$ by reversing the ordering of rows and columns. To compute the refinement matrix corresponding to the dual scaling functions, we need to identify first the structure of refinement matrices $\mathbf{M}^{\Theta}_{j,0}$ corresponding to Θ:

$$\mathbf{M}^{\Theta}_{j,0} = \begin{pmatrix} \mathbf{M}^{\Theta}_L & & \\ & \tilde{\mathbf{A}}_j & \\ & & \mathbf{M}^{\Theta}_R \end{pmatrix}, \tag{29}$$

where $\mathbf{M}^{\Theta}_L$ and $\mathbf{M}^{\Theta}_R$ are blocks 21×7 and $\tilde{\mathbf{A}}_j$ is a matrix of the size $(2^{j+1}-13)\times(2^j-13)$ given by

$$\begin{aligned} \left(\tilde{\mathbf{A}}_j\right)_{m,n} &= \frac{\tilde{h}_{m-2n-4}}{\sqrt{2}}, \quad n = 1,\dots 2^j-13,\ -1 \le m-2n \le 13, \\ &= 0, \qquad\qquad \text{otherwise,} \end{aligned} \tag{30}$$

where $\tilde{h}_m$ are dual scaling coefficients (11). The refinement coefficients for the left boundary functions of the first type are computed according to the proof of Lemma 3.1 in [17]. The refinement coefficients for the left boundary functions of the second type are given by definition (16). The matrix $\mathbf{M}^{\Theta}_R$ can be computed by the similar way. Since

$$\tilde{\Phi}_j = \mathbf{Q}_j^{-T}\Theta_j = \mathbf{Q}_j^{-T}\left(\mathbf{M}^{\Theta}_{j,0}\right)^T\Theta_{j+1} = \mathbf{Q}_j^{-T}\left(\mathbf{M}^{\Theta}_{j,0}\right)^T\mathbf{Q}^T_{j+1}\tilde{\Phi}_{j+1}, \tag{31}$$

the refinement matrix $\tilde{\mathbf{M}}_{j,0}$ corresponding to the dual scaling basis $\tilde{\Phi}_j$ is given by

$$\tilde{\mathbf{M}}_{j,0} = \mathbf{Q}_{j+1}\mathbf{M}^{\Theta}_{j,0}\mathbf{Q}_j^{-1}. \tag{32}$$

5. Construction of wavelets

Our next goal is to determine the corresponding single-scale wavelet bases Ψ_j. It is directly connected to the task of determining an appropriate matrices $\mathbf{M}_{j,1}$ such that

$$\Psi_j = \mathbf{M}_{j,1}^T \Phi_{j+1}. \tag{33}$$

We follow a general principle called *stable completion* which was proposed in [3]. This approach was already used in [5, 17, 26]. In our case, however, the initial stable completion can not be found by the same way, because it leads to singular matrices.

Definition 1. Any $\mathbf{M}_{j,1} : l^2(\mathcal{J}_j) \to l^2(\mathcal{I}_{j+1})$ is called a *stable completion* of $\mathbf{M}_{j,0}$, if

$$\|\mathbf{M}_j\|_{l^2(\mathcal{I}_{j+1}) \to l^2(\mathcal{I}_{j+1})} = O(1), \quad \left\|\mathbf{M}_j^{-1}\right\|_{l^2(\mathcal{I}_{j+1}) \to l^2(\mathcal{I}_{j+1})} = O(1), \quad j \to \infty, \tag{34}$$

where $\mathbf{M}_j := (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$.

The idea is to determine first an initial stable completion and then to project it to the desired complement space W_j. This is summarized in the following theorem [3].

Theorem 2. *Let Φ_j and $\tilde{\Phi}_j$ be a primal and a dual scaling basis, respectively. Let $\mathbf{M}_{j,0}$ and $\tilde{\mathbf{M}}_{j,0}$ be refinement matrices corresponding to these bases. Suppose that $\check{\mathbf{M}}_{j,1}$ is some stable completion of $\mathbf{M}_{j,0}$ and $\check{\mathbf{G}}_j = \check{\mathbf{M}}_j^{-1}$. Then*

$$\mathbf{M}_{j,1} := \left(\mathbf{I} - \mathbf{M}_{j,0} \tilde{\mathbf{M}}_{j,0}^T\right) \check{\mathbf{M}}_{j,1} \tag{35}$$

is also a stable completion and $\mathbf{G}_j = \mathbf{M}_j^{-1}$ has the form

$$\mathbf{G}_j = \begin{pmatrix} \tilde{\mathbf{M}}_{j,0}^T \\ \check{\mathbf{G}}_{j,1} \end{pmatrix}. \tag{36}$$

Moreover, the collections

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j := \check{\mathbf{G}}_{j,1} \tilde{\Phi}_{j+1}, \tag{37}$$

form biorthogonal systems

$$\left\langle \Psi_j, \tilde{\Psi}_j \right\rangle = \mathbf{I}, \quad \left\langle \Phi_j, \tilde{\Psi}_j \right\rangle = \left\langle \Psi_j, \tilde{\Phi}_j \right\rangle = \mathbf{0}. \tag{38}$$

To find the initial stable completion we use a factorization $\mathbf{M}_{j,0} = \mathbf{H}_j \mathbf{C}_j$, where

$$\mathbf{H}_j := \begin{pmatrix} \mathbf{H}_L & & \\ & \mathbf{H}_j^I & \\ & & \mathbf{H}_R \end{pmatrix}, \tag{39}$$

$$\mathbf{H}_L := \begin{pmatrix} 0.25 & 0 & 0 & 0 & 0 \\ 0.875 & 1 & 8 & 0 & 0 \\ 0.25 & 6 & 1 & 0 & 0 \\ 0 & 4.8 & 0 & 1 & 0 \\ 0 & 1.2 & 0 & 1.8125 & 2 \\ 0 & 0 & 0 & 1.25 & 1 \\ 0 & 0 & 0 & 0.3125 & 0 \end{pmatrix}, \quad \mathbf{H}_R := \mathbf{H}_L^{\updownarrow}, \tag{40}$$

Matrix $\left(\mathbf{H}_j^I\right)$ has the size $(2^{j+1}-7)\times(2^{j+1}-9)$. Its elements are given by:

$$\begin{aligned} \left(\mathbf{H}_j^I\right)_{mn} &:= 1, \quad 1 \le n \le 2^{j+1}-9,\ n \text{ odd}, m = n+1 \\ &:= h^I_{2,m-n+2}, \quad 1 \le n \le 2^{j+1}-9,\ n \text{ even}, -1 \le m-n \le 3, \\ &:= 0, \qquad \text{otherwise}, \end{aligned} \tag{41}$$

where $h^I_{11} = h^I_{15} = 0.25$, $h^I_{12} = h^I_{14} = 1$, $h^I_{13} = 1.5$, and

$$\mathbf{C}_j := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{C}_L & & \\ & \mathbf{C}_j^I & \\ & & \mathbf{C}_R \end{pmatrix}, \quad \mathbf{C}_L := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} \end{pmatrix}, \tag{42}$$

$$\mathbf{C}_R := \mathbf{C}_L^{\updownarrow}, \quad \mathbf{C}_j^I := \begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & \\ b & 0 & & \\ 0 & 0 & & \\ 0 & b & & \\ \vdots & 0 & \ddots & \\ & & & b \\ & & & 0 \\ 0 & & & 0 \end{pmatrix}, \quad b := \frac{7}{8}. \tag{43}$$

The factorization corresponding to inner and boundary blocks is not the same as the factorization in [15]. Therefore by our approach we obtain new inner and boundary wavelets. We define

$$\mathbf{B}_j := \sqrt{2} \begin{pmatrix} \mathbf{B}_L & & \\ & \mathbf{B}_j^I & \\ & & \mathbf{B}_R \end{pmatrix}, \mathbf{B}_L := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{2} \end{pmatrix}, \mathbf{B}_R := \mathbf{B}_L^{\updownarrow}, \tag{44}$$

$$\mathbf{B}_j^I := \begin{pmatrix} 0 & 0 & b^{-1} & 0 & 0 & 0 & \dots & & & 0 \\ 0 & 0 & 0 & 0 & b^{-1} & 0 & \dots & & & 0 \\ & & & & & & \ddots & & & \\ & & & & & & & b^{-1} & 0 & 0 \end{pmatrix}, \tag{45}$$

and

$$\mathbf{F}_j := \begin{pmatrix} \mathbf{F}_L & & \\ & \mathbf{F}_j^I & \\ & & \mathbf{F}_R \end{pmatrix}, \tag{46}$$

$$\mathbf{F}_L := \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{F}_R := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{F}_j^I := \begin{pmatrix} 1 & 0 & & \\ 0 & 0 & & \\ 0 & 1 & & \\ \vdots & 0 & \ddots & \\ & & & 1 \end{pmatrix}. \tag{47}$$

The above findings can be summarized as follows.

Lemma 3. *The following relations hold:*

$$\mathbf{B}_j \mathbf{C}_j = \mathbf{I}_{\#\mathcal{I}_j}, \quad \mathbf{F}_j^T \mathbf{F}_j = \mathbf{I}_{2^j}, \quad \mathbf{B}_j \mathbf{F}_j = \mathbf{0}, \quad \mathbf{F}_j^T \mathbf{C}_j = \mathbf{0}. \tag{48}$$

Now we are able to define the initial stable completions of the refinement matrices $\mathbf{M}_{j,0}$.

Lemma 4. *Under the above assumptions, the matrices*

$$\check{\mathbf{M}}_{j,1} := \mathbf{H}_j \mathbf{F}_j, \quad j \geq j_0, \tag{49}$$

are uniformly stable completions of the matrices $\mathbf{M}_{j,0}$. Moreover, the inverse

$$\check{\mathbf{G}}_j = \begin{pmatrix} \check{\mathbf{G}}_{j,0} \\ \check{\mathbf{G}}_{j,1} \end{pmatrix} \tag{50}$$

of $\check{\mathbf{M}}_j = \left(\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1}\right)$ is given by $\check{\mathbf{G}}_{j,0} = \mathbf{B}_j \mathbf{H}_j^{-1}, \quad \check{\mathbf{G}}_{j,1} = \mathbf{F}_j^T \mathbf{H}_j^{-1}$.

The proof of this lemma is straightforward and similar to the proof in [17]. Then using the initial stable completion $\check{\mathbf{M}}_{j,1}$ we are already able to contruct wavelets according to the Theorem 2. Left boundary wavelets are displayed at the Figure 5.

5.1. Decomposition of a scaling basis on a coarse scale

In the previous sections we assumed that the supports of the left and right boundary functions do not overlap and therefore the coarsest level was four. It might be too restrictive, especially in higher dimensions, because then there are many scaling functions. Here we decompose scaling basis Φ_4 into two parts Φ_3 and Ψ_3. It also improves the condition number of the basis. We construct wavelets on the level three to have four vanishing moments. Note that wavelets on other levels have six vanishing moments, but there the vanishing moments guaranties the smoothness of dual functions [10], and four vanishing moments for wavelets are sufficient in the most of the applications. Scaling functions in Φ_3 are defined by (10) for $j = 3$. Functions in Ψ_3 are defined by

$$\psi_{3,k}(x) := \frac{(B_{t_k}^8)^{(4)}(x)}{\left\|(B_{t_k}^8)^{(4)}\right\|}, \quad k = 1, \ldots, 8, \; x \in [0,1], \tag{51}$$

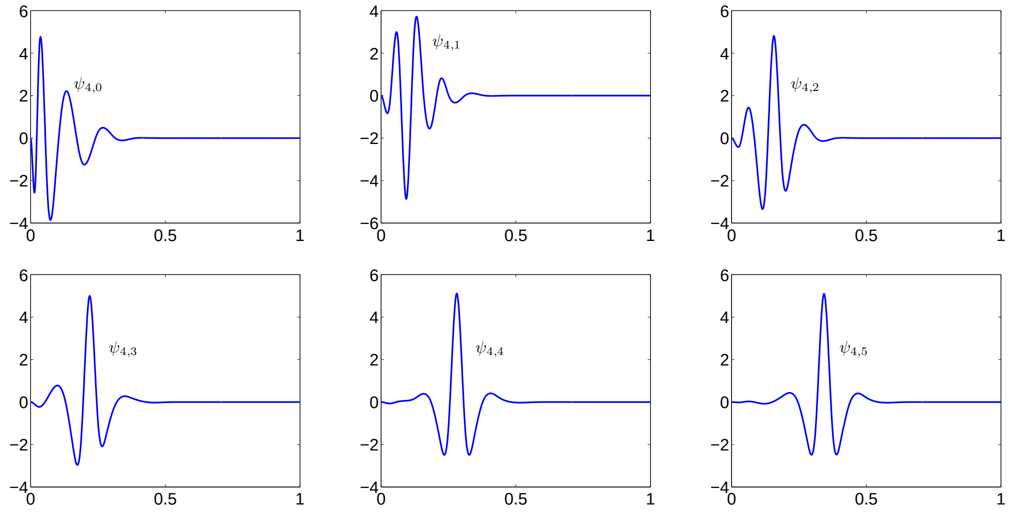

Figure 3: Left boundary wavelets for the scale $j = 4$.

where $B^8_{t_k}$ is a B-spline of order eight on the sequence of knots t_k and $^{(4)}$ denotes the fourth derivative. The sequences of knots t_k are given by:

$$\begin{aligned} t_1 &= [0, 0, 1/32, 1/16, 1/8, 2/8, 3/8, 4/8, 5/8]; \\ t_2 &= [0, 1/32, 1/16, 1/8, 3/16, 2/8, 3/8, 4/8, 5/8]; \\ t_3 &= [1/32, 1/16, 1/8, 2/8, 5/16, 3/8, 4/8, 5/8, 6/8]; \\ t_4 &= [1/16, 1/8, 2/8, 3/8, 7/16, 4/8, 5/8, 6/8, 7/8]; \\ t_5 &= [1/8, 2/8, 3/8, 4/8, 9/16, 5/8, 6/8, 7/8, 15/16]; \\ t_6 &= [2/8, 3/8, 4/8, 5/8, 11/16, 6/8, 7/8, 15/16, 31/32]; \\ t_7 &= [3/8, 4/8, 5/8, 6/8, 13/16, 7/8, 15/16, 31/32, 1]; \\ t_8 &= [3/8, 4/8, 5/8, 6/8, 7/8, 15/16, 31/32, 1, 1]; \end{aligned} \tag{52}$$

Lemma 5. *Functions from the set $\Phi_3 \cup \Psi_3$ generate the same space as functions from the set Φ_4, i.e.* span $\Phi_3 \cup \Psi_3 =$ span Φ_4. *Functions $\psi_{3,k}$, $k = 1, \ldots, 8$, have four vanishing wavelet moments.*

Proof. Since Φ_4 is a basis of the space of all cubic splines on the knots

$$\mathbf{t}^4 = [0, 0, 1/32, 1/16, 2/16, \ldots, 15/16, 31/32, 1, 1]. \tag{53}$$

Functions in Φ_3 are cubic splines on the subsets of these knots. Functions in Ψ_3 are also cubic splines, because they are fourth derivative of the spline of order eight, and they are defined on the subsets of knots $\mathbf{t}^4$. Therefore $\Phi_3 \cup \Psi_3 \subset$ span Φ_4.

Functions in Φ_3 are linearly independent. Function $\psi_{3,i}$ cannot be written as linear combination of functions from $\Phi_3 \cup \Psi_3 \backslash \{\psi_{3,i}\}$, because it is a cubic spline on sequence of the knots t_i containing an additional knot. Hence, $\Psi_3 \cup \Phi_3$ is a linearly independent subset of span Φ_4, which proves the first assertion.

To prove that the functions $\psi_{3,k}$, $k = 1, \ldots, 8$, have four vanishing moments, we use the integration by parts. We obtain for $n = 0, \ldots, 3$:

$$\int_0^1 x^n (B^8_{t_k})^{(4)}(x)\, dx = \left[x^n \left(B^8_{t_k}\right)^{(3)}(x) \right]_0^1 - \int_0^1 n x^{n-1} \left(B^8_{t_k}\right)^{(3)}(x)\, dx. \tag{54}$$

Since $\left(B_{t_k}^8\right)^{(n)}$ is the spline of order $8-n$ on the knots of multiplicity at most two in points 0 and 1, we have

$$\left(B_{t_k}^8\right)^{(n)}(0) = \left(B_{t_k}^8\right)^{(n)}(1) = 0, \quad n = 0, \dots 4, \tag{55}$$

and thus

$$\int_0^1 (B_{t_k}^8)^{(4)}(x)\, dx = 0 \tag{56}$$

and

$$\int_0^1 x^n (B_{t_k}^8)^{(4)}(x)\, dx = -\int_0^1 n x^{n-1} \left(B_{t_k}^8\right)^{(3)}(x)\, dx, \quad n = 1, \dots, 3. \tag{57}$$

Using (55) and the integration by parts three times, we obtain:

$$\int_0^1 x^n (B_{t_k}^8)^{(4)}(x)\, dx = (-1)^n\, n! \left[\left(B_{t_k}^8\right)^{(4-n)}(1) - \left(B_{t_k}^8\right)^{(4-n)}(0)\right] = 0, \tag{58}$$

for $n = 1, \dots, 3$, which proves the assertion. □

Remark 2. In some constructions, the condition number of the wavelet basis is improved by orthogonalization of boundary wavelets or by the orthogonalization of scaling functions on the coarsest level. In our case, the improvement was insignificant.

5.2. Norm equivalences

It remains to prove that Ψ and $\tilde{\Psi}$ are Riesz bases for the space $L^2([0,1])$ and that properly normalized basis Ψ is a Riesz basis for Sobolev space $H^s([0,1])$ for some s specified below. The proofs are based on the theory developed in [13] and [17].

For a function f defined on the real line a Sobolev exponent of smoothness is defined as $\sup\{s : f \in H^s(\mathbb{R})\}$. It is known that primal scaling functions extended to the real line by zero have the Sobolev regularity at least $\gamma = \frac{5}{2}$ and that dual scaling functions extended to the real line by zero have the Sobolev regularity at least $\tilde{\gamma} = 0.344$.

Theorem 6. *i) The sets* $\{\Phi_j\} := \{\Phi_j\}_{j \geq j_0}$ *and* $\left\{\tilde{\Phi}_j\right\} := \left\{\tilde{\Phi}_j\right\}_{j \geq j_0}$ *are uniformly stable, i.e.*

$$c \left\|b\right\|_{l_2(\mathcal{I}_j)} \leq \left\|\sum_{k \in \mathcal{I}_j} b_k \phi_{j,k}\right\| \leq C \left\|b\right\|_{l_2(\mathcal{I}_j)} \quad \text{for all } b = \{b_k\}_{k \in \mathcal{I}_j} \in l^2(\mathcal{I}_j),\ j \geq j_0. \tag{59}$$

ii) For all $j \geq j_0$*, the Jackson inequalities hold, i.e.*

$$\inf_{v_j \in S_j} \left\|v - v_j\right\| \lesssim 2^{-sj} \left\|v\right\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < N, \tag{60}$$

and

$$\inf_{v_j \in \tilde{S}_j} \left\|v - v_j\right\| \lesssim 2^{-sj} \left\|v\right\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < \tilde{N}. \tag{61}$$

iii) For all $j \geq j_0$*, the Bernstein inequalities hold, i.e.*

$$\left\|v_j\right\|_{H^s([0,1])} \lesssim 2^{sj} \left\|v_j\right\| \quad \text{for all } v_j \in S_j \text{ and } s < \gamma, \tag{62}$$

and

$$\left\|v_j\right\|_{H^s([0,1])} \lesssim 2^{sj} \left\|v_j\right\| \quad \text{for all } v_j \in \tilde{S}_j \text{ and } s < \tilde{\gamma}. \tag{63}$$

Proof. i) Due to Lemma 2.1 in [17], the collections $\{\Phi_j\} := \{\Phi_j\}_{j \geq j_0}$ and $\left\{\tilde{\Phi}_j\right\} := \left\{\tilde{\Phi}_j\right\}_{j \geq j_0}$ are uniformly stable, if Φ_j and $\tilde{\Phi}_j$ are biorthogonal,

$$\|\phi_{j,k}\| \lesssim 1, \left\|\tilde{\phi}_{j,k}\right\| \lesssim 1, \quad k \in \mathcal{I}_j, \ j \geq j_0, \tag{64}$$

and Φ_j and $\tilde{\Phi}_j$ are locally finite, i.e.

$$\# \{k' \in \mathcal{I}_j : \Omega_{j,k'} \cap \Omega_{j,k} \neq \emptyset\} \lesssim 1, \quad \text{for all } k \in \mathcal{I}_j, \ j \geq j_0, \tag{65}$$

and

$$\# \left\{k' \in \mathcal{I}_j : \tilde{\Omega}_{j,k'} \cap \tilde{\Omega}_{j,k} \neq \emptyset\right\} \lesssim 1, \quad \text{for all } k \in \mathcal{I}_j, \ j \geq j_0, \tag{66}$$

where $\Omega_{j,k} := \text{supp } \phi_{j,k}$ and $\tilde{\Omega}_{j,k} := \text{supp } \tilde{\phi}_{j,k}$. By (24) the sets Φ_j and $\tilde{\Phi}_j$ are biorthogonal. The properties (64), (65), and (66) follow from (10), (12), and (19).

ii) By Lemma 2.1 in [17], the Jackson inequalities are the consequences of i) and the polynomial exactness of primal and dual multiresolution analyses.

iii) The Bernstein inequalities follow from i) and the regularity of basis functions, for details see [17].

□

The following fact follows from [13].

Corollary 1. *We have the norm equivalences*

$$\|v\|_{H^s}^2 \sim 2^{2sj_0} \left\| \sum_{k \in \mathcal{I}_{j_0}} \left\langle v, \tilde{\phi}_{j_0,k} \right\rangle \phi_{j_0,k} \right\|^2 + \sum_{j=j_0}^{\infty} 2^{2sj} \left\| \sum_{k \in \mathcal{J}_j} \left\langle v, \tilde{\psi}_{j,k} \right\rangle \psi_{j,k} \right\|^2, \tag{67}$$

where $v \in H^s([0,1])$ *and* $s \in (-\tilde{\gamma}, \gamma)$.

The norm equivalence for $s = 0$, Theorem 2, and Lemma 4, imply that

$$\Psi := \Phi_{j_0} \cup \bigcup_{j=j_0}^{\infty} \Psi_j \quad \text{and} \quad \tilde{\Psi} := \tilde{\Phi}_{j_0} \cup \bigcup_{j=j_0}^{\infty} \tilde{\Psi}_j \tag{68}$$

are biorthogonal Riesz bases of the space $L^2([0,1])$. Let us define

$$\mathbf{D} = \left(\mathbf{D}_{\lambda,\tilde{\lambda}}\right)_{\lambda,\tilde{\lambda} \in \mathcal{J}}, \quad \mathbf{D}_{\lambda,\tilde{\lambda}} := \delta_{\lambda,\tilde{\lambda}} 2^{|\lambda|}, \quad \lambda, \tilde{\lambda} \in \mathcal{J}. \tag{69}$$

The relation (67) implies that $\mathbf{D}^{-s}\Psi$ is a Riesz basis of the Sobolev space $H^s([0,1])$ for $s \in (-\tilde{\gamma}, \gamma)$.

6. Quantitative properties of constructed bases

In this section, we compare quantitative properties of bases constructed in this paper, cubic spline-wavelet basis from [26] and cubic spline multiwavelet basis recently adapted to homogeneous boundary conditions in [28]. The condition of multi-scale wavelet bases is shown in Table 1. Our wavelet basis is denoted by CF, a basis from

[28] is denoted by Schneider and a basis from [26] adapted to complementary boundary conditions by method from [18] is denoted by Primbs. The last basis is the same as the basis from [5] adapted to complementary boundary conditions by method from [18], see Remark 1.

Other criteria for the effectiveness of wavelet bases is the condition number of a corresponding stiffness matrix. Here, let us consider the stiffness matrix:

$$\mathbf{A}_{j_0,s} = \left(\langle \psi''_{j,k}, \psi''_{l,m} \rangle \right)_{\psi_{j,k}, \psi_{l,m} \in \Psi_{j_0,s}}. \tag{70}$$

It is well-known that the condition number of $\mathbf{A}_{j_0,s}$ increases quadratically with the matrix size. To remedy this, we use a diagonal matrix for preconditioning

$$\mathbf{A}^{prec}_{j_0,s} = \mathbf{D}^{-1}_{j_0,s} \mathbf{A}_{j_0,s} \mathbf{D}^{-1}_{j_0,s}, \tag{71}$$

where

$$\mathbf{D}_{j_0,s} = \mathrm{diag} \left(\langle \psi''_{j,k}, \psi''_{j,k} \rangle^{1/2} \right)_{\psi_{j,k} \in \Psi_{j_0,s}}. \tag{72}$$

In [7] the anisotropic wavelet basis were used for solving fourth-order problems. Here, we use isotropic wavelet basis, i.e. we define multiscale wavelet basis on the unit square by

$$\Psi^{2D}_{3,s} = \Phi^{2D}_3 \cup \bigcup_{j=3}^{s} \Psi^{2D}_j, \tag{73}$$

where

$$\Phi^{2D}_3 = \Phi_3 \otimes \Phi_3, \quad \Psi^{2D}_j = \Phi_j \otimes \Psi_j \cup \Psi_j \otimes \Phi_j \cup \Psi_j \otimes \Psi_j. \tag{74}$$

The symbol $\otimes$ denotes the tensor product. The preconditioned stiffness matrix ${}^{2D}\mathbf{A}^{prec}_{j_0,s}$ for the biharmonic equation defined on the unit square is similar to the one dimensional case. Condition numbers of the stiffness matrices are listed in Table 1 and Table 2. The condition number of the stiffness matrix corresponding to wavelet basis by Primbs exceeds 10^4 already for number of levels $j = 3$. Wavelet basis from [17] adapted to complementary boundary conditions by method from [18] is very badly conditioned, its quantitative properties can be found in [28].

7. Numerical example

Now, we compare the quantitative behaviour of the adaptive wavelet method with our bases and bases from [28]. Both bases are formed by cubic splines and have local

Table 1: The condition numbers of wavelet bases and stiffness matrices, $j_0 = 3$ for CF and Schneider, $j_0 = 4$ for Primbs.

	$\Psi_{j_0,j}$			$\mathbf{A}^{prec}_{j_0,j}$		
j	CF	Schneider	Primbs	CF	Schneider	Primbs
1	8.3	1.9	14.9	64.8	472.0	1111.0
3	12.5	2.4	45.9	66.5	569.5	1116.9
5	15.3	2.6	69.8	66.6	640.8	1117.0
7	18.0	2.7	85.8	66.7	693.0	1117.0

Quadratic spline wavelets with short support for fourth-order problems

Dana Černá · Václav Finěk

Abstract In the paper, we propose constructions of new quadratic spline-wavelet bases on the interval and the unit square satisfying homogeneous Dirichlet boundary conditions of the second order. The basis functions have small supports and wavelets have one vanishing moment. We show that stiffness matrices arising from discretization of the biharmonic problem using a constructed wavelet basis have uniformly bounded condition numbers and these condition numbers are very small.

Keywords Wavelet · Quadratic spline · Homogeneous Dirichlet boundary conditions · Condition number · Biharmonic equation

Mathematics Subject Classification (2000) 46B15 · 65N12 · 65T60

1 Introduction

In this paper, we propose a construction of quadratic spline wavelet bases on the interval that are well-conditioned, adapted to homogeneous Dirichlet boundary conditions of the second order, the wavelets have one vanishing moment and the shortest possible support. The wavelet basis of the space $H_0^2\left((0,1)^2\right)$ is then obtained by an isotropic tensor product.

Wavelet bases are useful for solving the fourth-order problems. In [11], a construction of cubic spline wavelet basis was proposed and it was shown that the Galerkin method based on this wavelet basis is very efficient even in comparison with multigrid methods. We show that our wavelet basis is even better conditioned than basis in [11]. Moreover, since our wavelets have vanishing moments, they can be used in adaptive wavelet methods.

First of all, we summarize the desired properties of a constructed basis:

- *Riesz basis property.* We construct Riesz bases of the space $H_0^2\,(0,1)$ and $H_0^2\left((0,1)^2\right)$.
- *Polynomial exactness.* Since the primal basis functions are quadratic B-splines, the primal multiresolution analysis has polynomial exactness of order three.
- *Vanishing moments.* The inner wavelets have one vanishing moment, the wavelets near the boundary do not need to have vanishing moments.

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D. Černá
Department of Mathematics and Didactics of Mathematics, Technical University of Liberec,
Studentská 2, 461 17 Liberec, Czech Republic
Tel.: +420-48-535-2407
Fax: +420-48-535-2332
E-mail: dana.cerna@tul.cz

V. Finěk
Department of Mathematics and Didactics of Mathematics, Technical University of Liberec,
Studentská 2, 461 17 Liberec, Czech Republic

- *Short support.* The wavelets have the shortest possible support among quadratic spline wavelets with one vanishing moment.
- *Locality.* The primal basis functions are local.
- *Closed form.* The primal scaling functions and wavelets are known in the closed form.
- *Homogeneous Dirichlet boundary conditions.* Our wavelet bases satisfy homogeneous Dirichlet boundary conditions of second order.
- *Well-conditioned bases.* Our objective is to construct a well conditioned wavelet basis.

Moreover, in a comparison with constructions in [2], [8], [12], [13] that are quite long and technical, the construction in this paper is very simple. Many constructions of spline wavelet or multiwavelet bases on the interval have been proposed in recent years. In [1], [2], [8], [12] cubic spline wavelets on the interval were constructed. In [7] cubic spline multiwavelet bases were designed and they were adapted to complementary boundary conditions of second order in [13]. In these cases dual functions are known and are local. Spline wavelet or multiwavelet bases whose duals are not local were constructed in [4], [9], [10], [11]. Some of these bases were already adapted to boundary conditions. The advantage of our construction is the shortest possible support for a given number of required vanishing moments. Vanishing moments are necessary in some applications such as adaptive wavelet methods [5], [6]. Originally, these methods were designed for wavelet bases with local duals. However, it was shown in [14] that wavelet bases without local dual basis can be used if the solved equation is linear.

2 Wavelet bases

This section provides a short introduction to the concept of wavelet bases in Sobolev spaces. In this paper, we consider the domain $\Omega = (0,1)$ or $\Omega = (0,1)^2$. We consider the Sobolev space or its subspace by $H \subset H^s(\Omega)$ for nonnegative integer s and the corresponding inner product by $\langle \cdot, \cdot \rangle_H$, a norm by $\|\cdot\|_H$ and a seminorm by $|\cdot|_H$. In case $s = 0$ we consider the space $L^2(\Omega)$ and we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the L^2-inner product and the L^2-norm, respectively. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j,k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale* or a *level*. Let

$$\|\mathbf{v}\|_2 = \sqrt{\sum_{\lambda \in \mathcal{J}} |v_\lambda|^2}, \quad \text{for} \quad \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \tag{1}$$

and

$$l^2(\mathcal{J}) = \left\{ \mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \|\mathbf{v}\|_2 < \infty \right\}. \tag{2}$$

A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *(primal) wavelet basis* of H, if

i) Ψ is a *Riesz basis* for H, i.e. the closure of the span of Ψ is H and there exist constants $c, C \in (0, \infty)$ such that

$$c\|\mathbf{b}\|_2 \le \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \le C \|\mathbf{b}\|_2, \quad \mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{3}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies } (3)\}$, $C_\psi := \inf\{C : C \text{ satisfies } (3)\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi / c_\psi$ is called the *condition number* of Ψ.

ii) The functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \le C 2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap at any point $x \in \Omega$.

By the Riesz representation theorem, to any basis of the space H there exists a unique family $\tilde{\Psi} = \left\{ \tilde{\psi}_\lambda, \lambda \in \tilde{\mathcal{J}} \right\} \subset H$ biorthogonal to Ψ, i.e.

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle_H = \delta_{i,j} \delta_{k,l}, \quad \text{for all} \quad (i,k) \in \mathcal{J}, \quad (j,l) \in \tilde{\mathcal{J}}, \tag{4}$$

where $\delta_{i,j}$ denotes the Kronecker delta, i.e. $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. This family is a Riesz basis for H if and only if the primal basis is a Riesz basis for H. The functions $\tilde{\psi}_{j,l}$ do not need to be local, therefore $\tilde{\Psi}$ do not need to be a wavelet basis in the sense of the above definition. The basis $\tilde{\Psi}$ is called a *dual* basis.

Wavelets are usually constructed using a function ψ called a mother-wavelet by

$$\psi_{j,k} = 2^{j/2}\psi\left(2^j x - k + n\right), n \in \mathbb{N}.$$

Also the inner wavelets in this paper are constructed by this way. This does not implicate that the dual basis has a mother-wavelet.

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis. A sequence $\mathcal{V} = \{V_j\}_{j \geq j_0}$, of closed linear subspaces $V_j \subset H$ is called a *multiresolution* or *multiscale analysis*, if

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots H \tag{5}$$

and $\cup_{j \geq j_0} V_j$ is complete in H.

The nestedness and the closedness of the multiresolution analysis implies the existence of the *complement spaces* W_j such that $V_{j+1} = V_j \oplus W_j$.

We now assume that V_j and W_j are spanned by sets of basis functions

$$\Phi_j = \{\phi_{j,k}, k \in \mathcal{I}_j\}, \quad \Psi_j = \{\psi_{j,k}, k \in \mathcal{J}_j\}, \tag{6}$$

where $\mathcal{I}_j$ and $\mathcal{J}_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis and the wavelet basis of H are given by

$$\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j, \quad \Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j. \tag{7}$$

Let us denote

$$\tilde{\Phi}_j = \left\{\tilde{\phi}_{j,k}, k \in \mathcal{I}_j\right\}, \quad \tilde{\Psi}_j = \left\{\tilde{\psi}_{j,k}, k \in \mathcal{J}_j\right\}, \tag{8}$$

and

$$\tilde{V}_j = \operatorname{span} \tilde{\Phi}_j, \quad \tilde{W}_j = \operatorname{span} \tilde{\Psi}_\mathrm{j}. \tag{9}$$

The spaces $\tilde{V}_j$ are also nested:

$$\tilde{V}_j \subset \tilde{V}_{j+1}, \quad j \geq j_0. \tag{10}$$

Most common way of construction of wavelet bases is using dual functions. In our paper, we use a different approach and construct scaling functions $\phi_{j,k}$ as quadratic splines and we derive wavelets $\psi_{j,k}$ directly as linear combinations of functions $\phi_{j+1,k}$, where the coefficients of the linear combinations are chosen such that wavelets have vanishing moments.

Polynomial exactness of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that

$$\mathbb{P}_{N-1}(\Omega) \subset V_j, \quad \mathbb{P}_{\tilde{N}-1}(\Omega) \subset \tilde{V}_j, \quad j \geq j_0, \tag{11}$$

where $\mathbb{P}_m(\Omega)$ is the space of all algebraic polynomials on Ω of degree less or equal to m.

The polynomial exactness of order $\tilde{N}$ on the dual side is equivalent to $\tilde{N}$ vanishing wavelet moments on the primal side, i.e.

$$\int_\Omega P(x)\,\psi_\lambda(x)\,dx = 0, \quad \text{for any } P \in \mathbb{P}_{\tilde{N}-1}, \ \psi_\lambda \in \bigcup_{j \geq j_0} \Psi_j. \tag{12}$$

3 Primal scaling basis

A primal scaling basis is generated from function ϕ. Let ϕ be a quadratic B-spline defined on knots $[0,1,2,3]$. It can be written explicitly as:

$$\phi(x) = \begin{cases} \frac{x^2}{2}, & x \in [0,1], \\ -x^2 + 3x - \frac{3}{2}, & x \in [1,2], \\ \frac{x^2}{2} - 3x + \frac{9}{2}, & x \in [2,3], \\ 0, & \text{otherwise}, \end{cases} \tag{13}$$

The function ϕ satisfies a scaling equation [8]:

$$\phi\left(x\right) = \frac{\phi\left(2x\right)}{4} + \frac{3\phi\left(2x-1\right)}{4} + \frac{3\phi\left(2x-2\right)}{4} + \frac{\phi\left(2x-3\right)}{4}. \tag{14}$$

For $j \geq 2$ and $x \in [0,1]$ we set

$$\phi_{j,k}\left(x\right) = 2^{j/2}\phi(2^j x - k + 1),\; k = 1, \dots 2^j - 2. \tag{15}$$

The graphs of the functions $\phi_{j,k}$ on the coarsest level $j = 2$ are displayed in Figure 1.

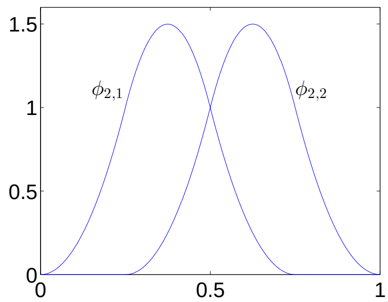

Fig. 1 Primal scaling basis for $j = 2$.

We define a wavelet ψ as

$$\psi(x) = -\frac{1}{2}\phi(2x-1) + \frac{1}{2}\phi(2x-2). \tag{16}$$

Then $\operatorname{supp}\psi = [0.5, 2.5]$ and ψ has one vanishing wavelet moments, i.e.

$$\int_{-\infty}^{\infty} \psi(x)dx = 0. \tag{17}$$

The graph of ψ is shown in Figure 2.

We define a boundary wavelet ψ_b by:

$$\psi_b(x) = a\,\phi(2x) + b\,\phi(2x-1), \tag{18}$$

where a and b are real parameters. Since we want to have wavelets with the shortest possible support for a given number of vanishing moments, we will consider two choices of the parameters:

a) $a = -\frac{1}{2}$, $b = \frac{1}{2}$,
b) $a = 1$, $b = 0$.

The properties of these wavelets are summarized in the following lemma.

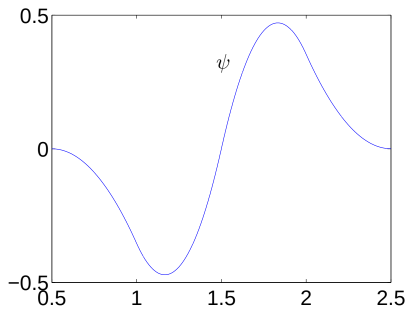

Fig. 2 Wavelet ψ.

Lemma 1 *i) The function* $\psi_b(x)$ *defined by (18) with the choice of parameters a) satisfies* $\operatorname{supp}\psi_b = [0,2]$ *and*

$$\int_{-\infty}^{\infty} \psi_b(x)dx = 0. \tag{19}$$

ii) The function $\psi_b(x)$ *defined by (18) with the choice of parameters b) satisfies* $\operatorname{supp}\psi_b = \left[0,\frac{3}{2}\right]$.

Proof The length of the support of the function ψ_b is derived from the lengths of the supports of the functions $\phi(2x)$ and $\phi(2x-1)$. By (13) we have

$$\operatorname{supp}\phi(2x) = [0,1.5] \quad \text{and} \operatorname{supp}\phi(2x-1) = [0.5,2]\,. \tag{20}$$

Since the functions $\phi(2x)$ and $\phi(2x-1)$ are given in the closed form, the formula (19) can be verified easily.

Thus, we can choose boundary wavelet with one vanishing moment and larger support or boundary wavelets with shorter supports but without vanishing moments. If $f \in H_0^2(0,1)$ and f is constant at the interval $[0,\epsilon]$, $0<\epsilon<1$, then f has to be zero at $[0,\epsilon]$. The same holds for the interval $[1-\epsilon,1]$. Hence $f \in H_0^2(0,1)$ can not be nonzero constant near the boundary and therefore in some applications such as adaptive wavelet methods the vanishing moment does not play the significant role for boundary wavelets. The graphs of boundary wavelets ψ_b are displayed in Figure 3. All the following lemmas and theorems are valid for both choices of parameters.

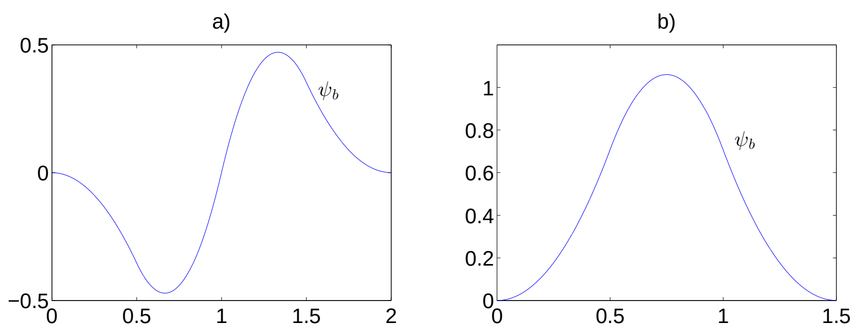

Fig. 3 Boundary wavelet ψ_b for a) and b), respectively.

For $j \geq 2$ and $x \in [0,1]$ we define

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k + 2), k = 2,...,2^j - 1, \tag{21}$$
$$\psi_{j,1}(x) = 2^{j/2}\psi_b(2^j x), \quad \psi_{j,2^j}(x) = 2^{j/2}\psi_b(2^j(1-x)).$$

We denote

$$\Phi_j = \left\{ \phi_{j,k} / \left| \phi_{j,k} \right|_{H_0^2(0,1)}, k = 1, \ldots, 2^j - 2 \right\}, \tag{22}$$
$$\Psi_j = \left\{ \psi_{j,k} / \left| \psi_{j,k} \right|_{H_0^2(0,1)}, k = 1, \ldots, 2^j \right\}.$$

In Section 5 we show that the sets

$$\Psi^s = \Phi_2 \cup \bigcup_{j=2}^{1+s} \Psi_j \quad \text{and} \quad \Psi = \Phi_2 \cup \bigcup_{j=2}^{\infty} \Psi_j \tag{23}$$

are a multiscale wavelet basis and a wavelet basis of the space $H_0^2(0,1)$, respectively. We use $u \otimes v$ to denote the tensor product of functions u and v, i.e. $(u \otimes v)(x_1, x_2) = u(x_1)\, v(x_2)$. We set

$$F_j = \left\{ \phi_{j,k} \otimes \phi_{j,l} \,/\, \left| \phi_{j,k} \otimes \phi_{j,l} \right|_{H_0^2(\Omega)}, k, l = 1, \ldots, 2^j - 2 \right\}$$
$$G_j^1 = \left\{ \phi_{j,k} \otimes \psi_{j,l} \,/\, \left| \phi_{j,k} \otimes \psi_{j,l} \right|_{H_0^2(\Omega)}, k = 1, \ldots, 2^j - 2, l = 1, \ldots 2^j \right\}$$
$$G_j^2 = \left\{ \psi_{j,k} \otimes \phi_{j,l} \,/\, \left| \psi_{j,k} \otimes \phi_{j,l} \right|_{H_0^2(\Omega)}, k = 1, \ldots, 2^j, l = 1, \ldots 2^j - 2 \right\}$$
$$G_j^3 = \left\{ \psi_{j,k} \otimes \psi_{j,l} \,/\, \left| \psi_{j,k} \otimes \psi_{j,l} \right|_{H_0^2(\Omega)}, k, l = 1, \ldots, 2^j \right\}$$

where $\Omega = (0,1)^2$. We show that the sets defined by

$$\Psi_s^{2D} = F_2 \cup \bigcup_{j=2}^{1+s} \left(G_j^1 \cup G_j^2 \cup G_j^3 \right), \quad \Psi^{2D} = F_2 \cup \bigcup_{j=2}^{\infty} \left(G_j^1 \cup G_j^2 \cup G_j^3 \right) \tag{24}$$

are a wavelet basis and a multiscale wavelet basis of the space $H_0^2(\Omega)$.

4 Refinement matrices

From the nestedness and the closedness of multiresolution spaces it follows that there exist *refinement matrices* $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that

$$\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}, \quad \Psi_j = \mathbf{M}_{j,1}^T \Phi_{j+1}. \tag{25}$$

By (14), the entries of the refinement matrix $\mathbf{M}_{j,0}$ satisfy:

$$(\mathbf{M}_{j,0})_{m,n} = \begin{cases} \frac{h_{m+2-2n}}{\sqrt{2}}, & n = 1, \ldots, 2^j - 2, \ 1 \le m + 2 - 2n \le 4, \\ 0, & \text{otherwise}, \end{cases} \tag{26}$$

where

$$\mathbf{h} = [h_1, h_2, h_3, h_4] = \left[\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4} \right] \tag{27}$$

is a vector of coefficients from scaling equation (14).

It follows from the equations (16) and (18) that the matrix $\mathbf{M}_{j,1}$ is of the size $\left(2^{j+1} - 2\right) \times 2^j$ and has the structure

$$\mathbf{M}_{j,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} a & b & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & & 0 & 0 \\ \vdots & \vdots & & & & & & & \vdots \\ 0 & \ldots & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \ldots & 0 & 0 & 0 & 0 & 0 & b & a \end{pmatrix}^T \tag{28}$$

There also exist refinement matrices $\tilde{\mathbf{M}}_{j,0}$ and $\tilde{\mathbf{M}}_{j,1}$ corresponding to dual spaces that satisfy:

$$\tilde{\Phi}_j = \tilde{\mathbf{M}}_{j,0}^T \tilde{\Phi}_{j+1}, \quad \tilde{\Psi}_j = \tilde{\mathbf{M}}_{j,1}^T \tilde{\Phi}_{j+1}. \tag{29}$$

The structure of the matrix $\tilde{\mathbf{M}}_{j,0}$ is derived in the proof of Lemma 2. We do not need to know the structure of the matrix $\tilde{\mathbf{M}}_{j,1}$ in this paper.

The Euclidean norm of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|_2$ and the spectral norm of the matrix $\mathbf{M}$ is denoted as $\|\mathbf{M}\|_2$. The following lemma is crucial for the proof of a Riesz basis property.

Lemma 2 *The norm of the matrix* $\tilde{\mathbf{M}}_{j,0}$ *satisfies* $\left\|\tilde{\mathbf{M}}_{j,0}\right\|_2 \leq 2^p, p = \frac{\ln 6}{\ln 4}$.

Proof We prove the lemma for the choice a) of parameters for the boundary wavelet, for the choice b) the proof is similar. We denote the entries of the matrix $\tilde{\mathbf{M}}_{j,0}$ as $\tilde{M}_{k,l}$, $k = 1, \ldots 2^{j+1} - 2$, $l = 1, \ldots, 2^j - 2$.

Due to the biorthogonality of the sets $\Psi_j \cup \Phi_j$ and $\tilde{\Psi}_j \cup \tilde{\Phi}_j$ we have

$$\mathbf{M}_{j,0}^T \tilde{\mathbf{M}}_{j,0} = \mathbf{I}_j \tag{30}$$

and

$$\mathbf{M}_{j,1}^T \tilde{\mathbf{M}}_{j,0} = \mathbf{0}_j, \tag{31}$$

where $\mathbf{I}_j$ denotes the identity matrix and $\mathbf{0}_j$ denotes the zero matrix of the appropriate size.

From (28) and (31) we have for $l = 1, \ldots, 2^j - 2$:

$$\tilde{M}_{1,l} = \tilde{M}_{2,l}, \quad \tilde{M}_{2^{j+1}-2,l} = \tilde{M}_{2^{j+1}-3,l}, \tag{32}$$

and

$$\tilde{M}_{k,l} = \tilde{M}_{k+1,l}, \quad \text{for } k \text{ even}, \quad k = 2, \ldots, 2^{j+1} - 4. \tag{33}$$

We substitute these relations into (30) and we obtain a new system of equations $\mathbf{A}_j \mathbf{B}_j = \mathbf{I}_j$, where

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{7}{4} & \frac{1}{4} & 0 & \ldots & & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & & \ldots & 0 & \frac{1}{4} & \frac{7}{4} \end{pmatrix} \tag{34}$$

and $\mathbf{B}_j$ contains $\tilde{M}_{k,l}$ for k even, i.e. the entries $B_{k,l}$ of the matrix $\mathbf{B}_j$ satisfy:

$$B_{k,l} = \tilde{M}_{2k,l}, \quad k, l = 1, \ldots 2^j - 2. \tag{35}$$

We factorize the matrix $\mathbf{A}_j$ as $\mathbf{A}_j = \mathbf{C}_j \mathbf{D}_j$, where

$$\mathbf{C}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{3+2\sqrt{2}}{4} & \frac{1}{4} & 0 & 0 & \ldots & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & \ldots & 0 & 0 & \frac{1}{4} & \frac{3+2\sqrt{2}}{4} \end{pmatrix} \tag{36}$$

and

$$\mathbf{D}_j = \begin{pmatrix} \frac{7-a}{3+2\sqrt{2}} & 0\;0 \ldots 0\;0 & \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-4}} \\ a & 1\;0 \quad\; 0\;0 & \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-5}} \\ \frac{a}{-3-2\sqrt{2}} & 0\;1 \quad\; 0\;0 & \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-6}} \\ \vdots & \vdots \;\; \ddots \;\; \vdots & \vdots \\ \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-6}} & 0\;0 \quad\; 1\;0 & \frac{a}{-3-2\sqrt{2}} \\ \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-5}} & 0\;0 \quad\; 0\;1 & a \\ \frac{a}{\left(-3-2\sqrt{2}\right)^{2^j-4}} & 0\;0 \ldots 0\;0 & \frac{7-a}{3+2\sqrt{2}} \end{pmatrix}, \tag{37}$$

with

$$a = \frac{1-\sqrt{2}}{6+4\sqrt{2}}. \tag{38}$$

More precisely, the entries $D_{k,l}$ of the matrix $\mathbf{D}_j$ are given by:

$$\begin{aligned} D_{1,1} = D_{2^j-2,2^j-2} &= \frac{7-a}{3+2\sqrt{2}}, \\ D_{k,1} = D_{2^j-1-k,2^j-2} &= \frac{a}{\left(-3-2\sqrt{2}\right)^{k-2}}, \quad \text{for } k = 2, \ldots, 2^j-2, \\ D_{k,k} &= 1, \quad \text{for } k = 2, \ldots, 2^j-3, \\ D_{k,l} &= 0, \quad \text{otherwise.} \end{aligned} \tag{39}$$

It is easy to verify that $\tilde{\mathbf{C}}_j = \mathbf{C}_j^{-1}$ has entries:

$$\tilde{C}_{k,l} = \frac{1}{\left(-3-2\sqrt{2}\right)^{|k-l|}}, \tag{40}$$

and the matrix $\mathbf{D}_j^{-1}$ has the structure:

$$\mathbf{D}_j^{-1} = \begin{pmatrix} d_1 & 0 \ldots 0 & d_n \\ d_2 & 1 \quad\; 0 & d_{n-1} \\ \vdots & \ddots & \vdots \\ d_{n-1} & 0 \quad\; 1 & d_2 \\ d_n & 0 \ldots 0 & d_1 \end{pmatrix}, \tag{41}$$

with $n = 2^j - 2$ and

$$\begin{aligned} d_1 &= \frac{\left(3+2\sqrt{2}\right)\,\alpha_n}{7-a} \\ d_k &= \frac{a\,\alpha_n}{(7-a)\left(-3-2\sqrt{2}\right)^{k-3}} - \frac{\alpha_n\,\beta_n}{\left(-3-2\sqrt{2}\right)^{n-k}}, \quad k = 2, \ldots, n-1, \\ d_n &= \frac{-a\,\alpha_n}{(7-a)^2\,\left(-3-2\sqrt{2}\right)^{2^j-6}}, \end{aligned} \tag{42}$$

where the constants α_n and β_n are given by

$$\alpha_n = \left(1 - \frac{a^2}{(7-a)^2\left(-3-2\sqrt{2}\right)^{2n-6}}\right)^{-1} \tag{43}$$

and

$$\beta_n = \frac{a^2}{(7-a)^2\left(-3-2\sqrt{2}\right)^{n-5}}. \tag{44}$$

Note that $\alpha_n \approx 1$ and $\beta_n \approx 0$.

Since the matrices $\mathbf{C}_j$ and $\mathbf{D}_j$ are invertible, we can define $\mathbf{B}_j = \mathbf{A}_j^{-1} = \mathbf{D}_j^{-1}\mathbf{C}_j^{-1}$. Substituting this into (31) we obtain the entries of the matrix $\tilde{\mathbf{M}}_{j,0}$:

$$\tilde{M}_{1,l} = \frac{d_1}{\left(-3-2\sqrt{2}\right)^{|1-l|}} + \frac{d_n}{\left(-3-2\sqrt{2}\right)^{|n-l|}}, \tag{45}$$

$$\tilde{M}_{1,2^j-l} = \tilde{M}_{2,2^j-l} = \tilde{M}_{3,2^j-l} = \tilde{M}_{2^{j+1}-4,l} = \tilde{M}_{2^{j+1}-3,l} = \tilde{M}_{2^{j+1}-2,l},$$

and for $k = 1, \ldots, 2^j - 2$, $l = 1, \ldots, 2^j - 2$, we have

$$\tilde{M}_{2k,l} = B_{k,l} = \frac{1}{\left(-3-2\sqrt{2}\right)^{|k-l|}} + \frac{d_k}{\left(-3-2\sqrt{2}\right)^{|1-l|}} + \frac{d_{n+1-k}}{\left(-3-2\sqrt{2}\right)^{|n-l|}} \tag{46}$$

The entries $\tilde{M}_{2k-1,l}$ are given by (33).

It is well-known that for any matrix $\mathbf{M}$ of the size $m \times n$ with entries $M_{k,l}$:

$$\|\mathbf{M}\|_2 \le \sqrt{\|\mathbf{M}\|_1 \|\mathbf{M}\|_\infty}, \tag{47}$$

where

$$\|\mathbf{M}\|_1 = \max_{l=1,\ldots,n} \sum_{k=1}^{m} |M_{k,l}|, \quad \|\mathbf{M}\|_\infty = \max_{k=1,\ldots,m} \sum_{l=1}^{n} |M_{k,l}|. \tag{48}$$

In our case, from (45), (48), and a formula for a sum of a geometric sequence we obtain:

$$\left\|\tilde{\mathbf{M}}_{j,0}\right\|_1 \le 3\sqrt{2} \quad \text{and} \quad \left\|\tilde{\mathbf{M}}_{j,0}\right\|_\infty \le \sqrt{2}. \tag{49}$$

Thus

$$\left\|\tilde{\mathbf{M}}_{j,0}\right\|_2 \le \sqrt{6} = 2^p \quad \text{for} \quad p = \frac{\ln 6}{\ln 4}. \tag{50}$$

5 Riesz basis on Sobolev space

In this section, we show that Ψ and Ψ^{2D} are Riesz bases. We use Theorem 5.3. from [11]. It says that if P_j is a linear projection from V_{j+1} onto V_j and for $0 < p < q$ there exists a constant C such that

$$\|P_m P_{m+1} \ldots P_{n-1}\| \le C\, 2^{p\,(n-m)}, \tag{51}$$

then

$$\left\{2^{-4}\phi_{2,k}, k = 1, 2\right\} \cup \left\{2^{-2j}\psi_{j,k}, j \ge 2, k = 1, \ldots 2^j\right\} \tag{52}$$

is a Riesz basis of $H_0^q(0,1)$.

