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Boundary value problems for second-order differential inclusions on compact and non-compact intervals in the Euclidean and abstract spaces

HABILITATION THESIS

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Abstract

This thesis is devoted to the study of boundary value problems for second-order differential inclusions. As the title of the thesis indicates, several types of problems will be discussed – vector problems on compact or non-compact intervals, problems in Banach spaces and finally also vector problems involving impulses.

The key tool that is used in the thesis is an appropriate continuation principle that contains besides other the transversality condition which verification is very complicated. Therefore, the second substantial part of the thesis deals with the bound sets technique which can be used as a tool for its guaranteeing.

The thesis is submitted as the collection of 12 scholarly works published in international journals with nonzero impact factors with the commentary.

Contents

Preface

Boundary value problems for second-order differential inclusions occur naturally in many applications. The applications concern, for instance, population genetics, combustion models, power law fluids, unsteady flows of gas through semi-infinite porous media, etc.

An investigation of linear oscillators with weak interactions leads to vector second-order systems. If the friction (damping) is not viscous but dry, then the mathematical model can be described by the system

$$\ddot{x} + A \operatorname{sgn} \dot{x} + Bx = P(t), \; x \in \mathbb{R}^n,$$

where $A,\ B$ are regular $(n \times n)$-matrices and P is a locally Lebesgue integrable vector forcing term. Because of discontinuity at $y = 0$ in sgn y, the Filippov solutions should be considered that can be identified as Carathéodory solutions of the relevant inclusion

$$\ddot{x} + Bx \in P(t) - A \operatorname{Sgn} \dot{x}, \; x \in \mathbb{R}^n,$$

where

$$\operatorname{Sgn} y := \begin{cases} -1, & \text{for } y < 0, \\ [-1, 1], & \text{for } y = 0, \\ 1, & \text{for } y > 0. \end{cases}$$

Another stimulation for studying of the boundary value problems for the second-order differential inclusions comes from control problems

$$\left.\begin{array}{l} \ddot{x} = f(t, x, \dot{x}, u), \; t \in J, \; u \in U, \\ x \in S, \end{array}\right\} \tag{1}$$

where S is a suitable constraint (e.g. boundary conditions) and $u \in U$ are control parameters such that $u(t) \in \mathbb{R}^n$, for all $t \in J$. Defining the multivalued mapping $F(t, x, y) := \{f(t, x, y, u)\}_{u \in U}$, solutions of the original problem (1) coincide with those of

$$\left.\begin{array}{l} \ddot{x} \in F(t, x, \dot{x}), \\ x \in S. \end{array}\right\}$$

Systems like two models described above are the motivations for the problems that will be studied in the thesis.

The thesis will be submitted as the collection of 12 previously published scholarly works with the commentary that will be organized as follows. In the first part of the commentary, the short historical overview concerning boundary value problems for second-order differential equations and inclusions will be mentioned. The second and the third part of the commentary will be devoted to the continuation principles and the bound sets technique which are two key tools that have been applied for obtaining the thesis' results. Finally, the particular contributions to the theory of the boundary value problems for second-order differential inclusions will be described in the fourth section.

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1 Boundary value problems for second-order differential inclusions

In the first part of the commentary, the short historical summary concerning boundary value problems for the second-order differential equations and inclusions will be presented. The attention will be firstly focused on vector problems both on compact and non-compact intervals. Subsequently, the brief historical overview dealing with boundary value problems in abstract spaces will be mentioned, and finally also the history of impulsive vector problems will be shortly observed.

1.1 Boundary value problems for vector second-order differential inclusions on compact and non-compact intervals

The boundary value problems for second-order vector systems on compact intervals have been systematically studied since the 70's (see, e.g., [30, 33, 41, 42, 51, 54, 55, 58, 62, 66, 72]). In these papers and monographs, different methods have been used like an upper and lower solutions technique, degree arguments or topological approach for obtaining the existence of a solution of the boundary value problems on compact intervals.

If we shift our attention to the problems on non-compact intervals, we realize that there are much less publications devoted to the asymptotic boundary value problems for the second-order inclusions or equations (see, e.g., [3, 7, 32, 36, 37, 38, 39, 40, 53, 57, 61, 77], and the references therein). In mentioned papers and monographs, various fixed point theorems, topological degree theory, shooting methods, upper and lower solution technique, etc., have been applied for the solvability of given problems.

The difficulties related to asymptotic problems are mainly caused by the fact that the application of degree arguments in Fréchet spaces is always very delicate (see, e.g., [1, 2, 5, 9, 46, 47, 48, 71]). Perhaps the most difficult concerning asymptotic problems seems to be the study of a topological structure of solution sets to nonlinear problems (see, e.g., [4, 8, 10], [11, Chapter III.3], [27, 49]). Let us point out that the study of the topological structure of the solution set can be regarded as a very interesting problem itself, but it also helps us to have "good" values of the associated solution operators of partly linearized systems for which the structure of solution sets is investigated.

In comparison with mentioned publications, the right-hand sides that are considered in the thesis satisfy quite slight regularity assumptions. More concretely, the problems occurring in the thesis have usually multivalued upper-Carathéodory right-hand sides. The related existence and localization results in the thesis have been obtained by combination of topological methods with the inverse limit technique (for asymptotic problems) and by combination of topological methods together with bound sets technique and Scorza-Dragoni type results for the problems on compact intervals. This approaches allows to obtain not only the existence results but also to get the information about the localization of a solution.

The theory of the bound sets which is used in the thesis for problems on compact intervals was introduced 40 years ago by Robert Gaines and Jean Mawhin in the book [50], where the existence of periodic solutions for nonlinear differential systems was studied. The theory of bound sets was subsequently extended to other types of boundary value problems by Jean Mawhin, e.g., in papers [64] and [65].

In the papers quoted so far, the bounding functions which guarantee the existence of a bound set were taken for the first-order problems of class C^1 and of class C^2 for the second-order problems. The less regular bounding functions were employed, e.g., in the papers [79], [83] for the first-order single-valued problems and in [80] for the single-valued second-order problems. Variants of this concept were used in the paper [43] for the Picard problem and in papers [44], [45] for the Sturm–Liouville boundary conditions.

Concerning the boundary value problems for differential inclusions, the bound sets theory was employed for multivalued first-order Floquet problems in $\mathbb{R}^n$ in papers [19]-[21]. In the first one, the r.h.s. of the studied differential inclusion was upper semi-continuous which allowed the authors to put the conditions ensuring the existence of a bound set K directly on the boundary of K. The second paper dealt with the upper-Carathéodory r.h.s. This caused that the conditions had to be satisfied at some vicinity of the boundary of K. Finally, in the third paper, the Scorza-Dragoni type technique was employed which allowed to put the conditions directly on the boundary also in the case of upper-Carathéodory r.h.s.

By the same strategy (consisting in strict and non-strict localization of bounding functions), the theory of bound sets was developed for the second-order multivalued Dirichlet and Floquet problems in $\mathbb{R}^n$ in the papers [12], [13], [18] and [73].

1.2 Boundary value problems for second-order differential inclusions in Banach spaces

Although the general theory of ordinary differential equations and inclusions in Banach spaces has been developed at a satisfactory level for quite long time (see, e.g., the monograph [81] and the references therein), there were not so many contributions to boundary value problems in infinite dimensional spaces until recently (see, e.g., [6], [22], [35], [67], [69], [70], and [78]). Lately, the theory of boundary value problems in Banach spaces has started to attract the increasing attention (see, e.g., [14]-[17], [59], [60], [82]). The boundary value problems in Banach spaces were in recent papers considered together with various generalizations (like impulse effects or the usage of fractional derivatives) and it is expectable that the increased attention to the boundary value problems in Banach spaces will be given also in the future.

1.3 Impulsive boundary value problems for second-order differential inclusions

Boundary value problems with impulses have been widely studied because of their applications in areas, where the parameters are subject to certain perturbations in time. For instance, in the treatment of some diseases, impulses may correspond to administration of a drug treatment or in environmental sciences, they can describe the seasonal changes or harvesting.

While the theory of single-valued impulsive problems is deeply examined (see, e.g., [28, 29, 63], and the references therein), the theory dealing with multivalued impulsive problems has not been studied so much yet (for the overview of known results see, e.g., the monographs [31, 52], and the references therein). However, it is worth to study also the multivalued case, since the multivalued problems come e.g. from single-valued problems with discontinuous right-hand sides, or from control theory like it has been mentioned in the preface.

2 Continuation principle

The key tool that is used in the thesis is an appropriate continuation principle. Therefore, in this section, the continuation principle will be at first described in short for multivalued problems on non-compact intervals in $\mathbb{R}^n$. Subsequently, its modifications for problems on compact intervals, for problems in Banach spaces and for the impulsive problems will be discussed.

2.1 Continuation principle for non-impulsive problems in $\mathbb{R}^n$ on non-compact and compact intervals

Let us, at first, consider the second-order b.v.p. in $\mathbb{R}^n$

$$\left.\begin{array}{c}\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \text{ for a.a. } t \in I,\\ x \in S,\end{array}\right\} \tag{2}$$

where

- I is a given (possibly noncompact) real interval,
- $S \subset AC^1_{loc}(I, \mathbb{R}^n)$, where $AC^1_{loc}(I, \mathbb{R}^n)$, denotes the space of functions $x : I \to \mathbb{R}^n$ with locally absolutely continuous first derivatives,
- $F : I \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping, i.e. $F(\cdot, x, y) : I \multimap \mathbb{R}^n$ is measurable on every compact subinterval of I, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, the map $F(t, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is upper semicontinuous, for almost all $t \in I$, and the set $F(t, x, y)$ is compact and convex, for all $(t, x, y) \in I \times \mathbb{R}^n \times \mathbb{R}^n$.

In order to develop the suitable continuation principle, the family of associated problems $P(q, \lambda)$ has to be assigned to the original problem (2). The associated problems have the following form

$$\left.\begin{array}{c}\ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in I,\\ x \in S_1,\end{array}\right\} P(q, \lambda)$$

where

- $q \in Q$; Q is a retract of $C^1(I, \mathbb{R}^n)$,
- S_1 is a closed subset of S,
- $\lambda \in [0, 1]$,

- $H : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping such that

$$H(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in I \times \mathbb{R}^n \times \mathbb{R}^n. \tag{3}$$

The inclusion (3) guarantees that if q^* would be a solution of the problem $P(q^*, 1)$, then it is also the solution of the original problem (2). Therefore, it is possible to transform the b.v.p. (2) into the fixed point problem

$$q^* \in T(q^*, 1), \tag{4}$$

where $T : Q \times [0,1] \multimap C^1(I, \mathbb{R}^n)$ is a solution mapping that assigns to each $q \in Q$ and $\lambda \in [0,1]$ the set of solutions of $P(q, \lambda)$.

After the transformation of the original problem (2) into the fixed point problem (4) and by using of fixed point index technique in Fréchet spaces, the following continuation principle has been developed in [23].

Theorem 2.1 *Let us consider the b.v.p.* (2) *together with the family of associated problems* $P(q, \lambda)$ *and assume that*

(i) *for each* $(q, \lambda) \in Q \times [0,1]$, *the associated problem* $P(q, \lambda)$ *is solvable with an* R_δ*-set of solutions,*

(ii) *there exists a nonnegative, locally integrable function* $\alpha : I \to \mathbb{R}$ *such that*

$$|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)| \le \alpha(t)(1 + |x(t)| + |\dot{x}(t)|), \text{ a.e. in } I,$$

for any $(q, \lambda, x) \in \Gamma_T$,

(iii) $T(Q \times \{0\}) \subset Q$,

(iv) *there exist a point* $t_0 \in I$ *and constants* $M_0 \ge 0$, $M_1 \ge 0$ *such that* $|x(t_0)| \le M_0$ *and* $|\dot{x}(t_0)| \le M_1$, *for any* $x \in T(Q \times [0,1])$,

(v) *if* $q_j, q \in Q$, $q_j \to q$, $q \in T(q, \lambda)$, *for some* $\lambda \in [0,1]$, *then there exists* $j_0 \in \mathbb{N}$ *such that, for every* $j \ge j_0$, $\theta \in [0,1]$ *and* $x \in T(q_j, \theta)$, *we have* $x \in Q$.

Then the b.v.p. (2) *has a solution in* $S_1 \cap Q$.

Remark 2.1 It was proven in [23] that the solution mapping T has compact values. Therefore, the condition (i) concerning R_δ-values in Theorem 2.1 is satisfied if, e.g., $T(q, \lambda)$ is, for all $(q, \lambda) \in Q \times [0,1]$, convex or an AR-space or contractible.

Remark 2.2 Let us note that in the single-valued case of Carathéodory ordinary differential equations, we can only assume in Theorem 2.1 (i) that the associated problems are uniquely solvable.

Remark 2.3 If the associated problems $P(q,\lambda)$ are fully linearized, i.e. if they take the form

$$\left.\begin{array}{c}\ddot{x}(t) \in \lambda F(t, q(t), \dot{q}(t)), \text{ for a.a. } t \in I, \\ x \in S_1,\end{array}\right\} P(q,\lambda)^L$$

where S_1 is a closed convex subset of S, then it is possible to modify conditions (i) and (ii) in Theorem 2.1 as follows:

(i^L) $T(q,\lambda) \neq \emptyset$, for all $(q,\lambda) \in Q \times [0,1]$,

(ii^L) there exists a nonnegative, locally integrable function $\alpha : I \to \mathbb{R}$ such that
$$|F(t, q(t), \dot{q}(t))| \leq \alpha(t), \text{ a.e. in } I, \text{ for any } q \in Q.$$

It is obvious that the most problematic assumption appearing in the continuation principle is so called "pushing" condition (v). Therefore, the attention will be given now to this condition and its possible simplifications will be shown in subsequent remarks.

Remark 2.4 If the set Q is convex, then the condition (v) of Theorem 2.1 can be replaced by

(v') if $\partial Q \times [0,1] \supset \{(q_j,\lambda_j)\}$ converges to $(q,\lambda) \in \partial Q \times [0,1], q \in T(q,\lambda)$, then there exists $j_0 \in \mathbb{N}$ such that, for every $j \geq j_0$ and $x_j \in T(q_j,\lambda_j)$, we have $x_j \in Q$.

Remark 2.5 Moreover, if the associated problems $P(q,\lambda)^L$ are uniquely solvable, for each $(q,\lambda) \in Q \times [0,1]$, then, by continuity of T, we can reformulate the "pushing" condition (v') from Remark 2.4 as follows:

(v'') if $\{(x_j,\lambda_j)\}$ is a sequence in $S_1 \times [0,1]$, with $\lambda_j \to \lambda$ and x_j converging to a solution $x \in Q$ of $P(q,\lambda)^L$, for $q = x$ and $\lambda = \lambda$, then x_j belongs to Q, for j sufficiently large.

Remark 2.6 If the interval I would be compact, then it is possible to simplify significantly "pushing" condition into the following form:

(v^C) Let $Q \backslash \partial Q$ be nonempty and let the solution map T has no fixed points on the boundary ∂Q of Q, for every $(q,\lambda) \in Q \times [0,1)$.

It is obvious that the verification of (v^C) is easier than of (v), (v') or (v''), but still quite complicated. The fulfilment of condition (v^C) for the boundary value problems on compact intervals can be guaranteed by the bound sets theory, and that's why the following Section 3 will be devoted to this approach.

Remark 2.7 All of the conditions (v), (v'), (v'') and (v^C) can be omitted if

$$S_1 \subseteq S \cap Q,$$

which significantly simplifies the usage of the continuation principle in particular practical applications.

2.2 Continuation principle for impulsive problems in $\mathbb{R}^n$ on compact intervals

If we would consider that the b.v.p. (2) contains impulses at fixed times t_1, t_2, ..., t_p, $p \in \mathbb{N}$, then it is possible to obtain the related modification of the continuation principle for non-impulsive problems. As you can see in the next proposition that has been proven for impulsive problems on compact intervals in [74], the changes that are necessary in case of impulsive problems are only minor.

Proposition 2.1 *Let us consider the b.v.p.* (2), *where*

- $I = [0, T]$,
- $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper-Carathéodory mapping,*
- S *is a subset of* $PAC^1([0, T], \mathbb{R}^n)$ *which is the space of functions* $x : [0, T] \to \mathbb{R}^n$ *such that*

$$x(t) = \begin{cases} x_{[0]}(t), & \text{for } t \in [0, t_1], \\ x_{[1]}(t), & \text{for } t \in (t_1, t_2], \\ . \\ . \\ . \\ x_{[p]}(t), & \text{for } t \in (t_p, T], \end{cases}$$

with $x_{[0]} \in AC^1([0, t_1], \mathbb{R}^n), x_{[i]} \in AC^1((t_i, t_{i+1}], \mathbb{R}^n), x(t_i^+) = \lim_{t \to t_i^+} x(t) \in \mathbb{R}$ *and* $\dot{x}(t_i^+) = \lim_{t \to t_i^+} \dot{x}(t) \in \mathbb{R}$, *for every* $i = 1, ..., p$.

Together with (2)*, let us consider the family of associated problems* $P(q,\lambda)$*, where* $H : [0,T] \times \mathbb{R}^{4n} \times [0,1] \multimap \mathbb{R}^n$ *is an upper-Carathéodory mapping satisfying the inclusion* (3)*. Moreover, assume that*

(i) *there exists a retract* Q *of* $PC^1([0,T],\mathbb{R}^n)$*, with* $Q \setminus \partial Q \neq \emptyset$*, and a closed subset* S_1 *of* S *such that the associated problem* $P(q,\lambda)$ *has, for each* $(q,\lambda) \in Q \times [0,1]$*, a nonempty and convex set of solutions* $T(q,\lambda)$*,*

(ii) $-$ (iv) *assumptions* (ii) $-$ (iv) *from Theorem 2.1 hold,*

(v) *the solution map* $T(\cdot,\lambda)$ *has no fixed points on the boundary* ∂Q *of* Q*, for every* $\lambda \in [0,1)$*.*

Then the b.v.p. (2) *has a solution in* $S_1 \cap Q$*.*

Remark 2.8 Let us note that although Theorem 2.1 and Proposition 2.1 contain the same assumption

(iii) $T(Q \times \{0\}) \subset Q$

its verification is in practical applications much more complicated in case of the impulsive problems considered in Proposition 2.1.

To be more specific, let us consider, e.g., the non-impulsive homogeneous Dirichlet problem

$$\left.\begin{array}{c} \ddot{x}(t) = 0, \text{ for a.a. } t \in [0,T], \\ x(T) = x(0) = 0. \end{array}\right\} \tag{5}$$

This problem has only the trivial solution, and so the assumption (iii) is trivially satisfied if $0 \in Q$.

On the other hand, if we consider the impulsive homogeneous Dirichlet problem

$$\left.\begin{array}{c} \ddot{x}(t) = 0, \text{ for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i), \ i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), \ i = 1,\dots,p, \end{array}\right\} \tag{6}$$

where A_i, B_i, $i = 1,\dots,p$, are real $n \times n$ matrices, it is not so easy to obtain that it possesses only the trivial solution. It is valid, e.g., for the antiperiodic impulses, i.e. for $A_i = B_i = -I$, for every $i = 1,...,p$. Other possibility that guarantees the existence of the exclusively trivial solution is if $p = 1$, $A_1 = -I$ and $B_1 = I$ provided $T \neq 2t_1$.

2.3 Continuation principle for problems in abstract spaces

If we consider, instead of $\mathbb{R}^n$, an abstract space, the particular conditions appearing in the continuation principle become significantly more complicated.

As one of the motivations for studying boundary value problems in abstract spaces, the following *abstract nonlinear wave equations* in Hilbert spaces can be used: Let $t \in [0, T]$ and let $\xi \in \Omega$, where Ω is a nonempty, bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. Consider the functional evolution equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + \tilde{B}u(t,\cdot) + \beta||u(t,\cdot)||^{p-2}u = \varphi(t,u), \tag{7}$$

where $u = u(t,\xi)$, subject to boundary conditions

$$u(T,\cdot) = Mu(0,\cdot), \quad \frac{\partial u(T,\cdot)}{\partial t} = N\frac{\partial u(0,\cdot)}{\partial t}. \tag{8}$$

Assume that $a \geq 0$, $\beta \geq 0$, $p > 1$ are constants, $\tilde{B}\colon L^2(\Omega) \to L^2(\Omega)$ is a linear operator and that $\varphi\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is sufficiently regular. The problem under consideration can be still restricted by a constraint:

$$u(t,\cdot) \in \overline{K} := \{e \in L^2(\Omega) \mid ||e|| \leq r\},\ t \in [0,T].$$

Taking $x(t) := u(t,\cdot)$ with $x \in AC^1([0,T], L^2(\Omega))$, $A(t) \equiv A := a$, $B(t)\colon L^2(\Omega) \to L^2(\Omega)$ defined by $x = u(t,\cdot) \to \tilde{B}x$, $f\colon [0,T] \times L^2(\Omega) \to L^2(\Omega)$ defined by $(t,v) \to \varphi(t,v(\cdot))$, and $F(t,x,y) \equiv F(t,x) := -\beta||x||^{p-2}x + f(t,x)$, the above problem can be rewritten into the form

$$\left.\begin{array}{r} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = F(t,x(t),\dot{x}(t)), \quad \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0). \end{array}\right\} \tag{9}$$

possibly together with $x(t) \in \overline{K}, t \in [0,T]$, where $K \subset L^2(\Omega)$ is a nonempty, open, convex subset of $L^2(\Omega)$.

If $\varphi(t,\cdot)$ is e.g. bounded, but discontinuous at finitely many points, then the Filippov regularization $\tilde{\varphi}$ of $\varphi(t,\cdot)$ (see, e.g., [26, 34]) can lead to a multivalued problem

$$\left.\begin{array}{r} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t,x(t),\dot{x}(t)), \quad \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0). \end{array}\right\} \tag{10}$$

For boundary value problems in Banach spaces like (10), the following continuation principle has been developed in [16] and [17] by using a suitable topological degree technique.

Theorem 2.2 *Let us consider the b.v.p.*

$$\left.\begin{array}{c}\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \text{ for a.a. } t \in [0, T], \\ x \in S,\end{array}\right\} \tag{11}$$

where

- *E is a separable Banach space with the norm $||\cdot||$ satisfying the Radon-Nikodym property,*
- *$F\colon [0, T] \times E \times E \multimap E$ is an upper-Carathéodory mapping,*
- *$S \subset AC^1([0, T], E)$.*

Let $H : [0, T] \times E \times E \times E \times E \times [0, 1] \multimap E$ be an upper-Carathéodory mapping such that

$$H(t, c, d, c, d, 1) \subset F(t, c, d), \text{ for all } (t, c, d) \in [0, T] \times E \times E.$$

Moreover, assume that the following conditions hold:

(i) There exist a closed set $S_1 \subset S$ and a closed, convex set $Q \subset C^1([0, T], E)$ with a nonempty interior $Int\ Q$ such that each associated problem

$$\left.\begin{array}{c}\ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in [0, T], \\ x \in S_1,\end{array}\right\} \quad P(q, \lambda)$$

where $q \in Q$ and $\lambda \in [0, 1]$, has a nonempty, convex set of solutions (denoted by $T(q, \lambda)$).

(ii) For every nonempty, bounded set $\Omega \subset E \times E \times E \times E$, there exists $\nu_\Omega \in L^1([0, T], [0, \infty))$ such that

$$||H(t, x, y, u, v, \lambda)|| \leq \nu_\Omega(t),$$

for a.a. $t \in [0, T]$ and all $(x, y, u, v) \in \Omega$ and $\lambda \in [0, 1]$.

(iii) The solution mapping T is quasi-compact and μ-condensing with respect to a monotone and nonsingular measure of noncompactness μ defined on $C^1([0, T], E)$.

(iv) For each $q \in Q$, the set of solutions of the problem $P(q, 0)$ is a subset of $Int\ Q$, i.e. $T(q, 0) \subset Int\ Q$, for all $q \in Q$.

(v) For each $\lambda \in (0, 1)$, the solution mapping $T(\cdot, \lambda)$ has no fixed points on the boundary ∂Q of Q.

Then the b.v.p. (11) *has a solution in* $Q \cap S_1$.

Remark 2.9 Let us point out that one of the differences between continuation principles in $\mathbb{R}^n$ and in abstract spaces lies in the fact that it would be extremely difficult to avoid the convexity of given set of candidate solutions Q, provided the degree arguments are applied for noncompact maps (for more details, see, e.g., [25]). For this reason, the set Q is considered to be convex in the case of continuation principle in Banach spaces.

Remark 2.10 Similarly as in finite-dimensional Euclidean spaces, the geometry concerning second-order problems in Banach spaces, reflecting the behaviour of controlled trajectories, is much more sophisticated than for first-order problems. On the other hand, the sufficient existence conditions are again better than those for equivalent first-order problems (see [22]).

3 Bound sets approach

One of the most problematic assumptions that has appeared in continuation principles in the previous section for problems on compact intervals is the following transversality condition:

(v) the solution map $T(\cdot, \lambda)$ has no fixed points on the boundary ∂Q of Q, for every $\lambda \in [0, 1)$.

It has occurred in Remark 2.6, Proposition 2.1 as well as in Theorem 2.2 dealing with the problems in Banach spaces.

Since its direct verification is generally quite complicated, this section will be devoted to the bound sets technique which can be used as a tool for its guaranteeing.

Let us point out that the bound sets technique cannot be applied jointly with the degree arguments for problems on non-compact intervals, because bounded subsets of non-normable Fréchet spaces are equal to their boundaries.

3.1 Bound sets approach for non-impulsive problems in $\mathbb{R}^n$

For the developing of the bound sets theory for the second-order boundary value problems, let us at first consider the easiest case, i.e. non-impulsive vector problems. More concretely, let us consider the vector Floquet semi-linear problem

$$\left.\begin{array}{r} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t, x(t), \dot{x}(t)), \text{ for a.a. } t \in [0, T], \\ x(T) = Mx(0), \\ \dot{x}(T) = N\dot{x}(0), \end{array}\right\} \tag{12}$$

where

(i^U) $A, B : [0, T] \to \mathbb{R}^{n\times n}$ are measurable matrix functions such that $|A(t)| \leq a(t)$ and $|B(t)| \leq b(t)$, for all $t \in [0, T]$ and suitable integrable functions $a, b : [0, T] \to [0, \infty)$,

(ii^U) $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping,

(iii^U) M and N are real $n \times n$ matrices with M non-singular.

Moreover, let (in the whole Sections 3.1 and 3.2) $K \subset \mathbb{R}^n$ be a nonempty open set and $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function with ∇V locally Lipschitzian and satisfying

$(H1)$ $V|_{\partial K} = 0$,

$(H2)$ $V(x) \leq 0$, for all $x \in \overline{K}$.

Furthermore, let (in the whole Sections 3.1 and 3.2) M be such that $M\partial K = \partial K$.

Definition 3.1 A nonempty open set $K \subset \mathbb{R}^n$ is called a *bound set* for the b.v.p. (12) if every solution x of (12) such that $x(t) \in \overline{K}$, for each $t \in [0, T]$, does not satisfy $x(t^*) \in \partial K$, for any $t^* \in [0, T]$.

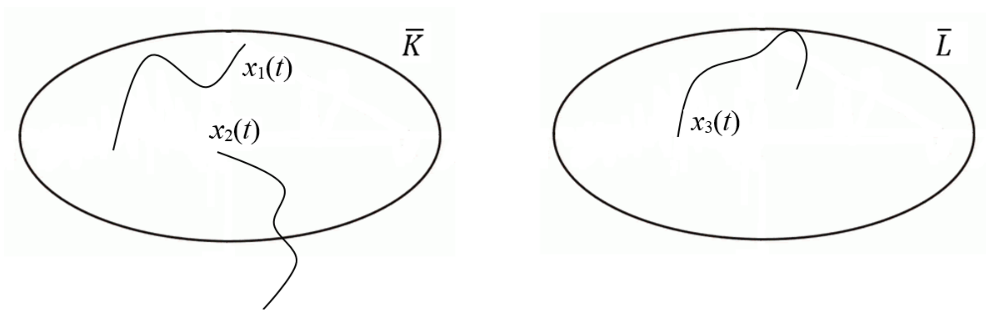

Figure 1: Solutions of the b.v.p. (12) and the sets $\overline{K}$ and $\overline{L}$ such that K is a bound set for (12) and L is not.

In [12], the following result guaranteeing the existence of the bound set K for the Floquet b.v.p. with an upper-Carathéodory r.h.s. has been proven.

Proposition 3.1 *Let us consider the b.v.p.* (12) *and suppose that there exists* $\varepsilon > 0$ *such that, for all* $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0, T)$ *and* $v \in \mathbb{R}^n$, *the following condition*

$$\limsup_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle - \langle \nabla V(x), v \rangle}{h} > 0 \tag{13}$$

holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$, *and that*

$$\langle \nabla V(My), Nw \rangle \cdot \langle \nabla V(y), w \rangle \geq 0, \tag{14}$$

for all $y \in \partial K$ *and* $w \in \mathbb{R}^n$. *Then all possible solutions* $x : [0, T] \to \overline{K}$ *of problem* (12) *are such that* $x(t) \in K$, *for every* $t \in [0, T]$, *i.e.* K *is a bound set for the Floquet problem* (12).

Remark 3.1 If condition (13) is replaced by the following one

$$\limsup_{h \to 0^+} \frac{\langle \nabla V(x+hv), v+hw \rangle - \langle \nabla V(x), v \rangle}{h} > 0, \tag{15}$$

for all $x \in \overline{K} \cap N_\varepsilon(\partial K),\ t \in (0,T),\ v \in \mathbb{R}^n$ and $w \in F(t,x,v) - A(t)v - B(t)x$, while all the other assumptions of Proposition 3.1 remain valid, then the same conclusion holds.

Definition 3.2 A function $V : \mathbb{R}^n \to \mathbb{R}$ from Proposition 3.1 satisfying conditions $(H1)$, $(H2)$, (14) and at least one of conditions (13), (15) is called a *bounding function* for the set K relative to (12).

It was shown in [12] that:

- if Q is defined as follows
$$Q := \{q \in C^1([0,T], \mathbb{R}^n) \mid q(t) \in \overline{K}, \quad \text{for all } t \in [0,T]\} \tag{16}$$
and if K is a bound set for each associated problem $P(q, \lambda)$ (defined consistently like in Section 2), then condition (v^C) from continuation principle on compact intervals (see Remark 2.6) is satisfied,

- if $K \subset \mathbb{R}^n$ is a nonempty open set whose closure $\overline{K}$ is a retract of $\mathbb{R}^n$, then the set Q defined by formula (16) is a retract of the space $C^1([0,T], \mathbb{R}^n)$.

Summing up, the two particular conditions appearing in the continuation principle for problems on compact intervals can be guaranteed by the bound sets theory as has been just indicated.

Since the verification of the conditions (13), (15) from Proposition 3.1 and Remark 3.1 is in general still not very easy, we will turn our attention to the more regular bounding functions which will lead to the simplification and practical applicability. More concretely, in the case when $V \in C^2(\mathbb{R}^n, \mathbb{R})$, the following corollary immediately follows.

Corollary 3.1 *Let us consider the b.v.p.* (12) *and assume that there exists a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfying conditions $(H1)$ and $(H2)$. Moreover, assume that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K),\ t \in (0,T)$ and $v \in \mathbb{R}^n$, condition*

$$\langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0 \tag{17}$$

holds, for all $w \in F(t,x,v) - A(t)v - B(t)x$, where H denotes the Hesse second-order differential operator. Furthermore, let condition (14) *holds, for all $y \in \partial K$ and $w \in \mathbb{R}^n$. Then K is a bound set for problem* (12).

The easiest way how the set K can be defined is assumed it as an open ball centered at the origin. How in such a case the conditions would be simplified is described in the following example.

Example 3.1 Given $R > 0$, put $K := \{x \in \mathbb{R}^n \mid |x| < R\}$. Let the function $V : \mathbb{R}^n \to \mathbb{R}$ be defined, for all $x \in \overline{K}$, as follows:

$$V(x) = \frac{1}{2}\left(|x|^2 - R^2\right). \tag{18}$$

Then V satisfies conditions $(H1)$ and $(H2)$. Moreover, for each $x \in \mathbb{R}^n$, $\nabla V(x) = x$ and $HV(x) = I$.

Therefore, condition (13) can be reformulated in the following way: there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0, T)$ and $v \in \mathbb{R}^n$, the inequality

$$\langle v, v\rangle + \langle x, w\rangle > 0$$

holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$.

Remark 3.2 Let us note that condition (14) depends both on the boundary conditions in (12) and on the gradient ∇V of the bounding function V. In particular, (14) is trivially satisfied if $M = N = I$, where I denotes the $n \times n$ unit matrix. This case corresponds to the investigation of periodic solutions of the inclusion in (12).

In the case if $My = my$ and $Nw = nw$, for all y, $w \in \mathbb{R}^n$, where $m, n \in \mathbb{R}$, and if V is defined by formula (18), it is easy to see that condition (14) is satisfied if and only if $mn \geq 0$.

Conditions (13), (15), (17) are not strictly localized on the boundary of the bound set K, but assumed at some vicinity $\overline{K} \cap N_\varepsilon(\partial K)$ of it. This is caused by the fact that the r.h.s. of the considered b.v.p. was an upper-Carathéodory mapping. If the r.h.s. would be more regular, then it is possible to put conditions ensuring the existence of the bound set directly on the boundary of the set K. To be more concrete, it was shown in [13] that if in (12)

(i^C) $A, B : [0, T] \to \mathbb{R}^{n \times n}$ are continuous matrix functions,

(ii^C) M and N are $n \times n$ matrices, M is non-singular,

(iii^C) $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper semicontinuous multivalued mapping with nonempty, compact, convex values,

then it is possible to localize the conditions for the bounding function directly on the boundary of the bound set as the following theorem shows.

Theorem 3.1 *Let us consider the b.v.p.* (12) *satisfying* (i^C)-(iii^C) *and suppose that, for all* $x \in \partial K$, $t \in (0,T)$ *and* $v \in \mathbb{R}^n$ *with*

$$\langle \nabla V(x), v\rangle = 0, \tag{19}$$

the following condition holds

$$\liminf_{h\to 0} \frac{\langle \nabla V(x+hv), v+hw\rangle}{h} > 0, \tag{20}$$

for all $w \in F(t,x,v) - A(t)v - B(t)x$.

Moreover, suppose that, for all $x \in \partial K$ *and* $v \in \mathbb{R}^n$ *with*

$$\langle \nabla V(x), v\rangle \le 0 \le \langle \nabla V(Mx), Nv\rangle, \tag{21}$$

at least one of the following conditions

$$\liminf_{h\to 0^+} \frac{\langle \nabla V(x+hv), v+hw_1\rangle}{h} > 0 \tag{22}$$

or

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(Mx+hNv), Nv+hw_2\rangle}{h} > 0 \tag{23}$$

holds, for all $w_1 \in F(0,x,v) - A(0)v - B(0)x$, *or, for all* $w_2 \in F(T,Mx,Nv) - A(T)Nv - B(T)Mx$. *Then* K *is a bound set for problem* (12).

Remark 3.3 Let us note that if condition (22) holds, for some $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (21) and $w_1 \in F(0,x,v) - A(0)v - B(0)x$ then, according to the continuity of ∇V, $\langle \nabla V(x), v\rangle = 0$. Similarly, if (23) holds, for some $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (21) and $w_2 \in F(T,Mx,Nv) - A(T)Nv - B(T)Mx$, then $\langle \nabla V(Mx), Nv\rangle = 0$.

Therefore, the validity of (21), (22) and (23) implies, in particular, that

$$\langle \nabla V(x), v\rangle = \langle \nabla V(Mx), Nv\rangle = 0. \tag{24}$$

As well as in the case of upper-Carathéodory r.h.s., also now the practically applicable version of the bound sets theory can be obtain if the more regular bounding function V is considered.

Remark 3.4 If a bounding function V is of class C^2, conditions (20), (22) and (23) can be rewritten in terms of gradients and Hessian matrices. Concretely, (20) takes the form

$$\langle HV(x)v,\, v\rangle + \langle \nabla V(x), w\rangle > 0,$$

for all $x \in \partial K,\ v \in \mathbb{R}^n$ satisfying (19), $t \in (0,T)$ and $w \in F(t,x,v) - A(t)v - B(t)x$.

For the sake of simplicity, in order to discuss (22) and (23), let us restrict ourselves to those V, M and N for which (21) implies (24). In such a case, it is easy to see that (22) and (23) are equivalent to

$$\max\{\langle HV(x)v,\, v\rangle\ +\ \langle \nabla V(x), w_1\rangle,$$
$$\langle HV(Mx)\cdot Nv,\, Nv\rangle\ +\ \langle \nabla V(Mx), w_2\rangle\} > 0,$$

for all $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (21), $w_1 \in F(0,x,v) - A(0)v - B(0)x$ and $w_2 \in F(T,Mx,Nv) - A(T)Nv - B(T)Mx$.

In particular, observe that (21) always implies (24) under conditions (22) and (23) (see Remark 3.3). The same is true if one of the following possibilities takes place:

(i) $M = N = I$, i.e. for the periodic problem associated to the inclusion in (12),

(ii) $M = N = -I$, i.e. for the anti-periodic b.v.p. associated to the inclusion in (12), and for $\nabla V(-x) = -\nabla V(x)$, for all $x \in \partial K$,

(iii) $M = a \cdot I$, $N = b \cdot I$, where $a \cdot b > 0$, and $\nabla V(ax) = a\nabla V(x)$, for all $x \in \partial K$.

At the beginning of this section, the Floquet b.v.p. (12) with an upper-Carathéodory r.h.s. was studied via non-strictly localized bounding functions, i.e. in the case when the conditions concerning (Liapunov-like) bounding functions were not imposed directly on the boundaries of bound sets, but at some vicinity of them.

As was shown afterwards, this problem of non-strict localization does not occur for Marchaud systems, i.e. for systems with globally upper semicontinuous r.h.s.

Finally, it will be displayed now that also the case of upper-Carathéodory systems can be treated by the strictly localized bounding functions when the Scorza-Dragoni type approach is applied. The original idea of applying the Scorza-Dragoni technique comes from [68], where guiding functions were employed for vector first-order Carathéodory differential equations.

Approximating the original problem by a sequence of problems satisfying non-strictly localized conditions of Proposition 3.1 and applying the Scorza-Dragoni type result (see [34, Proposition 8], and [18, Proposition 2.1] for multivalued mappings), the following result has been obtained in [18]. The condition for bounding function is now required only on the boundary ∂K of the set K, and not on the whole neighborhood $\overline{K} \cap N_\varepsilon(\partial K)$, as in Proposition 3.1. On the other hand, the more regular bounding function of class C^2 is directly considered now.

Theorem 3.2 *Let us consider the Floquet b.v.p.* (12) *satisfying* (i^U)-(iii^U) *and assume that* $V \in C^2(\mathbb{R}^n, \mathbb{R})$. *Moreover, assume that*

(i) *for all* $x \in \partial K,\ t \in (0,T)$ *and* $v \in \mathbb{R}^n$, *it holds that*

$$\langle \nabla V(x), w \rangle > 0, \tag{25}$$

for all $w \in F(t,x,v) - A(t)v - B(t)x$,

(ii) *there exists* $\varepsilon > 0$ *such that* $HV(x)$ *is positive semi-definite, for all* $x \in \overline{K} \cap N_\varepsilon(\partial K)$,

(iii) *for all* $x \in \partial K$ *and* $v \in \mathbb{R}^n$,

$$\langle \nabla V(Mx), Nv \rangle \cdot \langle \nabla V(x), v \rangle > 0 \ \text{or}\ \langle \nabla V(Mx), Nv \rangle = \langle \nabla V(x), v \rangle = 0.$$

Then K *is a bound set for problem* (12).

Remark 3.5 Let us note that particular conditions appearing in bound sets results - (14), (22) and (23) as well as condition (iii) from Theorem 3.2 depend on the boundary conditions in (12). All of them can be omitted in the case when the Floquet boundary conditions would be replaced by other types of boundary conditions, e.g. by the Dirichlet conditions $x(T) = x(0) = 0$, together with the additional assumption $0 \in K$.

3.2 Bound sets approach for impulsive problems in $\mathbb{R}^n$

If we consider, instead of non-impulsive problems, the second-order impulsive ones, it is again possible to develop the bound sets technique that can be used for guaranteeing of the transversality condition (v) in Proposition 2.1. For this purpose, let us consider the Dirichlet impulsive problem

$$\left.\begin{array}{c}\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0,T],\\ x(T) = x(0) = 0,\\ x(t_i^+) = A_i x(t_i), \ \ i = 1, \dots, p,\\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), \ \ i = 1, \dots, p,\end{array}\right\} \tag{26}$$

where

- $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory or a globally upper semicontinuous multivalued mapping,
- $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T, \ p \in \mathbb{N}$,
- $A_i, \ B_i, \ i = 1, \ldots, p$, are real $n \times n$ matrices.
- $x(a^+) = \lim\limits_{t \to a^+} x(t)$.

By a *solution* of problem (26) we shall mean a function $x \in PAC^1([0, T], \mathbb{R}^n)$ satisfying (26).

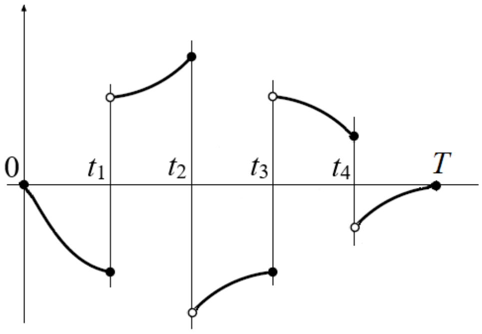

Figure 2: A solution of the impulsive b.v.p. (26) for $n = 1$ and $p = 4$

Moreover, let, in the whole Section 3.2, $0 \in K$ and $A_i, \ i = 1, \ldots, p$, satisfy $A_i \partial K = \partial K$.

In [74], the following proposition has been proven for impulsive problems with the upper-Carathéodory r.h.s. via non-strictly localized bounding function.

Proposition 3.2 *Let us consider the impulsive b.v.p.* (26)*, where* $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper-Carathéodory multivalued mapping. Suppose, moreover, that there exists* $\varepsilon > 0$ *such that, for all* $x \in \overline{K} \cap N_\varepsilon(\partial K), \ t \in (0, T)$ *and* $v \in \mathbb{R}^n$*, the following condition*

$$\limsup_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle - \langle \nabla V(x), v \rangle}{h} > 0 \tag{27}$$

holds, for all $w \in F(t, x, v)$*, and that*

$$\langle \nabla V(A_i x), B_i v \rangle \cdot \langle \nabla V(x), v \rangle > 0, \tag{28}$$

for all $i = 1, \ldots, p, \ x \in \partial K$ *and* $v \in \mathbb{R}^n$ *with* $\langle \nabla V(x), v \rangle \neq 0$.

Then K *is a bound set for the impulsive Dirichlet problem* (26)*.*

Remark 3.6 Let us note that the condition (27) is the same as the corresponding one (13) for the problems without impulses.

Remark 3.7 When the bounding function V is of class C^2, the condition (27) can be rewritten (analogously like in the non-impulsive case) in terms of gradients and Hessian matrices as follows:
There exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K),\ t \in (0,T)$ and $v \in \mathbb{R}^n$, condition

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0 \tag{29}$$

holds, for all $w \in F(t,x,v)$.

Condition (27), (29) were not strictly localized on the boundary of the bound set K, but assumed at some vicinity $\overline{K} \cap N_\varepsilon(\partial K)$ of it since the r.h.s. of the considered impulsive Dirichlet problem was an upper-Carathéodory mapping. If the r.h.s. would be more regular, then it is possible (analogously like in the non-impulsive case) to localize conditions ensuring the existence of the bound set directly on the boundary of the set K. To be more concrete, it was shown in [75] that if in (26), $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper semicontinuous multivalued mapping with nonempty, compact, convex values, then it is possible to localize the conditions for the bounding function directly on the boundary of the bound set as the following theorem shows.