First we define suitable projections P_j from V_{j+1} onto V_j and show that these projections satisfies (51). Then we show that Ψ which differs from (52) only by scaling is also a Riesz basis of $H_0^2(0,1)$. We denote

$$\mathcal{I}_j = \left\{1, 2, \ldots, 2^j - 2\right\} \quad \text{and} \quad \mathcal{J}_j = \left\{1, 2, \ldots, 2^j\right\} \tag{53}$$

and for $j \ge 2$ we define

$$\Gamma_j = \{\phi_{j,k}\}_{k \in \mathcal{I}_j} \cup \{\psi_{j,k}\}_{k \in \mathcal{J}_j} \quad \text{and} \quad \mathbf{F}_j = \langle \Gamma_j, \Gamma_j \rangle. \tag{54}$$

Let a set

$$\hat{\Gamma}_j = \left\{\hat{\phi}_{j,k}\right\}_{k \in \mathcal{I}_j} \cup \left\{\hat{\psi}_{j,k}\right\}_{k \in \mathcal{J}_j} \tag{55}$$

be given by

$$\hat{\Gamma}_j = \mathbf{F}_j^{-1} \Gamma_j. \tag{56}$$

Since obviously

$$\left\langle \Gamma_j, \hat{\Gamma}_j \right\rangle = \mathbf{I}_j, \tag{57}$$

functions from $\hat{\Gamma}_j$ are duals to functions from Γ_j in the space V_{j+1}. Since $\mathbf{F}_j^{-1}$ is not a sparse matrix, these duals are not local. We define a projection P_j from V_{j+1} onto V_j by

$$P_j f = \sum_{k \in \mathcal{I}_j} \left\langle f, \hat{\phi}_{j,k} \right\rangle \phi_{j,k}. \tag{58}$$

Lemma 3 *Let* $f \in V_{j+1}$, $a_k^j = \left\langle f, \hat{\phi}_{j,k} \right\rangle$, $\mathbf{a}_j = \left\{ a_k^j \right\}_{k \in \mathcal{I}_j}$, $j \geq 2$, *and* $\mathbf{S}_j : \mathbf{a}_{j+1} \mapsto \mathbf{a}_j$. *Then* $\|\mathbf{S}_j\|_2 \leq 2^p$, $p = \frac{\ln 6}{\ln 4}$.

Proof We have

$$\begin{aligned} P_j f &= \sum_{k \in \mathcal{I}_j} a_k^j \phi_{j,k} = \sum_{k \in \mathcal{I}_j} \left\langle f, \hat{\phi}_{j,k} \right\rangle \phi_{j,k} \\ &= \sum_{k \in \mathcal{I}_j} \sum_{l \in \mathcal{I}_{j+1}} a_l^{j+1} \left\langle \phi_{j+1,l}, \hat{\phi}_{j,k} \right\rangle \phi_{j,k}. \end{aligned} \tag{59}$$

Therefore

$$a_k^j = \sum_{l \in \mathcal{I}_{j+1}} a_l^{j+1} \left\langle \phi_{j+1,l}, \hat{\phi}_{j,k} \right\rangle. \tag{60}$$

Let us denote

$$S_{l,k}^j = \left\langle \hat{\phi}_{j,k}, \phi_{j+1,l} \right\rangle, \quad \mathbf{S}_j = \left\{ S_{l,k}^j \right\}_{l \in \mathcal{I}_{j+1}, k \in \mathcal{I}_j} \tag{61}$$

then we can write

$$\mathbf{a}_j = \mathbf{S}_j \mathbf{a}_{j+1}, \tag{62}$$

and

$$\mathbf{S}_j = \left\langle \hat{\Phi}_j, \Phi_{j+1} \right\rangle = \left\langle \hat{\Phi}_j, \tilde{\mathbf{M}}_{j,0} \Phi_j + \tilde{\mathbf{M}}_{j,1} \Psi_j \right\rangle = \tilde{\mathbf{M}}_{j,0}. \tag{63}$$

By Lemma 2 the assertion is proved.

Lemma 4 *A projection* P_j *satisfies*

$$\|P_m P_{m+1} \dots P_{n-1}\| \leq C\, 2^{p\,(n-m)}, \quad p = \frac{\ln 6}{\ln 4}, \tag{64}$$

for all $2 \leq m < n$ *and a constant* C *independent on* m *and* n.

Proof Let $f_n \in V_n$ and $f_m = P_m P_{m+1} \dots P_{n-1} f_n$. We represent f_j by $f_j = \sum_{k \in \mathcal{I}_j} a_k^j \phi_j$ for $j = m, n$ and we set $\mathbf{a}_j = \left\{ a_k^j \right\}_{k \in \mathcal{I}_j}$. It is known [1] that $\{\phi_{j,k}, k \in \mathcal{I}_j\}$ is a Riesz basis of $V_j = \operatorname{span} \Phi_j$ and there exist constants C_1 and C_2 independent of j such that:

$$C_1 \|\mathbf{a}_j\|_2 \leq \left\| \sum_{k \in \mathcal{I}_j} a_k^j \phi_{j,k} \right\| \leq C_2 \|\mathbf{a}_j\|_2. \tag{65}$$

By Lemma 3 we have for $p = \frac{\ln 6}{\ln 4}$:

$$\begin{aligned} \|f_m\| &\leq C_2 \|\mathbf{a}_m\|_2 \leq C_2 \|\mathbf{S}_m\|_2 \|\mathbf{S}_{m+1}\|_2 \dots \|\mathbf{S}_{n-1}\|_2 \|\mathbf{a}_n\|_2 \\ &\leq C_2 2^{p(n-m)} \|\mathbf{a}_n\|_2 \leq C_1^{-1} C_2 2^{p(n-m)} \|f_n\|. \end{aligned} \tag{66}$$

Thus (64) is proved.

Theorem 1 *The set*

$$\left\{2^{-4}\phi_{2,k}, k=1,2\right\} \cup \left\{2^{-2j}\psi_{j,k}, j \geq 2, k=1,\ldots 2^j\right\} \tag{67}$$

is a Riesz basis of $H_0^{\mu}(0,1)$ *for* $\frac{\ln 6}{\ln 4} < \mu < 2.5$.

Proof By Lemma 4 and Theorem 5.3. from [11], the set

$$\left\{2^{-4}\phi_{2,k}, k=1,2\right\} \cup \left\{2^{-2j}\psi_{j,k}, j \geq 2, k=1,\ldots 2^j\right\} \tag{68}$$

is a Riesz basis of the space $H_0^{\mu}(0,1)$ for $\frac{\ln 6}{\ln 4} < \mu < \nu$, where ν is the Sobolev exponent of smoothness of the basis, i.e. $\nu = 2.5$.

Theorem 2 *The set* Ψ *is a Riesz basis of* $H_0^2(0,1)$.

Proof From (21) there exist nonzero constants C_1 and C_2 such that

$$C_1 2^{2j} \leq |\psi_{j,k}|_{H_0^2(\Omega)} \leq C_2 2^{2j}, \quad \text{for } j \geq 2, \quad k=1,\ldots,2^j, \tag{69}$$

and

$$C_1 2^4 \leq |\phi_{2,k}|_{H_0^2(\Omega)} \leq C_2 2^4, \quad \text{for } k=1,2. \tag{70}$$

Let $\hat{\mathbf{b}} = \{\hat{a}_{2,k},\ k \in \mathcal{I}_2\} \cup \left\{\hat{b}_{j,k}, j \geq 2,\ k \in \mathcal{J}_j\right\}$ be such that

$$\left\|\hat{\mathbf{b}}\right\|_2^2 = \sum_{k\in\mathcal{I}_2} \hat{a}_{2,k}^2 + \sum_{k\in\mathcal{J}_j, j\geq 2} \hat{b}_{j,k}^2 < \infty. \tag{71}$$

We define

$$a_{2,k} = \frac{2^4 \hat{a}_{2,k}}{|\phi_{2,k}|_{H_0^2(0,1)}},\ k \in \mathcal{I}_2, \quad b_{j,k} = \frac{2^{2j}\hat{b}_{j,k}}{|\psi_{j,k}|_{H_0^2(0,1)}},\ j \geq 2,\ k \in \mathcal{J}_j, \tag{72}$$

and $\mathbf{b} = \{a_{2,k},\ k \in \mathcal{I}_2\} \cup \{b_{j,k}, j \geq 2,\ k \in \mathcal{J}_j\}$. Then

$$\|\mathbf{b}\|_2 \leq \frac{\left\|\hat{\mathbf{b}}\right\|_2}{C_1} < \infty. \tag{73}$$

Theorem 1 implies that there exist constants C_3 and C_4 such that

$$C_3 \|\mathbf{b}\|_2 \leq \left\| \sum_{k\in\mathcal{I}_2} a_{2,k} 2^{-4}\phi_{2,k} + \sum_{k\in\mathcal{J}_j, j\geq 2} b_{j,k} 2^{-2j}\psi_{j,k} \right\|_{H_0^2(0,1)} \leq C_4 \|\mathbf{b}\|_2 . \tag{74}$$

Therefore

$$\frac{C_4}{C_1}\left\|\hat{\mathbf{b}}\right\|_2 \geq C_4 \|\mathbf{b}\|_2 \geq \left\| \sum_{k\in\mathcal{I}_2} a_{2,k} 2^{-4}\phi_{2,k} + \sum_{k\in\mathcal{J}_j, j\geq 2} b_{j,k} 2^{-2j}\psi_{j,k} \right\|_{H_0^2(0,1)} \tag{75}$$

$$= \left\| \sum_{k\in\mathcal{I}_2} \frac{\hat{a}_{2,k}}{|\phi_{2,k}|_{H_0^2(0,1)}}\phi_{2,k} + \sum_{k\in\mathcal{J}_j, j\geq 2} \frac{\hat{b}_{2,k}}{|\psi_{j,k}|_{H_0^2(0,1)}}\psi_{j,k} \right\|_{H_0^2(0,1)}$$

and similarly

$$\frac{C_3}{C_2}\left\|\hat{\mathbf{b}}\right\|_2 \leq \left\| \sum_{k\in\mathcal{I}_2} \frac{\hat{a}_{2,k}}{|\phi_{2,k}|_{H_0^2(0,1)}}\phi_{2,k} + \sum_{k\in\mathcal{J}_j, j\geq 2} \frac{\hat{b}_{2,k}}{|\psi_{j,k}|_{H_0^2(0,1)}}\psi_{j,k} \right\|_{H_0^2(0,1)} . \tag{76}$$

Theorem 3 *The set* Ψ^{2D} *is a Riesz basis of* $H_0^2\left((0,1)^2\right)$.

Proof The theorem is a consequence of Lemma 1, (69), and Theorem 5.3. from [11].

6 Quantitative properties of constructed bases

In this section, we present the condition numbers of the stiffness matrices for the biharmonic problem in two dimensions. We consider the biharmonic equation

$$\Delta^2 u = f \quad \text{on} \quad \Omega = (0,1)^2, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{77}$$

where Δ is the Laplace operator. The variational formulation is $\mathbf{Au} = \mathbf{f}$, where $\mathbf{A} = \left\langle \Delta\Psi^{2D}, \Delta\Psi^{2D} \right\rangle$, $u = \mathbf{u}^T \Psi^{2D}$, and $\mathbf{f} = \left\langle f, \Psi^{2D} \right\rangle$. It is known that then $\text{cond}\, \mathbf{A} \leq C < \infty$. Since $\mathbf{A}$ is symmetric and positive definite, we have also

$$\text{cond}\, \mathbf{A}_s \leq C, \qquad \text{where } \mathbf{A}_s = \left\langle \Delta\Psi_s^{2D}, \Delta\Psi_s^{2D} \right\rangle. \tag{78}$$

The condition numbers of the stiffness matrices $\mathbf{A}_s$ are shown in Table 1. For the basis *b*) the condition number is even smaller than for a wavelet basis in [11].

Table 1 The condition numbers of the stiffness matrices $\mathbf{A}_s$ of the size $N \times N$ corresponding to multiscale wavelet bases with s levels of wavelets.

s	N	a)	b)
1	36	37.3	6.1
2	196	62.1	7.8
3	900	80.1	8.7
4	3844	92.3	9.8
5	15876	100.4	10.5
6	64516	106.3	11.1

7 Numerical example

We present the quantitative behaviour of the adaptive wavelet method using the bases constructed in this paper. We consider the equation (77) with a solution u given by

$$u(x,y) = v(x)\, v(y), \qquad v(x) = x^2 \left(1 - e^{10x-10}\right)^2. \tag{79}$$

The solution exhibits a sharp gradient near the point $[1,1]$. We solve the problem by the method designed in [6] with the approximate multiplication of the stiffness matrix with a vector proposed in [3]. We use wavelets up to the scale $|\lambda| \leq 10$. The convergence history is shown in Figure 4. In our experiments, the convergence rate, i.e. the slope of the

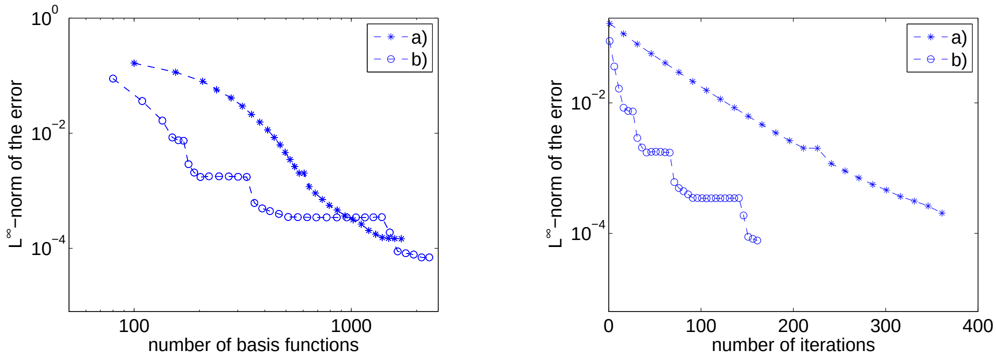

Fig. 4 The convergence history for adaptive wavelet scheme with various wavelet bases.

curve, is similar for both bases. However, bases a) and b) significantly differ in the number of iterations needed to resolve the problem with desired accuracy. The number of basis functions in both cases was about 10^3 for an error in L^∞-norm about 10^{-4}. The number of all basis functions for full grid, i.e. basis functions of the level ten or less, is about 10^6, therefore by using an adaptive method the significant compression was achieved. It can seem that the number of iterations is quite large, but one could take into account that at the beginning the iterations were done for a much smaller vector and the size of this vector increases successively.

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Cubic spline wavelets with short support for fourth-order problems

Dana Černá[a], Václav Finěk[a]

[a] *Department of Mathematics and Didactics of Mathematics, Technical University in Liberec, Studentská 2, 461 17 Liberec, Czech Republic*

Abstract

In the paper, we propose a construction of new cubic spline-wavelet bases on the unit cube satisfying homogeneous Dirichlet boundary conditions of the second order. The basis functions have small supports and wavelets have vanishing moments. We show that stiffness matrices arising from discretization of the biharmonic problem using a constructed wavelet basis have uniformly bounded condition numbers and these condition numbers are very small. We present quantitative properties of the constructed bases and we show a superiority of our construction in comparison to some other cubic spline wavelet bases satisfying boundary conditions of the same type.

Keywords: wavelet, cubic spline, homogeneous Dirichlet boundary conditions, condition number, biharmonic problem
2000 MSC: 46B15, 65N12, 65T60

1. Introduction

In recent years wavelets have been successfully used for solving various types of differential equations [8, 9] as well as integral equations [17, 19, 20]. The quantitative properties of wavelet methods strongly depend on the choice of a wavelet basis, in particular on its condition number. Therefore, a construction of a wavelet basis is an important issue.

In this paper, we propose a construction of cubic spline wavelet bases on the interval that are well-conditioned, adapted to homogeneous Dirichlet boundary conditions of the second order, the wavelets have vanishing moments and the shortest possible support. The wavelet basis of the space $H_0^2\left((0,1)^2\right)$ is then obtained by an isotropic tensor product. We compare the condition numbers of the corresponding stiffness matrices for various constructions. Finally, a quantitative behaviour of an adaptive wavelet method for several boundary-adapted cubic spline wavelet bases is studied.

First of all, we summarize the desired properties of a constructed basis:

- *Riesz basis property.* We construct Riesz bases of the space $H_0^2(0,1)$ and $H_0^2\left((0,1)^2\right)$.
- *Polymial exactness.* Since the primal basis functions are cubic B-splines, the primal multiresolution analysis has polynomial exactness of order four.

Email addresses: dana.cerna@tul.cz (Dana Černá), vaclav.finek@tul.cz (Václav Finěk)

- *Vanishing moments.* The inner wavelets have two vanishing moments, the wavelets near the boundary can have less vanishing moments.
- *Short support.* The wavelets have the shortest possible support for a given number of vanishing moments.
- *Locality.* The primal basis functions are local.
- *Closed form.* The primal scaling functions and wavelets are known in the closed form.
- *Homogeneous Dirichlet boundary conditions.* Our wavelet bases satisfy homogeneous Dirichlet boundary conditions of the second order.
- *Well-conditioned bases.* Our objective is to construct a well conditioned wavelet basis.

Moreover, in a comparison with constructions in [1, 4, 11, 21, 22] that are quite long and technical, the construction in this paper is very simple. Many constructions of cubic spline wavelet or multiwavelet bases on the interval have been proposed in recent years. In [2, 4, 11, 21] cubic spline wavelets on the interval were constructed. In [10] cubic spline multiwavelet bases were designed and they were adapted to complementary boundary conditions of the second order in [22]. In these cases dual functions are known and are local. Cubic spline wavelet or multiwavelet bases where duals are not local were constructed in [7, 14, 15, 16]. Some of these bases were already adapted to boundary conditions and used for solving differential equations [6, 18]. The advantage of our construction is the shortest possible support for a given number of required vanishing moments. Vanishing moments are necessary in some applications such as adaptive wavelet methods [8, 9]. Originally, these methods were designed for wavelet bases with local duals. However, it was shown in [12] that wavelet bases without local dual basis can be used if the solved equation is linear.

This paper is organized as follows: In Section 2 we briefly review the concept of wavelet bases. In Section 3 we propose a construction of primal and dual scaling bases. The refinement matrices are computed in Section 4. In Section 5 the properties of the projectors associated with constructed bases are derived and the proof that the bases are indeed Riesz bases is given. Quantitative properties of constructed bases and other known cubic spline wavelet and multiwavelet bases are studied in Section 6. In Section 7 we compare the number of basis functions and the number of iterations needed to resolve the problem with desired accuracy for bases constructed in this paper and bases from [4, 22]. A numerical example is presented for an equation with the biharmonic operator in two dimensions.

2. Wavelet bases

This section provides a short introduction to the concept of wavelet bases in Sobolev spaces. In this paper, we consider the domain $\Omega = (0,1)$ or $\Omega = (0,1)^2$. We denote the Sobolev space or its subspace by $H \subset H^s(\Omega)$ for nonnegative integer s and the corresponding inner product by $\langle \cdot, \cdot \rangle_H$, a norm by $\|\cdot\|_H$ and a seminorm by $|\cdot|_H$. In case $s = 0$ we consider the space $L^2(\Omega)$ and we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the L^2-inner product and the L^2-norm, respectively. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j,k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale* or a *level.* Let

$$\|\mathbf{v}\|_{l^2(\mathcal{J})} := \sqrt{\sum_{\lambda \in \mathcal{J}} |v_\lambda|^2}, \text{ for } \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \tag{1}$$

and

$$l^2(\mathcal{J}) := \left\{ \mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \|\mathbf{v}\|_{l^2(\mathcal{J})} < \infty \right\}. \tag{2}$$

A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *(primal) wavelet basis* of H, if

i) Ψ is a *Riesz basis* for H, i.e. the closure of the span of Ψ is H and there exist constants $c, C \in (0, \infty)$ such that

$$c \|\mathbf{b}\|_{l^2(\mathcal{J})} \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \leq C \|\mathbf{b}\|_{l^2(\mathcal{J})}, \quad \mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{3}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies (3)}\}$, $C_\psi := \inf\{C : C \text{ satisfies (3)}\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi / c_\psi$ is called the *condition number* of Ψ.

ii) The functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \leq \mathrm{C} 2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where Ω_λ is the support of ψ_λ, and at a given level j the supports of only finitely many wavelets overlap at any point $x \in \Omega$.

By the Riesz representation theorem, there exists a unique family

$$\tilde{\Psi} = \left\{ \tilde{\psi}_\lambda, \lambda \in \tilde{\mathcal{J}} \right\} \subset H \tag{4}$$

biorthogonal to Ψ, i.e.

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle_H = \delta_{i,j} \delta_{k,l}, \quad \text{for all} \quad (i,k) \in \mathcal{J}, \quad (j,l) \in \tilde{\mathcal{J}}, \tag{5}$$

where $\delta_{i,j}$ denotes the Kronecker delta, i.e. $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ for $i \neq j$. This family is also a Riesz basis for H, but the functions $\tilde{\psi}_{j,l}$ need not be local. The basis $\tilde{\Psi}$ is called a *dual* wavelet basis.

In many cases, the wavelet system Ψ is constructed with the aid of a multiresolution analysis. A sequence $\mathcal{V} = \{V_j\}_{j \geq j_0}$, of closed linear subspaces $V_j \subset H$ is called a *multiresolution* or *multiscale analysis*, if

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots H \tag{6}$$

and $\cup_{j \geq j_0} V_j$ is complete in H.

The nestedness and the closedness of the multiresolution analysis implies the existence of the *complement spaces* W_j such that $V_{j+1} = V_j \oplus W_j$.

We now assume that V_j and W_j are spanned by sets of basis functions

$$\Phi_j := \{\phi_{j,k}, k \in \mathcal{I}_j\}, \quad \Psi_j := \{\psi_{j,k}, k \in \mathcal{J}_j\}, \tag{7}$$

where $\mathcal{I}_j$ and $\mathcal{J}_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis and the wavelet basis of H are given by

$$\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j, \quad \Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j. \tag{8}$$

The dual wavelet system $\tilde{\Psi}$ generates a dual multiresolution analysis $\tilde{\mathcal{V}}$ with a dual scaling basis $\tilde{\Phi}_{j_0}$.

Polynomial exactness of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that $\mathbb{P}_{N-1}(\Omega) \subset V_j$ and $\mathbb{P}_{\tilde{N}-1}(\Omega) \subset \tilde{V}_j$, $j \geq j_0$, where $\mathbb{P}_m(\Omega)$ is the space of all algebraic polynomials on Ω of degree less or equal to m. The polynomial exactness of order $\tilde{N}$ on the dual side is equivalent to $\tilde{N}$ vanishing wavelet moments on the primal side, i.e.

$$\int_\Omega P(x)\,\psi_\lambda(x)\,dx = 0, \quad \text{for any } P \in \mathbb{P}_{\tilde{N}-1},\ \psi_\lambda \in \bigcup_{j \geq j_0} \Psi_j. \tag{9}$$

3. Primal scaling basis

A primal scaling basis is the same as the basis constructed in [4, 16]. This basis is generated from functions ϕ and ϕ_b. Let ϕ be a cubic B-spline defined on knots $[0, 1, 2, 3, 4]$. It can be written explicitly as:

$$\phi(x) = \begin{cases} \frac{x^3}{6}, & x \in [0,1], \\ -\frac{x^3}{2} + 2x^2 - 2x + \frac{2}{3}, & x \in [1,2], \\ \frac{x^3}{2} - 4x^2 + 10x - \frac{22}{3}, & x \in [2,3], \\ -\frac{x^3}{6} + 2x^2 - 8x + \frac{32}{3}, & x \in [3,4], \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

Then this function satisfies a scaling equation [16] :

$$\phi(x) = \frac{\phi(2x)}{8} + \frac{\phi(2x-1)}{2} + \frac{3\phi(2x-2)}{4} + \frac{\phi(2x-3)}{2} + \frac{\phi(2x-4)}{8}. \tag{11}$$

The function ϕ_b is a cubic B-spline defined on knots $[0, 0, 1, 2, 3]$. It is given by:

$$\phi_b(x) = \begin{cases} -\frac{11x^3}{12} + \frac{3x^2}{2}, & x \in [0,1], \\ \frac{7x^3}{12} - 3x^2 + \frac{9x}{2} - \frac{3}{2}, & x \in [1,2], \\ -\frac{x^3}{6} + \frac{3x^2}{2} - \frac{9x}{2} + \frac{9}{2}, & x \in [2,3], \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

The function ϕ_b satisfies a scaling equation [16]:

$$\phi_b(x) = \frac{\phi_b(2x)}{4} + \frac{11\phi(2x)}{16} + \frac{\phi(2x-1)}{2} + \frac{\phi(2x-2)}{8}. \tag{13}$$

For $j \in \mathbb{N}$ and $x \in [0, 1]$ we set

$$\begin{aligned} \phi_{j,k}(x) &= 2^{j/2}\phi(2^j x - k),\ k = 2, \ldots 2^j - 2, \\ \phi_{j,1}(x) &= 2^{j/2}\phi_b(2^j x), \quad \phi_{j,2^j-1}(x) = 2^{j/2}\phi_b(2^j(1-x)). \end{aligned} \tag{14}$$

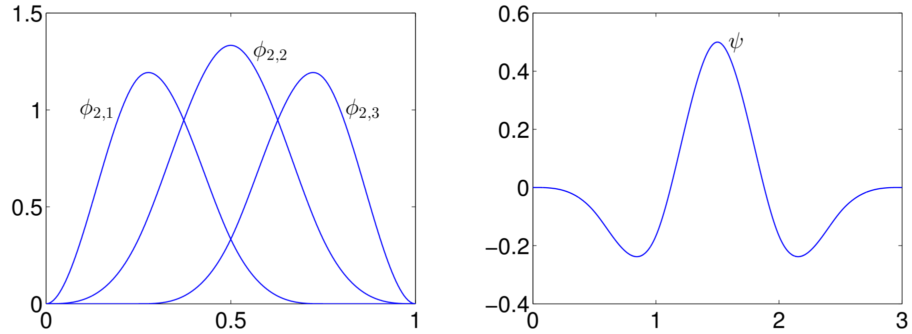

Figure 1: Primal scaling basis for $j = 2$ (left) and the wavelet ψ (right).

The graphs of the functions $\phi_{j,k}$ on the coarsest level $j = 2$ are displayed in Figure 1.

We define a wavelet ψ as

$$\psi(x) = -\frac{1}{2}\phi(2x) + \phi(2x-1) - \frac{1}{2}\phi(2x-2). \tag{15}$$

Then $\operatorname{supp}\psi = [0,3]$ and ψ has two vanishing wavelet moments, i.e.

$$\int_{-\infty}^{\infty} x^k \psi(x)dx = 0, \quad k = 0, 1. \tag{16}$$

The same wavelet was used in the construction of a wavelet basis for the space $L^2(\mathbb{R})$ in [13]. The graph of ψ is shown in Figure 1.

We define a boundary wavelet ψ_b by:

$$\psi_b(x) = \phi_b(2x) + m\phi(2x) + n\phi(2x-1), \tag{17}$$

where m and n are real parameters. In applications, the length of the support and the number of vanishing wavelet moments play a role. We consider four choices of parameters m and n:

a) $m = 0$, $n = 0$
b) $m = -0.75$, $n = 0$
c) $m = -0.45$, $n = 0$
d) $m = -1.35$, $n = 0.6$

These choices are optimal in the following sense: a) defines a wavelet with the shortest possible support, b) defines a wavelet with the shortest possible support among the wavelets of the form (17) with the first vanishing moment, c) corresponds to the wavelet with the shortest possible support among the wavelets of the form (17) with the second vanishing wavelet moment. Wavelet corresponding to d) has two vanishing moments. It is summarized in the following lemma.

Lemma 1. *a) The function* $\psi_b(x) = \phi_b(2x)$ *satisfies* $\operatorname{supp}\psi_b = [0, 1.5]$*.*
b) The function $\psi_b(x) = \phi_b(2x) - 0.75\,\phi(2x)$ *satisfies* $\operatorname{supp}\psi_b = [0,2]$ *and*

$$\int_{-\infty}^{\infty} \psi_b(x)dx = 0. \tag{18}$$

c) The function $\psi_b(x) = \phi_b(2x) - 0.45\,\phi(2x)$ *satisfies* $\operatorname{supp}\psi_b = [0,2]$ *and*

$$\int_{-\infty}^{\infty} x\psi_b(x)dx = 0. \tag{19}$$

d) The function $\psi_b(x) = \phi_b(2x) - 1.35\,\phi(2x) + 0.6\,\phi(2x-1)$ *satisfies* $\operatorname{supp}\psi_b = [0,2.5]$,

$$\int_{-\infty}^{\infty} \psi_b(x)dx = 0, \quad \text{and} \quad \int_{-\infty}^{\infty} x\psi_b(x)dx = 0. \tag{20}$$

Proof. The length of the support of the function ψ_b is derived from the lengths of the supports of functions $\phi_b(2x)$, $\phi(2x)$, and $\phi(2x-1)$. By (10) and (12) we have $\operatorname{supp}\phi_b(2x) = [0,1.5]$, $\operatorname{supp}\phi(2x) = [0,2]$, and $\operatorname{supp}\phi(2x-1) = [0.5,2.5]$. Since the functions $\phi_b(2x)$, $\phi(2x)$ and $\phi(2x-1)$ are given in the closed form, the formulas (18), (19), and (20) can be verified easily. □

Thus, we can choose boundary wavelet with two vanishing moments and larger support or boundary wavelets with shorter supports but only with one or zero vanishing moments. If $f \in H_0^2(0,1)$ and f is constant or linear at the interval $[0,\epsilon]$, then f have to be zero at $[0,\epsilon]$. The same holds for the interval $[1-\epsilon,1]$. Hence $f \in H_0^2(0,1)$ can not be nonzero constant or linear near the boundary and therefore in some applications such as adaptive wavelet methods the vanishing moments does not play the significant role for boundary wavelets. The graphs of boundary wavelets ψ_b are displayed in Figure 2. All the following lemmas and theorems are valid for the wavelet basis Ψ including the boundary wavelet with parameters m and n given by a), b), c), or d).

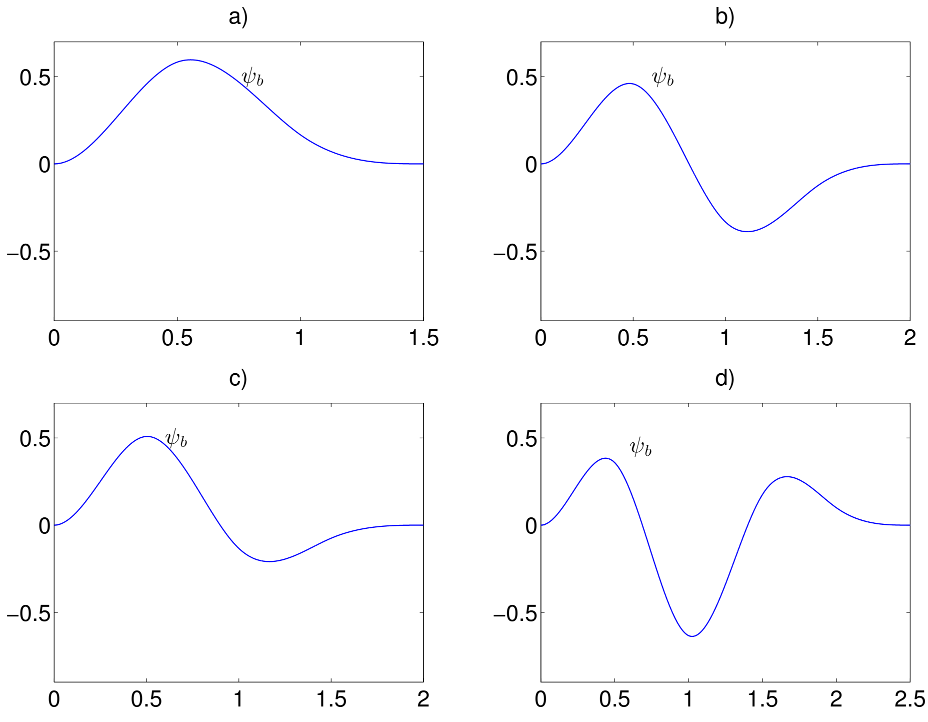

Figure 2: Boundary wavelet ψ_b for a), b), c), and d), respectively.

For $j \in \mathbb{N}$ and $x \in [0,1]$ we define

$$\begin{aligned} \psi_{j,k}(x) &= 2^{j/2}\psi(2^j x - k + 2), k = 2, ..., 2^j - 1, \\ \psi_{j,1}(x) &= 2^{j/2}\psi_b(2^j x), \quad \psi_{j,2^j}(x) = 2^{j/2}\psi_b(2^j(1-x)). \end{aligned} \tag{21}$$

We denote

$$\begin{aligned} \Phi_j &= \left\{ \phi_{j,k} / \left| \phi_{j,k} \right|_{H_0^2(0,1)}, k = 1, \ldots, 2^j - 1 \right\}, \\ \Psi_j &= \left\{ \psi_{j,k} / \left| \psi_{j,k} \right|_{H_0^2(0,1)}, k = 1, \ldots, 2^j \right\}. \end{aligned} \tag{22}$$

Then the sets

$$\Psi^s = \Phi_2 \cup \bigcup_{j=2}^{1+s} \Psi_j \quad \text{and} \quad \Psi = \Phi_2 \cup \bigcup_{j=2}^{\infty} \Psi_j \tag{23}$$

are a multiscale wavelet basis and a wavelet basis of the space $H_0^2(0,1)$, respectively. We use $u \otimes v$ to denote the tensor product of functions u and v, i.e. $(u \otimes v)(x_1, x_2) = u(x_1)\, v(x_2)$. We set

$$\begin{aligned} F_j &= \left\{ \phi_{j,k} \otimes \phi_{j,l} \,/\, \left| \phi_{j,k} \otimes \phi_{j,l} \right|_{H_0^2(\Omega)}, k, l = 1, \ldots, 2^j - 1 \right\} \\ G_j^1 &= \left\{ \phi_{j,k} \otimes \psi_{j,l} \,/\, \left| \phi_{j,k} \otimes \psi_{j,l} \right|_{H_0^2(\Omega)}, k = 1, \ldots, 2^j - 1, l = 1, \ldots 2^j \right\} \\ G_j^2 &= \left\{ \psi_{j,k} \otimes \phi_{j,l} \,/\, \left| \psi_{j,k} \otimes \phi_{j,l} \right|_{H_0^2(\Omega)}, k = 1, \ldots, 2^j, l = 1, \ldots 2^j - 1 \right\} \\ G_j^3 &= \left\{ \psi_{j,k} \otimes \psi_{j,l} \,/\, \left| \psi_{j,k} \otimes \psi_{j,l} \right|_{H_0^2(\Omega)}, k, l = 1, \ldots, 2^j \right\} \end{aligned}$$

where $\Omega = (0,1)^2$. A wavelet basis and a multiscale wavelet basis of the space $H_0^2(\Omega)$ are defined as

$$\Psi_s^{2D} = F_2 \cup \bigcup_{j=2}^{1+s} \left(G_j^1 \cup G_j^2 \cup G_j^3 \right), \quad \Psi^{2D} = F_2 \cup \bigcup_{j=2}^{\infty} \left(G_j^1 \cup G_j^2 \cup G_j^3 \right). \tag{24}$$

Remark 1. Wavelet basis of the space $H^2(\Omega)$ can be constructed in a similar way. We add two boundary functions ϕ_{b1} and ϕ_{b2} that are B-splines on sequences of knots $[0,0,0,0,1]$ and $[0,0,0,1,2]$, respectively. Then scaling basis is generated from the functions ϕ_{b1}, ϕ_{b2}, ϕ_b and ϕ as in (14), see also [2], and boundary wavelets are constructed as appropriate linear combinations of ϕ_{b1}, ϕ_{b2} and ϕ_b in a similar way as above.

4. Refinement matrices

From the nestedness and the closedness of multiresolution spaces it follows that there exist *refinement matrices* $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that

$$\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}, \quad \Psi_j = \mathbf{M}_{j,1}^T \Phi_{j+1}. \tag{25}$$

In these formulas we view the sets of functions Φ_j and Ψ_j as column vectors with entries $\phi_{j,k}$, $k = 1, \ldots, 2^j - 1$, and $\psi_{j,k}$, $k = 1, \ldots, 2^j$, respectively. Due to the length of the support of primal scaling functions, the refinement matrix $\mathbf{M}_{j,0}$ has the following structure:

$$\mathbf{M}_{j,0} = \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{M}_{j,0}^I & \\ & & \mathbf{M}_R \end{pmatrix}. \tag{26}$$

where $\mathbf{M}_{j,0}^I$ is a $(2^{j+1}-3) \times (2^j - 3)$ matrix given by

$$\left(\mathbf{M}_{j,0}^I\right)_{m,n} = \begin{cases} \frac{h_{m+1-2n}}{\sqrt{2}}, & n = 1, \ldots, 2^j - 3, \ 0 \le m+1-2n \le 4, \\ 0, & \text{otherwise}, \end{cases} \tag{27}$$

where

$$\mathbf{h} = [h_0, h_1, h_2, h_3, h_4] = \left[\frac{1}{8}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{8}\right] \tag{28}$$

is a vector of coefficients from scaling equation (11). We denote a vector of coefficients from scaling equation (13) by

$$\mathbf{h}_b = \left[h_0^b, h_1^b, h_2^b, h_3^b\right] = \left[\frac{1}{4}, \frac{11}{16}, \frac{1}{2}, \frac{1}{8}\right] \tag{29}$$

Then $\mathbf{M}_L = \frac{1}{\sqrt{2}} \mathbf{h}_b^T$ and the matrix $\mathbf{M}_R$ is obtained from a matrix $\mathbf{M}_L$ by reversing the ordering of rows.

It follows from the equations (15) and (17) that the matrix $\mathbf{M}_{j,1}$ is of the size $(2^{j+1}-1) \times 2^j$ and has the structure

$$\mathbf{M}_{j,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & m & n & 0 & 0 & 0 & \ldots & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & & 0 \\ \vdots & \vdots & & & & & & \vdots \\ 0 & \ldots & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & \ldots & 0 & 0 & 0 & n & m & 1 \end{pmatrix}^T \tag{30}$$

There also exist refinement matrices $\tilde{\mathbf{M}}_{j,0}$ and $\tilde{\mathbf{M}}_{j,1}$ corresponding to dual spaces that satisfy:

$$\tilde{\Phi}_j = \tilde{\mathbf{M}}_{j,0}^T \tilde{\Phi}_{j+1}, \quad \tilde{\Psi}_j = \tilde{\mathbf{M}}_{j,1}^T \tilde{\Phi}_{j+1}, \tag{31}$$

where the sets $\tilde{\Phi}_j$ and $\tilde{\Psi}_j$ are viewed as column vectors.

The Euclidean norm of a vector $\mathbf{v}$ is denoted by $\|\mathbf{v}\|_2$ and the spectral norm of the matrix $\mathbf{M}$ is denoted as $\|\mathbf{M}\|_2$. The following lemma is crucial for the proof of a Riesz basis property.

Lemma 2. *The norm of the matrix $\tilde{\mathbf{M}}_{j,0}$ satisfies $\left\|\tilde{\mathbf{M}}_{j,0}\right\|_2 \le 2^p, p = 1 + \frac{\ln 3}{\ln 4}$.*

Proof. We prove the lemma for the choice c) of parameters for boundary wavelet, for choices a), b), and d) the proof is similar. We denote the entries of the matrix $\tilde{\mathbf{M}}_{j,0}$ as $\tilde{M}^{j,0}_{k,l}$, $k = 1, \dots 2^{j+1} - 1$, $l = 1, \dots, 2^j - 1$.

Due to biorthogonality of the sets $\Psi_j \cup \Phi_j$ and $\tilde{\Psi}_j \cup \tilde{\Phi}_j$ we have

$$\mathbf{M}^T_{j,0} \tilde{\mathbf{M}}_{j,0} = \mathbf{I}_j \tag{32}$$

and

$$\mathbf{M}^T_{j,1} \tilde{\mathbf{M}}_{j,0} = \mathbf{0}_j, \tag{33}$$

where $\mathbf{I}_j$ denotes the identity matrix and $\mathbf{0}_j$ denotes the zero matrix of the appropriate size.

From (30) and (33) we have

$$\tilde{M}^{j,0}_{1,l} = 0.45\, \tilde{M}^{j,0}_{2,l}, \quad \tilde{M}^{j,0}_{2^{j+1}-1,l} = 0.45\, \tilde{M}^{j,0}_{2^{j+1}-2,l}, \tag{34}$$

and

$$\tilde{M}^{j,0}_{k,l} = \frac{\tilde{M}^{j,0}_{k-1,l} + \tilde{M}^{j,0}_{k+1,l}}{2}, \quad \text{for } k \text{ odd}, \quad k = 3, \dots, 2^{j+1} - 3. \tag{35}$$

We substitute these relations into (32) and we obtain a new system of equations $\mathbf{A}_j \mathbf{B}_j = \mathbf{I}_j$, where

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{21}{20} & \frac{3}{8} & 0 & \dots & & 0 \\ \frac{3}{8} & \frac{5}{4} & \frac{3}{8} & & & \vdots \\ 0 & \frac{3}{8} & \frac{5}{4} & \frac{3}{8} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{3}{8} & \frac{5}{4} & \frac{3}{8} \\ 0 & & \dots & 0 & \frac{3}{8} & \frac{21}{20} \end{pmatrix} \tag{36}$$

and $\mathbf{B}_j$ contains $\tilde{M}^{j,0}_{k,l}$ for k even, i.e. the entries $B^j_{k,l}$ of the matrix $\mathbf{B}_j$ satisfy:

$$B^j_{k,l} = \tilde{M}^{j,0}_{2k,l}, \quad k, l = 1, \dots 2^j - 1. \tag{37}$$

We factorize the matrix $\mathbf{A}_j$ as $\mathbf{A}_j = \mathbf{C}_j \mathbf{D}_j$, where

$$\mathbf{C}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{9}{8} & \frac{3}{8} & 0 & 0 & \dots & 0 \\ \frac{3}{8} & \frac{5}{4} & \frac{3}{8} & & & \vdots \\ 0 & \frac{3}{8} & \frac{5}{4} & \frac{3}{8} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{3}{8} & \frac{5}{4} & \frac{3}{8} \\ 0 & \dots & 0 & 0 & \frac{3}{8} & \frac{9}{8} \end{pmatrix} \tag{38}$$

and

$$\mathbf{D}_j = \begin{pmatrix} \frac{37}{40} & 0 & 0 & \dots & 0 & 0 & \frac{1}{40(-3)^{2^j-3}} \\ \frac{1}{40} & 1 & 0 & & 0 & 0 & \frac{1}{40(-3)^{2^j-4}} \\ \frac{-1}{40\cdot 3} & 0 & 1 & & 0 & 0 & \frac{1}{40(-3)^{2^j-5}} \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \frac{1}{40(-3)^{2^j-5}} & 0 & 0 & & 1 & 0 & \frac{-1}{40\cdot 3} \\ \frac{1}{40(-3)^{2^j-4}} & 0 & 0 & & 0 & 1 & \frac{1}{40} \\ \frac{1}{40(-3)^{2^j-3}} & 0 & 0 & \dots & 0 & 0 & \frac{37}{40} \end{pmatrix}. \qquad (39)$$

More precisely, the entries $D^j_{k,l}$ of the matrix $\mathbf{D}_j$ are given by:

$$\begin{aligned} D^j_{1,1} &= D^j_{2^j-1,2^j-1} = \frac{37}{40}, \\ D^j_{k,1} &= D^j_{2^j+1-k,2^j-1} = \frac{(-1)^k}{40\cdot 3^{k-2}}, \quad \text{for } k = 2,\dots,2^j-1, \\ D^j_{k,k} &= 1, \quad \text{for } k = 2,\dots,2^j-2, \\ D^j_{k,l} &= 0, \quad \text{otherwise.} \end{aligned} \qquad (40)$$

It is easy to verify that $\tilde{\mathbf{C}}_j = \mathbf{C}_j^{-1}$ has entries:

$$\tilde{C}^j_{k,l} = \sqrt{2}\left(-\frac{1}{3}\right)^{|k-l|}, \qquad (41)$$

and the matrix $\mathbf{D}_j^{-1}$ has the structure:

$$\mathbf{D}_j^{-1} = \begin{pmatrix} d^j_1 & 0 & \dots & 0 & d^j_n \\ d^j_2 & 1 & & 0 & d^j_{n-1} \\ \vdots & & \ddots & & \vdots \\ d^j_{n-1} & 0 & & 1 & d^j_2 \\ d^j_n & 0 & \dots & 0 & d^j_1 \end{pmatrix}, \qquad (42)$$

with $n = 2^j - 1$ and

$$\begin{aligned} d^j_1 &= \frac{40\,\alpha_n}{37}, \; d^j_n = \frac{(-1)^{n-1}\,40\,\alpha_n}{37^2\cdot 3^{n-2}}, \\ d^j_k &= (-1)^{k-1}\,\alpha_n\left(\frac{1}{37\cdot 3^{k-2}} + \frac{3^k}{\beta_n}\right), \quad k = 2,\dots,n-1. \end{aligned} \qquad (43)$$

where constants α_n and β_n are given by

$$\alpha_n = \left(1 - \frac{1}{37^2\cdot 3^{2n-4}}\right)^{-1}, \quad \beta_n = 37^2\cdot 3^{2n-3}. \qquad (44)$$

Therefore $\mathbf{B}_j = \mathbf{A}_j^{-1} = \mathbf{D}_j^{-1}\mathbf{C}_j^{-1}$ and substituting it into (33) we obtain the entries of the matrix $\tilde{\mathbf{M}}_{j,0}$:

$$\tilde{M}^{j,0}_{2,l} = \frac{40\sqrt{2}\alpha_n}{37}\left(\frac{-1}{3}\right)^{|1-l|} + \frac{(-1)^{|n-1|}\,40\sqrt{2}\,\alpha_n}{37^2\cdot 3^{n-2}}\left(\frac{-1}{3}\right)^{|n-l|}, \tag{45}$$

$$\tilde{M}^{j,0}_{2^{j+1}-2,l} = \tilde{M}^{j,0}_{2,2^j-l}, \tag{46}$$

and for $k \in (2, 2^{j+1}-2)$ even:

$$\tilde{M}^{j,0}_{k,l} = \frac{\sqrt{2}}{(-3)^{|k-l|}} + \frac{\sqrt{2}\alpha_n\,(-1)^{k-1}}{(-3)^{|1-l|}}\left(\frac{1}{37\cdot 3^{k-2}} + \frac{3^k}{\beta_n}\right) + \frac{\sqrt{2}\alpha_n\,(-1)^{n-k}}{(-3)^{|n-l|}}\left(\frac{1}{37\cdot 3^{n-k-1}} + \frac{3^{n+1-k}}{\beta_n}\right) \tag{47}$$

The entries $\tilde{M}^{j,0}_{k,l}$ for k odd are given by (34) and (35).

It is well-known that for any matrix $\mathbf{M}$ of the size $m \times n$ with entries $M_{k,l}$:

$$\|\mathbf{M}\|_2 \leq \sqrt{\|\mathbf{M}\|_1\,\|\mathbf{M}\|_\infty}, \tag{48}$$

where

$$\|\mathbf{M}\|_1 = \max_{l=1,\dots,n}\sum_{k=1}^{m}|M_{k,l}|, \quad \|\mathbf{M}\|_\infty = \max_{k=1,\dots,m}\sum_{l=1}^{n}|M_{k,l}|. \tag{49}$$

In our case, from (45), (49), and a formula for a sum of a geometric sequence we obtain:

$$\left\|\tilde{\mathbf{M}}_{j,0}\right\|_1 \leq 3\sqrt{2} \quad \text{and} \quad \left\|\tilde{\mathbf{M}}_{j,0}\right\|_\infty \leq 2\sqrt{2}. \tag{50}$$

Thus

$$\left\|\tilde{\mathbf{M}}_{j,0}\right\|_2 \leq 2\sqrt{3} = 2^p \quad \text{for} \quad p = 1 + \frac{\ln 3}{\ln 4}. \tag{51}$$

□

The consequence of the proof of Lemma 2 is that the matrix $\mathbf{M}_j = (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$ representing the discrete wavelet transform is invertible.

Lemma 3. *The matrix* $\mathbf{M}_j = (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$ *is invertible.*

Proof. We prove the lemma for the choice c) of parameters for boundary wavelet, for other choices the proof is similar. The matrix $\mathbf{M}_j$ is invertible if and only if the matrix $\tilde{\mathbf{M}}_j = (\tilde{\mathbf{M}}_{j,0}, \tilde{\mathbf{M}}_{j,1})$ satisfying $\mathbf{M}_j^T\tilde{\mathbf{M}}_j = \mathbf{I}_j$ exists and is unique. The existence and uniqueness of the matrix $\tilde{\mathbf{M}}_{j,0}$ is already shown in the proof of Lemma 2. The entries $\tilde{M}^{j,1}_{k,l}$ of the matrix $\tilde{\mathbf{M}}_{j,1}$ satisfy for $l = 1, \dots, 2^{j+1}$:

$$\tilde{M}^{j,1}_{1,l} = \delta_{l,1} - 0.45\,\tilde{M}^{j,1}_{2,l}, \quad \tilde{M}^{j,1}_{2^{j+1}-1,l} = \delta_{l,1} - 0.45\,\tilde{M}^{j,1}_{2^{j+1}-2,l}, \tag{52}$$

and

$$\tilde{M}^{j,1}_{k,l} = \delta_{k,2l-1} + \frac{\tilde{M}^{j,1}_{k-1,l} + \tilde{M}^{j,1}_{k+1,l}}{2}, \; k\,\text{odd}, \; k = 3, \dots, 2^{j+1}-3. \tag{53}$$

Using these relations we obtain a system of equations with the matrix $\mathbf{A}_j$ defined by (36). From the proof of Lemma 2 follows that $\mathbf{A}_j$ is invertible. Therefore the matrix $\tilde{\mathbf{M}}_{j,1}$ exists and is unique. □

5. Riesz basis on Sobolev spaces

For $j \geq 2$ we define a column vector

$$(\Gamma_j)_k = \begin{cases} \phi_{j,k}, & k = 1, 2, \ldots, 2^j - 1, \\ \psi_{j,k-2^j-1}, & k = 2^j, \ldots 2^{j+1} - 1. \end{cases} \tag{54}$$

The symbol $\langle \cdot, \cdot \rangle$ denotes the standard $L^2(\Omega)$ inner product. If $\mathbf{u}$ and $\mathbf{v}$ are two vectors of functions of the length n, then $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes matrix with entries $\langle \mathbf{u}_k, \mathbf{v}_l \rangle$, $k, l = 1, \ldots, n$. We set $\mathbf{F}_j = \langle \Gamma_j, \Gamma_j \rangle$, where $\hat{\Gamma}_j = \mathbf{F}_j^{-1} \Gamma_j$. We denote

$$\mathcal{I}_j = \left\{1, 2, \ldots, 2^j - 1\right\} \quad \text{and} \quad \mathcal{J}_j = \left\{1, 2, \ldots, 2^j\right\} \tag{55}$$

and the entries of $\hat{\Gamma}_j$ as

$$\hat{\phi}_{j,k} = \left(\hat{\Gamma}_j\right)_k, k \in \mathcal{I}_j, \ \hat{\psi}_{j,k} = \left(\hat{\Gamma}_j\right)_{k+2^j-1}, k \in \mathcal{J}_j. \tag{56}$$

Since obviously

$$\left\langle \Gamma_j, \hat{\Gamma}_j \right\rangle = \mathbf{I}_j, \tag{57}$$

functions from $\hat{\Gamma}_j$ are duals to functions from Γ_j in the space V_{j+1}. Since $\mathbf{F}_j^{-1}$ is not a sparse matrix, these duals are not local. We define a projection P_j from V_{j+1} onto V_j by

$$P_j f = \sum_{k \in \mathcal{I}_j} \left\langle f, \hat{\phi}_{j,k} \right\rangle \phi_{j,k}. \tag{58}$$

Lemma 4. *Let $f \in V_{j+1}$, $a_k^j = \left\langle f, \hat{\phi}_{j,k} \right\rangle$, $\mathbf{a}_j = \left\{a_k^j\right\}_{k \in \mathcal{I}_j}$, $j \geq 2$, and $\mathbf{S}_j : \mathbf{a}_{j+1} \mapsto \mathbf{a}_j$. Then $\|\mathbf{S}_j\|_2 \leq 2^p$, $p = 1 + \frac{\ln 3}{\ln 4}$.*

Proof. We have

$$\begin{aligned} P_j f &= \sum_{k \in \mathcal{I}_j} a_k^j \phi_{j,k} = \sum_{k \in \mathcal{I}_j} \left\langle f, \hat{\phi}_{j,k} \right\rangle \phi_{j,k} \\ &= \sum_{k \in \mathcal{I}_j} \sum_{l \in \mathcal{I}_{j+1}} a_l^{j+1} \left\langle \phi_{j+1,l}, \hat{\phi}_{j,k} \right\rangle \phi_{j,k}. \end{aligned} \tag{59}$$

Therefore

$$a_k^j = \sum_{l \in \mathcal{I}_{j+1}} a_l^{j+1} \left\langle \phi_{j+1,l}, \hat{\phi}_{j,k} \right\rangle. \tag{60}$$

Let us denote

$$S_{l,k}^j = \left\langle \hat{\phi}_{j,k}, \phi_{j+1,l} \right\rangle, \quad \mathbf{S}_j = \left\{S_{l,k}^j\right\}_{l \in \mathcal{I}_{j+1}, k \in \mathcal{I}_j} \tag{61}$$

then we can write $\mathbf{a}_j = \mathbf{S}_j \mathbf{a}_{j+1}$, and

$$\mathbf{S}_j = \left\langle \hat{\Phi}_j, \Phi_{j+1} \right\rangle = \left\langle \hat{\Phi}_j, \tilde{\mathbf{M}}_{j,0} \Phi_j + \tilde{\mathbf{M}}_{j,1} \Psi_j \right\rangle = \tilde{\mathbf{M}}_{j,0}. \tag{62}$$

By Lemma 2 the assertion is proved. □

Lemma 5. *A projection P_j satisfies*

$$\|P_m P_{m+1} \dots P_{n-1}\| \leq 2^{p\,(n-m)}, \quad p = 1 + \frac{\ln 3}{\ln 4}, \tag{63}$$

for all $2 \leq m < n$.