Theorem 3.3 *Let us consider the impulsive b.v.p.* (26)*, where* $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper semicontinuous multivalued mapping with nonempty, compact, convex values.*

Suppose moreover that, for all $x \in \partial K,\ t \in (0,T) \setminus \{t_1, \dots t_p\}$ *and* $v \in \mathbb{R}^n$ *with*

$$\langle \nabla V(x), v\rangle = 0, \tag{30}$$

the following condition holds

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle}{h} > 0, \tag{31}$$

for all $w \in F(t,x,v)$.
At last, suppose that, for all $x \in \partial K$ *and* $v \in \mathbb{R}^n$ *with*

$$\langle \nabla V(A_i x), B_i v\rangle \le 0 \le \langle \nabla V(x), v\rangle, \text{ for some } i = 1, \dots, p, \tag{32}$$

the following condition

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle}{h} > 0 \tag{33}$$

holds, for all $w \in F(t_i, x, v)$*. Then* K *is a bound set for the impulsive Dirichlet problem* (26)*.*

Remark 3.8 Let us now consider the particular case when the bounding function V is of class C^2. Then conditions (31) and (33) can be rewritten in terms of gradients and Hessian matrices as follows:
Suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ the following holds:

$$\text{if } \langle \nabla V(x), v\rangle = 0, \text{ then } \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0, \tag{34}$$

for all $t \in (0,T) \setminus \{t_1, \dots t_p\}$ and $w \in F(t,x,v)$, and

$$\text{if } \langle \nabla V(A_i x), B_i v\rangle \le 0 \le \langle \nabla V(x), v\rangle \text{ for some } i = 1, ..., p, \tag{35}$$

$$\text{then } \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0,$$

for all $w \in F(t_i, x, v)$.

If the Scorza-Dragoni type result ([34, Proposition 5.1]) is applied, and if the original problem is approximated by a sequence of problems satisfying non-strictly localized conditions, the relevant condition can be required directly on the boundary ∂K of the set K also in the case of upper-Carathéodory r.h.s., and not on the whole neighborhood $\overline{K} \cap N_\varepsilon(\partial K)$, as in Proposition 3.2. Analogously like in the non-impulsive case, the bounding function of class C^2 is considered now.

Theorem 3.4 *Let us consider the impulsive b.v.p.* (26)*, where* $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper-Carathéodory multivalued mapping, and let* $V \in C^2(\mathbb{R}^n, \mathbb{R})$.

Moreover, assume that

(*i*) *for all* $x \in \partial K$ *and* $v \in \mathbb{R}^n$*, the inequality*

$$\langle \nabla V(x), w\rangle > 0$$

holds, for all $t \in (0,T)$ *and* $w \in F(t,x,v)$,

(*ii*) *there exists* $h > 0$ *such that* $HV(x)$ *is positive semidefinite in* $N_h(\partial K)$,

(*iii*) *for all* $i = 1, \dots, p$, $x \in \partial K$ *and* $v \in \mathbb{R}^n$*, with* $\langle \nabla V(x), v\rangle \ne 0$*, it holds that*

$$\langle \nabla V(A_i x), B_i v\rangle \cdot \langle \nabla V(x), v\rangle > 0.$$

Then K *is a bound set for problem* (26).

Remark 3.9 Both Theorems 3.3 and 3.4 give an existence result for an impulsive Dirichlet boundary value problem with a strictly localized bounding function respectively for u.s.c and upper-Carathéodory multimap. However Theorem 3.4 doesn't represent an extension of Theorem 3.3, since the first one deals with a C^2-bounding function, while the second one is related to a C^1-bounding function and can not be easily extended to the Carathéodory case.
In the case when the multivalued mapping F is upper semicontinuous and the bounding function V is of class C^2, i.e. when it is possible to apply both theorems, conditions of Theorem 3.3 are weaker than assumptions of Theorem 3.4. In fact, in this case, according to Remark 3.8, condition (31) of the first theorem reads as

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0$$

for every $x \in \partial K$, $v \in \mathbb{R}^n$, and for every $t \in (0,T) \setminus \{t_1, ..., t_p\}, w \in F(t,x,v)$ if $\langle \nabla V(x), v\rangle \neq 0$, or for every $w \in F(t_i, x, v)$ if $\langle \nabla V(A_i x), B_i v\rangle \leq 0 \leq \langle \nabla V(x), v\rangle$, which are implied by assumptions (i) and (ii) of the second theorem.

Remark 3.10 In all conditions that have appeared in Sections 3.1 and 3.2, the element v has played the role of the first derivative of the solution x. If x is a solution of particular b.v.p. such that $x(t) \in \overline{K}$, for every $t \in [0,T]$, and if there exists a continuous increasing function $\psi : [0,\infty) \to [0,\infty)$ satisfying

$$\lim_{s\to\infty} \frac{s^2}{\psi(s)} ds = \infty, \tag{36}$$

and such that

$$|F(t,c,d)| \leq \psi(|d|), \tag{37}$$

for a.a. $t \in [0,T]$ and every $c, d \in \mathbb{R}^n$ with $|c| \leq R := \max\{|x| : x \in \overline{K}\}$, then, by means of the Nagumo-type result (see [78, Lemma 2.1] and [56, Lemma 5.1]), it holds that $|\dot{x}(t)| \leq B$, for every $t \in [0,T]$, where B is defined by

$$B = \psi^{-1}(\psi(2R) + 2R). \tag{38}$$

Hence, it is sufficient to require all conditions in previous propositions and theorems only for all $v \in \mathbb{R}^n$ with $|v| \leq B$ and not for all $v \in \mathbb{R}^n$.

3.3 Bound sets approach for problems in abstract spaces

Also the continuation principle in abstract spaces developed in the form of Theorem 2.2 has contained the transversality condition (v) that is not easily verifiable. Therefore, in this section, the bound sets technique in Banach spaces will be described in short that can be used for its guaranteeing.

For this purpose, let us consider the Floquet boundary value problem

$$\left.\begin{array}{r}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in F(t,x(t),\dot{x}(t)),\quad \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0),\end{array}\right\} \tag{39}$$

where

- E is a separable Banach space satisfying the Radon-Nikodym property (with the norm $||\cdot||$),
- $A, B\colon [0,T]\to\mathcal{L}(E)$ are Bochner integrable, where $\mathcal{L}(E)$ stands for the Banach space of all linear, bounded transformations $L\colon E\to E$ endowed with the sup-norm,
- $F\colon [0,T]\times E\times E\multimap E$ is an upper-Carathéodory (or upper semicontinuous) multivalued mapping,
- $M,\ N\in\mathcal{L}(E)$.

Moreover, let E' be the Banach space dual to E and let us denote by $\langle\cdot,\,\cdot\rangle$ the pairing (the duality relation) between E and E', i.e., for all $\Phi\in E'$ and $x\in E$, we put $\Phi(x){:=}\langle\Phi,\,x\rangle$.

Similarly like in the finite-dimensional case, two cases will be distinguished - namely when (i) A,B are Bochner integrable transformations and F is an upper-Carathéodory mapping, and (ii) A, B are continuous transformations and F is globally upper semicontinuous (i.e. a Marchaud mapping). Unlike in the first case, the second one allows to apply bounding functions which can be strictly localized on the boundaries of given bound sets. Finally, it will be shown that using Scorza-Dragoni type technique, the strict localization is possible also in the case of upper-Carathéodory r.h.s.

The geometry concerning second-order problems, reflecting the behaviour of controlled trajectories, is again much more sophisticated than for first-order problems. Moreover, to express desired transversality conditions in terms of bounding functions, it requires for second-order problems in Banach spaces to employ newly dual spaces. On the other hand, the sufficient conditions are better than those for equivalent first-order problems (see [22]).

Let (in the whole Section 3.3) K be a nonempty, open subset of E containing 0 and let $V: E\to\mathbb{R}$ be a C^1-function with a locally Lipschitz Frechét derivative $\dot{V}_x$ satisfying

$(H1)$ $V|_{\partial K} = 0$,

$(H2)$ $V(x) \leq 0$, for all $x \in \overline{K}$.

Moreover, let M be invertible and such that $M\partial K = \partial K$.

As the first possibility, let us consider the b.v.p. (39) with an upper-Carathéodory r.h.s. In [16], the bound sets theory for such a case has been developed in the form of the following proposition.

Proposition 3.3 *Suppose that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K),\ t \in (0,T)$ and $y \in E$, the following condition*

$$\limsup_{h \to 0^-} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y \rangle}{h} + \langle \dot{V}_{x+hy},\, w \rangle > 0 \tag{40}$$

holds, for all $w \in F(t,x,y) - A(t)y - B(t)x$, and that

$$\left\langle \dot{V}_{Mx}, Nz \right\rangle \cdot \left\langle \dot{V}_x, z \right\rangle > 0, \ \text{or} \ \left\langle \dot{V}_{Mx}, Nz \right\rangle = \left\langle \dot{V}_x, z \right\rangle = 0, \tag{41}$$

for all $x \in \partial K$ and $z \in E$. Then K is a bound set for the Floquet problem (39).

Remark 3.11 Condition (40) can be, analogously like in the finite-dimensional case replaced by

$$\limsup_{h \to 0^+} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y \rangle}{h} + \langle \dot{V}_{x+hy},\, w \rangle > 0, \tag{42}$$

for all $x \in \overline{K} \cap N_\varepsilon(\partial K),\ t \in (0,T)$, $y \in E$, and all $w \in F(t,x,y) - A(t)y - B(t)x$.

Remark 3.12 If we would consider different type of boundary conditions, the assumption (41) can be omitted. In particular, this case was studied for both upper-Carathéodory and globally upper semicontinuous r.h.s. in [15].

If the mapping $F(t,x,y) - A(t)y - B(t)x$ is globally upper semicontinuous in (t,x,y), then the conditions ensuring the existence of a bound set can be localized directly on the boundary of K, as will be shown in the following theorem that was proven in [16].

Theorem 3.5 *Let $F\colon [0,T]\times E\times E \multimap E$ be an upper semicontinuous mapping with nonempty, compact, convex values and A and B be continuous. Suppose moreover that, for all $x\in\partial K,\ t\in(0,T)$ and $y\in E$ with*

$$\langle \dot V_x, y\rangle = 0, \tag{43}$$

the following condition holds

$$\liminf_{h\to 0}\frac{\langle \dot V_{x+hy}, y+hw\rangle}{h} > 0, \tag{44}$$

for all $w\in F(t,x,y)-A(t)y-B(t)x$.

At last, assume that, for all $x\in\partial K$ and $y\in E$ with

$$\langle \dot V_x, y\rangle \le 0 \le \langle \dot V_{Mx}, Ny\rangle, \tag{45}$$

at least one of the following conditions

$$\liminf_{h\to 0^+}\frac{\langle \dot V_{x+hy}, y+hw_1\rangle}{h} > 0 \tag{46}$$

or

$$\liminf_{h\to 0^-}\frac{\langle \dot V_{Mx+hNy}, Ny+hw_2\rangle}{h} > 0 \tag{47}$$

holds, for all $w_1\in F(0,x,y)-A(0)y-B(0)x$, or, for all $w_2\in F(T,Mx,Ny)-A(T)Ny-B(T)Mx$, respectively. Then K is a bound set for problem (39).

Remark 3.13 One can readily check that, for $V\in C^2(E,\mathbb{R})$, the inequalities (40), (42), as well as (44), become

$$\langle \ddot V_x(y),\, y\rangle + \langle \dot V_x,\, w\rangle > 0,$$

with t,x,y,w as in Proposition 3.3 or in Theorem 3.5.

Remark 3.14 The typical case occurs when $E=H$ is a Hilbert space, $\langle\,,\,\rangle$ denotes the scalar product and

$$V(x) := \frac{1}{2}\left(\|x\|^2 - R^2\right) = \frac{1}{2}\left(\langle x,x\rangle - R^2\right),$$

for some $R>0$. In this case, $V\in C^2(H,\mathbb{R})$ and it is not difficult to see that conditions (40), (42), as well as (44) become

$$\langle y,\, y\rangle + \langle x,\, w\rangle > 0$$

with t,x,y and w as in Proposition 3.3 or in Theorem 3.5, where $K := \{x\in H|\ \|x\| < R\}$.

The conditions concerning bounding functions in abstract spaces were not in the commentary up to now imposed in the upper-Carathéodory case directly on the boundaries of bound sets, but at some vicinity of them. The strict localization is possible also in this case by means of the Scorza-Dragoni type technique developed in [76]. On the other hand, it is suitable to point out that the strict localization requires again a higher regularity of applied bounding functions.

Moreover, in this case, the bounding function V must satisfy (apart from previous conditions $(H1)$ and $(H2)$) also the following one:

$(H3)$ $||\dot{V}(x)|| \geq \delta$, for all $x \in \partial K$, where $\delta > 0$ is given.

The bound sets theory in abstract spaces for the upper-Carathéodory case via strictly localized conditions for bounding function was developed in [14] in the following form.

Theorem 3.6 *Consider the Floquet b.v.p.* (39) *with an upper-Carathéodory r.h.s. Assume that $K \subset E$ is an open, convex set containing* 0*. Furthermore, let there exist $\varepsilon > 0$ and a function $V \in C^2(E, \mathbb{R})$ satisfying $(H1) - (H3)$. Moreover, let there exist $h > 0$ such that*

$$\left\langle \ddot{V}_x(v), v \right\rangle \geq 0, \text{ for all } x \in N_h(\partial K),\ v \in E, \tag{48}$$

where $\ddot{V}_x(v)$ denotes the second Fréchet derivative of V at x in the direction $(v, v) \in E \times E$. Finally, let

$$\langle \dot{V}_x, w \rangle > 0, \tag{49}$$

and (41) *holds, for all $x \in \partial K$, $t \in (0, T)$, $z \in E$, and $w \in F(t, x, z) - A(t)z - B(t)x$. Then K is a bound set for problem* (39).

Remark 3.15 The result can be analogously like in the previous cases simplified, i.e. the assumption (41) can be omitted, when the different type of boundary conditions would be considered (see [17] for the Dirichlet problem).

4 Contribution to the theory of boundary value problems for second-order differential inclusions on compact and non-compact intervals in the Euclidean and abstract spaces

The topic of the thesis is the investigation of the boundary value problems on compact and non-compact intervals for second-order differential inclusions both in the Euclidean and abstract spaces. The thesis is submitted as the collection of 12 papers published in international journals.

The fundamental tools used in the collection of presented papers are suitable continuation principles described in Section 2 that contain apart from other assumptions so called "pushing" or transversality conditions whose verifying is in most cases very complicated. For its guaranteeing, the bound sets technique described in Section 3 is applied in the collection of submitted papers.

4.1 Non-impulsive boundary value problems for vector second-order differential inclusions on compact and non-compact intervals

Papers dealing with this topic are the following:

[23] J. Andres, M. Pavlačková, *Asymptotic boundary value problems for second-order differential systems.* Nonlin. Anal. **71** (5-6) (2009), 1462–1473.

[24] J. Andres, M. Pavlačková, *Topological structure of solution sets to asymptotic boundary value problems.* J. Diff. Eqns. **248** (1) (2010), 127—150.

[12] J. Andres, M. Kožušníková, L. Malaguti, *Bound sets approach to boundary value problems for vector second-order differential inclusions.* Nonlin. Anal. **71** (1-2) (2009), 28–44.

[13] J. Andres, M. Kožušníková, L. Malaguti, *On the Floquet problem for second-order Marchaud differential systems.* J. Math. Anal. Appl. **351** (2009), 360—372.

[18] J. Andres, L. Malaguti, M. Pavlačková, *Strictly localized bounding functions for vector second-order boundary value problems.* Nonlin. Anal. **71** (12) (2009), 6019–6028.

[73] M. Pavlačková, *A Scorza-Dragoni approach to Dirichlet problem with an upper-Carathéodory right-hand side*, Topol. Meth. Nonlin. Anal. **44** (1) (2014), 239–247.

In the first paper [23], the continuation principle has been developed for the vector second-order asymptotic multivalued boundary value problems (see Theorem 2.1). Afterwards, in paper [24], the topological structure of the solution sets of the studied asymptotic problems has been investigated by the inverse limit method, and the information about the structure has been employed, by virtue of the continuation principle from [23], for obtaining an existence result for nonlinear asymptotic problems.

In the first two mentioned papers [23] and [24], the continuation principle for asymptotic problems has been developed and applied. In paper [12], the version of the continuation principle for vector problems on compact interval has been specified and the appropriate transversality condition for problems on compact intervals has been stated (see Remark 2.6). In the compact case, the condition requires that the corresponding problems do not have solutions on the boundary of the sets of candidate solutions. Since this can be guaranteed by a bound sets approach, the rest of paper [12] has been devoted to the bound sets technique (see Proposition 3.1) for the following second-order vector Floquet boundary value problem

$$\left.\begin{array}{c}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in F(t,x(t),\dot{x}(t)),\ \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0),\end{array}\right\} \tag{50}$$

where

- $A, B : [0,T] \to \mathbb{R}^{n\times n}$ are measurable matrix functions such that $|A(t)| \leq a(t)$ and $|B(t)| \leq b(t)$, for all $t \in [0,T]$ and suitable integrable functions $a, b : [0,T] \to [0,\infty)$,
- M and N are $n \times n$ matrices, M is non-singular,
- $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping.

The bound sets approach has been in [12] combined with the continuation principle and the existence and the localization result has been obtained in this way. Let us note that since the right-hand side of the considered problem has been an upper-Carathéodory multivalued mapping, the related conditions guaranteeing the existence of a bound set have not been strictly localized on the boundary of the bound set but put at some vicinity of it.

This imperfection has been removed in paper [13], where the conditions ensuring the existence of a bound set have been localized directly on the boundary of the bound set (see Theorem 3.1). This has been possible since the right-hand side has been considered more regular; the Floquet problem with an upper semicontinuous r.h.s. has been studied there.

Combining previous result with Scorza-Dragoni type technique allowed to impose related conditions strictly on the boundaries of bound sets also in the case of Floquet problem with less regular upper-Carathéodory r.h.s. in [18] (see Theorem 3.2). The combination of Scorza-Dragoni approach with the bound sets technique has been applied also for the Dirichlet problem with upper-Carathéodory r.h.s. in [73].

4.2 Non-impulsive boundary value problems for second-order differential inclusions on compact intervals in abstract spaces

Papers dealing with this topic are the following:

[14] J. Andres, L. Malaguti, M. Pavlačková, *A Scorza-Dragoni approach to second-order boundary value problems in abstract spaces*. Appl. Math. Inf. Sci. **6** (2) (2012), 177–192.

[15] J. Andres, L. Malaguti, M. Pavlačková, *Dirichlet problem in Banach spaces: the bound sets approach*. Bound Value Probl 2013, 25 (2013). https://doi.org/10.1186/1687-2770-2013-25

[16] J. Andres, L. Malaguti, M. Pavlačková, *On second-order boundary value problems in Banach spaces: a bound sets approach*. Topol. Meth. Nonlin. Anal. **37** (2) (2011), 303–341.

[17] J. Andres, L. Malaguti, M. Pavlačková, *Scorza-Dragoni approach to Dirichlet problem in Banach spaces*. Bound Value Probl 2014, 23 (2014). https://doi.org/10.1186/1687-2770-2014-23

The papers [12], [13], [18], [23], [24], and [73] have dealt with the boundary value problems in $\mathbb{R}^n$. Besides this, we have been also studying the problems in abstract spaces. In [16], the existence and localization of strong (Carathéodory) solutions has been obtained for the second-order Floquet problem in a Banach space, i.e. for the problem

$$\left.\begin{array}{r}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in F(t,x(t),\dot{x}(t)),\qquad \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0),\qquad\qquad\end{array}\right\}\tag{51}$$

where

- E is a Banach space satisfying the Radon-Nikodym property,
- $A, B\colon [0,T]\to\mathcal{L}(E)$ are Bochner integrable,
- $F\colon [0,T]\times E\times E \multimap E$ is an upper-Carathéodory multivalued mapping,
- $M,\ N\in\mathcal{L}(E)$.

The result in [16] has been obtained by combination of continuation principle in abstract spaces (see Theorem 2.2) and bounding functions approach (see Proposition 3.3). The main theorem for upper-Carathéodory inclusions has been in [16] separately improved for Marchaud inclusions (see Theorem 3.5).

Using the Scorza-Dragoni approach, the results from [16] has been subsequently improved in [14], where the existence and localization of strong solutions of the second-order Floquet boundary value problems for upper-Carathéodory differential inclusions in Banach spaces has been obtained by strictly localized bounding functions (see Theorem 3.6).

The Dirichlet problem in abstract spaces, i.e. the problem

$$\left.\begin{array}{r}\ddot{x}(t)\in F(t,x(t),\dot{x}(t)),\qquad \text{for a.a. } t\in[0,T],\\ x(0)=x(T)=0,\qquad\qquad\end{array}\right\}\tag{52}$$

where

- E is a Banach space satisfying the Radon-Nikodym property,
- $F\colon [0,T]\times E\times E \multimap E$ is an upper-Carathéodory mapping or a globally upper semicontinuous mapping with compact, convex values,

has been studied by the combination of the continuation principle in abstract spaces together with the bounding-functions approach and the Scorza-Dragoni technique in [15] and [17]. Moreover, the main existence and localization result has been in [17] applied to a partial integro-differential equation involving possible discontinuities in state variables.

4.3 Impulsive boundary value problems for vector second-order differential inclusions on compact intervals

Papers dealing with this topic are the following:

[74] M. Pavlačková, V. Taddei, *A bounding function approach to impulsive Dirichlet problem with an upper-Carathéodory right-hand side*. Elec. J. Diff. Eqns **12** (2019), 1–18.

[75] M. Pavlačková, V. Taddei, *On the impulsive Dirichlet problem for second-order differential inclusions*, El. J. Qual. Th. Diff. Eqns **13** (2020), 1–22. https://doi.org/10.14232/ejqtde.2020.1.13

All up to now mentioned papers have dealt with the non-impulsive problems. Recently, we have started to develop the bound sets technique also for the Dirichlet impulsive problem

$$\left.\begin{array}{c} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0, T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i), \quad i = 1, \ldots, p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), \quad i = 1, \ldots, p, \end{array}\right\} \tag{53}$$

where

- $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory or a globally upper semicontinuous multivalued mapping,
- $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T,\ p \in \mathbb{N}$,
- $A_i,\ B_i,\ i = 1, \ldots, p$, are real $n \times n$ matrices.
- $x(a^+) = \lim\limits_{t \to a^+} x(t)$.

In [74], the existence and localization result for a vector impulsive Dirichlet problem with multivalued upper-Carathéodory right-hand side has been obtained by combining the continuation principle for impulsive problems (see Proposition 2.1) with a bound sets technique (see Proposition 3.1). The main theorem has been in [74] illustrated by an application to the forced pendulum equation with viscous damping term and dry friction coefficient.

The most recent paper [75] has been devoted to a vector impulsive Dirichlet problem with multivalued upper-Carathéodory or globally upper semicontinuous right-hand side. Its advantage in comparison with the previous paper

[74] lies in the usage of strictly localized bounding functions (see Theorem 3.3 and Theorem 3.4).

As for further research, we have started recently together with Italian colleagues to study also different types of boundary conditions in the impulsive problem (53). Moreover, we are also planing to developed the bound sets theory for impulsive problems in abstract spaces in the future.

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5 Attachments

The thesis is submitted as the collection of 12 published scholarly papers - [12], [13], [14], [15], [16], [17], [18], [23], [24], [73], [74], [75] which are attached in this chapter.

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Asymptotic boundary value problems for second-order differential systems[☆]

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ABSTRACT

Topological methods are developed for the solvability of vector second-order boundary value problems on noncompact intervals. The solutions are located in given sets and enjoy prescribed properties. The main theorem is supplied by two illustrating examples.

1. Introduction

Asymptotic boundary value problems (b.v.p.s) for second-order differential equations, inclusions and systems, occur naturally in many applications. Among the most popular asymptotic problems are those related to the following equations:

$\ddot{x} = \operatorname{sh} x$ (Boltzmann–Poisson),

$\sqrt{t}\ddot{x} = \sqrt{x^3}$ (Thomas–Fermi),

$\varphi(t)\ddot{x} = x^\beta$ (Emden–Fowler).

Some further applications concern, for instance, population genetics, combustion models, power law fluids, unsteady flows of gas through semi-infinite porous media etc.

An investigation of linear oscillators with weak interactions leads to vector second-order systems. If the friction (damping) is not viscous, but dry (i.e. when the isotropic Coulomb's law holds), then the mathematical model can be described by the system

$$\ddot{x} + A \operatorname{sgn} \dot{x} + Bx = P(t), \quad x \in \mathbb{R}^n,$$

where $A,\ B$ are regular $(n \times n)$-matrices and P is a locally Lebesgue integrable vector forcing term. Because of discontinuity at $y = 0$ in $\operatorname{sgn} y$, we can only consider Filippov solutions which can be identified as Carathéodory solutions of the inclusion

$$\ddot{x} + Bx \in P(t) - A \operatorname{Sgn} \dot{x}, \quad x \in \mathbb{R}^n,$$

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where

$$\operatorname{Sgn} y := \begin{cases} -1, & \text{for } y < 0,\\ [-1, 1], & \text{for } y = 0,\\ 1, & \text{for } y > 0. \end{cases}$$

Another stimulation comes from asymptotic control problems

$$\left.\begin{array}{l} \ddot{x} = f(t, x, \dot{x}, u),\ t \in [t_0, \infty),\ u \in U,\\ x \in S, \end{array}\right\}$$

where S is a suitable constraint (e.g. asymptotic boundary conditions) and $u \in U$ are control parameters such that $u(t) \in \mathbb{R}^n$, for all $t \geq t_0$. Defining the multivalued mapping $F(t, x, y) := \{f(t, x, y, u)\}_{u \in U}$, solutions of the original problem coincide with those of

$$\left.\begin{array}{l} \ddot{x} \in F(t, x, \dot{x}),\\ x \in S. \end{array}\right\}$$

Systems like two models described above will be the objects of our investigation in this paper. By their *solutions*, we shall always understand functions $x : I \to \mathbb{R}^n$ belonging to $AC^1_{loc}(I, \mathbb{R}^n)$, where I is a real (possibly noncompact) interval. For further practical motivations, see, for example, [1,2].

Although boundary value problems for second-order vector systems have been systematically studied since the 70's, there are only a few papers devoted to asymptotic ones (see, e.g., [3–9], and the references therein). There are essentially two ways to attack asymptotic b.v.p.s: (a) *sequentially* which concerns mainly bounded solutions or (b) *directly* which is associated with multivalued operators in Fréchet spaces. The difficulties related to first-order asymptotic problems (cf. [10–17]) are, in principle, of the same sort as for second-order problems. We can simply say that degree arguments in Fréchet spaces are always very delicate (cf. [18–20,12,21,16,17,22]).

The main purpose of the present paper is to verify the localization of solutions in given sets by means of a continuation principle in Fréchet spaces. General methods are developed in Section 3. The main existence and localization results are formulated in Section 4, where two illustrating examples are also supplied. Finally, we add one concluding remark.

2. Preliminaries

In the entire text, all spaces are at least metric. Our problems under consideration naturally lead to the notion of a Fréchet space. Let us recall that by a *Fréchet space*, we mean a complete (metrizable) locally convex vector space. Its topology can be generated by a countable family of seminorms or by a metric (see, e.g., [13, Chapter I.1]). Nevertheless, a topology of non-normable Fréchet spaces brings some problems. For instance, a contractivity of a given operator with respect to a metric need not follow from a contractivity with respect to each seminorm (for the related counter-example, see [13, Example II.2.12]). Other difficulties related to Fréchet spaces concern bounded subsets of non-normable Fréchet spaces which always have empty interiors. For more details concerning Fréchet spaces see, e.g., [20,12,13,16,17]. Let us note that if a Fréchet space is normable, then it becomes a *Banach space*. Fréchet spaces in our considerations below will be, in particular, the following:

- the space $C(I, \mathbb{R}^n)$ of continuous functions $x : I \to \mathbb{R}^n$ with the family of seminorms $p_i(q) : C(I, \mathbb{R}^n) \to \mathbb{R}$ defined by

  $$p_i(q) := \max_{t \in K_i} |q(t)|,$$

  where $\{K_i\}$ is a sequence of compact subintervals of I such that

  $$\bigcup_{i=1}^{\infty} K_i = I, \tag{1}$$

  $$K_i \subset K_{i+1}, \quad \text{for all } i \in \mathbb{N}, \tag{2}$$

- the space $C^1(I, \mathbb{R}^n)$ of smooth functions $x : I \to \mathbb{R}^n$ with the system of seminorms $p_i^*(q) : C^1(I, \mathbb{R}^n) \to \mathbb{R}$ defined by

  $$p_i^*(q) := \max_{t \in K_i} |q(t)| + \max_{t \in K_i} |\dot{q}(t)|,$$

  where $\{K_i\}$ is a sequence of compact subintervals of I satisfying (1) and (2),
- the space $AC^1_{loc}(I, \mathbb{R}^n)$ of functions $x : I \to \mathbb{R}^n$ with locally absolutely continuous first derivatives endowed with the family of seminorms $p_i^+(q) : AC^1_{loc}(I, \mathbb{R}^n) \to \mathbb{R}$ defined by

  $$p_i^+(q) := \max_{t \in K_i} |q(t)| + \max_{t \in K_i} |\dot{q}(t)| + \int_{K_i} |\ddot{q}(t)|\, \mathrm{d}t,$$

  where $\{K_i\}$ is a sequence of compact subintervals of I satisfying (1) and (2).

  The topologies in Fréchet spaces mentioned above can be generated by the metrics

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_i(x - y)}{1 + p_i(x - y)} \tag{3}$$

or

$$d(x,y)^* := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_i^*(x-y)}{1+p_i^*(x-y)} \tag{4}$$

or

$$d(x,y)^+ := \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_i^+(x-y)}{1+p_i^+(x-y)}, \tag{5}$$

respectively.

Lemma 2.1. *If $I \subset \mathbb{R}$ is arbitrary, then any subset $A \subset C(I,\mathbb{R}^n)$ or $A \subset C^1(I,\mathbb{R}^n)$ or $A \subset AC^1_{loc}(I,\mathbb{R}^n)$ is bounded with respect to the metric defined by (3) or (4) or (5), respectively.*

Proof. For any $f, g \in A \subset C(I,\mathbb{R}^n)$, it holds that

$$d(f,g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{p_i(f-g)}{1+p_i(f-g)} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \leq 1.$$

Quite analogous estimates can be obtained for $f, g \in A \subset C^1(I,\mathbb{R}^n)$ or $f, g \in A \subset AC^1_{loc}(I,\mathbb{R}^n)$. □

We also recall some geometric notions of subsets of metric spaces. If (X,d) is an arbitrary space and $A \subset X$, by $\mathrm{Int}(A)$, $\overline{A}$ and ∂A we mean the interior, the closure and the boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x,a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$. Similarly, A is called a *neighborhood retract* of X if there exists an open subset $U \subset X$ such that $A \subset U$ and A is a retract of U.

We say that a nonempty subset A of a space X is *contractible* if there exist a point $x_0 \in A$ and a homotopy $h : A \times [0,1] \to A$ such that $h(x,0) = x$ and $h(x,1) = x_0$, for every $x \in A$. A nonempty set $A \subset X$ is called an R_δ*-set* if there exists a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of compact, contractible sets such that

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Note that any R_δ-set is nonempty, compact and connected and that any convex compact set is obviously an R_δ-set.

A nonempty, compact subset A of a space X is called ∞*-proximally connected* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for every $n \in \mathbb{N}$ and for any map $g : \partial\Delta^n \to N_\delta(A)$, there exits a map $\tilde{g} : \Delta^n \to N_\varepsilon(A)$ such that $g(x) = \tilde{g}(x)$, for every $x \in \partial\Delta^n$, where $\partial\Delta^n := \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ and $\Delta^n := \{x \in \mathbb{R}^{n+1} \mid |x| \leq 1\}$. On neighborhood retracts of Fréchet spaces, the notions of ∞-proximally connected sets and R_δ-sets coincide. For more details about the above subsets of metric spaces, see, e.g., [13,23].

We also employ the following definitions and statements from the multivalued analysis in the sequel. Let X and Y be arbitrary metric spaces. We say that F is a multivalued mapping from X to Y (written $F : X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is prescribed. We associate with F its *graph* Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

A multivalued mapping $F : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X. Every upper semicontinuous map with closed values has a closed graph.

The reverse relation between upper semicontinuous mappings and those with closed graphs is expressed by the following proposition.

Proposition 2.1 (*cf., e.g., [13,23]*). *Let X,Y be metric spaces and $F : X \multimap Y$ be a multivalued mapping with the closed graph such that $F(X) \subset K$, where K is a compact set. Then F is u.s.c.*

A multivalued mapping $F : X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x\in X} F(x)$ is contained in a compact subset of Y and it is called *closed* if $F(B)$ is closed in Y, for every closed subset B of X. We say that a multivalued mapping $F : X \multimap Y$ is an R_δ*-mapping* if it is a u.s.c. mapping with R_δ-values.

We say that a multivalued map $\varphi : X \multimap Y$ is a J*-mapping* (written, $\varphi \in J(X,Y)$) if it is a u.s.c. mapping and $\varphi(x)$ is ∞-proximally connected, for every $x \in X$. If the space Y is a neighborhood retract of a Fréchet space, then $\varphi \in J(X,Y)$, provided φ is an R_δ-mapping, as already mentioned (cf. [13,23]).

Let Y be a separable metric space and $(\Omega, \mathcal{U}, \nu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $F : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid F(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$.

We say that mapping $F : I \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $I \subset \mathbb{R}$, is an *upper-Carathéodory mapping* if the map $F(\cdot,x) : I \multimap \mathbb{R}^n$ is measurable on every compact subinterval of I, for all $x \in \mathbb{R}^m$, the map $F(t,\cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all (a.a.) $t \in I$, and the set $F(t,x)$ is compact and convex, for all $(t,x) \in I \times \mathbb{R}^m$.

Proposition 2.2 (*cf., e.g., [24]*). *Let $F : [0, a] \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping satisfying $|y| \leq r(t)(1 + |x|)$, for every $(t, x) \in [0, a] \times \mathbb{R}^m$, and every $y \in F(t, x)$, where $r : [0, a] \to [0, \infty)$ is an integrable function. Then the composition $F(t, q(t))$ admits, for every $q \in C([0, a], \mathbb{R}^m)$, a single-valued measurable selection.*

If $X \cap Y \neq \emptyset$ and $F : X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed point* of F if $x \in F(x)$. The set of all fixed points of F will be denoted by $\mathrm{Fix}(F)$, i.e.

$$\mathrm{Fix}(F) := \{x \in X \mid x \in F(x)\}.$$

Assume that X is a retract of a Fréchet space E and D is an open subset of X. Let $G \in J(D, E)$ be locally compact, let $\mathrm{Fix}(G)$ be compact and let the following condition hold:

$$\forall x \in \mathrm{Fix}(G)\ \exists \text{ a set } U_x \text{ open in } D,\ x \in U_x, \quad \text{such that } G(U_x) \subset X. \tag{6}$$

The class of locally compact J-mappings from D to E with a compact fixed point set and satisfying (6) will be denoted by $J_A(D, E)$.

We say that $G_1, G_2 \in J_A(D, E)$ are *homotopic in $J_A(D, E)$* if

1. there exists a homotopy $H \in J(D \times [0, 1], E)$ such that $H(\cdot, 0) = G_1$ and $H(\cdot, 1) = G_2$,
2. for every $x \in D$, there exists an open neighborhood V_x of x in D such that $H|_{V_x \times [0,1]}$ is a compact mapping,
3. for every $x \in D$ and every $t \in [0, 1]$, the following condition holds:

$$\text{If } x \in H(x, t),\ \text{then there exists a set } U_x \text{ open in } D,\ x \in U_x, \quad \text{such that } H(U_x \times [0, 1]) \subset X. \tag{7}$$

Remark 2.1. Note that condition (7) is equivalent to the following one:

$$\text{If } \{x_j\}_{j=1}^{\infty} \subset D \text{ converges to } x \in H(x, t),\ \text{for some } t \in [0, 1], \quad \text{then } H(\{x_j\} \times [0, 1]) \subset X,\ \text{for } j \text{ sufficiently large.} \tag{8}$$

Remark 2.2. If $E = X$ is a Banach space, then condition (7) can be reduced to

$$\mathrm{Fix}(H) \cap \partial D = \emptyset,$$

for all $t \in [0, 1]$, where $\mathrm{Fix}(H) := \{x \in D \mid x \in H(x, t)\}$ (see, e.g., [11]).

The following proposition, which will be applied for the existence of a solution of the b.v.p., immediately follows from a result in [12,13].

Proposition 2.3. *Let X be a retract of a Fréchet space E, D be an open subset of X and H be a homotopy in $J_A(D, E)$ such that*
(i) *$H(\cdot, 0)(D) \subset X$,*
(ii) *there exists $H_0 \in J(X)$ such that $H_0|_D = H(\cdot, 0)$, H_0 is compact and*

$$\mathrm{Fix}(H_0) \cap (X \setminus D) = \emptyset.$$

Then there exists $x \in D$ such that $x \in H(x, 1)$.

As a direct consequence of Proposition 2.3, we obtain the following result.

Corollary 2.1. *Let X be a retract of a Fréchet space E, H be a homotopy in $J_A(X, E)$ such that $H(x, 0) \subset X$, for every $x \in X$, and $H(\cdot, 0)$ be compact. Then $H(\cdot, 1)$ has a fixed point.*

It will be also convenient to recall the following results.

Lemma 2.2 (*cf. [25, Theorem 0.3.4]*). *Let $[a, b] \subset \mathbb{R}$ be a compact interval. Assume that the sequence of absolutely continuous functions $x_k : [a, b] \to \mathbb{R}^n$ satisfies the following conditions:*
(i) *the set $\{x_k(t) \mid k \in \mathbb{N}\}$ is bounded, for every $t \in [a, b]$,*
(ii) *there exists a function $\alpha : [a, b] \to \mathbb{R}$, integrable in the sense of Lebesgue, such that*

$$|\dot{x}_k(t)| \leq \alpha(t), \quad \text{for a.a. } t \in [a, b] \text{ and for all } k \in \mathbb{N}.$$

Then there exists a subsequence of $\{x_k\}$ (for the sake of simplicity denoted in the same way as the sequence) converging to an absolutely continuous function $x : [a, b] \to \mathbb{R}^n$ in the following way:

1. *$\{x_k\}$ converges uniformly to x,*
2. *$\{\dot{x}_k\}$ converges weakly in $L^1([a, b], \mathbb{R}^n)$ to $\dot{x}$.*

The following lemma is a slight modification of the well-known result.

Lemma 2.3 (*cf. [26, p. 88]*). *Let $[a, b] \subset \mathbb{R}$ be a compact interval, E_1, E_2 be Euclidean spaces and $F : [a, b] \times E_1 \multimap E_2$ be an upper-Carathéodory mapping.*

Assume, in addition, that for every nonempty, bounded set $\mathcal{B} \subset E_1$, there exists $\nu = \nu(\mathcal{B}) \in L^1([a, b], [0, \infty))$ such that

$$|F(t, x)| \leq \nu(t),$$

for a.a. $t \in [a, b]$ and every $x \in \mathcal{B}$.

Let us define the Nemytskiǐ operator $\mathfrak{N}_F : C([a,b],E_1) \multimap L^1([a,b],E_2)$ in the following way:

$$\mathfrak{N}_F(x) := \{f \in L^1([a,b],E_2) \mid f(t) \in F(t,x(t)), \text{ a.e. on } [a,b]\},$$

for every $x \in C([a,b],E_1)$. Then, if sequences $\{x_i\} \subset C([a,b],E_1)$ and $\{f_i\} \subset L^1([a,b],E_2)$, $f_i \in \mathfrak{N}_F(x_i)$, $i \in \mathbb{N}$, are such that $x_i \to x$ in $C([a,b],E_1)$ and $f_i \to f$ weakly in $L^1([a,b],E_2)$, then $f \in \mathfrak{N}_F(x)$.

3. General methods

In this section, we consider the second-order b.v.p.

$$\left.\begin{array}{l}\ddot{x}(t) \in F(t,x(t),\dot{x}(t)), \quad \text{for a.a. } t \in I,\\ x \in S,\end{array}\right\} \tag{9}$$

where I is a given (possibly noncompact) real interval, $F : I \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and $S \subset AC^1_{loc}(I,\mathbb{R}^n)$.

For the main result of this section (cf. Theorem 3.1), the following proposition is crucial.

Proposition 3.1. *Let $H : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory map and assume that*

(i) *there exists a subset Q of $C^1(I,\mathbb{R}^n)$ such that, for any $q \in Q$, the set $T(q)$ of all solutions of the b.v.p.*

$$\left.\begin{array}{l}\ddot{x}(t) \in H(t,x(t),\dot{x}(t),q(t),\dot{q}(t)), \quad \textit{for a.a. } t \in I,\\ x \in S\end{array}\right\} \tag{10}$$

is nonempty,

(ii) *there exist a point $t_0 \in I$ and constants $M_0 \geq 0$, $M_1 \geq 0$ such that $|x(t_0)| \leq M_0$ and $|\dot{x}(t_0)| \leq M_1$, for any $x \in T(Q)$,*

(iii) *there exists a nonnegative, locally integrable function $\alpha : I \to \mathbb{R}$ such that*

$$|H(t,x(t),\dot{x}(t),q(t),\dot{q}(t))| \leq \alpha(t)(1+|x(t)|+|\dot{x}(t)|),$$

a.e. in I, for any $(q,x) \in \Gamma_T$.

Then $T(Q)$ is a relatively compact subset of $C^1(I,\mathbb{R}^n)$. Moreover, the solution operator $T : Q \multimap S$ is u.s.c. with compact values if and only if the following condition is satisfied:

(iv) *for each sequence $\{q_k, x_k\} \subset \Gamma_T$ satisfying $\{(q_k,\dot{q}_k,x_k)\} \to (q,\dot{q},x)$ uniformly on compact intervals, where $q \in Q$, it holds that $x \in S$.*

Proof. Let $t, t_0 \in I$ be arbitrary. We begin by showing the integral form of a solution of the inclusion

$$\ddot{x}(t) \in H(t,x(t),\dot{x}(t),q(t),\dot{q}(t)). \tag{11}$$

Integrating (11) in the sense of Aumann (see, e.g., [27]), we obtain

$$\dot{x}(t) - \dot{x}(t_0) \in \int_{t_0}^{t} H(s,x(s),\dot{x}(s),q(s),\dot{q}(s))\,\mathrm{d}s, \tag{12}$$

and consequently

$$x(t) - \dot{x}(t_0)\cdot(t-t_0) - x(t_0) \in \int_{t_0}^{t}\int_{t_0}^{s} H(\tau,x(\tau),\dot{x}(\tau),q(\tau),\dot{q}(\tau))\,\mathrm{d}\tau\,\mathrm{d}s. \tag{13}$$

Moreover, integrating (13) by parts, we get

$$x(t) - \dot{x}(t_0)\cdot(t-t_0) - x(t_0) \in \left[s\int_{t_0}^{s} H(\tau,x(\tau),\dot{x}(\tau),q(\tau),\dot{q}(\tau))\mathrm{d}\tau\right]_{t_0}^{t} - \int_{t_0}^{t} sH(s,x(s),\dot{x}(s),q(s),\dot{q}(s))\mathrm{d}s$$

and, therefore,

$$x(t) \in x(t_0) + \dot{x}(t_0)\cdot(t-t_0) + \int_{t_0}^{t}(t-s)\cdot H(s,x(s),\dot{x}(s),q(s),\dot{q}(s))\,\mathrm{d}s \tag{14}$$

is the integral form of a solution of the inclusion (11).

It follows from the well-known Arzelà–Ascoli lemma that the set $T(Q)$ is relatively compact if and only if it is bounded, and functions in $T(Q)$ as well as their first derivatives are equicontinuous. The set $T(Q)$ is bounded in $C^1(I,\mathbb{R}^n)$ with respect to the metric defined by (4), according to Lemma 2.1. Hence, for the relative compactness of $T(Q)$, it is sufficient to show that all elements of $T(Q)$ and their first derivatives are equicontinuous.

Nevertheless, it will be also convenient to have explicit estimates of solutions of (11) and their derivatives w.r.t. each seminorm in $C^1(I,\mathbb{R}^n)$. Let $x \in T(Q)$ be arbitrary, $t_0 \in I$ be such that $|x(t_0)| \leq M_0$ and $|\dot{x}(t_0)| \leq M_1$, for all $x \in T(Q)$, and $[t_1,t_2] \subset I$ be an arbitrary compact interval such that $t_0 \in [t_1,t_2]$. Then, according to (12) and (14), and (iii), we have, for a.a. $t \in [t_1, t_2]$,

$$
\begin{aligned}
|x(t)| + |\dot{x}(t)| &\le |x(t_0)| + |\dot{x}(t_0)| \cdot |t - t_0| + \left| \int_{t_0}^{t} |t-s| \cdot |H(s, x(s), \dot{x}(s), q(s), \dot{q}(s))| \, \mathrm{d}s \right| + |\dot{x}(t_0)| \\
&\quad + \left| \int_{t_0}^{t} |H(s, x(s), \dot{x}(s), q(s), \dot{q}(s))| \, \mathrm{d}s \right| \\
&\le M_0 + M_1 \cdot |t - t_0| + |t - t_0| \cdot \left| \int_{t_0}^{t} \alpha(s)(1 + |x(s)| + |\dot{x}(s)|) \, \mathrm{d}s \right| + M_1 \\
&\quad + \left| \int_{t_0}^{t} \alpha(s)(1 + |x(s)| + |\dot{x}(s)|) \, \mathrm{d}s \right| \\
&\le M_0 + M_1 \cdot [1 + |t_2 - t_1|] + [1 + |t_2 - t_1|] \int_{t_1}^{t} \alpha(s)(1 + |x(s)| + |\dot{x}(s)|) \mathrm{d}s \\
&\le M_0 + M_1 \cdot [1 + |t_2 - t_1|] + [1 + |t_2 - t_1|] \int_{t_1}^{t_2} \alpha(s) \, \mathrm{d}s + [1 + |t_2 - t_1|] \int_{t_1}^{t} \alpha(s)(|x(s)| + |\dot{x}(s)|) \, \mathrm{d}s.
\end{aligned}
$$

By the Gronwall lemma (cf. [28]), we obtain

$$
|x(t)| + |\dot{x}(t)| \le K_{[t_1,t_2]} \, \mathrm{e}^{[1+|t_2-t_1|] \int_{t_1}^{t} \alpha(s) \, \mathrm{d}s} \le K_{[t_1,t_2]} \, \mathrm{e}^{[1+|t_2-t_1|] \int_{t_1}^{t_2} \alpha(s) \, \mathrm{d}s}, \tag{15}
$$

where

$$
K_{[t_1,t_2]} := M_0 + [1 + |t_2 - t_1|] \left\{ M_1 + \int_{t_1}^{t_2} \alpha(s) \, \mathrm{d}s \right\}.
$$

Since $[t_1, t_2] \subset I$ is arbitrary, it immediately follows from the estimate (15) that $T(Q)$ is bounded in each seminorm.

Now, let us check the equicontinuity w.r.t. the family of seminorms of all elements of $T(Q)$ and their first derivatives, as mentioned above. Let $x \in T(Q)$ be arbitrary, let $t_0 \in I$ be such that $|x(t_0)| \le M_0$ and $|\dot{x}(t_0)| \le M_1$, for all $x \in T(Q)$, and let $t_1, t_2 \in I$ be arbitrary. Then, according to the integral representation (14), we obtain

$$
\begin{aligned}
|x(t_1) - x(t_2)| &\le |\dot{x}(t_0)| \cdot |t_1 - t_2| \\
&\quad + \left| \int_{t_0}^{t_1} (t_1 - s) \cdot H(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \, \mathrm{d}s - \int_{t_0}^{t_2} (t_2 - s) \cdot H(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \, \mathrm{d}s \right| \\
&= |\dot{x}(t_0)| \cdot |t_1 - t_2| \\
&\quad + \left| \int_{t_0}^{t_1} (t_1 - s) \cdot H(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \, \mathrm{d}s - \int_{t_0}^{t_1} (t_2 - s) \cdot H(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \, \mathrm{d}s \right. \\
&\quad + \left. \int_{t_0}^{t_1} (t_2 - s) \cdot H(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \, \mathrm{d}s - \int_{t_0}^{t_2} (t_2 - s) \cdot H(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \, \mathrm{d}s \right| \\
&\le |\dot{x}(t_0)| \cdot |t_1 - t_2| \\
&\quad + \left| \int_{t_0}^{t_1} (t_1 - t_2) \cdot H(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \, \mathrm{d}s \right| + \left| \int_{t_1}^{t_2} (t_2 - s) \cdot H(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \, \mathrm{d}s \right| \\
&\le |\dot{x}(t_0)| \cdot |t_1 - t_2| \\
&\quad + \left| \int_{t_0}^{t_1} |t_1 - t_2| \cdot |H(s, x(s), \dot{x}(s), q(s), \dot{q}(s))| \, \mathrm{d}s \right| + \left| \int_{t_1}^{t_2} |t_2 - s| \cdot |H(s, x(s), \dot{x}(s), q(s), \dot{q}(s))| \, \mathrm{d}s \right| \\
&\le M_1 \cdot |t_1 - t_2| + \left| \int_{t_0}^{t_1} |t_1 - t_2| \cdot \alpha(s)(1 + |x(s)| + |\dot{x}(s)|) \, \mathrm{d}s \right| \\
&\quad + \left| \int_{t_1}^{t_2} |t_2 - s| \cdot \alpha(s)(1 + |x(s)| + |\dot{x}(s)|) \, \mathrm{d}s \right| \\
&\le M_1 \cdot |t_1 - t_2| + \left| \int_{t_0}^{t_1} |t_1 - t_2| \cdot \alpha(s) \left(1 + K_{[\min\{t_0,t_1\},\ \max\{t_0,t_1\}]} \, \mathrm{e}^{[1+|t_1-t_0|] \cdot \left| \int_{t_0}^{t_1} \alpha(\tau) \, \mathrm{d}\tau \right|} \right) \mathrm{d}s \right| \\
&\quad + \left| \int_{t_1}^{t_2} |t_2 - s| \cdot \alpha(s) \left(1 + K_{[\min\{t_1,t_2\},\ \max\{t_1,t_2\}]} \, \mathrm{e}^{[1+|t_2-t_1|] \cdot \left| \int_{t_1}^{t_2} \alpha(\tau) \, \mathrm{d}\tau \right|} \right) \mathrm{d}s \right|.
\end{aligned} \tag{16}
$$

Moreover, according to (12) and (iii), we have

$$\begin{aligned}|\dot{x}(t_1)-\dot{x}(t_2)| &\le \left|\int_{t_0}^{t_1} H(s,x(s),\dot{x}(s),q(s),\dot{q}(s))\,\mathrm{d}s-\int_{t_0}^{t_2} H(s,x(s),\dot{x}(s),q(s),\dot{q}(s))\,\mathrm{d}s\right|\\ &\le \left|\int_{t_1}^{t_2} |H(s,x(s),\dot{x}(s),q(s),\dot{q}(s))|\,\mathrm{d}s\right| \le \left|\int_{t_1}^{t_2}\alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,\mathrm{d}s\right|.\\ &\le \left|\int_{t_1}^{t_2}\alpha(s)\left(1+K_{[\min\{t_1,t_2\},\ \max\{t_1,t_2\}]}\,\mathrm{e}^{[1+|t_2-t_1|]\cdot\left|\int_{t_1}^{t_2}\alpha(\tau)\,\mathrm{d}\tau\right|}\right)\mathrm{d}s\right|.\end{aligned} \tag{17}$$

Taking into account estimates (16) and (17), x and $\dot{x}$ are equicontinuous, for each $x\in T(Q)$, because $\alpha(\cdot)\in L^1_{loc}(I,\mathbb{R})$. Thus, $T(Q)$ is relatively compact.

Now, we show that the graph Γ_T of the operator T is closed. Let $\{(q_k,x_k)\}\subset\Gamma_T$ be such that $\{(q_k,\dot{q}_k,x_k)\}\to(q,\dot{q},x)$, uniformly on compact intervals. Let $[t_1,t_2]\subset I$ be an arbitrary compact interval such that $t_0\in[t_1,t_2]$. According to (12) and (iii), we have, for a.a. $t\in[t_1,t_2]$ and all $k\in\mathbb{N}$,

$$\begin{aligned}|\dot{x}_k(t)| &\le |\dot{x}_k(t_0)|+\left|\int_{t_0}^{t}|H(s,x_k(s),\dot{x}_k(s),q_k(s),\dot{q}_k(s))|\,\mathrm{d}s\right|\\ &\le |\dot{x}_k(t_0)|+\left|\int_{t_0}^{t}\alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,\mathrm{d}s\right|.\end{aligned}$$

Moreover, it follows from condition (iii) that, for each $[t_1,t_2]\subset I$, there exists a constant $L_{[t_1,t_2]}$ such that $|\int_{t_1}^{t_2}\alpha(s)\,\mathrm{d}s|\le L_{[t_1,t_2]}$. Therefore, in view of the assumption (ii) and estimate (15), we have, for a.a. $t\in[t_1,t_2]$ and all $k\in\mathbb{N}$,

$$|\dot{x}_k(t)|\le M_1+L_{[t_1,t_2]}P_{[t_1,t_2]}, \tag{18}$$

where

$$P_{[t_1,t_2]}:=1+\left\{M_0+(1+|t_2-t_1|)\left(M_1+L_{[t_1,t_2]}\right)\right\}\mathrm{e}^{[1+|t_2-t_1|]L_{[t_1,t_2]}}.$$

By condition (iii) and estimates (15) and (18), the sequence $\{y_k:=\dot{x}_k\}$ satisfies the assumptions of Lemma 2.2. Therefore, there exists a subsequence of $\{\dot{x}_k\}$, for the sake of simplicity denoted in the same way as the sequence, uniformly convergent to $\dot{x}$ on $[t_1,t_2]$ and such that $\{\ddot{x}_k\}$ converges weakly to $\ddot{x}$ in $L^1([t_1,t_2],\ \mathbb{R}^n)$.

If we set $z_k:=(x_k,y_k)$, then $\dot{z}_k=(\dot{x}_k,\dot{y}_k)=(\dot{x}_k,\ddot{x}_k)\to(\dot{x},\ddot{x})$, weakly in $L^1([t_1,t_2],\ \mathbb{R}^n)$. Let us now consider the following system

$$\dot{z}_k(t)\in H^*(t,z_k(t),q_k(t),\dot{q}_k(t)),\qquad \text{for a.a. } t\in[t_1,t_2], \tag{19}$$

where

$$\dot{z}_k(t)=(\dot{x}_k(t),\dot{y}_k(t))$$

and

$$H^*(t,z_k(t),q_k(t),\dot{q}_k(t))=(y_k(t),H(t,x_k(t),\dot{x}_k(t),q_k(t),\dot{q}_k(t))).$$

Using Lemma 2.3, for $f_i:=\dot{z}_k,\ f:=(\dot{x},\ddot{x}),\ x_i:=(z_k,q_k,\dot{q}_k)$, it follows that

$$(\dot{x}(t),\ddot{x}(t))\in H^*(t,x(t),\dot{x}(t),q(t),\dot{q}(t)),$$

for a.a. $t\in[t_1,t_2]$, i.e.

$$\ddot{x}(t)\in H(t,x(t),\dot{x}(t),q(t),\dot{q}(t)),\qquad \text{for a.a. } t\in[t_1,t_2].$$

Since $[t_1,t_2]$ is arbitrary,

$$\ddot{x}(t)\in H(t,x(t),\dot{x}(t),q(t),\dot{q}(t)),\qquad \text{for a.a. } t\in I.$$

Condition (iv) implies that $x\in S$, by which Γ_T is closed. Moreover, it follows immediately from Proposition 2.1 that the operator T is u.s.c.

Since T is a compact mapping, $T(q)$ is, for each $q\in Q$, a relatively compact set. Moreover, the operator T has a closed graph which implies that $T(q)$ is, for each $q\in Q$, closed, and so T has compact values. □

Remark 3.1. The estimate for the solution can sometimes imply the one for its derivative, provided the right-hand side (r.h.s.) of a given inclusion satisfies suitable growth restrictions. For instance, if the r.h.s. is entirely bounded by a constant, then the boundedness of derivatives follows directly from the boundedness of solutions by means of the well-known Landau inequality:

$$|\dot{x}(t)|\le 2\,[|x(t)|\,|\ddot{x}(t)|]^{\frac{1}{2}}\,.$$

As the main result of this section, we are ready to formulate the following theorem.

Theorem 3.1. *Let us consider the b.v.p. (9), where I is a given real interval, $F : I \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of $AC^1_{loc}(I, \mathbb{R}^n)$.*

Let $H : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that

$$H(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in I \times \mathbb{R}^n \times \mathbb{R}^n. \tag{20}$$

Assume that

(i) *there exists a retract Q of $C^1(I, \mathbb{R}^n)$ and a closed subset S_1 of S such that the associated problem*

$$\left.\begin{array}{l}\ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in I,\\ x \in S_1\end{array}\right\} \tag{21}$$

is solvable with an R_δ-set of solutions, for each $(q, \lambda) \in Q \times [0, 1]$,

(ii) *there exists a nonnegative, locally integrable function $\alpha : I \to \mathbb{R}$ such that*

$$|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)| \le \alpha(t)(1 + |x(t)| + |\dot{x}(t)|), \quad \text{a.e. in } I,$$

for any $(q, \lambda, x) \in \Gamma_T$, where T denotes the multivalued map which assigns to any $(q, \lambda) \in Q \times [0, 1]$ the set of solutions of (21),

(iii) $T(Q \times \{0\}) \subset Q$,

(iv) *there exist a point $t_0 \in I$ and constants $M_0 \ge 0$, $M_1 \ge 0$ such that $|x(t_0)| \le M_0$ and $|\dot{x}(t_0)| \le M_1$, for any $x \in T(Q \times [0, 1])$,*

(v) *if $q_j, q \in Q$, $q_j \to q$, $q \in T(q, \lambda)$, then there exists $j_0 \in \mathbb{N}$ such that, for every $j \ge j_0$, $\theta \in [0, 1]$ and $x \in T(q_j, \theta)$, we have $x \in Q$.*

Then the b.v.p. (9) has a solution in $S_1 \cap Q$.

Proof. At first, we show that all the assumptions of Proposition 3.1 are satisfied. Conditions (i), (ii) and (iv) in Proposition 3.1 guarantee conditions (i), (ii) and (iii) in Proposition 3.1.

Let $\{(q_k, \lambda_k, x_k)\} \subset \Gamma_T$, $(q_k, \lambda_k, x_k) \to (q, \lambda, x)$, $(q, \lambda) \in Q \times [0, 1]$ be arbitrary. Then, since $x_k \in S_1$, $x_k \to x$ and S_1 is closed, it holds that $x \in S_1$. Therefore, condition (iv) from Proposition 3.1 holds as well. Thus, $T : Q \times [0, 1] \multimap S_1$ is, according to Proposition 3.1, a compact u.s.c. mapping with compact values.

According to assumption (i), T has R_δ-values, and so it belongs to the class $J(Q \times [0, 1], C^1(I, \mathbb{R}^n))$. Assumption (v) implies that T is a homotopy in $J_A(Q, C^1(I, \mathbb{R}^n))$. From Corollary 2.1, it follows that there exists a fixed point of $T(\cdot, 1)$ in Q. Moreover, by the inclusion (20) and since $S_1 \subset S$, the fixed point of $T(\cdot, 1)$ is a solution of the original b.v.p. (9). □

Remark 3.2. According to Proposition 3.1, the solution operator T has compact values. Therefore, the condition concerning R_δ-values in Theorem 3.1 is satisfied if, e.g., $T(q, \lambda)$ is, for all $(q, \lambda) \in Q \times [0, 1]$, convex or contractible.

Remark 3.3. Let us note that in the single-valued case of Carathéodory ordinary differential equations, we can only assume in Theorem 3.1 (i) that the associated problems are uniquely solvable.

4. Main existence and localization results

Let us consider the b.v.p.

$$\left.\begin{array}{l}\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in I,\\ x \in S,\end{array}\right\} \tag{22}$$

where

(i) $I \subset \mathbb{R}$,

(ii) $A, B \in L^1_{loc}(I, \mathbb{R}^n \times \mathbb{R}^n)$ are such that $|A(t)| \le a(t)$ and $|B(t)| \le b(t)$, for a.a. $t \in I$ and suitable locally integrable functions $a, b : I \to [0, \infty)$,

(iii) $F : I \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping,

(iv) S is a subset of $AC^1_{loc}(I, \mathbb{R}^n)$.

If the problems associated to (22) are fully linearized, we obtain the following result.

Theorem 4.1. *Let us consider the b.v.p. (22) and assume that*

(i) *there exists a nonnegative, locally integrable function $\alpha : I \to \mathbb{R}$ such that*

$$|F(t, q(t), \dot{q}(t))| \le \alpha(t), \quad \text{a.e. in } I,$$

for any $q \in Q$, where Q is a retract of $C^1(I, \mathbb{R}^n)$,

(ii) *there exist a point* $t_0 \in I$ *and constants* $M_0 \geq 0$, $M_1 \geq 0$ *such that* $|x(t_0)| \leq M_0$ *and* $|\dot{x}(t_0)| \leq M_1$, *for any* $x \in T(Q \times [0, 1])$, *where* T *denotes the mapping which assigns to each* $(q, \lambda) \in Q \times [0, 1]$ *the set of solutions of fully linearized problems*

$$\left.\begin{array}{l}\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in \lambda F(t, q(t), \dot{q}(t)), \quad \text{for a.a. } t \in I,\\ x \in S_1,\end{array}\right\} \tag{23}$$

(iii) S_1 *is a closed convex subset of* S,

(iv) $T(q, \lambda) \neq \emptyset$, *for all* $(q, \lambda) \in Q \times [0, 1]$, *and* $T(Q \times \{0\}) \subset Q$,

(v) *("pushing" condition) if* $q_j, q \in Q$, $q_j \to q$, $q \in T(q, \lambda)$, *then there exists* $j_0 \in \mathbb{N}$ *such that, for every* $j \geq j_0$, $\theta \in [0, 1]$ *and* $x \in T(q_j, \theta)$, *we have* $x \in Q$.

Then the b.v.p. (22) *has a solution in* $S_1 \cap Q$.

Proof. Let $(q, \lambda) \in Q \times [0, 1]$, and $t_0 \in I$ be arbitrary. If x_1, x_2 are solutions of problem (23), then it follows from the integral representation of a solution (cf. the proof of Proposition 3.1) that, for a.a. $t \in I$, we have

$$x_1(t) \in x_1(t_0) + \dot{x}_1(t_0) \cdot (t - t_0) + \int_{t_0}^{t} (t - s) \cdot [-A(s)x_1(s) - B(s)\dot{x}_1(s) + C(s, q(s), \dot{q}(s), \lambda)] \, ds,$$

$$x_2(t) \in x_2(t_0) + \dot{x}_2(t_0) \cdot (t - t_0) + \int_{t_0}^{t} (t - s) \cdot [-A(s)x_2(s) - B(s)\dot{x}_2(s) + C(s, q(s), \dot{q}(s), \lambda)] \, ds.$$

Let $\theta \in [0, 1]$ be arbitrary. Then

$$\begin{aligned}\theta x_1(t) + (1-\theta)x_2(t) &\in \theta \cdot x_1(t_0) + (1-\theta) \cdot x_2(t_0) + [\theta \cdot \dot{x}_1(t_0) + (1-\theta) \cdot \dot{x}_2(t_0)] \cdot (t - t_0)\\ &\quad + \int_{t_0}^{t} (t - s) \cdot \theta \cdot [-A(s)x_1(s) - B(s)\dot{x}_1(s) + C(s, q(s), \dot{q}(s), \lambda)] \, ds\\ &\quad + \int_{t_0}^{t} (t - s) \cdot (1-\theta) \cdot [-A(s)x_2(s) - B(s)\dot{x}_2(s) + C(s, q(s), \dot{q}(s), \lambda)] \, ds\\ &= \theta \cdot x_1(t_0) + (1-\theta) \cdot x_2(t_0) + [\theta \cdot \dot{x}_1(t_0) + (1-\theta) \cdot \dot{x}_2(t_0)] \cdot (t - t_0)\\ &\quad + \int_{t_0}^{t} (t - s) \{-A(s)[\theta x_1(s) + (1-\theta)x_2(s)] - B(s)[\theta \dot{x}_1(s) + (1-\theta)\dot{x}_2(s)] + C(s, q(s), \dot{q}(s), \lambda)\} \, ds.\end{aligned}$$

Moreover, because of convexity of S, we obtain that

$$\theta x_1 + (1-\theta)x_2 \in S,$$

i.e., for any $(q, \lambda) \in Q \times [0, 1]$, the set of solutions of (23) is convex.

Since all assumptions of Theorem 3.1 are satisfied, the problem (22) has a solution in $S_1 \cap Q$. □

Remark 4.1. If the set Q is convex, then the "pushing" condition (v) of Theorem 4.1 can be replaced by

(v′) if $\partial Q \times [0, 1] \supset \{(q_j, \lambda_j)\}$ converges to $(q, \lambda) \in \partial Q \times [0, 1]$, $q \in T(q, \lambda)$, then there exists $j_0 \in \mathbb{N}$ such that, for every $j \geq j_0$ and $x_j \in T(q_j, \lambda_j)$, we have $x_j \in Q$.

Remark 4.2. If the associated problems (23) are uniquely solvable, for each $(q, \lambda) \in Q \times [0, 1]$, then, by continuity of T, we can reformulate the "pushing" condition (v′) from Remark 4.1 as follows:

(v″) if $\{(x_j, \lambda_j)\}$ is a sequence in $S_1 \times [0, 1]$, with $\lambda_j \to \lambda \in [0, 1)$ and x_j converging to a solution $x \in Q$ of (23), for $q = x$ and $\lambda = \lambda$, then x_j belongs to Q, for j sufficiently large.

As an application of Theorem 4.1, we can give the following nontrivial example, where the "pushing" condition (v'') will be employed. It is a vector generalization of an illustrating example briefly indicated in [17].

Example 4.1. *Let us consider the second-order b.v.p.*

$$\left.\begin{array}{l}\ddot{x}(t) + \dot{x}(t) = F(t, x(t)), \quad \text{for a.a. } t \in [0, \infty),\\ x(0) = 0,\\ \lim\limits_{t\to\infty} x(t) = 0,\end{array}\right\} \tag{24}$$

where $F = (F_1, F_2, \ldots, F_n) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ *is a Carathéodory function satisfying:*

1. *there exists a constant* $M > 0$ *such that, for each* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *with* $|x_i| > M$, *it holds that* $F_i(t, x) \cdot \operatorname{sgn} x_i > 0$, *for each* $t \in [0, \infty)$,

2. *there exists a constant* $N > 0$ *such that if*

$$\alpha_K(t) := \sup_{|x|\leq K} |F(t,x)|,$$

then

$$\int_0^\infty \alpha_K(t)\,\mathrm{d}t \leq N \cdot K$$

and

$$\lim_{t\to\infty} \alpha_K(t) = 0,$$

for all $K > 0$.

At first, let us show that all solutions $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ *of the b.v.p.* (24) *must satisfy* $|x_i(t)| \leq M$, *for all* $t \in [0,\infty)$ *and all* $i = 1,\dots,n$. *Assume by a contradiction that there exist* $t_0 \in [0,\infty)$ *and* $i \in \{1,2,\dots,n\}$ *such that*

$$|x_i(t_0)| = \max_{t\in[0,\infty)} |x_i(t)| > M.$$

Then $\dot{x}_i(t_0) = 0$ *and, according to assumption* 1,

$$0 \geq \ddot{x}_i(t_0)\cdot \operatorname{sgn} x_i(t_0) = F_i(t_0, x(t_0))\cdot \operatorname{sgn} x_i(t_0) > 0,$$

which is a contradiction. Therefore, all possible solutions $x(\cdot) = (x_1(\cdot), x_2(\cdot), \dots, x_n(\cdot))$ *of the b.v.p.* (24) *satisfy, for all* $t \in [0,\infty)$ *and all* $i \in \{1,2,\dots,n\}$,

$$|x_i(t)| \leq M,$$

as claimed.

Let us define a closed, convex set Q *in the following way*

$$Q := \left\{ q \in C^1([0,\infty),\mathbb{R}^n) \mid \|q\| = \max_{t\in[0,\infty)} |q(t)| \leq \sqrt{n}(M+1) \right\}$$

and let us consider the associated linear problems

$$\left.\begin{array}{l} \ddot{x}(t) = H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in [0,\infty), \\ x(0) = 0, \\ \lim\limits_{t\to\infty} x(t) = 0, \end{array}\right\} \tag{25}$$

where, for each $(q,\lambda) \in Q \times [0,1]$ *and* $t \in [0,\infty)$, *the function* $H : [0,\infty) \times \mathbb{R}^{4n} \times [0,1] \to \mathbb{R}^n$ *takes the form*

$$H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda) := \lambda F(t, q(t)) - \dot{x}(t).$$

For each $(q,\lambda) \in Q \times [0,1]$, *the b.v.p.* (25) *has the unique solution* $x = x(q,\lambda)$ *given, for a.a.* $t \in [0,\infty)$, *by the formula*

$$x(t) = -\lambda \left[\int_t^\infty F(\tau, q(\tau))\,\mathrm{d}\tau + \mathrm{e}^{-t}\left(\int_0^t \mathrm{e}^{\tau} F(\tau, q(\tau))\,\mathrm{d}\tau - \int_0^\infty F(\tau, q(\tau))\,\mathrm{d}\tau \right) \right].$$

Therefore, for all $t \in [0,\infty)$,

$$|x(t)| \leq (1-\mathrm{e}^{-t}) \int_t^\infty \alpha_{\sqrt{n}(M+1)}(\tau)\,\mathrm{d}\tau + \mathrm{e}^{-t} \int_0^t (\mathrm{e}^{\tau}+1)\alpha_{\sqrt{n}(M+1)}(\tau)\,\mathrm{d}\tau := \gamma(t). \tag{26}$$

Moreover, $\gamma(0) = 0$ *and* $\lim_{t\to\infty}\gamma(t) = 0$. *Observe that, for* $\lambda = 0$ *and an arbitrary* $q \in Q$, $x(t) \equiv 0$ *is the only solution of* (25).

Furthermore,

$$\begin{aligned} |\dot{x}(t)| &= \left| \lambda \left(\mathrm{e}^{-t} \int_0^\infty F(\tau, q(\tau))\,\mathrm{d}\tau - \mathrm{e}^{-t} \int_0^t \mathrm{e}^{\tau} F(\tau, q(\tau))\,\mathrm{d}\tau \right) \right| \\ &\leq \left| \mathrm{e}^{-t} \int_0^\infty \alpha_{\sqrt{n}(M+1)}(\tau)\,\mathrm{d}\tau + \mathrm{e}^{-t} \int_0^t \mathrm{e}^{\tau} \alpha_{\sqrt{n}(M+1)}(\tau)\,\mathrm{d}\tau \right| := \gamma_1(t). \end{aligned}$$

Therefore, all possible solutions of the b.v.p. (25) *are located in the set*

$$S_1 := \left\{ x \in C^1([0,\infty),\mathbb{R}^n) \mid |x(t)| \leq \gamma(t),\ |\dot{x}(t)| \leq \gamma_1(t),\ \text{for all } t \in [0,\infty) \right\}.$$

Since Q *and* S_1 *are convex, bounded, closed subsets of* $C^1([0,\infty),\mathbb{R}^n)$, *it only remains to verify the "pushing" condition* (v'') *from Remark* 4.2, *in order to apply Theorem* 4.1.

Let $\{(x^j, \lambda_j)\}_{j=1}^{\infty} \subset S_1 \times [0, 1]$ *be an arbitrary sequence such that* $\lambda_j \to \lambda$ *and* $x^j \to x^\lambda$, *where* $x^\lambda = (x_1^\lambda, x_2^\lambda, \ldots, x_n^\lambda)$ *is a solution of the b.v.p.*

$$\left.\begin{array}{l} \ddot{x}(t) + \dot{x}(t) = \lambda F(t, x(t)), \quad \text{for a.a. } t \in [0, \infty), \\ x(0) = 0, \\ \lim\limits_{t\to\infty} x(t) = 0. \end{array}\right\} \tag{27}$$

Since $\gamma(0) = \lim_{t\to\infty} \gamma(t) = 0$, *there exist points* $t_0, t_1 \in [0, \infty)$ *such that* $|x^j(t)| \le \gamma(t) \le \sqrt{n}(M + 1)$, *for all* $t \in [0, t_0) \cup (t_1, \infty)$ *and* $j \in \mathbb{N}$. *On the other hand,* $\{x^j\}$ *converges uniformly to* x^λ, *on the compact interval* $[t_0, t_1]$, *and an estimate similar to the one obtained above for the solutions of* (24) *yields that* $|x_i^\lambda(t)| \le M$, *for all* $t \in [0, \infty)$ *and* $i = 1, 2, \ldots, n$. *Therefore,* $|x^j(t)| \le \sqrt{n}(M + 1)$, *for all* $t \in [t_0, t_1]$ *and* $j \in \mathbb{N}$ *sufficiently large, which implies that* $x^j \in Q$, *for j large enough. Now, all assumptions of Theorem 4.1 are satisfied, by which problem (24) admits a solution in* $Q \cap S_1$.

Observe that condition (v) in Theorem 4.1 holds if $S_1 \subset Q$ by which Theorem 4.1 can be simplified in the following way.

Corollary 4.1. *Let us consider the b.v.p. (22), where I is a given real interval,* $F : I \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper-Carathéodory mapping and S is a subset of* $AC^1_{loc}(I, \mathbb{R}^n)$.

Assume that

(i) *there exists a retract Q of* $C^1(I, \mathbb{R}^n)$ *such that the associated problem*

$$\left.\begin{array}{l} \ddot{x}(t) \in F(t, q(t), \dot{q}(t)), \quad \text{for a.a. } t \in I, \\ x \in S \cap Q \end{array}\right\} \tag{28}$$

is solvable, for each $q \in Q$,

(ii) *there exists a nonnegative, locally integrable function* $\alpha : I \to \mathbb{R}$ *such that*

$$|F(t, q(t), \dot{q}(t))| \le \alpha(t), \text{ a.e. in } I,$$

for any $(q, x) \in \Gamma_T$, *where T denotes the multivalued map which assigns to any* $q \in Q$ *the set of solutions of* (28),

(iii) $\overline{T(Q)} \subset S$,

(iv) *there exist a point* $t_0 \in I$ *and constants* $M_0 \ge 0$, $M_1 \ge 0$ *such that* $|x(t_0)| \le M_0$ *and* $|\dot{x}(t_0)| \le M_1$, *for any* $x \in T(Q)$.

Then the b.v.p. (22) has a solution in $S \cap Q$.

Example 4.2. *Let us consider the second-order target problem*

$$\left.\begin{array}{l} \ddot{x}(t) \in F(t, x(t)), \quad \text{for a.a. } t \in [0, \infty), \\ \lim\limits_{t\to\infty} x(t) = l, \end{array}\right\} \tag{29}$$

where $F : [0, \infty) \times \mathbb{R} \multimap \mathbb{R}$ *is an upper-Carathéodory mapping and* $l \in \mathbb{R}^n$.

Moreover, let

$$\int_0^\infty t \cdot \alpha_K(t)\, \mathrm{d}t < K - |l|,$$

for some $K > 0$, *where* $\alpha_K(t) := \sup_{|x|\le K} |F(t, x)|$.

In order to apply Corollary 4.1, let us define the set Q of candidate solutions as

$$Q := \left\{q \in C^1([0, \infty), \mathbb{R}) \mid q(t) \le K, \text{ for all } t \in [0, \infty)\right\},$$

and let us consider the family of associated problems

$$\left.\begin{array}{l} \ddot{x}(t) \in F(t, q(t)), \quad \text{for a.a. } t \in [0, \infty), \\ \lim\limits_{t\to\infty} x(t) = l. \end{array}\right\} \tag{30}$$

Let $q \in Q$ *be arbitrary. Then* $F(t, q(t))$ *admits, according to Proposition 2.2, a single-valued selection* $f_q(t)$, *measurable on every compact subinterval of* $[0, \infty)$. *The problem*

$$\left.\begin{array}{l} \ddot{x}(t) = f_q(t), \quad \text{for a.a. } t \in [0, \infty), \\ \lim\limits_{t\to\infty} x(t) = l \end{array}\right\} \tag{31}$$

admits the unique solution

$$x(t) = l + \int_t^\infty (s - t) \cdot f_q(s)\, \mathrm{d}s,$$

which belongs to Q. Therefore, the set of solutions of (30) *is a non-empty subset of Q.*

Assumptions (iii) *and* (iv) *in Corollary* 4.1 *hold as well, because all solutions of* (30) *belong, for arbitrary* $q \in Q$, *to the following closed, bounded subset of* $C^1([0,\infty),\mathbb{R}^n$, *namely*

$$\left\{ x \in C^1([0,\infty),\mathbb{R}^n) \mid |x(t)-l| \le \int_t^\infty s\cdot\alpha_K(s)\,\mathrm{d}s;\ |\dot{x}(t)| \le t\cdot\alpha_K(t),\ t\in[0,\infty) \right\}.$$

All assumptions of previous corollary are satisfied, by which the target problem (29) *admits a solution in* Q.

5. Concluding remarks

Remark 5.1. The parameter set Q of candidate solutions can be taken everywhere, without any loss of generality, as a subset of $AC^1_{loc}(I,\mathbb{R}^n)$, where $I \subset \mathbb{R}$ is either an arbitrary interval or can be specified according to the context. On the other hand, if Q is only taken as a subset of $C(I,\mathbb{R}^n)$, then the solution derivatives can behave in a more liberal way. This can be an advantage if, for instance, the growth conditions concerning generating multivalued vector fields (r.h.s.) are independent of derivatives, provided we are exclusively interested in solutions, but not necessarily in their derivatives. Moreover, the obtained results need not be then available by means of methods developed for equivalent first-order differential systems, where derivatives are taken into account automatically.

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Topological structure of solution sets to asymptotic boundary value problems ☆

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ABSTRACT

Topological structure is investigated for second-order vector asymptotic boundary value problems. Because of indicated obstructions, the R_δ-structure is firstly studied for problems on compact intervals and then, by means of the inverse limit method, on non-compact intervals. The information about the structure is furthermore employed, by virtue of a fixed-point index technique in Fréchet spaces developed by ourselves earlier, for obtaining an existence result for nonlinear asymptotic problems. Some illustrating examples are supplied.

1. Introduction

Cauchy (initial value) problems for ordinary differential equations are, according to the result of Orlicz, generically solvable in a unique way. For the exceptional cases (non-uniqueness), Kneser firstly proved that the sets of their solutions are at every fixed time continua and then Hukuhara showed that the solution set (on a compact interval) itself is a continuum. Aronszajn improved this result in the sense that the solution sets are compact and acyclic, but in fact he specified these continua to be

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R_δ-sets. The analogous result was obtained for upper-Carathéodory differential inclusions by De Blasi and Myjak in [13]. For more details, historical remarks and related references, see [4, III.12.2].

Topological structure of solution sets to Cauchy problems on non-compact and, in particular, infinite intervals was studied by various techniques, e.g., in [1,4,9,12,16,21,28,29].

For boundary value problems, the situation is much more delicate and the related results are still very rare; see, e.g., [4, Chapter III.3], [8,14,24]. So far, topological structure of solution sets was investigated exclusively (as far as we know, with only one exception [19]) to boundary value problems on compact intervals. Moreover, because of the counter-examples in [2,16], [4, Example II.2.12], demonstrating the impossibility of asymptotic analogies to the situation on compact intervals, the main theorem in [19] might be empty. These troubles are due to an "unpleasant" related topology of non-normable Fréchet spaces. For instance, a contractivity of a given operator with respect to a metric need not follow from a contractivity with respect to each seminorm. Moreover, bounded subsets of non-normable Fréchet spaces have always empty interiors, etc.

Despite these difficulties, there is a chance to obtain some results for at least particular asymptotic problems like Kneser-type (Thomas–Fermi) problems. The key tool is for us the inverse limit method, sometimes also called the projective limit (see, e.g., [2–4,9,15,18,25]). We elaborated this technique for the needs of multivalued analysis in [1,2], [4, Chapter II.2]. We believe that this approach can bring further impulses in the field reflected in the title.

2. Preliminaries

At first, we recall some geometric notions of subsets of metric spaces, in particular, of retracts. For more details, see, e.g., [4,10,17].

For a subset $A \subset X$ of a metric space $X = (X, d)$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A\colon\ d(x,a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$.

We say that a metric space X is an *absolute retract* (*AR*-space) if, for each metric space Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over Y. Let us note that X is an *AR*-space if and only if it is a retract of some normed space. Moreover, if X is a retract of a convex set in a Fréchet space, then it is an *AR*-space. So, in particular, the spaces $C(J, \mathbb{R}^n)$, $C^1(J, \mathbb{R}^n)$, $AC^1_{loc}(J, \mathbb{R}^n)$ are *AR*-spaces as well as their convex subsets, where $J \subset \mathbb{R}$ is an arbitrary interval. The foregoing symbols denote, as usually, the spaces of functions $f : J \to \mathbb{R}^n$ which are continuous, smooth and those with locally absolutely continuous first derivatives, respectively, endowed with the respective topologies.

We say that a nonempty subset A of a metric space X is *contractible* if there exist a point $x_0 \in A$ and a homotopy $h : A \times [0,1] \to A$ such that $h(x,0) = x$ and $h(x,1) = x_0$, for every $x \in A$. A nonempty set $A \subset X$ is called an *R_δ-set* if there exists a decreasing sequence $\{A_n\}_{n=1}^\infty$ of compact, *AR*-spaces (or, despite of the hierarchy (1) below, compact, contractible sets) such that

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Note that any R_δ-set is nonempty, compact and connected. The following hierarchy holds for nonempty subsets of a metric space:

$$\text{compact} + \text{convex} \subset \text{compact } AR\text{-space} \subset \text{compact} + \text{contractible} \subset R_\delta\text{-set}, \tag{1}$$

and all the above inclusions are proper.

We also employ the following definitions and statements from the multivalued analysis in the sequel. Let X and Y be arbitrary metric spaces. We say that F is a multivalued mapping from X to Y

(written $F : X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is prescribed. We associate with F its *graph* Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

A multivalued mapping $F : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X. Every upper semicontinuous map with closed values has a closed graph.

The reverse relation between upper semicontinuous mappings and those with closed graphs is expressed in the following proposition.

Proposition 2.1. *(Cf., e.g., [4,17].) Let X, Y be metric spaces and $F : X \multimap Y$ be a multivalued mapping with the closed graph such that $F(X) \subset K$, where K is a compact set. Then F is u.s.c.*

A multivalued mapping $F : X \multimap X$ with bounded values is called *Lipschitzian* if there exists a constant $L > 0$ such that

$$d_H\big(F(x), F(y)\big) \leqslant L\,d(x, y),$$

for every $x, y \in X$, where

$$d_H(A, B) := \inf\{r > 0 \mid A \subset N_r(B) \text{ and } B \subset N_r(A)\}$$

stands for the Hausdorff distance; for its properties, see, e.g., [4,17].

We say that a multivalued mapping $F : X \multimap X$ with bounded values is a *contraction* if it is Lipschitzian with a Lipschitz constant $L \in [0, 1)$.

Let Y be a separable metric space and $(\Omega, \mathcal{U}, \nu)$ be a *measurable space*, i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $F : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid F(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$.

We say that mapping $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$, is an *upper-Carathéodory mapping* if the map $F(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable on every compact subinterval of J, for all $x \in \mathbb{R}^m$, the map $F(t, \cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all (a.a.) $t \in J$, and the set $F(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

We will employ the following selection statement.

Proposition 2.2. *(Cf., e.g., [6].) Let $F : [a, b] \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping satisfying $|y| \leqslant r(t)(1 + |x|)$, for every $(t, x) \in [a, b] \times \mathbb{R}^m$, and every $y \in F(t, x)$, where $r : [a, b] \to [0, \infty)$ is an integrable function. Then the composition $F(t, q(t))$ admits, for every $q \in C([a, b], \mathbb{R}^m)$, a single-valued measurable selection.*

If $X \cap Y \neq \emptyset$ and $F : X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed-point* of F if $x \in F(x)$. The set of all fixed-points of F will be denoted by $\mathrm{Fix}(F)$, i.e.

$$\mathrm{Fix}(F) := \{x \in X \mid x \in F(x)\}.$$

It will be also convenient to recall the following results.

Proposition 2.3. *(Cf. [26].) Let X be a closed, convex subset of a Banach space E and let $\phi : X \multimap X$ be a contraction with compact, convex values. Then $\mathrm{Fix}(\phi)$ is a nonempty, compact AR-space.*

Lemma 2.1. *(Cf. [7, Theorem 0.3.4].) Let $[a, b] \subset \mathbb{R}$ be a compact interval. Assume that the sequence of absolutely continuous functions $x_k : [a, b] \to \mathbb{R}^n$ satisfies the following conditions:*

(i) *the set* $\{x_k(t) \mid k \in \mathbb{N}\}$ *is bounded, for every* $t \in [a,b]$,
(ii) *there exists a function* $\alpha : [a,b] \to \mathbb{R}$, *integrable in the sense of Lebesgue, such that*

$$\left|\dot{x}_k(t)\right| \leqslant \alpha(t), \quad \textit{for a.a. } t \in [a,b] \textit{ and for all } k \in \mathbb{N}.$$

Then there exists a subsequence of $\{x_k\}$ (*for the sake of simplicity denoted in the same way as the sequence*) *converging to an absolutely continuous function* $x : [a,b] \to \mathbb{R}^n$ *in the following way*:

1. $\{x_k\}$ *converges uniformly to* x,
2. $\{\dot{x}_k\}$ *converges weakly in* $L^1([a,b],\mathbb{R}^n)$ *to* $\dot{x}$.

The following lemma is a slight modification of the well-known result.

Lemma 2.2. *(Cf. [30, p. 88].) Let* $[a,b] \subset \mathbb{R}$ *be a compact interval,* E_1, E_2 *be Euclidean spaces and* $F : [a,b] \times E_1 \multimap E_2$ *be an upper-Carathéodory mapping.*

Assume in addition that, for every nonempty, bounded set $\mathcal{B} \subset E_1$, *there exists* $\nu = \nu(\mathcal{B}) \in L^1([a,b], [0,\infty))$ *such that*

$$\left|F(t,x)\right| \leqslant \nu(t),$$

for a.a. $t \in [a,b]$ *and every* $x \in \mathcal{B}$.

Let us define the Nemytskiǐ operator $N_F : C([a,b],E_1) \multimap L^1([a,b],E_2)$ *in the following way*:

$$N_F(x) := \bigl\{ f \in L^1\bigl([a,b],E_2\bigr) \bigm| f(t) \in F\bigl(t,x(t)\bigr), \textit{ a.e. on } [a,b] \bigr\},$$

for every $x \in C([a,b],E_1)$. *Then, if sequences* $\{x_i\} \subset C([a,b],E_1)$ *and* $\{f_i\} \subset L^1([a,b],E_2)$, $f_i \in N_F(x_i)$, $i \in \mathbb{N}$, *are such that* $x_i \to x$ *in* $C([a,b],E_1)$ *and* $f_i \to f$ *weakly in* $L^1([a,b],E_2)$, *then* $f \in N_F(x)$.

3. Topological structure on compact intervals

Before investigating the asymptotic problems, it will be useful to study the topological structure of related solution sets on compact intervals.

At first, let us consider the problems for fully linearized systems

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C(t), \quad \text{for a.a. } t \in [0,m], \\ x \in S_m, \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C(t), \quad \text{for a.a. } t \in [0,m], \\ (x,\dot{x}) \in S'_m, \end{aligned}\right\} \tag{3}$$

where

(i) $A, B : [0,m] \to \mathbb{R}^{n\times n}$ are integrable matrix functions such that $|A(t)| \leqslant a(t)$, $|B(t)| \leqslant b(t)$, for a.a. $t \in [0,m]$ and suitable nonnegative functions $a, b \in L^1([0,m],\mathbb{R})$,
(ii) S_m is a closed, convex subset of $AC^1([0,m],\mathbb{R}^n)$ (S'_m is a closed, convex subset of $AC^1([0,m],\mathbb{R}^n) \times AC([0,m],\mathbb{R}^n)$),
(iii) $C : [0,m] \multimap \mathbb{R}^n$ is an integrable mapping with convex closed values such that $|C(t)| \leqslant c(t)$, for a.a. $t \in [0,m]$ and a suitable nonnegative function $c \in L^1([0,m],\mathbb{R})$,
(iv) there exist $t_0 \in [0,m]$ and constants M_0, M_1 such that $|x(t_0)| \leqslant M_0$ and $|\dot{x}(t_0)| \leqslant M_1$, for all solutions of problem (2) (all solutions of problem (3)).