Proof. Let $f_n \in V_n$, $f_m = P_m P_{m+1} \dots P_{n-1} f_n$, $f_j = \sum_{k \in \mathcal{I}_j} a_k^j \phi_j$, $\mathbf{a}_j = \left\{a_k^j\right\}_{k \in \mathcal{I}_j}$, $j = m, n$. Since Φ_j is a Riesz basis of V_j [2, 16], there exist constants C_1 and C_2 independent of j such that:

$$C_1 \|\mathbf{a}_j\|_2 \leq \left\| \sum_{k \in \mathcal{I}_j} a_k^j \phi_j \right\| \leq C_2 \|\mathbf{a}_j\|_2 . \tag{64}$$

By Lemma 4 we have for $p = 1 + \frac{\ln 3}{\ln 4}$:

$$\begin{aligned} \|f_m\| &\leq C_2 \|\mathbf{a}_m\|_2 \leq C_2 \|\mathbf{S}_m\|_2 \|\mathbf{S}_{m+1}\|_2 \dots \|\mathbf{S}_{n-1}\|_2 \|\mathbf{a}_n\|_2 \\ &\leq C_2 2^{p(n-m)} \|\mathbf{a}_n\|_2 \leq \frac{C_2}{C_1} 2^{p(n-m)} \|f_n\| . \end{aligned} \tag{65}$$

Thus (63) is proved. □

Theorem 6. *The set Ψ is a Riesz basis of $H_0^2(0,1)$.*

Proof. By Lemma 5 and Theorem 5.3. from [16], the set

$$\left\{2^{-2}\phi_{j,k}, j \geq 2, k = 1, \dots 2^j\right\} \cup \left\{2^{-2j}\psi_{j,k}, j \geq 2, k = 1, \dots 2^j\right\} \tag{66}$$

is a Riesz basis of the space $H_0^\mu(0,1)$ for $1 + \frac{\ln \sqrt{3}}{\ln 2} < \mu < 2.5$.

Since obviously

$$c 2^{2j} \leq |\psi_{j,k}|_{H_0^2(\Omega)} \leq C 2^{2j}, \quad \text{for } j \geq 2, \quad k = 1, \dots, 2^j, \tag{67}$$

the set Ψ defined by (23) is a Riesz basis of the space $H_0^2(0,1)$. □

Theorem 7. *The set Ψ^{2D} is a Riesz basis of $H_0^2\left((0,1)^2\right)$.*

Proof. The theorem is a consequence of Lemma 6, (67), and Theorem 5.3. from [16]. □

6. Quantitative properties of constructed bases

In this section, we compare the condition numbers of the stiffness matrices for the biharmonic problem in two dimensions for different wavelet bases. For $\Omega = (0,1)^2$ we consider the biharmonic equation

$$\Delta^2 u = f \text{ on } \Omega, \quad u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{68}$$

where Δ is the Laplace operator and n is the outer unit normal vector. The variational formulation is $\mathbf{A}\mathbf{u} = \mathbf{f}$, where $\mathbf{A} = \left\langle \Delta\Psi^{2D}, \Delta\Psi^{2D} \right\rangle$, $u = \mathbf{u}^T \Psi^{2D}$, and $\mathbf{f} = \left\langle f, \Psi^{2D} \right\rangle$. It is known that then $\operatorname{cond} \mathbf{A} \leq C < \infty$. Since $\mathbf{A}_s = \left\langle \Delta\Psi_s^{2D}, \Delta\Psi_s^{2D} \right\rangle$ is a part of the matrix $\mathbf{A}$ that is symmetric and positive definite, we have also $\operatorname{cond} \mathbf{A}_s \leq C$. The

s	N	a)	b)	c)	d)	JZ11	CF12	S09
1	49	14.6	13.6	8.5	50.7	34.0	128.1	484.4
2	225	21.4	18.1	14.3	72.3	34.9	141.3	583.4
3	961	23.7	20.2	17.5	84.4	35.1	212.0	626.9
4	3969	24.5	21.4	18.2	91.3	35.3	257.6	653.5
5	16129	24.8	22.2	18.4	95.3	35.5	281.2	673.2
6	65025	25.2	22.6	18.6	98.0	35.8	297.2	689.4

Table 1: The condition numbers of the stiffness matrices $\mathbf{A}_s$ of the size $N \times N$ corresponding to multiscale wavelet bases with s levels of wavelets.

condition numbers of the stiffness matrices $\mathbf{A}_s$ are shown in Table 1. A construction by Jia and Zhao from [16] is denoted as JZ11, a construction from [4] is denoted as CF12, a construction of multiwavelet basis from [22] is denoted as S09 and wavelet bases constructed in this paper are denoted as a), b), c), and d) according to the choice of parameters for the boundary wavelet. The size of the stifness matrix is $N \times N$ for wavelet bases constructed in this paper, it differs for other bases. The condition number for our wavelet bases is comparable to the wavelet basis from [16], but the difference is that wavelets from [16] have not vanishing moments and therefore can not be used in some applications such as adaptive wavelet methods. Wavelet bases from [4, 22] have significantly larger condition number.

Remark 2. We can also treat the fourth-order problem subject to nonhomogeneous Dirichlet boundary conditions:

$$\Delta^2 u = f \quad \text{on } \Omega, \quad u = g \text{ on } \partial\Omega, \quad \frac{\partial u}{\partial n} = h \text{ on } \partial\Omega. \tag{69}$$

Let $w \in H^2(\Omega)$ be a function such that

$$w = g \text{ on } \partial\Omega, \quad \frac{\partial w}{\partial n} = h \text{ on } \partial\Omega. \tag{70}$$

Then the solution u of the problem (69) can be computed as $u = w + \tilde{u}$, where $\tilde{u}$ solves the problem

$$\Delta^2 \tilde{u} = f - \Delta^2 w \text{ on } \Omega, \quad \tilde{u} = 0 \text{ on } \partial\Omega, \quad \frac{\partial \tilde{u}}{\partial n} = 0 \text{ on } \partial\Omega. \tag{71}$$

If $\Omega = (0,1)$, we can simply set w to be a Hermite cubic polynomial:

$$w(x) = Ax^3 + Bx^2 + Cx + D \tag{72}$$

with $A = 2g(0) - h(0) - 2g(1) + h(1)$, $B = -3g(0) + 2h(0) + 3g(1) - h(1)$, $C = -h(0)$, $D = g(0)$.

The case $\Omega = (0,1)^2$ can be treated in a similar way. Since in formulation (69), the values of normal derivative of u are not well defined at corners, we will consider more precise formulation:

$$\begin{aligned} w &= g \text{ on } \partial\Omega,\ \frac{\partial w}{\partial y}(x,0) = h_1(x),\ \frac{\partial w}{\partial x}(1,y) = h_2(y), \\ \frac{\partial w}{\partial y}(x,1) &= h_3(x),\ \frac{\partial w}{\partial x}(0,y) = h_4(y),\ x,y \in [0,1]. \end{aligned} \tag{73}$$

If $w \in C^2\left(\bar{\Omega}\right)$ then

$$\begin{aligned} h_1(0) &= \frac{\partial w}{\partial y}(0,0) = \frac{\partial g}{\partial y}(0,0), \\ \frac{\mathrm{d}h_1}{\mathrm{d}x}(0) &= \frac{\partial^2 w}{\partial x \partial y}(0,0) = \frac{\partial^2 w}{\partial y \partial x}(0,0) = \frac{\mathrm{d}h_4}{\mathrm{d}y}(0), \end{aligned} \tag{74}$$

and similarly at other corners. Therefore, we assume that

$$\begin{aligned} h_1(0) &= \frac{\partial g}{\partial y}(0,0),\ h_1(1) = \frac{\partial g}{\partial y}(1,0), h_3(0) = \frac{\partial g}{\partial y}(0,1), \\ h_3(1) &= \frac{\partial g}{\partial y}(1,1), \frac{\mathrm{d}h_1}{\mathrm{d}x}(0) = \frac{\mathrm{d}h_4}{\mathrm{d}y}(0),\ \frac{\mathrm{d}h_1}{\mathrm{d}x}(1) = \frac{\mathrm{d}h_2}{\mathrm{d}y}(0), \\ \frac{\mathrm{d}h_3}{\mathrm{d}x}(1) &= \frac{\mathrm{d}h_2}{\mathrm{d}y}(1),\ \frac{\mathrm{d}h_3}{\mathrm{d}x}(0) = \frac{\mathrm{d}h_4}{\mathrm{d}y}(1). \end{aligned} \tag{75}$$

We first construct a function u^1 that satisfies boundary conditions at the part of the boundary $\{[0,y], y \in [0,1]\} \cup \{[1,y], y \in [0,1]\}$. We set

$$u^1(x,y) = A(y)x^3 + B(y)x^2 + C(y)x + D(y) \tag{76}$$

with $A(y) = 2g(0,y) + h_4(y) - 2g(1,y) + h_2(y)$, $B(y) = -3g(0,y) - 2h_4(y) + 3g(1,y) - h_2(y)$, $C(y) = h_4(y)$, $D(y) = g(0,y)$. We define $\tilde{g} = g - u^1$ on $\partial\Omega$, $\tilde{h}_1(x) = h_i(x) - \frac{\partial u^1}{\partial y}(x,0)$ and $\tilde{h}_3(x) = h_3(x) - \frac{\partial u^1}{\partial y}(x,1)$. We construct a function u^2 that satisfies $u^2 = \tilde{g}$ on $\partial\Omega$ and

$$\frac{\partial u^2}{\partial y}(x,0) = \tilde{h}_1(x),\ \frac{\partial u^2}{\partial y}(x,1) = \tilde{h}_3(x),\ \frac{\partial u^2}{\partial x}(0,y) = \frac{\partial u^2}{\partial x}(1,y) = 0. \tag{77}$$

We set

$$u^2(x,y) = \tilde{A}(x)y^3 + \tilde{B}(x)y^2 + \tilde{C}(x)y + \tilde{D}(x) \tag{78}$$

with $\tilde{A}(x) = 2\tilde{g}(x,0) + \tilde{h}_1(x) - 2\tilde{g}(x,1) + \tilde{h}_3(x)$, $\tilde{B}(x) = -3\tilde{g}(x,0) - 2\tilde{h}_1(x) + 3\tilde{g}(x,1) - \tilde{h}_3(x)$, $\tilde{C}(x) = \tilde{h}_1(x)$, $\tilde{D}(x) = \tilde{g}(x,0)$. Due to (75) it can be simply verified that $w = u^1 + u^2$ satisfies (73).

7. Numerical example

In this section, we compare the quantitative behaviour of the adaptive wavelet method with a basis constructed in this paper and bases from [4, 22]. All these bases are formed by cubic splines. We first briefly review an adaptive wavelet method. The method was proposed by Cohen, Dahmen and DeVore in [8]. We use a slightly modified version from [3] with an adaptive matrix-vector multiplication from [5].

For $\Omega = (0,1)^2$ we consider the fourth-order problem (68). Let Ψ^{2D} be a wavelet basis constructed in this paper. As mentioned above, the original equation (68) can be reformulated as an equivalent biinfinite matrix equation $\mathbf{Au} = \mathbf{f}$, where $\mathbf{A} = \left\langle \Delta\Psi^{2D}, \Delta\Psi^{2D} \right\rangle$, $u = \mathbf{u}^T \Psi^{2D}$, and $\mathbf{f} = \left\langle f, \Psi^{2D} \right\rangle$. Thus the original problem is equivalent to the well-posed problem in l^2. While the classical adaptive methods uses refining and derefining a given mesh according to a-posteriori local error estimates, the wavelet approach is different. Instead of turning to a finite dimensional approximation, we try to devise a convergent iteration for the l^2-problem. Then all infinite-dimensional quantities have to be replaced by finitely supported ones and the routine for the application of the biinfinite-dimensional matrix $\mathbf{A}$ approximately have to be designed.

The simplest convergent iteration for the l^2-problem is a *Richardson iteration* which has the following form:

$$\mathbf{u}_0 := \mathbf{0}, \quad \mathbf{u}_{n+1} := \mathbf{u}_n + \omega \left(\mathbf{f} - \mathbf{A}\mathbf{u}_n \right), \quad n = 0, 1, \ldots. \tag{79}$$

For the convergence, the relaxation parameter ω has to satisfy

$$\rho := \left\| \mathbf{I} - \omega \mathbf{A} \right\|_2 < 1, \tag{80}$$

where $\left\| \cdot \right\|_2$ is a spectral norm. Then the iteration (79) convergences with an error reduction per step

$$\left\| \mathbf{u}_{n+1} - \mathbf{u} \right\|_2 \leq \rho \left\| \mathbf{u}_n - \mathbf{u} \right\|_2, \tag{81}$$

where $\left\| \cdot \right\|_2$ is the Euclidean norm. Condition (80) is satisfied if $0 < \omega < \frac{2}{\lambda_{max}}$, where λ_{max} is the largest eigenvalue of $\mathbf{A}$. It is known that the optimal relaxation parameter $\hat{\omega}$ and the corresponding error reduction can be computed as

$$\hat{\omega} = \frac{2}{\lambda_{min} + \lambda_{max}}, \quad \rho\left(\hat{\omega}\right) = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} = \frac{\kappa\left(\mathbf{A}\right) - 1}{\kappa\left(\mathbf{A}\right) + 1}. \tag{82}$$

where λ_{min} is the smallest eigenvalue of $\mathbf{A}$. Hence the estimate of the number of iterations needed to resolve the problem with desired accuracy depends on the condition number of the matrix $\mathbf{A}$ that can be estimated by $\mathbf{A}_{j_{max}}$, where j_{max} is the maximal level used in the computations.

In the algorithm the sparse representation of the vector $\mathbf{f}$ is needed. It can be found due to the relation:

$$\left\langle f, \psi_{j,k} \right\rangle \leq C 2^{-js} \left\| f \cdot \chi_{\operatorname{supp} \psi_{j,k}} \right\|_{H^s(\Omega)}, \tag{83}$$

where $f \in H^s\left(\operatorname{supp} \psi_{j,k}\right) \cap L^2\left(\Omega\right)$, $0 \leq s < d$, d is the number of vanishing moments of wavelet $\psi_{j,k}$, $\chi_{\operatorname{supp} \psi_{j,k}}$ is an indicator function and C is a constant independent of j. It ensures that in the regions where f is smooth the corresponding coefficients of $\mathbf{f}$ are very small and can be thresholded. For the proof see [23].

We provide a numerical example. We consider the equation (68) with a solution u given by

$$u\left(x,y\right) = v\left(x\right) v\left(y\right), \quad v\left(x\right) = x^2 \left(1 - e^{10x-10}\right)^2. \tag{84}$$

The relation (83) is not guaranteed for boundary wavelets corresponding to a), b) and c). However, since there is only a small number of boundary wavelets in comparison to a number of all wavelets up to some maximal level j_{max}, the sparse representation of $\mathbf{f}$ can be constructed. The vector $\tilde{\mathbf{f}}^{j_{max}}$ that includes the entries of $\mathbf{f}$ up to the level j_{max} was computed and its entries in absolute value were sorted. They are displayed in Figure 3 for the cases a) and d). The graphs almost coincides. The graphs for choices b) and d) lay between the graphs for a) and d). Since the structure of $\tilde{\mathbf{f}}^{j_{max}}$ is similar for all choices of parameters, the sparse representations that are obtained by thresholding the coefficients smaller than some threshold ϵ are similar for all choices of parameters and the choice d) does not give significantly better results even though boundary wavelets have two vanishing moments.

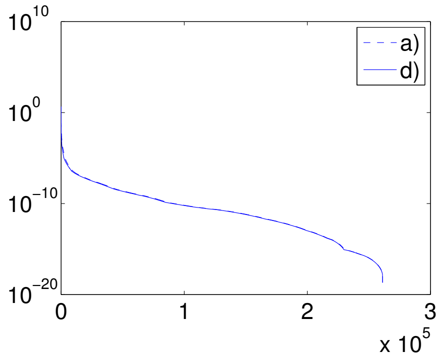

Figure 3: The sorted absolute values of entries of the vector $\tilde{\mathbf{f}}^9$ for the choices of parameters $a)$ and $d)$.

The solution exhibits a sharp gradient near the point $[1, 1]$. We solve the problem by the method designed in [9] with the approximate multiplication of the stiffness matrix with a vector proposed in [5]. We use wavelets up to the scale $|\lambda| \leq 10$. The convergence history is shown in Figure 4. In our experiments, the convergence rate,

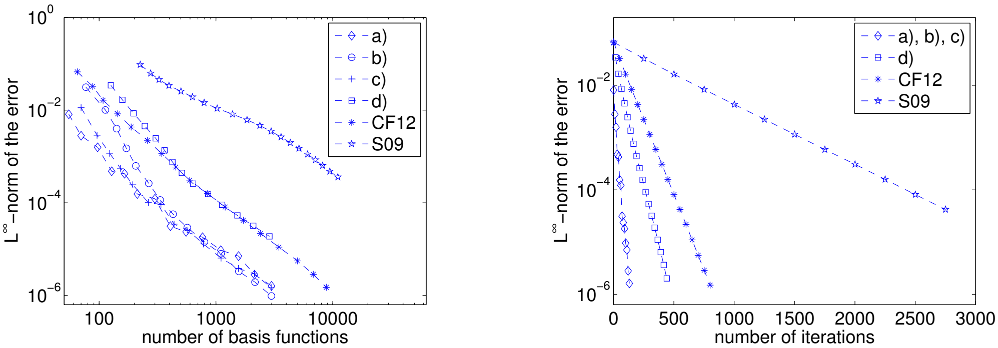

Figure 4: The convergence history for adaptive wavelet scheme with various wavelet bases.

i.e. the slope of the curve, is similar for all bases. Since the initial threshold depends on Riesz bounds of the wavelet basis Ψ, the initial approximations are different and the curves are not similar. Due to low condition number of the stiffness matrix, bases a), b), and c) are significantly better in the number of iterations needed to resolve the problem with desired accuracy. The number of basis functions in cases a), b), and c) was about 1200 for an error in L^∞-norm about 10^{-6}. The number of all basis functions for full grid, i.e. basis functions of the level ten or less, is about 10^6, therefore by using an adaptive method the significant compression was achieved. It can seem that the number of iterations is quite large, but one could take into account that in the beginning the iterations were done for much smaller vector and the size of the vector increases successively.

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WAVELET BASIS OF CUBIC SPLINES ON THE HYPERCUBE SATISFYING HOMOGENEOUS BOUNDARY CONDITIONS

DANA ČERNÁ * and VÁCLAV FINĚK

Department of Mathematics and Didactics of Mathematics, Technical University of Liberec
Studentská 2, 461 17 Liberec, Czech Republic
dana.cerna@tul.cz
vaclav.finek@tul.cz

In the paper, we propose a construction of a new cubic spline-wavelet basis on the hypercube satisfying homogeneous Dirichlet boundary conditions. Wavelets have two vanishing moments. Stiffness matrices arising from discretization of elliptic problems using a constructed wavelet basis have uniformly bounded condition numbers and we show that these condition numbers are small. We present quantitative properties of the constructed basis and we provide a numerical example to show the efficiency of the Galerkin method using the constructed basis.

Keywords: Construction; Wavelet; Cubic spline; Homogeneous Dirichlet boundary conditions; Condition number; Elliptic problem; Galerkin method; Conjugate gradient method.

AMS Subject Classification: 46B15, 65N12, 65T60

1. Introduction

In this paper, we propose a construction of a new cubic spline wavelet basis on the hypercube that is well-conditioned, adapted to homogeneous Dirichlet boundary conditions and the wavelets have two vanishing moments. The wavelet basis of the space $H_0^1(\Omega)$, where $\Omega = (0,1)^d$ and $d \in \mathbb{N}$, is then obtained by a tensor product and a proper normalization.

First of all, we summarize the desired properties of a wavelet basis:

- *Riesz basis property.* We construct a Riesz basis of the space $L^2(\Omega)$ that, when normalized with respect to H^1-seminorm, is also a Riesz basis of the space $H_0^1(\Omega)$.
- *Polymial exactness.* Since the primal basis functions are cubic B-splines, the primal multiresolution analysis has polynomial exactness of order four in $H_0^1(\Omega)$. This means that any $p \in H_0^1(\Omega)$ that is a polynomial of degree less than or equal to four belongs to the span of scaling functions at the given level.
- *Vanishing moments.* The wavelets have two vanishing moments.
- *Locality.* The primal basis functions are local in the sense of Definition 2.1 below.
- *Smoothness.* Primal basis functions belong to $C^2(\Omega)$ and dual basis functions belong to $C(\Omega)$, where $C(\Omega)$ is the space of continuous functions on domain Ω and $C^n(\Omega)$ is the space of functions on domain Ω that have continuous derivatives up to order $n \in \mathbb{N}$.

*Corresponding author

- *Explicit expression.* The primal scaling functions and wavelets have an explicit expression.
- *Homogeneous Dirichlet boundary conditions.* The wavelet basis satisfies homogeneous Dirichlet boundary conditions.
- *Well-conditioned bases.* Our objective is to construct a wavelet basis that is well conditioned with respect to the L_2-norm and is well conditioned with respect to the H^1-seminorm, when normalized appropriately.

Many constructions of spline wavelet or multiwavelet bases on the interval have been proposed in recent years.[4,5,10,16,19–22] In Ref. 2, 3, 12, 18 cubic spline wavelets on the interval were constructed. In these cases dual functions are known and are local. Spline wavelet or multiwavelet bases where duals are not local are also known.[6,13–16] The advantage of our construction in comparison with biorthogonal cubic spline wavelets with local duals[2,3,12,18] is that the support of the wavelets is shorter, condition numbers of the corressponding stiffness matrices are smaller and also a simple construction.

In this paper, scaling functions are the same as scaling functions in Ref. 2 and 18. The construction of wavelets in these papers and also in Ref. 3 and 12 is quite long and technical. It is based on the concept of stable completions.[1] Using this approach the Riesz basis property of the basis is a consequence of polynomial exactness of the primal multiresolution analysis, local supports of primal and dual basis functions and uniform stability of primal and dual multiresolution analysis. We use a different approach. We construct wavelets directly such that they have vanishing moments. Therefore the construction is very simple. Then we prove the Riesz basis property partly using the theory developed in Ref. 9.

It was observed that an original construction in Ref. 12 leads to badly conditioned stifness matrices. Therefore, the construction was optimized in Ref. 2, 3, 18. The wavelet bases from Ref. 2, 18 are adapted to homogeneous boundary conditions of the first order, i.e. they are of the same type as the basis in this paper. In Section 6 we compare the condition numbers. The length of the support of cubic wavelets in Ref. 2, 3, 12, 18 is at least seven, the length of the support of our wavelets is five.

2. Wavelet bases in Sobolev spaces

In this section, we recall the definition of a wavelet basis in a Hilbert space and the concept of Sobolev spaces.

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|\cdot\|_H$. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale*. For $p \in \mathbb{N}$ let

$$\|\mathbf{v}\|_p := \left(\sum_{\lambda \in \mathcal{J}} |v_\lambda|^p \right)^{1/p}, \quad \text{for} \quad \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \tag{2.1}$$

and

$$l^p(\mathcal{J}) := \left\{ \mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \|\mathbf{v}\|_p < \infty \right\}. \tag{2.2}$$

Our aim is to construct a wavelet basis in the sense of the following definition.

Definition 2.1. *A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a (primal) wavelet basis of H, if* *i)* *Ψ is a Riesz basis for H, i.e. the closure of the span of Ψ is H and there exist constants $c, C \in (0,\infty)$ such that*

$$c\,\|\mathbf{b}\|_2 \leq \left\| \sum_{\lambda\in\mathcal{J}} b_\lambda \psi_\lambda \right\|_H \leq C\,\|\mathbf{b}\|_2\,, \quad \textit{for all} \quad \mathbf{b} := \{b_\lambda\}_{\lambda\in\mathcal{J}} \in l^2(\mathcal{J})\,. \tag{2.3}$$

Constants $c_\Psi := sup\{c : c$ satisfies (2.3)$\}$, $C_\Psi := inf\{C : C$ satisfies (2.3)$\}$ are called Riesz bounds and the number $cond\ \Psi = C_\Psi/c_\Psi$ is called the condition number of Ψ.

ii) *The functions are local in the sense that* $\mathrm{diam}\,(\mathrm{supp}\,\psi_\lambda) \leq \tilde{\mathrm{C}}\,2^{-|\lambda|}$ *for all $\lambda \in \mathcal{J}$ and at a given level j the supports of only finitely many wavelets overlap at any point $x \in \Omega$.*

Remark 2.1. A Riesz basis for H is actually a (Schauder) basis for H. The condition that the closure of the span of Ψ is H implies that for any $f \in H$ there exists $\{a_\lambda\}_{\lambda\in\mathcal{J}} \in l^1(\mathcal{J})$ such that

$$f = \sum_{\lambda\in\mathcal{J}} a_\lambda \psi_\lambda. \tag{2.4}$$

If $\{b_\lambda\}_{\lambda\in\mathcal{J}} \in l^1(\mathcal{J})$ is such that

$$f = \sum_{\lambda\in\mathcal{J}} a_\lambda \psi_\lambda = \sum_{\lambda\in\mathcal{J}} b_\lambda \psi_\lambda, \tag{2.5}$$

then due to $l^1(\mathcal{J}) \subset l^2(\mathcal{J})$ and (2.3) we have

$$c \sum_{\lambda\in\mathcal{J}} |a_\lambda - b_\lambda|^2 \leq \left\| \sum_{\lambda\in\mathcal{J}} a_\lambda \psi_\lambda - \sum_{\lambda\in\mathcal{J}} b_\lambda \psi_\lambda \right\|_H^2 = 0. \tag{2.6}$$

Hence, $a_\lambda = b_\lambda$ and the expansion (2.4) is unique.

For the two countable sets of functions $\Gamma, \tilde{\Gamma} \subset H$, the symbol $\left\langle \Gamma, \tilde{\Gamma} \right\rangle_H$ denotes the matrix

$$\left\langle \Gamma, \tilde{\Gamma} \right\rangle_H := \{\langle \gamma, \tilde{\gamma} \rangle_H\}_{\gamma\in\Gamma, \tilde{\gamma}\in\tilde{\Gamma}}\,. \tag{2.7}$$

Remark 2.2. It is known that the constants c_Ψ and C_Ψ from Definition 2.1 satisfy

$$c_\Psi = \sqrt{\lambda_{min}\left(\langle \Psi, \Psi \rangle_H\right)}, \quad C_\Psi = \sqrt{\lambda_{max}\left(\langle \Psi, \Psi \rangle_H\right)}, \tag{2.8}$$

where $\lambda_{min}\left(\langle \Psi, \Psi \rangle_H\right)$ and $\lambda_{max}\left(\langle \Psi, \Psi \rangle_H\right)$ are the smallest and the largest eigenvalues of the matrix $\langle \Psi, \Psi \rangle_H$, respectively.

Let M be a Lebesgue measurable subset of $\mathbb{R}^d$. The space $L_2(M)$ is the space of all Lebesgue measurable functions on M such that the norm

$$\|f\| = \left(\int_M |f(x)|^2\, dx \right)^{1/2} \tag{2.9}$$

is finite. The space $L_2(M)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_M f(x)\, \overline{g(x)} dx, \quad f, g \in L_2(M)\,. \tag{2.10}$$

The Sobolev space $H^s\left(\mathbb{R}^d\right)$ for $s \geq 0$ is defined as the space of all functions $f \in L_2\left(\mathbb{R}^d\right)$ such that the seminorm

$$|f|_{H^s(\mathbb{R}^d)} = \left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \hat{f}(\xi) \right|^2 |\xi|^{2s} \, d\xi \right)^{1/2} \tag{2.11}$$

is finite. The symbol $\hat{f}$ denotes the Fourier transform of the function f defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^s} f(x) \, e^{-i\xi \cdot x} dx. \tag{2.12}$$

The space $H^s\left(\mathbb{R}^d\right)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^s(\mathbb{R}^d)} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) \, \overline{\hat{g}(\xi)} \left(1 + |\xi|^{2s}\right) d\xi, \quad f, g \in H^s\left(\mathbb{R}^d\right), \tag{2.13}$$

and the norm

$$\|f\|_{H^s(\mathbb{R}^d)} = \sqrt{\langle f, f \rangle_{H^s(\mathbb{R}^d)}}. \tag{2.14}$$

For an open set $M \subset \mathbb{R}^d$, $H^s(M)$ is the set of restrictions of functions from $H^s\left(\mathbb{R}^d\right)$ to M equipped with the norm

$$\|f\|_{H^s(M)} = \inf \left\{ \|g\|_{H^s(\mathbb{R}^d)} : g \in H^s(M) \text{ and } g|_M = f \right\}. \tag{2.15}$$

The space $H^{-s}(M)$ is defined as the dual space to $H^s(M)$. Let $C_0^\infty(M)$ be the space of all continuous functions with the support in M such that they have continuous derivatives of order r for any $r \in \mathbb{R}$. The space $H_0^s(M)$ is defined as the closure of $C_0^\infty(M)$ in $H^s\left(\mathbb{R}^d\right)$. It is known that

$$\|f\|_{H^1(M)} = |f|_{H^1(M)} + \|f\|, \tag{2.16}$$

where

$$|f|_{H^1(M)} = \sqrt{\langle \nabla f, \nabla f \rangle} \tag{2.17}$$

is the seminorm in $H^1(M)$ and ∇f denotes the gradient of f.

3. Construction of scaling functions

A primal scaling basis is the same as a scaling basis in Ref. 2, 18. It is generated from functions ϕ, ϕ_{b1} and ϕ_{b2} as follows. Let ϕ be a cubic B-spline defined on knots $\{0, 1, 2, 3, 4\}$. It can be written explicitly as

$$\phi(x) = \begin{cases} \frac{x^3}{6}, & x \in [0,1], \\ -\frac{x^3}{2} + 2x^2 - 2x + \frac{2}{3}, & x \in [1,2], \\ \frac{x^3}{2} - 4x^2 + 10x - \frac{22}{3}, & x \in [2,3], \\ \frac{(4-x)^3}{6}, & x \in [3,4], \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Then ϕ satisfies *the scaling equation*[2,18]

$$\phi(x) = \frac{\phi(2x)}{8} + \frac{\phi(2x-1)}{2} + \frac{3\phi(2x-2)}{4} + \frac{\phi(2x-3)}{2} + \frac{\phi(2x-4)}{8}. \tag{3.2}$$

Let ϕ_{b1} be a cubic B-spline defined on knots $\{0,0,0,1,2\}$ and ϕ_{b2} be a cubic B-spline defined on knots $\{0,0,1,2,3\}$, i.e.,

$$\phi_{b1}(x) = \begin{cases} \frac{7x^3}{4} - \frac{9x^2}{2} + 3x, & x \in [0,1], \\ \frac{(2-x)^3}{4}, & x \in [1,2], \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

and

$$\phi_{b2}(x) = \begin{cases} -\frac{11x^3}{12} + \frac{3x^2}{2}, & x \in [0,1], \\ \frac{7x^3}{12} - 3x^2 + \frac{9x}{2} - \frac{3}{2}, & x \in [1,2], \\ \frac{(3-x)^3}{6}, & x \in [2,3], \\ 0, & \text{otherwise}. \end{cases} \tag{3.4}$$

Then ϕ_{b1} and ϕ_{b2} satisfy the scaling equations[2, 18]

$$\begin{aligned} \phi_{b1}(x) &= \frac{\phi_{b1}(2x)}{2} + \frac{3\phi_{b2}(2x)}{4} + \frac{3\phi(2x)}{16}, \\ \phi_{b2}(x) &= \frac{\phi_{b2}(2x)}{4} + \frac{11\phi(2x)}{16} + \frac{\phi(2x-1)}{2} + \frac{\phi(2x-2)}{8}. \end{aligned} \tag{3.5}$$

For $j \in \mathbb{N}$, $j \geq 3$ and $x \in [0,1]$ we set

$$\begin{aligned} &\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k),\ k = 3, \ldots 2^j - 1, \\ &\phi_{j,1}(x) = 2^{j/2}\phi_{b1}(2^j x), \quad \phi_{j,2^j+1}(x) = 2^{j/2}\phi_{b1}(2^j(1-x)), \\ &\phi_{j,2}(x) = 2^{j/2}\phi_{b2}(2^j x), \quad \phi_{j,2^j}(x) = 2^{j/2}\phi_{b2}(2^j(1-x)). \end{aligned} \tag{3.6}$$

Furthermore, we define

$$\Phi_j = \left\{\phi_{j,k}/\left\|\phi_{j,k}\right\|, k = 1, \ldots, 2^j + 1\right\} \quad \text{and} \quad V_j = \operatorname{span}\Phi_j. \tag{3.7}$$

It was proved in Ref. 2 that the sets Φ_j are uniform Riesz bases of the space V_j. This means that the sets Φ_j are Riesz bases of the space V_j with Riesz bounds independent on j. The graphs of the functions $\phi_{j,k}$ on the coarsest level $j = 3$ are displayed in Figure 1.

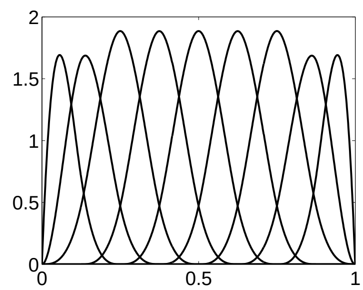

Fig. 1. Functions $\phi_{3,k}$, $k = 1, \ldots, 9$.

4. Construction of wavelets

In some applications such as adaptive wavelet methods,[7,8] vanishing moments of wavelets are needed. In our case, we construct wavelets with two vanishing moments, i.e.

$$\int_{\mathrm{supp}\,\psi} x^k \psi(x) dx = 0, \quad k = 0, 1. \tag{4.1}$$

For $k \geq 3$ we set $\tilde{V}_j$ as the space of continuous piecewise linear function:

$$\tilde{V}_j = \left\{ v \in C\,(0,1) : v|_{\left(\frac{k}{2^j}, \frac{k+1}{2^j}\right)} \in P_1 \left(\frac{k}{2^j}, \frac{k+1}{2^j} \right) \text{ for } k = 0, \ldots, 2^j - 1 \right\}, \tag{4.2}$$

where $P_1\,(a,b)$ is the space of all algebraic polynomials on (a,b) of degree less than or equal to 1. Clearly, with this choice the dimension of $\tilde{V}_j$ is $2^j + 1$ that is the same as the dimension of V_j. We construct wavelets $\psi_{j,k}$, $k = 1, \ldots, 2^j$, such that $\psi_{j,k} \in V_{j+1} \cap \tilde{V}_j^{\perp}$, where $\tilde{V}_j^{\perp}$ is the orthogonal complement of $\tilde{V}_j$ with respect to the L_2-norm. Then

$$\left\langle \psi_{j,k}, \tilde{\phi} \right\rangle = 0 \tag{4.3}$$

for all functions $\tilde{\phi} \in \tilde{V}_j$ and (4.1) is satisfied.

Since we want $\psi_{j,k} \in V_{j+1}$, we define a generator wavelet ψ as

$$\psi(x) = \sum_{k=0}^{M} g_k \phi(2x - k). \tag{4.4}$$

Then $\mathrm{supp}\,\psi = \left[0, \frac{M}{2} + 2\right]$. Let

$$\bar{V} = \left\{ v \in C\left(0, \frac{M}{2} + 2\right) : v|_{(k,k+1)} \in P_1\,(k, k+1)\,,\ k = 0, \ldots, \frac{M}{2} + 1 \right\}. \tag{4.5}$$

The dimension of $\bar{V}$ is $\frac{M}{2} + 3$ and ψ can be found as the solution of the system

$$\langle \psi, f_i \rangle = 0, \quad i = 1, \ldots \frac{M}{2} + 3, \tag{4.6}$$

where $\{f_i\}_{i=1}^{\frac{M}{2}+3}$ is a basis of $\bar{V}$. For $M \leq 5$, the system (4.6) has only trivial solution. Therefore, we choose $M = 6$ and compute

$$[g_0, \ldots, g_6] = \left[\frac{-1}{184}, \frac{7}{46}, \frac{-119}{184}, 1, \frac{-119}{184}, \frac{7}{46}, \frac{-1}{184} \right]. \tag{4.7}$$

Then for

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k + 2)|_{[0,1]},\ k = 3, ..., 2^j - 2,\ j \in \mathbb{N},\ j \geq 3, \tag{4.8}$$

the condition (4.3) is satisfied and the functions ψ and $\psi_{j,k}$ have two vanishing moments. The support of the wavelet ψ is $[0, 5]$. The graph of ψ is shown in Figure 2.

We define boundary wavelets ψ_{b1} and ψ_{b2} by:

$$\psi_{b1}(x) = g_0^{b1} \phi_{b1}(2x) + g_1^{b1} \phi_{b2}(2x) + \sum_{k=2}^{4} g_k^{b1} \phi(2x - k + 2), \tag{4.9}$$

$$\psi_{b2}(x) = g_0^{b2} \phi_{b1}(2x) + g_1^{b2} \phi_{b2}(2x) + \sum_{k=2}^{6} g_k^{b2} \phi(2x - k + 2),$$

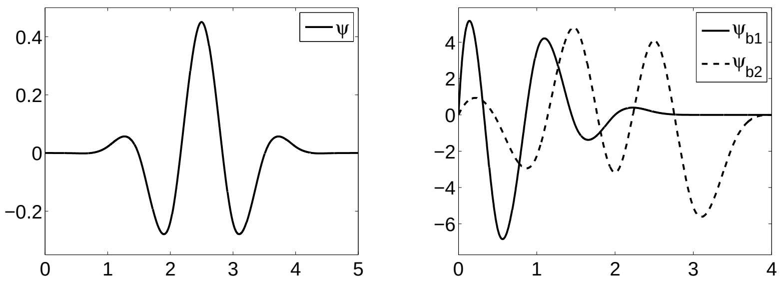

Fig. 2. Wavelets ψ, ψ_{b1} and ψ_{b2}.

where

$$[g_0^{b1},\dots,g_4^{b1}]=\left[\frac{939}{70},\frac{-393}{20},\frac{6233}{560},-4,1\right], \tag{4.10}$$
$$[g_0^{b2},\dots,g_6^{b2}]=\left[\frac{2770661}{1828560},\frac{256057}{457140},\frac{-493633}{76992},\frac{20761777}{1828560},\frac{-76369591}{7314240},7,-3\right].$$

Then $\operatorname{supp}\psi_{b1}=[0,3]$, $\operatorname{supp}\psi_{b2}=[0,4]$ and both boundary wavelets have two vanishing moments.

For $j\in\mathbb{N}$, $j\geq 3$ and $x\in[0,1]$ we define

$$\psi_{j,1}(x)=2^{j/2}\psi_{b1}(2^jx),\quad \psi_{j,2^j}(x)=2^{j/2}\psi_{b1}(2^j(1-x)), \tag{4.11}$$
$$\psi_{j,2}(x)=2^{j/2}\psi_{b2}(2^jx),\quad \psi_{j,2^j-1}(x)=2^{j/2}\psi_{b2}(2^j(1-x)).$$

and

$$\Psi_j=\left\{\psi_{j,k}/\left\|\psi_{j,k}\right\|,k=1,\dots,2^j\right\},\quad W_j=\operatorname{span}\Psi_j. \tag{4.12}$$

We denote

$$\Psi^s=\Phi_3\cup\bigcup_{j=3}^{2+s}\Psi_j\quad\text{and}\quad\Psi=\Phi_3\cup\bigcup_{j=3}^{\infty}\Psi_j. \tag{4.13}$$

In the following, we prove that Ψ is a Riesz basis of the space $L_2(0,1)$. The set Ψ^s is a finite dimensional subset of Ψ.

Theorem 4.1. *The sets Ψ_j, $j\geq 3$, are uniform Riesz bases of W_j.*

Proof. We compute the matrix

$$\mathbf{F}_j:=\langle\Psi_j,\ \Psi_j\rangle \tag{4.14}$$

using (4.4) and (4.9). For example, for $j=3$ we obtain

$$\mathbf{F}_3=\begin{pmatrix}
1.000 & 0.128 & 0.103 & 0.003 & 0 & 0 & 0 & 0\\
0.128 & 1.000 & 0.432 & -0.145 & -0.014 & 0 & 0 & 0\\
0.103 & 0.432 & 1.000 & -0.029 & -0.077 & 0.001 & 0 & 0\\
0.003 & -0.145 & -0.029 & 1.000 & -0.029 & -0.077 & -0.014 & 0\\
0 & -0.014 & -0.077 & -0.029 & 1.000 & -0.029 & -0.145 & 0.003\\
0 & 0 & 0.001 & -0.077 & -0.029 & 1.000 & 0.432 & 0.103\\
0 & 0 & 0 & -0.014 & -0.145 & 0.432 & 1.000 & 0.128\\
0 & 0 & 0 & 0 & 0.003 & 0.103 & 0.128 & 1.000
\end{pmatrix}, \tag{4.15}$$

where the numbers are rounded to three decimal digits. The matrix $\mathbf{F}_j$ for $j \geq 3$ has a similar structure. The first two rows and columns and the last two rows and columns corresponds to boundary wavelets and for $k, l = 3, \ldots 2^j - 2$:

$$(\mathbf{F}_j)_{k,l} = \begin{cases} 1, & k = l, \\ -0.029, & |k-l| = 1, \\ -0.077, & |k-l| = 2, \\ -0.001, & |k-l| = 3, \\ 0, & \text{otherwise.} \end{cases} \tag{4.16}$$

It is easy to see that $\mathbf{F}_j$ is banded and diagonally dominant. Estimates for the smallest eigenvalue λ^j_{min} and the largest eigenvalue λ^j_{max} of the matrix $\mathbf{F}_j$ can be computed using the Gershgorin circle theorem:

$$\lambda^j_{min} \geq \min \left(\left| F^j_{ii} \right| - \sum_{k=1}^{n} \left| F^j_{ik} \right| \right) > 0.2, \tag{4.17}$$

$$\lambda^j_{max} \leq \max \left(\left| F^j_{ii} \right| + \sum_{k=1}^{n} \left| F^j_{ik} \right| \right) < 1.8, \tag{4.18}$$

$$\tag{4.19}$$

where F^j_{ik} are the entries of the matrix $\mathbf{F}_j$. With the help of Remark 2.2 the assertion is proven. □

The proof that Ψ is a Riesz basis is based on the following theorem.[9,13]

Theorem 4.2. *Let $J \in \mathbb{N}$ and let V_j and $\tilde{V}_j$, $j \geq J$, be subspaces of $L_2(0,1)$ such that*

$$V_j \subset V_{j+1}, \quad \tilde{V}_j \subset \tilde{V}_{j+1}, \quad \dim V_j = \dim \tilde{V}_j < \infty, \quad j \geq J. \tag{4.20}$$

Let Φ_j be uniform Riesz bases of V_j, $\tilde{\Phi}_j$ be uniform Riesz bases of $\tilde{V}_j$, Ψ_j be uniform Riesz bases of $\tilde{V}_j^{\perp} \cap V_{j+1}$, where $\tilde{V}_j^{\perp}$ is the orthogonal complement of $\tilde{V}_j$ with respect to the L^2-inner product, and let

$$\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\} = \Phi_J \cup \bigcup_{j=J}^{\infty} \Psi_j. \tag{4.21}$$

Furthermore, let the matrix

$$\mathbf{G}_j := \left\langle \Phi_j, \tilde{\Phi}_j \right\rangle \tag{4.22}$$

be invertible and the spectral norm of $\mathbf{G}_j^{-1}$ is bounded independently on j. In addition, for some positive constants C, γ and d, $\gamma < d$, let

$$\inf_{v_j \in V_j} \|v - v_j\| \leq C 2^{-jd} \|v\|_{H^d(0,1)}, \ v \in H^d_0(0,1), \tag{4.23}$$

and for $0 \leq s < \gamma$ let

$$\|v_j\|_{H^d(0,1)} \leq C 2^{js} \|v_j\|, \ v_j \in V_j, \tag{4.24}$$

and let similar estimates (4.23) and (4.24) hold for $\tilde{\gamma}$ and $\tilde{d}$ on the dual side. Then there exist constants k and K, $0 < k \leq K < \infty$, such that

$$k \|\mathbf{b}\|_2 \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda 2^{-|\lambda| s} \psi_\lambda \right\|_{H^s(0,1)} \leq K \|\mathbf{b}\|_2, \quad \mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}) \tag{4.25}$$

holds for $s \in (-\tilde{\gamma}, \gamma)$.

Now we are ready to prove the Riesz basis property (2.3) for Ψ.

Theorem 4.3. *The set* Ψ *is a wavelet basis of the space* $L_2(0,1)$.

Proof. For $j \geq 3$ we consider the set

$$\bar{\Phi}_j = \left\{ \phi_{j,k}, k = 1, \ldots, 2^j + 1 \right\} \tag{4.26}$$

that is a Riesz basis of the space V_j. Recall that $\tilde{V}_j$ is defined by (4.2). Let

$$\tilde{\phi}(x) = \begin{cases} x+1, \ x \in [-1,0], \\ 1-x, \ x \in [0,1], \\ \quad 0, \quad \text{otherwise}, \end{cases} \tag{4.27}$$

and for $x \in [0,1]$ we define

$$\tilde{\phi}_{j,k}(x) = 2^{j/2} \tilde{\phi}\left(2^j x - k + 1\right), k = 2, \ldots, 2^j, \tag{4.28}$$

$$\tilde{\phi}_{j,k}(x) = 2^{(j+1)/2} \tilde{\phi}\left(2^j x - k + 1\right), k = 1, 2^j + 1. \tag{4.29}$$

It was proved in Ref. 2 that

$$\tilde{\Phi}_j = \left\{ \tilde{\phi}_{j,k}, k = 1, \ldots, 2^j + 1 \right\} \tag{4.30}$$

are uniform Riesz bases of $\tilde{V}_j$.

The matrix $\mathbf{G}_j = \left\langle \bar{\Phi}_j, \tilde{\Phi}_j \right\rangle$ is

$$\mathbf{G}_j = \begin{pmatrix}
\frac{17}{40} & \frac{11}{40} & \frac{11}{80} & 0 & 0 & 0 & 0 & & 0 \\
\frac{19}{120} & \frac{9}{20} & \frac{17}{80} & \frac{11}{120} & 0 & 0 & 0 & & 0 \\
\frac{1}{60} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & 0 & 0 & & 0 \\
0 & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & & & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & 0 \\
0 & & & & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{60} \\
0 & & & & & \frac{11}{120} & \frac{17}{80} & \frac{9}{20} & \frac{19}{120} \\
0 & & & & & & \frac{11}{80} & \frac{11}{40} & \frac{17}{40}
\end{pmatrix}. \tag{4.31}$$

It is easy to verify that the matrix $\mathbf{G}_j$ is banded and strictly diagonally dominant. Therefore, it is invertible and the spectral norm of $\mathbf{G}_j^{-1}$ is bounded independently on j. It is known[9] that when γ is the Sobolev exponent of smoothness of the basis functions and d is the polynomial exactness of V_j than (4.23) and (4.24) are satisfied. In our case, the Sobolev exponent of smoothness is $\gamma = 3.5$ and the polynomial exactness of V_j is $d = 4$. On the dual side, $\tilde{\gamma} = 1.5$ and $\tilde{d} = 2$. Therefore, due to Theorem 4.2, relation (4.25) is satisfied for $s \in (-1.5, 3.5)$. Since we proved that (4.25) holds for $s = 0$, the set Ψ is indeed a wavelet basis of the space $L_2(0,1)$. □

It remains to prove that when the wavelet basis Ψ is normalized in the H^1-seminorm, then it is a wavelet basis of the space $H_0^1(0,1)$. We denote

$$\mathcal{I}_3 := \{0, 1, \ldots, 8\} \quad \text{and} \quad \mathcal{J}_j := \left\{1, \ldots, 2^j\right\}. \tag{4.32}$$

Theorem 4.4. *The set*

$$\left\{\phi_{3,k}/\left|\phi_{3,k}\right|_{H_0^1(0,1)}, k\in\mathcal{I}_3\right\}\cup\left\{\psi_{j,k}/\left|\psi_{j,k}\right|_{H_0^1(0,1)}, j\geq 3, k\in\mathcal{J}_j\right\} \tag{4.33}$$

is a wavelet basis of the space $H_0^1(0,1)$.

Proof. We follow the Proof of Theorem 2 in Ref. 4. From the proof of Theorem 4.3, we know that relation (4.25) holds for $s=1$. Therefore the set

$$\left\{2^{-3}\phi_{3,k}, k\in\mathcal{I}_3\right\}\cup\left\{2^{-j}\psi_{j,k}, j\geq 3, k\in\mathcal{J}_j\right\} \tag{4.34}$$

is a wavelet basis of the space $H_0^1(0,1)$. From (3.6), (4.8) and (4.11) there exist nonzero constants C_1 and C_2 such that

$$C_1 2^j \leq \left|\psi_{j,k}\right|_{H_0^1(\Omega)} \leq C_2 2^j, \quad \text{for} \quad j\geq 3, \quad k\in\mathcal{J}_j, \tag{4.35}$$

and

$$C_1 2^3 \leq \left|\phi_{3,k}\right|_{H_0^1(\Omega)} \leq C_2 2^3, \quad \text{for}\, k\in\mathcal{I}_3. \tag{4.36}$$

Let $\hat{\mathbf{b}}=\{\hat{a}_{3,k},\ k\in\mathcal{I}_3\}\cup\left\{\hat{b}_{j,k}, j\geq 3,\ k\in\mathcal{J}_j\right\}$ be such that

$$\left\|\hat{\mathbf{b}}\right\|_2^2=\sum_{k\in\mathcal{I}_3}\hat{a}_{3,k}^2+\sum_{k\in\mathcal{J}_j, j\geq 3}\hat{b}_{j,k}^2<\infty. \tag{4.37}$$

We define

$$a_{3,k}=\frac{2^3\hat{a}_{3,k}}{\left|\phi_{3,k}\right|_{H_0^1(0,1)}},\ k\in\mathcal{I}_3,\quad b_{j,k}=\frac{2^j\hat{b}_{j,k}}{\left|\psi_{j,k}\right|_{H_0^1(0,1)}},\ j\geq 3,\ k\in\mathcal{J}_j, \tag{4.38}$$

and $\mathbf{b}=\{a_{3,k},\ k\in\mathcal{I}_3\}\cup\{b_{j,k}, j\geq 3,\ k\in\mathcal{J}_j\}$. Then

$$\|\mathbf{b}\|_2\leq\frac{\left\|\hat{\mathbf{b}}\right\|_2}{C_1}<\infty. \tag{4.39}$$

Since the set in (4.34) is a Riesz basis of $H_0^1(0,1)$ there exist constants C_3 and C_4 such that

$$C_3\|\mathbf{b}\|_2\leq\left\|\sum_{k\in\mathcal{I}_3}a_{3,k}2^{-3}\phi_{3,k}+\sum_{k\in\mathcal{J}_j, j\geq 3}b_{j,k}2^{-j}\psi_{j,k}\right\|_{H_0^1(0,1)}\leq C_4\|\mathbf{b}\|_2. \tag{4.40}$$

Therefore

$$\begin{aligned}\frac{C_4}{C_1}\left\|\hat{\mathbf{b}}\right\|_2\geq C_4\|\mathbf{b}\|_2&\geq\left\|\sum_{k\in\mathcal{I}_3}a_{3,k}2^{-3}\phi_{3,k}+\sum_{k\in\mathcal{J}_j, j\geq 3}b_{j,k}2^{-j}\psi_{j,k}\right\|_{H_0^1(0,1)}\\&=\left\|\sum_{k\in\mathcal{I}_3}\frac{\hat{a}_{3,k}}{\left|\phi_{3,k}\right|_{H_0^1(0,1)}}\phi_{3,k}+\sum_{k\in\mathcal{J}_j, j\geq 3}\frac{\hat{b}_{3,k}}{\left|\psi_{j,k}\right|_{H_0^1(0,1)}}\psi_{j,k}\right\|_{H_0^1(0,1)}\end{aligned} \tag{4.41}$$

and similarly

$$\frac{C_3}{C_2}\left\|\hat{\mathbf{b}}\right\|_2\leq\left\|\sum_{k\in\mathcal{I}_3}\frac{\hat{a}_{3,k}}{\left|\phi_{3,k}\right|_{H_0^1(0,1)}}\phi_{3,k}+\sum_{k\in\mathcal{J}_j, j\geq 3}\frac{\hat{b}_{3,k}}{\left|\psi_{j,k}\right|_{H_0^1(0,1)}}\psi_{j,k}\right\|_{H_0^1(0,1)}. \tag{4.42}$$

□

It is known[2,17] that an orthogonalization of the scaling functions on the coarsest level can lead to improved quantitative properties of the resulting wavelet basis. Therefore, we define the set

$$\Phi_3^{ort} = \left\{\phi_{3,k}^{ort},\ k \in \mathcal{I}_3\right\} \tag{4.43}$$

by

$$\Phi_3^{ort} := \mathbf{K}^{-1}\Phi_3, \quad \mathbf{K} = \langle \Phi_3, \Phi_3 \rangle \,. \tag{4.44}$$

Then the set of scaling functions Φ_3^{ort} is orthonormal and

$$\Psi^{ort} := \Phi_3^{ort} \cup \bigcup_{j=3}^{\infty} \Psi_j \tag{4.45}$$

is a wavelet basis of the space $L^2(0,1)$ and its appropriate rescaling is a wavelet basis of the space $H_0^1(0,1)$.

5. Multivariate wavelets

We present two well-known constructions of multivariate wavelet bases on the unit hypercube $\Omega = (0,1)^d$.[23] They are both based on tensorizing univariate wavelet bases and preserve Riesz basis property, locality of wavelets, vanishing moments and polynomial exactness.

5.1. *Anisotropic construction*

For notational simplicity, we denote

$$\psi_{2,k} := \phi_{3,k}^{ort}, \quad k \in \mathcal{J}_2 := \mathcal{I}_3 \tag{5.1}$$

and

$$\mathcal{J} := \{(j,k),\ j \geq 2,\ k \in \mathcal{J}_j\}\,. \tag{5.2}$$

Then we can write

$$\Psi^{ort} = \{\psi_{j,k},\ j \geq 2,\ k \in \mathcal{J}_j\} = \{\psi_\lambda,\ \lambda \in \mathcal{J}\}\,. \tag{5.3}$$

Recall that for $\lambda = (j,k)$ we denote $|\lambda| = j$. We use $u \otimes v$ to denote the tensor product of functions u and v, i.e. $(u \otimes v)(x_1, x_2) = u(x_1)\, v(x_2)$. We define multivariate basis functions as:

$$\psi_{\boldsymbol{\lambda}} = \otimes_{i=1}^d \psi_{\lambda_i}, \quad \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbf{J}, \quad \mathbf{J} = \mathcal{J}^d = \mathcal{J} \otimes \ldots \otimes \mathcal{J}. \tag{5.4}$$

Since Ψ^{ort} is a Riesz basis of $L_2(0,1)$ and Ψ^{ort} normalized with respect to H^1-seminorm is a Riesz basis of $H_0^1(0,1)$, the set

$$\Psi^{ani} := \{\psi_{\boldsymbol{\lambda}}, \boldsymbol{\lambda} \in \mathbf{J}\} \tag{5.5}$$

is a Riesz basis of $L_2(\Omega)$ and its normalization

$$\left\{ \frac{\psi_{\boldsymbol{\lambda}}}{|\psi_{\boldsymbol{\lambda}}|_{H^1\left((0,1)^d\right)}}, \boldsymbol{\lambda} \in \mathbf{J} \right\} \tag{5.6}$$

is a Riesz basis of $H_0^1(\Omega)$. The set

$$\Psi_s^{ani} := \{\psi_{\boldsymbol{\lambda}}, \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)\,,\ |\lambda_i| < 2 + s\} \tag{5.7}$$

is a finite-dimensional approximation of Ψ^{ani}.

5.2. *Isotropic construction*

We define for $j \geq 3$ and $\mathbf{k} = (k_1, \ldots k_d)$ multivariate scaling functions:

$$\phi_{j,\mathbf{k}} := \otimes_{i=1}^{d} \phi_{j,k_i}, \tag{5.8}$$

and

$$\Phi_j^{iso} := \{\phi_{j,\mathbf{k}},\ \mathbf{k} = (k_1, \ldots k_d),\ k_i \in \mathcal{I}_j,\ i = 1, \ldots, d\}. \tag{5.9}$$

For $e \in \{0, 1\}$ we define

$$\psi_{j,k,e} = \begin{cases} \phi_{j,k},\ e = 0, \\ \psi_{j,k},\ e = 1. \end{cases} \tag{5.10}$$

We denote the index set:

$$\mathcal{J}_{j,e} = \begin{cases} \mathcal{I}_j,\ e = 0, \\ \mathcal{J}_j,\ e = 1. \end{cases} \tag{5.11}$$

For $\mathbf{k} = (k_1, \ldots k_d)$ and $\mathbf{e} = (e_1, \ldots, e_d)$ we define multivariate functions

$$\psi_{j,\mathbf{k},\mathbf{e}} = \otimes_{i=1}^{d} \psi_{j,k_i,e_i} \tag{5.12}$$

and the set of wavelets on the level j as

$$\Psi_j^{iso} = \{\psi_{j,\mathbf{k},\mathbf{e}},\ k_i \in \mathcal{J}_{j,e_i},\ \mathbf{e} \in E\}, \quad \text{where } E = \{0,1\}^d \setminus \{\mathbf{0}\}. \tag{5.13}$$

It is known that then the set

$$\Psi_{iso} = \Phi_3^{iso} \cup \bigcup_{j=3}^{\infty} \Psi_j^{iso} \tag{5.14}$$

is a wavelet basis of $L_2(\Omega)$ and its normalization with respect to the $H^1(\Omega)$-seminorm is a Riesz basis of $H_0^1(\Omega)$. The set

$$\Psi_s^{iso} = \Phi_3^{iso} \cup \bigcup_{j=3}^{2+s} \Psi_j^{iso} \tag{5.15}$$

is a finite dimensional subset of Ψ^{iso}.

6. Quantitative properties

In this section, we present the condition numbers of the stiffness matrices for the following elliptic problem:

$$-\epsilon \Delta u + a u = f \quad \text{on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{6.1}$$

where Δ is the Laplace operator, ϵ and a are positive constants. The variational formulation for an anisotropic wavelet basis is

$$\mathbf{A}^{ani} \mathbf{u}^{ani} = \mathbf{f}^{ani}, \tag{6.2}$$

where

$$\begin{aligned} \mathbf{A}^{ani} &:= \epsilon \left\langle \nabla \Psi^{ani}, \nabla \Psi^{ani} \right\rangle + a \left\langle \Psi^{ani}, \Psi^{ani} \right\rangle, \\ u &:= \left(\mathbf{u}^{ani}\right)^T \Psi^{ani}, \quad \mathbf{f}^{ani} = \left\langle f, \Psi^{ani} \right\rangle. \end{aligned} \tag{6.3}$$

An advantage of discretization of elliptic equation (6.1) using a wavelet basis is that the system (6.2) can be simply preconditioned by a diagonal preconditioner.[11] Let $\mathbf{D}$ be a matrix of diagonal elements of the matrix $\mathbf{A}$, i.e. $\mathbf{D}_{\lambda,\mu} = \mathbf{A}_{\lambda,\mu}\delta_{\lambda,\mu}$, where $\delta_{\lambda,\mu}$ denotes Kronecker delta. Setting

$$\tilde{\mathbf{A}}^{ani} := \left(\mathbf{D}^{ani}\right)^{-1/2} \mathbf{A}^{ani} \left(\mathbf{D}^{ani}\right)^{-1/2}, \qquad (6.4)$$
$$\tilde{\mathbf{u}}^{ani} := \left(\mathbf{D}^{ani}\right)^{1/2} \mathbf{u}^{ani}, \quad \tilde{\mathbf{f}}^{ani} := \left(\mathbf{D}^{ani}\right)^{-1/2} \mathbf{f}^{ani}$$

we obtain the preconditioned system $\tilde{\mathbf{A}}^{ani}\tilde{\mathbf{u}}^{ani} = \tilde{\mathbf{f}}^{ani}$. It is known[11] that there exist constants C_1, C_2, and C such that

$$0 < C_1 \leq \left\| \tilde{\mathbf{A}}^{ani} \right\|_2 \leq C_2 \qquad (6.5)$$

and thus $\operatorname{cond} \tilde{\mathbf{A}}^{ani} \leq C < \infty$. Let

$$\mathbf{A}_s^{ani} = \epsilon \left\langle \nabla \Psi_s^{ani}, \nabla \Psi_s^{ani} \right\rangle + a \left\langle \Psi_s^{ani}, \Psi_s^{ani} \right\rangle, \qquad (6.6)$$
$$u_s^{ani} = \left(\mathbf{u}_s^{ani}\right)^T \Psi_s^{ani}, \quad \mathbf{f}_s^{ani} = \left\langle f, \Psi_s^{ani} \right\rangle.$$

and let $\mathbf{D}_s^{ani}$ be a matrix of diagonal elements of the matrix $\mathbf{A}_s^{ani}$, i.e. $\left(\mathbf{D}_s^{ani}\right)_{\lambda,\mu} = \left(\mathbf{A}_s^{ani}\right)_{\lambda,\mu} \delta_{\lambda,\mu}$. We set

$$\tilde{\mathbf{A}}_s^{ani} := \left(\mathbf{D}_s^{ani}\right)^{-1/2} \mathbf{A}_s^{ani} \left(\mathbf{D}_s^{ani}\right)^{-1/2}, \qquad (6.7)$$
$$\tilde{\mathbf{u}}_s^{ani} := \left(\mathbf{D}_s^{ani}\right)^{1/2} \mathbf{u}_s^{ani}, \quad \tilde{\mathbf{f}}_s^{ani} := \left(\mathbf{D}_s^{ani}\right)^{-1/2} \mathbf{f}_s^{ani}$$

and obtain preconditioned finite-dimensional system

$$\tilde{\mathbf{A}}_s^{ani}\tilde{\mathbf{u}}_s^{ani} = \tilde{\mathbf{f}}_s^{ani}. \qquad (6.8)$$

Since $\tilde{\mathbf{A}}_s^{ani}$ is a part of the matrix $\tilde{\mathbf{A}}^{ani}$ that is symmetric and positive definite, we have also

$$\operatorname{cond} \tilde{\mathbf{A}}_s^{ani} \leq C. \qquad (6.9)$$

The preconditioned system for an isotropic wavelet basis

$$\tilde{\mathbf{A}}_s^{iso}\tilde{\mathbf{u}}_s^{iso} = \tilde{\mathbf{f}}_s^{iso}. \qquad (6.10)$$

is derived in a similar way. The stiffness matrix $\tilde{\mathbf{A}}_s^{iso}$ also satisfies

$$0 < C_1 \leq \left\| \tilde{\mathbf{A}}_s^{iso} \right\|_2 \leq C_2, \quad \operatorname{cond} \tilde{\mathbf{A}}_s^{iso} \leq C. \qquad (6.11)$$

The eigenvalues and condition numbers of the stiffness matrices for one-dimensional problem are shown in Table 1. We denote the stiffness matrix for the bases Ψ_s and Ψ_s^{ort} preconditioned as in (6.7) by $\tilde{\mathbf{A}}_s$ and $\tilde{\mathbf{A}}_s^{ort}$, respectively. The consequence of Remark 2.2 is that the condition number with respect to the H^1-seminorm of the multiscale wavelet basis Ψ_s normalized with respect to the H^1-seminorm is equal to the square root of the condition number of the stiffness matrix $\tilde{\mathbf{A}}_s$. The eigenvalues and condition numbers of the stiffness matrices for two-dimensional and three-dimensional problems are shown in Table 2 and Table 3. Table 1, Table 2 and Table 3 correspond to the choice of parameters $\epsilon = 1$ and $a = 0$, i.e. for the Poisson equation.

In Table 4 we compare the condition numbers of the stifness matrices for the Poisson equation in one dimension for various constructions of cubic spline wavelet bases adapted

Table 1. The maximal eigenvalues, the minimal eigenvalues and the condition numbers numbers of the matrices $\tilde{\mathbf{A}}_s^{ort}$ and $\tilde{\mathbf{A}}_s$ of the size $N \times N$ corresponding to the one-dimensional problem.

s	N	λ_{max}^{ort}	λ_{min}^{ort}	cond$\tilde{\mathbf{A}}_s^{ort}$	λ_{max}	λ_{min}	cond$\tilde{\mathbf{A}}_s$
1	17	1.67e0	2.99e-1	5.57e0	1.67e0	2.15e-1	7.74e0
2	33	1.68e0	2.99e-1	5.60e0	1.68e0	2.15e-1	7.79e0
3	65	1.68e0	2.99e-1	5.61e0	1.68e0	2.15e-1	7.81e0
4	129	1.68e0	2.99e-1	5.62e0	1.68e0	2.15e-1	7.81e0
5	257	1.68e0	2.99e-1	5.62e0	1.68e0	2.15e-1	7.82e0
6	513	1.68e0	2.99e-1	5.62e0	1.68e0	2.15e-1	7.82e0
7	1 025	1.68e0	2.99e-1	5.62e0	1.68e0	2.15e-1	7.82e0
8	2 049	1.68e0	2.99e-1	5.62e0	1.68e0	2.15e-1	7.82e0

Table 2. The maximal eigenvalues, the minimal eigenvalues and the condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s^{ani}$ and $\tilde{\mathbf{A}}_s^{iso}$ of the size $N \times N$ corresponding to the two-dimensional problem.

s	N	λ_{max}^{ani}	λ_{min}^{ani}	cond$\tilde{\mathbf{A}}_s^{ani}$	λ_{max}^{iso}	λ_{min}^{iso}	cond$\tilde{\mathbf{A}}_s^{iso}$
1	289	2.46e0	1.51e-1	16.2e0	3.21e0	6.22e-2	51.6e0
2	1 089	2.67e0	1.39e-1	19.2e0	3.27e0	5.60e-2	58.4e0
3	4 225	2.80e0	1.18e-1	23.8e0	3.29e0	5.60e-2	58.8e0
4	16 641	2.88e0	9.75e-2	29.6e0	3.31e0	5.60e-2	59.0e0
5	66 049	2.92e0	8.25e-2	35.4e0	3.31e0	5.60e-2	59.2e0
6	263 169	2.94e0	7.15e-2	41.1e0	3.32e0	5.60e-2	59.2e0
7	1 058 841	2.95e0	6.38e-2	46.3e0	3.32e0	5.60e-2	59.3e0
8	4 231 249	2.96e0	5.82e-2	50.9e0	3.32e0	5.60e-2	59.3e0

Table 3. The maximal eigenvalues, the minimal eigenvalues and the condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s^{ani}$ and $\tilde{\mathbf{A}}_s^{iso}$ of the size $N \times N$ corresponding to the three-dimensional problem.

s	N	λ_{max}^{ani}	λ_{min}^{ani}	cond$\tilde{\mathbf{A}}_s^{ani}$	λ_{max}^{iso}	λ_{min}^{iso}	cond$\tilde{\mathbf{A}}_s^{iso}$
1	4 913	3.94e0	6.78e-2	58.2e0	6.34e0	7.65e-3	829.3e0
2	35 937	4.47e0	5.07e-2	88.0e0	6.47e0	7.42e-3	871.4e0
3	274 625	4.77e0	3.80e-2	125.4e0	6.52e0	7.41e-3	879.5e0
4	2 146 689	5.01e0	2.77e-2	181.2e0	6.56e0	7.41e-3	883.0e0
5	16 974 593	5.12e0	2.04e-2	250.7e0	6.56e0	7.41e-3	885.0e0

to homogeneous boundary conditions of the first order. The construction from this paper is denoted *new*. $P(m,n)$ denotes construction from Ref. 18 and $C\,(m,n)$ and $C\,(m,n)^{ort}$ denote the constructions from Ref. 2 without and with orthogonalization of the scaling functions on the coarsest level, respectively. Parameter (m,n) corresponds to standard notation of biorthogonal wavelets, where m is a polynomial exactness of the primal multiresolution analysis and n is a polynomial exactness of the dual multiresolution analysis. Parameter s denotes the number of levels of wavelets.

In Table 5 and Table 6 a dependence of the condition number on the parameter ϵ is shown. It is computed for the two-dimensional problem and $a = 1$. It can be seen that

Table 4. The condition numbers of the stifness matrices for various constructions.

s	*new*	$P(4,4)$	$P(4,6)$	$CF(4,4)$	$CF(4,6)$	$CF(4,4)^{ort}$	$CF(4,6)^{ort}$
1	5.6	49.1	52.0	47.0	15.4	49.0	15.3
4	5.6	199.8	134.9	50.0	18.1	51.6	16.2
7	5.6	216.8	138.4	50.3	18.9	50.3	16.3

if ϵ increases the condition number become close to the condtion number of the stiffness matrix for the Poisson problem and if ϵ decreases than the condition number become close to the condition number of Gramian matrix with respect to the L^2-inner product, i.e. the case $\epsilon = 0$ and $a = 1$. The condition numbers are even significantly lower than condition numbers for one-dimensional problem and periodized biorthogonal wavelets, see tables in Ref. 23.

Table 5. Condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s^{iso}$ of the size $N \times N$ for various values of ϵ corresponding to the two-dimensional problem .

s	N	$\epsilon = 10^3$	$\epsilon = 1$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-9}$	$\epsilon = 0$
1	289	51.6	51.6	145.3	393.1	393.1
2	1 089	58.4	58.4	146.7	447.8	447.8
3	4 225	58.8	58.8	146.8	471.3	471.4
4	16 641	59.0	59.0	146.8	484.0	484.0
5	66 049	59.2	59.2	146.8	491.1	491.1
6	263 169	59.2	59.2	146.8	494.8	494.9
7	1 058 841	59.3	59.3	146.8	496.8	496.9
8	4 231 249	59.3	59.3	146.8	497.8	497.9

Table 6. Condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s^{ani}$ of the size $N \times N$ for various values of ϵ corresponding to the two-dimensional problem .

s	N	$\epsilon = 10^3$	$\epsilon = 1$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-9}$	$\epsilon = 0$
1	289	16.2	16.2	15.1	16.2	16.2
2	1 089	19.2	19.2	19.0	30.8	30.8
3	4 225	23.8	23.8	23.5	46.9	46.9
4	16 641	29.6	29.6	29.4	63.9	63.9
5	66 049	35.6	35.5	35.4	81.2	81.3
6	263 169	41.3	41.1	41.1	98.0	98.1
7	1 058 841	46.4	46.3	46.3	113.6	113.9
8	4 231 249	51.0	51.0	51.0	127.2	128.9

7. Numerical example

The constructed wavelet basis can be used for solving various types of problems. Let us mention for example solving partial differential and integral equations by adaptive wavelet method.[7,8] In this section we use the constructed wavelet basis in a wavelet-Galerkin method. We consider the problem (6.1) with $\Omega = (0,1)^2$, $\epsilon = 1$ and $a = 0$. The right-hand side f is such that the solution u is given by:

$$u(x,y) = v(x)\, v(y), \quad v(x) = x\left(1 - e^{5x-5}\right). \tag{7.1}$$

We discretize the equation using a Galerkin method and the isotropic wavelet basis constructed in this paper and we obtain the discrete problem (6.10). We solve it by the conjugate gradient method using a simple multilevel approach:

1. Compute $\tilde{\mathbf{A}}_s^{iso}$ and $\tilde{\mathbf{f}}_s^{iso}$, choose $\mathbf{v}_0$ of the length 9^2.

2. For $j = 0, \ldots, s$ find the solution $\tilde{\mathbf{u}}_j$ of the system $\tilde{\mathbf{A}}_j^{iso}\tilde{\mathbf{u}}_j = \tilde{\mathbf{f}}_j^{iso}$ by the conjugate gradient method with initial vector $\mathbf{v}_j$ defined for $j \geq 1$ by

$$(\mathbf{v}_j) = \begin{cases} \tilde{\mathbf{u}}_{j-1}, \; i = 1, \ldots, k_j, \\ \;\; 0, \quad i = k_j, \ldots, k_{j+1}, \end{cases} \tag{7.2}$$

where $k_j = \left(2^{j+2} + 1\right)^2$.

The method for anisotropic wavelet basis is similar. A criterion $\|\mathbf{r}_j\| < \epsilon_j$, where $\mathbf{r}_j := \tilde{\mathbf{A}}_j^{iso}\tilde{\mathbf{u}}_j - \tilde{\mathbf{f}}_j^{iso}$, is used for terminating iterations of the conjugate gradient (CG) method at level j. It is possible to choose smaller ϵ_j on coarser levels,[14] but in our case we choose ϵ_j constant for all levels, because other choices of ϵ_j did not lead to significantly smaller number of iterations in our experiments. Namely, for the given number of levels s we set $\epsilon_j = 10^{-5}2^{-3s}$, $j = 0, \ldots, s$, for the isotropic case and $\epsilon_j = 10^{-4}2^{-3s}$, $j = 0, \ldots, s$, for the anisotropic case.

We explain the choice of ϵ_j. Let u be the exact solution of (6.1) and

$$u_s^* = \left(\mathbf{D}_s^{iso}\right)^{-1/2} \left(\tilde{\mathbf{u}}_s^*\right)^T \Psi_s, \tag{7.3}$$

where $\tilde{\mathbf{u}}_s^*$ is the exact solution of the discrete problem (6.10). It is known[23] that

$$\|u - u_s^*\|_{H^m(\Omega)} \leq C 2^{-(4-m)s}. \tag{7.4}$$

We have

$$\mathbf{r}_s := \tilde{\mathbf{A}}_s^{iso}\tilde{\mathbf{u}}_s - \tilde{\mathbf{f}}_s^{iso} = \tilde{\mathbf{A}}_s^{iso}\tilde{\mathbf{u}}_s - \tilde{\mathbf{A}}_s^{iso}\tilde{\mathbf{u}}_s^*. \tag{7.5}$$

where $\tilde{\mathbf{u}}_s$ is the approximate solution of (6.10) by the conjugate gradient method. The relation (6.11) implies

$$\frac{1}{C_2}\|\mathbf{r}_s\|_2 \leq \|\tilde{\mathbf{u}}_s - \tilde{\mathbf{u}}_s^*\|_2 \leq \frac{1}{C_1}\|\mathbf{r}_s\|_2 \tag{7.6}$$

Due to Theorem 4.3 we have

$$C_3 \|\tilde{\mathbf{u}}_s - \tilde{\mathbf{u}}_s^*\|_2 \leq \|\tilde{u}_s - \tilde{u}_s^*\|_{H^1(\Omega)} \leq C_4 \|\tilde{\mathbf{u}}_s - \tilde{\mathbf{u}}_s^*\|_2 . \tag{7.7}$$

Hence, by (7.4), (7.6) and (7.7)

$$\begin{aligned} \|u_s - u\|_{H^1(\Omega)} &= \|u_s - u_s^* + u_s^* - u\|_{H^1(\Omega)} && (7.8)\\ &\leq \|u_s - u_s^*\|_{H^1(\Omega)} + \|u_s^* - u\|_{H^1(\Omega)} && (7.9)\\ &\leq \frac{C_4}{C_1}\|\mathbf{r}_s\|_2 + C2^{-3s}. && (7.10) \end{aligned}$$

Therefore, if we choose the criterion $\|\mathbf{r}_s\|_2 \le \tilde{C}2^{-3s}$, we achieve for u_s the same convergence rate as for u_s^*.

Since span Ψ_s^{iso} = span Φ_{3+s} there exists a matrix $\mathbf{T}_s^{iso}$ such that $\Psi_s^{iso} = \mathbf{T}_s^{iso}\Phi_{3+s}$. This matrix represents the discrete wavelet transform and the multiplication of the matrix $\mathbf{T}_s^{iso}$ with a vector requires $\mathcal{O}(N)$ work, where $N \times N$ is the size of the matrix $\mathbf{T}_s^{iso}$. For details see e.g. Ref. 17. Thus

$$\mathbf{A}_s^{iso} = \left(\mathbf{D}_s^{iso}\right)^{-1/2} \mathbf{T}_s^{iso} \mathbf{A}_{s+3}^{\Phi} \left(\mathbf{T}_s^{iso}\right)^T \left(\mathbf{D}_s^{iso}\right)^{-1/2}, \tag{7.11}$$

where $\mathbf{A}_{s+3}^{\Phi}$ is the stiffness matrix with respect to the basis Φ_{3+s}. Since $\mathbf{A}_{s+3}^{\Phi}$ is banded and $\mathbf{D}_s^{iso}$ is diagonal, the multiplication of the matrix $\mathbf{A}_s^{iso}$ with a vector requires $\mathcal{O}(N)$ floating-point operations. We conclude that one CG iteration requires $\mathcal{O}(N)$ floating-point operations. We denote the number of iterations on the level j as M_j. The number of operations needed to compute one CG iteration on the level j requires about one quarter of operations needed to compute one CG iteration on the level $j+1$. Thus M_j iterations at level j is equivalent to $M_j\, 4^{j-s}$ iterations at level s. Therefore, we define the total number of equivalent iterations by

$$M = \sum_{j=0}^{s} \frac{M_j}{4^{s-j}}. \tag{7.12}$$

The results are listed in Table 7 and Table 8. The residuum is denoted $\mathbf{r}_s$, u is the exact solution of the given problem and u_s is an approximate solution obtained by multilevel Galerkin method with s levels of wavelets. It can be seen that the number of conjugate gradient iterations is quite small and that

$$\frac{\|u_s - u\|_\infty}{\|u_{s+1} - u\|_\infty} \approx \frac{\|u_s - u\|}{\|u_{s+1} - u\|} \approx \frac{1}{16}, \tag{7.13}$$

i.e. that order of convergence is 4. It confirms the theory.

Table 7. Number of iterations and error estimates for multilevel conjugate gradient method for isotropic wavelet basis.

s	N	M	$\|\mathbf{r}_s\|$	$\|u_s - u\|_\infty$	$\|u_s - u\|$	$\|u_s^* - u\|_\infty$	$\|u_s^* - u\|$
1	289	17.00	1.00e-6	1.02e-5	2.95e-6	1.02e-5	2.95e-6
2	1 089	17.06	1.51e-7	6.95e-7	2.49e-7	6.96e-7	2.49e-7
3	4 225	16.75	1.29e-8	4.83e-8	1.61e-8	4.82e-8	1.61e-8
4	16 641	15.31	1.78e-9	2.87e-9	9.92e-10	2.86e-9	9.92e-10
5	66 049	14.48	1.59e-10	1.79e-10	6.18e-11	1.77e-10	6.18e-11
6	263 169	12.77	3.21e-11	1.12e-11	3.77e-12	1.10e-11	3.75e-12

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Table 8. Number of iterations and error estimates for multilevel conjugate gradient method for anisotropic wavelet basis.

s	N	M	$\|\mathbf{r}_s\|$	$\|u_s - u\|_\infty$	$\|u_s - u\|$	$\|u_s^* - u\|_\infty$	$\|u_s^* - u\|$
1	289	9.25	8.15e-6	1.03e-5	2.97e-6	1.02e-5	2.95e-6
2	1 089	11.13	1.16e-6	7.10e-7	2.49e-7	6.96e-7	2.49e-7
3	4 225	11.42	1.33e-7	4.91e-8	1.62e-8	4.82e-8	1.61e-8
4	16 641	12.05	1.32e-8	2.90e-9	9.93e-10	2.86e-9	9.92e-10
5	66 049	12.14	1.31e-9	1.76e-10	6.20e-11	1.77e-10	6.18e-11
6	263 169	11.95	1.32e-10	1.14e-11	3.78e-12	1.10e-11	3.75e-12

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Article

Sparse wavelet representation of differential operators with piecewise polynomial coefficients

Dana Černá [1], and Václav Finěk [1]

[1] Department of Mathematics and Didactics of Mathematics, Technical University in Liberec, Studentská 2, Liberec 461 17, Czech Republic;

* Correspondence: dana.cerna@tul.cz; Tel.: +420-48-535-2837

Abstract: We propose a construction of a Hermite cubic spline-wavelet basis on the interval and hypercube. The basis is adapted to homogeneous Dirichlet boundary conditions. The wavelets are orthogonal to piecewise polynomials of degree at most seven on a uniform grid. Therefore the wavelets have eight vanishing moments and the matrices arising from discretization of differential equations with coefficients that are piecewise polynomials of degree at most four on uniform grids are sparse. Numerical examples demonstrate the efficiency of an adaptive wavelet method with the constructed wavelet basis for solving one-dimensional elliptic equation and the two-dimensional Black-Scholes equation with a quadratic volatility.

Keywords: Riesz basis; wavelet; spline; interval; differential equation; sparse matrix, Black-Scholes equation

1. Introduction

Wavelets are a powerful and useful tool for analysing signals, detection of singularities, data compression and numerical solution of partial differential and integral equations. One of the most important properties of wavelets is that they have vanishing moments. Vanishing wavelet moments ensure so called compression property of wavelets. It means that a function f that is smooth, except at some isolated singularities, typically has a sparse representation in a wavelet basis, i.e. only a small number of wavelet coefficients carry most of the information on f. Similarly as functions also certain differential and integral operators have sparse or quasi-sparse representation in a wavelet basis. This compression property of wavelets leads to design of many multiscale wavelet-based methods for the solution of differential equations. First wavelet methods used orthogonal wavelets, e.g. Daubechies wavelets or coiflets [1,34]. Their disadvantage is that the most orthogonal wavelets are usually not known in a closed form and that their smoothness is typically dependent on the length of the support. The orthogonal wavelets that are known in a closed form are Haar wavelets. They were succesfully used for solving differential equations e.g. in [21,31,32]. Another useful tool is the short Haar wavelet transform that was derived and used for solving differential equations in [3–5]. Since spline wavelets are known in a closed form and they are smoother and have more vanishing moments than orthogonal wavelets of the same length of support, many wavelet methods using spline wavelets were proposed [27,28,30]. For a review of wavelet methods for solving differential equations see also [17,29].

It is known that spectral methods can be used to study singularity formation for PDE solution [2,20,39]. Due to their compression property wavelets can also be used to study singularity formation for PDE solutions. The wavelet approach simply insists in analyzing wavelet coefficients that are large in regions where the singularity occurs and very small in regions where the function is smooth and derivatives are relatively small. Many adaptive wavelet methods are based on this property [13,14].

We focus on an adaptive wavelet method that was originally designed in [13,14] and later modified in many papers [24,25,37], because it has the following advantages:

- *Optimality.* For a large class of differential equations, both linear and nonlinear, it was shown that this method converges and is asymptotically optimal in the sense that storage and number of floating point operations, needed to resolve the problem with desired accuracy, depend linearly on the number of parameters representing the solution and the number of these parameters is small. Thus, the computational complexity for all steps of the algorithm is controlled.
- *High order-approximation.* The method enables high order approximation. The order of approximation depends on the order of the spline wavelet basis.
- *Sparsity.* The solution and the right-hand side of the equation have sparse representation in a wavelet basis, i.e. they are represented by a small number of numerically significant parameters. In the beginning iterations start for a small vector of parameters and the size of the vector increases successively until the required tolerance is reached. The differential operator is represented by a sparse or quasi-sparse matrix and a procedure for computing the product of this matrix with a finite-length vector with linear complexity is known.
- *Preconditioning.* For a large class of problems the matrices arising from a discretization using wavelet bases can be simply preconditioned by a diagonal preconditioner and the condition numbers of these preconditioned matrices are uniformly bounded. It is important that the preconditioner is simple such as the diagonal preconditioner, because in some implementations only nonzero elements in columns of matrices corresponding to significant coefficients of solutions are stored and used.

It should be noted that also other spline wavelet methods utilize some of these features, but up to our knowledge there are not other wavelet methods than adaptive wavelet methods based on ideas from [13,14] that have all these properties. For more details about adaptive wavelet methods see Section 6 and [13,14,19,24,25,37,38].

In this paper, we are concerned with the wavelet discretization of the partial differential equation

$$-\sum_{k,l=1}^{d} \frac{\partial}{\partial x_k}\left(p_{k,l}\frac{\partial u}{\partial x_l}\right) + \sum_{k=1}^{d} q_k \frac{\partial u}{\partial x_k} + p_0 u = f \quad \text{on } \Omega = (0,1)^d, \quad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

We assume that $q_k(x) \geq Q > 0$, the functions $p_{k,l}$, q_k, p_0 and f are sufficiently smooth and bounded on Ω, and that $p_{k,l}$ satisfy the uniform ellipticity condition

$$\sum_{k=1}^{d}\sum_{l=1}^{d} p_{k,l}(x)\, x_k x_l \geq C \sum_{k=1}^{d} x_k^2, \quad x = (x_1, \ldots, x_d), \tag{2}$$

where $C > 0$ is independent on x. The discretization matrix for wavelet bases is typically not sparse, but only quasi-sparse, i.e. the matrix of the size $N \times N$ has $\mathcal{O}(N \times \log N)$ nonzero entries. For multiplication of this matrix with a vector a routine called APPLY have to be used [6,14,24]. However, it was observed in several papers, e.g. in [23] that "quantitatively the application of the APPLY routine is very demanding, where this routine is also not easy to implement". Therefore, in [23] a wavelet basis was constructed with respect to which the discretization matrix is sparse, i.e. it has $\mathcal{O}(N)$ nonzero entries, for equation (1) if the coefficients are constant. The construction from [23] was modified in [10,11] with the aim to improve the condition number of the discretization matrices. Some numerical experiments with these bases can be found in [9,15]. In this paper, our aim is to construct a wavelet basis such that the discretization matrix corresponding to (1) is sparse if the coefficients $p_{k,l}$, q_k and p_0 are piecewise polynomial functions of degree at most n on the uniform grid, where $n = 6$ for $p_{k,l}$, $n = 5$ for q_k and $n = 4$ for p_0. Our construction is based on Hermite cubic splines. Let us mention that cubic Hermite wavelets were constructed also in [10,11,18,23,26,33,35,36,40].

Example 1. We have recently implemented adaptive wavelet method for solving the Black-Scholes equation

$$\frac{\partial V}{\partial t} - \sum_{k,l=1}^{d} \frac{\rho_{k,l}}{2} \sigma_k \sigma_l S_k S_l \frac{\partial^2 V}{\partial S_k \partial S_l} - r \sum_{k=1}^{d} S_k \frac{\partial V}{\partial S_k} + rV = 0, \tag{3}$$

where $(S_1, \ldots, S_d, t) \in \left(0, S_1^{max}\right) \times \ldots \times \left(0, S_d^{max}\right) \times (0, T)$. We used the θ-scheme for time discretization and tested the performance of the adaptive method with respect to the choice of a wavelet basis for $d = 1, 2, 3$. Some results can be found in [9]. In the case of cubic spline wavelets, the smallest number of iteration was required for the wavelet basis from [8]. The discretization matrix for most spline wavelet bases is not sparse, but only quasi-sparse and thus the above mentioned routine APPLY have to be used. For wavelet bases from [10,11,23] the discretization matrix corresponding to the Black-Scholes operator is sparse if volatilities σ_i are constant. However, in more realistic models, volatilities are represented by non-constant functions, e.g. piecewise polynomial functions [41]. For the basis that will be constructed in this paper the discretization matrix is sparse also for the Black-Scholes equation with volatilities σ_i that are piecewise quadratic.

2. Wavelet bases

In this section, we briefly review the concept of a wavelet basis in Sobolev spaces and introduce notations, for more details see e.g. [38]. Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|\cdot\|_H$ and let $\langle \cdot, \cdot \rangle$ denote the L^2-inner product. Let $\mathcal{J}$ be an index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$, where $|\lambda| := j \in \mathbb{Z}$ is a level. For $\mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, v_\lambda \in \mathbb{R}$, we define

$$\|\mathbf{v}\|_2 := \left(\sum_{\lambda \in \mathcal{J}} |v_\lambda|^2 \right)^{1/2}, \quad l^2(\mathcal{J}) := \{\mathbf{v} : \|\mathbf{v}\|_2 < \infty\}. \tag{4}$$

Our aim is to construct a wavelet basis in the sense of the following definition.

Definition 1. A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *wavelet basis* of H, if

i) Ψ is a *Riesz basis* for H, i.e. the closure of the span of Ψ is H and there exist constants $c, C \in (0, \infty)$ such that

$$c \|\mathbf{b}\|_2 \le \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \le C \|\mathbf{b}\|_2, \quad \text{for all} \quad \mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{5}$$

ii) The functions are *local* in the sense that $\operatorname{diam}(\operatorname{supp} \psi_\lambda) \le \tilde{C} 2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$ and at a given level j the supports of only finitely many wavelets overlap at any point x.

For the two countable sets of functions $\Gamma, \tilde{\Gamma} \subset H$, the symbol $\langle \Gamma, \tilde{\Gamma} \rangle_H$ denotes the matrix

$$\langle \Gamma, \tilde{\Gamma} \rangle_H := \{\langle \gamma, \tilde{\gamma} \rangle_H\}_{\gamma \in \Gamma, \tilde{\gamma} \in \tilde{\Gamma}}. \tag{6}$$

The constants $c_\Psi := \sup\{c : c \text{ satisfies } (5)\}$ and $C_\Psi := \inf\{C : C \text{ satisfies } (5)\}$ are called *Riesz bounds* and the number *cond* $\Psi = C_\Psi / c_\Psi$ is called the *condition number* of Ψ. It is known that

$$c_\Psi = \sqrt{\lambda_{min}(\langle \Psi, \Psi \rangle_H)}, \quad C_\Psi = \sqrt{\lambda_{max}(\langle \Psi, \Psi \rangle_H)}, \tag{7}$$

where $\lambda_{min}(\langle \Psi, \Psi \rangle_H)$ and $\lambda_{max}(\langle \Psi, \Psi \rangle_H)$ are the smallest and the largest eigenvalues of the matrix $\langle \Psi, \Psi \rangle_H$, respectively.

Let M be a Lebesgue measurable subset of $\mathbb{R}^d$. The space $L^2(M)$ is the space of all Lebesgue measurable functions on M such that the norm

$$\|f\| = \left(\int_M |f(x)|^2 \, dx \right)^{1/2} \tag{8}$$

is finite. The space $L^2(M)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_M f(x) \overline{g(x)} dx, \quad f, g \in L^2(M). \tag{9}$$

The Sobolev space $H^s\left(\mathbb{R}^d\right)$ for $s \geq 0$ is defined as the space of all functions $f \in L^2\left(\mathbb{R}^d\right)$ such that the seminorm

$$|f|_{H^s(\mathbb{R}^d)} = \left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \hat{f}(\xi) \right|^2 |\xi|^{2s} \, d\xi \right)^{1/2} \tag{10}$$

is finite. The symbol $\hat{f}$ denotes the Fourier transform of the function f defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^s} f(x) \, e^{-i\xi \cdot x} dx. \tag{11}$$

The space $H^s\left(\mathbb{R}^d\right)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{H^s(\mathbb{R}^d)} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) \overline{\hat{g}(\xi)} \left(1 + |\xi|^{2s}\right) d\xi, \quad f, g \in H^s\left(\mathbb{R}^d\right), \tag{12}$$

and the norm

$$\|f\|_{H^s(\mathbb{R}^d)} = \sqrt{\langle f, f \rangle_{H^s(\mathbb{R}^d)}}. \tag{13}$$

For an open set $M \subset \mathbb{R}^d$, $H^s(M)$ is the set of restrictions of functions from $H^s\left(\mathbb{R}^d\right)$ to M equipped with the norm

$$\|f\|_{H^s(M)} = \inf \left\{ \|g\|_{H^s(\mathbb{R}^d)} : g \in H^s(M) \text{ and } g|_M = f \right\}. \tag{14}$$

Let $C_0^\infty(M)$ be the space of all continuous functions with the support in M such that they have continuous derivatives of order r for any $r \in \mathbb{R}$. The space $H_0^s(M)$ is defined as the closure of $C_0^\infty(M)$ in $H^s\left(\mathbb{R}^d\right)$. It is known that

$$\|f\|_{H^1(M)} = |f|_{H^1(M)} + \|f\|, \tag{15}$$

where

$$|f|_{H^1(M)} = \sqrt{\langle \nabla f, \nabla f \rangle} \tag{16}$$

is the seminorm in $H^1(M)$ and ∇f denotes the gradient of f.

3. Construction of scaling functions

We start with the same scaling functions as in [10,11,18,23,26,33,35,36]. Let

$$\phi_1(x) = \begin{cases} (x+1)^2 (1-2x), & x \in [-1, 0], \\ (1-x)^2 (1+2x), & x \in [0, 1], \\ 0, & \text{otherwise}, \end{cases} \quad \phi_2(x) = \begin{cases} (x+1)^2 x, & x \in [-1, 0], \\ (1-x)^2 x, & x \in [0, 1], \\ 0, & \text{otherwise}. \end{cases} \tag{17}$$

For $j \geq 3$ and $x \in [0,1]$ we define

$$\phi_{j,2k+l-1}(x) = 2^{j/2}\phi_l\left(2^j x - k\right) \text{ for } k = 1,\ldots,2^j-1, \quad l = 1,2, \tag{18}$$
$$\phi_{j,1}(x) = 2^{j/2}\phi_2\left(2^j x\right), \quad \phi_{j,2^{j+1}}(x) = 2^{j/2}\phi_2\left(2^j(x-1)\right),$$

and

$$\Phi_j = \left\{\phi_{j,k}, k = 1,\ldots,2^{j+1}\right\}, \quad V_j = \text{span}\,\Phi_j. \tag{19}$$

Then the spaces V_j form a multiresolution analysis. We choose dual space $\tilde{V}_j$ as the set of all functions

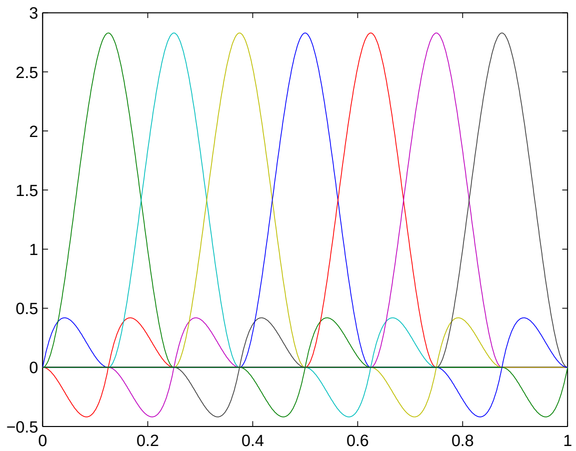

Figure 1. Scaling functions on the level $j = 3$.

$v \in L^2(0,1)$ such that v restricted to the interval $\left(\frac{k-1}{2^{j-2}}, \frac{k}{2^{j-2}}\right)$ is a polynomial of degree less than 8 for any $k = 1,\ldots,2^{j-2}$, i.e.

$$\tilde{V}_j = \left\{v \in L^2(0,1) : v|_{\left(\frac{k-1}{2^{j-2}},\frac{k}{2^{j-2}}\right)} \in \Pi_8\left(\frac{k-1}{2^{j-2}}, \frac{k}{2^{j-2}}\right) \text{ for } k = 0,\ldots,2^{j-2}\right\}, \tag{20}$$

where $\Pi_8(a,b)$ denotes the set of all polynomials on (a,b) of degree less than 8. Let

$$W_j = \tilde{V}_j^{\perp} \cap V_{j+1}, \tag{21}$$

where $\tilde{V}_j^{\perp}$ is the orthogonal complement of $\tilde{V}_j$ with respect to the L^2-inner product. If a function g is a piecewise polynomial of degree n we write $\deg g = n$.

Lemma 2. *Let the spaces W_j, $j \geq 3$, are defined as above. Then all functions $g \in W_i$ and $h \in W_j$, $i,j \geq 3$, $|i-j| > 2$, satisfy*

$$\langle a\,g, h\rangle = 0, \quad \langle b\,g', h\rangle = 0, \quad \langle c\,g', h'\rangle = 0, \tag{22}$$

where a,b,c are piecewise polynomial functions such that $a,b,c \in \tilde{V}_p$, $p \leq \max(i,j)$, $\deg a \leq 4$, $\deg b \leq 5$, and $\deg c \leq 6$.

Proof of Lemma 2. Let us assume that $j > i+2$. We have $g \in W_i \subset V_{i+1} \subset V_{j-2} \subset \tilde{V}_j$, $\deg g \leq 3$, $a \in \tilde{V}_j$, $\deg a \leq 4$, and thus $ag \in \tilde{V}_j$. Since $h \in W_j$ and W_j is orthogonal to $\tilde{V}_j$ we obtain $\langle a\,g, h\rangle = 0$.

Similarly, the relation $\langle b\, g', h\rangle = 0$ is the consequence of the fact that $bg' \in \tilde{V}_j$ and $h \in W_j$. Using integration by parts we obtain

$$\langle c\, g', h' \rangle = -\langle c'\, g' + c\, g'', h \rangle . \tag{23}$$

Since $c'\, g' + c\, g'' \in \tilde{V}_j$ and $h \in W_j$ we have $\langle c\, g', h' \rangle = 0$. The situation for $j < i + 2$ is similar. □

Therefore the discretization matrix for the equation (1) is sparse. Let Ψ_j be a basis of W_j. The proof that

$$\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\} = \Phi_3 \cup \bigcup_{j=3}^{\infty} \Psi_j. \tag{24}$$

is a Riesz basis of the space $L^2(0,1)$ and that Ψ is a Riesz basis of the space $H_0^1(0,1)$ when normalized with respect to the H^1-norm is based on the following theorem [16,23].

Theorem 3. *Let $J \in \mathbb{N}$ and let V_j and $\tilde{V}_j$, $j \geq J$, be subspaces of $L^2(0,1)$ such that*

$$V_j \subset V_{j+1}, \quad \tilde{V}_j \subset \tilde{V}_{j+1}, \quad \dim V_j = \dim \tilde{V}_j < \infty, \quad j \geq J. \tag{25}$$

Let Φ_j be bases of V_j, $\tilde{\Phi}_j$ be bases of $\tilde{V}_j$, Ψ_j be bases of $\tilde{V}_j^{\perp} \cap V_{j+1}$, such that Riesz bounds with respect to the L^2-norm of Φ_j, $\tilde{\Phi}_j$, and Ψ_j are uniformly bounded, and let Ψ be given by (24). Furthermore, let the matrix

$$\mathbf{G}_j := \langle \Phi_j, \tilde{\Phi}_j \rangle \tag{26}$$

be invertible and the spectral norm of $\mathbf{G}_j^{-1}$ is bounded independently on j. In addition, for some positive constants C, γ and d, such that $\gamma < d$, let

$$\inf_{v_j \in V_j} \|v - v_j\| \leq C 2^{-jd} \|v\|_{H^d(0,1)}, \; v \in H_0^d(0,1), \tag{27}$$

and for $0 \leq s < \gamma$ let

$$\|v_j\|_{H^d(0,1)} \leq C 2^{js} \|v_j\|, \; v_j \in V_j, \tag{28}$$

and let similar estimates (27) and (28) hold for $\tilde{\gamma}$ and $\tilde{d}$ on the dual side. Then there exist constants k and K, $0 < k \leq K < \infty$, such that

$$k \|\mathbf{b}\|_2 \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda 2^{-|\lambda| s} \psi_\lambda \right\|_{H^s(0,1)} \leq K \|\mathbf{b}\|_2, \quad \mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}) \tag{29}$$

holds for $s \in (-\tilde{\gamma}, \gamma)$.

We focus on the spaces V_j and $\tilde{V}_j$ defined by (19) and (20), respectively, and we show that they satisfy the assumptions of the Theorem 3.

Theorem 4. *There exist uniform Riesz bases $\hat{\Phi}_j$ of V_j and $\tilde{\Phi}_j$ of $\tilde{V}_j$ such that the matrix*

$$\mathbf{G}_j = \langle \hat{\Phi}_j, \tilde{\Phi}_j \rangle \tag{30}$$

is invertible and the spectral norm of $\mathbf{G}_j^{-1}$ is bounded independently on j.

Proof of Theorem 4. Let Φ_j, V_j and $\tilde{V}_j$ be defined as above. For $i = 0, \ldots, 7$ we define

$$p_i(x) \begin{cases} (x - 1/2)^i, & x \in [0,1], \\ 0, & \text{otherwise}, \end{cases} \tag{31}$$

and

$$\theta_{j,8k+i+1} = 2^{(j-2)/2} p_i \left(2^{j-2}x - k\right), \quad k \in \mathbb{Z}, \quad i = 0, \ldots, 7. \tag{32}$$

Then the set $\Theta_j = \left\{\theta_{j,k}, k = 1, \ldots, 2^{j+1}\right\}$ is a basis of $\tilde{V}_j$ and the matrix $\mathbf{A}_j = \langle \Phi_j, \Theta_j \rangle$, $j \geq 3$, has the structure:

$$\mathbf{A}_j = \begin{pmatrix} \boxed{\mathbf{A}^L} & & & & & \\ & \boxed{\mathbf{A}} & & & & \\ & & \boxed{\mathbf{A}} & & & \\ & & & \ddots & & \\ & & & & \boxed{\mathbf{A}} & \\ & & & & & \boxed{\mathbf{A}^R} \end{pmatrix}, \tag{33}$$

where $\mathbf{A}$ is the matrix of the size 10×8. Our aim is to apply several transforms on Φ_j and Θ_j such that new bases $\hat{\Phi}_j$ of V_j and $\tilde{\Phi}_j$ of $\tilde{V}_j$ are local and the matrix $\mathbf{G}_j$ defined by (30) and its transpose $\mathbf{G}_j^T$ are both strictly diagonally dominant. First, we replace functions $\theta_{j,k}$ by functions $g_{j,k}$ in such a way that the matrix of L^2-inner products of $\phi_{j,k}$ and $g_{j,l}$ is tridiagonal. Therefore we define

$$g_{j,8k+i+1} = \sum_{l=1}^{8} c_{j,l}^i \theta_{j,8k+l}, \quad i = 0, \ldots, 7, \quad k = 0, \ldots, 2^{j-2} - 1, \tag{34}$$

where the coefficients $c_{j,l}^i$ are chosen such that

$$\langle \phi_{j,p}, g_{j,q} \rangle = 0, \quad |p - q| > 1, \quad \langle \phi_{j,p}, g_{j,p} \rangle = 1, \quad p = 1, \ldots, 2^{j+1} - 1. \tag{35}$$

For $m = 8k + i + 1$, $i = 0, \ldots, 7$, $k = 1, \ldots, 2^{j-2} - 2$, we substitute (34) into (35) and using $\operatorname{supp} \phi_{j,8k+l+1} \cap \operatorname{supp} g_{j,m} = 0$ for $l \neq 0, \ldots, 9$, we obtain systems of 8 linear algebraic equations with 8 unknown coefficents:

$$\mathbf{A}^i \mathbf{c}^i = \mathbf{e}^i, \quad \text{for } k = 2, \ldots, 2^{j-2} - 2, \quad \mathbf{c}^i = \left(c_{j,l}^i\right)_{l=1}^{8}, \tag{36}$$

the system matrices $\mathbf{A}^i$ that are submatrices of $\mathbf{A}$ containing all rows of $\mathbf{A}$ except i-th and $(i+2)$-th rows, and $\mathbf{e}^i$ are unit vectors such that $\left(\mathbf{e}^i\right)_l = \delta_{i,l}$. The symbol $\delta_{i,l}$ denotes the Kronecker delta. We computed all of the system matrices precisely using symbolic computations and verified that they are regular. Thus the coefficients $c_{j,l}^i$ exist and are unique.

The matrix $\mathbf{B}_j$ defined by $\left(\mathbf{B}_j\right)_{k,l} = \left\langle \phi_{j,k}, g_{j,m} \right\rangle$, $k, l = 1, \ldots, 2^{j+1}$, is tridiagonal and has the structure

$$\mathbf{B}_j = \begin{pmatrix} \boxed{\mathbf{B}^L} & & & & & \\ & \boxed{\mathbf{B}} & & & & \\ & & \boxed{\mathbf{B}} & & & \\ & & & \ddots & & \\ & & & & \boxed{\mathbf{B}} & \\ & & & & & \boxed{\mathbf{B}^R} \end{pmatrix}, \tag{37}$$

where

$$\mathbf{B} = \begin{pmatrix} 13.199 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.000 & 0.098 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.185 & 1.000 & -24.781 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.138 & 1.000 & 0.104 & 0 & 0 & 0 & 0 \\ 0 & 0 & 13.887 & 1.000 & -6.026 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.074 & 1.000 & 0.041 & 0 & 0 \\ 0 & 0 & 0 & 0 & 34.953 & 1.000 & 8.824 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.018 & 1.000 & 0.023 \\ 0 & 0 & 0 & 0 & 0 & 0 & -9.423 & 1.000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.092 \end{pmatrix}, \tag{38}$$

and

$$\mathbf{B}^L = \mathbf{B}_{\{2,\ldots,10\}}, \quad \mathbf{B}^R = \mathbf{B}_{\{1,\ldots,8,10\}}. \tag{39}$$

The symbol $\mathbf{B}_M$ denotes the submatrix of the matrix $\mathbf{B}$ containing rows from $\mathbf{B}$ with indices from M. In (38) the numbers are rounded to three decimal digits.

We apply several transforms on $\phi_{j,k}$ and denote the new functions by $\phi^i_{j,k}$. In the following, let

$$B_{j,k,l} = \left(\mathbf{B}_j\right)_{k,l}, \quad B^i_{j,k,l} = \left\langle \phi^i_{j,l}, g_{j,k} \right\rangle, \quad i = 1, \ldots, 4. \tag{40}$$

We define

$$\begin{aligned}
\phi^1_{j,k} &= \phi_{j,k} - \frac{B_{j,k,k+1}}{B_{j,k+1,k+1}} \phi_{j,k+1} \text{ for } k \text{ even}, \quad \phi^1_{j,k} = \phi_{j,k} \text{ for } k \text{ odd}, \\
\phi^2_{j,k} &= \phi^1_{j,k} - \frac{B^1_{j,k,k-1}}{B^1_{j,k-1,k-1}} \phi^1_{j,k-1} \text{ for } k \text{ even}, \quad \phi^2_{j,k} = \phi^1_{j,k} \text{ for } k \text{ odd}, \\
\phi^3_{j,4+8k} &= \phi^2_{j,4+8k} - \frac{B^2_{j,4+8k,2+8k}}{B^2_{j,2+8k,2+8k}} \phi^1_{j,2+8k} \text{ for } k = 1, \ldots, 2^{j-2}, \phi^3_{j,l} = \phi^2_{j,l} \text{ otherwise}, \\
\phi^4_{j,6+8k} &= \phi^3_{j,6+8k} - \frac{B^3_{j,6+8k,4+8k}}{B^3_{j,4+8k,4+8k}} \phi^1_{j,4+8k} \text{ for } k = 1, \ldots, 2^{j-2}, \phi^4_{j,l} = \phi^3_{j,l} \text{ otherwise},
\end{aligned} \tag{41}$$

and

$$\hat{\phi}_{j,l} = \begin{cases} 2.1\,\phi^4_{j,l} & l = 4 + 8k, \\ 10\,\phi^4_{j,l} & l = 2^{j+1}, \\ \phi^4_{j,l} & \text{otherwise}. \end{cases} \tag{42}$$

Furthermore, we set $\tilde{\phi}_{j,2+8k} = 1.3 g_{j,2+8k}$ for $k = 0, \ldots, 2^{j-2}$ and $\tilde{\phi}_{j,l} = g_{j,l}$ for $l \neq 2 + 8k$. Let $\hat{\Phi}_j = \left\{\hat{\phi}_{j,l}, l = 1, \ldots 2^{j+1}\right\}$ and $\tilde{\Phi}_j = \left\{\tilde{\phi}_{j,l}, l = 1, \ldots 2^{j+1}\right\}$. The matrix $\mathbf{G}_j$ defined by (30) has the same structure as $\mathbf{A}_j$ and $\mathbf{B}_j$, i.e.

$$\begin{aligned}
\left(\mathbf{G}_j\left(8i + k, 8i + l\right)\right)_{k=0,\ldots,14, l=1,\ldots,8} &= \mathbf{G}, \quad i = 2, \ldots, 2^{j-2} - 2 \\
\left(\mathbf{G}_j\left(k, l\right)\right)_{k=1,\ldots,14, l=1,\ldots,8} &= \mathbf{G}^L, \\
\left(\mathbf{G}_j\left(2^{j+1} - 8 + k, 2^{j+1} - 8 + l\right)\right)_{k=0,\ldots,8, l=1,\ldots,8} &= \mathbf{G}^R,
\end{aligned} \tag{43}$$

where

$$\mathbf{G} = \begin{pmatrix} 0 & -1.6863 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.0000 & 0.1278 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.8555 & 0 & 2.5773 & 0 & 0 & 0 & 0 \\ 0 & -0.1790 & 1.0000 & 0.1040 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.8443 & 0 & 0.5229 & 0 & 0 \\ 0 & 0 & 0 & -0.0737 & 1.0000 & 0.0413 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.7599 & 0 & -0.2011 \\ 0 & 0 & 0 & 0 & 0 & -0.0179 & 1.0000 & 0.0228 \\ 0 & 0 & 0 & 0 & 0 & -0.1689 & 0 & 2.4295 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0920 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2011 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3675 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3330 \end{pmatrix} \tag{44}$$

and

$$\mathbf{G}^L = \mathbf{G}_{\{2,\ldots,15\}}, \quad \mathbf{G}^R = \begin{pmatrix} & \mathbf{G}_{\{1,\ldots,8\}} & & \\ 0 & \ldots & 0 & -0.920 \end{pmatrix} \tag{45}$$

Thus the matrices $\mathbf{G}_j$ and $\mathbf{G}_j^T$ are diagonally dominant and invertible and due to the Johnson's lower bound for the smallest singular value [22], we have

$$\sigma_{min}\left(\mathbf{G}_j\right) \geq 0.117, \quad \left\|\mathbf{G}_j^{-1}\right\|_2 = \frac{1}{\sigma_{min}\left(\mathbf{G}_j\right)} \leq 8.527. \tag{46}$$

It remains to prove that $\hat{\Phi}_j$ are uniform Riesz bases of V_j and $\tilde{\Phi}_j$ are uniform Riesz bases of $\tilde{V}_j$. Since $\hat{\phi}_{j,k}$ are locally supported and there exists M independent of j and k such that $\left\|\hat{\phi}_{j,k}\right\|_{L^2(0,1)} \leq M$, we have

$$\left\|\sum_k c_{j,k}\hat{\phi}_{j,k}\right\|^2_{L^2(0,1)} = \sum_k \sum_l c_{j,k}c_{j,l} \int_0^1 \hat{\phi}_{j,k}(x)\,\hat{\phi}_{j,l}(x)\,dx \leq C\,\|c\|_2^2. \tag{47}$$

and similarly for $\tilde{\phi}_{j,k}$ we have

$$\left\|\sum_k c_{j,k}\tilde{\phi}_{j,k}\right\|^2_{L^2(0,1)} \leq C\,\|c\|_2^2. \tag{48}$$

By the same argument as in the Proof of Theorem 3.3. in [23], from (47), (48), invertibility of $\mathbf{G}_j$ and (46) we can conclude that $\hat{\Phi}_j$ and $\tilde{\Phi}_j$ are uniform Riesz bases of their spans.