Lemma 3.1. *Under the above assumptions* (i)–(iv), *the solution set of problem* (2) (*the set of solutions and their first derivatives of problem* (3)) *is convex and compact.*

Proof. Let us prove that the set of solutions and their first derivatives of the b.v.p. (3) is convex and compact. By the similar reasoning, it is possible to obtain that the solution set of problem (2) is convex and compact as well.

Let us denote by $P(t, x(t), \dot{x}(t)) := C(t) - A(t)\dot{x}(t) - B(t)x(t)$. If x_1, x_2 are solutions of problem (3), then it follows from the integral representation of a solution and its derivative that, for a.a. $t \in [0, m]$, we have

$$x_1(t) \in x_1(t_0) + \dot{x}_1(t_0) \cdot (t - t_0) + \int_{t_0}^{t} (t - s) \cdot P\big(s, x_1(s), \dot{x}_1(s)\big)\, ds,$$

$$x_2(t) \in x_2(t_0) + \dot{x}_2(t_0) \cdot (t - t_0) + \int_{t_0}^{t} (t - s) \cdot P\big(s, x_2(s), \dot{x}_2(s)\big)\, ds$$

and

$$\dot{x}_1(t) \in \dot{x}_1(t_0) + \int_{t_0}^{t} P\big(s, x_1(s), \dot{x}_1(s)\big)\, ds,$$

$$\dot{x}_2(t) \in \dot{x}_2(t_0) + \int_{t_0}^{t} P\big(s, x_2(s), \dot{x}_2(s)\big)\, ds.$$

Let $\theta \in [0, 1]$ be arbitrary. Then

$$\begin{aligned}
&\theta x_1(t) + (1 - \theta)x_2(t) \\
&\quad \in \theta \cdot x_1(t_0) + (1 - \theta) \cdot x_2(t_0) + \big[\theta \cdot \dot{x}_1(t_0) + (1 - \theta) \cdot \dot{x}_2(t_0)\big] \cdot (t - t_0) \\
&\qquad + \int_{t_0}^{t} (t - s) \cdot \theta \cdot P\big(s, x_1(s), \dot{x}_1(s)\big)\, ds + \int_{t_0}^{t} (t - s) \cdot (1 - \theta) \cdot P\big(s, x_2(s), \dot{x}_2(s)\big)\, ds \\
&\quad = \theta \cdot x_1(t_0) + (1 - \theta) \cdot x_2(t_0) + \big[\theta \cdot \dot{x}_1(t_0) + (1 - \theta) \cdot \dot{x}_2(t_0)\big] \cdot (t - t_0) \\
&\qquad + \int_{t_0}^{t} (t - s) \cdot P\big(s, \theta x_1(s) + (1 - \theta)x_2(s), \theta\dot{x}_1(s) + (1 - \theta)\dot{x}_2(s)\big)\, ds.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\theta\dot{x}_1(t) + (1 - \theta)\dot{x}_2(t) &\in \theta\dot{x}_1(t_0) + (1 - \theta)\dot{x}_2(t_0) \\
&\quad + \int_{t_0}^{t} P\big(s, \theta x_1(s) + (1 - \theta)x_2(s), \theta\dot{x}_1(s) + (1 - \theta)\dot{x}_2(s)\big)\, ds.
\end{aligned}$$

Finally, because of convexity of S'_m, we obtain that

$$\big(\theta x_1 + (1-\theta)x_2, \theta \dot{x}_1 + (1-\theta)\dot{x}_2\big) \in S'_m$$

and, therefore, the set of solutions of (3) and their derivatives is convex.

Let us also prove that the set of solutions of (3) and their derivatives is relatively compact. It follows from the well-known Arzelà–Ascoli lemma that the set of solutions is relatively compact in $C^1([0,m],\mathbb{R}^n)$ if and only if it is bounded and all solutions and their first derivatives are equi-continuous.

At first, let us show that the set of solutions of (3) is bounded in $C^1([0,m],\mathbb{R}^n)$. Let x be a solution of (3) and let $t \in [0,m]$ be arbitrary. Then

$$\begin{aligned}
\big|x(t)\big| + \big|\dot{x}(t)\big| &\leqslant \big|x(t_0)\big| + \big|\dot{x}(t_0)\big| \cdot |t-t_0| + \left|\int_{t_0}^{t} |t-s| \cdot \big|P\big(s, x(s), \dot{x}(s)\big)\big|\, ds\right| + \big|\dot{x}(t_0)\big| \\
&\quad + \left|\int_{t_0}^{t} \big|P\big(s, x(s), \dot{x}(s)\big)\big|\, ds\right| \\
&\leqslant M_0 + M_1 \cdot |t-t_0| + |t-t_0| \cdot \left|\int_{t_0}^{t} c(s) + a(s)\big|\dot{x}(s)\big| + b(s)\big|x(s)\big|\, ds\right| + M_1 \\
&\quad + \left|\int_{t_0}^{t} c(s) + a(s)\big|\dot{x}(s)\big| + b(s)\big|x(s)\big|\, ds\right| \\
&\leqslant M_0 + M_1 \cdot [1+m] + [1+m]\int_0^t c(s) + a(s)\big|\dot{x}(s)\big| + b(s)\big|x(s)\big|\, ds \\
&\leqslant M_0 + M_1 \cdot [1+m] + [1+m]\int_0^m c(s)\, ds + [1+m]\int_0^t k(s)\big(\big|x(s)\big| + \big|\dot{x}(s)\big|\big)\, ds,
\end{aligned}$$

where, for all $s \in [0,m]$, $k(s) := \max\{a(s), b(s)\}$.

By the Gronwall lemma (cf. [22]), we obtain that

$$\big|x(t)\big| + \big|\dot{x}(t)\big| \leqslant K \cdot e^{[1+m]\int_0^m k(s)\, ds}, \tag{4}$$

where

$$K := M_0 + [1+m]\left\{M_1 + \int_0^m c(s)\, ds\right\}.$$

Therefore, the set of solutions of (3) and their derivatives is bounded in $C^1([0,m],\mathbb{R}^n)$.

Let us now show that all solutions of (3) and their first derivatives are also equi-continuous. Let x be a solution of (3) and $t_2, t_3 \in [0,m]$ be arbitrary. Then, we have

$$
\begin{aligned}
&\big|x(t_3)-x(t_2)\big| \\
&\quad\leqslant \big|\dot{x}(t_0)\big|\cdot|t_3-t_2| + \left|\int_{t_0}^{t_3}(t_3-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds - \int_{t_0}^{t_2}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right| \\
&\quad= \big|\dot{x}(t_0)\big|\cdot|t_3-t_2| + \left|\int_{t_0}^{t_3}(t_3-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds - \int_{t_0}^{t_3}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right. \\
&\qquad\left. + \int_{t_0}^{t_3}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds - \int_{t_0}^{t_2}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right| \\
&\quad\leqslant \big|\dot{x}(t_0)\big|\cdot|t_3-t_2| + \left|\int_{t_0}^{t_3}(t_3-t_2)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right| + \left|\int_{t_3}^{t_2}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right| \\
&\quad\leqslant \big|\dot{x}(t_0)\big|\cdot|t_3-t_2| + \left|\int_{t_0}^{t_3}|t_3-t_2|\cdot\big|P\big(s,x(s),\dot{x}(s)\big)\big|\,ds\right| + \left|\int_{t_3}^{t_2}|t_2-s|\cdot\big|P\big(s,x(s),\dot{x}(s)\big)\big|\,ds\right| \\
&\quad\leqslant M_1\cdot|t_3-t_2| + \left|\int_{t_0}^{t_3}|t_3-t_2|\cdot\big(c(s)+k(s)\cdot K\cdot e^{[1+m]\int_0^m k(u)\,du}\big)\,ds\right| \\
&\qquad + \left|\int_{t_3}^{t_2}|t_2-s|\cdot\big(c(s)+k(s)\cdot K\cdot e^{[1+m]\int_0^m k(u)\,du}\big)\,ds\right|.
\end{aligned}
\tag{5}
$$

Moreover,

$$
\begin{aligned}
\big|\dot{x}(t_3)-\dot{x}(t_2)\big| &\leqslant \left|\int_{t_0}^{t_3}P\big(s,x(s),\dot{x}(s)\big)\,ds - \int_{t_0}^{t_2}P\big(s,x(s),\dot{x}(s)\big)\,ds\right| \\
&\leqslant \left|\int_{t_3}^{t_2}\big|P\big(s,x(s),\dot{x}(s)\big)\big|\,ds\right| \\
&\leqslant \left|\int_{t_3}^{t_2}\big(c(s)+k(s)\cdot K\cdot e^{[1+m]\int_0^m k(u)\,du}\big)\,ds\right|.
\end{aligned}
\tag{6}
$$

Taking into account estimates (5) and (6), x and $\dot{x}$ are equi-continuous, because $c(\cdot), k(\cdot) \in L^1([0,m],\mathbb{R})$. Thus, the set of solutions of (3) and their derivatives is relatively compact.

We will still show that the set of solutions of (3) and their derivatives is closed. Let $\{x_k\}$ be a sequence of solutions of (3) such that $(x_k,\dot{x}_k) \to (x,\dot{x})$. For all $k \in \mathbb{N}$ and a.a. $t \in [0,m]$, we have

$$\begin{aligned}|\dot{x}_k(t)| &\leqslant |\dot{x}_k(t_0)| + \left| \int_{t_0}^{t} \left| P\big(s, x_k(s), \dot{x}_k(s)\big) \right| ds \right| \\ &\leqslant M_1 + \left| \int_{t_0}^{t} \left(c(s) + k(s) \cdot K \cdot e^{[1+m]\int_0^m k(u)\,du} \right) ds \right|.\end{aligned}$$

Since $c(\cdot), k(\cdot) \in L^1([0,m], \mathbb{R})$, there exists a constant L such that, for a.a. $t \in [0,m]$,

$$\left| \int_{t_0}^{t} \left(c(s) + k(s) \cdot K \cdot e^{[1+m]\int_0^m k(u)\,du} \right) ds \right| \leqslant L.$$

Therefore, for all $k \in \mathbb{N}$ and for a.a. $t \in [0,m]$,

$$|\dot{x}_k(t)| \leqslant M_1 + L. \tag{7}$$

Moreover, since, for all $k \in \mathbb{N}$ and a.a. $t \in [0,m]$, $|\ddot{x}_k(t)| \leqslant c(t) + k(t) \cdot K \cdot e^{[1+m]\int_0^m k(u)\,du}$, the sequence $\{y_k := \dot{x}_k\}$ satisfies all assumptions of Lemma 2.1.

Thus, applying Lemma 2.1 to the sequence $\{\dot{x}_k\}$, we get that there exists a subsequence of $\{\dot{x}_k\}$, for the sake of simplicity denoted in the same way as the sequence, which converges uniformly to $\dot{x}$ on $[0,m]$ and such that $\{\ddot{x}_k\}$ converges weakly to $\ddot{x}$ in $L^1([0,m], \mathbb{R}^n)$.

If we set $z_k := (x_k, y_k)$, then $\dot{z}_k = (\dot{x}_k, \dot{y}_k) = (\dot{x}_k, \ddot{x}_k) \to (\dot{x}, \ddot{x})$ weakly in $L^1([0,m], \mathbb{R}^n)$. Let us now consider the system

$$\dot{z}_k(t) \in H\big(t, z_k(t)\big), \quad \text{for a.a. } t \in [0,m], \tag{8}$$

where $\dot{z}_k(t) = (\dot{x}_k(t), \dot{y}_k(t))$ and $H(t, z_k(t)) = (y_k(t), P(t, x_k(t), y_k(t)))$.

Applying Lemma 2.2, for $f_i := \dot{z}_k$, $f := (\dot{x}, \ddot{x})$, $x_i := z_k$, it follows that

$$\big(\dot{x}(t), \ddot{x}(t)\big) \in H\big(t, x(t), \dot{x}(t)\big),$$

for a.a. $t \in [0,m]$, i.e.

$$\ddot{x}(t) \in P\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0,m].$$

Moreover, since the set S'_m is closed, $(x_k, \dot{x}_k) \in S'_m$, for all $k \in \mathbb{N}$, and $(x_k, \dot{x}_k) \to (x, \dot{x})$, it also holds that $(x, \dot{x}) \in S'_m$. After all, the set of solutions of (3) and their derivatives is convex and compact, as claimed. □

Remark 3.1. If still $k \cdot B(t) \in C(t)$, for a.a. $t \in [0,m]$, and $k = (k_1, k_2, \ldots, k_n) \in S_m$, then constant k is obviously a solution of (2), and consequently the set of solutions of (2) is also nonempty. Nontrivial examples of solvability of (3), where S'_m corresponds to Kneser-type boundary conditions, are, for instance, in the scalar case ($n = 1$) the conditions $A(t) \equiv 1, C(t) - B(t)x \geqslant 0$, for $t \in [0,m], x \in [0,\infty)$ (cf. [20]) or $C(t) \equiv 0$ and $B(t) \not\equiv 0$, $B(t) \leqslant 0$, for $t \in [0,m]$ (cf. Hartman–Wintner type results, e.g., in [27]).

Furthermore, let us study the structure of a solution set, on a compact interval, to a semi-linear problem.

Hence, let $m \in \mathbb{N}$ and let us consider the b.v.p.

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C\bigl(t, x(t), \dot{x}(t)\bigr), \quad &\text{for a.a. } t \in [0, m], \\ l(x, \dot{x}) = 0, \quad & \end{aligned}\right\} \tag{P_m}$$

where

(i) $A, B \in L^1([0, m], \mathbb{R}^{n\times n})$ are such that $|A(t)| \leqslant a(t)$ and $|B(t)| \leqslant b(t)$, for all $t \in [0, m]$ and suitable integrable functions $a, b : [0, m] \to [0, \infty)$,
(ii) $l : C^1([0, m], \mathbb{R}^n) \times C([0, m], \mathbb{R}^n) \to \mathbb{R}^{2n}$ is a linear bounded operator,
(iii) the associated homogeneous problem

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = 0, \quad &\text{for a.a. } t \in [0, m], \\ l(x, \dot{x}) = 0 \quad & \end{aligned}\right\} \tag{H_m}$$

has only the trivial solution,
(iv) $C : [0, m] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping,
(v) there exists an integrable function $\alpha : [0, m] \to [0, \infty)$, with $\int_0^m \alpha(t)\,dt$ sufficiently small, such that

$$d_H\bigl(C(t, x_1, y_1), C(t, x_2, y_2)\bigr) \leqslant \alpha(t) \cdot \bigl(|x_1 - x_2| + |y_1 - y_2|\bigr),$$

for a.a. $t \in [0, m]$ and all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$,
(vi) there exist a point $(x_0, y_0) \in \mathbb{R}^{2n}$ and a constant $C_0 \geqslant 0$ such that

$$\bigl|C(t, x_0, y_0)\bigr| \leqslant C_0 \cdot \alpha(t)$$

holds, for a.a. $t \in [0, m]$ ($\overset{(v)}{\Longrightarrow} |C(t, x, y)| := \sup\{|z| \mid z \in C(t, x, y)\} \leqslant \alpha(t)(C_0 + |x_0| + |y_0| + |x| + |y|)$ holds, for a.a. $t \in [0, m]$ and all $x, y \in \mathbb{R}^n$).

Lemma 3.2. *Under the above assumptions* (i)–(vi)*, the set of solutions of the b.v.p.* (P_m) *is a nonempty, compact AR-space.*

Proof. Problem (P_m) is equivalent to the first-order problem

$$\left.\begin{aligned} \dot{\xi}(t) + D(t)\xi(t) \in K\bigl(t, \xi(t)\bigr), \quad &\text{for a.a. } t \in [0, m], \\ l(\xi) = 0, \quad & \end{aligned}\right\} \tag{$\tilde{P}_m$}$$

where

$$\xi(t)_{2n\times 1} = \bigl(x(t), \dot{x}(t)\bigr)^T,$$
$$D(t)_{2n\times 2n} = \begin{pmatrix} 0 & -I \\ B(t) & A(t) \end{pmatrix}$$

and

$$K(t, \xi)_{2n\times 1} = \bigl(0, C(t, x, \dot{x})\bigr)^T.$$

Similarly, the associated homogeneous problem (H_m) is equivalent to the first-order problem

$$\left.\begin{aligned} \dot{\xi}(t) + D(t)\xi(t) = 0, \quad & \text{for a.a. } t \in [0,m], \\ l(\xi) = 0. & \end{aligned}\right\} \qquad (\tilde{H}_m)$$

The Fredholm alternative implies (see, e.g., [22]) that there exists the Green function $\tilde{G}$ for the homogeneous problem $(\tilde{H}_m)$ such that each solution $\xi(\cdot)$ of $(\tilde{P}_m)$ can be expressed by the formula $\xi(t) = \int_0^m \tilde{G}(t,s)k(s)\,ds$, where $k(\cdot)$ is a suitable measurable selection of $K(\cdot,\xi(\cdot))$ (cf. Proposition 2.2). If we denote by $\tilde{G}$ the block matrix

$$\tilde{G}_{2n\times 2n} = \begin{pmatrix} \tilde{G}^{11}_{n\times n} & \tilde{G}^{12}_{n\times n} \\ \tilde{G}^{21}_{n\times n} & \tilde{G}^{22}_{n\times n} \end{pmatrix}, \tag{9}$$

then each solution $x(\cdot)$ of (P_m) and its derivative $\dot{x}(\cdot)$ can be expressed as

$$x(t) = \int_0^m \tilde{G}^{12}(t,s)c(s)\,ds$$

and

$$\dot{x}(t) = \int_0^m \tilde{G}^{22}(t,s)c(s)\,ds,$$

where $c(\cdot)$ is a suitable measurable selection of $C(\cdot, x(\cdot), \dot{x}(\cdot))$. Moreover, in view of (v) and (vi),

$$\left|x(t)\right| + \left|\dot{x}(t)\right| \leqslant \int_0^m \left\{\left|\tilde{G}^{12}(t,s)\right| + \left|\tilde{G}^{22}(t,s)\right|\right\}\alpha(s)\left[C_0 + |x_0| + |y_0| + \left|x(s)\right| + \left|\dot{x}(s)\right|\right]ds,$$

for a.a. $t \in [0,m]$. If we denote by $\overline{G} := \sup_{(t,s)\in[0,m]\times[0,m]}\{|\tilde{G}^{12}(t,s)| + |\tilde{G}^{22}(t,s)|\}$, then

$$\max_{t\in[0,m]}\left\{\left|x(t)\right| + \left|\dot{x}(t)\right|\right\} \leqslant \overline{G}\int_0^m \alpha(s)\Big[C_0 + |x_0| + |y_0| + \max_{t\in[0,m]}\left\{\left|x(t)\right| + \left|\dot{x}(t)\right|\right\}\Big]ds.$$

Therefore,

$$\max_{t\in[0,m]}\left\{\left|x(t)\right| + \left|\dot{x}(t)\right|\right\} \leqslant \frac{\overline{G}\cdot(C_0 + |x_0| + |y_0|)\cdot\int_0^m \alpha(s)\,ds}{1 - \overline{G}\int_0^m \alpha(s)\,ds} := M,$$

provided

$$\int_0^m \alpha(s)\,ds < \frac{1}{\overline{G}}. \tag{10}$$

Therefore, if $\int_0^m \alpha(s)\,ds$ is small enough, namely if the inequality (10) holds, then the set of solutions of (P_m) is equal to the set of solutions of the problem

$$\left.\begin{array}{ll}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in C^*\big(t,x(t),\dot{x}(t)\big), & \text{for a.a. } t\in[0,m],\\ l(x,\dot{x})=0, & \end{array}\right\} \qquad (R_m)$$

where C^* satisfies conditions (iv)–(v) in Lemma 3.2 with C replaced by C^*, but this time

$$C^*(t,x,y):=\begin{cases} C(t,x,y), & \text{for } |x|\leqslant M \text{ and } |y|\leqslant M,\\ C(t,M_0,M_1), & \text{otherwise,}\end{cases}$$

where M_0, M_1 are suitable vectors such that $|M_0|=|M_1|=M$. It follows immediately from its definition that C^* satisfies

$$\begin{aligned}\big|C^*(t,x,y)\big| &:= \sup\big\{|z| \,\big|\, z\in C^*(t,x,y)\big\} = \sup\big\{|z| \,\big|\, z\in C(t,x,y),\ \text{where } |x|\leqslant M,\ |y|\leqslant M\big\}\\ &\leqslant \alpha(t)\big(C_0^*+\big|x_0^*\big|+\big|y_0^*\big|+2M\big):=\beta(t),\end{aligned} \tag{11}$$

where $(x_0^*, y_0^*)\in\mathbb{R}^{2n}$ is such that $|C^*(t,x_0^*,y_0^*)|\leqslant C_0^*\alpha(t)$, for a.a. $t\in[0,m]$.

Let us denote by $G(\cdot,\cdot\cdot):=\tilde{G}^{12}(\cdot,\cdot\cdot)$ the Green function associated to the second-order homogeneous problem (H_m) and define the Nemytskiĭ operator

$$N: C^1\big([0,m],\mathbb{R}^n\big)\multimap C^1\big([0,m],\mathbb{R}^n\big)$$

by the formula

$$Nx:=\Bigg\{h\in C^1\big([0,m],\mathbb{R}^n\big)\,\Bigg|\, h(\cdot)=\int_0^m G(\cdot,s)f(s)\,ds,\ \text{where } f\in L^1\big([0,m],\mathbb{R}^n\big),$$
$$f(t)\in C^*\big(t,x(t),\dot{x}(t)\big),\ \text{for a.a. } t\in[0,m]\Bigg\}.$$

Let us note that $Nx\neq\emptyset$, for all $x\in C^1([0,m],\mathbb{R}^n)$, because, for all $x\in C^1([0,m],\mathbb{R}^n)$, $C^*(t,x(t),\dot{x}(t))$ possesses a measurable selection (again, according to Proposition 2.2).

It is evident that the set of solutions of problem (R_m) is equal to the set of fixed-points of the operator N. In order to show that $\mathrm{Fix}(N)$ is, by means of Proposition 2.3, a nonempty, compact *AR*-space, we will proceed in several steps.

(1) At first, let us show that the operator N has convex values. If $h_1,h_2\in Nx$, then there exist integrable selections $f_1(\cdot), f_2(\cdot)$ of $C^*(\cdot,x(\cdot),\dot{x}(\cdot))$ such that, for a.a. $t\in[0,m]$,

$$h_1(t)=\int_0^m G(t,s)f_1(s)\,ds$$

and

$$h_2(t)=\int_0^m G(t,s)f_2(s)\,ds.$$

Let $\lambda \in [0,1]$ be arbitrary. Then, for a.a. $t \in [0,m]$,

$$\lambda h_1(t) + (1-\lambda)h_2(t) = \int_0^m G(t,s)\big[\lambda f_1(s) + (1-\lambda) f_2(s)\big]\,ds.$$

Since mapping C^* has convex values, $\lambda f_1(s) + (1-\lambda) f_2(s) \in C^*(s, x(s), \dot{x}(s))$, for a.a. $s \in [0,m]$. Therefore, $\lambda h_1 + (1-\lambda)h_2 \in Nx$, i.e. the operator N has convex values, as claimed.

(2) Secondly, let us show that the operator N has compact values. Let $x \in C^1([0,m], \mathbb{R}^n)$ be arbitrary and let v be an arbitrary integrable function such that $v(t) \in C^*(t, x(t), \dot{x}(t))$, for a.a. $t \in [0,m]$.

Let us consider the element h of Nx defined, for a.a. $t \in [0,m]$, by

$$h(t) := \int_0^m G(t,s)v(s)\,ds.$$

If $t, \tau \in [0,m]$ are arbitrary, then

$$\begin{aligned} \big|h(t) - h(\tau)\big| &= \left| \int_0^m G(t,s)v(s)\,ds - \int_0^m G(\tau,s)v(s)\,ds \right| \\ &\leqslant \int_0^m \big|G(t,s) - G(\tau,s)\big| \cdot \big|v(s)\big|\,ds \leqslant \int_0^m \big|G(t,s) - G(\tau,s)\big| \cdot \beta(s)\,ds. \end{aligned} \tag{12}$$

Since $\beta(\cdot)$ is, by the definition, an integrable function, estimate (12) implies the equi-continuity of h. Moreover, it immediately follows from condition (11) and properties of the Green function that h is also bounded. Therefore, the well-known Arzelà–Ascoli lemma implies that the set Nx is relatively compact.

The relative compactness of values follows also alternatively from the contractivity of N which will be proved in the next step (3). It is namely well known that contractivity implies condensity.

The closedness of values follows from the fact that, according to [23], N can be expressed as the closed graph composition of operators $\phi \circ S_{C^*}$, where $S_{C^*} : C^1([0,m], \mathbb{R}^n) \multimap L^1([0,m], \mathbb{R}^n)$ and $\phi : L^1([0,m], \mathbb{R}^n) \to C^1([0,m], \mathbb{R}^n)$ are defined by

$$S_{C^*}(x) := \big\{ f \in L^1\big([0,m], \mathbb{R}^n\big) \;\big|\; f(t) \in C^*\big(t, x(t), \dot{x}(t)\big), \text{ for a.a. } t \in [0,m] \big\}$$

and

$$\phi(f) := \left\{ h \in C^1\big([0,m], \mathbb{R}^n\big) \;\middle|\; h(t) = \int_0^m G(t,s) f(s)\,ds, \text{ for a.a. } t \in [0,m] \right\}.$$

(3) In order to show that the operator N is a contraction, let us consider the Banach space $C^1([0,m], \mathbb{R}^n)$ endowed with the norm

$$|x|_{C^1} := \sup_{t \in [0,m]} \big\{ \big|x(t)\big| + \big|\dot{x}(t)\big| \big\},$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$. If $x, y \in C^1([0,m],\mathbb{R}^n)$ are arbitrary, then there exist $h_x \in Nx$, $h_y \in Ny$ and integrable selections $f_x(\cdot)$ of $C^*(\cdot, x(\cdot), \dot{x}(\cdot))$ and $f_y(\cdot)$ of $C^*(\cdot, y(\cdot), \dot{y}(\cdot))$ (cf. Proposition 2.2) such that

$$\begin{aligned}
d_H(Nx, Ny) &= |h_x - h_y|_{C^1} \\
&= \left| \int_0^m G(t,s) f_x(s)\,ds - \int_0^m G(t,s) f_y(s)\,ds \right|_{C^1} \\
&= \sup_{t\in[0,m]} \left\{ \left| \int_0^m G(t,s)\big[f_x(s) - f_y(s)\big]\,ds \right| + \left| \int_0^m \frac{\partial}{\partial t} G(t,s)\big[f_x(s) - f_y(s)\big]\,ds \right| \right\} \\
&\leqslant \sup_{t\in[0,m]} \int_0^m \left\{ \big|G(t,s)\big| + \left| \frac{\partial}{\partial t} G(t,s) \right| \right\} \cdot \big|f_x(s) - f_y(s)\big|\,ds \\
&\leqslant \sup_{t\in[0,m]} \big\{ \big|x(t) - y(t)\big| + \big|\dot{x}(t) - \dot{y}(t)\big| \big\} \\
&\quad \cdot \sup_{(t,s)\in[0,m]\times[0,m]} \left\{ \big|G(t,s)\big| + \left| \frac{\partial}{\partial t} G(t,s) \right| \right\} \cdot \int_0^m \alpha(t)\,dt \\
&= \sup_{(t,s)\in[0,m]\times[0,m]} \left\{ \big|G(t,s)\big| + \left| \frac{\partial}{\partial t} G(t,s) \right| \right\} \cdot \int_0^m \alpha(t)\,dt \cdot |x - y|_{C^1}.
\end{aligned}$$

If the integral $\int_0^m \alpha(t)\,dt$ is small enough, namely if

$$\mathfrak{L} := \sup_{(t,s)\in[0,m]\times[0,m]} \left\{ \big|G(t,s)\big| + \left| \frac{\partial}{\partial t} G(t,s) \right| \right\} \cdot \int_0^m \alpha(t)\,dt < 1, \tag{13}$$

then the operator N is a desired contraction with a Lipschitz constant $\mathfrak{L} \in [0,1)$.

Finally, since N is a contraction with compact and convex values, the set $\mathrm{Fix}(N)$ is, according to Proposition 2.3, a nonempty, compact *AR*-space which completes the proof. □

Remark 3.2. The conclusion of Lemma 3.2 can be deduced from the main result for first-order systems in [8]. Our proof is, however, much more transparent and especially allows us to express explicitly the smallness of the integral $\int_0^m \alpha(t)\,dt$ in conditions (v) and (vi). It is given by the identical inequalities (10) and (13), namely

$$\int_0^m \alpha(t)\,dt < \frac{1}{\sup_{(t,s)\in[0,m]\times[0,m]}\{|G(t,s)| + |\frac{\partial}{\partial t} G(t,s)|\}}. \tag{14}$$

Remark 3.3. If the mapping $C(t,\cdot,\cdot)$ is Lipschitzian with a sufficiently small constant L, i.e. if condition (v) takes the form

(v') there exists a sufficiently small constant $L \geqslant 0$, such that

$$d_H\big(C(t,x_1,y_1),C(t,x_2,y_2)\big) \leqslant L\cdot\big(|x_1-x_2|+|y_1-y_2|\big),$$

for a.a. $t \in [0,m]$ and all $x_1,x_2,y_1,y_2 \in \mathbb{R}^n$, then the same conclusion holds, provided

$$L < \frac{1}{\sup_{t\in[0,m]}\int_0^m |G(t,s)|+|\frac{\partial}{\partial t}G(t,s)|\,ds}. \tag{15}$$

Example 3.1. Let us consider the Dirichlet problem

$$\left.\begin{aligned} \ddot{x}(t) \in C\big(t,x(t),\dot{x}(t)\big), \quad &\text{for a.a. } t\in[0,1],\\ x(0)=0, \quad x(1)=0, \quad & \end{aligned}\right\} \tag{16}$$

where $C : [0,m]\times\mathbb{R}^n\times\mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping such that

$$d_H\big(C(t,x_1,y_1),C(t,x_2,y_2)\big) \leqslant \alpha(t)\cdot\big(|x_1-x_2|+|y_1-y_2|\big),$$

for a.a. $t\in[0,1]$, and all $x_1,x_2,y_1,y_2 \in \mathbb{R}^n$, where $\alpha \in L^1([0,1],[0,\infty))$ is such that (for better conditions, see [14,24] and cf. also [4, Theorem III.3.18])

$$\int_0^1 \alpha(t)\,dt < \frac{1}{2}. \tag{17}$$

Moreover, let there exist $C_0 > 0$ such that

$$\big|C(t,0,0)\big| \leqslant C_0\cdot\alpha(t), \quad \text{for a.a. } t\in[0,1]. \tag{18}$$

We will show that, under the above assumptions, the set of solutions of (16) is a nonempty, compact *AR*-space. The homogeneous problem associated to (16), i.e.

$$\left.\begin{aligned} \ddot{x}(t)=0, \quad &\text{for a.a. } t\in[0,1],\\ x(0)=0, \quad x(1)=0, & \end{aligned}\right\}$$

has only the trivial solution and the related Green function G and its derivative $\frac{\partial G}{\partial t}$ take the forms

$$G(t,s) := \begin{cases} (t-1)s, & 0\leqslant t\leqslant s\leqslant 1,\\ (s-1)t, & 0\leqslant s\leqslant t\leqslant 1,\end{cases}$$

and

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} s, & 0\leqslant t\leqslant s\leqslant 1,\\ s-1, & 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

Since

$$\sup_{(t,s)\in[0,1]\times[0,1]}\left\{\big|G(t,s)\big|+\left|\frac{\partial}{\partial t}G(t,s)\right|\right\} \leqslant 2,$$

condition (17) ensures that the problem (16) is, according to Lemma 3.2 (cf. condition (14)), solvable with a compact *AR*-space of solutions.

4. Topological structure on non-compact intervals

Because of counter-examples (see [1], [4, Example II.2.12], [16]), there is no chance to make a straightforward extension of Lemma 3.2 to b.v.p.s on non-compact intervals. On the other hand, the information concerning the solution sets on compact intervals can be sometimes useful for obtaining the topological structure of the set of solutions to asymptotic problems.

One of the efficient methods which can be used for studying b.v.p.s on non-compact intervals is an *inverse limit method*. Let us recall that by the *inverse system*, we mean a family $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$, where Σ is a set directed by the relation $\leqslant$, X_α is, for all $\alpha \in \Sigma$, a metric space and $\pi_\alpha^\beta : X_\beta \to X_\alpha$ is a continuous function, for all $\alpha, \beta \in \Sigma$ such that $\alpha \leqslant \beta$. Moreover, $\pi_\alpha^\alpha = \mathrm{id}_{X_\alpha}$ and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$, for all $\alpha \leqslant \beta \leqslant \gamma$. The *limit of inverse system* S is denoted by $\lim_\leftarrow S$ and it is defined by

$$\lim_{\leftarrow} S := \left\{ (x_\alpha) \in \prod_{\alpha \in \Sigma} X_\alpha \;\middle|\; \pi_\alpha^\beta(x_\beta) = x_\alpha, \text{ for all } \alpha \leqslant \beta \right\}.$$

If we denote by $\pi_\alpha : \lim_\leftarrow S \to X_\alpha$ the restriction of the projection $p_\alpha : \prod_{\alpha \in \Sigma} X_\alpha \to X_\alpha$ onto α-th axis, then it holds $\pi_\alpha = \pi_\alpha^\beta \pi_\beta$, for all $\alpha \leqslant \beta$.

Let us now consider two inverse systems $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $S' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$. By a *multivalued mapping of the system* S *into the system* S', we mean a family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma : \Sigma' \to \Sigma$ and multivalued mappings $\varphi_{\sigma(\alpha')} : X_{\sigma(\alpha')} \multimap Y_{\alpha'}$ such that, for all $\alpha' \leqslant \beta'$,

$$\pi_{\alpha'}^{\beta'} \varphi_{\sigma(\beta')} = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}^{\sigma(\beta')}.$$

Mapping $\{\sigma, \varphi_{\sigma(\alpha')}\}$ induces a *limit mapping* $\varphi : \lim_\leftarrow S \multimap \lim_\leftarrow S'$ satisfying, for all $\alpha' \in \Sigma'$,

$$\pi_{\alpha'} \varphi = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}.$$

We will make use of the following result. For more details about the inverse limit method, see, e.g., [2–4,9,15].

Proposition 4.1. *(Cf. [3,18,25].) Let $S = \{X_m, \pi_m^p, \mathbb{N}\}$ and $S' = \{Y_m, \pi_m^p, \mathbb{N}\}$ be two inverse systems such that $X_m \subset Y_m$. If $\varphi : \lim_\leftarrow S \multimap \lim_\leftarrow S'$ is a limit map induced by a mapping $\{\mathrm{id}, \varphi_m\}$, where $\varphi_m : X_m \multimap Y_m$, and if $\mathrm{Fix}(\varphi_m)$ are, for all $m \in \mathbb{N}$, R_δ-sets, then the fixed-point set $\mathrm{Fix}(\varphi)$ of φ is an R_δ-set, too.*

The following corollary is a direct consequence of Proposition 4.1.

Corollary 4.1. *Let us consider the sequence of b.v.p.s $\{(K_m)\}_{m=1}^\infty$, where*

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C\big(t, x(t), \dot{x}(t)\big), &\quad \text{for a.a. } t \in [t_0, t_0 + m], \\ k(x, \dot{x})|_{t \in [t_0, t_0+m]} = 0, & \end{aligned}\right\} \tag{K_m}$$

and let us assume that each problem (K_m), $m \in \mathbb{N}$, has an R_δ-set of solutions which corresponds to a fixed-point of the associated integral operator. Moreover, let the boundary condition be such that, for all $m \in \mathbb{N}$, the following holds:

If $x : [t_0, t_0 + m] \to \mathbb{R}^n$ is a solution of problem (K_m), then $x|_{[t_0, t_0+m-1]} : [t_0, t_0 + m - 1] \to \mathbb{R}^n$ is a solution of problem (K_{m-1}).

Then the set of solutions of the problem

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C\big(t, x(t), \dot{x}(t)\big), &\quad \text{for a.a. } t \in [t_0, \infty), \\ k(x, \dot{x}) = 0 & \end{aligned}\right\} \tag{K_∞}$$

is an R_δ-set.

As an illustration, we can give the following simple example.

Example 4.1. Consider the problem

$$\left.\begin{aligned} \ddot{x}(t) \in C\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0,\infty),\\ x(0) = 0, \quad x(1) = 0, \end{aligned}\right\} \tag{19}$$

where $C : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping such that

$$d_H\big(C(t, x_1, y_1), C(t, x_2, y_2)\big) \leqslant \alpha(t) \cdot \big(|x_1 - x_2| + |y_1 - y_2|\big),$$

for a.a. $t \in [0,1]$, and all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, where $\alpha \in L^1([0,1],[0,\infty))$ is such that

$$\int_0^1 \alpha(t)\,dt < \frac{1}{2}. \tag{20}$$

Moreover, let there exist $C_0 > 0$ such that

$$\big|C(t,0,0)\big| \leqslant C_0 \cdot \alpha(t), \quad \text{for a.a. } t \in [0,1]. \tag{21}$$

We will show that, under the above assumptions, the set of solutions of (19) can be expressed as a special union of R_δ-sets.

In order to solve (19), we will consider separately the Dirichlet problem

$$\left.\begin{aligned} \ddot{x}(t) \in C\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0,1],\\ x(0) = 0, \quad x(1) = 0 \end{aligned}\right\} \tag{22}$$

and the Cauchy (initial value) problem

$$\left.\begin{aligned} \ddot{x}(t) \in C\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [1,\infty),\\ x(1) = 0, \quad \dot{x}(1) = x_1. \end{aligned}\right\} \tag{23}$$

According to Lemma 3.2, the b.v.p. (22) is solvable with an R_δ-set of solutions (cf. Example 3.1). In fact, the set of solutions of (22) is, according to Lemma 3.2, a nonempty, compact *AR*-space.

Let $x(\cdot)$ be a solution of the Dirichlet problem (22) and let us put $x_1 := \dot{x}(1)$. Now, let us consider, for this inter-face value of the derivative, the problem (23). The Cauchy problem, considered on an arbitrary compact interval $[1,m]$, $m \in \mathbb{N}$, has an R_δ-set of solutions (cf. [13]). Using the inverse limit method, we can conclude that, for the fixed $x_1 = \dot{x}(1)$, the Cauchy problem (23) has, according to Corollary 4.1, an R_δ-set of solutions on $[1,\infty)$ which, in particular, implies that the related solution set is nonempty. If we denote by $x_D : [0,1] \to \mathbb{R}^n$ the solution of the Dirichlet problem (22) satisfying $\dot{x}_D(1) = x_1$ and by $x_2 : [1,\infty] \to \mathbb{R}^n$ the solution of the Cauchy problem (23), then

$$x(t) := \begin{cases} x_D(t), & \text{for all } t \in [0,1],\\ x_2(t), & \text{for all } t \in [1,\infty), \end{cases}$$

is the solution of the original problem (19).

Although the solution set of each separate problem was proved to be an R_δ-set, the solution set of the whole problem can be more complex. Nevertheless, if, for instance, the Dirichlet problem is uniquely solvable, then the solution set of the whole problem is an R_δ-set, too.

Combining Corollary 4.1 with Lemma 3.1, we obtain immediately the following result.

Proposition 4.2. *Let us consider the problems for fully linearized systems on compact intervals* (2) *and* (3) *together with the asymptotic problems*

$$\left.\begin{array}{c} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C(t), \quad \text{for a.a. } t \in [0,\infty), \\ x \in S, \end{array}\right\} \tag{24}$$

$$\left.\begin{array}{c} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C(t), \quad \text{for a.a. } t \in [0,\infty), \\ (x,\dot{x}) \in S', \end{array}\right\} \tag{25}$$

where

(i) $A, B : [0,\infty) \to \mathbb{R}^{n\times n}$ *are locally integrable matrix functions such that* $|A(t)| \leqslant a(t)$, $|B(t)| \leqslant b(t)$, *for a.a.* $t \in [0,\infty)$ *and a suitable nonnegative functions* $a, b \in L^1_{loc}([0,\infty),\mathbb{R})$,

(ii) S *and* S_m *are, for all* $m \in \mathbb{N}$, *closed, convex subsets of* $AC^1_{loc}([0,\infty),\mathbb{R}^n)$ *and* $AC^1([0,m],\mathbb{R}^n)$ (S' *and* S'_m *are, for all* $m \in \mathbb{N}$, *closed, convex subsets of* $AC^1_{loc}([0,\infty),\mathbb{R}^n) \times AC_{loc}([0,\infty),\mathbb{R}^n)$ *and* $AC^1([0,m],\mathbb{R}^n) \times AC([0,m],\mathbb{R}^n)$),

(iii) $C : [0,\infty) \multimap \mathbb{R}^n$ *is a locally integrable mapping with convex closed values such that* $|C(t)| \leqslant c(t)$, *for a.a.* $t \in [0,m]$ *and a suitable nonnegative function* $c \in L^1_{loc}([0,\infty),\mathbb{R})$,

(iv) *there exists* $t_0 \in [0,\infty)$ *such that, for all* $m \in \mathbb{N}$, *we are able to find constants* M_{m_0}, M_{m_1} *such that* $|x(t_0)| \leqslant M_{m_0}$ *and* $|\dot{x}(t_0)| \leqslant M_{m_1}$, *for all solutions* $x(\cdot)$ *of problem* (2) (*all solutions* $x(\cdot)$ *of problem* (3)).

Moreover, let, for all $m \in \mathbb{N}$, *the set of solutions of* (2) (*the set of solutions of* (3) *and their derivatives*) *be nonempty and correspond to fixed-points of the associated integral operator. Furthermore, let the boundary condition be such that, for all* $m \in \mathbb{N}$, *the following holds*:

If $x : [0,m] \to \mathbb{R}^n$ *belongs to* S_m, *then* $x|_{[0,m-1]} : [0,m-1] \to \mathbb{R}^n$ *belongs to* S_{m-1}.

(*If* $(x,\dot{x}) : [0,m]\times[0,m] \to \mathbb{R}^{2n}$ *belongs to* S'_m, *then* $(x|_{[0,m-1]}, \dot{x}|_{[0,m-1]}) : [0,m-1]\times[0,m-1] \to \mathbb{R}^{2n}$ *belongs to* S'_{m-1}.)

Then the set of solutions of the problem (24) (*the set of solutions of the problem* (25) *and their derivatives*) *is an* R_δ*-set.*

As an application of Proposition 4.2, let us consider the second-order asymptotic (Kneser-type) b.v.p. with a multivalued vector perturbation

$$\left.\begin{array}{c} \ddot{x}(t) + \dot{x}(t) + b(t)x(t) \in F(t), \quad \text{for a.a. } t \in [0,\infty), \\ x(0) = 1, \\ x(t) \geqslant 0, \quad \dot{x}(t) \leqslant 0, \quad \text{for all } t \in [0,\infty), \end{array}\right\} \tag{P}$$

where

- $b : [0,\infty) \to \mathbb{R}$ is a locally integrable function such that $|b(t)| \leqslant a(t)$, for all $t \in [0,\infty)$, where $a \in L^1_{loc}([0,\infty),\mathbb{R})$,
- $F : [0,\infty) \multimap \mathbb{R}$ is a locally integrable mapping with convex closed values such that $|F(t)| \leqslant \alpha(t)$, for all $t \in [0,\infty)$, where $\alpha \in L^1_{loc}([0,\infty),\mathbb{R})$ and that

$$-b(t) \notin F(t), \quad \text{for all } t \text{ in a right neighbourhood of } 0.$$

Moreover, let $v(t) - b(t)x \geqslant 0$, for all $t \in [0,\infty)$, $x \in [0,1]$ and each measurable selection $v(\cdot)$ of $F(\cdot)$.