□

4. Construction of wavelets

Now we construct a basis Ψ_j of the space $W_j = \tilde{V}_j^{\perp} \cap V_{j+1}$ such that $cond\ \Psi_j \leq C$, where C is a constant independent on j, and functions from Ψ_j are translations and dilations of some generators. We propose one boundary generator ψ^b and functions ψ^i, $i = 1, \ldots, 8$, generating inner wavelets such that the sets

$$\Psi_j = \left\{\psi_{j,k}, k = 1, \ldots, 2^{j+1}\right\}, \quad j \geq 3, \tag{49}$$

contain functions $\psi_{j,k}$ defined for $x \in [0,1]$ by

$$\begin{aligned}\psi_{j,1}(x) &= 2^{j/2}\psi^b\left(2^j x\right), \quad \psi_{j,2^{j+1}}(x) = 2^{j/2}\psi^b\left(2^j(1-x)\right), \\ \psi_{j,8k+l+1}(x) &= 2^{j/2}\psi^l\left(2^j x - 4k\right), \quad l = 1,\ldots 8,\ 1 < 8k+l+1 < 2^{j+1}.\end{aligned} \tag{50}$$

We denote the scaling functions on the level $j = 1$ by

$$\phi_{1,2k+l-2}(x) = 2^{1/2}\phi_l(2x-k) \text{ for } k \in \mathbb{Z}, \quad l = 1,2, \quad x \in \mathbb{R}. \tag{51}$$

For $l = 1,\ldots,6$ let the functions ψ^l have the form

$$\psi^l = \sum_{k=1}^{14} h_{l,k}\phi_{1,k} \tag{52}$$

and be such that ψ^1, ψ^2 and ψ^3 are antisymmetric and ψ^4, ψ^5 and ψ^6 are symmetric. Thus supp $\psi^l = [0,4]$ for $l = 1,\ldots,6$. Let p_i be polynomials defined by (31). It is clear that if

$$\left\langle \psi^l(x), p_i\left(\frac{x}{4}\right)\right\rangle = 0, \quad i = 0,\ldots,7, \quad l = 1,\ldots 6, \tag{53}$$

then $\left\langle \psi^l\left(2^j x - k\right), p_i\left(2^{j-2}x - m\right)\right\rangle = 0$, for $k, m \in 4\mathbb{Z}$, and thus $\left\langle \psi_{j,8i+l}, g\right\rangle = 0$ for any $g \in \tilde{V}_j$, $i = 0,\ldots,2^{j-2}-1$, and $l = 1,\ldots,6$. Substituting (52) into (53) we obtain the system of linear algebraic equations with the solution $\mathbf{h}^l = \{h_{l,k}\}_{k=1}^{14}$ of the form

$$\mathbf{h}^l = a_{l,1}\mathbf{u}_1 + a_{l,2}\mathbf{u}_2 + a_{l,3}\mathbf{u}_3, \quad l = 1,2,3, \tag{54}$$

and

$$\mathbf{h}^l = b_{l-3,1}\mathbf{v}_1 + b_{l-3,2}\mathbf{v}_2 + b_{l-3,3}\mathbf{v}_3, \quad l = 4,5,6, \tag{55}$$

where $a_{l,k}$ and $b_{l,k}$ are chosen real parameters and

$$[\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3] = \begin{pmatrix} -\frac{29}{95361} & \frac{120}{31787} & \frac{5716}{31787} \\ 0 & 0 & 1 \\ \frac{592}{95361} & \frac{13477}{95361} & -\frac{56300}{95361} \\ 0 & 1 & 0 \\ \frac{13456}{95361} & -\frac{39892}{95361} & \frac{49671}{31787} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{11708}{4541} & -\frac{26022}{4541} & \frac{116428}{4541} \\ -\frac{13456}{95361} & \frac{39892}{95361} & -\frac{49671}{31787} \\ 1 & 0 & 0 \\ -\frac{592}{95361} & -\frac{13477}{95361} & \frac{56300}{95361} \\ 0 & 1 & 0 \\ \frac{29}{95361} & -\frac{120}{31787} & -\frac{5716}{31787} \\ 0 & 0 & 1 \end{pmatrix}, \quad [\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3] = \begin{pmatrix} \frac{11}{10500} & -\frac{17}{2625} & -\frac{1727}{10500} \\ 0 & 0 & -1 \\ -\frac{13}{875} & -\frac{293}{2625} & \frac{1123}{2625} \\ 0 & -1 & 0 \\ -\frac{1}{12} & \frac{5}{21} & -\frac{53}{84} \\ -1 & 0 & 0 \\ \frac{34}{175} & -\frac{6}{25} & \frac{386}{525} \\ 0 & 0 & 0 \\ -\frac{1}{12} & \frac{5}{21} & -\frac{53}{84} \\ 1 & 0 & 0 \\ -\frac{13}{875} & -\frac{293}{2625} & \frac{1123}{2625} \\ 0 & 1 & 0 \\ \frac{11}{10500} & -\frac{17}{2625} & -\frac{1727}{10500} \\ 0 & 0 & 1 \end{pmatrix}. \tag{56}$$

For $l \in \{7,8\}$ let the functions ψ^l have the form

$$\psi^l = \sum_{k=1}^{28} h_{l,k}\phi_{1,k}. \quad (57)$$

These functions are uniquely determined by imposing that ψ^7 is symmetric, ψ^8 is antisymmetric, both ψ^7 and ψ^8 are L^2–orthogonal to the functions ψ^l, $l = 1,\dots,6$, they are normalized with respect to the L^2–norm and

$$\left\langle \psi^l\left(x\right), p_i\left(\frac{x}{4}\right)\right\rangle = 0, \quad \text{for } i = 0,\dots,7. \quad (58)$$

It remains to construct boundary function ψ^b. Let

$$\psi^b = \sum_{k=0}^{14} h_{b,k}\phi_{1,k}|_{[0,\infty)}. \quad (59)$$

Substituting (59) into

$$\left\langle \psi^b\left(x\right), p_i\left(\frac{x}{4}\right)\right\rangle = 0, \quad \text{for } i = 0,\dots,7, \quad (60)$$

we obtain the system of 8 equations for 15 unknown coefficients. The solution $\mathbf{h}^b = \{h_{b,k}\}_{k=0}^{14}$ is the linear combination of vectors $\mathbf{w}_i$ given by

$$\mathbf{w}_1 = \left(\frac{150}{83},\frac{1429}{1992},\frac{4509}{664},\frac{74}{249},\frac{2839}{166},-\frac{1897}{1992},\frac{6741}{664},-\frac{53}{249},1,0,0,0,0,0,0\right)^T, \quad (61)$$

$$\mathbf{w}_l = \begin{pmatrix} 0 \\ \mathbf{u}_{l-1}\end{pmatrix}, \quad l = 2,3,4, \quad \mathbf{w}_l = \begin{pmatrix} 0 \\ \mathbf{v}_{l-4}\end{pmatrix}, \quad l = 5,6,7, \quad (62)$$

i.e.

$$\mathbf{h}^b = \sum_{i=1}^{7} d_i W_i, \quad (63)$$

where d_i are chosen real parameters.

Hence the set Ψ_j depends on the choice of $a_{k,l}$, $b_{k,l}$ and d_i. However, it is not true that $cond\,\Psi_j \leq C$ for all possible choices of these parameters. Moreover, for some choices the condtition numbers of Ψ_j are uniformly bounded, but the condition number of the resulting basis Ψ is large, e.g. 10^6.

Therefore, we optimize the construction to improve the condition number of Ψ. We choose $a_{k,1}$ and then we set $a_{k,2}$ and $a_{k,3}$ such that $\langle \psi^i, \psi^j\rangle = \delta_{i,j}$ for $i,j = 1,2,3$ and similarly we we choose $b_{k,1}$ and then set $b_{k,2}$ and $b_{k,3}$ such that $\langle \psi^i, \psi^j\rangle = \delta_{i,j}$ for $i,j = 4,5,6$. Moreover, the functions ψ^7 and ψ^8 are constructed such that they are orthogonal to ψ^i for $i = 1,\dots,6$ and due to the symmetry and antisymmetry we have $\langle \psi^i, \psi^j\rangle = \delta_{i,j}$ for $i = 1,2,3$ and $j = 4,5,6$ and $\langle \psi^7, \psi^8\rangle = 0$. In summary ψ^i is orthogonal to ψ^j with respect to the L^2–norm for $i,j = 1,\dots 6$, $i \neq j$. To further improve the condition number we orthogonalize scaling functions on the coarsest level $j = 3$, i.e. we determine the set

$$\Phi_3^{ort} := \mathbf{K}^{-1}\Phi_3, \quad \mathbf{K} = \langle \Phi_3, \Phi_3\rangle, \quad (64)$$

and we redefine Φ_3 as $\Phi_3 := \Phi_3^{ort}$.

Furthermore we wrote a program that computes the condition number of the wavelet basis containing all wavelets up to the level 7 with respect to the both L^2–norm and H^1–norm for given parameters $a_{k,l}$, $b_{k,l}$ and d_i and performed extensive numerical experiments. In the following we consider the parameters that lead to good results:

$$\begin{aligned}
\mathbf{a}_1 &= (a_{1,1}, a_{1,2}, a_{1,3}) = (4.62, 4.43, 0.67), \\
\mathbf{a}_2 &= (a_{2,1}, a_{2,2}, a_{2,3}) = (7.196227729728021, -4.658487033189625, -2.279869518963229), \\
\mathbf{a}_3 &= (a_{3,1}, a_{3,2}, a_{3,3}) = (-0.775021413514386, 0.613425421561151, 0.151825757948663), \\
\mathbf{b}_1 &= (b_{1,1}, b_{1,2}, b_{1,3}) = (0.24, -3.92, 4.17), \\
\mathbf{b}_2 &= (b_{2,1}, b_{2,2}, b_{2,3}) = (4.214132381596882, -2.612399654970785, -1.411579368326525), \\
\mathbf{b}_3 &= (b_{3,1}, b_{3,2}, b_{3,3}) = (-0.601286696663076, -0.778487053796787, -0.180033928710130), \\
\mathbf{d} &= (d_1, \ldots, d_7) = (-0.075, -0.363, -0.616, -0.134, 0.344, 0.580, 0.099),
\end{aligned} \tag{65}$$

and after computing ψ^b and ψ^i, $i = 1, \ldots, 8$, using these parameters, we normalize them with respect to the L^2-norm, i.e. we redefine $\psi^b := \psi^b / \left\|\psi^b\right\|$ and $\psi^i := \psi^i / \left\|\psi^i\right\|$. The wavelets $\psi_{3,1}, \ldots, \psi_{3,9}$ that are dilations of $\psi^b, \psi^1, \ldots, \psi^8$ are displayed in Figure 2.

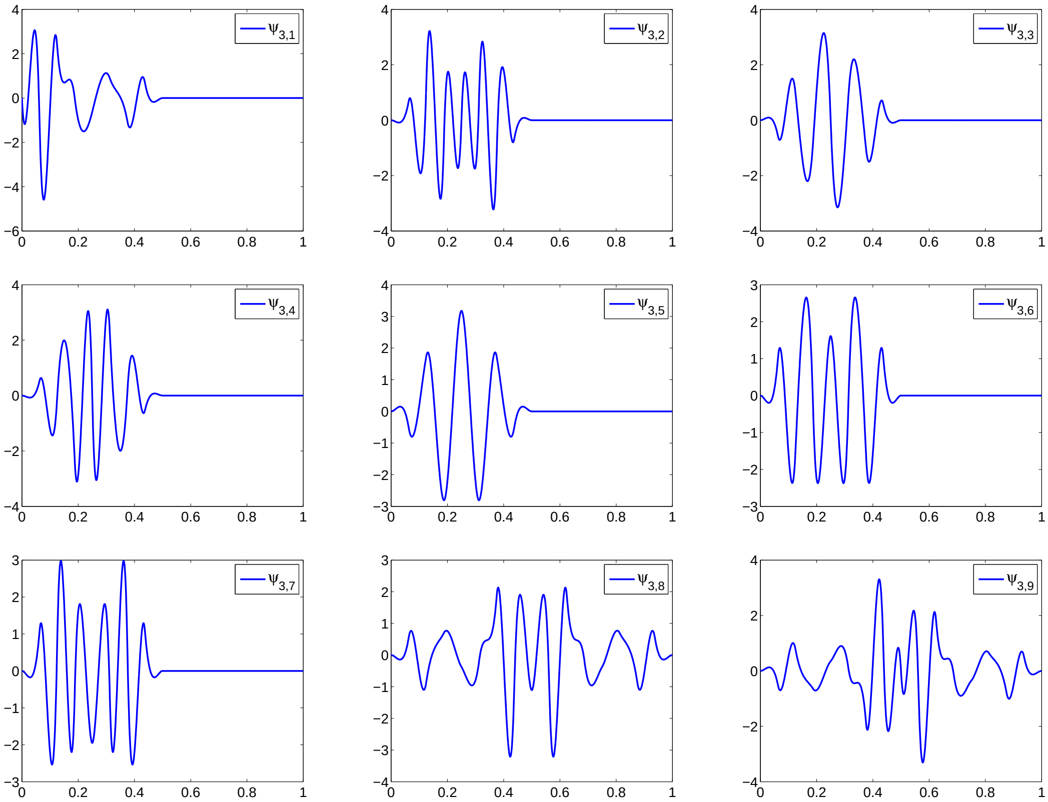

Figure 2. Wavelets $\psi_{3,1}, \ldots, \psi_{3,9}$.

Theorem 5. *The sets Ψ_j with the parameters given by (65) are uniform Riesz bases of W_j for $j \geq 3$.*

Proof of Theorem 5. Since we constructed wavelets such that many of them are orthogonal, there is only small number of nonzero entries in $\mathbf{N}_j$. Since wavelets are normalized with respect to the L^2-norm, we have

$$\left(\mathbf{N}_j\right)_{k,k} = 1. \tag{66}$$

Direct computation yields that

$$\begin{aligned} \left(\mathbf{N}_j\right)_{k=1,l=2,\ldots,9} &= \left(\mathbf{N}_j\right)_{k=2^j,l=2^j-1,\ldots,2^j-8} = \mathbf{z}, \\ \left(\mathbf{N}_j\right)_{k=2,\ldots,9,l=1} &= \left(\mathbf{N}_j\right)_{k=2^j-1,\ldots,2^j-8,l=2^j} = \mathbf{z}^T, \end{aligned} \tag{67}$$

where

$$\mathbf{z} = (0.0022, -0.0927, -0.0166, -0.0339, -0.0075, 0.0045, -0.2652, 0.2439)\,, \tag{68}$$

and for $i = 1, \ldots, 2^{j-2} - 2$ we have

$$\begin{aligned} \left(\mathbf{N}_j\right)_{k=8i,8i+1,l=8i+8,8i+9} &= \mathbf{N} \\ \left(\mathbf{N}_j\right)_{k=8i+8,8i+9,l=8i,8i+1} &= \mathbf{N}^T, \end{aligned} \tag{69}$$

where

$$\mathbf{N} = \begin{pmatrix} -0.2048 & 0.1885 \\ -0.1885 & 0.1734 \end{pmatrix}. \tag{70}$$

The numbers in (68) and (70) are rounded to four decimal digits. All other entries of $\mathbf{N}_j$ are zero. The structure of the Gram matrix $\mathbf{N}_j = \langle \Psi_j, \Psi_j \rangle$ is displayed in Figure 3. Using Gershgorin theorem the

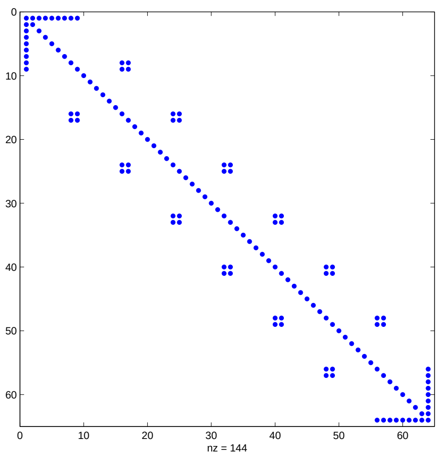

Figure 3. The structure of the matrix $\mathbf{N}_j$.

smallest eigenvalue $\lambda_{min}\left(\mathbf{N}_j\right) \geq 0.21$ and the largest eigenvalue $\lambda_{max}\left(\mathbf{N}_j\right) \leq 1.79$. Therefore Ψ_j are uniform Riesz bases of their spans. □

Theorem 6. *The set Ψ is a Riesz basis of $L^2(0,1)$ and when normalized with respect to the H^1-norm it is a Riesz basis of $H_0^1(0,1)$.*

Proof of Theorem 6. Due to the Theorem 3, Theorem 4 and Theorem 5, the relation (29) holds both for $s = 0$ and $s = 1$. Hence, Ψ is a Riesz basis of $L^2(0,1)$ and

$$\left\{2^{-3}\phi_{3,k}, k = 1, \ldots, 16\right\} \cup \left\{2^{-j}\psi_{j,k}, j \geq 3, k = 1, \ldots, 2^{j+1}\right\} \tag{71}$$

is a Riesz basis of $H_0^1(0,1)$. To show that also

$$\left\{\frac{\phi_{3,k}}{\|\phi_{3,k}\|_{H^1(0,1)}}, k=1,\ldots,16\right\} \cup \left\{\frac{\psi_{j,k}}{\left\|\psi_{j,k}\right\|_{H^1(0,1)}}, j\geq 3, k=1,\ldots,2^{j+1}\right\} \quad (72)$$

is a Riesz basis of $H_0^1(0,1)$, we follow the Proof of Theorem 2 in [7]. From (18) and (50) there exist nonzero constants C_1 and C_2 such that

$$C_1 2^j \leq \left\|\psi_{j,k}\right\|_{H_0^1(\Omega)} \leq C_2 2^j, \quad \text{for} \quad j\geq 3, \quad k=1,\ldots,2^{j+1}, \quad (73)$$

and

$$C_1 2^3 \leq \|\phi_{3,k}\|_{H_0^1(\Omega)} \leq C_2 2^3, \quad \text{for}\, k=1,\ldots,16. \quad (74)$$

Let $\hat{\mathbf{b}} = \{\hat{a}_{3,k},\ k=1,\ldots,16\} \cup \left\{\hat{b}_{j,k}, j\geq 3,\ k=1,\ldots,2^{j+1}\right\}$ be such that

$$\left\|\hat{\mathbf{b}}\right\|_2^2 = \sum_{k=1}^{16} \hat{a}_{3,k}^2 + \sum_{j=3}^{\infty}\sum_{k=1}^{2^{j+1}} \hat{b}_{j,k}^2 < \infty. \quad (75)$$

We define

$$a_{3,k} = \frac{2^3 \hat{a}_{3,k}}{\|\phi_{3,k}\|_{H_0^1(0,1)}},\ k=1,\ldots,16, \quad b_{j,k} = \frac{2^j \hat{b}_{j,k}}{\left\|\psi_{j,k}\right\|_{H_0^1(0,1)}},\ j\geq 3,\ k=1,\ldots,2^{j+1}, \quad (76)$$

and $\mathbf{b} = \{a_{3,k},\ k=1,\ldots,16\} \cup \left\{b_{j,k}, j\geq 3,\ k=1,\ldots,2^{j+1}\right\}$. Then

$$\|\mathbf{b}\|_2 \leq \frac{\left\|\hat{\mathbf{b}}\right\|_2}{C_1} < \infty. \quad (77)$$

Since the set (71) is a Riesz basis of $H_0^1(0,1)$ there exist constants C_3 and C_4 such that

$$C_3\|\mathbf{b}\|_2 \leq \left\|\sum_{k=1}^{16} a_{3,k}2^{-3}\phi_{3,k} + \sum_{j=3}^{\infty}\sum_{k=1}^{2^{j+1}} b_{j,k}2^{-j}\psi_{j,k}\right\|_{H_0^1(0,1)} \leq C_4\|\mathbf{b}\|_2\,. \quad (78)$$

Therefore

$$\begin{aligned}\frac{C_4}{C_1}\left\|\hat{\mathbf{b}}\right\|_2 &\geq C_4\|\mathbf{b}\|_2 \geq \left\|\sum_{k=1}^{16} a_{3,k}2^{-3}\phi_{3,k} + \sum_{j=3}^{\infty}\sum_{k=1}^{2^{j+1}} b_{j,k}2^{-j}\psi_{j,k}\right\|_{H_0^1(0,1)} \\ &= \left\|\sum_{k=1}^{16}\frac{\hat{a}_{3,k}}{\|\phi_{3,k}\|_{H_0^1(0,1)}}\phi_{3,k} + \sum_{j=3}^{\infty}\sum_{k=1}^{2^{j+1}}\frac{\hat{b}_{j,k}}{\left\|\psi_{j,k}\right\|_{H_0^1(0,1)}}\psi_{j,k}\right\|_{H_0^1(0,1)}\end{aligned} \quad (79)$$

and similarly

$$\frac{C_3}{C_2}\left\|\hat{\mathbf{b}}\right\|_2 \leq \left\|\sum_{k=1}^{16}\frac{\hat{a}_{3,k}}{\|\phi_{3,k}\|_{H_0^1(0,1)}}\phi_{3,k} + \sum_{j=3}^{\infty}\sum_{k=1}^{2^{j+1}}\frac{\hat{b}_{j,k}}{\left\|\psi_{j,k}\right\|_{H_0^1(0,1)}}\psi_{j,k}\right\|_{H_0^1(0,1)}. \quad (80)$$

□

The condition number of the resulting wavelet basis with wavelets up to the level 10 with respect to the L^2-norm is 17.2 and the condition number of this basis normalized with respect to the H^1-norm is 6.0. The sparsity patterns of the matrix arising from a discretization using a wavelet basis constructed in this paper and a wavelet basis from [23] for the one-dimensional Black-Scholes equation with quadratic volatilities from Example 1 is displayed in Figure 4.

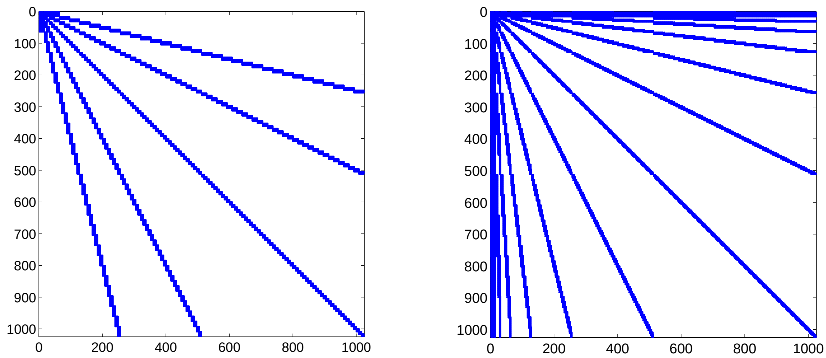

Figure 4. The sparsity pattern of the matrices arising from a discretization using a wavelet basis constructed in this paper (left) and a wavelet basis from [23] (right) for the Black-Scholes equation with quadratic volatilities.

5. Wavelets on the hypercube

We present a well-known construction of a multivariate wavelet basis on the unit hypercube $\Omega = (0,1)^d$, for more details see e.g. [38]. It is based on tensorizing univariate wavelet bases and preserves Riesz basis property, locality of wavelets, vanishing moments and polynomial exactness. This approach is known as an anisotropic approach.

For notational simplicity, we denote $\mathcal{J}_j = \{1, \ldots, 2^{j+1}\}$ for $j \geq 3$, and

$$\psi_{2,k} := \phi_{3,k}, \quad k \in \mathcal{J}_2 := \mathcal{J}_3, \quad \mathcal{J} := \left\{(j,k)\,,\ j \geq 2,\ \in \mathcal{J}_j\right\}. \tag{81}$$

Then we can write

$$\Psi = \left\{\psi_{j,k},\ j \geq 2,\ k \in \mathcal{J}_j\right\} = \{\psi_\lambda,\ \lambda \in \mathcal{J}\}. \tag{82}$$

We use $u \otimes v$ to denote the tensor product of functions u and v, i.e. $(u \otimes v)(x_1, x_2) = u(x_1)\, v(x_2)$. We define multivariate basis functions as:

$$\psi_{\boldsymbol{\lambda}} = \otimes_{i=1}^{d} \psi_{\lambda_i}, \quad \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbf{J}, \quad \mathbf{J} = \mathcal{J}^d = \mathcal{J} \otimes \ldots \otimes \mathcal{J}. \tag{83}$$

Since Ψ is a Riesz basis of $L^2(0,1)$ and Ψ normalized with respect to H^1-norm is a Riesz basis of $H_0^1(0,1)$, the set

$$\Psi^{ani} := \{\psi_{\boldsymbol{\lambda}}, \boldsymbol{\lambda} \in \mathbf{J}\} \tag{84}$$

is a Riesz basis of $L^2(\Omega)$ and its normalization with respect to the H^1-norm is a Riesz basis of $H_0^1(\Omega)$. Using the same argument as in the Proof of Lemma 2 we conclude that for this basis the discretization matrix is sparse for the equation (1) with piecewise polynomial coefficients on uniform meshes such that $\deg p_{k,l} \leq 6$, $\deg q_k \leq 5$, and $\deg a_0 \leq 4$.

6. Numerical examples

In this section, we solve the elliptic equation (1) and the equation with the Black-Scholes operator from Example 1 by an adaptive wavelet method with the basis constructed in this paper. We briefly describe the algorithm. While the classical adaptive methods typically uses refining a mesh according to a-posteriori local error estimates, the wavelet approach is different and it comprises the following steps [13,14,17]:

1. One starts with a variational formulation for a suitable wavelet basis but instead of turning to a finite dimensional approximation, the continuous problem is transformed into an infinite-dimensional l^2-problem.
2. Then one proposes a convergent iteration for the l^2-problem.
3. Finally, one derives an implementable version of this idealized iteration, where all infinite-dimensional quantities are replaced by finitely supported ones.

To the left-hand side of the equation (1) we associate the following bilinear form

$$a(v,w) := \int_\Omega \sum_{k,l=1}^{d} p_{k,l} \frac{\partial v}{\partial x_k} \frac{\partial w}{\partial x_l} + \sum_{k=1}^{d} q_k \frac{\partial v}{\partial x_k} w + p_0 \, v \, w \, \mathrm{d}x. \tag{85}$$

The weak formulation of (1) reads as folows: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \langle f, v \rangle \quad \text{for all } v \in H_0^1(\Omega). \tag{86}$$

Instead of turning to a finite dimensional approximation, the equation (86) is reformulated as an equivalent biinfinite matrix equation $\mathbf{Au} = \mathbf{f}$, where

$$(\mathbf{A})_{\lambda,\mu} = a(\psi_\lambda, \psi_\mu), \quad (\mathbf{f})_\lambda = \langle f, \psi_\lambda \rangle, \tag{87}$$

for $\psi_\lambda, \psi_\mu \in \Psi$, and Ψ is a wavelet basis of $H_0^1(\Omega)$.

We use the standard Jacobi diagonal preconditioner $\mathbf{D}$ for preconditioning this equation, i.e. $\mathbf{D}_{\lambda,\mu} = \mathbf{D}_{\lambda,\mu}\delta_{\lambda,\mu}$. If the coefficients are constant one can also use an efficient diagonal preconditioner from [12]. The algorithm for solving the l^2-problem is the following:

1. Compute sparse representation $\mathbf{f}_j$ of the right-hand side $\mathbf{f}$ such that $\|\mathbf{f} - \mathbf{f}_j\|_2$ is smaller than a given tolerance ϵ_j^1. The computation of a sparse representation insists in thresholding the smallest coefficients and working only with the largest ones. We denote the routine as $\mathbf{f}_j := \mathbf{RHS}[\mathbf{f}, \epsilon_j^1]$.
2. Compute K steps of GMRES for solving the system $\mathbf{Av} = \mathbf{f}_j$ with the initial vector $\mathbf{v}_j$. Each iteration of GMRES requires multiplication of the infinite-dimensional matrix with a finitely supported vector. Since for the wavelet basis constructed in this paper, the matrix is sparse, it can be computed exactly. Otherwise, it is computed approximately with the given tolerance ϵ_j^2 by the method from [6]. We denote the routine $\mathbf{z} = \mathbf{GMRES}[\mathbf{A}, \mathbf{f}_j, \mathbf{v}_j, K]$.
3. Compute sparse representation $\mathbf{v}_{j+1}$ of $\mathbf{z}$ with the error smaller than ϵ_j^2. We denote the routine $\mathbf{v}_{j+1} := \mathbf{COARSE}[\mathbf{z}, \epsilon_j^2]$. It insists in thresholding the coefficients.

We repeat the steps 1., 2., and 3. until the norm of the residual $\mathbf{r}_j = \|\mathbf{f} - \mathbf{Av}_j\|_2$ is not smaller than the required tolerance $\tilde{\epsilon}$. Since we work with the sparse representation of the right-hand side and the sparse representation of the vector representing the solution, the method is adaptive. It is known that the coefficients in the wavelet basis are small in regions where the function is smooth and large in regions where the function has some singularity. Therefore, by this method the singularities are automatically detected.

We use the following algorithm that is modified version of the original algorithm from [13,14]:

Algorithm 7. u: =**SOLVE** [**A**, **f**, $\tilde{\epsilon}$]

1. Choose $k_0, k_1, k_2 \in (0,1)$, $K \in \mathbb{N}$.
2. Set $j := 0$, $\mathbf{v}_0 := 0$ and $\epsilon := \|\mathbf{f}\|_2$.
3. While $\epsilon > \tilde{\epsilon}$

    $j := j + 1,$
    $\epsilon := k_0 \epsilon,$
    $\epsilon_j^1 := k_1 \epsilon,$
    $\epsilon_j^2 := k_2 \epsilon,$
    $\mathbf{f}_j := \mathbf{RHS}[\mathbf{f}, \epsilon_j^1],$
    $\mathbf{z} := \mathbf{GMRES}[\mathbf{A}, \mathbf{f}_j, \mathbf{v}_{j-1}, K]$
    $\mathbf{v}_j := \mathbf{COARSE}[\mathbf{z}, \epsilon_j^2],$
    Estimate $\mathbf{r}_j = \mathbf{f} - \mathbf{A}\mathbf{v}_j$ and set $\epsilon := \|\mathbf{r}_j\|_2$.

end while,
4. $\mathbf{u} := \mathbf{v}_j$,
5. Compute approximate solution $\tilde{u} = \sum_{u_\lambda \in \mathbf{u}} u_\lambda \psi_\lambda$.

For an appropriate choice of parameters k_0, k_1, k_2 and K and more details about the routines **RHS** and **COARSE** we refer to [13,14,38].

Example 2. We solve the equation

$$-\epsilon u'' + x^2 u' + u = f \quad \text{on } (0,1), \quad u(0) = u(1) = 0, \tag{88}$$

where $\epsilon = 0.001$ and the right-hand side f is corresponding to the solution

$$u(x) = x\left(1 - e^{50x-50}\right) \quad \text{for } x \in [0,1]. \tag{89}$$

We solve this equation using the adaptive wavelet method described above with the wavelet basis constructed in this paper. The approximate solution and the derivative of the approximate solution that were computed using only 79 coefficients are displayed in Figure 5. The significant coefficients were located near the point 1, because the solution has a large derivative near this point.

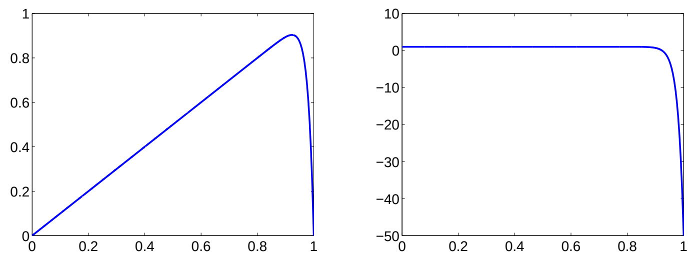

Figure 5. The approximate solution (left) and the derivative of the approximate solution (right) for Example 1.

The sparsity patterns of the matrices arising from discretization of equation (88) using wavelets constructed in this paper and wavelets from [23] are the same as the sparsity patterns of matrices for Example 1 that are displayed in Figure 4. Convergence history is displayed in Figure 6. The number of iterations equals the parameter j from Algorithm 7, the number of basis functions determining the approximate solution in j-th iteration is the same as the number of nonzero entries of the vector $\mathbf{v}_j$ and the L^∞-norm of the error is given by

$$\|u - \tilde{u}\|_\infty = \max_{x\in[0,1]} |u(x) - \tilde{u}(x)| . \tag{90}$$

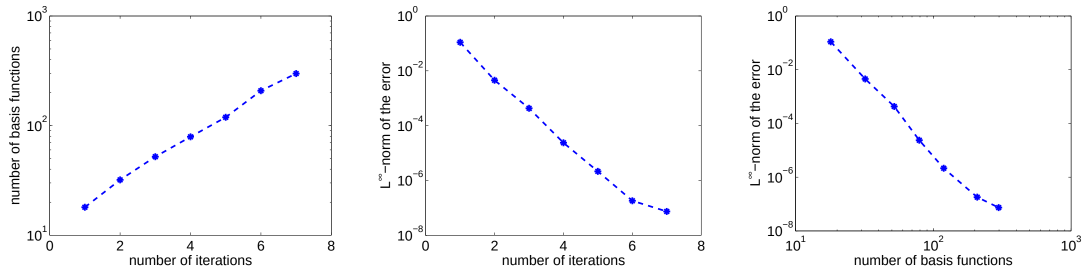

Figure 6. Convergence history for Example 1. The number of basis functions and the L^∞-norm of the error are in logarithmic scaling.

Example 3. We consider the equation

$$\frac{\partial V}{\partial t} - \sum_{k,l=1}^{2} \frac{\rho_{k,l}}{2}\sigma_k\sigma_l S_k S_l \frac{\partial^2 V}{\partial S_k \partial S_l} - r\sum_{k=1}^{2} S_k \frac{\partial V}{\partial S_k} + rV = f, \tag{91}$$

for $(S_1, S_2) \in \Omega := (0,1)^2$ and $t \in (0,1)$. We choose parameters of the Black-Scholes operator as $\rho_{1,1} = \rho_{2,2} = 1$, $\rho_{1,2} = \rho_{2,1} = 0.88$, $\sigma_1(x) = 0.1x^2 - 0.1x + 0.66$, $\sigma_2(x) = 0.1x^2 - 0.1x + 0.97$, $r = 0.02$, and we set the right-hand side f, the initial and boundary conditions such that the solution V is given by

$$V(S_1, S_2, t) = e^{-rt} S_1 S_2 \left(1 - e^{20S_1 - 20}\right)\left(1 - e^{20S_2 - 20}\right) \tag{92}$$

for $(S_1, S_2, t) \in \Omega \times (0,1)$. We use the Crank-Nicolson scheme for the semidiscretization of the equation (91) in time. Let $M \in \mathbb{N}$, $\tau = M^{-1}$, $t_l = l\tau$, $l = 0, \ldots, M$, and denote $V_l(S_1, S_2) = V(S_1, S_2, t_l)$ and $f_l(S_1, S_2) = f(S_1, S_2, t_l)$. The Crank-Nicolson scheme has the form:

$$\frac{V_{l+1} - V_l}{\tau} - \sum_{k,l=1}^{2} \frac{\rho_{k,l}}{4}\sigma_k\sigma_l S_k S_l \frac{\partial^2 (V_{l+1} + V_l)}{\partial S_k \partial S_l} - \frac{r}{2}\sum_{k=1}^{2} S_k \frac{\partial (V_{l+1} + V_l)}{\partial S_k} + \frac{r(V_{l+1} + V_l)}{2} = \frac{f_{l+1} + f_l}{2}. \tag{93}$$

In this scheme, the function V_l is known from the equation on the previous time level and the function V_{l+1} is an unknown solution. Thus, for the given time level t_l the equation (93) is of the form (1) and we can use the adaptive wavelet method for solving it. The approximate solution V_1 for $\tau = 1/365$ that was computed using 731 coefficients is displayed in Figure 7.

It can be seen that the gradient of the solution V_1 has largest values near the point $[1,1]$. Therefore the largest wavelet coefficients correspond to wavelets with supports in regions near this point and wavelet coefficients are small for wavelets that are not located in these regions. Thus many wavelet coefficients are ommited and the representation of the solution is sparse. Convergence history is shown in Figure 8.

7. Conclusions

In this paper, we constructed a new cubic spline multiwavelet basis on the unit interval and unit cube. The basis is adapted to homogeneous Dirichlet boundary conditions and wavelets have eight vanishing moments. The main advantage of this basis is that the matrices arising from a discretization

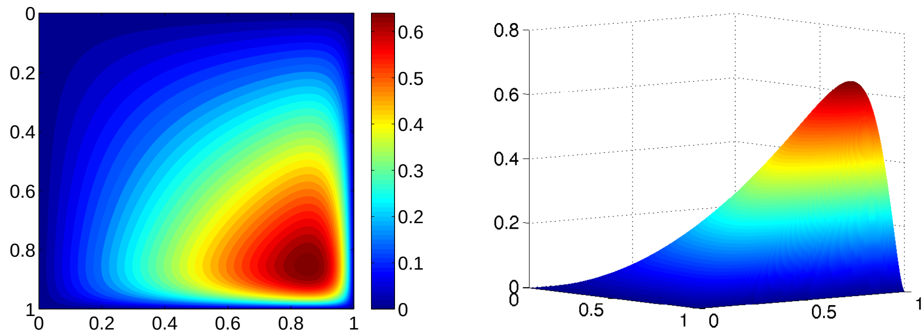

Figure 7. Contour plot (left) and 3D plot (right) of the approximate solution V_1 for Example 2.

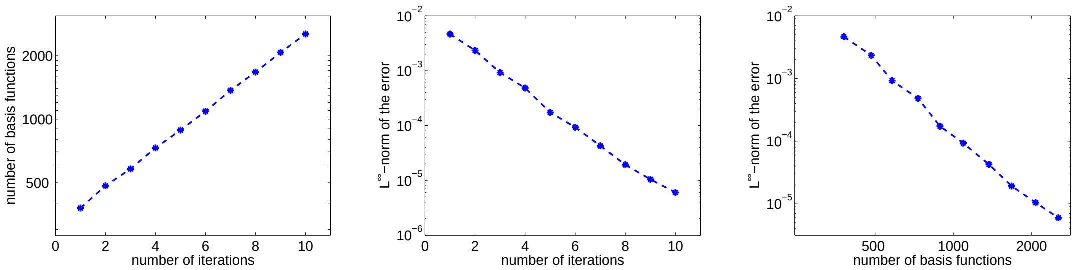

Figure 8. Convergence history for Example 2. The number of basis functions and the L^∞-norm of the error are in logarithmic scaling.

of a differential equation (1) with piecewise polynomial coefficients on uniform meshes such that $\deg p_{k,l} \leq 6$, $\deg q_k \leq 5$, and $\deg a_0 \leq 4$, are sparse and not only quasi-sparse. We proved that the constructed basis is indeed a wavelet basis, i.e. Riesz basis property (5) is satisfied. We performed extensive numerical experiments and present the construction that leads to the wavelet basis that is well-conditioned with respect to the L^2-norm as well as the H^1-norm.

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Author Contributions: The authors contributed equally to this work.

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QUADRATIC SPLINE WAVELETS WITH SHORT SUPPORT SATISFYING HOMOGENEOUS BOUNDARY CONDITIONS *

DANA ČERNÁ † AND VÁCLAV FINĚK ‡

Abstract. In the paper, we construct a new quadratic spline-wavelet basis on the interval and on a unit square satisfying homogeneous Dirichlet boundary conditions of the first order. The wavelets have one vanishing moment and the shortest support among quadratic spline wavelets with at least one vanishing moments adapted to the same type of boundary conditions. The stiffness matrices arising from a discretization of the second-order elliptic problems using the constructed wavelet basis have uniformly bounded condition numbers and the condition numbers are small. We present some quantitative properties of the constructed basis. We provide numerical examples to show that the Galerkin method and the adaptive wavelet method using our wavelet basis require smaller number of iterations than these methods with other quadratic spline wavelet bases. Moreover, due to the short support of the wavelets, one iteration requires smaller number of floating point operations.

Key words. wavelet, quadratic spline, homogeneous Dirichlet boundary conditions, condition number, elliptic problem

AMS subject classifications. 46B15, 65N12, 65T60

1. Introduction. Wavelets are a powerful tool in signal analysis, image processing, and engineering applications. They are also used for the numerical solution of various types of equations. Wavelet methods are used especially for preconditioning of systems of linear algebraic equations arising from the discretization of elliptic problems [9], adaptive solving of operator equations [6, 7], solving of certain type of partial differential equations with a dimension independent convergence rate [12], and a sparse representation of operators [2].

The quantitative properties of any wavelet method strongly depend on the used wavelet basis, namely on its condition number, the length of the support of wavelets, the number of vanishing wavelet moments and the smoothness of basis functions. Therefore, a construction of appropriate wavelet bases is an important issue.

In this paper, we construct a quadratic spline wavelet basis on the interval and on the unit square that is well-conditioned and adapted to homogeneous Dirichlet boundary conditions of the first order. The wavelets have one vanishing moment and we show that the support is the shortest among all quadratic spline wavelets with one vanishing moment. The condition numbers of the stiffness matrices arising from the discretization of elliptic problems using the constructed basis are uniformly bounded and small. Let $\Omega_d = (0,1)^d$, $d = 1, 2$. The wavelet basis of the space $H_0^1(\Omega_2)$ is then obtained by an isotropic tensor product. More precisely, our aim is to propose a wavelet basis on Ω_d that satisfies the following properties:

- *Riesz basis property.* We construct Riesz bases of the space $H_0^1(\Omega_d)$.
- *Locality.* The primal basis functions are local in the sense of Definition 2.1.
- *Vanishing moments.* The wavelets have one vanishing moment.
- *Polynomial exactness.* Since the scaling basis functions are quadratic B-splines, the primal multiresolution analysis has polynomial exactness of order three.
- *Short support.* The wavelets have the shortest possible support among quadratic spline wavelets with one vanishing moment.
- *Closed form.* The primal scaling functions and wavelets have an explicit expression.
- *Homogeneous Dirichlet boundary conditions.* The wavelet basis satisfies homogeneous Dirichlet boundary conditions of the first order.
- *Well-conditioned bases.* The wavelet basis is well-conditioned with respect to the $H^1(\Omega_d)$-seminorm.

In [8, 10], a construction of a spline-wavelet biorthogonal wavelet basis on the interval was proposed. Both the primal and dual wavelets are local. A disadvantage of these bases was their relatively large condition number. Therefore many modifications of this construction were proposed [1, 3, 4, 15]. The construction in [20] outperforms the previous constructions for the linear and quadratic spline-wavelet bases with respect to conditioning of the wavelet bases. In [22, 23, 11] the construction was significantly improved also for cubic spline wavelet basis.

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†Department of Mathematics and Didactics of Mathematics, Technical University in Liberec, Studentská 2, 461 17 Liberec, Czech Republic. (dana.cerna@tul.cz)

‡Department of Mathematics and Didactics of Mathematics, Technical University in Liberec, Studentská 2, 461 17 Liberec, Czech Republic. (vaclav.finek@tul.cz)

Spline wavelet bases with nonlocal duals were also constructed and adapted to various types of boundary conditions [26, 25, 27, 5, 13, 18, 17, 19]. The main advantage of these types of bases in comparison to bases with local duals are usually the shorter support of wavelets, the lower condition number of the basis and the corresponding stiffness matrices and the simplicity of the construction. The cubic spline multiwavelet basis from [13] has an additional advantage that the discretization of the second order elliptic equations with constant coefficients leads to truly sparse matrices, i.e., the number of all nonzero entries in any row is bounded by some constant c independent on the matrix size, whereas the discretization matrices for other wavelet bases have typically $\mathcal{O}(N \log N)$ nonzero entries, where $N \times N$ is the matrix size. It enables to simplify and improve an adaptive wavelet method, because a routine called APPLY for the multiplication of the discretization matrix with a vector can be avoided.

The constructed basis can be used in many applications, e.g., the wavelet Galerkin method and an adaptive wavelet method for solving second order elliptic equations, parabolic equations and partial integro-differential equations on tensor product domains and domains that are images of tensor product domains under continuous mapping. These problems arise for example in financial mathematics for valuation of options under the Black-Scholes model, stochastic volatility models and Lévy model, see [16]. Wavelet methods seem to be superior to classical methods especially for solution of partial integro-differential equations, because they enable to represent the integral term by sparse or almost sparse matrices while the classical methods typically lead to the full matrices. Due to the short support and a small condition number the constructed basis can lead to improved efficiency of these methods.

Wavelet bases of the same type as the basis in this paper are the bases from [22, 11, 20]. The constructions in [22, 11, 20] are based on the constructions of boundary dual scaling functions that are linear combinations of restrictions of dual functions on the real line to $[0, 1]$ such that the boundary dual scaling functions preserve the polynomial exactness. Then boundary wavelets are constructed by the method of stable completion. In this paper the construction is much simpler, because we construct boundary wavelets directly without using dual scaling functions. The constructions from [22] and [20] lead to the same basis in the case of quadratic spline wavelet bases adapted to homogeneous Dirichlet boundary conditions of the first order. Therefore in Section 5 we compare our basis with bases from [11, 20]. Furthermore, we adapt bases from [3, 5] to homogeneous boundary conditions and compare the resulting bases with our basis.

2. Construction of quadratic-spline wavelets.

In this section we propose a construction of a new quadratic spline wavelet basis on the unit interval and on the unit square. The proposed wavelets have one vanishing moment and we show that their support is the smallest possible. First, we briefly review a definition of a wavelet basis, for more details about wavelet bases see [21]. Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|\cdot\|_H$. Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the L^2-inner product and the L^2-norm, respectively. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale*. We define

$$\|\mathbf{v}\|_2 := \sqrt{\sum_{\lambda \in \mathcal{J}} v_\lambda^2}, \quad \text{for} \quad \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R},$$

and

$$l^2(\mathcal{J}) := \left\{ \mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, \, v_\lambda \in \mathbb{R}, \|\mathbf{v}\|_2 < \infty \right\}.$$

Our aim is to construct a wavelet basis of H in the sense of the following definition.

DEFINITION 2.1. *A family* $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ *is called a wavelet basis of* H, *if*

i) Ψ *is a Riesz basis for* H, *i.e., the closure of the span of* Ψ *is* H *and there exist constants* $c, C \in (0, \infty)$ *such that*

$$c \|\mathbf{b}\|_2 \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \leq C \|\mathbf{b}\|_2, \tag{2.1}$$

for all $\mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J})$.

ii) *The functions are local in the sense that diam* $\operatorname{supp} \psi_\lambda \leq C2^{-|\lambda|}$ *for all* $\lambda \in \mathcal{J}$, *and at a given level* j *the supports of only finitely many wavelets overlap at any point* x.

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For the two countable sets of functions $\Gamma, \Theta \subset H$, the symbol $\langle \Gamma, \Theta \rangle_H$ denotes the matrix

$$\langle \Gamma, \Theta \rangle_H := \{ \langle \gamma, \theta \rangle_H \}_{\gamma \in \Gamma, \theta \in \Theta} .$$

REMARK 2.2. The constants

$$c_\Psi := \sup \{ c : c \text{ satisfies } (2.1) \} \quad \text{and} \quad C_\Psi := \inf \{ C : C \text{ satisfies } (2.1) \}$$

are called *Riesz bounds* and the number $cond\, \Psi = C_\Psi / c_\Psi$ is called the *condition number* of Ψ. It is known that the constants c_Ψ and C_Ψ satisfy:

$$c_\Psi = \sqrt{\lambda_{min} \left(\langle \Psi, \Psi \rangle_H \right)}, \quad C_\Psi = \sqrt{\lambda_{max} \left(\langle \Psi, \Psi \rangle_H \right)},$$

where $\lambda_{min} \left(\langle \Psi, \Psi \rangle_H \right)$ and $\lambda_{max} \left(\langle \Psi, \Psi \rangle_H \right)$ are the smallest and the largest eigenvalues of the matrix $\langle \Psi, \Psi \rangle_H$, respectively.

We define a scaling basis as a basis of quadratic B-splines in the same way as in [22, 5, 20]. Let ϕ be a quadratic B-spline defined on knots $[0, 1, 2, 3]$. It can be written explicitly as:

$$\phi(x) = \begin{cases} \frac{x^2}{2}, & x \in [0,1], \\ -x^2 + 3x - \frac{3}{2}, & x \in [1,2], \\ \frac{x^2}{2} - 3x + \frac{9}{2}, & x \in [2,3], \\ 0, & \text{otherwise}. \end{cases} \tag{2.2}$$

The function ϕ satisfies a scaling equation [5]

$$\phi(x) = \frac{\phi(2x)}{4} + \frac{3\phi(2x-1)}{4} + \frac{3\phi(2x-2)}{4} + \frac{\phi(2x-3)}{4}. \tag{2.3}$$

Let ϕ_b be a multiple of the quadratic B-spline defined on knots $[0, 0, 1, 2]$ such that $\|\phi_b\|_{L^1} = \|\phi\|_{L^1}$, i.e.

$$\phi_b(x) = \begin{cases} -\frac{9x^2}{4} + 3x, & x \in [0,1], \\ \frac{3x^2}{4} - 3x + 3, & x \in [1,2], \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

The function ϕ_b satisfies a scaling equation [5]

$$\phi_b(x) = \frac{\phi_b(2x)}{2} + \frac{9\phi(2x)}{8} + \frac{3\phi(2x-1)}{8}. \tag{2.5}$$

The graphs of the functions ϕ_b and ϕ are displayed in Figure 2.1.

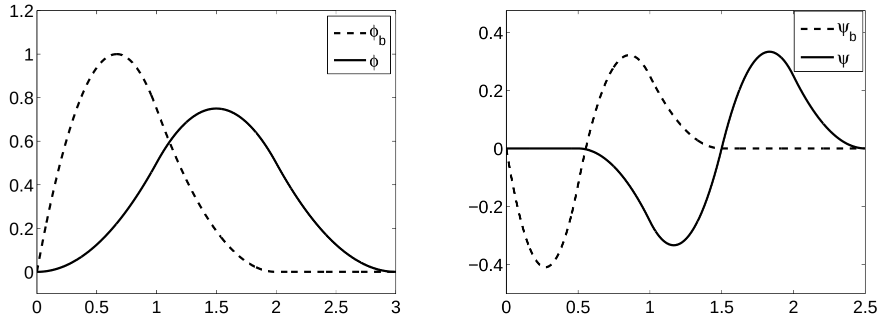

FIG. 2.1. *The scaling functions ϕ and ϕ_b and the wavelets ψ and ψ_b.*

For $j \geq 2$ and $x \in [0, 1]$ we set

$$\begin{aligned} \phi_{j,k}(x) &= 2^{j/2} \phi(2^j x - k + 2), \quad k = 2, ..., 2^j - 1, \\ \phi_{j,1}(x) &= 2^{j/2} \phi_b(2^j x), \quad \phi_{j,2^j}(x) = 2^{j/2} \phi_b(2^j (1 - x)). \end{aligned} \tag{2.6}$$

We define a wavelet ψ and a boundary wavelet ψ_b as

$$\psi(x) = -\frac{1}{2}\phi(2x-1) + \frac{1}{2}\phi(2x-2) \quad \text{and} \quad \psi_b(x) = \frac{-\phi_b(2x)}{2} + \frac{\phi(2x)}{2}. \tag{2.7}$$

Due to the normalization of ϕ_b, the coefficients in these two equations are the same which will simplify the proofs in the next section. Then $\operatorname{supp}\psi = [0.5, 2.5]$, $\operatorname{supp}\psi_b = [0, 1.5]$, and both wavelets have one vanishing moment, i.e.

$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \psi_b(x)dx = 0. \tag{2.8}$$

The graphs of the wavelet ψ and the boundary wavelet ψ_b are displayed in Figure 2.1. In the following lemma we show that the support of the wavelet ψ is the shortest among all quadratic spline wavelets with one vanishing moment.

LEMMA 1. *Let ϕ be defined by (2.2). If $\psi \in \overline{\operatorname{span}}\{\phi(2\cdot -k), k \in \mathbb{Z}\}$ and ψ satisfies (2.8), then the length of the support of ψ is at least 2.*

Proof. Since $\psi \in \overline{\operatorname{span}}\{\phi(2\cdot -k), k \in \mathbb{Z}\}$ we have

$$\psi(x) = \sum_{k\in\mathbb{Z}} a_k \phi(2x-k),$$

for some coefficients $a_k \in \mathbb{R}$. Let us suppose that the length of the support of ψ is at most 2. Then $\operatorname{supp}\psi \subset [j/2, (j+4)/2]$ for some $j \in \mathbb{Z}$. Since $\psi(x) = 0$ for all $x \in [k/2, (k+1)/2]$, where $k \in \mathbb{Z}\backslash\{j, j+1, j+2, j+3\}$, the coefficients $a_k = 0$ for all $k \in \mathbb{Z}\backslash\{j, j+1\}$. Due to (2.8) we have $a_j + a_{j+1} = 0$. Thus up to a multiplication by a constant and shifting by $k/2$, $k \in \mathbb{Z}$, there is only one wavelet that has the length of support at most 2 and this wavelet is a wavelet defined by (2.7). □

Using the similar argument as in the proof of Lemma 1 it is easy to see that also the boundary wavelet ψ_b has the shortest possible support among all boundary wavelets with one vanishing moment corresponding to scaling functions defined by (2.6).

For $j \geq 2$ and $x \in [0,1]$ we define

$$\begin{aligned} \psi_{j,k}(x) &= 2^{j/2}\psi(2^j x - k + 2), k = 2, ..., 2^j - 1, \\ \psi_{j,1}(x) &= 2^{j/2}\psi_b(2^j x), \quad \psi_{j,2^j}(x) = -2^{j/2}\psi_b(2^j(1-x)). \end{aligned} \tag{2.9}$$

We denote the index sets by

$$\mathcal{I}_j = \left\{k \in \mathbb{Z} : 1 \leq k \leq 2^j\right\}.$$

We define

$$\Phi_j = \{\phi_{j,k}, k \in \mathcal{I}_j\}, \quad \Psi_j = \{\psi_{j,k}, k \in \mathcal{I}_j\},$$

and

$$\Psi = \Phi_2 \cup \bigcup_{j=2}^{\infty} \Psi_j, \quad \Psi^s = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j, \quad j_0 = 2. \tag{2.10}$$

In Section 4 we prove that Ψ, when normalized with respect to the H^1–seminorm, forms a wavelet basis of the Sobolev space $H_0^1(0,1)$.