We will prove the following theorem.

Theorem 4.1. *Under the above assumptions, the set of solutions of* (P) *and their first derivatives is an* R_δ*-set.*

Proof. Together with the b.v.p. (P), let us consider the family of associated problems on compact intervals

$$\left.\begin{aligned} \ddot{x}(t)+\dot{x}(t)+b(t)\cdot x(t)\in F(t), \quad \text{for a.a. } t\in[0,m], \\ x(0)=1, \\ x(t)\geqslant 0, \quad \dot{x}(t)\leqslant 0, \quad \text{for all } t\in[0,m], \end{aligned}\right\} \tag{P_m}$$

where $m\in\mathbb{N}$.

It was shown in [11] (see Lemma 2.1 in [11] and the remarks below) that under the above assumptions imposed on b, the following two norms in $AC^1([0,m],\mathbb{R})$, where $m>0$ is arbitrary, are equivalent:

$$\|x\| := \sup_{t\in[0,m]}\big|x(t)\big| + \sup_{t\in[0,m]}\big|\dot{x}(t)\big| + \int_0^m \big|\ddot{x}(t)\big|\,dt;$$

$$\|x\|_* := \sup_{t\in[0,m]}\big|x(t)\big| + \int_0^m \big|\ddot{x}(t)+b(t)\cdot x(t)+\dot{x}(t)\big|\,dt.$$

If x is a solution of the b.v.p. (P_m), for some $m\in\mathbb{N}$, then

$$\|x\|_* = 1+\int_0^m \alpha(t)\,dt := M_m,$$

because α is a locally integrable function.

Since $\sup_{t\in[0,m]}|\dot{x}(t)|\leqslant\|x\|$, and the equivalence of norms $\|x\|_*$ and $\|x\|$, there exists the sequence $\{k_m\}$ of positive numbers such that all solutions of the b.v.p. (P_m), for fixed $m\in\mathbb{N}$, satisfy $|\dot{x}(t)|\leqslant k_m\cdot M_m$.

Since the sets

$$\begin{aligned} S'_m := \big\{&(x,\dot{x})\in AC^1\big([0,m],\mathbb{R}\big)\times AC\big([0,m],\mathbb{R}\big),\ x(0)=1, \\ &x(t)\geqslant 0,\ \dot{x}(t)\leqslant 0,\ \text{for all } t\in[0,m]\big\}, \\ S' := \big\{&(x,\dot{x})\in AC^1_{loc}\big([0,\infty),\mathbb{R}\big)\times AC_{loc}\big([0,\infty),\mathbb{R}\big),\ x(0)=1, \\ &x(t)\geqslant 0,\ \dot{x}(t)\leqslant 0,\ \text{for all } t\in[0,\infty)\big\} \end{aligned}$$

are closed and convex, the b.v.p.s (P), (P_m) satisfy assumptions (i)–(iv) of Proposition 4.2 (with $M_{m_0}=1, M_{m_1}=k_m\cdot M_m$).

The non-emptiness of the set of solutions of (P_m) follows from Corollary 2 in [20] and the fact that $F(\cdot)$ admits (according to Proposition 2.2) a single-valued measurable selection $v(\cdot)$ such that $v(t)-b(t)x\geqslant 0$, for all $t\in[0,\infty)$ and $x\in[0,\infty)$.

If we denote by $P(t,x(t),\dot{x}(t)) := F(t)-b(t)x(t)-\dot{x}(t)$, then $x(\cdot)$ is a solution of (P_m) if and only if, for a.a. $t\in[0,m]$,

$$x(t)\in x(u)-\big|x(u)\big|+1+\dot{x}(0)\cdot t+\int_0^t (t-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds, \tag{26}$$

$$\dot{x}(t) \in \dot{x}(u) + \left|\dot{x}(u)\right| + \dot{x}(0) + \int_0^t P\bigl(s, x(s), \dot{x}(s)\bigr)\,ds, \tag{27}$$

for each $u \in [0, m]$, provided

$$-b(t) \notin F(t), \tag{28}$$

on a subset of $[0, m]$ with a non-zero measure. Indeed. Since the constraint in (P_m) can be equivalently expressed as

$$\left.\begin{array}{c} x(0) = 1, \\ x(u) - \left|x(u)\right| = 0, \quad \dot{x}(u) + \left|\dot{x}(u)\right| = 0, \quad \text{for all } u \in [0, m], \end{array}\right\} \tag{29}$$

every solution $x(\cdot)$ of (P_m) and its derivative $\dot{x}(\cdot)$ obviously satisfy (26) and (27). Reversely, derivating (26) and (27), we obtain

$$\dot{x}(t) \in \dot{x}(0) + \int_0^t P\bigl(s, x(s), \dot{x}(s)\bigr)\,ds,$$

$$\ddot{x}(t) \in P\bigl(t, x(t), \dot{x}(t)\bigr).$$

Moreover, $x(0) \in x(u) - |x(u)| + 1$, $\dot{x}(0) \in \dot{x}(u) + |\dot{x}(u)| + \dot{x}(0)$, for each $u \in [0, m]$, i.e. $\dot{x}(u) + |\dot{x}(u)| = 0$ and, in particular, for $u = 0$, $|x(0)| = 1$. Thus, for $x(0) = 1$, we also have $x(u) - |x(u)| = 0$, by which (29) (i.e. the constraint in (P_m)) is satisfied. On the other hand, if $x(0) = -1$, we arrive at $x(u) - |x(u)| = -2$, i.e. $x(u) = -1$, for all $u \in [0, m]$, and subsequently $-b(t) \in F(t)$, for a.a. $t \in [0, m]$, which is a contradiction with (28).

The set of solutions of (P_m) and their first derivatives is a fixed-point set of the map $\varphi_m : C^1([0, m], \mathbb{R}) \times C([0, m], \mathbb{R}) \multimap C^1([0, m], \mathbb{R}) \times C([0, m], \mathbb{R})$, where, for all $t \in [0, m]$,

$$\begin{aligned} \varphi_m(x, \dot{x})(t) := \Biggl\{ \Biggl(& \bigcup_{u \in [0,m]} x(u) - \left|x(u)\right| + 1 + \dot{x}(0) \cdot t + \int_0^t (t - s) \cdot f(s)\,ds, \\ & \bigcup_{u \in [0,m]} \dot{x}(u) + \left|\dot{x}(u)\right| + \dot{x}(0) + \int_0^t f(s)\,ds \Biggr) \Bigm| f \in L^1\bigl([0, m], \mathbb{R}^n\bigr) \text{ and} \\ & f(s) \in P\bigl(t, x(s), \dot{x}(s)\bigr), \text{ for a.a. } s \in [0, m] \Biggr\}. \end{aligned}$$

It can be easily seen that $\{\varphi_m\}_{m=1}^{\infty}$ is a map of the inverse system

$$\bigl\{C^1\bigl([0, m], \mathbb{R}\bigr) \times C\bigl([0, m], \mathbb{R}\bigr), \pi_m^p, \mathbb{N}\bigr\}$$

into itself, where, for all $p \geqslant m$, $x \in C^1([0, p], \mathbb{R}) \times C([0, p], \mathbb{R})$, $\pi_m^p(x, \dot{x}) = (x|_{[0,m]}, \dot{x}|_{[0,m]})$. Mappings $\{\varphi_m\}_{m=1}^{\infty}$ induce the limit mapping $\varphi : C^1([0, \infty), \mathbb{R}) \times C([0, \infty), \mathbb{R}) \multimap C^1([0, \infty), \mathbb{R}) \times C([0, \infty), \mathbb{R})$, where, for all $t \geqslant 0$,

$$\varphi(x,\dot{x})(t) := \Biggl\{ \Biggl(\bigcup_{u\in[0,\infty)} x(u) - \bigl|x(u)\bigr| + 1 + \dot{x}(0)\cdot t + \int_0^t (t-s)\cdot f(s)\,ds,$$

$$\bigcup_{u\in[0,\infty)} \dot{x}(u) + \bigl|\dot{x}(u)\bigr| + \dot{x}(0) + \int_0^t f(s)\,ds \Biggr) \Bigm| f \in L^1\bigl([0,m],\mathbb{R}^n\bigr) \text{ and}$$

$$f(s) \in P\bigl(t, x(s), \dot{x}(s)\bigr),\ \text{for a.a. } s\in[0,\infty) \Biggr\}.$$

The fixed-point set of the mapping φ is the set of solutions and their derivatives of the problem (P). Applying Proposition 4.2, the set of solutions and their first derivatives of the original problem (P) is therefore an R_δ-set, as claimed. □

Remark 4.1. One can readily check that if $(1,\ldots,1)\cdot B(t) \in C(t)$, for a.a. $t\in[0,\infty)$, then constant vector $(1,\ldots,1)$ is a stationary solution of the Kneser-type asymptotic problem

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)\cdot x(t) \in C(t), &\quad \text{for a.a. } t\in[0,\infty), \\ x_i(0) = 1, &\quad \text{for all } i\in\{1,2,\ldots,n\}, \\ x_i(t)\geqslant 0,\quad \dot{x}_i(t)\leqslant 0, &\quad \text{for all } i\in\{1,2,\ldots,n\} \text{ and } t\in[0,\infty), \end{aligned}\right\} \tag{P^1}$$

where

- $A:[0,\infty)\to\mathbb{R}^{n\times n}$ is a locally integrable matrix function such that $|A(t)|\leqslant a(t)$, for all $t\in[0,\infty)$, where $a\in L^1_{loc}([0,\infty),\mathbb{R})$,
- $B:[0,\infty)\to\mathbb{R}^{n\times n}$ is a locally integrable matrix function such that $|B(t)|\leqslant b(t)$, for all $t\in[0,\infty)$, where $b\in L^1_{loc}([0,\infty),\mathbb{R})$,
- $C:[0,\infty)\multimap\mathbb{R}^n$ is a locally integrable mapping with convex closed values such that $|C(t)|\leqslant c(t)$, for all $t\in[0,\infty)$, where $c\in L^1_{loc}([0,\infty),\mathbb{R})$.

If still

$$(-1,\ldots,-1)\cdot B(t)\notin C(t),\quad \text{for a.a. } t \text{ in a right neighborhood of } 0,$$

then it can be proved quite analogously as in Theorem 4.1 that the set of solutions of (P^1) and their first derivatives is an R_δ-set.

Remark 4.2. Similarly, if $a,b\in L^1_{loc}([0,\infty),\mathbb{R})$ are locally integrable functions such that $b(t)\leqslant 0$, for a.a. $t\in[0,\infty)$, and $b(t)\neq 0$, for a.a. t in a right neighborhood of 0, then (cf. Remark 3.1) it can be proved quite analogously as in Theorem 4.1 that the set of solutions of the Kneser-type asymptotic problem

$$\left.\begin{aligned} \ddot{x}(t) + a(t)\dot{x}(t) + b(t)\cdot x(t) = 0, &\quad \text{for a.a. } t\in[0,\infty), \\ x(0) = 1, & \\ x(t)\geqslant 0,\quad \dot{x}(t)\leqslant 0, &\quad \text{for all } t\in[0,\infty), \end{aligned}\right\}$$

and their first derivatives is an R_δ-set.

Remark 4.3. As a particular case of Corollary 4.1, we are theoretically able to obtain the result which can be proved combining Lemma 3.2 and Proposition 4.1. In such a case an integrable function α in condition (v) in Lemma 3.2 should however satisfy conditions (cf. (14))

$$\int_{t_0}^{t_0+m} \alpha(t)\,dt < \frac{1}{\sup_{(t,s)\in[t_0,t_0+m]\times[t_0,t_0+m]}\{|G_m(t,s)| + |\frac{\partial}{\partial t}G_m(t,s)|\}}, \tag{30}$$

for sufficiently large $m \in \mathbb{N}$, which is probably not very realistic.

5. Application to existence result

As we could see, the investigation of a topological structure of solution sets to asymptotic problems is sufficiently interesting itself. Nevertheless, the main advantage consists in its further application to existence results for nonlinear asymptotic problems.

This application will be demonstrated here by means of the following proposition developed by ourselves in [5, Theorem 3.1 and Corollary 4.1].

Proposition 5.1. *Let us consider the b.v.p.*

$$\left.\begin{array}{c} \ddot{x}(t) \in F\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in J, \\ x \in S, \end{array}\right\} \tag{31}$$

where J is a given (possibly infinite) real interval, $F : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of $AC^1_{loc}(J, \mathbb{R}^n)$.

Let $H : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory map such that

$$H(t, c, d, c, d) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in J \times \mathbb{R}^n \times \mathbb{R}^n.$$

Assume that

(i) *there exists a retract Q of $C^1(J, \mathbb{R}^n)$ such that the associated problem*

$$\left.\begin{array}{c} \ddot{x}(t) \in H\big(t, x(t), \dot{x}(t), q(t), \dot{q}(t)\big), \quad \text{for a.a. } t \in J, \\ x \in S \cap Q, \end{array}\right\} \tag{32}$$

is solvable with an R_δ-set of solutions, for each $q \in Q$,

(ii) *there exists a nonnegative, locally integrable function $\alpha : J \to \mathbb{R}$ such that*

$$\big|H\big(t, x(t), \dot{x}(t), q(t), \dot{q}(t)\big)\big| \leqslant \alpha(t)\big(1 + |x(t)| + |\dot{x}(t)|\big), \quad \text{a.e. in } J,$$

for any $(q, x) \in \Gamma_T$, where T denotes the multivalued map which assigns to any $q \in Q$ the set of solutions of (32),

(iii) $\overline{T(Q)} \subset S$,

(iv) *there exist a point $t_0 \in J$ and constants $M_0 \geqslant 0$, $M_1 \geqslant 0$ such that $|x(t_0)| \leqslant M_0$ and $|\dot{x}(t_0)| \leqslant M_1$, for any $x \in T(Q)$.*

Then the b.v.p. (31) has a solution in $S \cap Q$.

Hence, let us consider the second-order nonlinear (Kneser-type) asymptotic b.v.p.

$$\left.\begin{array}{c}\ddot{x}(t)+\dot{x}(t)+B\big(t,x(t),\dot{x}(t)\big)\cdot x(t)\in F\big(t,x(t),\dot{x}(t)\big),\quad \text{for a.a. } t\in[0,\infty),\\ x_i(0)=1,\quad \text{for all } i\in\{1,2,\dots,n\},\\ x_i(t)\geqslant 0,\quad \dot{x}_i(t)\leqslant 0,\quad \text{for all } i\in\{1,2,\dots,n\} \text{ and } t\in[0,\infty),\end{array}\right\}\tag{33}$$

where

- $x(t)=(x_1(t),x_2(t),\dots,x_n(t))$,
- $B:[0,\infty)\times\mathbb{R}^{2n}\to\mathbb{R}^{n\times n}$ is a diagonal Carathéodory matrix function, i.e.

$$B\big(t,x(t),\dot{x}(t)\big)=\begin{pmatrix} b_1(t,x(t),\dot{x}(t)) & 0 & \dots & 0\\ 0 & b_2(t,x(t),\dot{x}(t)) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & b_n(t,x(t),\dot{x}(t))\end{pmatrix}$$

  with $|B(t,x,y)|\leqslant\beta(t)(1+|x|)$, for all $(x,y)\in\mathbb{R}^{2n}$ and $t\in[0,\infty)$, where $\beta\in L^1_{loc}([0,\infty),\mathbb{R})$,
- $F=(F_1,F_2,\dots,F_n):[0,\infty)\times\mathbb{R}^{2n}\multimap\mathbb{R}^n$ is an upper-Carathéodory mapping such that $|F(t,x,y)|\leqslant\alpha(t)(1+|x|)$, for all $(x,y)\in\mathbb{R}^{2n}$ and $t\in[0,\infty)$, where $\alpha\in L^1_{loc}([0,\infty),\mathbb{R})$, and that

$$-b_i(t,x,y)\notin F_i(t,x,y),\quad \text{for all } i\in\{1,\dots,n\},\ (x,y)\in[0,1]^n\times(-\infty,0]^n$$

  and for t in a right neighbourhood of 0.

For applying Proposition 5.1, let us define the set of candidate solutions as follows

$$Q:=\big\{(x_1,\dots,x_n)\in C^1\big([0,\infty),\mathbb{R}^n\big)\ \big|\ x_i(0)=1,\ x_i(t)\geqslant 0,\ \dot{x}_i(t)\leqslant 0,\\ \text{for all } i\in\{1,\dots,n\},\ t\in[0,\infty)\big\}.$$

Let us still consider the associated problems

$$\left.\begin{array}{c}\ddot{x}(t)+\dot{x}(t)+B\big(t,q(t),\dot{q}(t)\big)\cdot x(t)\in F\big(t,q(t),\dot{q}(t)\big),\quad \text{for a.a. } t\in[0,\infty),\\ x_i(0)=1,\quad \text{for all } i\in\{1,2,\dots,n\},\\ x_i(t)\geqslant 0,\quad \dot{x}_i(t)\leqslant 0,\quad \text{for all } i\in\{1,2,\dots,n\} \text{ and } t\in[0,\infty).\end{array}\right\}\tag{P_q}$$

If $v_i(t)-b_i(t,q(t),\dot{q}(t))\cdot x_i\geqslant 0$, for all $i\in\{1,2,\dots,n\}$, $q\in Q$, $t\in[0,\infty)$, $x_i\in[0,1]$ and each measurable selection $v_i(t)$ of $F_i(t,q(t),\dot{q}(t))$, then we will check that the b.v.p. (33) has a solution.

More concretely, let us verify, that the b.v.p. (P_q) satisfies, for all $q\in Q$, all assumptions of Proposition 5.1.

ad (i) Since (P_q) represents n separate problems on a diagonal, it can be proved exactly in the same way as in Theorem 4.1 that the b.v.p. (P_q) has, for each $q\in Q$, an R_δ-set of solutions.

ad (ii) $|F(t,q(t),\dot{q}(t))-B(t,q(t),\dot{q}(t))\cdot x-y|\leqslant\alpha(t)(1+\sqrt{n})+\beta(t)\cdot(1+\sqrt{n})\cdot|x|+|y|$, for a.a. $t\in[0,\infty)$, all $(x,y)\in\mathbb{R}^{2n}$ and $q\in Q$.

ad (iii) Since the set $S:=Q$ is closed and each solution of the b.v.p. (P_q) belongs to Q, it holds that $\overline{T(Q)}\subset S$, where the map T is the solution mapping that assigns to each $q\in Q$ the set of solutions of (P_q).

ad (iv) Let $t_0 = 0$. Then each solution $x(\cdot)$ of (P_q) satisfies, for an arbitrary $q \in Q$, $|x(0)| = \sqrt{n}$. Moreover, the fact that $x(\cdot)$ is a solution of (P_q) implies that $x(\cdot)$ is also a solution of the b.v.p.

$$\left.\begin{aligned} \ddot{x}(t) + \dot{x}(t) + B\big(t, q(t), \dot{q}(t)\big) \cdot x(t) \in F\big(t, q(t), \dot{q}(t)\big), &\quad \text{for a.a. } t \in [0, 1], \\ x_i(0) = 1, &\quad \text{for all } i \in \{1, 2, \dots, n\}, \\ x_i(t) \geqslant 0, \quad \dot{x}_i(t) \leqslant 0, &\quad \text{for all } i \in \{1, 2, \dots, n\} \text{ and } t \in [0, 1]. \end{aligned}\right\} \qquad (P_{q,1})$$

Thus, it follows from the arguments in the proof of Theorem 4.1 that $|\dot{x}(0)| \leqslant k_1 \cdot M_1$, where k_1 is a suitable positive constant and $M_1 := \sqrt{n} + \int_0^1 \alpha(t)\, dt$.

Since all assumptions of Proposition 5.1 are satisfied, we are ready to formulate the last theorem.

Theorem 5.1. *Under the above assumptions, the b.v.p.* (33) *admits a solution* $x(\cdot) = (x_1(\cdot), \dots, x_n(\cdot))$ *such that* $0 \leqslant x_i(t) \leqslant 1$, *for all* $i \in \{1, 2, \dots, n\}$ *and* $t \in [0, \infty)$.

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Bound sets approach to boundary value problems for vector second-order differential inclusions

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ABSTRACT

A continuation principle is established for the solvability of vector second-order boundary value problems associated with upper-Carathéodory differential inclusions. For Floquet second-order problems, this principle is combined with a bound sets approach. The viability result is also obtained in this way.

1. Introduction

In this paper, we will formulate a general principle for the solvability of the second-order boundary value problem (b.v.p.)

$$\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in J, x \in S, \tag{1}$$

where $J = [t_0, t_1]$ is a compact interval, $F : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of $AC^1(J, \mathbb{R}^n)$.

Moreover, for the Floquet semi-linear problem

$$\begin{aligned} &\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in J, \\ &x(t_1) = Mx(t_0), \\ &\dot{x}(t_1) = N\dot{x}(t_0), \end{aligned} \tag{2}$$

where $A, B : J \to \mathbb{R}^n \times \mathbb{R}^n$ are integrable matrix functions and M and N are real $n \times n$ matrices with M non-singular, the viability result will be obtained by means of a bound sets technique.

Vector second-order boundary value problems for differential equations were studied by many authors (see, e.g., [6, 11,12,17,19,20,25,31,34,37,38]). The papers dealing with multivalued second-order vector problems are a little more rare

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(see [14,16,18,21,23,26,27,29,30,32,36]). For multivalued vector second-order Dirichlet problems, the existence results were obtained, e.g., in [21,32] and generalized to the case when F takes its values in a Hilbert space in [36]. Second-order differential inclusions with Sturm–Liouville boundary conditions were studied in [18]. In [16,23,26,27,30], second-order multivalued problems with general nonlinear boundary conditions which include the classical Dirichlet, Neumann and periodic boundary conditions were under consideration.

The bound sets approach which we use in Sections 4 and 5 of our paper was initiated by Gaines and Mawhin in [22] for obtaining the existence of periodic solutions of first-order as well as second-order systems of differential equations (see also the references therein). For periodic boundary value problems associated with second-order differential equations, i.e. for F single-valued and $M = N = I$, various approaches (such as an upper and lower solutions technique) were employed in [11,12,25,31]. In these papers, a ball centered at the origin plays in fact the role of a bound set and a bounding function is of class C^2. A bound sets technique was also applied for single-valued periodic problems in [33].

Bound sets theory for multivalued first-order Floquet problems was employed in [3–5]. For single-valued Floquet second-order problems with continuous right hand sides, see, e.g., [17,38].

In this paper, we consider at first b.v.p. (1) and develop a continuation principle for its solvability using the fixed point index arguments (cf. Theorem 1). One of the assumptions which must be satisfied in order the continuation principle can be applied is so-called transversality condition (cf. hypothesis (v) of Theorem 1). It requires that the related solution operator has no fixed points on the boundary of a set of candidate solutions. The transversality condition can be guaranteed by means of Liapunov-like bounding functions to which the second part of the paper is devoted. In the third part of our paper, we investigate the Floquet problem (2). The existence result (cf. Theorem 2) is obtained when combining the bound sets approach with the mentioned continuation principle developed in the previous sections.

The paper is organized as follows. In the second section, we recall suitable definitions and statements from multivalued analysis and fixed point index theory. Subsequently, in Section 3, we develop by means of fixed point index arguments a general method for the solvability of multivalued vector second-order b.v.p. (1). In Section 5, the general method is combined with a bound sets technique (developed in Section 4), and the viability result is obtained in this way for the Floquet problem (2). Finally, an illustrative example is supplied.

2. Preliminaries

Let us start with notations we use in the paper. In the entire text, all spaces are at least metric and all multivalued mappings $F : X \multimap Y$ have at least nonempty values, i.e. $F : X \to 2^Y \setminus \{\emptyset\}$. At first, we recall some geometric properties of metric spaces, in particular, the notions of retracts. If X is an arbitrary space and $A \subset X$, by $Int\ A$, $\overline{A}$ and ∂A, we shall mean the interior, the closure and the boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x, a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$. Similarly, A is called a *neighbourhood retract* of X if there exists an open subset $U \subset X$ such that $A \subset U$ and A is a retract of U.

Let X, Y be two spaces. We say that X is an *absolute retract* (AR-space) if, for each Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over Y. If f is extendable over some neighborhood of A, for each closed $A \subset Y$ and each continuous mapping $f : A \to X$, then X is an *absolute neighborhood retract* (ANR-space). Let us note that X is an ANR-space if and only if it is a retract of an open subset of a normed space and that X is an AR-space if and only if it is a retract of some normed space. For more details, see, e.g., [13].

We say that a nonempty subset $A \subset X$ is *contractible* if there exist $x_0 \in A$ and a homotopy $h : A \times [0, 1] \to A$ such that $h(x, 0) = x$ and $h(x, 1) = x_0$, for every $x \in A$. A nonempty subset $A \subset X$ is called an R_δ*-set* if there exists a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of compact, AR-spaces such that

$$A = \cap\{A_n; n = 1, 2, \ldots\}.$$

Note that any R_δ-set is nonempty, compact and connected. The following hierarchies hold for nonempty subsets of a metric space:

$$\text{convex} \subset AR \subset ANR, \tag{3}$$

$$\text{compact} + \text{convex} \subset \text{compact}\, AR \subset \text{compact+contractible} \subset R_\delta\text{-set}, \tag{4}$$

and all the above inclusions are proper. For more details, see, e.g., [1,2,24].

For a given compact real interval J, we denote by $C(J, \mathbb{R}^n)$ (by $C^1(J, \mathbb{R}^n)$) the set of all functions $x : J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $AC^1(J, \mathbb{R}^n)$, we shall mean the set of all functions $x : J \to \mathbb{R}^n$ with an absolutely continuous first derivative on J. A set $S \subset C^1(J, \mathbb{R}^n)$ is called *bounded* if there exists a function $\varphi \in C(J, \mathbb{R})$ such that

$$|x(t)| < \varphi(t) \quad \text{and} \quad |\dot{x}(t)| < \varphi(t), \quad \text{for all } x \in S \text{ and all } t \in J.$$

Functions in S are called *equi-continuous* on J if, for all $t \in J$ and every $\varepsilon > 0$, there exists $\delta = \delta(t, \varepsilon) > 0$ such that, for all $t^* \in J$ satisfying

$$|t - t^*| < \delta,$$

it holds that

$$|x(t) - x(t^*)| < \varepsilon,$$

for all $x \in S$.

In the sequel, we shall also need the following definitions and notions from multivalued analysis. Let X, Y be two spaces. We say that F is a *multivalued mapping* from X to Y (written $F : X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

A multivalued mapping $F : X \multimap Y$ is called *upper semi-continuous* (shortly, u.s.c.) if, for each open set $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

The relationship between upper semi-continuous mappings and mappings with closed graphs is expressed by the following propositions (see, e.g., [2,24,28]).

Proposition 1. *Let X, Y be metric spaces and $F : X \multimap Y$ be a multivalued mapping with the closed graph such that $F(X) \subset K$, where K is a compact set. Then F is u.s.c.*

Proposition 2. *Let X, Y be metric spaces and $F : X \multimap Y$ be an upper semi-continuous multivalued mapping with closed values. Then the graph Γ_F of F is a closed subset of $X \times Y$.*

A multivalued mapping $F : X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x \in X} F(x)$ is contained in a compact subset of Y and it is called *closed* if the set $F(B)$ is closed in Y, for every closed subset B of X.

A multivalued mapping $F : X \multimap X$ with bounded values is called *Lipschitzian* if there exists a constant $k > 0$ such that, for every $x, y \in X$,

$$d_H(F(x), F(y)) \leq kd(x, y),$$

where

$$d_H(A, B) := \inf\{r > 0 \mid A \subset N_r(B) \text{ and } B \subset N_r(A)\}$$

stands for the Hausdorff distance.

Let Y be a metric space and $(\Omega, \mathcal{U}, \nu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $F : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid F(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$.

If $X \cap Y \neq \emptyset$ and $F : X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed point* of F if $x \in F(x)$. The set of all fixed points of F is denoted by $\mathrm{Fix}(F)$, i.e.

$$\mathrm{Fix}(F) := \{x \in X \mid x \in F(x)\}.$$

For more details concerning the topics of multivalued analysis, see, e.g., [2,7,8,24,28].

Now, we recall the notion of an appropriate fixed point index which will be used in the sequel. Let X be an *ANR*-space and let $G : X \multimap X$ be a compact R_δ*-mapping,* i.e. a compact u.s.c. mapping with R_δ-values.

Let $D \subset X$ be an open subset of X with no fixed points of G on its boundary ∂D. Then it is possible to define an integer $\mathrm{ind}(G, X, D)$, called a *fixed point index* over X w.r.t. D. In the following proposition, we collect the most important properties of such a fixed point index. For more details, see, e.g., [1,2,10,24].

Proposition 3 (*Properties of the Fixed Point Index*).

(i) *(Existence) If* $\mathrm{ind}(G, X, D) \neq 0$, *then G has a fixed point in D.*
(ii) *(Localization) If $D' \subseteq D$ is an open subset of X such that* $\mathrm{Fix}(G) \subset D'$, *then*

$$\mathrm{ind}(G, X, D) = \mathrm{ind}(G|_{D'}, X, D').$$

(iii) *(Homotopy) If there exists a compact homotopy $H : X \times [0, 1] \multimap X$ (in the same class of mappings under consideration) with $H(\cdot, 0) = G_1$ and $H(\cdot, 1) = G_2$ and if ∂D is fixed point free w.r.t. H, then*

$$\mathrm{ind}(G_1, X, D) = \mathrm{ind}(G_2, X, D).$$

(iv) *(Normalization) If $X = D$, then*

$$\mathrm{ind}(G, X, D) = \mathrm{ind}(G, X, X) = \Lambda(G),$$

where $\Lambda(G)$ is a generalized Lefschetz number of G (for its definition, see, e.g., [1,2,24]*).*
(v) *(Contraction) If $X' \subset X$ are ANR-spaces such that $G(X) \subset X'$ and $G|_{X'}$ is a compact R_δ-mapping such that* $\mathrm{Fix}(G|_{X'}) \cap \partial(D \cap X') = \emptyset$, *then*

$$\mathrm{ind}(G, X, D) = \mathrm{ind}(G|_{X'}, X', D \cap X').$$

Remark 1. If $G : X \multimap X$ is a compact R_δ-mapping and X is an AR-space, then $\Lambda(G) = 1$ (see, e.g., [1,2,24]).

Now, it will be convenient to recall some results which are needed in the sequel.

Lemma 1 *(Cf. [7, Theorem 0.3.4]). Let $K \subset \mathbb{R}$ be a compact interval. Assume that the sequence of absolutely continuous functions $x_k : K \to \mathbb{R}^n$ satisfies the following conditions:*

(i) *the set $\{x_k(t) \mid k \in \mathbb{N}\}$ is bounded, for every $t \in K$,*
(ii) *there exists a function $\alpha : K \to \mathbb{R}$, integrable in the sense of Lebesgue, such that*

$$|\dot{x}_k(t)| \le \alpha(t), \quad \text{for a.a. } t \in K \text{ and for all } k \in \mathbb{N}.$$

Then there exists a subsequence of $\{x_k\}$ (for the sake of simplicity denoted in the same way as the sequence) converging to an absolutely continuous function $x : K \to \mathbb{R}^n$ in the following way:

(iii) *$\{x_k\}$ converges uniformly to x,*
(iv) *$\{\dot{x}_k\}$ converges weakly in $L^1(K, \mathbb{R}^n)$ to $\dot{x}$.*

Lemma 2 *(Mazur's Lemma, Cf. e.g.,[40, p. 120]). Let E be a normed space and let the sequence $\{x_i\} \subset E$ be weakly convergent to $x \in E$. Then, for each $i \in \mathbb{N}$, there exists $N(i) \in \mathbb{N}$ such that a sequence $\{y_i\}$ of linear combinations of $\{x_i\}$,*

$$y_i = \sum_{k=i}^{N(i)} a_{i_k} x_k, \quad \text{where } a_{i_k} \ge 0, \text{ for each } i \le k \le N(i), \quad \text{and } \sum_{k=i}^{N(i)} a_{i_k} = 1,$$

is strongly convergent to x.

The following lemma is well-known for a Banach space $E = E_1 = E_2$ (see, e.g., [39, p. 88]). For the sake of completeness, we shall prove its slight modification for E_1 not necessarily equal to E_2.

Lemma 3. *Let $[a, b] \subset \mathbb{R}$ be a compact interval, let E_1, E_2 be Banach spaces and let $F : [a, b] \times E_1 \multimap E_2$ be a multivalued mapping satisfying the following conditions:*

(i) *$F(\cdot, x)$ has a strongly measurable selection, for every $x \in E_1$,*
(ii) *$F(t, \cdot)$ is u.s.c., for a.a. $t \in [a, b]$,*
(iii) *the set $F(t, x)$ is compact and convex, for all $(t, x) \in [a, b] \times E_1$.*

Assume in addition that, for every nonempty, bounded set $\Omega \subset E_1$, there exists $\nu = \nu(\Omega) \in L^1([a, b], \mathbb{R}^+)$ such that

$$|F(t, x)| \le \nu(t),$$

for a.a. $t \in [a, b]$ and every $x \in \Omega$.

Let us define the Nemytskiĭ operator $N_F : C([a, b], E_1) \multimap L^1([a, b], E_2)$ in the following way

$$N_F(x) := \{f \in L^1([a, b], E_2) \mid f(t) \in F(t, x(t)), \text{ a.e. on } [a, b]\},$$

for every $x \in C([a, b], E_1)$. Then, if sequences $\{x_i\} \subset C([a, b], E_1)$ and $\{f_i\} \subset L^1([a, b], E_2)$, $f_i \in N_F(x_i)$, $i \in \mathbb{N}$, are such that $x_i \to x$ in $C([a, b], E_1)$ and $f_i \to f$ weakly in $L^1([a, b], E_2)$, then $f \in N_F(x)$.

Proof. According to Mazur's Lemma, the weak convergence of $\{f_i\}_{i=1}^\infty$ to f implies the existence of a sequence of nonnegative numbers $\{a_{i_k}\}_{i=1}^\infty$ such that

- $\sum_{k=i}^\infty a_{i_k} = 1$, for all $i \in \mathbb{N}$,
- for every $i \in \mathbb{N}$, there exists a number $k_0(i)$ such that $a_{i_k} = 0$, for all $k \ge k_0(i)$,
- the sequence $\{\tilde{f}_i\}_{i=1}^\infty$, where $\tilde{f}_i$ are defined by

$$\tilde{f}_i(t) = \sum_{k=i}^\infty a_{i_k} f_k(t),$$

converges to f with respect to the norm of the space $L^1([a, b], E_2)$.

Passing to a subsequence, if necessary, we can assume that $\{\tilde{f}_i\}_{i=1}^\infty$ converges to f almost everywhere on $[a, b]$ (see, e.g., [15, p. 34]).

It follows from assumption (ii) that, for a.a. $t \in [a, b]$, and for a given $\varepsilon > 0$, there exists an integer $i_0 = i_0(\varepsilon, t)$ such that

$$F(t, x_i(t)) \subset N_\varepsilon(F(t, x(t))), \quad \text{for all } i \ge i_0.$$

Then

$$f_i(t) \in N_\varepsilon(F(t, x(t))), \quad \text{for all } i \ge i_0,$$

and hence also

$$\tilde{f}_i(t) \in N_\varepsilon(F(t, x(t))), \quad \text{for all } i \geq i_0.$$

Therefore,

$$f(t) \in F(t, x(t)), \quad \text{for a.a. } t \in [a, b],$$

i.e. $f \in N_F(x)$. $\square$

3. General method

In this section, we consider the second-order boundary value problem of the following general form

$$\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in J, x \in S, \tag{5}$$

where $J = [t_0, t_1]$ is a given compact interval, $F : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping, i.e. the map $F(\cdot, x, y) : J \multimap \mathbb{R}^n$ is measurable, for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, the map $F(t, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is u.s.c., for almost all $t \in J$, and the set $F(t, x, y)$ is compact and convex, for all $(t, x, y) \in J \times \mathbb{R}^n \times \mathbb{R}^n$. Furthermore, we assume that $S \subset AC^1(J, \mathbb{R}^n)$.

By a *solution* of problem (5), we mean a function $x : J \to \mathbb{R}^n$ belonging to $AC^1(J, \mathbb{R}^n)$ and satisfying (5), for almost all $t \in J$.

For the main result of this section (see Theorem 1), the following proposition is crucial.

Proposition 4. *Let $J = [t_0, t_1]$, $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping and let S be a nonempty subset of $AC^1(J, \mathbb{R}^n)$. Assume that*

(i) *there exists a subset Q of $C^1(J, \mathbb{R}^n)$ such that, for any $q \in Q$, the set $\mathfrak{T}(q)$ of all solutions of the boundary value problem*

$$\ddot{x}(t) \in G(t, x(t), \dot{x}(t), q(t), \dot{q}(t)), \quad \text{for a.a. } t \in J, x \in S \tag{6}$$

is nonempty,

(ii) *there exist constants $M_0 > 0$, $M_1 > 0$ such that $|x(t_0)| \leq M_0$ and $|\dot{x}(t_0)| \leq M_1$, for any $x \in \mathfrak{T}(Q)$,*

(iii) *there exists a nonnegative, integrable function $\alpha : J \to \mathbb{R}$ such that*

$$|G(t, x(t), \dot{x}(t), q(t), \dot{q}(t))| := \sup\{|y| \mid y \in G(t, x(t), \dot{x}(t), q(t), \dot{q}(t))\} \leq \alpha(t), \quad \text{a.e. in } J, \text{ for any } (q, x) \in \Gamma_{\mathfrak{T}}.$$

Then $\mathfrak{T}(Q)$ is a relatively compact subset of $C^1(J, \mathbb{R}^n)$. Moreover, the multivalued operator $\mathfrak{T} : Q \multimap S$ is u.s.c. with compact values if and only if the following condition is satisfied:

(iv) *for each sequence $\{q_k, x_k\} \subset \Gamma_{\mathfrak{T}}$ satisfying $\{(q_k, \dot{q}_k, x_k)\} \to (q, \dot{q}, x)$, where $q \in Q$, it holds that $x \in S$.*

Proof. We begin with showing the integral form of a solution of the inclusion

$$\ddot{x}(t) \in G(t, x(t), \dot{x}(t), q(t), \dot{q}(t)). \tag{7}$$

Integrating (7) in the sense of Aumann (see, e.g., [8,28]), we get

$$\dot{x}(t) - \dot{x}(t_0) \in \int_{t_0}^{t} G(s, x(s), \dot{x}(s), q(s), \dot{q}(s))\mathrm{d}s, \tag{8}$$

and consequently

$$x(t) - \dot{x}(t_0) \cdot (t - t_0) - x(t_0) \in \int_{t_0}^{t} \int_{t_0}^{s} G(\tau, x(\tau), \dot{x}(\tau), q(\tau), \dot{q}(\tau))\mathrm{d}\tau\mathrm{d}s. \tag{9}$$

Moreover, integrating (9) by parts, we obtain

$$x(t) - \dot{x}(t_0) \cdot (t - t_0) - x(t_0) \in \left[s \int_{t_0}^{s} G(\tau, x(\tau), \dot{x}(\tau), q(\tau), \dot{q}(\tau))\mathrm{d}\tau\right]_{t_0}^{t} - \int_{t_0}^{t} sG(s, x(s), \dot{x}(s), q(s), \dot{q}(s))\mathrm{d}s \tag{10}$$

and, therefore,

$$x(t) \in x(t_0) + \dot{x}(t_0) \cdot (t - t_0) + \int_{t_0}^{t} (t - s) \cdot G(s, x(s), \dot{x}(s), q(s), \dot{q}(s))\mathrm{d}s \tag{11}$$

is the integral form of a solution of the inclusion (7).

It follows from the well-known Arzelà–Ascoli lemma that the set $\mathfrak{T}(Q)$ is relatively compact in $C^1([a, b], \mathbb{R}^n)$ if and only if it is bounded and functions in $\mathfrak{T}(Q)$ and their first derivatives are equi-continuous.

At first, we show that the set $\mathfrak{T}(Q)$ is bounded. Let $x \in \mathfrak{T}(Q)$ and $t \in [t_0, t_1]$ be arbitrary. Then, according to (11) and (iii), we have

$$\begin{aligned} |x(t)| &\leq |x(t_0)| + |\dot{x}(t_0)| \cdot |t - t_0| + \int_{t_0}^{t} |t - s| \cdot |G(s, x(s), \dot{x}(s), q(s), \dot{q}(s))| \mathrm{d}s \\ &\leq M_0 + M_1 \cdot |t_1 - t_0| + |t_1 - t_0| \int_{t_0}^{t} \alpha(s) \mathrm{d}s. \end{aligned} \tag{12}$$

Furthermore, according to (8) and (iii),

$$|\dot{x}(t)| \leq |\dot{x}(t_0)| + \int_{t_0}^{t} |G(s, x(s), \dot{x}(s), q(s), \dot{q}(s))| \mathrm{d}s \leq M_1 + \int_{t_0}^{t} \alpha(s) \mathrm{d}s. \tag{13}$$

It immediately follows from estimates (12) and (13) that $\mathfrak{T}(Q)$ is bounded.

Therefore, for the relative compactness of $\mathfrak{T}(Q)$, it is sufficient to show that all elements of $\mathfrak{T}(Q)$ and their first derivatives are equi-continuous.

Let $x \in \mathfrak{T}(Q)$ and $t_2, t_3 \in J$ be arbitrary. Then, according to (11), we obtain

$$\begin{aligned} |x(t_3) - x(t_2)| &\leq |\dot{x}(t_0)| \cdot |t_3 - t_2| \\ &\quad + \left| \int_{t_0}^{t_3} (t_3 - s) \cdot G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s - \int_{t_0}^{t_2} (t_2 - s) \cdot G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s \right| \\ &= |\dot{x}(t_0)| \cdot |t_3 - t_2| \\ &\quad + \left| \int_{t_0}^{t_3} (t_3 - s) \cdot G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s - \int_{t_0}^{t_3} (t_2 - s) \cdot G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s \right. \\ &\quad \left. + \int_{t_0}^{t_3} (t_2 - s) \cdot G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s - \int_{t_0}^{t_2} (t_2 - s) \cdot G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s \right| \\ &\leq |\dot{x}(t_0)| \cdot |t_3 - t_2| \\ &\quad + \left| \int_{t_0}^{t_3} (t_3 - t_2) \cdot G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s \right| + \left| \int_{t_3}^{t_2} (t_2 - s) \cdot G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s \right| \\ &\leq |\dot{x}(t_0)| \cdot |t_3 - t_2| \\ &\quad + \int_{t_0}^{t_3} |t_3 - t_2| \cdot |G(s, x(s), \dot{x}(s), q(s), \dot{q}(s))| \mathrm{d}s + \left| \int_{t_3}^{t_2} |t_2 - s| \cdot |G(s, x(s), \dot{x}(s), q(s), \dot{q}(s))| \mathrm{d}s \right| \\ &\leq M_1 \cdot |t_3 - t_2| + \int_{t_0}^{t_3} |t_3 - t_2| \cdot \alpha(s)\, \mathrm{d}s + \left| \int_{t_3}^{t_2} |t_2 - s| \cdot \alpha(s) \mathrm{d}s \right|. \end{aligned} \tag{14}$$

Moreover, according to (8), we have

$$\begin{aligned} |\dot{x}(t_3) - \dot{x}(t_2)| &\leq \left| \int_{t_0}^{t_3} G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s - \int_{t_0}^{t_2} G(s, x(s), \dot{x}(s), q(s), \dot{q}(s)) \mathrm{d}s \right| \\ &\leq \left| \int_{t_3}^{t_2} |G(s, x(s), \dot{x}(s), q(s), \dot{q}(s))| \mathrm{d}s \right| \leq \left| \int_{t_3}^{t_2} \alpha(s) \mathrm{d}s \right|. \end{aligned} \tag{15}$$

Taking into account estimates (14) and (15), x and $\dot{x}$ are equi-continuous, for each $x \in \mathfrak{T}(Q)$, because $\alpha(\cdot) \in L^1(J, \mathbb{R})$. Thus, $\mathfrak{T}(Q)$ is relatively compact.

We now show that the graph of the operator $\mathfrak{T}$ is closed. Let $\{(q_k, x_k)\} \subset \Gamma_{\mathfrak{T}}$ be such that $\{(q_k, \dot{q}_k, x_k)\} \to (q, \dot{q}, x)$, where $q \in Q$. For all $k \in \mathbb{N}$ and a.a. $t \in [t_0, t_1]$, we have

$$|\dot{x}_k(t)| \leq |\dot{x}_k(t_0)| + \int_{t_0}^{t} |G(s, x_k(s), \dot{x}_k(s), q_k(s), \dot{q}_k(s))| \mathrm{d}s \leq |\dot{x}_k(t_0)| + \int_{t_0}^{t} \alpha(s) \mathrm{d}s.$$

From condition (iii), it follows that there exists a constant $K_{[t_0,t_1]}$ such that

$$\int_{t_0}^{t_1} \alpha(s) \mathrm{d}s \leq K_{[t_0,t_1]}.$$

Therefore, for all $k \in \mathbb{N}$ and for a.a. $t \in [t_0, t_1]$,

$$|\dot{x}_k(t)| \leq |\dot{x}_k(t_0)| + \int_{t_0}^{t_1} \alpha(s) \mathrm{d}s \leq M_1 + K_{[t_0,t_1]}, \tag{16}$$

according to assumption (ii). By conditions (iii) and (16), the sequence $\{y_k := \dot{x}_k\}$ satisfies assumptions of Lemma 1.

Thus, applying Lemma 1 to the sequence $\{y_k := \dot{x}_k\}$, we get that there exists a subsequence of $\{\dot{x}_k\}$, for the sake of simplicity denoted in the same way as the sequence, which converges uniformly to $\dot{x}$ on $[t_0, t_1]$ and such that $\{\ddot{x}_k\}$ converges weakly to $\ddot{x}$ in $L^1([t_0, t_1], \mathbb{R}^n)$.

If we set $z_k := (x_k, y_k)$, then $\dot{z}_k = (\dot{x}_k, \dot{y}_k) = (\dot{x}_k, \ddot{x}_k) \to (\dot{x}, \ddot{x})$ weakly in $L^1([t_0, t_1], \mathbb{R}^n)$. Let us now consider the system

$$\dot{z}_k(t) \in H(t, z_k(t), q_k(t), \dot{q}_k(t)), \quad \text{for a.a. } t \in [t_0, t_1], \tag{17}$$

where $\dot{z}_k(t) = (\dot{x}_k(t), \dot{y}_k(t))$ and $H(t, z_k(t), q_k(t), \dot{q}_k(t)) = (y_k(t), G(t, x_k(t), \dot{x}_k(t), q_k(t), \dot{q}_k(t)))$.

Applying Lemma 3, for $f_i := \dot{z}_k, f := (\dot{x}, \ddot{x}),\ x_i := (z_k, q_k, \dot{q}_k)$, it follows that

$$(\dot{x}(t), \ddot{x}(t)) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t)),$$

for a.a. $t \in [t_0, t_1]$, i.e.

$$\ddot{x}(t) \in G(t, x(t), \dot{x}(t), q(t), \dot{q}(t)), \text{ for a.a. } t \in [t_0, t_1].$$

Condition (iv) implies that $x \in S$, and therefore $\Gamma_{\mathfrak{T}}$ is closed. Moreover, it immediately follows from Proposition 1 that the operator $\mathfrak{T}$ is u.s.c.

Since $\mathfrak{T}$ is a compact mapping, $\mathfrak{T}(q)$ is, for each $q \in Q$, a relatively compact set. Moreover, the operator $\mathfrak{T}$ has a closed graph which implies that $\mathfrak{T}(q)$ is, for each $q \in Q$, closed, and therefore $\mathfrak{T}$ has compact values. □

Remark 2. Sometimes, the estimate for the solution can imply the one for its derivative, provided the right-hand side (shortly, r.h.s.) of a given inclusion satisfies suitable growth restrictions. For instance, if the r.h.s. is entirely bounded by a constant, then the boundedness of derivatives follows from the boundedness of solutions by means of the well-known Landau inequality:

$$|\dot{x}(t)| \le 2\left[|x(t)||\ddot{x}(t)|\right]^{\frac{1}{2}}.$$

The same conclusion holds if, more generally, the r.h.s. satisfies the Bernstein–Nagumo type condition. For more details about these conditions, see, e.g., [11] for periodic b.v.p., [18] for Sturm–Liouville b.v.p. or [34] for Dirichlet b.v.p.

As the main result of this section, we can now formulate the following theorem ensuring the existence of a solution of the boundary value problem (5).

Theorem 1. *Let us consider the boundary value problem (5), where $J = [t_0, t_1]$ is a compact interval, $F : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a multivalued upper-Carathéodory mapping and S is a subset of $AC^1(J, \mathbb{R}^n)$.*

Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ be an upper-Carathéodory map such that

$$G(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in J \times \mathbb{R}^n \times \mathbb{R}^n, \tag{18}$$

and assume that

(i) *there exists a retract Q of $C^1(J, \mathbb{R}^n)$ such that $Q \setminus \partial Q$ is nonempty and a closed subset S_1 of S such that the associated problem*

$$\ddot{x}(t) \in G(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in J, x \in S_1 \tag{19}$$

is solvable with an R_δ-set of solutions, for each $(q, \lambda) \in Q \times [0, 1]$,

(ii) *there exists a nonnegative, integrable function $\alpha : J \to \mathbb{R}$ such that*

$$|G(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)| \le \alpha(t), \quad \text{a.e. in } J,$$

for any $(q, \lambda, x) \in \Gamma_{\mathfrak{T}}$, where $\mathfrak{T}$ denotes the multivalued mapping which assigns to any $(q, \lambda) \in Q \times [0, 1]$ the set of solutions of (19),

(iii) $\mathfrak{T}(Q \times \{0\}) \subset Q$,

(iv) *there exist constants $M_0 > 0,\ M_1 > 0$ such that $|x(t_0)| \le M_0$ and $|\dot{x}(t_0)| \le M_1$, for any $x \in \mathfrak{T}(Q \times [0, 1])$,*

(v) *the solution map $\mathfrak{T}$ has no fixed points on the boundary ∂Q of Q, for every $(q, \lambda) \in Q \times [0, 1]$.*

Then problem (5) has a solution.

Proof. At first, we show that all assumptions of Proposition 4 are satisfied. Conditions (i), (ii) and (iv) guarantee the assumptions (i), (ii) and (iii) of Proposition 4.

Let $\{(q_k, \lambda_k, x_k)\} \subset \Gamma_{\mathfrak{T}}, \{(q_k, \lambda_k, x_k)\} \to (q, \lambda, x), (q, \lambda) \in Q \times [0, 1]$ be arbitrary. Then, since $x_k \in S_1, x_k \to x$ and S_1 is closed, it holds that $x \in S_1$. Therefore, assumption (iv) from Proposition 4 is satisfied as well. Thus, $\mathfrak{T} : Q \times [0, 1] \multimap C^1(J, \mathbb{R}^n)$ is, according to Proposition 4, a compact u.s.c. mapping with compact values. Moreover, according to assumptions (i) and (v), $\mathfrak{T}$ has R_δ-values and it does not have fixed points on the boundary of Q.

Since Q is a retract of the space $C^1(J, \mathbb{R}^n)$, there exists an extension $\tilde{\mathfrak{T}} : C^1(J, \mathbb{R}^n) \times [0, 1] \multimap C^1(J, \mathbb{R}^n)$ of mapping $\mathfrak{T} : Q \times [0, 1] \multimap C^1(J, \mathbb{R}^n)$. Precisely, $\tilde{\mathfrak{T}}$ can be defined in such a way that it is a compact u.s.c. mapping with R_δ-values without any fixed points on ∂Q and such that $\tilde{\mathfrak{T}}|_{Q \times [0,1]} = \mathfrak{T}$ and $\tilde{\mathfrak{T}}(C^1(J, \mathbb{R}^n), \{0\}) \subset Q$.

Since $\tilde{\mathfrak{T}}$ is a homotopy, we obtain, using the homotopy property of the index (cf. condition (iii) in Proposition 3), for $X = C^1(J, \mathbb{R}^n)$, $D = \operatorname{int} Q$ and $H(\cdot, \cdot\cdot) = \tilde{\mathfrak{T}}(\cdot, \cdot\cdot)$, that

$$\operatorname{ind}(\tilde{\mathfrak{T}}(\cdot, 1), C^1(J, \mathbb{R}^n), \operatorname{int} Q) = \operatorname{ind}(\tilde{\mathfrak{T}}(\cdot, 0), C^1(J, \mathbb{R}^n), \operatorname{int} Q).$$

Moreover, thanks to the contraction property of the index (cf. condition (v) in Proposition 3), for $X = C^1(J, \mathbb{R}^n)$, $X' = Q$, $D = \operatorname{int} Q$ and $G(\cdot) = \tilde{\mathfrak{T}}(\cdot, 0)$, we have

$$\operatorname{ind}(\tilde{\mathfrak{T}}(\cdot, 0), C^1(J, \mathbb{R}^n), \operatorname{int} Q) = \operatorname{ind}(\mathfrak{T}(\cdot, 0), Q, \operatorname{int} Q).$$

Using the localization property of the index (cf. condition (ii) in Proposition 3), for $D = D' = \operatorname{int} Q$, $X = Q$ and $G(\cdot) = \mathfrak{T}(\cdot, 0)$, we get

$$\operatorname{ind}(\mathfrak{T}(\cdot, 0), Q, \operatorname{int} Q) = \operatorname{ind}(\mathfrak{T}(\cdot, 0)|_{\operatorname{int} Q}, Q, \operatorname{int} Q).$$

Applying the localization property once again, this time for $D' = \operatorname{int} Q$, $D = X = Q$ and $G(\cdot) = \mathfrak{T}(\cdot, 0)$, we obtain

$$\operatorname{ind}(\mathfrak{T}(\cdot, 0)|_{\operatorname{int} Q}, Q, \operatorname{int} Q) = \operatorname{ind}(\mathfrak{T}(\cdot, 0), Q, Q).$$

Since Q is a retract of the Banach space $C^1(J, \mathbb{R}^n)$, i.e. an *AR*-space, and $\mathfrak{T}(\cdot, 0)$ is a compact, u.s.c. mapping with R_δ-values, we arrive at

$$\operatorname{ind}(\tilde{\mathfrak{T}}(\cdot, 1), C^1(J, \mathbb{R}^n), \operatorname{int} Q) = \operatorname{ind}(\mathfrak{T}(\cdot, 0), Q, Q) = \Lambda(\mathfrak{T}(\cdot, 0)) = 1.$$

Existence property of the index (cf. condition (i) in Proposition 3) implies that there exists a fixed point of $\tilde{\mathfrak{T}}(\cdot, 1)$ in $\operatorname{int} Q$, i.e a fixed point of $\mathfrak{T}(\cdot, 1)$. By inclusion (18), it is a solution of problem (5) which completes the proof. □

Remark 3. The condition that Q is a retract of $C^1(J, \mathbb{R}^n)$ in Theorem 1 can be replaced by an assumption that Q is an absolute neighborhood retract and $\operatorname{ind}(\mathfrak{T}(\cdot, 0), Q, Q) \neq 0$. It is therefore possible to assume alternatively that Q is a retract of a convex subset of $C^1(J, \mathbb{R}^n)$ or of an open subset of $C^1(J, \mathbb{R}^n)$ together with $\operatorname{ind}(\mathfrak{T}(\cdot, 0), Q, Q) \neq 0$.

Remark 4. It can be readily checked that the assumption of Theorem 1 concerning R_δ-values can be replaced by any of its particular cases in (4). Moreover, for Dirichlet problems, explicit conditions were obtained in [14], and still improved in [35], for the topological structure of solutions to (19) to be a compact *AR*-space.

The following corollary deals with the special case when $S_1 \subset S \cap Q$.

Corollary 1. *Let us consider the boundary value problem (5), where $J = [t_0, t_1]$ is a given compact interval, $F : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a multivalued upper-Carathéodory mapping and S is a subset of $AC^1(J, \mathbb{R}^n)$.*

Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory map such that

$$G(t, c, d, c, d) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in J \times \mathbb{R}^n \times \mathbb{R}^n, \tag{20}$$

and assume that

(i) *there exists a retract Q of $C^1(J, \mathbb{R}^n)$ such that the associated problem*

$$\ddot{x}(t) \in G(t, x(t), \dot{x}(t), q(t), \dot{q}(t)), \quad \text{for a.a. } t \in J, x \in S \cap Q \tag{21}$$

is solvable with an R_δ-set of solutions, for each $q \in Q$,

(ii) *there exists a nonnegative, integrable function $\alpha : J \to \mathbb{R}$ such that*

$$|G(t, x(t), \dot{x}(t), q(t), \dot{q}(t))| \le \alpha(t), \quad \text{a.e. in } J,$$

for any $(q, x) \in \Gamma_{\mathfrak{T}}$, where $\mathfrak{T}$ denotes the multivalued map which assigns to any $q \in Q$ the set of solutions of (21),

(iii) $\overline{\mathfrak{T}(Q)} \subset S$,

(iv) *there exist constants $M_0 > 0$, $M_1 > 0$ such that $|x(t_0)| \le M_0$ and $|\dot{x}(t_0)| \le M_1$, for any $x \in \mathfrak{T}(Q)$.*

Then problem (5) has a solution.

Proof. Let $\mathfrak{T} : Q \multimap S \cap Q$ be the solution operator defined as in Proposition 4. Conditions (i), (ii) and (iv) guarantee the assumptions (i)–(iii) of Proposition 4. Since $\overline{\mathfrak{T}(Q)} \subset S$, hypothesis (iv) of Proposition 4 is satisfied as well. Hence, $\mathfrak{T}$ is, according to Proposition 4 and assumption (i), a compact u.s.c. mapping with R_δ-values. Moreover, since $\mathfrak{T} : Q \multimap Q$, where Q is an *AR*-space, the generalized Lefschetz number satisfies $\Lambda(\mathfrak{T}) = 1$ (cf. Remark 1). Applying the Lefschetz Theorem (cf., e.g., [2, p. 96]), we conclude that $\mathfrak{T}$ has a fixed point which is, by inclusion (20), a solution of problem (5). □

Remark 5. Let us note that in the single-valued case of ordinary differential equations, it is sufficient to assume in Theorem 1(i) and Corollary 1(i) that the related linearized problems are uniquely solvable.

4. Bound sets for Floquet problem

In this section, we are interested in constructing a Liapunov-like function V, usually called a *bounding function*, guaranteeing suitable transversality conditions which assure that there does not exist a solution of the boundary value problem lying in a closed set $\overline{K}$ and being tangential at some point of the boundary ∂K of K.

Let us consider the Floquet boundary value problem

$$\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [t_0, t_1], \tag{22}$$

$$x(t_1) = Mx(t_0), \tag{23}$$
$$\dot{x}(t_1) = N\dot{x}(t_0),$$

where

(i) $A, B : [t_0, t_1] \to \mathbb{R}^n \times \mathbb{R}^n$ are Lebesque integrable matrix functions,
(ii) $F : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping,
(iii) M and N are real $n \times n$ matrices with M non-singular.

Let $K \subset \mathbb{R}^n$ be a nonempty open set and let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying

(H1) $V|_{\partial K} = 0$,
(H2) $V(x) \leq 0$, for all $x \in \overline{K}$.

Definition 1. A nonempty open set $K \subset \mathbb{R}^n$ is called a *bound set* for the b.v.p. (22) and (23) if every solution x of (22) and (23) such that $x(t) \in \overline{K}$, for each $t \in [t_0, t_1]$, does not satisfy $x(t^*) \in \partial K$, for any $t^* \in [t_0, t_1]$.

Proposition 5. *Let $K \subset \mathbb{R}^n$ be a nonempty open set, $F : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping and let A and B be integrable matrix functions. Let M and N be real $n \times n$ matrices with M non-singular and such that $M\partial K = \partial K$. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with ∇V locally Lipschitzian and satisfying conditions* (H1) *and* (H2). *Suppose, moreover, that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (t_0, t_1)$ and $v \in \mathbb{R}^n$, the following condition*

$$\limsup_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle - \langle \nabla V(x), v \rangle}{h} > 0 \tag{24}$$

holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$, and that

$$\langle \nabla V(My), Nw \rangle \cdot \langle \nabla V(y), w \rangle > 0, \quad \text{or} \quad \langle \nabla V(My), Nw \rangle = \langle \nabla V(y), w \rangle = 0, \tag{25}$$

for all $y \in \partial K$ and $w \in \mathbb{R}^n$. Then all possible solutions $x : [t_0, t_1] \to \overline{K}$ of problem (22) *and* (23) *are such that $x(t) \in K$, for every $t \in [t_0, t_1]$, i.e. K is a bound set for the Floquet problem* (22) *and* (23).

Proof. Let $x : [t_0, t_1] \to \overline{K}$ be a solution of problem (22) and (23). We assume, by a contradiction, that there exists $t^* \in [t_0, t_1]$ such that $x(t^*) \in \partial K$. According to the boundary condition (23) and since $M\partial K = \partial K$, we can take, without any loss of generality, $t^* \in (t_0, t_1]$.

Since ∇V is locally Lipschitzian, there exist a bounded set $U \subset \mathbb{R}^n$ with $x(t^*) \in U$ and a constant $L > 0$ such that $\nabla V|_U$ is Lipschitzian with constant L. Let $\delta > 0$ be such that $x(t) \in U \cap N_\varepsilon(\partial K)$, for each $t \in [t^* - \delta, t^*]$.

Let us define the function $g : [t_0, t_1] \to \mathbb{R}$ by the following formula: $g(t) := V(x(t))$. According to the regularity properties of x and V, $g \in C^1([t_0, t_1], \mathbb{R})$. Since $g(t^*) = 0$ and $g(t) \leq 0$, for all $t \in [t_0, t_1]$, t^* is a local maximum point for g. Therefore, $\dot{g}(t^*) \geq 0$ and $\dot{g}(t^*) = 0$ when $t^* \in (t_0, t_1)$. Moreover, there exists a point $t^{**} \in (t^* - \delta, t^*)$ such that $\dot{g}(t^{**}) \geq 0$.

According to boundary conditions, if $t^* = t_1$, then also $x(t_0) \in \partial K$ and

$$\dot{g}(t_0) = \langle \nabla V(x(t_0)), \dot{x}(t_0) \rangle \leq 0.$$

Moreover, since $x(t_1) = Mx(t_0)$ and $\dot{x}(t_1) = N\dot{x}(t_0)$, we have

$$\dot{g}(t_1) = \langle \nabla V(x(t_1)), \dot{x}(t_1) \rangle = \langle \nabla V(Mx(t_0)), N\dot{x}(t_0) \rangle \geq 0.$$

Condition (25) then implies

$$\langle \nabla V(x(t_0)), \dot{x}(t_0) \rangle = \langle \nabla V(Mx(t_0)), N\dot{x}(t_0) \rangle = 0.$$

This is equivalent to $\dot{g}(t_0) = \dot{g}(t_1) = 0$.

Since $\dot{g}(t) = \langle \nabla V(x(t)), \dot{x}(t) \rangle$, where $\nabla V(x(t))$ is locally Lipschitzian and $\dot{x}(t)$ is absolutely continuous on $[t^* - \delta, t^*]$, $\ddot{g}(t)$ exists, for a.a. $t \in [t^* - \delta, t^*]$. Consequently,

$$0 \geq -\dot{g}(t^{**}) = \dot{g}(t^*) - \dot{g}(t^{**}) = \int_{t^{**}}^{t^*} \ddot{g}(s)\mathrm{d}s. \tag{26}$$

Let $t \in (t^{**}, t^*)$ be such that $\ddot{g}(t)$ and $\ddot{x}(t)$ exist. Then,

$$\lim_{h\to 0} \frac{\dot{x}(t+h) - \dot{x}(t)}{h} = \ddot{x}(t)$$

and, therefore, there exists a function $a(h)$, $a(h) \to 0$ as $h \to 0$, such that, for each h,

$$\dot{x}(t+h) = \dot{x}(t) + h[\ddot{x}(t) + a(h)]. \tag{27}$$

Moreover, since $x \in C^1([t_0, t_1], \mathbb{R}^n)$, there exists a function $b(h)$, $b(h) \to 0$ as $h \to 0$, such that, for each h,

$$x(t+h) = x(t) + h[\dot{x}(t) + b(h)]. \tag{28}$$

Consequently, we obtain

$$\begin{aligned}
\ddot{g}(t) &= \lim_{h\to 0} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} = \limsup_{h\to 0^-} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} \\
&= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t+h)), \dot{x}(t+h)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h[\dot{x}(t) + b(h)]), \dot{x}(t) + h[\ddot{x}(t) + a(h)]\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&\geq \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h[\ddot{x}(t) + a(h)]\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&\quad - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| \\
&= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&\quad - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| + \langle \nabla V(x(t) + h\dot{x}(t)), a(h)\rangle .
\end{aligned}$$

Since

$$\langle \nabla V(x(t) + h\dot{x}(t)), a(h)\rangle - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| \to 0 \quad \text{as } h \to 0,$$

$$\ddot{g}(t) \geq \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} > 0,$$

according to assumption (24), which leads to a contradiction with the inequality (26). □

Remark 6. If condition (24) is replaced by the following one

$$\limsup_{h\to 0^+} \frac{\langle \nabla V(x + hv), v + hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0, \tag{29}$$

for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (t_0, t_1)$, $v \in \mathbb{R}^n$ and $w \in F(t, x, v) - A(t)v - B(t)x$, while all the other assumptions of Proposition 5 remain valid, then the same conclusion holds.

In this case, in fact, we can take, without any loss of generality, $t^* \in [t_0, t_1)$. By a similar reasoning as in the proof of Proposition 5, we are able to find a point $t^{***} \in (t^*, t^* + \delta)$ such that $\dot{g}(t^{***}) \leq 0$.

Consequently,

$$0 \geq \dot{g}(t^{***}) = \dot{g}(t^{***}) - \dot{g}(t^*) = \int_{t^*}^{t^{***}} \ddot{g}(s)\mathrm{d}s. \tag{30}$$

Let $t \in (t^*, t^{***})$ be such that $\ddot{g}(t)$ and $\ddot{x}(t)$ exist. Using the same procedure as before, we can prove that

$$\ddot{g}(t) \geq \limsup_{h\to 0^+} \frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} > 0,$$

which leads to a contradiction with the inequality (30).

Definition 2. A function $V : \mathbb{R}^n \to \mathbb{R}$ from Proposition 5 satisfying conditions (H1), (H2), (25) and at least one of conditions (24) and (29) is called a *bounding function* for the set K relative to (22) and (23).

In the case when $V \in C^2(\mathbb{R}^n, \mathbb{R})$, $\ddot{g}(t) = \langle HV(x(t))\dot{x}(t), \dot{x}(t)\rangle + \langle \nabla V(x(t)), \ddot{x}(t)\rangle$, where H denotes the Hesse second-order differential operator, and the following corollary immediately follows.

Corollary 2. *Let $K \subset \mathbb{R}^n$ be a nonempty open set, $F : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping and let A and B be integrable matrix functions. Let M and N be real $n \times n$ matrices with M non-singular and such that $M\partial K = \partial K$. Assume that there exists a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfying conditions* (H1) *and* (H2). *Moreover, assume that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (t_0, t_1)$ and $v \in \mathbb{R}^n$, condition*

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0$$

holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$, and that

$$\langle \nabla V(My), Nw\rangle \cdot \langle \nabla V(y), w\rangle > 0, \quad or \quad \langle \nabla V(My), Nw\rangle = \langle \nabla V(y), w\rangle = 0,$$

for all $y \in \partial K$ and $w \in \mathbb{R}^n$. Then K is a bound set for problem (22) and (23).

Let us now consider the case when, instead of one bounding function $V : \mathbb{R}^n \to \mathbb{R}$, there exists a one-parametric family of functions $V_\xi : \mathbb{R}^n \to \mathbb{R}$, $\xi \in \partial K$, satisfying

(H1′) $V_\xi(\xi) = 0$, for all $\xi \in \partial K$,
(H2′) $V_\xi(x) \le 0$, for all $x \in \overline{K}$, where x is located in a neighborhood of ξ.

In such a case, the proofs remain almost the same, when replacing V by $V_{x(t^*)}$, and Proposition 5 can be easily reformulated as follows.

Corollary 3. *Let $K \subset \mathbb{R}^n$ be a nonempty open set, $F : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping and let A and B be integrable matrix functions. Let M and N be real $n \times n$ matrices with M non-singular and such that $M\partial K = \partial K$. Assume that, for each $\xi \in \partial K$, there exists a function $V_\xi \in C^1(\mathbb{R}^n, \mathbb{R})$ with ∇V_ξ locally Lipschitzian and satisfying conditions* (H1′) *and* (H2′). *Moreover, let*

$$\left\langle \nabla V_{My}(My), Nw\right\rangle \cdot \left\langle \nabla V_y(y), w\right\rangle > 0, \quad or \quad \left\langle \nabla V_{My}(My), Nw\right\rangle = \left\langle \nabla V_y(y), w\right\rangle = 0, \tag{31}$$

for all $y \in \partial K$ and $w \in \mathbb{R}^n$. Furthermore, assume that, for all V_ξ, $\xi \in \partial K$, $x \in \overline{K}$ with x in a neighborhood of ξ, $t \in (t_0, t_1)$ and $v \in \mathbb{R}^n$, condition

$$\limsup_{h\to 0^-} \frac{\langle \nabla V_\xi(x + hv), v + hw\rangle - \langle \nabla V_\xi(x), v\rangle}{h} > 0 \tag{32}$$

holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$. Then K is a bound set for problem (22) and (23).

Remark 7. Let us note that it is obviously possible to replace condition (32) of the previous corollary by the following one

$$\limsup_{h\to 0^+} \frac{\langle \nabla V_\xi(x + hv), v + hw\rangle - \langle \nabla V_\xi(x), v\rangle}{h} > 0, \tag{33}$$

where x, ξ, v and w are the same as in Corollary 3.

Definition 3. A function $V_\xi : \mathbb{R}^n \to \mathbb{R}$ from Corollary 3 satisfying conditions (H1′), (H2′), (31) and at least one of conditions (32) and (33) is called a *bounding function*for the set K at ξ relative to (22) and (23).

We now supply illustrating examples and demonstrate how conditions ensuring the existence of a bound set change in particular cases.

Example 1. Given $R > 0$, put

$$K = \{x \in \mathbb{R}^n \mid |x| < R\}.$$

Let the function $V : \mathbb{R}^n \to \mathbb{R}$ be defined, for all $x \in \overline{K}$, as follows:

$$V(x) = \frac{1}{2}\left(|x|^2 - R^2\right). \tag{34}$$

Then V satisfies conditions (H1) and (H2), because

$$V(\xi) = 0$$

and

$$V(x) \le 0,$$

for all $\xi \in \partial K$ and all $x \in \overline{K}$. Moreover, for each $x \in \mathbb{R}^n$,

$$\nabla V(x) = x$$

and

$$HV(x) = I,$$

where I denotes the unit matrix.

Therefore, condition (24) can be reformulated in the following way: there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (t_0, t_1)$ and $v \in \mathbb{R}^n$, the inequality

$$\langle v, v\rangle + \langle x, w\rangle > 0$$

holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$.

Example 2. Let $K \subset \mathbb{R}^n$ be convex. Geometrically, it means that, for each $\xi \in \partial K$, there exist an outer normal n_ξ, not necessarily unique, and a neighborhood U_ξ of ξ such that

$$\langle n_\xi, (x - \xi)\rangle \le 0,$$

for each $x \in U_\xi \cap \overline{K}$.

Let, for each $\xi \in \partial K$, $V_\xi : \mathbb{R}^n \to \mathbb{R}$ be the C^2-function defined, for each $x \in \overline{K}$, by

$$V_\xi(x) = \langle n_\xi, (x - \xi)\rangle.$$

It follows immediately that V_ξ satisfies, for each $\xi \in \partial K$, conditions (H1′) and (H2′).

Moreover, for each $\xi \in \partial K$,

$$\nabla V_\xi(x) = n_\xi$$

and

$$HV_\xi(x) = 0.$$

Therefore, condition (32) takes the following form: for all V_ξ, $\xi \in \partial K$, $x \in \overline{K}$ with x in a neighborhood of ξ, $t \in (t_0, t_1)$ and $v \in \mathbb{R}^n$, the inequality

$$\langle n_\xi, w\rangle > 0$$

holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$.

Remark 8. Let us note that condition (25) depends both on the boundary conditions (23) and on the gradient ∇V of the bounding function V. In particular, (25) is trivially satisfied if $M = N = I$, where I denotes the $n \times n$ unit matrix. This case corresponds to the investigation of periodic solutions of the inclusion (22).

In the case if $My = my$ and $Nw = nw$, for all $y, w \in \mathbb{R}^n$, where $m, n \in \mathbb{R}$, and if V is defined by formula (34), it is easy to see that condition (25) is satisfied if and only if $mn > 0$.

5. Floquet boundary value problem

In this section, we investigate the boundary value problem (22) and (23) when combining the bound sets approach, developed in the previous section, with the continuation principle from Section 3.

For the main result concerning the existence of a solution of the second-order Floquet problem, we need to ensure that a set $Q \subset C^1([t_0, t_1], \mathbb{R}^n)$ is a retract of the space $C^1([t_0, t_1], \mathbb{R}^n)$ and also that no solution of the given b.v.p. is located on ∂Q. Firstly, we show that if Q is defined as follows

$$Q := \{q \in C^1([t_0, t_1], \mathbb{R}^n) \mid q(t) \in \overline{K},\ \text{for all } t \in [t_0, t_1]\} \tag{35}$$

and if all assumptions of Proposition 5 are satisfied, then no solution of the boundary value problem (22) and (23) belongs to ∂Q.

Proposition 6. *Let $K \subset \mathbb{R}^n$ be a nonempty open bounded set, $F : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping and let A and B be integrable matrix functions. Let M and N be real $n \times n$ matrices with M non-singular and such that $M\partial K = \partial K$. Furthermore, let $Q \subset C^1([t_0, t_1], \mathbb{R}^n)$ be defined by formula (35). Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with ∇V locally Lipschitzian and satisfying conditions* (H1) *and* (H2). *Suppose, moreover, that there exists $\varepsilon > 0$ such that, for every $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (t_0, t_1)$ and $v \in \mathbb{R}^n$, condition (24) holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$, and that condition (25) holds, for all $y \in \partial K$ and $w \in \mathbb{R}^n$. Then problem (22) and (23) has no solution on ∂Q.*

Proof. Let x be a solution of problem (22) and (23). At first, we will show that

$$x \in \partial Q \Rightarrow \exists t_x \in [t_0, t_1] \quad \text{such that } x(t_x) \in \partial K.$$

For this purpose, assume that $x(t) \in K$, for each $t \in [t_0, t_1]$, and define the function $d : [t_0, t_1] \to [0, \infty)$ in the following way

$$d(t) := dist(x(t), \partial K).$$

Let $t^* \in [t_0, t_1]$ be arbitrary and let $\{t_k\}_{k=1}^{\infty} \subset [t_0, t_1]$ be a sequence converging to t^* such that $d(t_k)$ converges to a real number l.

Since $\overline{K}$ is bounded, ∂K is compact. Therefore, for each $k \in \mathbb{N}$, there exists $y_k \in \partial K$ such that

$$d(t_k) = |x(t_k) - y_k|,$$

and the sequence $\{y_k\}_{k=1}^{\infty}$ has a subsequence, for the sake of simplicity denoted in the same way as the sequence, which converges to a point $y_0 \in \partial K$. Since $y_k \to y_0$, $t_k \to t^*$ and $d(t_k) \to l$,

$$l = |x(t^*) - y_0| \geq d(t^*),$$

according to the definition of function d.

This means that d is a lower semi-continuous function on $[t_0, t_1]$ in a single-valued sense, i.e., for each $t^* \in [t_0, t_1]$ and each $\varepsilon > 0$, there exists a neighborhood U of t^* such that $d(t) > d(t^*) - \varepsilon$, for all $t \in U$. Therefore, d has on $[t_0, t_1]$ its minimum d_0. Since $x(t) \in K$, for each $t \in [t_0, t_1]$, d_0 must be positive. Hence, for each $t \in [t_0, t_1]$, $B(x(t), d_0) := \{y \in \mathbb{R}^n \mid |y - x(t)| < d_0\} \subset K$, and therefore, for all $y \in C^1([t_0, t_1], \mathbb{R}^n)$ satisfying $|y(t) - x(t)| < d_0$, for all $t \in [t_0, t_1]$, it holds that $y \in Q$. Thus, $x \in \text{Int}\, Q$.

Let us now consider a function $x : [t_0, t_1] \to \overline{K}$ in ∂Q. As a consequence of the first part of the proof, there must exist a point $t_x \in [t_0, t_1]$ such that $x(t_x) \in \partial K$. But then x cannot be a solution of problem (22) and (23), according to Proposition 5. □

As mentioned before, we are also interested in conditions imposed on the set K ensuring that the set $Q \subset C^1([t_0, t_1], \mathbb{R}^n)$ is a retract of the Banach space $C^1([t_0, t_1], \mathbb{R}^n)$.

Lemma 4. *Let $K \subset \mathbb{R}^n$ be a nonempty open set whose closure $\overline{K}$ is a retract of $\mathbb{R}^n$. Then the set Q defined by formula (35) is a retract of the space $C^1([t_0, t_1], \mathbb{R}^n)$.*

Proof. Since $\overline{K}$ is a retract of $\mathbb{R}^n$, there exists a continuous function $\Phi : \mathbb{R}^n \to \overline{K}$ satisfying $\Phi(x) = x$, for each $x \in \overline{K}$. Let us define a function $\tilde{\Phi} : C^1([t_0, t_1], \mathbb{R}^n) \to Q$ in the following way: for each $x \in C^1([t_0, t_1], \mathbb{R}^n)$, put $\tilde{\Phi}(x) = \tilde{x}$, where $\tilde{x} : [t_0, t_1] \to \overline{K}$ satisfies

$$\tilde{x}(t) = \Phi(x(t)),$$

for every $t \in [t_0, t_1]$.

From the definition of Q and the properties of Φ it immediately follows that $\tilde{\Phi}$ is well defined and that $\tilde{\Phi}(q) = q$, for each $q \in Q$. Let $\{x_k\}_{k=1}^{\infty} \subset C^1([t_0, t_1], \mathbb{R}^n)$ converges to $x \in C^1([t_0, t_1], \mathbb{R}^n)$. Since Φ is continuous, $\{\tilde{\Phi}(x_k)\}_{k=1}^{\infty}$ converges to $\tilde{\Phi}(x)$ in $C^1([t_0, t_1], \mathbb{R}^n)$. Therefore, $\tilde{\Phi}$ is continuous and Q is a retract of $C^1([t_0, t_1], \mathbb{R}^n)$. □

Remark 9. The set Q defined by formula (35) is an example of an *AR*-space, because it is a retract of the Banach space $C^1([t_0, t_1], \mathbb{R}^n)$.

Now, let us specify problem (5) as the Floquet b.v.p. (22) and (23). Under appropriate assumptions imposed on F, A, B, M and N, we will prove its solvability by means of a continuation principle developed in the form of Theorem 1. Defining the set Q of candidate solutions by formula (35), we are able to verify, for each $(q, \lambda) \in [0, 1) \times Q$, the transversality condition (v) of Theorem 1.

Theorem 2. *Let us consider the boundary value problem (22) and (23), where $F : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping, A, B are integrable matrix functions which satisfy $|A(t)| \leq a(t)$ and $|B(t)| \leq b(t)$, for all $t \in [t_0, t_1]$ and suitable integrable functions $a, b : [t_0, t_1] \to [0, \infty)$. Let M and N be real $n \times n$ matrices with M non-singular and such that $M\partial K = \partial K$.*

Furthermore, assume that

(i) *there exists a nonempty open bounded set $K \subset \mathbb{R}^n$ whose closure $\overline{K}$ is a retract of $\mathbb{R}^n$,*
(ii) *there exists an upper-Carathéodory mapping $G : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ such that*

$$G(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n,$$

(iii) *$G(t, \cdot, \cdot, r_1, r_2, \lambda)$ is Lipschitzian with a sufficiently small Lipschitz constant, for each $t \in [t_0, t_1]$, $r_1 \in \overline{K}$, $r_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,*

(iv) *there exists a nonnegative, integrable function* $\alpha : [t_0, t_1] \to \mathbb{R}$ *such that*

$$|G(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)| \le \alpha(t), \quad \text{a.e. in } [t_0, t_1],$$

for each $(q, \lambda, x) \in \Gamma_{\mathfrak{T}}$, *where* Q *is defined by formula* (35) *and* $\mathfrak{T}$ *denotes the multivalued mapping which assigns to any* $(q, \lambda) \in Q \times [0, 1]$ *the set of solutions of*

$$\begin{aligned} &\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in G(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in [t_0, t_1],\\ &x(t_1) = Mx(t_0),\\ &\dot{x}(t_1) = N\dot{x}(t_0), \end{aligned} \tag{36}$$

(v) *the associated homogenous problem*

$$\begin{aligned} &\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = 0, \quad \text{for a.a. } t \in [t_0, t_1],\\ &x(t_1) = Mx(t_0),\\ &\dot{x}(t_1) = N\dot{x}(t_0) \end{aligned}$$

has only the trivial solution,

(vi) $\mathfrak{T}(Q \times \{0\}) \subset Q$,

(vii) *there exists a function* $V \in C^1(\mathbb{R}^n, \mathbb{R})$ *with* ∇V *locally Lipschitzian satisfying conditions (H1) and* (H2) *such that* $\langle \nabla V(My), Nw\rangle \cdot \langle \nabla V(y), w\rangle > 0$, *or* $\langle \nabla V(My), Nw\rangle = \langle \nabla V(y), w\rangle = 0$, *for all* $y \in \partial K$ *and* $w \in \mathbb{R}^n$,

(viii) *there exists* $\varepsilon > 0$ *such that, for all* $\lambda \in [0, 1)$, $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (t_0, t_1)$ *and* $y \in \mathbb{R}^n$, *the following condition*

$$\limsup_{h \to 0^-} \frac{\langle \nabla V(x + hy), y + hw\rangle - \langle \nabla V(x), y\rangle}{h} > 0 \tag{37}$$

holds, for all $w \in G(t, x, y, x, y, \lambda) - A(t)y - B(t)x$.

Then problem (22) *and* (23) *has a solution in* Q.

Proof. We will check that all the assumptions of Theorem 1 are satisfied. Since A and B are integrable matrix functions and G is an upper-Carathéodory mapping, $G(t, x, \dot{x}, q, \dot{q}, \lambda) - A(t)\dot{x} - B(t)x$ is also an upper-Carathéodory multivalued mapping, for each $(q, \lambda) \in Q \times [0, 1]$.

Problem (36) is equivalent to the following first-order problem:

$$\begin{aligned} &\dot{\xi}(t) + C(t)\xi(t) \in H(t, \xi(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in [t_0, t_1],\\ &\xi(t_1) = D\xi(t_0), \end{aligned} \tag{38}$$

where

$$\begin{aligned} &\xi_{2n\times 1} = (x, y)^{\mathrm{T}} = (x, \dot{x})^{\mathrm{T}},\\ &C(t)_{2n\times 2n} = \begin{pmatrix} 0 & -I \\ B(t) & A(t) \end{pmatrix},\\ &D_{2n\times 2n} = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \end{aligned}$$

and

$$H(t, \xi, q, \dot{q}, \lambda) = (0, G(t, x, y, q, \dot{q}, \lambda))^{\mathrm{T}}.$$

According to assumption (v), the associated homogenous problem to (38)

$$\begin{aligned} &\dot{\xi}(t) + C(t)\xi(t) = 0, \quad \text{for a.a. } t \in [t_0, t_1],\\ &\xi(t_1) = D\xi(t_0), \end{aligned}$$

has only the trivial solution. Moreover, the multivalued mapping $H : [t_0, t_1] \times \mathbb{R}^{4n} \times [0, 1] \multimap \mathbb{R}^{2n}$ has the same properties as mapping G, i.e. it is an upper-Carathéodory mapping and there exists a sufficiently small constant $L > 0$ such that, for each $r_1 \in \overline{K}$, $r_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$d_H(H(t, \xi_1, r_1, r_2, \lambda), H(t, \xi_2, r_1, r_2, \lambda)) \le L \cdot |\xi_1 - \xi_2|,$$

for a.a. $t \in [t_0, t_1]$ and every $\xi_1, \xi_2 \in \mathbb{R}^{2n}$. It is easy to see that $|C(t)| \le a(t) + b(t)$, for all $t \in [t_0, t_1]$. Therefore, we are able to apply, for each $(q, \lambda) \in Q \times [0, 1]$, Theorem 4 in [9] (for $\alpha(t) \equiv t$, $L\xi = \xi(t_1) - D\xi(t_0)$, $\theta = 0$) to problem (38) (see also Theorem 3.3.8 in [2]) and obtain its solvability with a compact AR-space of solutions. Since a Cartesian product of two sets is an AR-space if and only if both sets are AR-spaces (see, e.g., [13, p. 92]), it follows that the set of solutions of problem (36) must be also a compact AR-space and, in particular, an R_δ-set (cf. (4)), as required.

Let us denote by X the set of all solutions of the b.v.p. (38), for any choice of $q \in Q$ and $\lambda \in [0, 1]$, and let $\xi \in X$ be arbitrary. Since ξ is a solution of problem (38), there exists $h \in L^1([t_0, t_1], \mathbb{R}^{2n})$ such that

$$h(t) \in H(t, \xi(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in [t_0, t_1], \tag{39}$$

and that

$$\dot{\xi}(t) + C(t)\xi(t) = h(t), \quad \text{for a.a. } t \in [t_0, t_1], \tag{40}$$

$$\xi(t_1) = D\xi(t_0).$$

Let us define

$$Y := \left\{ \hat{h}(t) = \int_{t_0}^{t} h(s)\mathrm{d}s \mid h(\cdot) \text{ satisfies (39) and (40), for some } \xi \in X \right\}.$$

We now show that, according to condition (iv) in Theorem 2 (and the definition of H), the set $Y \subset C([t_0, t_1], \mathbb{R}^{2n})$ is bounded and equi-continuous. In fact, for each $\hat{h} \in Y$, it holds that

$$|\hat{h}(t)| = \left| \int_{t_0}^{t} h(s)\mathrm{d}s \right| \le \int_{t_0}^{t} \alpha(s)\,\mathrm{d}s \le \int_{t_0}^{t_1} \alpha(s)\mathrm{d}s, \quad \text{for all } t \in [t_0, t_1],$$

i.e. Y is bounded.

Now, let us take $t, \hat{t} \in [t_0, t_1]$. With no loss of generality, we can assume that $t < \hat{t}$. For each $\hat{h} \in Y$, we obtain

$$|\hat{h}(\hat{t}) - \hat{h}(t)| = \left| \int_{t_0}^{\hat{t}} h(s)\mathrm{d}s - \int_{t_0}^{t} h(s)\mathrm{d}s \right| = \left| \int_{t}^{\hat{t}} h(s)\mathrm{d}s \right| \le \int_{t}^{\hat{t}} \alpha(s)\mathrm{d}s.$$

This implies that the set Y is also equi-continuous. Consequently, according to Arzelà–Ascoli lemma, Y is a relatively compact subset of the Banach space $C([t_0, t_1], \mathbb{R}^{2n})$.

It is easy to see that problem (38) satisfies all the assumptions required in Lemma 2 in [9]. In particular, for $\alpha(t) \equiv t$, $L\xi = \xi(t_1) - D\xi(t_0)$ and $\theta = 0$. According to Lemma 2 in the quoted paper, we obtain that $X = \mathcal{K}(Y)$, where $\mathcal{K}$ denotes a continuous operator from $C([t_0, t_1], \mathbb{R}^{2n})$ into itself. Therefore, $X \subset \mathcal{K}(\overline{Y})$. Since $\overline{Y}$ is compact, $\mathcal{K}(\overline{Y})$ is also compact. Thus, in particular, it is bounded, and consequently X is bounded in $C([t_0, t_1], \mathbb{R}^{2n})$. Hence, there exists $R > 0$ such that the set of all solutions of problem (36) is located in

$$\overline{B(0, R)} = \left\{ x \in C([t_0, t_1], \mathbb{R}^n) \mid \max_{t \in [t_0, t_1]} |x(t)| \le R, \ \max_{t \in [t_0, t_1]} |\dot{x}(t)| \le R \right\},$$

for each $(q, \lambda) \in Q \times [0, 1]$.