A basis on $\Omega_d = (0,1)^d$ is built from the univariate wavelet basis by a tensor product [21]. Let $j \geq 2$, $\mathbf{k} = (k_1, \ldots, k_d)$, $\mathbf{k} \in \mathcal{I}_j^d := \mathcal{I}_j \times \ldots \times \mathcal{I}_j$, and $\mathbf{x} = (x_1, \ldots, x_d) \in \Omega_d$. We define the multivariate scaling functions by

$$\phi_{j,\mathbf{k}}^d(\mathbf{x}) = \prod_{l=1}^{d} \phi_{j,k_l}(x_l),$$

and for any $\mathbf{e} = (e_1, \ldots, e_d) \in E^d := \{0,1\}^d \setminus (0, \ldots, 0)$, we define the multivariate wavelet

$$\psi^d_{j,\mathbf{e},\mathbf{k}}(\mathbf{x}) = \prod_{l=1}^{d} \psi_{j,e_l,k_l}(x_l),$$

where

$$\psi_{j,e_l,k_l} = \begin{cases} \phi_{j,k_l}, & e_l = 1, \\ \psi_{j,k_l}, & e_l = 0. \end{cases}$$

The basis on the unit cube Ω_d is then given by

$$\Psi^{dD} = \left\{\phi^d_{2,\mathbf{k}},\ \mathbf{k} \in \mathcal{I}^d_j\right\} \cup \left\{\psi^d_{j,\mathbf{e},\mathbf{k}}, \mathbf{e} \in E^d, \mathbf{k} \in \mathcal{I}^d_j, j \geq 2\right\}.$$

This approach is called an isotropic approach. It preserves the regularity and polynomial exactness. Another approach is an anisotropic approach. The anisotropic basis on the unit square is $\Psi \otimes \Psi$.

3. Refinement matrices. In this section we present refinement matrices $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ corresponding to primal scaling functions and wavelets. We show that the matrix $\mathbf{M}_j = (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$ is invertible and thus there exist matrices $\tilde{\mathbf{M}}_{j,0}$ and $\tilde{\mathbf{M}}_{j,1}$ of the same sizes as $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$, respectively, such that

$$\left(\tilde{\mathbf{M}}_{j,0}, \tilde{\mathbf{M}}_{j,1}\right) = \mathbf{M}_j^{-1}. \tag{3.1}$$

We derive an explicit form of the matrix $\tilde{\mathbf{M}}_{j,0}$ and an estimate for the norm of the product $\tilde{\mathbf{M}}^T_{m,0}\tilde{\mathbf{M}}^T_{m+1,0} \ldots \tilde{\mathbf{M}}^T_{n,0}$, because this estimate is crucial for the proof of the Riesz basis property that will be presented in Section 4.

By (2.3), (2.5), (2.6), (2.7) and (2.9), there exist *refinement matrices* $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that

$$\Phi_j = \mathbf{M}^T_{j,0}\Phi_{j+1}, \quad \Psi_j = \mathbf{M}^T_{j,1}\Phi_{j+1}. \tag{3.2}$$

In these formulas we view the sets Φ_j and Ψ_j as column vectors with entries $\phi_{j,k}$ and $\psi_{j,k}$, $k \in \mathcal{I}_j$, respectively.

By the Riesz representation theorem there exist dual functions $\tilde{\phi}_{j,k}$ and $\tilde{\psi}_{j,k}$ such that

$$\left\langle \phi_{j,k}, \tilde{\phi}_{j,m} \right\rangle = \delta_{k,m}, \left\langle \phi_{j,k}, \tilde{\psi}_{l,m} \right\rangle = 0, \left\langle \psi_{l,m}, \tilde{\phi}_{j,k} \right\rangle = 0, \left\langle \psi_{j,k}, \tilde{\psi}_{l,m} \right\rangle = \delta_{j,k}\delta_{k,m},$$

for all $j, l \geq 2$, $l \geq j$, $k \in \mathcal{I}_j$, $m \in \mathcal{I}_l$. Let us denote

$$\tilde{\Phi}_j = \left\{\tilde{\phi}_{j,k}, k \in \mathcal{I}_j\right\}, \quad \tilde{\Psi}_j = \left\{\tilde{\psi}_{j,k}, k \in \mathcal{I}_j\right\}$$

and view these sets as column vectors. Then $\tilde{\Phi}_j, \tilde{\Psi}_j \subset \operatorname{span} \tilde{\Phi}_{j+1}$ and the matrices $\tilde{\mathbf{M}}_{j,0}$ and $\tilde{\mathbf{M}}_{j,1}$ defined by (3.1) are the refinement matrices for the dual system, i.e.,

$$\tilde{\Phi}_j = \tilde{\mathbf{M}}_{j,0}\tilde{\Phi}_{j+1}, \quad \tilde{\Psi}_j = \tilde{\mathbf{M}}_{j,1}\tilde{\Phi}_{j+1}.$$

Due to Remark 2.2, the Riesz bounds for the multiscale systems are related to the spectral norms of refinement matrices and products of these matrices.

Due to (2.3) and (2.5), the refinement matrix $\mathbf{M}_{j,0}$ has the following structure:

$$\mathbf{M}_{j,0} = \begin{bmatrix} \mathbf{M}_L & & \\ & \mathbf{M}^I_{j,0} & \\ & & \mathbf{M}_R \end{bmatrix}.$$

where $\mathbf{M}^I_{j,0}$ is a $2^{j+1} \times 2^j$ matrix given by

$$\left(\mathbf{M}^I_{j,0}\right)_{m,n} = \begin{cases} \frac{h_{m+2-2n}}{\sqrt{2}}, & n = 1, \ldots, 2^j,\ 1 \leq m + 2 - 2n \leq 4, \\ 0, & \text{otherwise}, \end{cases}$$

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where

$$\mathbf{h} = [h_1, h_2, h_3, h_4] = \left[\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right]$$

is a vector of coefficients from the scaling equation (2.3). The matrix $\mathbf{M}_L$ is given by

$$\mathbf{M}_L = \frac{1}{\sqrt{2}}\mathbf{h}_b^T, \quad \text{where} \quad \mathbf{h}_b = [h_1^b, h_2^b, h_3^b] = \left[\frac{1}{2}, \frac{9}{8}, \frac{3}{8}\right]$$

is a vector of coefficients from the scaling equation (2.5). The matrix $\mathbf{M}_R$ is obtained from a matrix $\mathbf{M}_L$ by reversing the ordering of rows.

It follows from (2.7) that the matrix $\mathbf{M}_{j,1}$ is of the size $2^{j+1} \times 2^j$ and has the structure

$$\mathbf{M}_{j,1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & & & 0 & 0 \\ \vdots & \vdots & & & & & & & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^T . \tag{3.3}$$

The following lemmas are crucial for the proof of a Riesz basis property.

LEMMA 2. *Let $j \geq 2$ and the entries $\tilde{M}_{k,l}^{j,0}$, $k \in \mathcal{I}_{j+1}$, $l \in \mathcal{I}_j$, of the matrix $\tilde{\mathbf{M}}_{j,0}$ be given by:*

$$\tilde{M}_{2,l}^{j,0} = \tilde{M}_{1,l}^{j,0} = \frac{d_1^j}{a^{|1-l|}} + \frac{d_n^j}{a^{|n-l|}},$$
$$\tilde{M}_{2^{j+1},l}^{j,0} = \tilde{M}_{2^{j+1}-1,l}^{j,0} = \frac{d_1^j}{a^{|n-l|}} + \frac{d_n^j}{a^{|1-l|}},$$

where $n = 2^j$, $a = -3 - 2\sqrt{2}$,

$$d_1^j = \frac{6\alpha_n}{3+\sqrt{2}}, \quad d_n^j = \frac{-36b\,\alpha_n\,a^{2-n}}{11+6\sqrt{2}}, \tag{3.4}$$
$$\alpha_n = \left(1 - \frac{36\,b^2\,a^{4-2n}}{11+6\sqrt{2}}\right)^{-1}, \quad b = \frac{13-9\sqrt{2}}{6},$$

and for $k = 2, \dots, n-1$ and $l \in \mathcal{I}_j$ let

$$\tilde{M}_{2k,l}^{j,0} = \tilde{M}_{2k-1,l}^{j,0} = \frac{1}{a^{|k-l|}} + \frac{d_k^j}{a^{|1-l|}} + \frac{d_{n+1-k}^j}{a^{|n-l|}},$$

where

$$d_k^j = \frac{-6b\,\alpha_n\,a^{2-k}}{3+\sqrt{2}} - \frac{36b\,\alpha_n\,a^{k+3-2n}}{11+6\sqrt{2}}. \tag{3.5}$$

Then

$$\mathbf{M}_{j,0}^T \tilde{\mathbf{M}}_{j,0} = \mathbf{I}_j, \quad \textit{and} \quad \mathbf{M}_{j,1}^T \tilde{\mathbf{M}}_{j,0} = \mathbf{0}_j, \tag{3.6}$$

where $\mathbf{I}_j$ denotes the identity matrix and $\mathbf{0}_j$ denotes the zero matrix of the appropriate size.

Proof. By similar approach as in [26, 25] we derive the explicit form of the entries $\tilde{M}_{k,l}^{j,0}$, $k \in \mathcal{I}_{j+1}$, $l \in \mathcal{I}_j$, of the matrix $\tilde{\mathbf{M}}_{j,0}$ such that (3.6) is satisfied. From (3.3) we obtain

$$\tilde{M}_{2k-1,l} = \tilde{M}_{2k,l}, \quad \text{for } k = 1, \dots, 2^j. \tag{3.7}$$

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We substitute (3.7) into (3.6) and we obtain a new system $\mathbf{A}_j\mathbf{B}_j = \mathbf{I}_j$, where

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{13}{8} & \frac{3}{8} & 0 & \dots & & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & & \dots & 0 & \frac{3}{8} & \frac{13}{8} \end{bmatrix} = \frac{\mathbf{H}_j}{\sqrt{2}} \begin{bmatrix} \frac{13}{12} & \frac{1}{4} & 0 & \dots & & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & & \dots & 0 & \frac{1}{4} & \frac{13}{12} \end{bmatrix},$$

where

$$(\mathbf{H}_j)_{k,l} = \begin{cases} \frac{3}{2}, & (k,l) = (1,1)\,,\ (k,l) = \left(2^j, 2^j\right) \\ 1, & k = l, k \neq 1, k \neq 2^j, \\ 0, & \text{otherwise}, \end{cases}$$

and $\mathbf{B}_j$ is the $2^j \times 2^j$ matrix with entries $B^j_{k,l} = \tilde{M}^{j,0}_{2k,l}$, $k, l \in \mathcal{I}_j$. We factorize the matrix $\mathbf{A}_j$ as $\mathbf{A}_j = \mathbf{H}_j\mathbf{C}_j\mathbf{D}_j$, where

$$\mathbf{C}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{3+2\sqrt{2}}{4} & \frac{1}{4} & 0 & 0 & \dots & 0 \\ \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & & \vdots \\ 0 & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \frac{1}{4} & \frac{3}{2} & \frac{1}{4} \\ 0 & \dots & 0 & 0 & \frac{1}{4} & \frac{3+2\sqrt{2}}{4} \end{bmatrix},$$

and

$$\mathbf{D}_j = \begin{bmatrix} \frac{3+\sqrt{2}}{6} & 0 & 0 & \dots & 0 & 0 & \frac{b}{a^{n-2}} \\ b & 1 & 0 & & 0 & 0 & \frac{b}{a^{n-3}} \\ \frac{b}{a} & 0 & 1 & & 0 & 0 & \frac{b}{a^{n-4}} \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ \frac{b}{a^{n-4}} & 0 & 0 & & 1 & 0 & \frac{b}{a} \\ \frac{b}{a^{n-3}} & 0 & 0 & & 0 & 1 & b \\ \frac{b}{a^{n-2}} & 0 & 0 & \dots & 0 & 0 & \frac{3+\sqrt{2}}{6} \end{bmatrix},$$

More precisely, the entries $D^j_{k,l}$ of the matrix $\mathbf{D}_j$ are given by:

$$\begin{aligned} D^j_{1,1} &= D^j_{n,n} = \frac{3+\sqrt{2}}{6}, \\ D^j_{k,1} &= D^j_{n+1-k,n} = \frac{b}{a^{k-2}}, \quad \text{for } k = 2,\dots,n, \\ D^j_{k,k} &= 1, \quad \text{for } k = 2,\dots,n-1, \\ D^j_{k,l} &= 0, \quad \text{otherwise}. \end{aligned}$$

It is easy to verify that $\tilde{\mathbf{C}}_j = \mathbf{C}_j^{-1}$ has entries $\tilde{C}^j_{k,l} = a^{-|k-l|}$, and the matrix $\mathbf{D}_j^{-1}$ has the structure:

$$\mathbf{D}_j^{-1} = \begin{bmatrix} d^j_1 & 0 & \dots & 0 & d^j_n \\ d^j_2 & 1 & & 0 & d^j_{n-1} \\ \vdots & & \ddots & & \vdots \\ d^j_{n-1} & 0 & & 1 & d^j_2 \\ d^j_n & 0 & \dots & 0 & d^j_1 \end{bmatrix},$$

with d_k^j given by (3.4) and (3.5). Since the matrices $\mathbf{C}_j$, $\mathbf{D}_j$ and $\mathbf{H}_j$ are invertible, we can define

$$\mathbf{B}_j = \mathbf{A}_j^{-1} = \mathbf{D}_j^{-1}\mathbf{C}_j^{-1}\mathbf{H}_j^{-1}. \tag{3.8}$$

Substituting (3.8) into (3.7) the lemma is proved. ∎

LEMMA 3. *There exist unique matrices $\tilde{\mathbf{M}}_{j,1}$, $j \geq 2$, such that*

$$\mathbf{M}_{j,0}^T\tilde{\mathbf{M}}_{j,1} = \mathbf{0}_j, \quad and \quad \mathbf{M}_{j,1}^T\tilde{\mathbf{M}}_{j,1} = \mathbf{I}_j. \tag{3.9}$$

Proof. For $l \in \mathcal{I}_{j+1}$ and $k \in \mathcal{I}_j$ the entries $\tilde{M}_{k,l}^{j,1}$ of the matrix $\tilde{\mathbf{M}}_{j,1}$ satisfy

$$\tilde{M}_{2k-1,l}^{j,1} = 2\delta_{2k-1,2l-1} + \tilde{M}_{2k,l}^{j,1}.$$

Using these relations we obtain a system of equations with the matrix $\mathbf{A}_j$ defined in the proof of Lemma 5. Since the matrix $\mathbf{A}_j$ is invertible, the matrix $\tilde{\mathbf{M}}_{j,1}$ exists and is unique. ∎

LEMMA 4. *We have $\Phi_{j+1} = \tilde{\mathbf{M}}_{j,0}\Phi_j + \tilde{\mathbf{M}}_{j,1}\Psi_j$ for all $j \geq 2$.*

Proof. Due to (3.2) we have

$$\begin{bmatrix}\Phi_j\\ \Psi_j\end{bmatrix} = \begin{bmatrix}\mathbf{M}_{j,0}^T\\ \mathbf{M}_{j,1}^T\end{bmatrix}\Phi_{j+1}, \quad j \geq 2.$$

Multiplying this equation by the matrix $\left[\tilde{\mathbf{M}}_{j,0}, \tilde{\mathbf{M}}_{j,1}\right]$ from the left-hand side and using (3.6) and (3.9) the lemma is proved. ∎

For any matrix $\mathbf{M}$ of the size $m \times n$ we set

$$\|\mathbf{M}\|_2 = \sup_{\mathbf{v}\in\mathbb{R}^n, \mathbf{v}\neq\mathbf{0}} \frac{\|\mathbf{M}\mathbf{v}\|_2}{\|\mathbf{v}\|_2}$$

and

$$\|\mathbf{M}\|_1 = \max_{l=1,\dots,n}\sum_{k=1}^m |M_{k,l}|, \quad \|\mathbf{M}\|_\infty = \max_{k=1,\dots,m}\sum_{l=1}^n |M_{k,l}|.$$

It is well-known that

$$\|\mathbf{M}\|_2 \leq \sqrt{\|\mathbf{M}\|_1\|\mathbf{M}\|_\infty}. \tag{3.10}$$

LEMMA 5. *The matrices $\tilde{\mathbf{M}}_{j,0}$, $j \geq 2$, have uniformly bounded norms, i.e., there exists $C \in \mathbb{R}$ independent of j such that $\left\|\tilde{\mathbf{M}}_{j,0}\right\|_2 \leq C$ for all $j \geq 2$.*

Proof. Since the matrices $\tilde{\mathbf{M}}_{j,0}$ are known in the explicit form, they have a regular structure and the values in each column and row are exponentially decreasing, we compute upper bounds for the 1-norm and ∞-norm such that we compute several largest values in each row and column and estimate the sum of the remaining entries. We obtain

$$\left\|\tilde{\mathbf{M}}_{j,0}\right\|_1 \leq 1.42, \quad \left\|\tilde{\mathbf{M}}_{j,0}\right\|_\infty \leq 2.91,$$

and due to (3.10) we have $\left\|\tilde{\mathbf{M}}_{j,0}\right\|_2 \leq 2.04$. ∎

For comparison we computed the norms of the matrices $\tilde{\mathbf{M}}_{j,0}$ numerically and we found that $\left\|\tilde{\mathbf{M}}_{j,0}\right\|_2 \leq 2$ for $j = 3, \dots, 12$, and $\left\|\tilde{\mathbf{M}}_{12,0}\right\|_2 = 1.9999997$.

LEMMA 6. *Let $\mathbf{S}_j = \tilde{\mathbf{M}}_{j,0}^T\tilde{\mathbf{M}}_{j+1,0}^T$, $j \geq 3$, and $\tilde{\mathbf{S}}_j$ be the matrix given by*

$$\left(\tilde{\mathbf{S}}_j\right)_{k,l} = (\mathbf{S}_j)_{2k-1,l} + (\mathbf{S}_j)_{2k,l}, \quad k \in \mathcal{I}_{j-1}, l \in \mathcal{I}_{j+2}.$$

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Then there exists a constant C independent of j such that $\left\|\tilde{\mathbf{S}}_j\right\|_2 < C < 2\sqrt{2}$.

Proof. Let $\mathbf{K}_j$ be a $2^j \times 2^{j+1}$ matrix with entries

$$(\mathbf{K}_j)_{k,2l-1} = (\mathbf{K}_j)_{k,2l} = a^{-|k-l|}, \quad k,l \in \mathcal{I}_j, \quad a = -3-2\sqrt{2}, \tag{3.11}$$

and let $\mathbf{L}_j = \tilde{\mathbf{M}}_{j,0}^T - \mathbf{K}_j$. We know the explicit expression of the matrix $\mathbf{L}_j$, because the explicit expressions of both $\tilde{\mathbf{M}}_{j,0}$ and $\mathbf{K}_j$ are known. We have

$$\mathbf{S}_j = \tilde{\mathbf{M}}_{j,0}^T \tilde{\mathbf{M}}_{j+1,0}^T = \mathbf{K}_j\mathbf{K}_{j+1} + \mathbf{K}_j\mathbf{L}_{j+1} + \mathbf{L}_j\mathbf{K}_{j+1} + \mathbf{L}_j\mathbf{L}_{j+1}.$$

Let us denote

$$\mathbf{N}_j = \mathbf{K}_j\mathbf{K}_{j+1}, \mathbf{O}_j = \mathbf{K}_j\mathbf{L}_{j+1}, \mathbf{P}_j = \mathbf{L}_j\mathbf{K}_{j+1}, \mathbf{Q}_j = \mathbf{L}_j\mathbf{L}_{j+1},$$

and $\tilde{\mathbf{N}}_j$, $\tilde{\mathbf{O}}_j$, $\tilde{\mathbf{P}}_j$, and $\tilde{\mathbf{Q}}_j$ be derived from $\mathbf{N}_j$, $\mathbf{O}_j$, $\mathbf{P}_j$ and $\mathbf{Q}_j$ by similar way as $\tilde{\mathbf{S}}_j$ from $\mathbf{S}_j$. Then $\tilde{\mathbf{S}}_j = \tilde{\mathbf{N}}_j + \tilde{\mathbf{O}}_j + \tilde{\mathbf{P}}_j + \tilde{\mathbf{Q}}_j$. From (3.11) we have for $k \in \mathcal{I}_j$, $l \in \mathcal{I}_{j+1}$

$$(\mathbf{N}_j)_{k,2l-1} = (\mathbf{N}_j)_{k,2l} = \mathbf{u}_k^T \mathbf{v}_l,$$

where

$$\mathbf{u}_k = \left[\frac{1}{a^{k-1}}, \frac{1}{a^{k-1}}, \frac{1}{a^{k-2}}, \dots, \frac{1}{a}, \frac{1}{a}, 1, 1, \frac{1}{a}, \frac{1}{a}, \dots, \frac{1}{a^{n-k}}, \frac{1}{a^{n-k}}\right]^T,$$
$$\mathbf{v}_l = \left[\frac{1}{a^{l-1}}, \frac{1}{a^{l-2}}, \dots, \frac{1}{a}, 1, \frac{1}{a}, \dots, \frac{1}{a^{2n-l}}\right]^T,$$

$n = 2^j$. Due to the structure of the vector $\mathbf{u}_k$ we can write

$$(\mathbf{N}_j)_{k,l} = \frac{a+1}{a}\,\tilde{\mathbf{u}}_k^T \tilde{\mathbf{v}}_l,$$

where

$$\tilde{\mathbf{u}}_k = \left[\frac{1}{a^{k-1}}, \frac{1}{a^{k-2}}, \dots, \frac{1}{a}, 1, \frac{1}{a}, \dots, \frac{1}{a^{n-k}}\right]^T,$$
$$\tilde{\mathbf{v}}_l = \begin{cases} \left[\frac{1}{a^{l-2}}, \frac{1}{a^{l-4}}, \dots, \frac{1}{a^2}, 1, \frac{1}{a}, \frac{1}{a^3} \dots, \frac{1}{a^{2n-l-1}}\right]^T, & l \text{ even},\\ \left[\frac{1}{a^{l-2}}, \frac{1}{a^{l-4}}, \dots, \frac{1}{a}, 1, \frac{1}{a^2}, \frac{1}{a^4} \dots, \frac{1}{a^{2n-l-1}}\right]^T, & l \text{ odd}.\end{cases}$$

For $k > \frac{l}{2}$, $l \in \mathcal{I}_{j+1}$, l even, we have

$$\begin{aligned}(\mathbf{N}_j)_{k,2l} &= \frac{a+1}{a}\left(\sum_{m=1}^{\frac{l}{2}} a^{3m-k-l} + \sum_{m=\frac{l}{2}+1}^{k} a^{l+1-k-m} + \sum_{m=k+1}^{n} a^{l+k+1-3m}\right)\\ &= \frac{a+1}{a}\left(a^{\frac{l}{2}-k}\frac{1-\left(\frac{1}{a^3}\right)^{\frac{l}{2}}}{1-\frac{1}{a^3}} + a^{\frac{l}{2}-k}\frac{1-\left(\frac{1}{a}\right)^{k-\frac{l}{2}}}{1-\frac{1}{a}} + a^{l-2-2k}\frac{1-\left(\frac{1}{a^3}\right)^{n-k}}{1-\frac{1}{a^3}}\right).\end{aligned}$$

Similarly for $k > \frac{l-1}{2}$, $l \in \mathcal{I}_{j+1}$, l odd, we obtain

$$\begin{aligned}(\mathbf{N}_j)_{k,2l} &= \frac{a+1}{a}\left(\sum_{m=1}^{\frac{l-1}{2}} a^{3m-k-l} + \sum_{m=\frac{l+1}{2}}^{k} a^{l+1-k-m} + \sum_{m=k+1}^{n} a^{l+k+1-3m}\right)\\ &= \frac{a+1}{a}\left(a^{\frac{l}{2}-k-\frac{3}{2}}\frac{1-\left(\frac{1}{a^3}\right)^{\frac{l-1}{2}}}{1-\frac{1}{a^3}} + a^{\frac{l+1}{2}-k}\frac{1-\left(\frac{1}{a^3}\right)^{k-\frac{l-1}{2}}}{1-\frac{1}{a}} + a^{l-2-2k}\frac{1-\left(\frac{1}{a^3}\right)^{n-k}}{1-\frac{1}{a^3}}\right).\end{aligned}$$

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If $k \leq \frac{l}{2}$, $l \in \mathcal{I}_{j+1}$, l even, then we have

$$\begin{aligned}(\mathbf{N}_j)_{k,2l} &= \frac{a+1}{a}\left(\sum_{m=1}^{k} a^{3m-k-l} + \sum_{k+1}^{\frac{l}{2}} a^{m+k-l} + \sum_{m=\frac{l}{2}+1}^{n} a^{l+k+1-3m}\right) \\ &= \frac{a+1}{a}\left(a^{2k-l}\frac{1-\left(\frac{1}{a^3}\right)^k}{1-\frac{1}{a^3}} + a^{k-\frac{l}{2}}\frac{1-\left(\frac{1}{a}\right)^{\frac{l}{2}-k}}{1-\frac{1}{a}} + a^{k-\frac{l}{2}-2}\frac{1-\left(\frac{1}{a^3}\right)^{n-\frac{l}{2}}}{1-\frac{1}{a^3}}\right).\end{aligned}$$

If $k \leq \frac{l-1}{2}$, $l \in \mathcal{I}_{j+1}$, l odd, then we have

$$\begin{aligned}(\mathbf{N}_j)_{k,2l} &= \frac{a+1}{a}\left(\sum_{m=1}^{k} a^{3m-k-l} + \sum_{k+1}^{\frac{l-1}{2}} a^{m+k+2-l} + \sum_{m=\frac{l+1}{2}+1}^{n} a^{l+k+1-3m}\right) \\ &= \frac{a+1}{a}\left(a^{2k-l}\frac{1-\left(\frac{1}{a^3}\right)^k}{1-\frac{1}{a^3}} + a^{k-\frac{l}{2}-\frac{1}{2}}\frac{1-\left(\frac{1}{a}\right)^{\frac{l-1}{2}-k}}{1-\frac{1}{a}} + a^{k-\frac{l}{2}-\frac{1}{2}}\frac{1-\left(\frac{1}{a^3}\right)^{n-\frac{l-1}{2}}}{1-\frac{1}{a^3}}\right).\end{aligned}$$

To compute an upper bound for the norm of the matrix $\tilde{\mathbf{S}}_j$, we compute bounds for the sums of the absolute values of the entries in rows and columns for matrices $\tilde{\mathbf{N}}_j$, $\tilde{\mathbf{O}}_j$, $\tilde{\mathbf{P}}_j$, and $\tilde{\mathbf{Q}}_j$. Since the values in the columns of the matrix $\tilde{\mathbf{N}}_j$ are exponentially decreasing, we can compute several largest values in each column and estimate the sum of absolute values of the remaining entries. We denote

$$\bar{\mathcal{I}}_{j+2} = \left\{1, 2, 3, 4, 2^{j+2}-3, 2^{j+2}-2, 2^{j+2}-1, 2^{j+2}\right\}, \quad \check{\mathcal{I}}_{j+2} = \mathcal{I}_{j+2} \backslash \bar{\mathcal{I}}_{j+2}$$

and we set

$$\left(\tilde{\mathbf{N}}_j\right)_{k,l} = 0, \quad \text{for } k \notin \mathcal{I}_{j-1}.$$

For l such that $l \mod 8 \in \{0, 1, 6, 7\}$ and $l \in \check{\mathcal{I}}_{j+2}$ we obtain

$$\begin{aligned}\sum_{k=1}^{2^{j-1}} \left|\left(\tilde{\mathbf{N}}_j\right)_{k,l}\right| &\leq \left|\left(\tilde{\mathbf{N}}_j\right)_{\lfloor \frac{l}{8}\rfloor-1,l}\right| + \left|\left(\tilde{\mathbf{N}}_j\right)_{\lfloor \frac{l}{8}\rfloor,l}\right| + \left|\left(\tilde{\mathbf{N}}_j\right)_{\lfloor \frac{l}{8}\rfloor+1,l}\right| \\ &\quad + \sum_{k=1}^{\lfloor \frac{l}{8}\rfloor-2} \left|\left(\tilde{\mathbf{N}}_j\right)_{k,l}\right| + \sum_{k=\lfloor \frac{l}{8}\rfloor+2}^{2^{j-1}} \left|\left(\tilde{\mathbf{N}}_j\right)_{k,l}\right| \\ &\leq 0.018 + 0.727 + 0.239 + 0.007 + 0.001 \leq 1.\end{aligned}$$

For l such that $l \mod 8 \in \{2, 3, 4, 5\}$ and $l \in \check{\mathcal{I}}_{j+2}$ we obtain

$$\begin{aligned}\sum_{k=1}^{2^{j-1}} \left|\left(\tilde{\mathbf{N}}_j\right)_{k,l}\right| &\leq \left|\left(\tilde{\mathbf{N}}_j\right)_{\lfloor \frac{l}{8}\rfloor-1,l}\right| + \left|\left(\tilde{\mathbf{N}}_j\right)_{\lfloor \frac{l}{8}\rfloor,l}\right| + \left|\left(\tilde{\mathbf{N}}_j\right)_{\lfloor \frac{l}{8}\rfloor+1,l}\right| \\ &\quad + \sum_{k=1}^{\lfloor \frac{l}{8}\rfloor-2} \left|\left(\tilde{\mathbf{N}}_j\right)_{k,l}\right| + \sum_{k=\lfloor \frac{l}{8}\rfloor+2}^{2^{j-1}} \left|\left(\tilde{\mathbf{N}}_j\right)_{k,l}\right| \\ &\leq 0.101 + 0.566 + 0.037 + 0.002 + 0.004 \leq 1.\end{aligned}$$

For $l \in \bar{\mathcal{I}}_{j+2}$ we have

$$\sum_{k=1}^{2^{j-1}} \left|\left(\tilde{\mathbf{N}}_j\right)_{k,l}\right| \leq 0.5.$$

We use the similar approach for computing the sums of absolute values of the entries in rows. We obtain

$$\sum_{k=1}^{2^{j-1}} \left| \left(\tilde{\mathbf{N}}_j \right)_{k,l} \right| \leq \begin{cases} 0.73, & l \in \bar{\mathcal{I}}_{j+2}, \\ 1.00, & l \in \check{\mathcal{I}}_{j+2}, \end{cases} \quad \sum_{l=1}^{2^{j+2}} \left| \left(\tilde{\mathbf{N}}_j \right)_{k,l} \right| \leq \begin{cases} 5.95, & k = 1, 2^{j-1}, \\ 6.80, & \text{otherwise}. \end{cases}$$

Similarly, we obtain

$$\sum_{k=1}^{2^{j-1}} \left| \left(\tilde{\mathbf{O}}_j \right)_{k,l} \right| \leq \begin{cases} 0.13, & l \in \bar{\mathcal{I}}_{j+2}, \\ 0.04, & l \in \check{\mathcal{I}}_{j+2}, \end{cases} \quad \sum_{l=1}^{2^{j+2}} \left| \left(\tilde{\mathbf{O}}_j \right)_{k,l} \right| \leq \begin{cases} 0.30, & k = 1, 2^{j-1}, \\ 0.02, & \text{otherwise}, \end{cases}$$

$$\sum_{k=1}^{2^{j-1}} \left| \left(\tilde{\mathbf{P}}_j \right)_{k,l} \right| \leq \begin{cases} 0.15, & l \in \bar{\mathcal{I}}_{j+2}, \\ 0.05, & l \in \check{\mathcal{I}}_{j+2}, \end{cases} \quad \sum_{l=1}^{2^{j+2}} \left| \left(\tilde{\mathbf{P}}_j \right)_{k,l} \right| \leq \begin{cases} 0.68, & k = 1, 2^{j-1}, \\ 0.04, & \text{otherwise}, \end{cases}$$

$$\sum_{k=1}^{2^{j-1}} \left| \left(\tilde{\mathbf{Q}}_j \right)_{k,l} \right| \leq \begin{cases} 0.03, & l \in \bar{\mathcal{I}}_{j+2}, \\ 0.01, & l \in \check{\mathcal{I}}_{j+2}, \end{cases} \quad \sum_{l=1}^{2^{j+2}} \left| \left(\tilde{\mathbf{Q}}_j \right)_{k,l} \right| \leq \begin{cases} 0.06, & k = 1, 2^{j-1}, \\ 0.01, & \text{otherwise}. \end{cases}$$

Therefore using (3.10) we have

$$\left\| \tilde{\mathbf{S}}_j \right\|_2 \leq \sqrt{1.1 \cdot 7} < 2\sqrt{2}. \qquad \square$$

For comparison we computed the norms of matrices $\tilde{\mathbf{S}}_j$ numerically and we found that $\left\| \tilde{\mathbf{S}}_j \right\|_2 \leq 2.27$ for $j = 1, \ldots, 12$, $\left\| \tilde{\mathbf{S}}_{12} \right\|_2 \approx 2.2623$ and it seems that this value does not further increase with increasing j.

LEMMA 7. *Let $m, n \geq 2$, $m < n$, then there exists a constant $C < 2$ such that*

$$\left\| \tilde{\mathbf{M}}_{m,0}^T \tilde{\mathbf{M}}_{m+1,0}^T \ldots \tilde{\mathbf{M}}_{n,0}^T \tilde{\mathbf{M}}_{n+1,0}^T \right\|_2 \leq C \left\| \tilde{\mathbf{M}}_{m,0}^T \tilde{\mathbf{M}}_{m+1,0}^T \ldots \tilde{\mathbf{M}}_{n-1,0}^T \right\|_2.$$

Proof. For m and n fixed such that $m, n \geq 2$, $m < n$, we use the notation:

$$\mathbf{R} = \tilde{\mathbf{M}}_{m,0}^T \tilde{\mathbf{M}}_{m+1,0}^T \ldots \tilde{\mathbf{M}}_{n-1,0}^T, \quad \mathbf{S} = \tilde{\mathbf{M}}_{n,0}^T \tilde{\mathbf{M}}_{n+1,0}^T.$$

Due to the structure of the matrices $\tilde{\mathbf{M}}_{j,0}$ given in Lemma 5 we have

$$\mathbf{R}_{k,2l} = \mathbf{R}_{k,2l-1}, \quad k \in \mathcal{I}_m, l \in \mathcal{I}_{n-1}.$$

Therefore, we can write $\mathbf{RS} = \tilde{\mathbf{R}}\tilde{\mathbf{S}}$, where the matrix $\tilde{\mathbf{R}}$ is $2^m \times 2^{n-1}$ matrix containing the even columns of the matrix $\mathbf{R}$, i.e. $\tilde{\mathbf{R}}_{k,l} = \mathbf{R}_{k,2l}$, and the matrix $\tilde{\mathbf{S}}$ is given by

$$\tilde{\mathbf{S}}_{k,l} = \mathbf{S}_{2k-1,l} + \mathbf{S}_{2k,l}, \quad k \in \mathcal{I}_{n-1}, l \in \mathcal{I}_{n+2}.$$

We have

$$\left\| \tilde{\mathbf{R}} \right\|_2 = \sup_{\mathbf{x} \in \mathbb{R}, \mathbf{x} \neq 0} \frac{\left\| \tilde{\mathbf{R}}\mathbf{x} \right\|_2}{\|\mathbf{x}\|_2} = \sup_{\mathbf{x} \in \mathbb{R}, \mathbf{x} \neq 0} \frac{\left(\sum_{k \in \mathcal{I}_m} \left(\sum_{l \in \mathcal{I}_{n-1}} \tilde{\mathbf{R}}_{k,l} \mathbf{x}_l \right)^2 \right)^{1/2}}{\|\mathbf{x}\|_2}.$$

Let $\tilde{\mathbf{x}}$ be a vector of the length $q = 2^n$ such that $\tilde{x}_{2j-1} = \tilde{x}_{2j} = x_j$ and let

$$\tilde{X} = \left\{ \tilde{\mathbf{x}} \in \mathbb{R}^q : \tilde{x}_{2j-1} = \tilde{x}_{2j}, \tilde{\mathbf{x}} \neq \mathbf{0} \right\}.$$

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Then $\|\tilde{\mathbf{x}}\|_2 = \sqrt{2}\,\|\mathbf{x}\|_2$ and we have

$$\left\|\tilde{\mathbf{R}}\right\|_2 = \sup_{\tilde{\mathbf{x}}\in\tilde{X}} \frac{\left(\sum\limits_{k\in\mathcal{I}_m}\left(\sum\limits_{l\in\mathcal{I}_n} 2^{-1}\mathbf{R}_{k,l}\tilde{\mathbf{x}}_l\right)^2\right)^{1/2}}{2^{-1/2}\,\|\tilde{\mathbf{x}}\|_2}$$

$$\leq \sup_{\tilde{\mathbf{x}}\in\mathbb{R}^q,\tilde{\mathbf{x}}\neq 0} \frac{2^{-1}\left(\sum\limits_{k\in\mathcal{I}_m}\left(\sum\limits_{l\in\mathcal{I}_n} \mathbf{R}_{k,l}\tilde{\mathbf{x}}_l\right)^2\right)^{1/2}}{2^{-1/2}\,\|\tilde{\mathbf{x}}\|_2} = \frac{\|\mathbf{R}\|_2}{\sqrt{2}}.$$

Using Lemma 6 we obtain

$$\|\mathbf{RS}\|_2 = \left\|\tilde{\mathbf{R}}\tilde{\mathbf{S}}\right\|_2 \leq \left\|\tilde{\mathbf{R}}\right\|_2 \left\|\tilde{\mathbf{S}}\right\|_2 \leq C\,\|\mathbf{R}\|_2$$

with $C < 2$. ∎

LEMMA 8. *There exist constants $C \in \mathbb{R}$ and $p < 0.5$ such that for all $m, n \geq 2$, $m < n$, we have*

$$\left\|\tilde{\mathbf{M}}^T_{m,0}\tilde{\mathbf{M}}^T_{m+1,0}\dots\tilde{\mathbf{M}}^T_{n-1,0}\right\|_2 \leq C\,2^{p(n-m)}. \tag{3.12}$$

Proof. The assertion of the lemma is a direct consequence of Lemma 5 and Lemma 7. ∎

4. Riesz basis on Sobolev space. In this section, we prove that Ψ is a Riesz basis of $H^1_0(\Omega_1)$ and Ψ^{2D} is a Riesz basis of $H^1_0(\Omega_2)$. The proof is based on the lemmas from Section 3 and on the theory developed in [19] that is summarized in the following theorem.

THEOREM 9. *Let H be a Hilbert space and let V_j, $j \geq J$, be the closed subspaces of $L_2(\Omega)$ such that $V_j \subset V_{j+1}$ and $\cup_{j=J}^{\infty} V_j$ is dense in H. Let H_q for fixed $q > 0$ be a linear subspace of H that is itself a normed linear space and assume that there exist positive constants A_1 and A_2 such that*

a) If $f \in H_q$ has decomposition $f = \sum_{j\geq J} f_j$, $f_j \in V_j$ then

$$\|f\|^2_{H_q} \leq A_1 \sum_{j\geq J} 2^{qj}\,\|f_j\|^2_H\,. \tag{4.1}$$

b) For each $f \in H_q$ there exists a decomposition $f = \sum_{j\geq J} f_j$, $f_j \in V_j$, such that

$$\sum_{j\geq J} 2^{qj}\,\|f_j\|^2_H \leq A_2\,\|f\|^2_{H_q}\,. \tag{4.2}$$

Furthermore, suppose that P_j is a linear projection from V_{j+1} onto V_j, W_j is the kernel space of P_j, $\Phi_j = \{\phi_{j,k}, k \in \mathcal{I}_j\}$ are Riesz bases of V_j with respect to the L_2–norm with uniformly bounded condition numbers and $\Psi_j = \{\psi_{j,k}, k \in \mathcal{I}_j\}$ are Riesz bases of W_j with uniformly bounded condition numbers. If there exist constants C and p such that $0 < p < q$ and

$$\|P_m P_{m+1}\dots P_{n-1}\| \leq C\,2^{p\,(n-m)}, \tag{4.3}$$

then

$$\left\{2^{-Jq}\phi_{J,k}, k \in \mathcal{I}_J\right\} \cup \left\{2^{-jq}\psi_{j,k}, j \geq J, k \in \mathcal{I}_j\right\} \tag{4.4}$$

is a Riesz basis of H_q.

Now we define suitable projections P_j from V_{j+1} onto V_j and show that these projections satisfy (4.3). Then we show that Ψ which differs from (4.4) only by scaling is also a Riesz basis of $H^1_0(0,1)$. For $j \geq 2$ we define

$$\Gamma_j = \{\phi_{j,k}\}_{k\in\mathcal{I}_j} \cup \{\psi_{j,k}\}_{k\in\mathcal{I}_j} \quad \text{and} \quad \mathbf{F}_j = \langle\Gamma_j, \Gamma_j\rangle\,.$$

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Let a set

(4.5)
$$\hat{\Gamma}_j = \left\{ \hat{\phi}_{j,k} \right\}_{k \in \mathcal{I}_j} \cup \left\{ \hat{\psi}_{j,k} \right\}_{k \in \mathcal{J}_j}$$

be given by

(4.6)
$$\hat{\Gamma}_j = \mathbf{F}_j^{-1} \Gamma_j .$$

Since obviously

$$\left\langle \Gamma_j, \hat{\Gamma}_j \right\rangle = \mathbf{I}_j ,$$

functions from $\hat{\Gamma}_j$ are duals to functions from Γ_j in the space V_{j+1}. Since $\mathbf{F}_j^{-1}$ is not a sparse matrix, these duals are not local. We define a projection P_j from V_{j+1} onto V_j by

$$P_j f = \sum_{k \in \mathcal{I}_j} \left\langle f, \hat{\phi}_{j,k} \right\rangle \phi_{j,k} .$$

LEMMA 10. *There exist $p < 0.5$ such that a projection P_j satisfies*

(4.7)
$$\| P_m P_{m+1} \dots P_{n-1} \| \leq C \, 2^{p\,(n-m)},$$

for all $2 \leq m < n$ and a constant C independent on m and n.

Proof. Let $f \in V_{j+1}$, $a_k^j = \left\langle f, \hat{\phi}_{j,k} \right\rangle$, $\mathbf{a}_j = \left\{ a_k^j \right\}_{k \in \mathcal{I}_j}$, $j \geq 2$, and $\mathbf{S}_j : \mathbf{a}_{j+1} \mapsto \mathbf{a}_j$. Then

$$\begin{aligned} P_j f &= \sum_{k \in \mathcal{I}_j} a_k^j \phi_{j,k} = \sum_{k \in \mathcal{I}_j} \left\langle f, \hat{\phi}_{j,k} \right\rangle \phi_{j,k} \\ &= \sum_{k \in \mathcal{I}_j} \sum_{l \in \mathcal{I}_{j+1}} a_l^{j+1} \left\langle \phi_{j+1,l}, \hat{\phi}_{j,k} \right\rangle \phi_{j,k} . \end{aligned}$$

Therefore

$$a_k^j = \sum_{l \in \mathcal{I}_{j+1}} a_l^{j+1} \left\langle \phi_{j+1,l}, \hat{\phi}_{j,k} \right\rangle .$$

Let us denote

$$S_{l,k}^j = \left\langle \hat{\phi}_{j,k}, \phi_{j+1,l} \right\rangle , \quad \mathbf{S}_j = \left\{ S_{l,k}^j \right\}_{l \in \mathcal{I}_{j+1}, k \in \mathcal{I}_j}$$

then we can write $\mathbf{a}_j = \mathbf{S}_j \mathbf{a}_{j+1}$ and due to Lemma 4 we have

$$\mathbf{S}_j = \left\langle \hat{\Phi}_j, \Phi_{j+1} \right\rangle = \left\langle \hat{\Phi}_j, \tilde{\mathbf{M}}_{j,0} \Phi_j + \tilde{\mathbf{M}}_{j,1} \Psi_j \right\rangle = \tilde{\mathbf{M}}_{j,0} .$$

Now, let us consider $f_n \in V_n$ and $f_m = P_m P_{m+1} \dots P_{n-1} f_n$. Then f_j can be represented by $f_j = \sum_{k \in \mathcal{I}_j} a_k^j \phi_j$ for $j = m, n$ and we set $\mathbf{a}_j = \left\{ a_k^j \right\}_{k \in \mathcal{I}_j}$. Since Φ_j is a Riesz basis of V_j, see [22], there exist constants C_1 and C_2 independent of j such that

$$C_1 \, \| \mathbf{a}_j \|_2 \leq \left\| \sum_{k \in \mathcal{I}_j} a_k^j \phi_{j,k} \right\| \leq C_2 \, \| \mathbf{a}_j \|_2 .$$

Due to Lemma 8 we have

$$\begin{aligned} \| f_m \| &\leq C_2 \, \| \mathbf{a}_m \|_2 \leq C_2 \, \| \mathbf{S}_m \, \mathbf{S}_{m+1} \dots \mathbf{S}_{n-1} \|_2 \, \| \mathbf{a}_n \|_2 \\ &= C_2 \left\| \tilde{\mathbf{M}}_{m,0}^T \, \tilde{\mathbf{M}}_{m+1,0}^T \dots \tilde{\mathbf{M}}_{n-1,0}^T \right\|_2 \| \mathbf{a}_n \|_2 \\ &\leq C_2 \, 2^{p(n-m)} \, \| \mathbf{a}_n \|_2 \leq C_1^{-1} \, C_2 \, 2^{p(n-m)} \, \| f_n \| . \end{aligned}$$

Thus (4.7) is proved. ∎

THEOREM 11. *The sets* Ψ_j *are Riesz bases of the spaces* $W_j = \mathrm{span}\ \Psi_j$, $j \geq 2$, *with the condition numbers bounded independently on* j, *namely* $\mathrm{cond}\ \Psi_j \leq 2$.

Proof. The matrix $\mathbf{U}_j = \langle \Psi_j, \Psi_j \rangle$ is tridiagonal with entries

$$\begin{aligned}
(\mathbf{U}_j)_{1,1} &= (\mathbf{U}_j)_{2^j,2^j} = \frac{27}{320}, \\
(\mathbf{U}_j)_{2,1} &= (\mathbf{U}_j)_{1,2} = (\mathbf{U}_j)_{2^j-1,2^j} = (\mathbf{U}_j)_{2^j,2^j-1} = \frac{47}{1920}, \\
(\mathbf{U}_j)_{k,k} &= \frac{1}{12}, \quad k = 2, \ldots, 2^j - 1, \\
(\mathbf{U}_j)_{k,k+1} &= (\mathbf{U}_j)_{k+1,k} = -\frac{1}{40}, \quad k = 2, \ldots, 2^j - 2, \\
(\mathbf{U}_j)_{k,l} &= 0, \quad \text{otherwise}.
\end{aligned}$$

Thus, $\mathbf{U}_j$ is strictly diagonally dominant and using Gershgorin circle theorem we obtain $\lambda_{min}(\mathbf{U}_j) \geq \frac{1}{30} \approx 0.0333$, $\lambda_{max}(\mathbf{U}_j) \leq \frac{2}{15} \approx 0.1333$, and $\mathrm{cond}\ \Psi_j \leq 2$. ∎

We also computed the eigenvalues of the matrix $\mathbf{U}_j$ numerically and the numerical values $\lambda_{min} \approx 0.0333$ and $\lambda_{max} \approx 0.1333$ correspond to the values computed using Gershgorin theorem. Thus the inequality in Theorem 11 seems to be sharp.

THEOREM 12. *The set*

$$\left\{2^{-2}\phi_{2,k}, k \in \mathcal{I}_2\right\} \cup \left\{2^{-j}\psi_{j,k}, j \geq 2, k \in \mathcal{I}_j\right\}$$

is a Riesz basis of $H_0^1(0,1)$.

Proof. Using the same argument as in [19] we conclude that (4.1) and (4.2) follows from the polynomial exactness of the scaling basis and the smoothness of basis functions and are satisfied for $H = L^2(0,1)$ and $H_q = H_0^q(0,1)$, $0 < q < 1.5$. Due to Lemma 10 the condition (4.3) is fulfilled. Therefore by Theorem 9 the assertion of Theorem 12 is proved. ∎

THEOREM 13. *The set*

$$\left\{\phi_{2,k}/\left|\phi_{2,k}\right|_{H_0^1(0,1)}, k \in \mathcal{I}_2\right\} \cup \left\{\psi_{j,k}/\left|\psi_{j,k}\right|_{H_0^1(0,1)}, j \geq 2, k \in \mathcal{I}_j\right\},$$

where $|\cdot|_{H_0^1(0,1)}$ *denotes the* $H_0^1(0,1)$*–seminorm, is a Riesz basis of* $H_0^1(0,1)$.

Proof. We follow the proof of Lemma 2 in [25]. From (2.9) there exist constants C_1 and C_2 such that

$$C_1 2^j \leq \left|\psi_{j,k}\right|_{H_0^1(\Omega)} \leq C_2 2^j, \quad \text{for } j \geq 2, \quad k \in \mathcal{I}_j, \tag{4.8}$$

and

$$C_1 2^2 \leq \left|\phi_{2,k}\right|_{H_0^1(\Omega)} \leq C_2 2^2, \quad \text{for } k \in \mathcal{I}_2. \tag{4.9}$$

Theorem 12 implies that there exist constants C_3 and C_4 such that

$$C_3 \left\|\mathbf{b}\right\|_2 \leq \left\| \sum_{k \in \mathcal{I}_2} a_{2,k} 2^{-2}\phi_{2,k} + \sum_{k \in \mathcal{I}_j, j \geq 2} b_{j,k} 2^{-j}\psi_{j,k} \right\|_{H_0^1(0,1)} \leq C_4 \left\|\mathbf{b}\right\|_2, \tag{4.10}$$

for any $\mathbf{b} = \{a_{2,k},\ k \in \mathcal{I}_2\} \cup \{b_{j,k}, j \geq 2,\ k \in \mathcal{J}_j\}$. Using (4.8), (4.9), and (4.10) we obtain

$$\left\|\mathbf{b}\right\|_2 \leq \frac{C_2}{C_3} \left\| \sum_{k \in \mathcal{I}_2} a_{2,k} \frac{\phi_{2,k}}{\left|\phi_{2,k}\right|_{H_0^1(\Omega)}} + \sum_{k \in \mathcal{J}_j, j \geq 2} b_{j,k} \frac{\psi_{j,k}}{\left|\psi_{j,k}\right|_{H_0^1(\Omega)}} \right\|_{H_0^1(0,1)}$$

and

$$\left\|\mathbf{b}\right\|_2 \geq \frac{C_1}{C_4} \left\| \sum_{k \in \mathcal{I}_2} a_{2,k} \frac{\phi_{2,k}}{\left|\phi_{2,k}\right|_{H_0^1(\Omega)}} + \sum_{k \in \mathcal{J}_j, j \geq 2} b_{j,k} \frac{\psi_{j,k}}{\left|\psi_{j,k}\right|_{H_0^1(\Omega)}} \right\|_{H_0^1(0,1)}.$$

∎

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REMARK 4.1. By Theorem 9 and the proof of Lemma 10 if p satisfies (3.12) then the norm equivalence (2.1) for Ψ from Section 2 normalized with respect to the H^s-norm is satisfied for $H = H^s$, where $s \in (p, 1.5)$. Since we proved in Section 3 that there exists p satisfying (3.12) such that $p < 0.5$ we proved the norm equivalence (2.1) for H^s with $s \in (0.5, 1.5)$. We computed the norms in (3.12) also numerically and we found that this theoretical estimate of p is not sharp. It seems that (3.12) holds also for any $p > 0$.

THEOREM 14. *The set Ψ^{2D} normalized with respect to the H^1–seminorm is a Riesz basis of $H_0^1\left((0,1)^2\right)$.*

Proof. Recall that $\hat{\phi}_{j,k}$ are defined by (4.5) and (4.6). For $\mathbf{k} = (k_1, k_2)$ let us define $\hat{\phi}^2_{j,\mathbf{k}} = \hat{\phi}_{j,k_1} \otimes \hat{\phi}_{j,k_2}$. Then for $\mathbf{k} = (k_1, k_2)$ and $\mathbf{l} = (l_1, l_2)$ we have

$$\left\langle \phi^2_{j,\mathbf{k}}, \hat{\phi}^2_{j,\mathbf{l}} \right\rangle = \delta_{k_1,l_1}\delta_{k_2,l_2},$$

and P_j^{2D} defined by

$$P_j^{2D} f = \sum_{\mathbf{k}\in\mathcal{I}_j\times\mathcal{I}_j} \left\langle f, \hat{\phi}^2_{j,\mathbf{k}} \right\rangle \phi^2_{j,\mathbf{k}}$$

is a projection from V^2_{j+1} onto V^2_j, where $V^2_j = V_j \otimes V_j$ for $j \geq 2$. We denote $\mathbf{S}^{2D}_j = \tilde{\mathbf{M}}^T_{j,0} \otimes \tilde{\mathbf{M}}^T_{j,0}$. It is well-known that for any matrix $\mathbf{B}$ we have $\|\mathbf{B}\otimes\mathbf{B}\|_2 = \|\mathbf{B}\|_2^2$. Using this relation and the same arguments as in the proof of Lemma 10 we obtain for $f_n \in V^2_n$ and $f_m = P^{2D}_m P^{2D}_{m+1} \ldots P^{2D}_{n-1} f_n$ the estimate:

$$\begin{aligned}\|f_m\| &\leq C_1 \|\mathbf{a}_m\|_2 \leq C_2 \left\|\mathbf{S}^{2D}_m \mathbf{S}^{2D}_{m+1} \ldots \mathbf{S}^{2D}_{n-1}\right\|_2 \|\mathbf{a}_n\|_2 \\ &= C_2 \left\| \left(\tilde{\mathbf{M}}^T_{m,0} \ldots \tilde{\mathbf{M}}^T_{n-1,0}\right) \otimes \left(\tilde{\mathbf{M}}^T_{m,0} \ldots \tilde{\mathbf{M}}^T_{n-1,0}\right) \right\|_2 \|\mathbf{a}_n\|_2 \\ &\leq C_3\, 2^{2p(n-m)}\, \|\mathbf{a}_n\|_2 \leq C_4\, 2^{2p(n-m)}\, \|f_n\|\end{aligned}$$

with $2p < 1$. Hence by Theorem 9 the assertion of the theorem is proved. ∎

5. Quantitative properties of constructed bases. In this section, we present the condition numbers of the stiffness matrices for the Helmholtz equation

$$-\epsilon\Delta u + au = f \quad \text{on} \quad \Omega_d, \qquad u = 0 \text{ on } \partial\Omega_d, \tag{5.1}$$

where Δ is the Laplace operator, ϵ and a are positive constants. We also study the case $\epsilon = 1$ and $a = 0$, i.e., the Poisson equation, and the case $\epsilon = 0$ and $a = 1$.