Putting

$$S_1 = \overline{B(0, R)} \cap \left\{ x \in AC^1([t_0, t_1], \mathbb{R}^n) \mid x(t_1) = Mx(t_0), \dot{x}(t_1) = N\dot{x}(t_0) \right\},$$

it is obvious that assumption (iv) of Theorem 1 is satisfied.

Moreover, according to condition (i) and Lemma 4, Q is a retract of $C^1([t_0, t_1], \mathbb{R}^n)$. Since $Q \setminus \partial Q$ is nonempty, condition (i) from Theorem 1 holds.

We will now show that condition (v) of Theorem 1 is also satisfied. Let us assume that $x \in Q$ is a fixed point of the mapping $\mathfrak{T}(\cdot, \lambda)$, for some $\lambda \in [0, 1]$. This implies that x is a solution of the problem

$$\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in G(t, x(t), \dot{x}(t), x(t), \dot{x}(t), \lambda), \quad \text{for a.a. } t \in [t_0, t_1], \tag{41}$$
$$x(t_1) = Mx(t_0),$$
$$\dot{x}(t_1) = N\dot{x}(t_0).$$

Hypotheses (i), (vii) and (viii) guarantee that K is a bound set for problem (41), for all $\lambda \in [0, 1)$. If $x \in \partial Q$ is a fixed point of $\mathfrak{T}(\cdot, 1)$ (i.e., for $\lambda = 1$), problem (22) and (23) has, according to assumption (ii), a solution in Q, and we are done. Otherwise, according to Proposition 6, $x \notin \partial Q$, i.e. condition (v) in Theorem 1 holds as well which completes the proof. □

Remark 10. In some particular cases, e.g. if $\mathfrak{T}(q, 0) = \{0\} \in \text{Int}\, Q$, for all $q \in Q$, it is convenient to verify directly that $\{q \in Q \mid q \in \mathfrak{T}(q, 0)\} \cap \partial Q = \emptyset$, and then require the hypothesis $(viii)$ only for all $\lambda \in (0, 1)$.

Remark 11. Because of the applied method, we have obtained, in fact, the viability result (cf. [7]) for the Floquet problem

$$\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [t_0, t_1],$$
$$x(t_1) = Mx(t_0),$$
$$\dot{x}(t_1) = N\dot{x}(t_0),$$
$$x(t) \in K, \quad \text{for all } t \in [t_0, t_1].$$

Let us conclude by the following illustrating example.

Example 3. Let us consider the second-order anti-periodic b.v.p.

$$\ddot{x}(t) \in F_1(t, x(t), \dot{x}(t)) + F_2(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [t_0, t_1], \tag{42}$$

$$x(t_1) = -x(t_0), \tag{43}$$
$$\dot{x}(t_1) = -\dot{x}(t_0),$$

where $F_1, F_2 : [t_0, t_1] \times \mathbb{R}^{2n} \multimap \mathbb{R}^n$ are upper-Carathéodory mappings such that

$$|F_1(t, x, y)| \le \alpha_1(t)$$

and

$$|F_2(t, x, y)| \le \alpha_2(t),$$

for each $(t, x, y) \in [t_0, t_1] \times \mathbb{R}^{2n}$, where $\alpha_1, \alpha_2 \in L^1([t_0, t_1], \mathbb{R})$.

Moreover, assume that $F_1(t, \cdot, \cdot)$ is Lipschitzian with a sufficiently small Lipschitz constant L, for all $t \in [t_0, t_1]$.

Let $R > 0$ and $\varepsilon > 0$ be such constants that the inequality

$$\langle x, w\rangle + \langle y, y\rangle > 0 \tag{44}$$

holds, for all $t \in (t_0, t_1), \lambda \in (0, 1), y \in \mathbb{R}^n, x \in \mathbb{R}^n$ with $|x| = R - \tilde{\varepsilon}$, where $\tilde{\varepsilon} \in [0, \varepsilon]$ is arbitrary, and all $w \in \lambda\left[F_1(t, x, y) + F_2(t, x, y)\right]$.

Let us define the set K in the following way:

$$K := \{x \in \mathbb{R}^n \mid |x| < R\}$$

and the set Q by formula (35). In order to apply Theorem 2 for the solvability of problem (42) and (43), let us consider the associated problem

$$\ddot{x}(t) \in \lambda(F_1(t, x(t), \dot{x}(t)) + F_2(t, q(t), \dot{q}(t))), \quad \text{for a.a. } t \in [t_0, t_1], \tag{45}$$

$$x(t_1) = -x(t_0), \tag{46}$$
$$\dot{x}(t_1) = -\dot{x}(t_0),$$

where $\lambda \in [0, 1]$ and $q \in Q$.

We show that the assumptions of Theorem 2 are satisfied by means of the C^2-function $V(x) := \frac{1}{2}(|x|^2 - R^2)$. Since $V(x) = 0$, for all $x \in \partial K$, and $V(x) \le 0$, for all $x \in \overline{K}$, V satisfies conditions (H1) and (H2). Moreover, since, for each $x \in \partial K$, $\nabla V(x) = x$ and $HV(x) = I$, condition (44) ensures the assumption (viii) of Theorem 2, for all $\lambda \in (0, 1)$.

Because $\overline{K}$ is convex, it is an AR-space, and so a retract of $\mathbb{R}^n$, which guarantee the validity of assumption (i) of Theorem 2.

Since $G(t, x, \dot{x}, q, \dot{q}, \lambda) = \lambda(F_1(t, x, \dot{x}) + F_2(t, q, \dot{q}))$, it holds that

$$G(t, c, d, c, d, 1) = F_1(t, c, d) + F_2(t, c, d),$$

and so condition (ii) is satisfied as well.

As concerns condition (vii), take $y \in \mathbb{R}^n$ with $|y| = R$, i.e. $y \in \partial K$, and $w \in \mathbb{R}^n$. Then

$$\begin{aligned}\langle \nabla V(My), Nw\rangle \cdot \langle \nabla V(y), w\rangle &= \langle \nabla V(-y), -w\rangle \cdot \langle \nabla V(y), w\rangle = \langle -y, -w\rangle \cdot \langle y, w\rangle \\ &= \langle y, w\rangle^2,\end{aligned}$$

which obviously leads to satisfying the required conditions.

Furthermore, the associated homogenous problem

$$\ddot{x}(t) = 0,$$
$$x(t_1) = -x(t_0),$$
$$\dot{x}(t_1) = -\dot{x}(t_0)$$

has only the trivial solution $x(t) = \mathfrak{T}(q, 0) \equiv 0$, for each $q \in Q$, and therefore $\mathfrak{T}(Q \times \{0\}) \equiv 0 \in \operatorname{Int} Q$, i.e. condition (v), (vi) and (viii), for $\lambda = 0$ (cf. Remark 10), are satisfied.

The assumption $M\partial K = \partial K$ is also satisfied, because the positive invariance of ∂K with respect to $M = -I$ is equivalent to the symmetry of ∂K with respect to the origin. Hence, the b.v.p. (42) and (43) admits, according to Theorem 2, a solution.

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On the Floquet problem for second-order Marchaud differential systems

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ABSTRACT

Solutions in a given set of the Floquet boundary value problem are investigated for second-order Marchaud systems. The methods used involve a fixed point index technique developed by ourselves earlier with a bound sets approach. Since the related bounding (Liapunov-like) functions are strictly localized on the boundaries of parameter sets of candidate solutions, some trajectories are allowed to escape from these sets. The main existence and localization theorem is illustrated by two examples for periodic and anti-periodic problems.

1. Introduction

Let us consider the boundary value problem (b.v.p.)

$$\left.\begin{aligned} &\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0,T],\\ &x(T) = Mx(0), \quad \dot{x}(T) = N\dot{x}(0), \end{aligned}\right\} \tag{1}$$

where

(i) $A, B : [0,T] \to \mathbb{R}^{n\times n}$ are continuous matrix functions,
(ii) M and N are $n \times n$ matrices, M is nonsingular,
(iii) $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper semicontinuous multivalued mapping with nonempty, compact, convex values,
(iv) there exists an integrable function $c : [0,T] \to [0,\infty)$ such that

$$\big|F(t,x,y)\big| \leqslant c(t)\big(1 + |x| + |y|\big)$$

holds, for a.a. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^{2n}$.

Mapping F satisfying conditions (iii) and (iv) is said to be a *Marchaud map*.

By a *solution* of problem (1), we mean a vector function $x : [0,T] \to \mathbb{R}^n$ with an absolutely continuous first derivative (i.e. $x \in AC^1([0,T],\mathbb{R}^n)$) which satisfies (1), for almost all $t \in [0,T]$.

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The main purpose of this paper is to develop a bound sets technique which consists in constructing so-called bounding (Liapunov-like) functions which make boundaries of prescribed sets of candidate solutions fixed point free. This will be done for even more general systems than (1) in Section 3. The related transversality condition is namely a typical requirement in application of topological (relative) degree arguments (cf. [1]).

On the other hand, a nonstrict localization of bounding functions, which is usual for Carathéodory systems (see [3]), makes parameter sets of candidate solutions "only" positively invariant. To eliminate this unpleasant handicap requires a completely different approach whose first-order analogy was already employed by ourselves in [4].

Let us note that, unlike the majority of comparable results obtained by other authors (see e.g. [8,10,12,15,17,18,20]), the main theorem (Theorem 4.1) gives explicitly the additional information concerning the localization of solutions of (1).

In the (single-valued) case of vector differential equations, the classical results in this field were already obtained by means of C^2-bounding functions in the 70s (see [7,13]). The usage of less regular functions is much more delicate (cf. [19]). Moreover, multivalued generating vector-fields are naturally associated with the notion of a Carathéodory solution. Nevertheless, Theorem 3.1 cannot be simply reduced to the main results in [7,13,19] because of application of different topological methods.

2. Some preliminaries

Let us recall at first some geometric notions of subsets of metric spaces, in particular, of retracts. If (X,d) is an arbitrary metric space and $A \subset X$, by $Int(A)$, $\overline{A}$ and ∂A we mean the interior, the closure and the boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A\colon\ d(x,a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$.

We say that a space X is an *absolute retract* (*AR*-space) if, for each space Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over Y. If f is extendable only over some neighborhood of A, for each closed $A \subset Y$ and each continuous mapping $f : A \to X$, then X is called an *absolute neighborhood retract* (*ANR*-space).

Let us note that X is an *ANR*-space if and only if it is a retract of an open subset of a normed space and that X is an *AR*-space if and only if it is a retract of some normed space. In particular, if X is a retract (of an open subset) of a convex set in a Banach space, then it is an *AR*-space (*ANR*-space). So, the space $C^1(J,\mathbb{R}^n)$, where $J \subset \mathbb{R}$ is a compact interval, is an *AR*-space as well as its convex subsets or retracts, while its open subsets are *ANR*-spaces.

A nonempty set $A \subset X$ is called an R_δ*-set* if there exists a decreasing sequence $\{A_n\}_{n=1}^\infty$ of compact *AR*-spaces such that

$$A = \bigcap_{n=1}^{\infty} A_n.$$

The following hierarchy holds for nonempty subsets of a metric space:

$$\text{compact} + \text{convex} \subset \text{compact } AR\text{-space} \subset R_\delta\text{-set}, \tag{2}$$

and all the above inclusions are proper. For more details concerning the theory of retracts, see [9].

We also employ the following definitions from the multivalued analysis in the sequel. Let X and Y be arbitrary metric spaces. We say that φ is a multivalued mapping from X to Y (written $\varphi : X \multimap Y$) if, for every $x \in X$, a nonempty subset $\varphi(x)$ of Y is prescribed.

A multivalued mapping $\varphi : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \subset U\}$ is open in X.

Let Y be a metric space and $(\Omega, \mathcal{U}, \nu)$ be a *measurable space*, i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $\varphi : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$. Obviously, every u.s.c. mapping is measurable.

We say that mapping $\varphi : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $\varphi(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $\varphi(t,\cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all (a.a.) $t \in J$, and the set $\varphi(t,x)$ is compact and convex, for all $(t,x) \in J \times \mathbb{R}^m$.

A multivalued mapping $\varphi : X \multimap X$ with bounded values is called *Lipschitzian* if there exists a constant $k > 0$ such that

$$d_H\big(\varphi(x), \varphi(y)\big) \leqslant k d(x,y),$$

for every $x, y \in X$, where

$$d_H(A,B) := \inf\big\{r > 0 \bigm| A \subset N_r(B) \text{ and } B \subset N_r(A)\big\}$$

stands for the Hausdorff distance.

If $X \cap Y \neq \emptyset$ and $\varphi : X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed point* of φ if $x \in \varphi(x)$. The set of all fixed points of φ is denoted by $Fix(\varphi)$, i.e.

$$Fix(\varphi) := \big\{x \in X \bigm| x \in \varphi(x)\big\}.$$

For more information and details concerning multivalued analysis, see, e.g., [1,5,14,16].

3. Bound sets for Floquet problems

In this section, we are interested in introducing a Liapunov-like function V, usually called a *bounding function*, verifying suitable transversality conditions which assure that there does not exist a solution of the b.v.p. lying in a closed set $\overline{K}$ and touching the boundary ∂K of K at some point.

We proceed in two steps. At first, we take into account only the interior points of the interval $[0,T]$ (see Proposition 3.1 below). Then we also consider the end points 0 and T (see Theorem 3.1 below).

Let $K \subset \mathbb{R}^n$ be a nonempty open set and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

(H1) $V|_{\partial K} = 0$,
(H2) $V(x) \leqslant 0$, for all $x \in \overline{K}$.

Definition 3.1. A nonempty open set $K \subset \mathbb{R}^n$ is called a *bound set* for problem (1) if there does not exist a solution x of (1) such that $x(t) \in \overline{K}$, for each $t \in [0,T]$, and $x(t_0) \in \partial K$, for some $t_0 \in [0,T]$.

Firstly, we show sufficient conditions for the existence of a bound set for the second-order Floquet problem (1) in the case of a smooth bounding function V with a locally Lipschitzian gradient.

Proposition 3.1. *Let $K \subset \mathbb{R}^n$ be a nonempty open set, $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values and A and B be continuous matrix functions. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies conditions* (H1) *and* (H2). *Suppose moreover that, for all $x \in \partial K$, $t \in (0,T)$ and $v \in \mathbb{R}^n$ with*

$$\bigl\langle \nabla V(x), v \bigr\rangle = 0, \tag{3}$$

the following condition holds

$$\liminf_{h \to 0} \frac{\langle \nabla V(x+hv), v+hw \rangle}{h} > 0, \tag{4}$$

for all $w \in F(t,x,v) - A(t)v - B(t)x$. Then all solutions $x : [0,T] \to \overline{K}$ of problem (1) *satisfy $x(t) \in K$, for every $t \in (0,T)$.*

Proof. Let $x : [0,T] \to \overline{K}$ be a solution of problem (1). We assume by a contradiction that there exists $t_0 \in (0,T)$ such that $x(t_0) \in \partial K$.

Let us define the function $g : [-t_0, T-t_0] \to (-\infty, 0]$ in the following way $g(h) := V(x(t_0+h))$. Then $g(0) = 0$ and $g(h) \leqslant 0$, for all $h \in [-t_0, T-t_0]$, i.e., there is a local maximum for g at the point 0, and so $\dot{g}(0) = \langle \nabla V(x(t_0)), \dot{x}(t_0) \rangle = 0$. Consequently, $v := \dot{x}(t_0)$ satisfies condition (3).

Since ∇V is locally Lipschitzian, there exist a set $U \subset \mathbb{R}^n$ with $x(t_0) \in U$ and a constant $L > 0$ such that $\nabla V|_U$ is Lipschitzian with constant L.

Let $\{h_k\}_{k=1}^{\infty}$ be an arbitrary decreasing sequence of positive numbers such that $h_k \to 0^+$ as $k \to \infty$, $x(t_0+h) \in U$, for all $h \in (0, h_1)$.

Since $g(0) = 0$ and $g(h) \leqslant 0$, for all $h \in (0, h_k]$, there exists, for each $k \in \mathbb{N}$, $h_k^* \in (0, h_k)$ such that $\dot{g}(h_k^*) \leqslant 0$.

Since $x \in C^1([0,T], \mathbb{R}^n)$, for each $k \in \mathbb{N}$,

$$x\bigl(t_0 + h_k^*\bigr) = x(t_0) + h_k^*\bigl[\dot{x}(t_0) + b_k^*\bigr], \tag{5}$$

where $b_k^* \to 0$ as $k \to \infty$.

If we define, for each $t \in [0,T]$,

$$P\bigl(t, x(t), \dot{x}(t)\bigr) := -A(t)\dot{x}(t) - B(t)x(t) + F\bigl(t, x(t), \dot{x}(t)\bigr), \tag{6}$$

then (1) can be written in the form

$$\ddot{x}(t) \in P\bigl(t, x(t), \dot{x}(t)\bigr), \quad \text{for a.a. } t \in [0,T].$$

Since $x([0,T])$ and $\dot{x}([0,T])$ are compact sets and P is globally upper semicontinuous with compact values, $P(t, x(t), \dot{x}(t))$ must be bounded on $[0,T]$, by which $\dot{x}$ is Lipschitzian on $[0,T]$. Thus, there exists a constant λ such that, for all $k \in \mathbb{N}$,

$$\left| \frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*} \right| \leqslant \lambda,$$

i.e. the sequence $\{\frac{\dot{x}(t_0+h_k^*)-\dot{x}(t_0)}{h_k^*}\}_{k=1}^{\infty}$ is bounded. Therefore, there exist a subsequence, for the sake of simplicity denoted as the sequence, of $\{\frac{\dot{x}(t_0+h_k^*)-\dot{x}(t_0)}{h_k^*}\}$ and $w \in \mathbb{R}^n$ such that

$$\frac{\dot{x}(t_0+h_k^*)-\dot{x}(t_0)}{h_k^*} \to w \tag{7}$$

as $k \to \infty$.

Let $\varepsilon > 0$ be given. Then, as a consequence of the regularity assumptions on F, A and B and of the continuity of both x and $\dot{x}$, there exists $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$ such that, for each $t \in (0, T)$, $|t - t_0| \leqslant \bar{\delta}$, it follows that

$$P\big(t, x(t), \dot{x}(t)\big) \subset P\big(t_0, x(t_0), \dot{x}(t_0)\big) + \varepsilon \overline{B}_n,$$

where B_n denotes the unit open ball in $\mathbb{R}^n$ centered at the origin. Subsequently, according to the Mean-Value Theorem (see [5, Theorem 0.5.3]), there exists $k_\varepsilon \in \mathbb{N}$ such that, for each $k \geqslant k_\varepsilon$,

$$\frac{\dot{x}(t_0+h_k^*)-\dot{x}(t_0)}{h_k^*} = \frac{1}{h_k^*}\int_{t_0}^{t_0+h_k^*} \ddot{x}(s)\,ds \in P\big(t_0, x(t_0), \dot{x}(t_0)\big) + \varepsilon \overline{B}_n.$$

Since P has compact values and $\varepsilon > 0$ is arbitrary,

$$w \in P\big(t_0, x(t_0), \dot{x}(t_0)\big).$$

As a consequence of property (7), there exists a sequence $\{a_k^*\}_{k=1}^\infty$, $a_k^* \to 0$ as $k \to \infty$, such that

$$\dot{x}\big(t_0 + h_k^*\big) = \dot{x}(t_0) + h_k^*\big[w + a_k^*\big], \tag{8}$$

for each $k \in \mathbb{N}$.

Let $k \geqslant k_\varepsilon$ be arbitrary. Since $h_k^* > 0$ and $\dot{g}(h_k^*) \leqslant 0$, in view of (5) and (8),

$$0 \geqslant \frac{\dot{g}(h_k^*)}{h_k^*} = \frac{\langle \nabla V(x(t_0+h_k^*)), \dot{x}(t_0+h_k^*)\rangle}{h_k^*} = \frac{\langle \nabla V(x(t_0)+h_k^*[\dot{x}(t_0)+b_k^*]), \dot{x}(t_0)+h_k^*[w+a_k^*]\rangle}{h_k^*}. \tag{9}$$

Since $b_k^* \to 0$ when $k \to +\infty$, it is possible to find $k_0 \in \mathbb{N}$ such that, for all $k \geqslant k_0$, it holds that $x(t_0) + \dot{x}(t_0)h_k^* \in U$. By means of the local Lipschitzianity of ∇V, for all $k \geqslant \max\{k_\varepsilon, k_0\}$,

$$\begin{aligned} 0 &\geqslant \frac{\dot{g}(h_k^*)}{h_k^*} \geqslant \frac{\langle \nabla V(x(t_0)+h_k^*\dot{x}(t_0)), \dot{x}(t_0)+h_k^*[w+a_k^*]\rangle}{h_k^*} - L\cdot\big|b_k^*\big|\cdot\big|\dot{x}(t_0)+h_k^*\big[w+a_k^*\big]\big| \\ &= \frac{\langle \nabla V(x(t_0)+h_k^*\dot{x}(t_0)), \dot{x}(t_0)+h_k^* w\rangle}{h_k^*} - L\cdot\big|b_k^*\big|\cdot\big|\dot{x}(t_0)+h_k^*\big[w+a_k^*\big]\big| + \big\langle \nabla V\big(x(t_0)+h_k^*\dot{x}(t_0)\big), a_k^*\big\rangle. \end{aligned}$$

Since $\langle \nabla V(x(t_0)+h_k^*\dot{x}(t_0)), a_k^*\rangle - L\cdot|b_k^*|\cdot|\dot{x}(t_0)+h_k^*[w+a_k^*]| \to 0$ as $k \to \infty$,

$$\liminf_{h\to 0^+} \frac{\langle \nabla V(x(t_0)+h\dot{x}(t_0)), \dot{x}(t_0)+hw\rangle}{h} \leqslant 0. \tag{10}$$

If we consider, instead of the sequence $\{h_k\}_{k=1}^\infty$, an increasing sequence $\{\bar{h}_k\}_{k=1}^\infty$ of negative numbers such that $\bar{h}_k \to 0^-$ as $k \to \infty$, $x(t_0 + \bar{h}_k) \in U$, for all $k \in \mathbb{N}$, and $\bar{h}_1 \in [-\delta, 0)$, we are able to find, for each $k \in \mathbb{N}$, $\bar{h}_k^* \in (\bar{h}_k, 0)$ such that $\dot{g}(\bar{h}_k^*) \geqslant 0$. Therefore, using the same procedure as in the first part of the proof, we obtain, for $k \in \mathbb{N}$ sufficiently large, that

$$0 \geqslant \frac{\dot{g}(\bar{h}_k^*)}{\bar{h}_k^*} \geqslant \frac{\langle \nabla V(x(t_0)+\bar{h}_k^*\dot{x}(t_0)), \dot{x}(t_0)+\bar{h}_k^*\overline{w}\rangle}{\bar{h}_k^*} - L\cdot\big|\bar{b}_k^*\big|\cdot\big|\dot{x}(t_0)+\bar{h}_k^*\big[\overline{w}+\bar{a}_k^*\big]\big| + \big\langle \nabla V\big(x(t_0)+\bar{h}_k^*\dot{x}(t_0)\big), \bar{a}_k^*\big\rangle,$$

where $\bar{a}_k^* \to 0$, $\bar{b}_k^* \to 0$ as $k \to \infty$ and $\overline{w} \in P(t_0, x(t_0), \dot{x}(t_0))$.

This means that $\langle \nabla V(x(t_0)+\bar{h}_k^*\dot{x}(t_0)), \bar{a}_k^*\rangle - L\cdot|\bar{b}_k^*|\cdot|\dot{x}(t_0)+\bar{h}_k^*[\overline{w}+\bar{a}_k^*]| \to 0$ as $k \to \infty$ which implies

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(x(t_0)+h\dot{x}(t_0)), \dot{x}(t_0)+h\overline{w}\rangle}{h} \leqslant 0. \tag{11}$$

Inequalities (10) and (11) are in a contradiction with condition (4), because $x(t_0) \in \partial K$, $\dot{x}(t_0)$ satisfies condition (3) and $w, \overline{w} \in P(t_0, x(t_0), \dot{x}(t_0))$. □

Theorem 3.1. *Let $K \subset \mathbb{R}^n$ be a nonempty open set, $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous mapping with nonempty, compact, convex values and A and B be continuous matrix functions. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies conditions* (H1) *and* (H2)*. Furthermore, assume that M and N are $n \times n$ matrices with M regular and satisfying*

$$M(\partial K) = \partial K. \tag{12}$$

Moreover, let, for all $x \in \partial K$, $t \in (0, T)$ and $v \in \mathbb{R}^n$ satisfying (3)*, condition* (4) *holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$.*

At last, suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ with

$$\big\langle \nabla V(x), v \big\rangle \leqslant 0 \leqslant \big\langle \nabla V(Mx), Nv \big\rangle, \tag{13}$$

at least one of the following conditions

$$\liminf_{h\to 0^+} \frac{\langle \nabla V(x+hv), v+hw_1\rangle}{h} > 0 \tag{14}$$

or

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(Mx+hNv), Nv+hw_2\rangle}{h} > 0 \tag{15}$$

holds, for all $w_1 \in F(0,x,v) - A(0)v - B(0)x$, or, for all $w_2 \in F(T,Mx,Nv) - A(T)Nv - B(T)Mx$. Then K is a bound set for problem (1).

Proof. Applying Proposition 3.1, we only need to show that if $x : [0,T] \to \overline{K}$ is a solution of problem (1), then $x(0) \in K$ and $x(T) \in K$. As in the proof of Proposition 3.1, we argue by a contradiction. Since $x(0) \in \partial K$ if and only if $x(T) \in \partial K$ (according to condition (12) and the regularity of M), we can take, without any loss of generality, a solution of (1) satisfying $x(0) \in \partial K$. Following the same reasoning as in the proof of Proposition 3.1, for $t_0 = 0$, we obtain

$$\big\langle \nabla V\big(x(0)\big), \dot{x}(0)\big\rangle \leqslant 0,$$

because $V(x(0)) = 0$ and $V(x(t)) \leqslant 0$, for all $t \in [0,T]$.

Moreover, since $V(x(T)) = 0$, it holds that

$$0 \leqslant \big\langle \nabla V\big(x(T)\big), \dot{x}(T)\big\rangle = \big\langle \nabla V\big(Mx(0)\big), N\dot{x}(0)\big\rangle,$$

by virtue of the boundary conditions in (1). Therefore, $v := \dot{x}(0)$ satisfies condition (13).

Using the same procedure as in the proof of Proposition 3.1, for $t_0 = 0$, $h_k \to 0^+$ and for $t_0 = T$, $\bar{h}_k \to 0^-$, respectively, we obtain the existence of a sequence of positive numbers $\{h_k^*\}_{k=1}^\infty$, $h_k^* \in (0,h_k)$, of a sequence of negative numbers $\{\bar{h}_k^*\}_{k=1}^\infty$, $\bar{h}_k^* \in (\bar{h}_k, 0)$, and of points $w_0 \in P(0,x(0),\dot{x}(0))$, $w_T \in P(T,x(T),\dot{x}(T))$ (P is defined by formula (6)) such that

$$\frac{\dot{x}(h_k^*) - \dot{x}(0)}{h_k^*} \to w_0 \quad \text{as } k \to \infty,$$

and

$$\frac{\dot{x}(T+\bar{h}_k^*) - \dot{x}(T)}{\bar{h}_k^*} \to w_T \quad \text{as } k \to \infty.$$

By the same arguments as in the previous proof, we get

$$\liminf_{h\to 0^+} \frac{\langle \nabla V(x(0)+h\dot{x}(0)), \dot{x}(0)+hw_0\rangle}{h} \leqslant 0 \tag{16}$$

and

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(x(T)+h\dot{x}(T)), \dot{x}(T)+hw_T\rangle}{h} \leqslant 0. \tag{17}$$

Moreover, using the boundary conditions in (1), the inequality (17) can be written in the form

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(Mx(0)+hN\dot{x}(0)), N\dot{x}(0)+hw_T\rangle}{h} \leqslant 0. \tag{18}$$

Inequalities (16) and (18) are in a contradiction with conditions (14) and (15), which completes the proof. □

Remark 3.1. Let us note that Theorem 3.1 can be particularly applied, when $A(t)$ or $B(t)$ (or both) are identically equal to zero matrices. Thus, this theorem gives sufficient conditions to have a bound set for a Floquet b.v.p. associated with a (not necessarily semi-linear) second-order Marchaud system.

Remark 3.2. If condition (14) holds, for some $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (13) and $w_1 \in F(0,x,v) - A(0)v - B(0)x$ then, according to the continuity of ∇V, $\langle \nabla V(x), v\rangle = 0$. Similarly, if (15) holds, for some $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (13) and $w_2 \in F(T,Mx,Nv) - A(T)Nv - B(T)Mx$, then $\langle \nabla V(Mx), Nv\rangle = 0$.

Therefore, the validity of (13), (14) and (15) implies, in particular, that

$$\big\langle \nabla V(x), v\big\rangle = \big\langle \nabla V(Mx), Nv\big\rangle = 0. \tag{19}$$

In the entire text, we deal with problem (1), where M is an invertible matrix satisfying (12). These conditions on M enable us to simplify some computations. On the other hand, when imposing more restrictions on V, the general case can be treated as well. More precisely, the following result is true and its proof easily follows from the one of Theorem 3.1.

Corollary 3.1. *Consider the b.v.p.* (1), *where M and N are arbitrary matrices. Let F, A, B, K and V be the same as in Theorem* 3.1. *Let condition* (4) *be valid, for all $x \in \partial K$, $t \in (0,T)$, $v \in \mathbb{R}^n$ with* (3) *and $w \in F(t,x,v) - A(t)v - B(t)x$.*

Moreover, assume that (14) *holds, for all $x \in \partial K$, $v \in \mathbb{R}^n$ with $\langle \nabla V(x), v\rangle \leqslant 0$, and $w_1 \in F(0,x,v) - A(0)v - B(0)x$, and that* (15) *holds, for all $x \in \partial K$, $v \in \mathbb{R}^n$ with $\langle \nabla V(x), v\rangle \geqslant 0$, and $w_2 \in F(T,x,v) - A(T)v - B(T)x$. Then K is a bound set for problem* (1).

Definition 3.2. A function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying all assumptions of Theorem 3.1 is called a *bounding function* for the set K relative to (1).

If the gradient of a smooth bounding function V is not locally Lipschitzian, then we can give the following corollary.

Corollary 3.2. *Let $K \subset \mathbb{R}^n$ be a nonempty open set, $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous mapping with nonempty, compact, convex values and A and B be continuous matrix functions. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ satisfying conditions* (H1) *and* (H2). *Furthermore, assume that M and N are $n \times n$ matrices with M regular and satisfying* (12). *Moreover, let, for all $x \in \partial K$, $t \in (0,T)$ and $v \in \mathbb{R}^n$ with*

$$\langle \nabla V(x), v\rangle = 0,$$

the following condition holds

$$\liminf_{h \to 0,\, y \to v,\, l \to w} \frac{\langle \nabla V(x+hy), v+hl\rangle}{h} > 0, \tag{20}$$

for all $w \in F(t,x,v) - A(t)v - B(t)x$.

At last, suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ with

$$\langle \nabla V(x), v\rangle \leqslant 0 \leqslant \langle \nabla V(Mx), Nv\rangle,$$

at least one of the following conditions

$$\liminf_{h \to 0^+,\, y \to v,\, l_1 \to w_1} \frac{\langle \nabla V(x+hy), v+hl_1\rangle}{h} > 0 \tag{21}$$

or

$$\liminf_{h \to 0^-,\, y \to v,\, l_2 \to w_2} \frac{\langle \nabla V(Mx+hNy), Nv+hl_2\rangle}{h} > 0 \tag{22}$$

holds, for all $w_1 \in F(0,x,v) - A(0)v - B(0)x$, or, for all $w_2 \in F(T,Mx,Nv) - A(T)Nv - B(T)Mx$. Then K is a bound set for problem (1).

Proof. Let $x : [0,T] \to \overline{K}$ be a solution of (1). We assume, by a contradiction, that there exists $t_0 \in [0,T]$ such that $x(t_0) \in \partial K$. We begin with the case when $t_0 \in (0,T)$. It is easy to see that $\langle \nabla V(x(t_0)), \dot{x}(t_0)\rangle = 0$. Reasoning as in the proof of Proposition 3.1, for each given decreasing sequence $h_k \to 0^+$, there exists $\{h_k^*\}$ with $h_k^* \in (0,h_k)$, $\dot{g}(h_k^*) \leqslant 0$, for all $k \in \mathbb{N}$, such that (9) is satisfied. Therefore, since $a_k^* \to 0$ and $b_k^* \to 0$ as $k \to +\infty$, we obtain

$$\liminf_{h \to 0^+,\, y \to \dot{x}(t_0),\, l \to w} \frac{\langle \nabla V(x(t_0)+hy), \dot{x}(t_0)+hl\rangle}{h} \leqslant \liminf_{k \to +\infty} \frac{\langle \nabla V(x(t_0)+h_k^*[\dot{x}(t_0)+b_k^*]), \dot{x}(t_0)+h_k^*[w+a_k^*]\rangle}{h_k^*} \leqslant 0$$

which is a contradiction with (20). By a similar argument, we also get a contradiction with (20) when taking into account an increasing sequence $h_k \to 0^-$. Finally, in view of (12) and the boundary conditions in (1), we arrive at a contradiction with (21) or (22), when taking $t_0 = 0$ or $t_0 = T$. □

Remark 3.3. If a bounding function V is of class C^2, conditions (4), (14) and (15) can be rewritten in terms of gradients and Hessian matrices. Concretely, (4) takes the form

$$\langle HV(x)\cdot v, v\rangle + \langle \nabla V(x), w\rangle > 0,$$

for all $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (3), $t \in (0,T)$ and $w \in F(t,x,v) - A(t)v - B(t)x$.

For the sake of simplicity, in order to discuss (14) and (15), let us restrict ourselves to those V, M and N for which (13) implies (19). In such a case, it is easy to see that (14) and (15) are equivalent to

$$\max\bigl\{\bigl\langle HV(x)\cdot v, v\bigr\rangle + \bigl\langle \nabla V(x), w_1\bigr\rangle, \bigl\langle HV(Mx)\cdot Nv, Nv\bigr\rangle + \bigl\langle \nabla V(Mx), w_2\bigr\rangle\bigr\} > 0, \tag{23}$$

for all $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (13), $w_1 \in F(0, x, v) - A(0)v - B(0)x$ and $w_2 \in F(T, Mx, Nv) - A(T)Nv - B(T)Mx$.

In particular, observe that (13) always implies (19) under conditions (14) and (15) (see Remark 3.2). The same is true if one of the following possibilities takes place:

(i) $M = N = Id$, i.e. for the periodic problem associated to the inclusion in (1),
(ii) $M = N = -Id$, i.e. for the anti-periodic b.v.p. associated to the inclusion in (1), and $\nabla V(-x) = -\nabla V(x)$, for all $x \in \partial K$,
(iii) $M = a \cdot Id$, $N = b \cdot Id$, where $a \cdot b > 0$, and $\nabla V(ax) = a\nabla V(x)$, for all $x \in \partial K$.

Sometimes it is convenient to take, instead of one function $V : \mathbb{R}^n \to \mathbb{R}$, the whole family of bounding functions. More precisely, we will assume that, for each $x \in \partial K$, there exists a function $V_x : \mathbb{R}^n \to \mathbb{R}$ satisfying

(H1′) $V_x(x) = 0$,
(H2′) $V_x(\xi) \leqslant 0$, for all $\xi \in \overline{K}$ with ξ in a neighborhood of x.

In this case, the proofs remain almost the same, after replacing V by $V_{x(t_0)}$. Therefore, Theorem 3.1 can be easily reformulated as follows.

Corollary 3.3. *Let $K \subset \mathbb{R}^n$ be a nonempty open set, $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous mapping with nonempty, compact, convex values and A and B be continuous matrix functions. Assume that there exists a family of C^1-functions $\{V_x\}_{x\in\partial K}$, $V_x : \mathbb{R}^n \to \mathbb{R}$, with locally Lipschitzian gradients satisfying conditions* (H1′) *and* (H2′)*.*

Furthermore, assume that M and N are $n \times n$ matrices with M regular and satisfying (12)*. Moreover, let, for all $x \in \partial K$, $t \in (0, T)$ and $v \in \mathbb{R}^n$ with*

$$\bigl\langle \nabla V_x(x), v\bigr\rangle = 0, \tag{24}$$

the following condition holds

$$\liminf_{h\to 0} \frac{\langle \nabla V_x(x + hv), v + hw\rangle}{h} > 0, \tag{25}$$

for all $w \in F(t, x, v) - A(t)v - B(t)x$.

At last, suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ with

$$\bigl\langle \nabla V_x(x), v\bigr\rangle \leqslant 0 \leqslant \bigl\langle \nabla V_{Mx}(Mx), Nv\bigr\rangle, \tag{26}$$

at least one of the following conditions

$$\liminf_{h\to 0^+} \frac{\langle \nabla V_x(x + hv), v + hw_1\rangle}{h} > 0 \tag{27}$$

or

$$\liminf_{h\to 0^-} \frac{\langle \nabla V_{Mx}(Mx + hNv), Nv + hw_2\rangle}{h} > 0 \tag{28}$$

holds, for all $w_1 \in F(0, x, v) - A(0)v - B(0)x$, or, for all $w_2 \in F(T, Mx, Nv) - A(T)Nv - B(T)Mx$. Then K is a bound set for problem (1)*.*

Definition 3.3. A function $V_x : \mathbb{R}^n \to \mathbb{R}$ satisfying related conditions in Corollary 3.3 is called a *bounding function* for the set K *at* x relative to (1).

The following illustrative example demonstrates how a family of C^2-bounding functions can easily guarantee the existence of a bound set for periodic b.v.p.s.

Example 3.1. Let us consider the periodic b.v.p.

$$\left.\begin{aligned} &\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F\bigl(t, x(t), \dot{x}(t)\bigr), \quad \text{for a.a. } t \in [0, T],\\ &x(T) = x(0), \quad \dot{x}(T) = \dot{x}(0) \end{aligned}\right\} \tag{29}$$

and let $K \subset \mathbb{R}^n$ be convex. This geometrically means that besides another, for each $x \in \partial K$, there exist an outer normal n_x, not necessarily unique, and a neighborhood U_x of x such that

$$\bigl\langle n_x, (y - x)\bigr\rangle \leqslant 0,$$

for each $y \in \overline{K} \cap U_x$.

Let, for each $x \in \partial K$, $V_x : \mathbb{R}^n \to \mathbb{R}$ be the C^2-function defined, for each $y \in \overline{K}$, by

$$V_x(y) := \langle n_x, (y-x) \rangle.$$

It immediately follows that V_x satisfies, for each $x \in \partial K$, conditions (H1′) and (H2′). Moreover, for each $x \in \partial K$,

$$\nabla V_x(x) = n_x$$

and

$$H V_x(x) = 0.$$

Therefore, if, for all $x \in \partial K$, $t \in (0, T)$ and $v \in \mathbb{R}^n$ with

$$\langle n_x, v \rangle = 0, \tag{30}$$

the following condition holds

$$\langle n_x, w \rangle > 0, \tag{31}$$

for all $w \in F(t, x, v) - A(t)v - B(t)x$, and if, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ satisfying (30), at least one of the following conditions

$$\langle n_x, w_1 \rangle > 0 \tag{32}$$

or

$$\langle n_x, w_2 \rangle > 0 \tag{33}$$

holds, for all $w_1 \in F(0, x, v) - A(0)v - B(0)x$, or, for all $w_2 \in F(T, x, v) - A(T)v - B(T)x$, respectively, then K is a bound set for problem (29).

For our main result concerning the existence and localization of a solution of the Floquet b.v.p., we need to ensure that no solution of given b.v.p.s lies on the boundary ∂Q of a parameter set Q of candidate solutions. We will finally show that if the set Q is defined as follows

$$Q := \left\{ q \in C^1\left([0, T], \mathbb{R}^n\right) \;\middle|\; q(t) \in \overline{K}, \text{ for all } t \in [0, T] \right\} \tag{34}$$

and if all assumptions of Theorem 3.1 are satisfied, then solutions of the b.v.p. (1) behave as indicated.

Proposition 3.2. *Let $K \subset \mathbb{R}^n$ be a nonempty open bounded set and let $Q \subset C^1([0, T], \mathbb{R}^n)$ be defined by the formula* (34)*. Assume that M and N are $n \times n$ matrices with M regular and satisfying condition* (12)*. Moreover, let there exists a C^1-function $V : \mathbb{R}^n \to \mathbb{R}$ with locally Lipschitzian gradient and satisfying conditions* (H1) *and* (H2)*. Furthermore, suppose that, for all $x \in \partial K$, $t \in (0, T)$ and $v \in \mathbb{R}^n$ satisfying* (3)*, condition* (4) *holds, for all $w \in F(t, x, v) - A(t)v - B(t)x$, and that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ satisfying* (13)*, at least one of conditions* (14)*,* (15) *holds, for all $w_1 \in F(0, x, v) - A(0)v - B(0)x$, or, for all $w_2 \in F(T, Mx, Nv) - A(T)Nv - B(T)Mx$. Then problem* (1) *has no solution on ∂Q.*

Proof. One can readily check that if $x \in \partial Q$, then there exists a point $t_x \in [0, T]$ such that $x(t_x) \in \partial K$. But then, according to Theorem 3.1, x cannot be a solution of (1). □

4. Main existence and localization result

The following topological method was developed by ourselves in [3] (cf. also [2]).

Proposition 4.1. *Let us consider the b.v.p.*

$$\left.\begin{aligned} &\ddot{x}(t) \in F\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in J, \\ &x \in S, \end{aligned}\right\} \tag{35}$$

where $J = [a, b]$ is a compact interval, S is a subset of $AC^1(J, \mathbb{R}^n)$ and $F : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping. Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that

$$G(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in J \times \mathbb{R}^n \times \mathbb{R}^n.$$

Assume that

(i) *there exists a retract Q of $C^1(J,\mathbb{R}^n)$ such that $Q\setminus\partial Q$ is nonempty and a closed subset S_1 of S such that the associated problem*

$$\left.\begin{aligned}&\ddot{x}(t)\in G\big(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda\big),\quad \text{for a.a. } t\in J,\\ &x\in S_1\end{aligned}\right\} \tag{36}$$

is solvable with an R_δ-set of solutions, for each $(q,\lambda)\in Q\times[0,1]$,

(ii) *there exists a nonnegative, integrable function $\alpha: J\to\mathbb{R}$ such that*

$$\big|G\big(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda\big)\big|\leqslant\alpha(t)\big(1+\big|x(t)\big|+\big|\dot{x}(t)\big|\big),\quad \text{a.e. in } J,$$

for each $(q,\lambda,x)\in\Gamma_{\mathfrak{T}}$, where $\mathfrak{T}$ denotes the multivalued mapping which assigns to any $(q,\lambda)\in Q\times[0,1]$ the set of solutions of (36) *and $\Gamma_{\mathfrak{T}}$ its graph,*

(iii) $\mathfrak{T}(Q\times\{0\})\subset Q$,

(iv) *there exist $t_0\in J$ and constants $M_0\geqslant 0$, $M_1\geqslant 0$ such that $|x(t_0)|\leqslant M_0$ and $|\dot{x}(t_0)|\leqslant M_1$, for any $x\in\mathfrak{T}(Q\times[0,1])$,*

(v) *the solution map $\mathfrak{T}$ has no fixed points on the boundary ∂Q of Q, for each $(q,\lambda)\in Q\times[0,1]$.*

Then problem (35) *has a solution in Q.*

Remark 4.1. As pointed out in [3], the condition that Q is a retract of $C^1(J,\mathbb{R}^n)$ in Proposition 4.1 can be replaced by the assumption that Q is an absolute neighborhood retract and $ind(\mathfrak{T}(\cdot,0),Q,Q)\neq 0$ (for the definition of the related fixed point index, see [1]). It is therefore possible to assume alternatively that Q is a retract of a convex subset of $C^1(J,\mathbb{R}^n)$ or of an open subset of $C^1(J,\mathbb{R}^n)$ together with $ind(\mathfrak{T}(\cdot,0),Q,Q)\neq 0$.