The variational formulation is

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \tag{5.2}$$

where

$$\mathbf{A} = \epsilon \langle \nabla\Psi, \nabla\Psi \rangle + a \langle \Psi, \Psi \rangle, \quad u = (\mathbf{u})^T \Psi, \quad \mathbf{f} = \langle f, \Psi \rangle.$$

An advantage of discretizing the elliptic equation (5.1) using a wavelet basis is that the system (5.2) can be simply preconditioned by a diagonal preconditioner [9]. Let $\mathbf{D}$ be a matrix of diagonal elements of the matrix $\mathbf{A}$, i.e., $\mathbf{D}_{\lambda,\mu} = \mathbf{A}_{\lambda,\mu}\delta_{\lambda,\mu}$, where $\delta_{\lambda,\mu}$ denotes Kronecker delta. Setting

$$\tilde{\mathbf{A}} = (\mathbf{D})^{-1/2} \mathbf{A} (\mathbf{D})^{-1/2}, \quad \tilde{\mathbf{u}} = (\mathbf{D})^{1/2} \mathbf{u}, \quad \tilde{\mathbf{f}} = (\mathbf{D})^{-1/2} \mathbf{f},$$

we obtain the preconditioned system

$$\tilde{\mathbf{A}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}}. \tag{5.3}$$

It is known [9] that there exist a constant C such that $\operatorname{cond} \tilde{\mathbf{A}} \leq C < \infty$.

Let Ψ^s be defined by (2.10) for $d = 1$ and similarly for $d > 1$. We define

$$\mathbf{A}_s = \epsilon \langle \nabla\Psi^s, \nabla\Psi^s \rangle + a \langle \Psi^s, \Psi^s \rangle, \quad u_s = (\mathbf{u}_s)^T \Psi^s, \quad \mathbf{f}_s = \langle f, \Psi^s \rangle.$$

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Let $\mathbf{D}_s$ be a matrix of diagonal elements of the matrix $\mathbf{A}_s$, i.e., $(\mathbf{D}_s)_{\lambda,\mu} = (\mathbf{A}_s)_{\lambda,\mu}\,\delta_{\lambda,\mu}$. We set

$$\tilde{\mathbf{A}}_s = (\mathbf{D}_s)^{-1/2}\,\mathbf{A}_s\,(\mathbf{D}_s)^{-1/2}, \quad \tilde{\mathbf{u}}_s = (\mathbf{D}_s)^{1/2}\,\mathbf{u}_s, \quad \tilde{\mathbf{f}}_s = (\mathbf{D}_s)^{-1/2}\,\mathbf{f}_s$$

and we obtain the preconditioned finite-dimensional system

$$\tilde{\mathbf{A}}_s\tilde{\mathbf{u}}_s = \tilde{\mathbf{f}}_s. \tag{5.4}$$

Since $\tilde{\mathbf{A}}_s$ is a part of the matrix $\tilde{\mathbf{A}}$ that is symmetric and positive definite, we have also

$$\text{cond}\,\tilde{\mathbf{A}}_s \leq C.$$

The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ for $\epsilon = 1$, $a = 0$, and $d = 1, 2$, are shown in Table 5.1. By Remark 2.2 these numbers correspond to the squares of the condition numbers of Ψ^s with respect to the H^1-seminorm. We computed also the condition numbers of Ψ^s with respect to the H^1-norm. The values were very close to the values presented in Table 5.1 (the difference was less than 1%).

For comparison, we computed also the condition numbers for other wavelet bases and displayed them in Figure 5.1 and Figure 5.2. The bases CF_2 and CF_3 refer to the wavelet bases from this paper with the coarsest level 2 and 3, respectively. D_{j_0} and P_{j_0} refer to the quadratic spline wavelet basis with 3 vanishing moments and the coarsest level j_0 from [11] and [20], respectively. We modified the construction from [3] to homogeneous boundary conditions. The resulting quadratic spline wavelet basis with three vanishing wavelet moments with the coarsest level j_0 is denoted as B_{j_0}. We found that bases D_{j_0}, P_{j_0}, and B_{j_0} lead to the same results and we realized that they contain the same wavelets up to a multiplication with a constant factor. Semiorthogonal quadratic spline wavelets with three vanishing moments on the interval were constructed in [5]. In Appendix A we show that the semiorthogonal quadratic spline wavelet basis corresponding to scaling functions that are B-splines on the Schoenberg sequence of knots such that wavelets have three vanishing moments and the basis is adapted to homogeneous boundary conditions do not exist. Therefore, we adapt this basis such that semiorthogonality is preserved and $2^j - 2$ wavelets on the level j have three vanishing moments and 2 wavelets on the level j are without vanishing moments. We denote the resulting basis as CQ. We also tested wavelet bases from [11, 20] with 5 vanishing moments, but the condition numbers were larger than for bases with 3 vanishing moments. All wavelets used in numerical experiments are presented in Appendix A.

Although it was not proved in this paper that using appropriate tensorising of 1D wavelet basis we obtain the wavelet basis in 3D, we listed the condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ for 3D case in Table 5.2. The condition numbers for several constructions of quadratic spline wavelet bases and various values of parameters ϵ and a are compared in Table 5.3.

TABLE 5.1
The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ of the size $N \times N$ corresponding to multiscale wavelet bases with s levels of wavelets for the one-dimensional and the two-dimensional Poisson equation.

	1D				2D			
s	N	λ_{min}	λ_{max}	cond$\tilde{\mathbf{A}}_s$	N	λ_{min}	λ_{max}	cond$\tilde{\mathbf{A}}_s$
1	8	0.50	1.38	2.77	64	0.25	1.88	7.5
2	16	0.50	1.41	2.83	256	0.19	2.08	11.1
3	32	0.50	1.42	2.83	1 024	0.16	2.17	13.7
4	64	0.50	1.42	2.84	4 096	0.14	2.20	15.4
5	128	0.50	1.42	2.84	16 384	0.13	2.22	16.6
6	256	0.50	1.42	2.84	65 536	0.13	2.23	17.4
7	512	0.50	1.42	2.84	262 144	0.12	2.23	17.9
8	1024	0.50	1.42	2.84	1 048 576	0.12	2.23	18.3

We computed also the condition numbers for the discretization matrices $\tilde{\mathbf{A}}_s$ corresponding to $\epsilon = 0$, $a = 1$, and $d = 1$. By Remark 2 these condition numbers represent the squares of the L^2 condition numbers of Ψ^s normalized with respect to the L^2-norm. The results are displayed in Figure 5.1. In this paper, we proved that the constructed basis is a Riesz basis in $H_0^1(0, 1)$. The condition numbers of matrices $\tilde{\mathbf{A}}_s$ corresponding to $\epsilon = 0$ and $a = 1$ for the new basis seems to be unbounded and thus it seems that the new basis is not a

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TABLE 5.2
The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ of the size $N \times N$ corresponding to multiscale wavelet bases with s levels of wavelets for the three-dimensional Poisson equation.

s	N	λ_{min}	λ_{max}	$\text{cond}\tilde{\mathbf{A}}_s$
1	512	0.15	3.23	47.4
2	4096	0.04	3.69	85.0
3	32768	0.03	3.83	113.8
4	262144	0.03	3.87	132.9
5	2097152	0.03	3.89	145.3

TABLE 5.3
The condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s$ of the size 65536×65536 for several choices of ϵ and a for our bases and bases from [11, 20].

ϵ	a	CF_2	CF_3	CF_2^{ort}	CF_3^{ort}	CQ	D_2	D_3
1000	1	17.4	16.3	17.1	16.4	62.0	116.3	98.4
1	0	17.4	16.7	17.1	16.4	62.0	116.3	98.4
1	1	17.4	16.7	17.1	16.4	62.0	116.6	98.5
10^{-3}	1	72.1	35.9	35.6	22.5	61.1	328.1	139.2
10^{-6}	1	746.0	577.0	425.7	287.6	46.3	1878.0	1115.4
0	1	872.6	687.4	511.0	351.5	46.4	2034.6	1251.4

Riesz basis in $L^2(0,1)$, see also Remark 4.1. Since the condition numbers of matrices $\tilde{\mathbf{A}}_s$ for $\epsilon = 1$ and $a = 0$ corresponding to the anisotropic basis $\Psi \otimes \Psi$ with respect to the H^1-seminorm depend on the condition numbers of Ψ^s both with respect to the L^2-norm and the H^1-seminorm, they are also increasing, see Figure 5.2. Thus in our case an isotropic wavelet basis from Section 2 has bounded and significantly smaller condition number than an anisotropic basis. We performed numerical experiments with both types of bases, but since the isotropic system lead to significantly better results we present in Section 6 only the experiments with the isotropic wavelet bases.

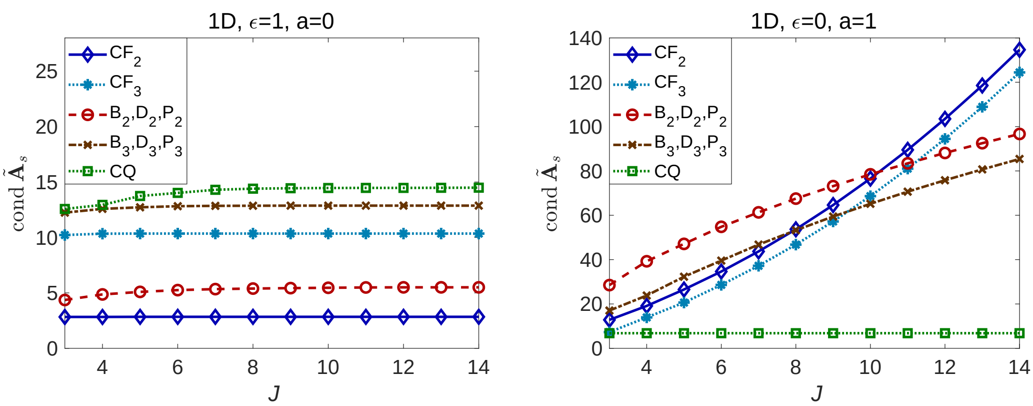

FIG. 5.1. *The condition numbers of the matrices $\tilde{\mathbf{A}}_s$, $s = J - j_0 + 1$, for the one-dimensional problem (5.1) with parameters $\epsilon = 1$, $a = 0$, and $\epsilon = 0$, $a = 1$. The parameter J denotes the finest level and j_0 denotes the coarsest level.*

6. Numerical examples. In this section we use the constructed wavelet basis in the wavelet-Galerkin method and the adaptive wavelet method.

6.1. Multilevel Galerkin method. We consider the problem (5.1) with Ω_2, $\epsilon = 1$ and $a = 0$. The right-hand side f is such that the solution u is given by:

$$u(x,y) = v(x)\, v(y), \quad v(x) = x\left(1 - e^{50x-50}\right).$$

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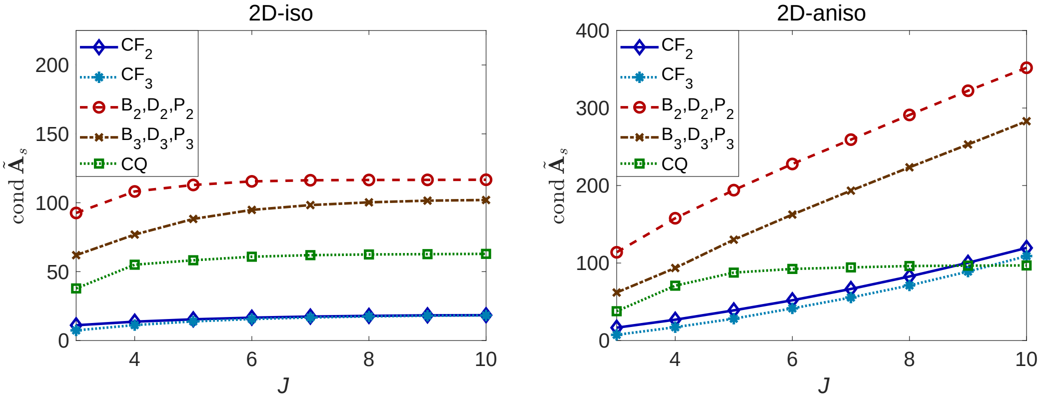

FIG. 5.2. *The condition numbers of the matrices* $\tilde{\mathbf{A}}_s$, $s = J - j_0 + 1$, *for* $\epsilon = 1$, $a = 0$ *and two-dimensional wavelet bases constructed using an isotropic approach and an anisotropic approach. The parameter* J *denotes the finest level and* j_0 *denotes the coarsest level.*

We discretize the equation using the Galerkin method with wavelet basis constructed in this paper and we obtain discrete problem $\tilde{\mathbf{A}}_s \tilde{\mathbf{u}}_s = \tilde{\mathbf{f}}_s$. We solve it by conjugate gradient method using a simple multilevel approach similarly as in [27, 19]:

1. Compute $\tilde{\mathbf{A}}_s$ and $\tilde{\mathbf{f}}_s$, choose $\mathbf{v}_0$ of the length 4^2.

2. For $j = 0, \ldots, s$ find the solution $\tilde{\mathbf{u}}_j$ of the system $\tilde{\mathbf{A}}_j \tilde{\mathbf{u}}_j = \tilde{\mathbf{f}}_j$ by conjugate gradient method with initial vector $\mathbf{v}_j$ defined for $j \geq 1$ by

$$(\mathbf{v}_j) = \begin{cases} \tilde{\mathbf{u}}_{j-1}, & i = 1, \ldots, k_j, \\ 0, & i = k_j, \ldots, k_{j+1}, \end{cases}$$

where $k_j = 2^{2(j+1)}$.

Let u be the exact solution of (5.1) and

$$u_s^* = \left(\tilde{\mathbf{u}}_s^*\right)^T \left(\mathbf{D}_s\right)^{-1/2} \Psi^s,$$

where $\tilde{\mathbf{u}}_s^*$ is the exact solution of the discrete problem (5.4). It is known [21] that due to the polynomial exactness of the spaces span Ψ^s there exists a constant C independent of s such that

$$\|u - u_s^*\| \leq C 2^{-3s}, \quad \|u - u_s^*\|_{H^1(\Omega_d)} \leq C 2^{-2s}, \tag{6.1}$$

for $u \in H^3(\Omega_d)$. Let u_s be an approximate solution obtained by multilevel Galerkin method with s levels of wavelets. It was shown in [27] that if we use the criterion for terminating iterations $\|\mathbf{r}_s\|_2 \leq C 2^{-2s}$, where $\mathbf{r}_s := \tilde{\mathbf{A}}_s \tilde{\mathbf{u}}_s - \tilde{\mathbf{f}}_s$, then we achieve for u_s the same convergence rate as for u_s^*. In our example, for the given number of levels s we use the criterion $\|\mathbf{r}_j\|_2 \leq 10^{-4} 2^{-2s}$, $j = 0, \ldots, s$, for terminating iterations in each level.

We denote the number of iterations on the level j as M_j. It is known [21] that employing the discrete wavelet transform one CG iteration can be performed with complexity of the order $\mathcal{O}(N)$, where $N \times N$ is the size of the matrix. Therefore the number of operations needed to compute one CG iteration on the level j requires about one quarter of operations needed to compute one CG iteration on the level $j+1$, we compute the total number of equivalent iterations by

$$M = \sum_{j=0}^{s} \frac{M_j}{4^{s-j}}.$$

The results are listed in Table 6.1. It can be seen that the number of conjugate gradient iterations is quite small and that

$$\frac{\|u_s - u\|_\infty}{\|u_{s+1} - u\|_\infty} \approx \frac{\|u_s - u\|}{\|u_{s+1} - u\|} \approx \frac{1}{8},$$

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i.e., that the order of convergence is 3. It corresponds to (6.1). The parameters r_2 and r_∞ in Table 6.1 are the experimental rates of convergence, i.e.

$$(r_2)_s = \frac{\log\left(\|u_{s-1} - u\| \,/\, \|u_s - u\|\right)}{\log 2}, \; (r_\infty)_s = \frac{\log\left(\|u_{s-1} - u\|_\infty \,/\, \|u_s - u\|_\infty\right)}{\log 2}.$$

We presented also the wall clock time in Table 6.1. It includes the computation of the right-hand side, the system matrix, iterations and evaluation of the solution on the grid with the step size 2^{-j_0-s}, where j_0 is the coarsest level.

TABLE 6.1
Number of iterations and error estimates for multilevel conjugate gradient method.

s	N	M	$\|u_s-u\|_\infty$	r_∞	$\|u_s-u\|$	r_2	time $[s]$
			CF_2				
1	64	18.50	3.19e-1		4.54e-2		0.04
2	256	21.63	1.32e-1	1.27	1.26e-3	5.17	0.05
3	1 024	23.66	2.60e-2	2.34	2.02e-3	2.64	0.06
4	4 096	23.00	2.91e-3	3.16	2.45e-4	3.04	0.09
5	16 384	20.89	4.06e-4	2.84	2.89e-5	3.08	0.16
6	65 536	18.37	5.35e-5	2.92	3.41e-6	3.08	0.30
7	262 144	15.68	6.82e-6	2.97	4.23e-7	3.01	0.99
8	1 048 576	13.02	8.63e-7	2.98	5.28e-8	3.00	3.89
9	4 194 304	10.35	1.08e-7	3.00	6.59e-9	3.00	14.87
10	16 777 216	8.85	1.41e-8	2.94	8.25e-10	3.00	58.12
			D_2, P_2, B_2				
s	N	M	$\|u_s-u\|_\infty$	r_∞	$\|u_s-u\|$	r_2	time $[s]$
1	64	27.50	3.19e-1		4.54e-2		0.04
2	256	48.88	1.32e-1	1.27	1.26e-3	5.17	0.07
3	1 024	59.22	2.60e-2	2.34	2.02e-3	2.64	0.11
4	4 096	59.38	2.91e-3	3.16	2.45e-4	3.04	0.19
5	16 384	50.76	4.06e-4	2.84	2.89e-5	3.08	0.33
6	65 536	39.44	5.35e-5	2.92	3.41e-6	3.08	0.68
7	262 144	29.92	6.84e-6	2.97	4.23e-7	3.01	2.20
8	1 048 576	21.50	8.64e-7	2.98	5.29e-8	3.00	9.53
9	4 194 304	17.66	1.09e-7	2.99	6.73e-9	2.97	47.39
10	16 777 216	15.79	1.38e-8	2.98	9.43e-10	2.84	248.41
			CQ				
s	N	M	$\|u_s-u\|_\infty$	r_∞	$\|u_s-u\|$	r_2	time $[s]$
0	64	13.00	3.19e-1		4.54e-2		0.03
1	256	30.25	1.32e-1	1.27	1.26e-3	5.17	0.05
3	1 024	35.06	2.60e-2	2.34	2.02e-3	2.64	0.07
4	4 096	33.82	2.91e-3	3.16	2.45e-4	3.04	0.14
5	16 384	30.30	4.06e-4	2.84	2.89e-5	3.08	0.21
6	65 536	25.32	5.35e-5	2.92	3.41e-6	3.08	0.41
7	262 144	20.74	6.84e-6	2.97	4.23e-7	3.01	1.39
8	1 048 576	17.87	8.64e-7	2.98	5.29e-8	3.00	5.55
9	4 194 304	14.82	1.08e-7	3.00	6.73e-9	2.97	21.62
10	16 777 216	12.36	1.36e-8	2.99	8.56e-10	2.97	83.54

6.2. Adaptive wavelet method. We compare the quantitative behavior of the adaptive wavelet method with our wavelet basis, the wavelet basis from [11], and the wavelet basis that is a modification of the basis from [5], see Appendix A. We consider the equation (5.1) with $d = 1$, $\epsilon = 1$, $a = 0$, and the solution

$$u(x) = e^{-\left|\frac{x}{4}-\frac{1}{8}\right|} - e^{-\frac{1}{8}} + \sin 3\pi x, \quad x \in [0,1].$$

Note that u is the sum of the infinitely differentiable function and the function

$$g(x) = e^{-\left|\frac{x}{4}-\frac{1}{8}\right|}$$

which has not derivative in the point 0.5. Let $\hat{g}$ be the Fourier transform of g, i.e.

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x)\, e^{-ix\xi} dx.$$

Since

$$\int_{\mathbb{R}} |\xi|^{2\mu} |\hat{g}(\xi)|^2 \, d\xi = \int_{\mathbb{R}} \frac{64\,|\xi|^{2\mu}}{(16\xi^2+1)^2} d\xi$$

is finite for $\mu < 3/2$ and it is not finite for $\mu \geq 3/2$, the solution u belongs to the Sobolev space $u \in H^s(0,1)$ only for $s < 3/2$. Therefore it is not guarantied that (6.1) holds and that the Galerkin method converges with the optimal rate. Since u is continuous and piecewise smooth, it can be shown that u belongs to the Besov space $B^s_{\tau,\tau}(0,1)$ for any $s > 0$ and $\tau = (s+1/2)^{-1}$. It is therefore convenient to solve this problem with the adaptive wavelet method proposed in [6, 7], because it is proved that this method converges with the optimal rate for functions from such spaces. More precisely, let u_j be the approximate solution in the jth step and let ρ_j denote the error in the energy norm which is in this example the same as the H^1-seminorm, i.e., $\rho_j = |u - u_j|_{H^1}$. Let $\mathbf{u}_j$ be the vector of coefficients corresponding to u_j and let N_j be the number of nonzero entries of $\mathbf{u}_j$. It follows from the theory developed in [7] that if the used basis is a quadratic spline wavelet basis then there exists a constant C independent on j such that

$$\rho_j \leq C N_j^{-r} \quad \text{for any } r < 2. \tag{6.2}$$

The method insists in solving the infinite preconditioned system (5.3) with Richardson iterations. The algorithm contains the routine **COARSE** that is based on thresholding the coefficients and the routine **RHS** that approximate the vector of the right-hand side that is infinite by a finite vector with a prescribed accuracy. For details about these two routines we refer to [7]. It is possible to modify the algorithm such that the routine **COARSE** is avoided, see [14]. Furthermore, it is necessary to have a routine that enables to compute a multiplication of the biinfinite matrix $\tilde{\mathbf{A}}$ with a finitely supported vector. This routine called **APPLY** was proposed in [7] and modified in [24, 12]. We use the version from [24]. We use the similar version of the method and notations that is presented as **CDD02SOLVE** in [14]. We compute the relaxation parameter ω and the error reduction factor ρ by

$$\omega = \frac{2}{\lambda_{max}\left(\tilde{\mathbf{A}}\right) + \lambda_{mix}\left(\tilde{\mathbf{A}}\right)}, \qquad \rho = \frac{\operatorname{cond} \tilde{\mathbf{A}} - 1}{\operatorname{cond} \tilde{\mathbf{A}} + 1},$$

and we set $\theta = 0.3$ and $K \in \mathbb{N}$ such that $2\rho^K/\theta < 0.6$.

We use the following version of the method:

ALGORITHM 15. **SOLVE** $[\tilde{\mathbf{A}}, \mathbf{f}, \epsilon] \rightarrow \mathbf{u}_\epsilon$

1. Set $j := 0$, $\mathbf{u}_0 := 0$, *and* $\epsilon_0 \geq \|\tilde{\mathbf{u}}\|_2$.

2. While $\epsilon_j > \epsilon$ *do*

  $\mathbf{z}_0 := \mathbf{u}_j$,

  For $l = 1, \ldots, K$ *do*

    $\mathbf{z}_l := \mathbf{z}_{l-1} + \omega\left(\mathbf{RHS}[\mathbf{f}, \frac{\epsilon_j \rho^l}{2\omega K}] - \mathbf{APPLY}[\tilde{\mathbf{A}}, \mathbf{z}_{l-1}, \frac{\epsilon_j \rho^l}{2\omega K}]\right)$,

  end for,

  $j := j + 1$

  $\epsilon_j := \frac{2\rho^K \epsilon_{j-1}}{\theta}$,

  $\mathbf{u}_j := \mathbf{COARSE}[\mathbf{z}_K, (1-\theta)\,\epsilon_j]$,

end while,

$\mathbf{u}_\epsilon := \mathbf{u}_j$.

We use the following parameters in the numerical experiments:

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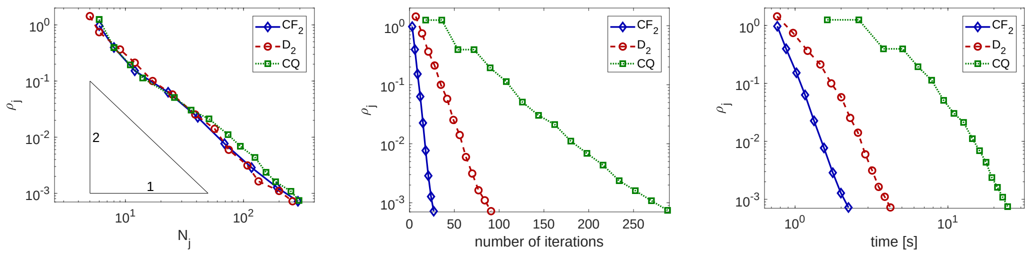

FIG. 6.1. *The convergence history for adaptive wavelet scheme with various wavelet bases.*

- CF_2: $\omega = 1.04$, $\rho = 0.48$, $K = 4$,
- D_2: $\omega = 0.89$, $\rho = 0.70$, $K = 7$,
- CQ_3: $\omega = 0.95$, $\rho = 0.87$, $K = 18$.

The convergence history is shown in Figure 6.1. Since the entries of the matrix $\tilde{\mathbf{A}}$, the estimates of eigenvalues of $\tilde{\mathbf{A}}$ and the parameters ω, ρ and K were precomputed for every basis, the wall clock time includes the computation of the right-hand side and the computation of iterations. The experimental convergence rate, i.e., the parameter r from (6.2) estimated for the observed values (N_j, ρ_j) by the least square method, for bases CF_2, D_2, and CQ was $r \approx 1.87$, $r \approx 1.95$, and $r \approx 1.77$, respectively. It can be seen that the number of iterations and the computational time needed to resolve the problem with desired accuracy is significantly smaller for the new wavelet basis. Moreover, due to the shorter support of the wavelets, the stiffness matrix is sparser and thus one iteration requires smaller number of operations.

Appendix A. Quadratic spline wavelet bases.

In this section we present inner and boundary scaling functions and wavelets that were used in numerical experiments in Section 6. The wavelet bases is generated from these function by the similar way as in (2.6) and (2.9). Let ϕ be given by (2.2) and $\breve{\phi}_b = 2\phi^b/3$, where ϕ_b is given by (2.4). Since diagonal preconditioning (5.4) is similar to the normalization of the basis with respect to the energy norm, the multiplication of ϕ_b with a constant has no effect on resulting condition numbers presented in Section 5 and numerical results in Section 6. The wavelets are given by

$$\breve{\psi}(x) = \sum_{k=0}^{7} g_k \phi(2x-k), \quad \breve{\psi}^i = g_{-1}^i \breve{\phi}^b(2x) + \sum_{k=0}^{5} g_k^i \phi(2x-k),$$

for $i = 1, 2$. The values of the parameters g_k and g_k^i are presented for several constructions below.

A.1. Primbs wavelet basis. The parameters for the construction from [20] are given by

$$[g_0, \ldots, g_7] = [-3, -9, 7, 45, -45, -7, 9, 3]/64,$$
$$[g_{-1}^1, \ldots, g_6^1] = \left[-10, \frac{65}{6}, -\frac{9}{14}, -\frac{31}{7}, -\frac{11}{21}, \frac{15}{14}, \frac{5}{14}\right]/64,$$
$$[g_{-1}^2, \ldots, g_6^2] = \left[-\frac{10}{3}, -\frac{5}{6}, \frac{65}{6}, -\frac{25}{3}, -\frac{13}{9}, \frac{3}{2}, \frac{1}{2}\right]/64.$$

More precisely, in [20] the parameters are multiplications of these parameters, but as we already mentioned different normalization does not play a role, because we use diagonal preconditioning (5.4) in our experiments.

A.2. Dijkema wavelet basis. There are several constructions in [11]. We used the parameters that are listed in the file mats.zip attached to [11], but we found that in the case of quadratic spline wavelets with three vanishing moments and homogeneous boundary conditions these parameters are multiples of the parameters from [20] and thus they lead to the same results.

A.3. Modification of Chui-Quak wavelet basis. In [5] the semiorthogonal quadratic spline wavelets with three vanishing moments were adapted to the interval. We adapt these wavelets to homogeneous boundary conditions. Since wavelets on the level j are linear combinations of scaling functions on the level $j+1$, they are given by 2^{j+1} parameters. We want to preserve semiorthogonality, therefore we have 2^j conditions on orthogonality to scaling functions on the level j. Furthermore, we want to preserve three vanishing moments. We obtain homogeneous system with $2^j + 2$ independent equations with 2^{j+1} variables that has only $2^j - 2$ independent solutions. Therefore there exists only $2^j - 2$ wavelets with three vanishing moments that are semiorthogonal. We add two wavelets on each level that are semiorthogonal but without vanishing moments. We obtain wavelets with parameters:

$$\begin{aligned} [g_0, \dots, g_7] &= [-1, 29, -147, 303, -303, 147, -29, 1]/480, \\ [g^1_{-1}, \dots, g^1_6] &= [450, -332, 148, -29, 1, 0, 0]/480, \\ [g^2_{-1}, \dots, g^2_6] &= \left[\frac{780}{11}, -\frac{1949}{11}, \frac{3481}{11}, -\frac{3362}{11}, \frac{1618}{11}, -29, 1\right]/480. \end{aligned}$$

A.4. Modification of Bittner wavelet basis. In [3] spline wavelet bases on the interval were constructed. We use the similar approach as in [3], but for quadratic spline wavelets with three vanishing moments satisfying homogeneous boundary conditions. The inner wavelet is the third derivative of the sixth-order B-spline on knots $[0, 1, 2, 5/2, 3, 4, 5]$. The boundary wavelets are the third derivatives of the sixth-order B-splines on knots $[0, 0, 1/2, 1, 2, 3, 4]$ and $[0, 0, 1, 3/2, 2, 3, 4]$, respectively. We found that by this approach we again obtain the same wavelets up to a constant factor as in [11, 20].

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CUBIC SPLINE WAVELETS WITH FOUR VANISHING MOMENTS ON THE INTERVAL AND THEIR APPLICATIONS TO OPTION PRICING UNDER KOU MODEL

DANA ČERNÁ

Department of Mathematics and Didactics of Mathematics, Technical University of Liberec
Studentská 2, 461 17 Liberec, Czech Republic
dana.cerna@tul.cz

The paper is concerned with the construction of a cubic spline wavelet basis on the unit interval and an adaptation of this basis to the first-order homogeneous Dirichlet boundary conditions. The wavelets have four vanishing moments and they have the shortest possible support among all cubic spline wavelets with four vanishing moments corresponding to B-spline scaling functions. We provide a rigorous proof of the stability of the basis in the space $L^2(0,1)$ or its subspace incorporating boundary conditions. To illustrate the applicability of the constructed bases we apply the wavelet-Galerkin method to option pricing under the double exponential jump-diffusion model and we compare the results with other cubic spline wavelet bases and with other methods.

Keywords: Wavelet; cubic spline; short support; Galerkin method; option pricing; Kou model.

AMS Subject Classification: 65T60, 65D07, 47G20, 65M60, 91G80

1. Introduction

Wavelets have already found applications in numerous fields, including signal analysis, image processing, approximation theory, engineering applications, and numerical simulations. They have been used for the numerical solution of various types of partial differential and integral equations. Wavelet methods are suitable for preconditioning of systems of linear algebraic equations arising from the discretization of elliptic problems,[19] adaptive solution of operator equations,[16] the numerical solution of certain types of partial differential equations with a dimension independent convergence rate.[23] Wavelet methods seem to be superior to classical methods especially for the solution of integral and partial integro-differential equations, because they enable to represent the integral term by sparse or almost sparse matrices while the classical methods suffer from the fact that the matrices arising from discretization are full.[3, 12]

The quantitative properties of any wavelet method strongly depend on the used wavelet basis, namely on the length of the support of basis functions, the number of vanishing wavelet moments, the smoothness of basis functions and the condition number of the basis. Hence, a construction of appropriate wavelet bases is an important issue.

In this paper, we construct a cubic spline wavelet basis on the interval and we adapt this basis to homogeneous Dirichlet boundary conditions of the first order. The bases are well-conditioned, the wavelets have four vanishing moments and we show that the support is the shortest among all cubic spline wavelets with four vanishing moment corresponding to B-spline scaling functions. The wavelet basis is composed of scaling functions and inner and boundary wavelets. The inner wavelets are the same as wavelets constructed in Ref. 11, 28. We provide a rigorous proof of the Riesz basis property. While such proofs are usually based on semiorthogonality of the wavelets,[15,34] the local supports of the biorthogonal wavelets,[18,20] the estimates of the norms of certain projections[7,36] or spectral properties of the matrices of the inner products of primal and dual scaling functions,[8,9,24] we use a different approach that is based on analyzing the sets of inner and boundary wavelets separately and verifying the minimal angle condition between spaces generated by inner and boundary wavelets.

To illustrate the applicability of the bases we apply the Crank-Nicolson scheme combined with the Galerkin method with the constructed basis for option pricing under the double exponential jump-diffusion model that was proposed by Kou in Ref. 39. This model is represented by a nonstationary partial integro-differential equation. We show the decay of elements of the matrices arising from discretization of the integral term. Due to this decay the discretization matrices can be truncated and represented by quasi-sparse matrices while the most standard methods suffer from the fact that the discretization matrices are full. Since the basis functions are piecewise cubic we obtain a high order convergence and the problem can be resolved with the small number of degrees of freedom. We present numerical examples for European options and we compare the results with other cubic spline wavelet bases and with other methods. For more details about wavelet-Galerkin method and using this method for the numerical solution of various option pricing problems we refer the reader to Ref. 25, 31, 32, 44.

First, let us briefly recall constructions of cubic spline wavelet and multiwavelet bases on the interval. Several constructions of biorthogonal B-spline wavelet bases on the interval were proposed in Ref. 20. In these cases both the primal and dual wavelets are local, but the disadvantage of these bases is their relatively large condition number. Modifications of these constructions that lead to better conditioned bases can be found in Ref. 4, 5, 6, 22, 42. Biorthogonal cubic Hermite spline multiwavelet bases on the interval with local duals were designed in Ref. 18, 45. Several cubic spline wavelet and multiwavelet bases with nonlocal duals have been constructed and adapted to various types of boundary conditions in Ref. 7, 8, 9, 10, 15, 24, 30, 34, 35, 36, 38, 41, 43. The main advantages of these types of bases in comparison with bases with local duals are usually the shorter support of wavelets, the lower condition number of the basis and the corresponding stiffness matrices, and the simplicity of the construction.

We recall the concept of a wavelet basis and introduce the notation. Let (a,b) be a bounded interval. Let $L^2(a,b)$ be a Hilbert space of all Lebesgue measurable real-valued functions defined on (a,b) such that their L^2-norm

$$\|f\| = \left(\int_a^b f^2(x)\, dx \right)^{1/2} \tag{1.1}$$

is finite. This space is equipped with the inner product

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx. \tag{1.2}$$

Let $H^1(a,b)$ be the Sobolev space, i.e. the space of all functions from $L^2(a,b)$ for which their first-order weak derivatives also belong to $L^2(a,b)$.

We consider four spaces, the space $V^N(a,b) = L^2(a,b)$ and the subspaces of $L^2(a,b)$ incorporating homogeneous boundary conditions in one or both endpoints, namely the spaces

$$V^D(a,b) = \left\{v \in H^1(a,b) : v(a) = v(b) = 0\right\}, \tag{1.3}$$

$$V^L(a,b) = \left\{v \in H^1(a,b) : v(a) = 0\right\}, \tag{1.4}$$

$$V^R(a,b) = \left\{v \in H^1(a,b) : v(b) = 0\right\}, \tag{1.5}$$

equipped with inner product (1.2). We use the shorthand notation $V^r = V^r(0,1)$, $r = N, D, L, N$. We construct wavelet bases for these spaces and prove their L^2-stability.

Let $\mathcal{J}$ be a finite or countably infinite index set and let

$$\|\mathbf{v}\| = \sqrt{\sum_{\lambda\in\mathcal{J}} v_\lambda^2}, \quad \text{for} \quad \mathbf{v} = \{v_\lambda\}_{\lambda\in\mathcal{J}},\, v_\lambda \in \mathbb{R}, \tag{1.6}$$

and

$$l^2(\mathcal{J}) = \left\{\mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda\in\mathcal{J}},\, v_\lambda \in \mathbb{R}, \|\mathbf{v}\| < \infty\right\}. \tag{1.7}$$

For the operator $\mathbf{M} : l^2(\mathcal{J}) \longrightarrow l^2(\mathcal{J})$ we define its *spectral norm* as

$$\|\mathbf{M}\| = \sup_{0\neq\mathbf{v}\in l^2(\mathcal{J})} \frac{\|\mathbf{M}\mathbf{v}\|}{\|\mathbf{v}\|}. \tag{1.8}$$

Schur's Theorem implies that if M is symmetric and its 1-*norm* defined as

$$\|\mathbf{M}\|_1 = \sup_{j\in\mathcal{J}} \sum_{i\in\mathcal{J}} |\mathbf{M}_{i,j}| \tag{1.9}$$

is finite, then M is a bounded operator on $l^2(\mathcal{J})$ and $\|\mathbf{M}\| \le \|\mathbf{M}\|_1$. Let λ_i, $i \in \mathcal{J}$, be eigenvalues of $\mathbf{M}$ and let us denote

$$\lambda_{max}(\mathbf{M}) = \sup_{i\in\mathcal{J}} |\lambda_i|, \quad \lambda_{min}(\mathbf{M}) = \inf_{i\in\mathcal{J}} |\lambda_i|. \tag{1.10}$$

Let $H \subset L^2(a,b)$ be a real Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle_H$ and the norm $\|\cdot\|_H$. Our aim is to construct a wavelet basis for H in the sense of the following definition.

Definition 1.1. *Let $\mathcal{J}$ be at most countable index set where each index $\lambda \in \mathcal{J}$ takes the form $\lambda = (j,k)$ and let denote $|\lambda| = j \in \mathbb{Z}$. A family $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a wavelet basis of H, if*

i) Ψ is a Riesz basis for H, i.e. the span of Ψ is dense in H and there exist constants $c, C \in (0,\infty)$ such that

$$c\|\mathbf{b}\| \le \left\|\sum_{\lambda\in\mathcal{J}} b_\lambda \psi_\lambda\right\|_H \le C\|\mathbf{b}\|, \tag{1.11}$$

for all $\mathbf{b} = \{b_\lambda\}_{\lambda\in\mathcal{J}} \in l^2(\mathcal{J})$.

ii) The functions are local in the sense that

$$\textit{diam}\ \text{supp}\ \psi_\lambda \le C2^{-|\lambda|}, \quad \lambda \in \mathcal{J}, \tag{1.12}$$

where the constant C does not depend on λ, and at a given level j the supports of only finitely many wavelets overlap at any point x.

iii) The family Ψ *has the hierarchical structure*

$$\Psi = \Phi_{j_0} \cup \bigcup_{j=j_0}^{K} \Psi_j \tag{1.13}$$

for some $K \in \mathbb{N} \cup \{\infty\}$.

iv) There exists $L \geq 1$ *such that all functions* $\psi_\lambda \in \Psi_j$, $j_0 \leq j \leq K$, *have* L *vanishing moments, i.e.*

$$\int_a^b x^k \, \psi_\lambda(x) = 0, \quad k = 0, \ldots, L-1. \tag{1.14}$$

The definition of a wavelet basis is not unified in the mathematical literature and the conditions $i) - iv)$ from Definition 1.1 can be generalized. The functions from the set Φ_{j_0} are called *scaling functions* and the functions from the set Ψ_j, $j \geq j_0$, are called *wavelets* on the level j. Wavelets in the inner part of the interval are typically translations and dilations of one function ψ or several functions $\psi_1, \ldots, \psi_p$ also called *wavelets* and the functions near the boundary are derived from functions called *boundary wavelets*.

The Riesz basis property (1.11) is crucial for stability and accuracy of the computation and for many types of operator equations it guaranties that the diagonally preconditioned system matrix is well conditioned.[17, 19]

For the two countable sets of functions $\Gamma, \Theta \subset L^2(\Omega)$ the symbol $\langle \Gamma, \Theta \rangle$ denotes the matrix

$$\langle \Gamma, \Theta \rangle = \{\langle \gamma, \theta \rangle\}_{\gamma \in \Gamma, \theta \in \Theta}. \tag{1.15}$$

Remark 1.1. The constants

$$c_\Psi = \sup\{c : c \text{ satisfies (1.11)}\} \quad \text{and} \quad C_\Psi = \inf\{C : C \text{ satisfies (1.11)}\} \tag{1.16}$$

are called (optimal) *Riesz bounds* and the number *cond* $\Psi = C_\Psi / c_\Psi$ is called the *condition number* of Ψ. In some papers the squares of norms are used in (1.11) and Riesz bounds are defined as c_Ψ^2 and C_Ψ^2. The Gram matrix $\langle \Psi, \Psi \rangle$ can be finite or biinfinite and it is known that it represents a linear operator which is continuous, positive definite, and self-adjoint, and that the constants c_Ψ and C_Ψ satisfy

$$c_\Psi = \sqrt{\lambda_{min}(\langle \Psi, \Psi \rangle)}, \quad C_\Psi = \sqrt{\lambda_{max}(\langle \Psi, \Psi \rangle)}. \tag{1.17}$$

Remark 1.2. The set of functions is called *a Riesz sequence* in H if there exist positive constants c and C that satisfy (1.11) but the closure of this set is not necessarily H.

2. Construction of a cubic spline wavelet basis on the unit interval

We define a scaling basis as a basis of cubic B-splines in the same way as in Ref. 5, 15, 42. Let ϕ be a cubic B-spline defined on knots $[0, 1, 2, 3, 4]$. It can be written explicitly as

$$\phi(x) = \begin{cases} \frac{x^3}{6}, & x \in [0,1], \\ -\frac{x^3}{2} + 2x^2 - 2x + \frac{2}{3}, & x \in [1,2], \\ \frac{x^3}{2} - 4x^2 + 10x - \frac{22}{3}, & x \in [2,3], \\ \frac{(4-x)^3}{6}, & x \in [3,4], \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

Then ϕ satisfies *a scaling equation*

$$\phi(x) = \frac{\phi(2x)}{8} + \frac{\phi(2x-1)}{2} + \frac{3\phi(2x-2)}{4} + \frac{\phi(2x-3)}{2} + \frac{\phi(2x-4)}{8}. \tag{2.2}$$

We define three boundary scaling functions. Let ϕ_{b0} be a cubic B-spline defined on knots $[0,0,0,0,1]$, i.e.

$$\phi_{b0}(x) = \begin{cases} (1-x)^3, & x \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Furthermore, let ϕ_{b1} be a cubic B-spline defined on knots $[0,0,0,1,2]$ and ϕ_{b2} be a cubic B-spline defined on knots $[0,0,1,2,3]$. The explicit forms of ϕ_{b1} and ϕ_{b2} are

$$\phi_{b1}(x) = \begin{cases} \frac{7x^3}{4} - \frac{9x^2}{2} + 3x, & x \in [0,1], \\ \frac{(2-x)^3}{4}, & x \in [1,2], \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$

and

$$\phi_{b2}(x) = \begin{cases} -\frac{11x^3}{12} + \frac{3x^2}{2}, & x \in [0,1], \\ \frac{7x^3}{12} - 3x^2 + \frac{9x}{2} - \frac{3}{2}, & x \in [1,2], \\ \frac{(3-x)^3}{6}, & x \in [2,3], \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Then the functions ϕ_{b0}, ϕ_{b1}, and ϕ_{b2} satisfy scaling equations

$$\phi_{b0}(x) = \phi_{b0}(2x) + \frac{\phi_{b1}(2x)}{2}, \tag{2.6}$$

$$\phi_{b1}(x) = \frac{\phi_{b1}(2x)}{2} + \frac{3\phi_{b2}(2x)}{4} + \frac{3\phi(2x)}{16}, \tag{2.7}$$

$$\phi_{b2}(x) = \frac{\phi_{b2}(2x)}{4} + \frac{11\phi(2x)}{16} + \frac{\phi(2x-1)}{2} + \frac{\phi(2x-2)}{8}. \tag{2.8}$$

The graphs of the functions ϕ_{b0}, ϕ_{b1}, ϕ_{b2}, and ϕ are displayed in Figure 1.

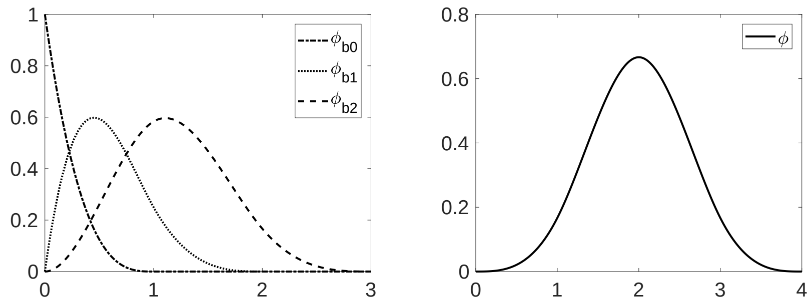

Fig. 1. The boundary scaling functions ϕ_{b0}, ϕ_{b1}, and ϕ_{b2} (left) and the scaling function ϕ (right).

For $j \geq 3$ and $x \in [0,1]$ we set

$$\phi_{j,k}(x) = \frac{2^{j/2}\phi(2^j x - k + 3)}{\|\phi\|}, \; k = 3, ..., 2^j, \tag{2.9}$$

$$\phi_{j,0}(x) = \frac{2^{j/2}\phi_{b0}(2^j x)}{\|\phi_{b0}\|}, \quad \phi_{j,2^j+3}(x) = \frac{2^{j/2}\phi_{b0}(2^j(1-x))}{\|\phi_{b0}\|},$$

$$\phi_{j,1}(x) = \frac{2^{j/2}\phi_{b1}(2^j x)}{\|\phi_{b1}\|}, \quad \phi_{j,2^j+2}(x) = \frac{2^{j/2}\phi_{b1}(2^j(1-x))}{\|\phi_{b1}\|},$$

$$\phi_{j,2}(x) = \frac{2^{j/2}\phi_{b2}(2^j x)}{\|\phi_{b2}\|}, \quad \phi_{j,2^j+1}(x) = \frac{2^{j/2}\phi_{b2}(2^j(1-x))}{\|\phi_{b2}\|}.$$

Hence, the basis functions $\phi_{j,k}$ are normalized with respect to the L^2-norm. Since there are four types of spaces, we define four types of scaling bases

$$\Phi_j^N = \left\{\phi_{j,k}, k = 0, \ldots, 2^j + 3\right\}, \tag{2.10}$$

$$\Phi_j^D = \left\{\phi_{j,k}, k = 1, \ldots, 2^j + 2\right\}, \tag{2.11}$$

$$\Phi_j^L = \left\{\phi_{j,k}, k = 1, \ldots, 2^j + 3\right\}, \tag{2.12}$$

$$\Phi_j^R = \left\{\phi_{j,k}, k = 0, \ldots, 2^j + 2\right\}, \tag{2.13}$$

for $j \geq 3$. We define a wavelet ψ as

$$\psi(x) = \phi(2x-1) - 4\phi(2x-2) + 6\phi(2x-3) - 4\phi(2x-4) + \phi(2x-5). \tag{2.14}$$

Then $\operatorname{supp} \psi = [0.5, 4.5]$ and ψ has four vanishing moments, i.e.

$$\int_{0.5}^{4.5} x^k \psi(x) dx = 0, \quad k = 0, 1, 2, 3. \tag{2.15}$$

It can be verified easily using substitution of (2.1) and (2.14) into (2.15).

In the following lemma we show that the wavelet ψ has the shortest possible support among all wavelets with four vanishing moments that are generated from cubic B-splines.

Lemma 2.1. *Let the function ϕ be given by (2.1). If*

$$\psi \in \overline{\operatorname{span}} \left\{\phi\left(2\cdot - k\right), \quad k \in \mathbb{Z}\right\} \tag{2.16}$$

and ψ has four vanishing moments, then the length of the support of ψ is at least four.

Proof. Since $\psi \in \overline{\operatorname{span}} \left\{\phi\left(2\cdot - k\right), k \in \mathbb{Z}\right\}$ we have

$$\psi\left(x\right) = \sum_{k \in \mathbb{Z}} h_k \phi\left(2x - k\right), \quad x \in \mathbb{R},$$

for some coefficients $h_k \in \mathbb{R}$. Let us suppose that the length of the support of ψ is at most four. Then

$$\operatorname{supp} \psi \subset \left[\frac{j}{2}, \frac{j+8}{2}\right] \tag{2.17}$$

for some $j \in \mathbb{Z}$. Since

$$\psi\left(x\right) = 0 \quad \text{for} \quad x \in \left[\frac{k}{2}, \frac{k+1}{2}\right], \quad k \in \mathbb{Z} \setminus \left\{j, j+1, \ldots, j+7\right\}, \tag{2.18}$$

and

$$\text{supp}\,\phi\,(2\cdot -k) = \left[\frac{k}{2}, \frac{k+4}{2}\right], \tag{2.19}$$

the coefficients satisfy

$$h_k = 0, \quad k \in \mathbb{Z}\backslash\left\{j, j+1, j+2, j+3, j+4\right\}. \tag{2.20}$$

Due to the four vanishing moments of the function ψ we obtain a homogeneous system of four linearly independent algebraic equations for the five parameters $h_j, \ldots, h_{j+4}$. Thus up to a multiplication by a constant and shifting by $k/2$, $k \in \mathbb{Z}$, there is only one wavelet that has the length of the support at most four and this wavelet is a wavelet defined by (2.14). □

The boundary wavelets ψ_{b1} and ψ_{b2} are defined by

$$\psi_{b1}(x) = \frac{25}{3}\phi_{b0}(2x) - \frac{385}{36}\phi_{b1}(2x) + \frac{1489}{216}\phi_{b2}(2x) - \frac{369}{160}\phi(2x) + \frac{2}{5}\phi(2x-1), \tag{2.21}$$

and

$$\psi_{b2}(x) = 6\phi_{b1}(2x) - \frac{57}{5}\phi_{b2}(2x) + \frac{919}{100}\phi(2x) - \frac{116}{25}\phi(2x-1) + \phi(2x-2). \tag{2.22}$$

Then $\text{supp}\,\psi_{b1} = [0, 2.5]$, $\text{supp}\,\psi_{b2} = [0, 3]$, and both wavelets have four vanishing moments, i.e.

$$\int_0^{2.5} x^k \psi_{b1}(x)dx = 0 \quad \text{and} \quad \int_0^{3} x^k \psi_{b2}(x)dx = 0, \tag{2.23}$$

for $k = 0, 1, 2, 3$.

Using the similar argument as in the proof of Lemma 2.1 it is easy to see that also the boundary wavelets ψ_{b1} and ψ_{b2} have the shortest possible supports among all boundary wavelets with four vanishing moments corresponding to scaling functions defined by (2.9) and that other such boundary wavelets with the supports in $[0, 3]$ are linear combinations of ψ_{b1} and ψ_{b2}.

Then the wavelet basis on the level $j \geq 3$ is defined as

$$\Psi_j^N = \left\{\psi_{j,k}, k = 1, \ldots, 2^j\right\}, \tag{2.24}$$

where

$$\psi_{j,k}(x) = \frac{2^{j/2}\psi(2^j x - k + 2)}{\|\psi\|}, \quad k = 3, ..., 2^j - 2, \tag{2.25}$$

$$\psi_{j,1}(x) = \frac{2^{j/2}\psi_{b1}(2^j x)}{\|\psi_{b1}\|}, \quad \psi_{j,2^j}(x) = \frac{2^{j/2}\psi_{b1}(2^j(1-x))}{\|\psi_{b1}\|}, \tag{2.26}$$

$$\psi_{j,2}(x) = \frac{2^{j/2}\psi_{b2}(2^j x)}{\|\psi_{b2}\|}, \quad \psi_{j,2^j-1}(x) = \frac{2^{j/2}\psi_{b2}(2^j(1-x))}{\|\psi_{b2}\|}. \tag{2.27}$$

Hence, the basis functions $\psi_{j,k}$ are also normalized with respect to the L^2-norm. To adapt the set Ψ_j^N to homogeneous Dirichlet boundary conditions we have to replace the function ψ_{b1} which is not equal to zero in the point 0 with another function. We denote this function as ψ_{b3} and we define it as

$$\psi_{b3}(x) = \frac{7}{3}\phi_{b2}(2x) - \frac{319}{60}\phi(2x) + \frac{101}{15}\phi(2x-1) - \frac{25}{6}\phi(2x-2) + \phi(2x-3). \tag{2.28}$$

Then the wavelet ψ_{b3} has also four vanishing moments and its support is $[0, 3.5]$. We define the boundary functions on the level $j \geq 3$ that are adapted to homogeneous Dirichlet boundary conditions as

$$\psi_{j,1}^D(x) = \frac{2^{j/2}\psi_{b2}(2^j x)}{\|\psi_{b2}\|}, \quad \psi_{j,2^j}^D(x) = \frac{2^{j/2}\psi_{b2}(2^j(1-x))}{\|\psi_{b2}\|}, \tag{2.29}$$
$$\psi_{j,2}^D(x) = \frac{2^{j/2}\psi_{b3}(2^j x)}{\|\psi_{b3}\|}, \quad \psi_{j,2^j-1}^D(x) = \frac{2^{j/2}\psi_{b3}(2^j(1-x))}{\|\psi_{b3}\|}.$$

We set

$$\Psi_j^D = \left\{\psi_{j,k}^D, k = 1, 2, 2^j - 1, 2^j\right\} \cup \left\{\psi_{j,k}, k = 3, \ldots, 2^j - 2\right\}, \tag{2.30}$$
$$\Psi_j^L = \left\{\psi_{j,k}^D, k = 1, 2\right\} \cup \left\{\psi_{j,k}, k = 3, \ldots, 2^j\right\}, \tag{2.31}$$
$$\Psi_j^R = \left\{\psi_{j,k}, k = 1, \ldots, 2^j - 2\right\} \cup \left\{\psi_{j,k}^D, k = 2^j - 1, 2^j\right\}. \tag{2.32}$$

The graphs of wavelets ψ_{b1}, ψ_{b2}, ψ_{b3}, and ψ are displayed in Figure 2.

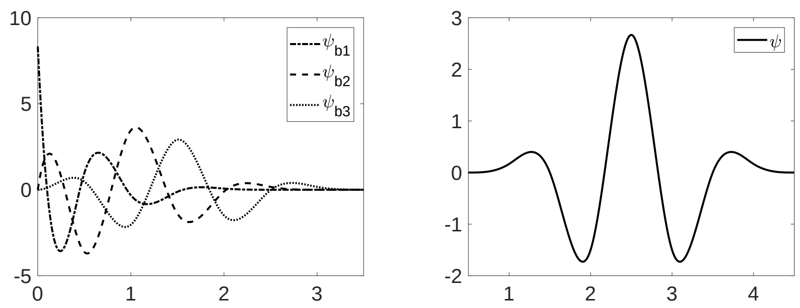

Fig. 2. The boundary wavelets ψ_{b1}, ψ_{b2}, ψ_{b3}, and the wavelet ψ.

Our aim is to show that for $r = N, D, L, R$, the set

$$\Psi^r = \Phi_3^r \cup \bigcup_{j=3}^{\infty} \Psi_j^r, \tag{2.33}$$

is a Riesz basis of the space V^r. We denote the finite-dimensional subset of Ψ^r with s levels of wavelets as

$$\Psi_s^r = \Phi_3^r \cup \bigcup_{j=3}^{2+s} \Psi_j^r. \tag{2.34}$$

First we define auxiliary bases $\check{\Psi}^r$ and $\check{\Psi}_j^r$, $r = N, D, L, R$, that contain the same inner wavelets and generate the same spaces, i.e. $\operatorname{span} \check{\Psi}_j^r = \operatorname{span} \Psi_j^r$, but boundary wavelets are different. The reason is that some constants characterizing the bases that are used in the proof of the Riesz basis property are too large for the bases Ψ^r. Hence, let us define

$$\check{\psi}_{b1}(x) = \psi_{b1}(x) - 0.3\psi_{b2}(x), \quad \check{\psi}_{b2}(x) = 0.3\psi_{b1}(x) + 0.4\psi_{b2}(x), \tag{2.35}$$

and

$$\check{\psi}_{j,1}(x) = \frac{2^{j/2}\check{\psi}_{b1}(2^j x)}{\|\check{\psi}_{b1}\|}, \quad \check{\psi}_{j,2^j}(x) = \frac{2^{j/2}\check{\psi}_{b1}(2^j(1-x))}{\|\check{\psi}_{b1}\|}, \tag{2.36}$$

$$\check{\psi}_{j,2}(x) = \frac{2^{j/2}\check{\psi}_{b2}(2^j x)}{\|\check{\psi}_{b2}\|}, \quad \check{\psi}_{j,2^j-1}(x) = \frac{2^{j/2}\check{\psi}_{b2}(2^j(1-x))}{\|\check{\psi}_{b2}\|}. \tag{2.37}$$

and the wavelet basis on the level $j \geq 3$ is defined as

$$\check{\Psi}_j^N = \left\{\check{\psi}_{j,k}, k = 1, 2, 2^j - 1, 2^j\right\} \cup \left\{\psi_{j,k}, k = 3, \ldots, 2^j - 2\right\}. \tag{2.38}$$

We define

$$\check{\psi}_{b3}(x) = 0.9\psi_{b2}(x) + 1.63\psi_{b3}(x), \quad \check{\psi}_{b4}(x) = 0.69\psi_{b2}(x) - 0.4\psi_{b3}(x), \tag{2.39}$$

and

$$\check{\psi}_{j,1}^D(x) = \frac{2^{j/2}\check{\psi}_{b3}(2^j x)}{\|\check{\psi}_{b3}\|}, \quad \check{\psi}_{j,2^j}^D(x) = \frac{2^{j/2}\check{\psi}_{b3}(2^j(1-x))}{\|\check{\psi}_{b3}\|}, \tag{2.40}$$

$$\check{\psi}_{j,2}^D(x) = \frac{2^{j/2}\check{\psi}_{b4}(2^j x)}{\|\check{\psi}_{b4}\|}, \quad \check{\psi}_{j,2^j-1}^D(x) = \frac{2^{j/2}\check{\psi}_{b4}(2^j(1-x))}{\|\check{\psi}_{b4}\|}. \tag{2.41}$$

and we set

$$\check{\Psi}_j^D = \left\{\check{\psi}_{j,1}^D, \check{\psi}_{j,2}^D, \check{\psi}_{j,2^j-1}^D, \check{\psi}_{j,2^j}^D\right\} \cup \left\{\psi_{j,k}, k = 3, \ldots, 2^j - 2\right\}, \tag{2.42}$$

$$\check{\Psi}_j^L = \left\{\check{\psi}_{j,1}^D, \check{\psi}_{j,2}^D, \check{\psi}_{j,2^j-1}, \check{\psi}_{j,2^j}\right\} \cup \left\{\psi_{j,k}, k = 3, \ldots, 2^j - 2\right\}, \tag{2.43}$$

$$\check{\Psi}_j^R = \left\{\check{\psi}_{j,1}, \check{\psi}_{j,2}, \check{\psi}_{j,2^j-1}^D, \check{\psi}_{j,2^j}^D\right\} \cup \left\{\psi_{j,k}, k = 3, \ldots, 2^j - 2\right\}, \tag{2.44}$$

and for $r = N, D, L, R$, we set

$$\check{\Psi}^r = \Phi_3^r \cup \bigcup_{j=3}^{\infty} \check{\Psi}_j^r, \quad \check{\Psi}_s^r = \Phi_3^r \cup \bigcup_{j=3}^{2+s} \check{\Psi}_j^r, \quad s \in \mathbb{N}. \tag{2.45}$$

Now we formulate sufficient conditions under which the sum of Riesz sequences is a Riesz sequence. We employ frame theory from Ref. 13, 14, 27. Let us recall that $\{f_k\}_{k\in\mathcal{I}}$ is a *frame sequence*, if there exist constants $\tilde{c}_f, \tilde{C}_f > 0$ such that

$$\tilde{c}_f \|f\|^2 \leq \sum_{k\in\mathcal{I}} |\langle f, f_k\rangle|^2 \leq \tilde{C}_f \|f\|^2 \tag{2.46}$$

hold for all functions $f \in \overline{\text{span}}\left(\{f_k\}_{k\in\mathcal{I}}\right)$. Supremum of $\{\tilde{c}_f : \tilde{c}_f \text{ satisfies (2.46)}\}$ is called a *lower frame bound*.

For the spaces F and G which are subspaces of the space $L^2(0,1)$, we define two notions of cosine:

$$\cos(F,G) = \sup_{0\neq f\in\tilde{F}, 0\neq g\in\tilde{G}} \frac{|\langle f, g\rangle|}{\|f\|\,\|g\|}, \quad \overline{\cos}(F,G) = \sup_{0\neq f\in F, 0\neq g\in G} \frac{|\langle f, g\rangle|}{\|f\|\,\|g\|}, \tag{2.47}$$

where $\tilde{F} = F \cap (F\cap G)^{\perp}$, $\tilde{G} = G \cap (F\cap G)^{\perp}$, and $M^{\perp}$ denotes the orthogonal complement of M. Clearly $0 \leq \cos(F,G) \leq \overline{\cos}(F,G) \leq 1$. The following theorem was derived in Ref. 27.

Theorem 2.1. *Let $\mathcal{I}$ and $\mathcal{J}$ be countable index sets and let $\{f_k\}_{k\in\mathcal{I}}$ and $\{g_l\}_{l\in\mathcal{J}}$ be frame sequences in $L^2(0,1)$ with lower frame bounds $\tilde{c}_f$ and $\tilde{c}_g$, respectively. If $\cos(F,G) < 1$*

for $F = \overline{\operatorname{span}}\left(\{f_k\}_{k\in\mathcal{I}}\right)$ *and* $G = \overline{\operatorname{span}}\left(\{g_l\}_{l\in\mathcal{J}}\right)$*, then* $\{f_k\}_{k\in\mathcal{I}} \cup \{g_l\}_{l\in\mathcal{J}}$ *is a frame sequence with a lower frame bound*

$$\tilde{c}_{f,g} = (1 - \cos(F,G)) \cdot \min(\tilde{c}_f, \tilde{c}_g). \tag{2.48}$$

The consequence of this theorem is the following theorem about the Riesz sequences.

Theorem 2.2. *Let* $\mathcal{I}$ *and* $\mathcal{J}$ *be countable index sets,* $\{f_k\}_{k\in\mathcal{I}}$ *be a Riesz sequence with a Riesz lower bound* c_f *and* $\{g_l\}_{l\in\mathcal{J}}$ *be a Riesz sequence with a Riesz lower bound* c_g*,* $F = \overline{\operatorname{span}}\left(\{f_k\}_{k\in\mathcal{I}}\right)$ *and* $G = \overline{\operatorname{span}}\left(\{g_l\}_{l\in\mathcal{J}}\right)$*. If* $\overline{\cos}(F,G) < 1$*, then* $\{f_k\}_{k\in\mathcal{I}} \cup \{g_l\}_{l\in\mathcal{J}}$ *is a Riesz sequence and its Riesz lower bound* c *satisfies*

$$c \geq \sqrt{1 - \overline{\cos}(F,G)} \cdot \min(c_f, c_g). \tag{2.49}$$

Proof. It is known that if $\{f_k\}_{k\in\mathcal{I}}$ is a Riesz sequence, then $\{f_k\}_{k\in\mathcal{I}}$ is a frame sequence.[14] From Theorem 2.1 and the fact that $\cos(F,G) \leq \overline{\cos}(F,G) < 1$ it follows that $\{f_k\}_{k\in\mathcal{I}} \cup \{g_l\}_{l\in\mathcal{J}}$ is a frame sequence. Furthermore, $\overline{\cos}(F,G) < 1$ implies that $F \cap G = \{0\}$ and thus the set $\{f_k\}_{k\in\mathcal{I}} \cup \{g_l\}_{l\in\mathcal{J}}$ is ω-independent, i.e.

$$\sum_{\lambda\in\mathcal{J}} c_\lambda f_\lambda = 0 \tag{2.50}$$

implies that $c_\lambda = 0$ for all $\lambda \in \mathcal{J}$. Indeed, if $f \in \{f_k\}_{k\in\mathcal{I}} \cup \{g_l\}_{l\in\mathcal{J}}$ is nonzero function, then

$$\overline{\cos}(F,G) = \sup_{0\neq u\in F, 0\neq v\in G} \frac{|\langle u,v\rangle|}{\|u\|\,\|v\|} \geq \frac{|\langle f,f\rangle|}{\|f\|\,\|f\|} = 1, \tag{2.51}$$

which is the contradiction. Since a frame sequence of ω-independent functions with a frame lower bound b is a Riesz sequence with a Riesz lower bound $\sqrt{b}$,[14] the Theorem 2.2 is proved. □

In the following theorem we derive the upper bound for $\overline{\cos}(F,G)$.

Theorem 2.3. *Let* $\mathcal{I}$ *and* $\mathcal{J}$ *be countable index sets,* $\{f_k\}_{k\in\mathcal{I}}$ *be a Riesz sequence with a Riesz lower bound* c_f *and* $\{g_l\}_{l\in\mathcal{J}}$ *be a Riesz sequence with a Riesz lower bound* c_g*. Then*

$$\overline{\cos}(F,G) \leq \frac{\|\mathbf{G}\|}{c_f c_g}, \tag{2.52}$$

where the entries of the matrix $\mathbf{G}$ *are defined by* $\mathbf{G}_{k,l} = \langle f_k, g_l\rangle$*,* $k \in \mathcal{I}$*,* $l \in \mathcal{J}$*.*

Proof. For $u = \sum_{k\in\mathcal{I}} (c_u)_k f_k$ and $v = \sum_{l\in\mathcal{J}} (c_v)_l g_l$ we have

$$\sup_{0\neq v\in G} \frac{|\langle u,v\rangle|}{\|v\|} \leq \sup_{0\neq c_v\in l^2(J)} \frac{|\langle \mathbf{G}c_u, c_v\rangle|}{c_g\,\|c_v\|} = \frac{\|\mathbf{G}c_u\|}{c_g} \leq \frac{\|\mathbf{G}\|\,\|c_u\|}{c_g} \leq \frac{\|\mathbf{G}\|\,\|u\|}{c_f c_g}. \tag{2.53}$$

Hence, we have

$$\overline{\cos}(F,G) = \sup_{0\neq u\in F, 0\neq v\in G} \frac{|\langle u,v\rangle|}{\|u\|\,\|v\|} \leq \frac{\|\mathbf{G}\|}{c_f c_g}. \tag{2.54}$$

This completes the proof. □

Now we focus on the inner wavelets that are translations and dilations of ψ. We show that the set of all these wavelets is a Riesz sequence and estimate its Riesz lower bound.

Theorem 2.4. *The set $\Psi_I = \left\{\psi_{j,k}, k = 3, \ldots, 2^j - 2, j \geq 3\right\}$ is a Riesz sequence with a Riesz lower bound $c_I > 0.783$.*

Proof. It was shown in Ref. 11 and Ref. 28 that

$$\left\{2^{j/2}\psi\left(2^j x - k\right), j, k \in \mathbb{Z}\right\} \tag{2.55}$$

is a Riesz basis of the space $L^2(\mathbb{R})$. Since Ψ_I is its subset, it is a Riesz sequence. We estimate its Riesz lower bound. Let c_I^K be a Riesz lower bound of the Riesz sequence $\Psi_I^K = \left\{\psi_{j,k}, k = 3, \ldots, 2^j - 2, 3 \leq j \leq K\right\}$ which is clearly a subset of Ψ_I. We estimate the Riesz lower bounds c_I^K using Remark 1.1. Due to the structure of the set Ψ_I^K, the Gram matrix $\mathbf{G}_I^K = \left\langle \Psi_I^K, \Psi_I^K \right\rangle$ has the block structure

$$\mathbf{G}_I^K = \begin{pmatrix} \mathbf{G}_{3,3} & \mathbf{G}_{3,4} & \ldots & \mathbf{G}_{3,K} \\ \mathbf{G}_{4,3} & \mathbf{G}_{4,4} & \ldots & \mathbf{G}_{4,K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}_{K,3} & \mathbf{G}_{K,4} & \ldots & \mathbf{G}_{K,K} \end{pmatrix}, \tag{2.56}$$

where $\mathbf{G}_{i,j} = \left\langle \Psi_i^I, \Psi_j^I \right\rangle$ for $\Psi_j^I = \left\{\psi_{j,k}, k = 3, \ldots, 2^j - 2\right\}$. The matrix $\mathbf{G}_I^K$ is symmetric and due to the normalization of basis functions it has ones on the diagonal. Let $\mathbf{F}_1^K$ be a matrix that contains the diagonal blocks and the blocks next to the diagonal blocks of the matrix $\mathbf{G}_I^K$, but without the diagonal entries, i.e.

$$\mathbf{F}_1^K = \begin{pmatrix} \mathbf{G}_{3,3} & \mathbf{G}_{3,4} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{G}_{4,3} & \mathbf{G}_{4,4} & \mathbf{G}_{4,5} & & \vdots \\ \mathbf{0} & \mathbf{G}_{5,4} & \mathbf{G}_{5,5} & \ddots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \mathbf{G}_{K-1,K} \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{G}_{K,K-1} & \mathbf{G}_{K,K} \end{pmatrix} - \mathbf{I}, \tag{2.57}$$

where $\mathbf{I}$ is an identity matrix and $\mathbf{0}$ are zero matrices of appropriate sizes. Let $\mathbf{F}_2^K = \mathbf{G}_I^K - \mathbf{F}_1^K - \mathbf{I}$. Let $\mathbf{x}$ be a normalized eigenvector corresponding to $\lambda_{min}\left(\mathbf{G}_I^K\right)$. Then

$$\begin{aligned} \lambda_{min}\left(\mathbf{G}_I^K\right) &= \mathbf{x}^T\mathbf{G}_I^K\mathbf{x} = \mathbf{x}^T\mathbf{I}\mathbf{x} + \mathbf{x}^T\mathbf{F}_1^K\mathbf{x} + \mathbf{x}^T\mathbf{F}_2^K\mathbf{x} \\ &\geq 1 - \left|\mathbf{x}^T\mathbf{F}_1^K\mathbf{x}\right| - \left|\mathbf{x}^T\mathbf{F}_2^K\mathbf{x}\right|. \end{aligned} \tag{2.58}$$

For $k \geq 3$ direct computation yields

$$\left(\mathbf{G}_{k,k}\right)_{i,j} = \begin{cases} 1, & i = j, \\ -\frac{7}{2652}, & |i-j| = 1, \\ -\frac{7}{78}, & |i-j| = 2, \\ \frac{7}{2652}, & |i-j| = 3, \\ 0, & \text{otherwise.} \end{cases} \tag{2.59}$$

The nonzero entries of the matrix $\mathbf{G}_{k+1,k}$ are given by

$$\left(\mathbf{G}_{k+1,k}\right)_{i,j} = h_{i-2j+1}, \tag{2.60}$$

for $i, j \in \mathbb{N}$ such that $1 \leq i \leq 2^{k+1}$, $1 \leq j \leq 2^k$, and $-3 \leq i - 2j + 1 \leq 8$, and

$$h_k = \int_{0.5}^{4.5} \sqrt{2}\psi(x)\,\psi(2x-k)\,dx. \tag{2.61}$$

Since ψ is a piecewise polynomial function, it is easy to compute the exact values of h_k. We obtain

$$[h_{-3}, \ldots, h_2] = a\left[\frac{1}{161280}, \frac{169}{23040}, -\frac{1187}{23040}, \frac{679}{4608}, -\frac{16643}{80640}, \frac{1189}{11520}\right] \tag{2.62}$$

with

$$a = \frac{105\sqrt{2}}{442}, \qquad \text{and} \quad h_k = h_{5-k},\ k = 3, \ldots, 8. \tag{2.63}$$

Due to the symmetry of the matrix $\mathbf{G}_I^K$ we have $\mathbf{G}_{k,k+1} = \mathbf{G}_{k+1,k}^T$.

The matrix $\mathbf{F}_1^K$ is symmetric and thus $\left|\mathbf{x}^T\mathbf{F}_1^K\mathbf{x}\right| \leq \lambda_{max}\left(\mathbf{F}_1^K\right)$ and

$$\lambda_{max}\left(\mathbf{F}_1^K\right) = \left\|\mathbf{F}_1^K\right\| = \left\|\left(\mathbf{F}_1^K\right)^m\right\|^{1/m} \tag{2.64}$$

for all $m \in \mathbb{N}$. Since the matrix $\mathbf{F}_1^K$ is known in the explicit form, the number of nonzero entries of $\mathbf{F}_1^K$ in any row and column is bounded by a constant C independent of K, and the matrix $\mathbf{F}_1^K$ has repeated structure, we are able to compute $\left\|\left(\mathbf{F}_1^K\right)^m\right\|_1$. For $m = 16$ we obtain

$$\lambda_{max}\left(\mathbf{F}_1^K\right) = \left\|\left(\mathbf{F}_1^K\right)^m\right\|^{1/m} \leq \left\|\left(\mathbf{F}_1^K\right)^m\right\|_1^{1/m} \leq 0.375. \tag{2.65}$$

Since $\mathbf{F}_2^K$ is also symmetric, we have $\left|\mathbf{x}^T\mathbf{F}_2^K\mathbf{x}\right| \leq \lambda_{max}\left(\mathbf{F}_2^K\right)$ and

$$\lambda_{max}\left(\mathbf{F}_2^K\right) = \left\|\mathbf{F}_2^K\right\| \leq \left\|\mathbf{F}_2^K\right\|_1 < 0.011. \tag{2.66}$$

In summary, we have $\lambda_{min}\left(\mathbf{G}_I^K\right) \geq 1 - 0.375 - 0.011 = 0.614$ and $c_I^K \geq \sqrt{0.614}$ for all $K \geq 3$ and thus $c_I \geq \sqrt{0.614} > 0.783$. □

Now we estimate Riesz lower bounds for the sets of boundary wavelets.

Theorem 2.5. *a) The set* $\check{\Psi}_L^N = \left\{\check{\psi}_{j,1}, \check{\psi}_{j,2}, j \geq 3\right\}$ *is a Riesz sequence with a Riesz lower bound* $\check{c}_L^N > 0.490$.
b) The set $\check{\Psi}_L^D = \left\{\check{\psi}_{j,1}^D, \check{\psi}_{j,2}^D, j \geq 3\right\}$ *is a Riesz sequence with a Riesz lower bound* $\check{c}_L^D > 0.380$.

Proof. a) According to Remark 1.1 the set $\check{\Psi}_L^N$ is a Riesz sequence if and only if the extremal eigenvalues of the (biinfinite) matrix $\mathbf{G}_L^N = \left\langle\check{\Psi}_L^N, \check{\Psi}_L^N\right\rangle$ satisfy

$$0 < \lambda_{min}\left(\mathbf{G}_L^N\right) \leq \lambda_{max}\left(\mathbf{G}_L^N\right) < \infty. \tag{2.67}$$

We denote

$$\check{\Psi}_L^{N,K} = \left\{\check{\psi}_{j,1}, \check{\psi}_{j,2}, 3 \leq j \leq K\right\}, \quad \mathbf{G}_L^{N,K} = \left\langle\check{\Psi}_L^{N,K}, \check{\Psi}_L^{N,K}\right\rangle. \tag{2.68}$$

Due to the length of the supports of the functions $\check{\psi}_{j,1}$ and $\check{\psi}_{j,2}$, and four vanishing moments of the wavelets, we have

$$\left(\mathbf{G}_L^{N,K}\right)_{i,j} = 0, \quad \text{if } |i-j| > 5. \tag{2.69}$$

Hence, the matrix $\mathbf{G}_L^{N,K}$ is banded. Moreover it is symmetric, it has repeated structure and it is known in an explicit form. Therefore, it is easy to compute an estimate of its 1-norm. We have

$$\lambda_{max}\left(\mathbf{G}_L^{N,K}\right) \leq \left\|\mathbf{G}_L^{N,K}\right\|_1 \leq 1.763. \tag{2.70}$$

Let $\mathbf{H}^K = \mathbf{G}_L^{N,K} - \mathbf{I}$, where $\mathbf{I}$ is the identity matrix of the appropriate size. We have

$$\lambda_{min}\left(\mathbf{G}_L^{N,K}\right) \geq 1 - \left\|\mathbf{H}^K\right\| \geq 1 - \left\|\left(\mathbf{H}^K\right)^m\right\|_1^{1/m} \geq 0.241 \tag{2.71}$$

for $m = 4$. Since the estimates in (2.70) and (2.71) do not depend on the maximal level K, the condition (2.67) is satisfied and therefore the set $\check{\Psi}_L^N$ is a Riesz sequence and its Riesz lower bound satisfies $\check{c}_L^N > \sqrt{0.241} > 0.490$.

b) The proof follows the lines of the proof of the part a) with $m = 8$. □

Corollary 2.1. *Since*

$$\check{\Psi}_R^N = \left\{\check{\psi}_{j,2^j-1}, \check{\psi}_{j,2^j}, j \geq 3\right\} \tag{2.72}$$

has similar structure as $\check{\Psi}_L^N$*, the set* $\check{\Psi}_R^N$ *is a Riesz sequence with a Riesz lower bound* $\check{c}_R^N = \check{c}_L^N$*. Due to the non-overlapping supports of* $\check{\psi}_{j,k}$*,* $k = 1, 2$*, and* $\check{\psi}_{j,l}$*,* $l = 2^j - 1, 2^j$*, the set* $\check{\Psi}_b^N = \check{\Psi}_L^N \cup \check{\Psi}_R^N$ *is also a Riesz sequence with a Riesz lower bound* $\check{c}_b^N = \check{c}_L^N = \check{c}_R^N$*. Similarly, we define*

$$\check{\Psi}_R^D = \left\{\check{\psi}_{j,2^j-1}^D, \check{\psi}_{j,2^j}^D, j \geq 3\right\}, \tag{2.73}$$

and

$$\check{\Psi}_b^D = \check{\Psi}_L^D \cup \check{\Psi}_R^D, \quad \check{\Psi}_b^L = \check{\Psi}_L^D \cup \check{\Psi}_R^N, \quad \check{\Psi}_b^R = \check{\Psi}_L^N \cup \check{\Psi}_R^D. \tag{2.74}$$

We denote their Riesz lower bounds by $\check{c}_b^D$*,* $\check{c}_b^L$*, and* $\check{c}_b^R$*, respectively. We have* $\check{c}_b^D = \check{c}_L^D$*,* $\check{c}_b^L = \check{c}_b^R = \min\left(\check{c}_L^N, \check{c}_L^D\right)$*.*

For $r = N, D, L, R$ we denote the set of all wavelets as $\check{\Psi}_m^r = \check{\Psi}_b^r \cup \Psi_I$. In the following theorem we prove that $\check{\Psi}_m^r$ is a Riesz sequence.

Theorem 2.6. *The sets* $\check{\Psi}_m^r$*,* $r = N, D, L, R$*, are Riesz sequences in the space* V^r*.*

Proof. From Theorem 2.4 and Corollary 2.1 we already know that Ψ_I and $\check{\Psi}_b^r$ for $r = N, D, L, R$, are Riesz sequences. Thus, due to Theorem 2.2 and Theorem 2.3 to prove that $\check{\Psi}_m^r$ is a Riesz sequence it remains to show that the matrix $\mathbf{H}^r = \left\langle \check{\Psi}_b^r, \Psi_I \right\rangle$ satisfies

$$\frac{\|\mathbf{H}^r\|}{\check{c}_b^r c_I} < 1. \tag{2.75}$$

Using the relation

$$\|\mathbf{H}^r\| = \sqrt{\left\|\mathbf{H}^r\left(\mathbf{H}^r\right)^T\right\|} \leq \left\|\left(\mathbf{H}^r\left(\mathbf{H}^r\right)^T\right)^m\right\|_1^{1/2m} \tag{2.76}$$

for $m = 8$ we obtain $\left\|\mathbf{H}^D\right\| < 0.293$ and $\left\|\mathbf{H}^N\right\| < 0.278$ and we may conclude that (2.75) is satisfied and that $\check{\Psi}_m^D$ and $\check{\Psi}_m^N$ are Riesz sequences. Hence, also $\check{\Psi}_L^D \cup \Psi_I$, $\check{\Psi}_L^N \cup \Psi_I$, $\Psi_I \cup \check{\Psi}_R^D$, and $\Psi_I \cup \check{\Psi}_L^D$, are Riesz sequences and due to the non-overlapping supports of the functions from $\check{\Psi}_L^r$ and $\check{\Psi}_R^r$, the sets $\check{\Psi}_m^L$ and $\check{\Psi}_m^R$ are also Riesz sequences. □

Theorem 2.7. *For $r = N, D, L, R$, the set Ψ^r is a Riesz basis of the space V^r.*

Proof. First we show that the set

$$\check{\Psi}^r = \Phi_3^r \cup \check{\Psi}_m^r \tag{2.77}$$

is a Riesz basis of the space V^r. We already know that $\check{\Psi}_m^r$ is a Riesz sequence. Since Φ_3^r is a finite set of functions, it is a Riesz sequence too. Therefore, the spaces

$$F = \overline{\mathrm{span}\check{\Psi}_m^a} \quad \text{and} \quad G = \mathrm{span}\Phi_3^a \tag{2.78}$$

are closed and since G is finite-dimensional, the set $F + G$ is also closed. The closedness of $F + G$ and the fact that $F \cap G = \{0\}$ implies $\overline{\cos}(F, G) < 1$.[21] Since it is known that the spline spaces $V_j^r = \mathrm{span}\,\Phi_j^r = \mathrm{span}\,\check{\Psi}_{j-3}^r$ are nested and their union is dense in V^r, see e.g. Ref. 5, 42, the set $\check{\Psi}^r$ is dense in V^r. Due to this fact and Theorem 2.2, the set $\check{\Psi}^r$ is a Riesz basis of V^r.

Since $\check{\Psi}^r = \mathbf{M}\Psi^r$, where $\mathbf{M}$ is a biinfinite block diagonal matrix such that the first diagonal block of $\mathbf{M}$ is an identity matrix and other diagonal blocks are of the form

$$\begin{pmatrix} a_1 & a_2 & 0 & \dots & 0 & 0 & 0 \\ a_3 & a_4 & 0 & & & & 0 \\ 0 & 0 & 1 & & & & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & & & & 1 & 0 & 0 \\ 0 & & & & 0 & b_1 & b_2 \\ 0 & 0 & 0 & \dots & 0 & b_3 & b_4 \end{pmatrix}, \tag{2.79}$$

where a_i and b_i, $i = 1, \dots, 4$, are determined by the relations (2.35) and (2.39). Since $\|\mathbf{M}\|$ and $\|\mathbf{M}^{-1}\|$ are bounded and

$$\langle \Psi^r, \Psi^r \rangle = \mathbf{M}^{-1} \langle \check{\Psi}^r, \check{\Psi}^r \rangle \left(\mathbf{M}^{-1}\right)^T \tag{2.80}$$

the Remark 1.1 implies that Ψ^r is a Riesz basis of V^r. □

Remark 2.1. A wavelet basis on a general domain can be constructed in the following way: First, the wavelet basis for the space $V^r(a, b)$ is derived from Ψ^r using a simple linear transformation $y = (x - a)/(b - a)$. Then a wavelet basis on the hyperrectangle can be constructed using an isotropic, anisotropic, or sparse tensor product. Finally, by splitting the domain into subdomains which are images of the hyperrectangle under appropriate parametric mappings one can obtain a wavelet basis on a fairly general domain.

3. Numerical results

In this section we use the Galerkin method with the constructed wavelets for valuation of options under a double exponential jump-diffusion model proposed by Kou in Ref. 39 and we compare the results with other approaches and other cubic spline wavelet bases.

Let S be the price of the underlying asset, t represent time to maturity, r be a risk-free rate and $U(S, t)$ be the market price of the option. Then the general jump-diffusion models are represented by the equation

$$\frac{\partial U}{\partial t} - \mathcal{D}(U) - \mathcal{I}(U) = 0,\; S > 0,\; t \in (0, T), \tag{3.1}$$

where the operators $\mathcal{D}$ and $\mathcal{I}$ are given by

$$\mathcal{D}(U) = \frac{\sigma^2 S^2}{2} \frac{\partial^2 U}{\partial S^2} + (r - \lambda \kappa) S \frac{\partial U}{\partial S} - (r + \lambda) U \tag{3.2}$$

and

$$\mathcal{I}(U) = \lambda \int_{-\infty}^{\infty} U(Se^x, t)\, g(x)\, dx. \tag{3.3}$$

The parameter λ is the intensity of the price jumps, i.e. the average number of jumps per unit time. The function g represents the probability density function which in the model of Kou is given by

$$g(x) = p\,\eta_1 e^{-\eta_1 x} H(x) + q\,\eta_2 e^{\eta_2 x} H(-x), \quad x \in \mathbb{R}, \tag{3.4}$$

where H denotes the Heaviside function, $p \in (0,1)$ represents the probability of the upward jump, $q = 1 - p$ represents the probability of the downward jump, $\eta_1 > 1$, and $\eta_2 > 0$. The parameter κ in this model is given by

$$\kappa = \frac{p\,\eta_1}{\eta_1 - 1} + \frac{q\,\eta_2}{\eta_2 + 1} - 1. \tag{3.5}$$

The initial and boundary conditions depend on the type of the option. We present the approach for a European put option. The value of a European call option can be computed using the put-call parity.[1] The initial condition for a European vanilla put option is given by

$$U(S, 0) = \max(K - S, 0), \tag{3.6}$$

where K is the strike price, and the boundary conditions have the form

$$U(0,t) = Ke^{-rt}, \quad \frac{\partial U}{\partial S}(S,t) \approx 0 \text{ for } S \to \infty. \tag{3.7}$$

We choose the maximal value S^{max} large enough and approximate the unbounded domain $(0, \infty)$ by a domain $\Omega = (0, S^{max})$. We replace the boundary conditions with

$$U(0,t) = Ke^{-rt}, \quad \frac{\partial U}{\partial S}(S_{max}, t) = 0. \tag{3.8}$$

Furthermore we have

$$\mathcal{I}(U) = \lambda \int_{-\infty}^{\infty} U(Se^x, t)\, g(x)\, dx = \lambda \int_{0}^{\infty} U(y,t) \frac{g\left(\log \frac{y}{S}\right)}{y} dy. \tag{3.9}$$

Since

$$U(y,t) \frac{g\left(\log \frac{y}{S}\right)}{y} \approx 0 \text{ for } y \to \infty, \tag{3.10}$$

we define

$$\tilde{\mathcal{I}}(U) = \lambda \int_{0}^{S_{max}} U(y,t) \frac{g\left(\log \frac{y}{S}\right)}{y} dy \tag{3.11}$$

and we approximate $\mathcal{I}(U) \approx \tilde{\mathcal{I}}(U)$.

Let $\tilde{U} = U - W$, where U is the solution of the equation (3.1) satisfying the initial and boundary conditions defined above and W is defined by $W(S,t) = Ke^{-rt}$ for $S \in [0, S^{max}]$ and $t \in [0,T]$. Then $\tilde{U} \in V^L(0, S^{max})$ and $\tilde{U}$ is the solution of the equation

$$\frac{\partial \tilde{U}}{\partial t} - \mathcal{D}\left(\tilde{U}\right) - \tilde{\mathcal{I}}\left(\tilde{U}\right) = f(W), \tag{3.12}$$

with

$$f(W) = -\frac{\partial W}{\partial t} + \mathcal{D}(W) + \tilde{\mathcal{I}}(W) \tag{3.13}$$

satisfying the initial condition

$$\tilde{U}(S,0) = U(S,0) - K, \quad S \in [0, S_{max}], \tag{3.14}$$

and boundary conditions

$$\tilde{U}(0,t) = 0, \quad \frac{\partial \tilde{U}}{\partial S}(S_{max}, t) = 0, \quad t \in [0,T]. \tag{3.15}$$

We use the Crank-Nicolson scheme for temporal discretization. Let

$$M \in \mathbb{N},\ \tau = \frac{T}{M},\ t_l = l\tau,\ l = 0, \ldots, M, \tag{3.16}$$

and let us denote

$$\tilde{U}_l(S) = \tilde{U}(S, t_l), \quad f_l(S) = f(W(S, t_l)). \tag{3.17}$$

The Crank-Nicolson scheme has the form

$$\frac{\tilde{U}_{l+1} - \tilde{U}_l}{\tau} - \frac{\mathcal{D}\left(\tilde{U}_{l+1}\right)}{2} - \frac{\tilde{\mathcal{I}}\left(\tilde{U}_{l+1}\right)}{2} = \frac{\mathcal{D}\left(\tilde{U}_l\right)}{2} + \frac{\tilde{\mathcal{I}}\left(\tilde{U}_l\right)}{2} + \frac{f_l + f_{l+1}}{2} \tag{3.18}$$

for $l = 0, \ldots, M-1$.

Let Ψ^L be a smooth enough wavelet basis for the space $V^L(0, S^{max})$, Ψ^L_s be its finite-dimensional subset with s levels of wavelets, and $V^L_s = \operatorname{span} \Psi^L_s$. We define

$$a(u,v) = \langle \mathcal{D}(u), v \rangle + \left\langle \tilde{\mathcal{I}}(u), v \right\rangle, \tag{3.19}$$

for all $u, v \in V^L_s$, $s \geq 1$. The Galerkin method consists in finding $\tilde{U}^s_{l+1} \in V^L_s$ such that

$$\frac{\left\langle \tilde{U}^s_{l+1}, v \right\rangle}{\tau} - \frac{a\left(\tilde{U}^s_{l+1}, v\right)}{2} = \frac{\left\langle \tilde{U}^s_l, v \right\rangle}{\tau} + \frac{a\left(\tilde{U}^s_l, v\right)}{2} + \left\langle \frac{f_l + f_{l+1}}{2}, v \right\rangle \tag{3.20}$$

for all $v \in V^L_s$. If we set $v = \psi_\mu \in \Psi^L_s$ and we expand $\tilde{U}^s_{l+1}$ in a basis Ψ^L_s, i.e.

$$\tilde{U}^s_{l+1} = \sum_{\psi_\lambda \in \Psi^L_s} u^s_\lambda \psi_\lambda, \tag{3.21}$$

then the vector of coefficients $\mathbf{u}^s = \{u^s_\lambda\}$ is the solution of the system of linear algebraic equations $\mathbf{A}^s \mathbf{u}^s = \mathbf{f}^s$, where

$$\mathbf{A}^s_{\mu,\lambda} = \frac{\langle \psi_\lambda, \psi_\mu \rangle}{\tau} - \frac{a(\psi_\lambda, \psi_\mu)}{2} \tag{3.22}$$

and

$$\mathbf{f}^s_\mu = \frac{\left\langle \tilde{U}^s_l, \psi_\mu \right\rangle}{\tau} + \frac{a\left(\tilde{U}^s_l, \psi_\mu\right)}{2} + \left\langle \frac{f_l + f_{l+1}}{2}, \psi_\mu \right\rangle. \tag{3.23}$$

It is clear that $\mathbf{f}^s$ and $\mathbf{u}^s$ depend on the time level t_l, but for simplicity we omit the index l.

For preconditioning we use the Jacobi diagonal preconditioner $\mathbf{D}^s$, where the diagonal elements of $\mathbf{D}^s$ satisfy

$$\mathbf{D}^s_{\lambda,\lambda} = \sqrt{\mathbf{A}^s_{\lambda,\lambda}}. \tag{3.24}$$

We obtain the preconditioned system

$$\tilde{\mathbf{A}}^s \tilde{\mathbf{u}}^s = \tilde{\mathbf{f}}^s \tag{3.25}$$

with

$$\tilde{\mathbf{A}}^s = \left(\mathbf{D}^s\right)^{-1} \mathbf{A}^s \left(\mathbf{D}^s\right)^{-1}, \quad \tilde{\mathbf{f}}^s = \left(\mathbf{D}^s\right)^{-1} \mathbf{f}^s, \quad \tilde{\mathbf{u}}^s = \mathbf{D}^s \mathbf{u}^s. \tag{3.26}$$

It is well-known that due to the compact support of the wavelets and a hierarchical structure of the wavelet basis the matrices arising from discretization of the differential operator $\mathcal{D}$ have so-called finger-band pattern.[44] Hence, we focus on the properties of the matrix $\mathbf{C}^s$ with entries

$$\mathbf{C}^s_{\mu,\lambda} = \left\langle \tilde{\mathcal{I}}\left(\psi_\lambda\right), \psi_\mu \right\rangle, \quad \psi_\lambda, \psi_\mu \in \Psi^L_s. \tag{3.27}$$

For the standard Galerkin method with the standard spline basis such matrix is full. However, it is known that for integral equations with some types of kernels and for wavelet bases with vanishing moments many entries of discretization matrices are small and can be thresholded and the matrices arising from discretization of the integral term can be approximated with a matrix that is sparse or quasi-sparse.[3, 12] The following theorem provides the decay estimates for the entries of the matrix $\mathbf{C}^s$ corresponding to a general wavelets with L vanishing moments.

Theorem 3.1. *Let $\psi_{i,k}$ and $\psi_{j,l}$ be wavelets with L vanishing moments, i.e. the conditions $i) - iv)$ from Definition 1.1 are satisfied, and let us denote*

$$\Omega_{i,k} = \operatorname{supp} \psi_{i,k}, \quad \Omega_{j,l} = \operatorname{supp} \psi_{j,l}, \quad \Omega_{i,j,k,l} = \Omega_{i,k} \times \Omega_{j,l}. \tag{3.28}$$

Let ϵ be some small parameter such that $0 < \epsilon < S^{max}$ and let us denote

$$\Omega_\epsilon = \left\{(S,y) : S \neq y, S, y \in [0, S^{max}]\right\} \setminus [0,\epsilon]^2. \tag{3.29}$$

If

$$interior\left(\Omega_{i,j,k,l}\right) \subset \Omega_\epsilon, \tag{3.30}$$

then

$$\left|\left\langle \tilde{\mathcal{I}}\left(\psi_\lambda\right), \psi_\mu \right\rangle\right| \leq C 2^{-\left(L+\frac{1}{2}\right)(i+j)}, \tag{3.31}$$

where C is a constant independent on i,j,k,l.

Proof. Let the centers of the supports of $\psi_{i,k}$ and $\psi_{j,l}$ be denoted by $S_{i,k}$ and $y_{j,l}$, respectively. Let us define

$$K(S,y) = \lambda \frac{g\left(\log \frac{y}{S}\right)}{y}, \quad S, y \in \left(0, S_{max}\right], \tag{3.32}$$

and $K(S,y) = 0$ if $y = 0$ or $S = 0$. If the condition (3.30) is satisfied, then $K \in C^\infty\left(\Omega_{i,k} \times \Omega_{j,l}\right)$. By Taylor Theorem there exists a function P that is a polynomial of degree at most $L-1$ with respect to S and there exists a function Q that is a polynomial of degree at most $L-1$ with respect to y such that

$$K(S,y) = P(S,y) + Q(S,y) + \frac{1}{(2L)!}\frac{\partial^{2L}K(\xi(S,y))}{\partial S^L \partial y^L}(S-S_{i,k})^L (y-y_{j,l})^L, \quad (3.33)$$

where

$$\xi(S,y) = (S_{i,k}, y_{j,l}) + \alpha((S,y) - (S_{i,k}, y_{j,l})) \quad (3.34)$$

for some $\alpha \in [0,1]$. Due to the L vanishing moments of the wavelets $\psi_{i,k}$ and $\psi_{j,l}$ we obtain

$$\iint\limits_{\Omega_{i,j,k,l}} P(S,y)\,\psi_{i,k}(S)\,\psi_{j,l}(y)\,dS\,dy = 0 \quad (3.35)$$

and

$$\iint\limits_{\Omega_{i,j,k,l}} Q(S,y)\,\psi_{i,k}(S)\,\psi_{j,l}(y)\,dS\,dy = 0. \quad (3.36)$$

Due to the property *ii)* from Definition 1.1 there exists a constant C_1 independent on i,j,k,l such that

$$|S-S_{i,k}|^L \le C_1 2^{-Li}, \quad |y-y_{j,l}|^L \le C_1 2^{-Lj}. \quad (3.37)$$

From (2.25) and (2.29) there exists a constant C_2 independent on i,j,k,l such that

$$\int\limits_{\Omega_{i,k}} |\psi_{i,k}(S)|\,dS \le C_2 2^{-i/2}, \quad \int\limits_{\Omega_{j,l}} |\psi_{j,l}(y)|\,dy \le C_2 2^{-j/2}. \quad (3.38)$$

Hence, we have

$$\begin{aligned} \left|\left\langle \tilde{\mathcal{I}}(\psi_{i,k}), \psi_{j,l}\right\rangle\right| &= \left| \iint\limits_{\Omega_{i,j,k,l}} K(S,y)\,\psi_{i,k}(S)\,\psi_{j,l}(y)\,dS\,dy \right| \quad (3.39) \\ &\le C \iint\limits_{\Omega_{i,j,k,l}} |S-S_{i,k}|^L\,|y-y_{j,l}|^L |\psi_{i,k}(S)\,\psi_{j,l}(y)|\,dSdy \\ &\le C\,2^{-Li-Lj-i/2-j/2}, \end{aligned}$$

with

$$C = \frac{C_1\,C_2}{(2L)!} \sup_{(S,y)\in\Omega_\epsilon} \left| \frac{\partial^L \partial^L K(S,y)}{\partial S^L \partial y^L} \right|. \quad (3.40)$$

This proves the theorem. □

Let

$$\tilde{\mathbf{C}}^s = (\mathbf{D}^s)^{-1}\,\mathbf{C}^s\,(\mathbf{D}^s)^{-1} \quad (3.41)$$

with $\mathbf{D}^s$ defined in the previous section. Then the discretization matrix $\tilde{\mathbf{A}}^s$ is the sum of the matrix $\tilde{\mathbf{C}}^s$ and the matrix arising from discretization of the differential operator. Due to Theorem 3.1 and the local support of wavelets, many entries of the matrix $\tilde{\mathbf{A}}^s$ are small and they can be thresholded. The structure of the truncated matrix $\tilde{\mathbf{A}}^s$ is presented in Figure 3. This matrix contains only entries larger than 10^{-10} and it was computed for the option with parameters from Example 3.1 and the wavelet basis from Section 2 with six levels of wavelets, but the structure is similar for options with other parameters and for other wavelet bases.

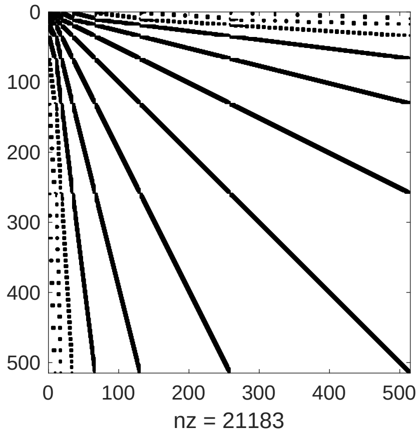

Fig. 3. The structure of the truncated discretization matrix $\tilde{\mathbf{A}}^s$.

Example 3.1. We use the proposed scheme for computing values of European vanilla options. We use the same parameters as in Ref. 26, 33, 37, 40, 46, i.e. option maturity $T = 0.25$ year, interest rate $r = 0.05$, volatility $\sigma = 0.15$, intensity $\lambda = 0.1$, $\eta_1 = 3.0465$, $\eta_2 = 3.0775$, probability of the upward jump $p = 0.3445$, and the strike price $K = 100$. We choose $S^{max} = 400$ and the threshold 10^{-10} for matrix compression. The resulting functions representing the values of the options are displayed in Figure 4.

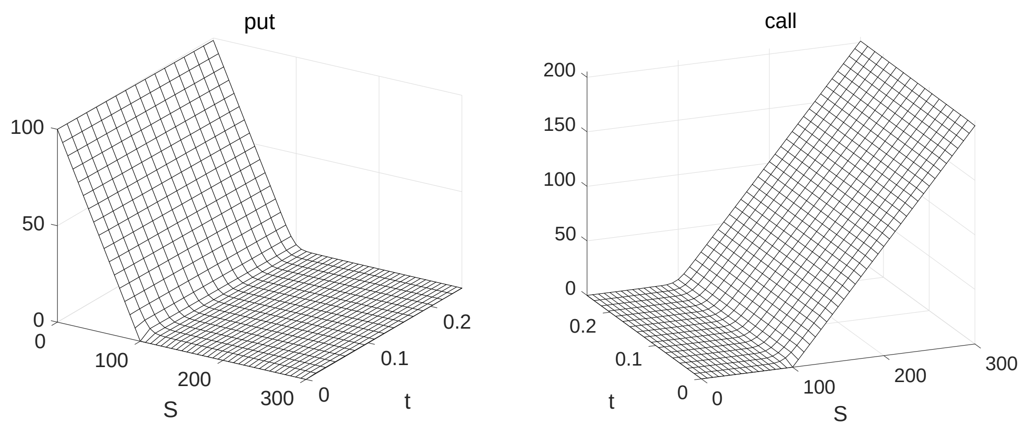

Fig. 4. Functions representing the values of a European put and call option for the Kou model.

The resulting values of the options for the asset prices $S = 90$, $S = 100$, and $S = 110$ are listed in Table 1. According to Ref. 40 the reference value is 9.430457 for $S = 90$, 2.731259 for $S = 100$, and 0.552363 for $S = 110$ for a put option. Reference values for call options are computed by put-call parity. In Table 1 we also present the pointwise errors, i.e. the differences between the computed values and the reference values, and the experimental rates of convergence computed as

$$\text{rate} = \frac{\log \text{error}\left(\frac{N+2}{2}, \frac{M}{4}\right) - \log \text{error}\left(N, M\right)}{\log 2}. \tag{3.42}$$

The optimal order for the Crank-Nicolson scheme is $\mathcal{O}\left(\tau^2\right)$ and optimal order for cubic spline approximation is $\mathcal{O}\left(h^4\right)$, where $h = 1/\left(N-2\right)$ represents the spatial step. It seems that the errors presented in Table 1 correspond to the optimal order $\mathcal{O}\left(h^4 + \tau^2\right)$.

Table 1. Values of a European vanilla put and call options, errors and rates of convergence.

S	N	M	put	call	error	rate
90	18	1	9.558719	0.800939	1.28e-1	
	34	4	9.465419	0.707639	3.50e-2	1.87
	66	16	9.429905	0.672125	5.52e-4	5.99
	130	64	9.430436	0.672656	2.21e-5	4.64
	258	256	9.430464	0.672684	6.88e-6	1.68
100	18	1	2.949505	4.191725	2.18e-1	
	34	4	2.707230	3.949449	2.40e-2	3.18
	66	16	2.729640	3.971860	1.62e-3	3.89
	130	64	2.731205	3.973425	5.31e-5	4.93
	258	256	2.731261	3.973481	1.58e-6	5.07
110	18	1	0.264414	11.506634	2.88e-1	
	34	4	0.550486	11.792706	1.88e-3	7.26
	66	16	0.551193	11.793413	1.17e-3	0.68
	130	64	0.552399	11.794619	3.53e-5	5.05
	258	256	0.552369	11.794588	5.78e-6	2.61

For the convenience of the reader we present in Table 2 the errors for values of European option with the same parameters as in this example, the Kou model and methods from Ref. 26, 37, 40. Other numerical results can be found in Ref. 33, 46. In comparison with methods from Ref. 26, 33, 37, 40, 46, the parameter N representing the number of basis functions needed to compute the solution with a desired accuracy is for our method significantly smaller. Thus significantly smaller matrices are involved in the computation.

The proposed scheme can be used also with other wavelet bases than those constructed in this paper. We compare the results with the results for wavelet bases of similar type, i.e. the wavelet basis that satisfies the following conditions:

i) It was proved that the basis is a Riesz basis of the space $V^L\left(0, S^{max}\right)$.
ii) All inner and boundary wavelets should have at least one vanishing moment.
iii) To achieve the similar rate of convergence, the basis functions should be piecewise cubic.
iv) The basis should be well-conditioned.

Table 2. Errors for values of European vanilla options for various methods.

	Kwon, Lee[40]			d'Halluin et al.[26]			Kadalbajoo et al.[37]		
S	N	M	error	N	M	error	N	M	error
90	128	25	3.63e-3	128	34	1.36e-3	128	12	8.42e-4
	256	50	8.78e-4	255	65	4.67e-4	256	24	3.06e-4
	512	100	2.24e-4	509	132	1.42e-4	512	48	8.48e-5
	1024	200	5.60e-5	1017	266	4.70e-5	1024	96	2.24e-5
	2048	400	1.40e-5	2033	533	7.00e-6	2048	192	5.79e-6
100	128	25	3.47e-2	128	34	3.51e-3	128	12	1.96e-3
	256	50	8.72e-3	255	65	1.00e-3	256	24	3.98e-4
	512	100	2.17e-3	509	132	3.72e-4	512	48	8.96e-5
	1024	200	5.42e-4	1017	266	1.57e-4	1024	96	2.15e-5
	2048	400	1.36e-4	2033	533	7.20e-5	2048	192	5.60e-6
110	128	25	8.15e-3	128	34	5.31e-3	128	12	1.85e-3
	256	50	2.10e-3	255	65	2.10e-3	256	24	4.99e-4
	512	100	5.28e-4	509	132	9.13e-4	512	48	1.28e-4
	1024	200	1.32e-4	1017	266	4.23e-4	1024	96	3.28e-5
	2048	400	3.30e-5	2033	533	1.03e-4	2048	192	8.51e-6

The bases that satisfy $i) - iii)$ were constructed e.g. in Ref. 4, 5, 22, 42. However, the cubic spline wavelet basis from Ref. 42 is not well-conditioned. Since the bases from Ref. 4, 5, 22 have the same inner wavelets and they differ only in the definition of boundary wavelets, we compare the results for the basis constructed in this paper with the results for the cubic spline wavelet basis from Ref. 5. Quite surprisingly for us most constructions of cubic spline wavelet or multiwavelet bases do not satisfy all conditions $i) - iv)$. Indeed, wavelets from Ref. 15 are not adapted to boundary conditions, the wavelet bases from Ref. 8, 10, 24, 34 satisfy only the first order Dirichlet boundary conditions, and bases from Ref. 6, 7, 45 satisfy only the second order Dirichlet boundary conditions. Thus, these bases satisfy $ii) - iv)$, but do not satisfy $i)$. Constructions of cubic spline wavelets without adaptation to boundary conditions and without the proof of the Riesz basis property can be found in Ref. 38, 43. The recently constructed cubic Hermite spline wavelets from Ref. 29 also have not yet been adapted to be the basis for $V^L(0, S^{max})$. Boundary wavelets from Ref. 30 and Ref. 41 do not have vanishing moments.

In Table 3 we present the number of nonzero elements (nnz) of the matrix truncated using the threshold 10^{-10} and the condition numbers (cond) of diagonally preconditioned discretization matrices for the Galerkin method with wavelet basis from Section 2 (short4), biorthogonal wavelet basis with 6 vanishing moments from Ref. 5 (bior4.6) and for the Galerkin method with B-splines (B-splines). Furthermore, we list the number of outer and the number of inner iterations needed to resolve the resulting system of equations by the generalized minimal residual method (GMRES) with the following input parameters: restart after ten iterations, maximum number of outer iterations is 100 and the iterations stop if the relative residual is less than 10^{-12}. As expected the discretization matrix corresponding to B-splines is well-conditioned but full. The truncated matrix corresponding to biorthogonal wavelets from Ref. 5 is quasi-sparse. For some problems, see e.g. Ref. 5, the discretization matrix is well-conditioned. However, it is known, that the boundary biorthogonal wavelets are typically highly oscillatory and have relatively large support and that this can lead to badly conditioned matrices. As can be seen in Table 3 this situation occurs in our case. We also computed the results for bior4.4 and bior4.8 wavelets from Ref. 5, but the condition numbers were even higher than for bior4.6 wavelets.

Table 3. The number of nonzero entries (nnz) and the condition number (cond) of the truncated discretization matrices, and the number of GMRES iterations (it).

basis	N	M	nnz	cond	it
short4	34	4	891	36	7(6)
	66	16	2105	38	7(8)
	130	64	4716	40	7(10)
	258	256	10042	40	7(9)
	514	1024	21183	41	7(6)
bior4.6	34	4	1061	2.0e3	83(5)
	66	16	2895	1.4e5	100(10)
	130	64	7474	6.6e6	100(10)
	258	256	19024	3.2e8	100(10)
B-spline	34	4	1156	17	5(5)
	66	16	4333	18	5(9)
	130	64	16616	18	5(7)
	258	256	64111	18	5(6)
	514	1024	244518	18	5(5)

4. Conclusion

We constructed the cubic spline wavelet basis on the interval with four vanishing wavelet moments and with short support and we adapted this basis to Dirichlet boundary conditions. We proved the Riesz basis property with respect to the L^2-norm. Using the tensor product and appropriate parametric mappings this basis can be adapted to higher-dimensional bounded domains. We used the Crank-Nicolson scheme and the Galerkin method with the constructed basis for the numerical solution of the partial integro-differential equations that represents the Kou's option pricing model. The advantage of the proposed method is the quasi-sparse structure of the discretization matrices and in comparison with methods from Ref. 26, 33, 37, 40, and 46, the presented method requires significantly smaller number of degrees of freedom needed to compute the solution with desired accuracy. We showed that our basis is more appropriate for the proposed scheme than biorthogonal cubic spline wavelet bases from Ref. 5 and than the basis of cubic B-splines and we discussed also other choices of cubic spline wavelet bases.

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