The solvability of (1) will be now proved, on the basis of Proposition 4.1. Defining namely the set Q of candidate solutions by the formula (34), we are able to verify, for each $(q,\lambda)\in Q\times(0,1]$, the transversality condition (v) in Proposition 4.1.

Let us consider the b.v.p. (1), where $F:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\multimap\mathbb{R}^n$ is an upper semicontinuous mapping with nonempty, compact, convex values, and A, B are continuous matrix functions such that $|A(t)|\leqslant a(t)$ and $|B(t)|\leqslant b(t)$, for all $t\in[0,T]$ and suitable integrable functions $a,b:[0,T]\to[0,\infty)$. Let M and N be $n\times n$ matrices with M nonsingular and satisfying (12).

Theorem 4.1. *Assume that*

(i) *there exists an upper semicontinuous mapping $C:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\multimap\mathbb{R}^n$ with nonempty, compact, convex values such that*

$$C(t,c,d,c,d)\subset F(t,c,d),\quad \text{for all } (t,c,d)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n,$$

(ii) *$C(t,\cdot,\cdot,r_1,r_2)$ is Lipschitzian with a sufficiently small Lipschitz constant L, for each $t\in[0,T]$, $r_1\in\overline{K}$ and $r_2\in\mathbb{R}^n$, where $K\subset\mathbb{R}^n$ is a nonempty open bounded set whose closure $\overline{K}$ is a retract of $\mathbb{R}^n$,*

(iii) *there exist a point $(x_0,y_0)\in\mathbb{R}^n\times\mathbb{R}^n$ and a constant $C_0\geqslant 0$ such that*

$$\big|C(t,x_0,y_0,r_1,r_2)\big|\leqslant C_0\cdot L$$

holds, for a.a. $t\in[0,T]$, all $r_1\in\overline{K}$ and $r_2\in\mathbb{R}^n$,

(iv) *the associated homogeneous problem*

$$\left.\begin{aligned}&\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)=0,\quad \text{for a.a. } t\in[0,T],\\ &x(T)=Mx(0),\quad \dot{x}(T)=N\dot{x}(0)\end{aligned}\right\}$$

has only the trivial solution such that $0\in K$,

(v) *there exists a function $V\in C^1(\mathbb{R}^n,\mathbb{R})$ with ∇V locally Lipschitzian and satisfying conditions* (H1) *and* (H2),

(vi) *for all $x\in\partial K$, $t\in(0,T)$, $\lambda\in(0,1)$ and $v\in\mathbb{R}^n$ with*

$$\big\langle\nabla V(x),v\big\rangle=0, \tag{37}$$

the following condition holds

$$\liminf_{h\to 0}\frac{\langle\nabla V(x+hv),v+hw\rangle}{h}>0, \tag{38}$$

for all $w\in\lambda C(t,x,v,x,v)-A(t)v-B(t)x$,

(vii) *for all* $x \in \partial K$, $\lambda \in (0,1)$ *and* $v \in \mathbb{R}^n$ *with*

$$\big\langle \nabla V(x), v \big\rangle \leqslant 0 \leqslant \big\langle \nabla V(Mx), Nv \big\rangle, \tag{39}$$

at least one of the following conditions

$$\liminf_{h \to 0^+} \frac{\langle \nabla V(x + hv), v + hw_1 \rangle}{h} > 0 \tag{40}$$

or

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(Mx + hNv), Nv + hw_2 \rangle}{h} > 0 \tag{41}$$

holds, for all $w_1 \in \lambda C(0, x, v, x, v) - A(0)v - B(0)x$*, or, for all* $w_2 \in -A(T)Nv - B(T)Mx + \lambda C(T, Mx, Nv, Mx, Nv)$.

Then the b.v.p. (1) *has a solution* $x(\cdot)$ *such that* $x(t) \in \overline{K}$*, for all* $t \in [0, T]$.

Proof. Let us define the set of candidate solution by the formula (34) and let us check that all the assumptions of Proposition 4.1 are satisfied.

First of all, observe that conditions (ii) and (iii) yield the inequality

$$\big|\lambda C\big(t, x(t), \dot{x}(t), q(t), \dot{q}(t)\big)\big| \leqslant L\big(C_0 + |x_0| + |y_0| + \big|x(t)\big| + \big|\dot{x}(t)\big|\big), \tag{42}$$

for each $(q, \lambda, x) \in \Gamma_{\mathfrak{T}}$ and a.a. $t \in [0, T]$, where $\mathfrak{T}$ denotes the mapping which assigns to any $(q, \lambda) \in Q \times [0, 1]$ the set of solutions of

$$\left.\begin{array}{l} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in \lambda C\big(t, x(t), \dot{x}(t), q(t), \dot{q}(t)\big), \quad \text{for a.a. } t \in [0, T], \\ x(T) = Mx(0), \quad \dot{x}(T) = N\dot{x}(0). \end{array}\right\} \tag{43}$$

This means that condition (ii) in Proposition 4.1 is satisfied with

$$\alpha(t) := L\big(\max\big\{C_0 + |x_0| + |y_0|, 1\big\}\big) + \max\big\{a(t), b(t)\big\}.$$

The properties of A, B and C and assumptions (ii)–(iv) guarantee (cf. [6] or [2, Lemma 3.1 and Remark 3.2]) that the set of solutions of the problem (43) must be, for all $(q, \lambda) \in Q \times [0, 1]$, a compact *AR*-space and, in particular, an R_δ-set, as required.

It follows from the main result in [6] (cf. the proof of Lemma 3.1 in [2]) and conditions (ii) and (iii) that there exists $R > 0$ such that the set of all solutions of problem (43) is a subset of

$$\overline{B(0,R)} := \Big\{x \in C^1\big([0, T], \mathbb{R}^n\big) \;\Big|\; \max_{t \in [0,T]} \big|x(t)\big| \leqslant R,\; \max_{t \in [0,T]} \big|\dot{x}(t)\big| \leqslant R\Big\},$$

for each $(q, \lambda) \in Q \times [0, 1]$.

Putting

$$S_1 := \overline{B(0,R)} \cap \big\{x \in AC^1\big([0, T], \mathbb{R}^n\big) \;\big|\; x(T) = Mx(0),\; \dot{x}(T) = N\dot{x}(0)\big\},$$

the boundedness of $\overline{B(0,R)}$ implies the same property for S_1, by which condition (iv) in Proposition 4.1 is trivially satisfied.

Furthermore, in view of the properties of K (cf. [3, Lemma 4]), Q is a retract of the space $C^1([0, T], \mathbb{R}^n)$. Since $Q \setminus \partial Q$ is nonempty and the boundary conditions in (43) define a closed set in $C^1([0, T], \mathbb{R}^n)$, condition (i) in Proposition 4.1 holds.

We will finally show that condition (v) in Proposition 4.1 is satisfied as well. Let us assume that $x \in Q$ is a fixed point of the mapping $\mathfrak{T}(\cdot, \lambda)$, for some $\lambda \in [0, 1]$. This implies that x is a solution of the problem

$$\left.\begin{array}{l} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in \lambda C\big(t, x(t), \dot{x}(t), x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0, T], \\ x(T) = Mx(0), \quad \dot{x}(T) = N\dot{x}(0). \end{array}\right\} \tag{44}$$

Properties of K and conditions (v)–(vii) imply that K is a bound set for problem (44), for all $\lambda \in (0, 1)$. If $x \in \partial Q$ is a fixed point of $\mathfrak{T}(\cdot, 1)$ (i.e., for $\lambda = 1$), problem (1) has, according to assumption (i), a solution in Q, and we are done.

Otherwise, according to Proposition 3.2, $x \notin \partial Q$, i.e. condition (v) in Proposition 4.1 holds, for all $(q, \lambda) \in Q \times (0, 1]$. Condition (iv) implies that also

$$Fix\big(\mathfrak{T}\big(Q \times \{0\}\big)\big) \cap \partial Q = \emptyset,$$

and so condition (v) from Proposition 4.1 is satisfied, for all $(q, \lambda) \in Q \times [0, 1]$, which completes the proof. □

Remark 4.2. In fact, $x \in Q \cap \overline{B(0,R)}$ holds for a solution $x(\cdot)$ of the b.v.p. (43), where $R > 0$ is a suitable constant implied by conditions (ii), (iii), as pointed out in the proof of Theorem 4.1. It particularly means that $\max_{t\in[0,T]} |\dot{x}(t)| \leqslant R$. Moreover, this estimate can be still improved by means of the Nagumo-type inequalities (cf. [10,11,17,20]). Thus, condition (ii) can be a bit relaxed in this way.

Remark 4.3. For fully linearized problems of the form

$$\left.\begin{aligned}&\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in \lambda F\big(t, q(t), \dot{q}(t)\big), \quad \text{for a.a. } t \in [0,T],\\ &x(T) = Mx(0), \quad \dot{x}(T) = N\dot{x}(0),\end{aligned}\right\}$$

condition (ii) in Theorem 4.1 is obviously trivially satisfied. Moreover, since F is convex-valued, one can easily check that $\mathfrak{T}(q,\lambda)$ is, for all $q \in Q$ and $\lambda \in [0,1]$, a convex set. The compactness of $\mathfrak{T}(q,\lambda)$ follows from the proof of Proposition 4.1. For more details, see [2,3].

We conclude by two illustrative examples of application of Theorem 4.1.

Example 4.1. Let us consider the anti-periodic b.v.p.

$$\left.\begin{aligned}&\ddot{x}(t) \in F_1\big(t, x(t), \dot{x}(t)\big) + F_2\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0,T],\\ &x(T) = -x(0), \quad \dot{x}(T) = -\dot{x}(0),\end{aligned}\right\} \tag{45}$$

where $F_1, F_2 : [0,T] \times \mathbb{R}^{2n} \multimap \mathbb{R}^n$ are upper semicontinuous mappings with compact and convex values such that $F_1(t,\cdot,\cdot)$ is Lipschitzian with a sufficiently small Lipschitz constant L, for all $t \in [0,T]$. Moreover, let

$$\big|F_1(t,0,0)\big| \leqslant C_0 \cdot L$$

and

$$\big|F_2(t,x,y)\big| \leqslant C_1 \cdot L\big(1 + |x|\big),$$

for each $(t,x,y) \in [0,T] \times \mathbb{R}^{2n}$, where $C_0, C_1 \geqslant 0$.

Let $R > 0$ be such that

$$\langle x, w\rangle + \langle v, v\rangle > 0 \tag{46}$$

holds, for all $t \in [0,T]$, $x \in \mathbb{R}^n$ with $|x| = R$, $\lambda \in (0,1)$, v satisfying $\langle x, v\rangle = 0$, and $w \in \lambda(F_1(t,x,v) + F_2(t,x,v))$. Put $K := \{x \in \mathbb{R}^n \mid |x| < R\}$ and let Q be defined by the formula (34).

In order to apply Theorem 4.1, for the solvability of problem (45), let us consider the associated problems

$$\left.\begin{aligned}&\ddot{x}(t) \in \lambda\big(F_1\big(t, x(t), \dot{x}(t)\big) + F_2\big(t, q(t), \dot{q}(t)\big)\big), \quad \text{for a.a. } t \in [0,T],\\ &x(T) = -x(0), \quad \dot{x}(T) = -\dot{x}(0),\end{aligned}\right\} \tag{47}$$

where $\lambda \in [0,1]$ and $q \in Q$.

We show that all assumptions of Theorem 4.1 are satisfied by means of the C^2-function $V(x) := \frac{1}{2}(|x|^2 - R^2)$. Since $V(x) = 0$, for all $x \in \partial K$, and $V(x) \leqslant 0$, for all $x \in \overline{K}$, V satisfies conditions (H1) and (H2). Moreover, since, for each $x \in \partial K$, $\nabla V(x) = x$ and $HV(x) = Id$, where H stands for the Hessian matrix, condition (46) ensures that K is a bound set for the problem

$$\left.\begin{aligned}&\ddot{x}(t) \in \lambda\big(F_1\big(t, x(t), \dot{x}(t)\big) + F_2\big(t, x(t), \dot{x}(t)\big)\big), \quad \text{for a.a. } t \in [0,T],\\ &x(T) = -x(0), \quad \dot{x}(T) = -\dot{x}(0),\end{aligned}\right\}$$

for each $(q,\lambda) \in Q \times (0,1)$.

Since $\overline{K}$ is convex, it is an AR-space, and so a retract of $\mathbb{R}^n$, as required. Moreover, because $C(t,x,\dot{x},q,\dot{q}) = F_1(t,x,\dot{x}) + F_2(t,q,\dot{q})$, it holds that

$$C(t,c,d,c,d) = F_1(t,c,d) + F_2(t,c,d), \quad \text{for all } (t,c,d) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n,$$

and so condition (i) is satisfied.

The associated homogeneous problem

$$\left.\begin{aligned}&\ddot{x}(t) = 0, \quad \text{for a.a. } t \in [0,T],\\ &x(T) = -x(0), \quad \dot{x}(T) = -\dot{x}(0)\end{aligned}\right\}$$

has only the trivial solution $x(t) = \mathfrak{T}(q,0) \equiv 0$, for each $q \in Q$, and $0 \in K$, by which condition (iv) holds.

Assumption (12) is obviously satisfied as well, because the invariance of ∂K with respect to $M = -Id$ is equivalent to the symmetry of ∂K with respect to the origin. The anti-periodic problem (45) therefore admits a solution in Q.

Example 4.2. Let us consider the periodic b.v.p.

$$\left.\begin{array}{l}\ddot{x}(t) \in F_1\big(t, x(t), \dot{x}(t)\big) + F_2\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0, T],\\ x(T) = x(0), \quad \dot{x}(T) = \dot{x}(0),\end{array}\right\} \tag{48}$$

where $F_1, F_2 : [0, T] \times \mathbb{R}^{2n} \multimap \mathbb{R}^n$ satisfy the same conditions as in Example 4.1. Given $\varphi(a) := 3 + \mathrm{e}^{-2aT} - 3\mathrm{e}^{-aT} - \mathrm{e}^{aT}$, take $a_0 > 0$ such that

$$\varphi(a_0) \neq 0. \tag{49}$$

Let $K \subset \mathbb{R}^n$ be a bounded open and convex set with $0 \in K$. For each $x \in \partial K$, let us denote by n_x an outer normal of K at x. Such an outer normal surely exists (see Example 3.1). Assume that, for each $x \in \partial K$ and $v \in \mathbb{R}^n$ satisfying (30), the following conditions hold

$$\langle n_x, w\rangle > 0, \tag{50}$$

for all $t \in (0, T)$, $\lambda \in (0, 1)$ and $w \in \lambda(F_1(t, x, v) + F_2(t, x, v)) + a_0^2(1 - \lambda)x$ and

$$\max\big\{\langle n_x, w_1\rangle, \langle n_x, w_2\rangle\big\} > 0, \tag{51}$$

for all $\lambda \in (0, 1)$ and $w_1 \in \lambda(F_1(0, x, v) + F_2(0, x, v)) + a_0^2(1 - \lambda)x$, $w_2 \in \lambda(F_1(T, x, v) + F_2(T, x, v)) + a_0^2(1 - \lambda)x$. Then it is possible to show that problem (48) is solvable.

Indeed, let us rewrite (48) as follows

$$\left.\begin{array}{l}\ddot{x}(t) - a_0^2 x(t) \in F_1\big(t, x(t), \dot{x}(t)\big) - a_0^2 x(t) + F_2\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0, T],\\ x(T) = x(0), \quad \dot{x}(T) = \dot{x}(0)\end{array}\right\}$$

and let us consider the associated problems

$$\left.\begin{array}{l}\ddot{x}(t) - a_0^2 x(t) \in \lambda\big(F_1\big(t, x(t), \dot{x}(t)\big) - a_0^2 q(t) + F_2\big(t, q(t), \dot{q}(t)\big)\big), \quad \text{for a.a. } t \in [0, T],\\ x(T) = x(0), \quad \dot{x}(T) = \dot{x}(0),\end{array}\right\} \tag{52}$$

where $\lambda \in [0, 1]$ and $q \in Q$ which is defined by the formula (34).

We will show that all assumptions of Theorem 4.1 are satisfied when considering the family of C^2-bounding functions $V_x(y) := \langle n_x, (y - x)\rangle$. According to (50) and (51) (see Example 3.1), conditions (vi), (vii) in Theorem 4.1 hold. Moreover, the family of bounding functions $\{V_x\}_{x \in \partial K}$ satisfies (H1′), (H2′) which are equivalent, for our aims, to (H1), (H2).

So, it only remains to consider the associated homogeneous problem

$$\left.\begin{array}{l}\ddot{x}(t) - a_0^2 x(t) = 0, \quad \text{for a.a. } t \in [0, T],\\ x(T) = x(0), \quad \dot{x}(T) = \dot{x}(0).\end{array}\right\} \tag{53}$$

It follows from (49) that the homogeneous problem (53) has only the trivial solution. Moreover, since $0 \in K$, condition (iv) is satisfied. Therefore, the periodic problem (48) has a solution in Q.

Let us note that in the special case, when K is a ball centered at the origin of some radius r, the following condition

$$\langle x, y\rangle \geqslant 0,$$

for all x with $|x| = r$, v satisfying (30), $t \in (0, T)$, $y \in F_1(t, x, v) + F_2(t, x, v)$, guarantees condition (50) for every $\lambda \in (0, 1)$. Consider $w \in \lambda(F_1(t, x, v) + F_2(t, x, v)) + a_0^2(1 - \lambda)x$ with λ, t, x and v as above. Then, $w = \lambda y + a_0^2(1 - \lambda)x$, and so $\langle x, w\rangle = \lambda\langle x, y\rangle + a_0^2(1 - \lambda)|x|^2 > 0$. Condition (51) can be reformulated in a similar way.

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Strictly localized bounding functions for vector second-order boundary value problems

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ABSTRACT

The solvability of the second-order Floquet problem in a given set is established by means of C^2-bounding functions for vector upper-Carathéodory systems. The applied Scorza–Dragoni type technique allows us to impose related conditions strictly on the boundaries of bound sets. An illustrating example is supplied for a dry friction problem.

1. Introduction

Let us consider the Floquet boundary value problem (b.v.p.)

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t, x(t), \dot{x}(t)), \qquad &\text{for a.a. } t \in [0, T], \\ x(T) = Mx(0), \qquad \dot{x}(T) = N\dot{x}(0), \qquad & \end{aligned}\right\} \tag{1}$$

where

(1_i) $A, B : [0, T] \to \mathbb{R}^{n\times n}$ are measurable matrix functions such that $|A(t)| \le a(t)$ and $|B(t)| \le b(t)$, for all $t \in [0, T]$ and suitable integrable functions $a, b : [0, T] \to [0, \infty)$,

(1_{ii}) M and N are $n \times n$ matrices, M is nonsingular,

(1_{iii}) $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping.

By a *solution* of problem (1), we mean a vector function $x : [0, T] \to \mathbb{R}^n$ with an absolutely continuous first derivative (i.e. $x \in AC^1([0, T], \mathbb{R}^n)$) which satisfies (1), for almost all $t \in [0, T]$.

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Problem (1) was studied by ourselves via a bound sets approach already in [1]. There the conditions concerning (Lyapunov-like) bounding functions were not imposed directly on the boundaries of bound sets, but at some vicinity of them.

This problem does not occur for Marchaud systems, i.e. for systems with globally upper semicontinuous right-hand sides (see [2]). On the other hand, the case of upper-Carathéodory systems must be furthermore elaborated, for the same goal, by means of suitable Scorza–Dragoni type theorems which is the main aim of this paper.

For the first-order systems, the situation is analogous, but less technical (see [3–7] and cf. [8, Chapter III.8]). Nevertheless, the second-order systems allow us some more flexibility in the sense that the derivatives need not necessarily be taken into account.

The original idea of applying the Scorza–Dragoni technique comes from [9], where guiding functions were employed for vector first-order Carathéodory differential equations. For further references concerning boundary value problems for second-order systems, see, e.g., [1,2,10–12] and the references therein.

Our main result (see Theorem 3.1) shows the solvability of the b.v.p. (1) in the upper-Carathéodory case with strictly localized bounding functions. We separated as much as possible the technicalities needed for its proof into Preliminaries. Its applicability is finally demonstrated by a simple illustrating example for a dry friction problem, when both periodic ($M = N = I$) and anti-periodic ($M = N = -I$) solutions coexist in a given set.

2. Preliminaries

If (X, d) is a metric space and $A \subset X$, by $\overline{A}$ and ∂A, we mean the closure and the boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x, a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X.

The symbol B_R denotes, as usually, the open ball in $\mathbb{R}^n$ with radius $R > 0$ centered at 0, i.e. $B_R := \{x \in \mathbb{R}^n \mid |x| < R\}$.

Let us recall the following definitions from the multivalued analysis. Let X and Y be arbitrary metric spaces. We say that φ is a multivalued mapping from X to Y (written $\varphi : X \multimap Y$) if, for every $x \in X$, a nonempty subset $\varphi(x)$ of Y is prescribed.

A multivalued mapping $\varphi : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \subset U\}$ is open in X.

Let Y be a metric space and $(\Omega, \mathcal{U}, \mu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure μ on $\mathcal{U}$. A multivalued mapping $\varphi : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$. In what follows, the symbol μ will exclusively denote the *Lebesgue measure* on $\mathbb{R}$.

We say that mapping $\varphi : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $\varphi(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $\varphi(t, \cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all (a.a.) $t \in J$, and the set $\varphi(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

Let $X = (X, d)$ be a metric space. A multivalued mapping $\varphi : X \multimap X$ with bounded values is called *Lipschitzian* if there exists a constant $L > 0$ such that

$$d_H(\varphi(x), \varphi(y)) \le Ld(x, y),$$

for every $x, y \in X$, where

$$d_H(A, B) := \inf\{r > 0 \mid A \subset N_r(B) \text{ and } B \subset N_r(A)\}$$

stands for the Hausdorff distance. For more information and details concerning multivalued analysis, see, e.g., [8,13–15].

We will also need the following slight modification of the Scorza–Dragoni type result for multivalued mappings.

Proposition 2.1 *(cf., e.g., [16, Proposition 8]). Let $X \subset \mathbb{R}^m$ be compact and let $F : [a, b] \times X \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping. Then there exists a multivalued mapping $F_0 : [a, b] \times X \multimap \mathbb{R}^n \cup \{\emptyset\}$ with compact, convex values and $F_0(t, x) \subset F(t, x)$, for all $(t, x) \in [a, b] \times X$, having the following properties:*

(i) *if $u, v : [a, b] \to \mathbb{R}^n$ are measurable functions with $v(t) \in F(t, u(t))$, on $[a, b]$, then $v(t) \in F_0(t, u(t))$, a.e. on $[a, b]$;*
(ii) *for every $\varepsilon > 0$, there exists a closed $I_\varepsilon \subset [a, b]$ such that $\mu([a, b] \setminus I_\varepsilon) < \varepsilon$, $F_0(t, x) \neq \emptyset$, for all $(t, x) \in I_\varepsilon \times X$, and F_0 is u.s.c. on $I_\varepsilon \times X$.*

It will be convenient to recall some basic facts concerning evolution systems. For a suitable introduction and further details, we refer, e.g., to [17].

Hence, let $C : [a, b] \to \mathbb{R}^{m\times m}$ be a measurable matrix function such that $|C(t)| \le c(t)$, for all $t \in [a, b]$, with $c \in L^1([a, b], [0, \infty))$ and let $f \in L^1([a, b], \mathbb{R}^m)$. Given $x_0 \in \mathbb{R}^m$, consider the linear initial value problem

$$\dot{x}(t) = C(t)x(t) + f(t), \quad x(a) = x_0. \tag{2}$$

It is well-known (see, e.g., [17]) that, for the uniquely solvable problem (2), there exists the evolution operator $\{U(t, s)\}_{(t,s)\in\Delta}$, where $\Delta := \{(t, s) : a \le s \le t \le b\}$, such that

$$|U(t, s)| \le \mathrm{e}^{\int_s^t |C(\tau)|\,\mathrm{d}\tau}, \quad \text{for all } (t, s) \in \Delta; \tag{3}$$

in addition, the unique solution $x(\cdot)$ of (2) is given by

$$x(t) = U(t, a)x_0 + \int_a^t U(t, s)f(s)\mathrm{d}s, \quad t \in [a, b]. \tag{4}$$

Given the $m \times m$ matrix D, the linear Floquet b.v.p.

$$\left.\begin{aligned} &\dot{x}(t) = C(t)x(t) + f(t),\\ &x(b) = Dx(a) \end{aligned}\right\} \tag{5}$$

associated to the equation in (2), satisfies the following property.

Lemma 2.1. *If $D - U(b, a)$ is invertible, then (5) admits a unique solution given by*

$$x(t) = U(t, a)\Big[D - U(b, a)\Big]^{-1} \int_a^b U(b, \tau)f(\tau)\,\mathrm{d}\tau + \int_a^t U(t, \tau)f(\tau)\,\mathrm{d}\tau. \tag{6}$$

Proof. In view of (4) and the Floquet boundary condition,

$$U(b, a)x_0 + \int_a^b U(b, s)f(s)\,\mathrm{d}s = Dx(a) = Dx_0.$$

Thus, if $D - U(b, a)$ is invertible, the foregoing equation is satisfied just for the unique value

$$x_0 = \Big[D - U(b, a)\Big]^{-1} \int_a^b U(b, \tau)f(\tau)\,\mathrm{d}\tau.$$

Substituting it into (4), we arrive at (6). □

Remark 2.1. Denoting

$$\Lambda := \mathrm{e}^{\int_a^b c(s)\,\mathrm{d}s}, \qquad \Gamma := |[D - U(b, a)]^{-1}|,$$

we obtain, in view of (3), (6) and the constraint for $C(t)$, the following estimate for the solution $x(\cdot)$ of (5)

$$|x(t)| \le \Lambda\,(\Lambda\Gamma + 1) \int_a^b |f(s)|\,\mathrm{d}s. \tag{7}$$

It is easy to give a necessary and sufficient condition for the invertibility of $D - U(b, a)$.

Lemma 2.2. *Operator $D - U(b, a)$ is invertible if and only if the associated homogeneous problem*

$$\left.\begin{aligned} &\dot{x}(t) = C(t)x(t), \quad \text{for a.a. } t \in [a, b],\\ &x(b) = Dx(a) \end{aligned}\right\} \tag{8}$$

has only the trivial solution.

Proof. Consider the b.v.p. (5) with $f \in L^1([a, b], \mathbb{R}^m)$. According to [6, Lemma 5.1], $D - U(b, a)$ is invertible if and only if (5) is uniquely solvable, for each $f \in L^1([a, b], \mathbb{R}^m)$. In the quoted lemma, this equivalence was obtained under the a priori assumption that D is invertible. It is however easy to see that the invertibility of D is not necessary to conclude. According to [18, Lemma 2], the property that (8) has only the trivial solution implies the unique solvability of (5), for each $f \in L^1([a, b], \mathbb{R}^m)$. Since the reversed implication is trivial, we have that the unique solvability of (5), for every $f \in L^1([a, b], \mathbb{R}^m)$, is, in fact, equivalent to the property that (8) has only the trivial solution which completes the proof. □

Let K be a nonempty bounded subset of $\mathbb{R}^n$ and put $Q := \{q \in C^1([0, T], \mathbb{R}^n) \mid q(t) \in \overline{K}, \text{for all } t \in [0, T]\}$. In the proof of our main result (Theorem 3.1 below), we shall need to consider the following family of b.v.p.s

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in G(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad &\text{for a.a. } t \in [0, T],\\ x(T) = Mx(0), \qquad \dot{x}(T) = N\dot{x}(0), \quad & \end{aligned}\right\} \tag{9}$$

where A and B satisfy conditions (1_i), $\lambda \in [0, 1]$ and $q \in Q$. In view of the above considerations, when assuming that the homogeneous problem associated to (9) has only the trivial solution and that

$$|G(t, x, y, r_1, r_2, \lambda)| \le \alpha(t), \quad \text{a.e. in } [0, T], \tag{10}$$

for all $x,\ y,\ r_2 \in \mathbb{R}^n$, $\lambda \in [0,1]$ and $r_1 \in \overline{K}$, we are able to show the existence of $R > 0$ such that $|x(t)| \le R$ and $|\dot{x}(t)| \le R$, for all $t \in [0,T]$ and every solution $x(\cdot)$ of (9). We can also prove that the constant R does not depend on G, but only on $\alpha(t),\ A(t),\ B(t),\ M,\ N$ and T. More precisely, given

$$C(t)_{2n\times 2n} := \begin{pmatrix} 0 & -I \\ B(t) & A(t) \end{pmatrix}, \qquad (11)$$
$$D_{2n\times 2n} := \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

and

$$\begin{aligned} &\tilde{\Lambda} := \mathrm{e}^{\int_0^T (a(s)+b(s)+\sqrt{n})\,\mathrm{d}s}, \qquad \tilde{\Gamma} := |[D-U(T,0)]^{-1}|, \\ &\eta := \tilde{\Lambda}\left(\tilde{\Lambda}\tilde{\Gamma}+1\right)\int_0^T \alpha(s)\,\mathrm{d}s, \end{aligned} \qquad (12)$$

where $\{U(t,s)\}_{(t,s)\in\Delta}$ is the evolution operator associated to $\dot{\xi}(t) = C(t)\xi(t)$, we can obtain the following result.

Lemma 2.3. *Let us consider the b.v.p. (9) and assume that conditions (1_i) and (10) are satisfied. Suppose, moreover, that*

(i) *the associated homogeneous problem*

$$\left.\begin{aligned} &\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = 0, \quad \text{for a.a. } t \in [0,T], \\ &x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0) \end{aligned}\right\}$$

has only the trivial solution,

(ii) *$G : [0,T] \times \mathbb{R}^{4n} \times [0,1] \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping.*

Then each solution $x(\cdot)$ of (9) satisfies $|x(t)| \le \eta$ and $|\dot{x}(t)| \le \eta$, for all $t \in [0,T]$, with η defined in (12).

Proof. Problem (9) is equivalent to the following first-order one:

$$\begin{aligned} &\dot{\xi}(t) + C(t)\xi(t) \in H(t,\xi(t),q(t),\dot{q}(t),\lambda), \quad \text{for a.a. } t \in [0,T], \\ &\xi(T) = D\xi(0), \end{aligned} \qquad (13)$$

where

$$\xi_{2n\times 1} = (x,y)^{\mathrm{T}} = (x,\dot{x})^{\mathrm{T}},$$

matrices $C(t)$ and D were defined above and

$$H(t,\xi,q,\dot{q},\lambda) = (0, G(t,x,y,q,\dot{q},\lambda))^{\mathrm{T}}.$$

According to the assumption (i), the associated homogeneous problem to (13)

$$\begin{aligned} &\dot{\xi}(t) + C(t)\xi(t) = 0, \quad \text{for a.a. } t \in [0,T], \\ &\xi(T) = D\xi(0), \end{aligned}$$

has only the trivial solution. In view of (1_i) and the definition of C, $|C(t)| \le a(t) + b(t) + \sqrt{n}$, for all $t \in [0,T]$. Let $\xi(t) = (x(t),\dot{x}(t))$ be a solution of problem (13) and let us take an arbitrary $h(t) \in H(t,\xi(t),q(t),\dot{q}(t),\lambda)$. From (10) and the definition of H, we obtain

$$|h(t)| \le \alpha(t), \quad \text{for a.a. } t \in [0,T].$$

Lemma 2.2 and (7) then imply that

$$|\xi(t)| \le \tilde{\Lambda}\left(\tilde{\Lambda}\tilde{\Gamma}+1\right)\int_0^T \alpha(s) = \eta$$

which completes the proof. □

3. Main existence and localization result

Now, it is time to consider the Floquet b.v.p. (1). For its solvability, Theorem 3.1 below will be formulated. As we shall see, the crucial requirement will consist of making a fixed-point free boundary of a parameter set Q of candidate solutions. In other words, we need to guarantee a transversality condition on the boundary of the associated bound set K. Therefore, in order to understand the geometric behaviour of trajectories, it is natural to start with the definition of a bound set K and to indicate two properties $(H1)$, $(H2)$ of the basic tool, a Lyapunov-like bounding function V.

Hence, let $K \subset \mathbb{R}^n$ be a nonempty open bounded set satisfying the following conditions:

(K_i) $0 \in K$,
(K_{ii}) $M\partial K = \partial K$,
(K_{iii}) the closure $\overline{K}$ of K is a retract of $\mathbb{R}^n$.

Moreover, let $V : \mathbb{R}^n \to \mathbb{R}$ be a suitable C^2-function such that

($H1$) $V|_{\partial K} = 0$,
($H2$) $V(x) \le 0$, for all $x \in \overline{K}$.

Definition 3.1. A nonempty open set $K \subset \mathbb{R}^n$ is called a *bound set* for the b.v.p. (9) if there does not exist a solution x of the b.v.p. (9) such that $x(t) \in \overline{K}$, for each $t \in [0, T]$, and $x(t_0) \in \partial K$, for some $t_0 \in [0, T]$.

The proof of Theorem 3.1 below will be based on the following proposition developed, in principle, by ourselves in [1, Theorem 2] jointly with the Scorza–Dragoni technique in Proposition 2.1.

Proposition 3.1 (*cf. [1, Theorem 2]*). *Let us consider the Floquet b.v.p. (1), the set K satisfying conditions (K_i)–(K_{iii}) and assume that*

(P_i) *there exists an upper-Carathéodory mapping $C : [0, T] \times \mathbb{R}^{4n} \times [0, 1] \multimap \mathbb{R}^n$ such that*

$$C(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n,$$

(P_{ii}) *for each $t \in [0, T]$, $r_1 \in \overline{K}$, $r_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, $C(t, \cdot, \cdot, r_1, r_2, \lambda)$ is Lipschitzian with a Lipschitz function $L(t) = L + l(t)$, where constant L as well as $\int_0^T l(t)\,dt$ are sufficiently small,*
(P_{iii}) *there exists a nonnegative, integrable function $\alpha : [0, T] \to \mathbb{R}$ such that*

$$|C(t, x, y, r_1, r_2, \lambda)| \le \alpha(t), \quad \text{for a.a. } t \in [0, T],$$

for all $x, y \in \mathbb{R}^n$, $r_1 \in \overline{K}$, $r_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,
(P_{iv}) *the associated homogeneous problem*

$$\left.\begin{array}{ll}\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = 0, & \text{for a.a. } t \in [0, T],\\ x(T) = Mx(0), \quad \dot{x}(T) = N\dot{x}(0) & \end{array}\right\}$$

has only the trivial solution,
(P_v) *$\mathfrak{T}(Q \times \{0\}) \subset Q$, where Q is defined by the formula*

$$Q := \{q \in C^1([0, T], \mathbb{R}^n) \mid q(t) \in \overline{K}, \quad \text{for all } t \in [0, T]\} \tag{14}$$

and $\mathfrak{T}$ denotes the mapping which assigns to any $(q, \lambda) \in Q \times [0, 1]$ the set of solutions of

$$\left.\begin{array}{ll}\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), & \text{for a.a. } t \in [0, T],\\ x(T) = Mx(0), \quad \dot{x}(T) = N\dot{x}(0), & \end{array}\right\} \tag{15}$$

(P_{vi}) *there exists $\varepsilon > 0$ such that, for all $\lambda \in (0, 1)$, $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0, T)$ and $v \in \mathbb{R}^n$ satisfying $|v| \le \eta$ with η defined in (12),*

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0, \tag{16}$$

for all $w \in C(t, x, v, x, v, \lambda) - A(t)v - B(t)x$,
(P_{vii}) *for all $x \in \partial K$ and $v \in \mathbb{R}^n$ satisfying $|v| \le \eta$,*

$$\langle \nabla V(Mx), Nv\rangle \cdot \langle \nabla V(x), v\rangle > 0$$

or

$$\langle \nabla V(Mx), Nv\rangle = \langle \nabla V(x), v\rangle = 0.$$

Then the Floquet b.v.p. (1) has a solution $x(\cdot)$ such that $x(t) \in \overline{K}$, for all $t \in [0, T]$.

Remark 3.1. We note that the analogous conclusions in [1, Proposition 5], which are crucial for proving [1, Theorem 2], are true even if conditions (24) and (25) in the quoted proposition are only valid for v satisfying $|v| \le \eta$, with η defined as in (12). Indeed, checking the proof of the quoted proposition, v plays the role of the solution derivative which, according to Lemma 2.3, always satisfies the mentioned inequality. Therefore, Proposition 3.1 represents a bit more general version of [1, Theorem 2]. It differs from the quoted theorem also because a sufficiently small Lipschitz constant L is here replaced by an integrable function $L(t)$ such that $\int_0^T L(t)\,dt$ is sufficiently small. This change is quite standard and it already appeared in [3, Proposition 2]. On the other hand, the bounding function V (taken here, for the sake of simplicity, from the C^2-class) can be only smooth with a locally Lipschitzian gradient ∇V, as in [1, Theorem 2].

Remark 3.2. Conditions (P_{vi}) and (P_{vii}) guarantee that K is a bound set for the b.v.p.

$$\left.\begin{array}{c}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in C(t,x(t),\dot{x}(t),x(t),\dot{x}(t),\lambda), \quad \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0), \qquad \dot{x}(T)=N\dot{x}(0),\end{array}\right\} \tag{17}$$

for all $\lambda \in (0, 1)$.

Approximating the original problem by a sequence of problems satisfying conditions of Proposition 3.1 and applying the Scorza–Dragoni type result (Proposition 2.1), we are already able to state the main result of the paper. The transversality condition is now required only on the boundary ∂K of the set K, but not on the whole neighborhood $\overline{K} \cap N_\varepsilon(\partial K)$, as in Proposition 3.1, whence the title.

Theorem 3.1. *Let us consider the Floquet b.v.p. (1) and assume that*

(i) *there exists an upper-Carathéodory mapping $C : [0, T] \times \mathbb{R}^{4n} \multimap \mathbb{R}^n$ such that*

$$C(t,c,d,c,d)\subset F(t,c,d), \quad \textit{for all } (t,c,d)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n,$$

(ii) *$C(t,\cdot,\cdot,r_1,r_2)$ is Lipschitzian with a sufficiently small Lipschitz constant L, for each $t \in [0, T]$, $r_1 \in \overline{K}$ and $r_2 \in \mathbb{R}^n$,*

(iii) *there exists a nonnegative, integrable function $\alpha : [0, T] \to \mathbb{R}$ such that*

$$|C(t,x,y,r_1,r_2)|\le\alpha(t), \quad \textit{for a.a. } t\in[0,T],$$

for all $x, y \in \mathbb{R}^n$, $r_1 \in \overline{K}$ and $r_2 \in \mathbb{R}^n$,

(iv) *the associated homogeneous problem*

$$\left.\begin{array}{c}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)=0, \quad \textit{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0)\end{array}\right\}$$

has only the trivial solution,

(v) *for all $\lambda \in (0, 1)$, $x \in \partial K$, $t \in (0, T)$ and $v \in \mathbb{R}^n$, it holds that*

$$\langle\nabla V(x),w\rangle>0, \tag{18}$$

for all $w \in \lambda C(t, x, v, x, v) - A(t)v - B(t)x$,

(vi) *there exists $h > 0$ such that $HV(x)$ is positive semi-definite, for all $x \in \overline{K} \cap N_h(\partial K)$,*

(vii) *for all $x \in \partial K$ and $v \in \mathbb{R}^n$,*

$$\langle\nabla V(Mx),Nv\rangle\cdot\langle\nabla V(x),v\rangle>0$$

or

$$\langle\nabla V(Mx),Nv\rangle=\langle\nabla V(x),v\rangle=0,$$

(viii) *for all $x \in \partial K$, $\nabla V(x) \neq 0$.*

Then the b.v.p. (1) has a solution $x(\cdot)$ such that $x(t) \in \overline{K}$, for all $t \in [0, T]$.

Proof. According to condition (viii) and since ∂K is compact, there exists $\delta > 0$ such that $\nabla V(x) \neq 0$, for all $x \in N_\delta(\partial K)$. Moreover, there exists $\gamma > 0$ such that $|\nabla V(x)| \geq \gamma$, for all $x \in \partial K$.

Let $\tau \in C^1(\mathbb{R}^n, [0, 1])$ be such that $\tau \equiv 1$ on $N_{\frac{\delta}{2}}(\partial K)$ and $\tau \equiv 0$ on $\mathbb{R}^n \setminus N_\delta(\partial K)$.

Conditions (i), (iii) and (iv) guarantee (cf. Lemma 2.3) that there exists $\eta > 0$, defined in (12), such that, for all $\lambda \in [0, 1]$ and $q \in Q$, where Q is defined by formula (14), an arbitrary solution $x(\cdot)$ of

$$\left.\begin{array}{c}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in\lambda C(t,x(t),\dot{x}(t),q(t),\dot{q}(t)), \quad \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0), \qquad \dot{x}(T)=N\dot{x}(0)\end{array}\right\} \tag{19}$$

satisfies $|x(t)| \leq \eta$ and $|\dot{x}(t)| \leq \eta$, for all $t \in [0, T]$.

Let us consider an open bounded set $K_0 \subset \mathbb{R}^n$ such that $\overline{K} \subset K_0$. Since C is an upper-Carathéodory mapping and A, B are measurable matrix functions, we can apply a Scorza–Dragoni type result (cf. Proposition 2.1). Consequently, there exists a decreasing sequence $\{\theta_m\}$ of subsets of $[0, T]$ and a measurable mapping $\overline{C} : [0, T] \times \overline{K_0} \times \overline{B_{2\eta}} \times \overline{K_0} \times \overline{B_{2\eta}} \multimap \mathbb{R}^n$ such that, for every $m \in \mathbb{N}$,

- $[0, T] \setminus \theta_m$ is compact and $\mu(\theta_m) < \frac{1}{m}$,
- $\overline{C}(t, x, y, u, v) \subset C(t, x, y, u, v)$, for all $(t, x, y, u, v) \in [0, T] \times \overline{K_0} \times \overline{B_{2\eta}} \times \overline{K_0} \times \overline{B_{2\eta}}$,
- $\overline{C}$ is u.s.c. on $([0, T] \setminus \theta_m) \times \overline{K_0} \times \overline{B_{2\eta}} \times \overline{K_0} \times \overline{B_{2\eta}}$,
- A, B are continuous matrix functions on $[0, T] \setminus \theta_m$.

It is obvious that $\bigcap_{m=1}^{\infty} \theta_m$ has zero Lebesgue measure and that $\lim_{m\to\infty} \chi_{\theta_m}(t) = 0$, for every $t \notin \bigcap_{m=1}^{\infty} \theta_m$. Therefore, $\overline{C}$ is an upper-Carathéodory mapping.

Let us define the mapping $\hat{C} : [0, T] \times \mathbb{R}^{4n} \multimap \mathbb{R}^n$ by the formula

$$\hat{C}(t, x, y, u, v) := \begin{cases} \overline{C}(t, x, y, u, v), & \text{for } (t, x, y, u, v) \in [0, T] \times K_0 \times B_{2\eta} \times K_0 \times B_{2\eta}, \\ C(t, x, y, u, v), & \text{otherwise.} \end{cases}$$

Since K_0 is open and $\hat{C}(t, x, y, u, v) \subset C(t, x, y, u, v)$, for all $(t, x, y, u, v) \in [0, T] \times \mathbb{R}^{4n}$, the mapping $\hat{C}$ is also an upper-Carathéodory mapping.

Let us define the function $p : [0, T] \to \mathbb{R}$ by the formula

$$p(t) := \left\{ \alpha(t) + 2a(t)\eta + b(t) \max_{k\in\overline{K}} |k| \right\}$$

and, for all $m \in \mathbb{N}$, let us consider the mth problem

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F_m(t, x(t), \dot{x}(t)), \quad & \text{for a.a. } t \in [0, T], \\ x(T) = Mx(0), \qquad \dot{x}(T) = N\dot{x}(0), & \end{aligned}\right\} \tag{P_m}$$

where $F_m : [0, T] \times \mathbb{R}^{2n} \multimap \mathbb{R}^n$ is defined by

$$F_m(t, x, y) := F(t, x, y) + \tau(x) \left(p(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|}.$$

Moreover, let us consider the problem (Π_m), associated to (P_m),

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C_m(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad & \text{for a.a. } t \in [0, T], \\ x(T) = Mx(0), \qquad \dot{x}(T) = N\dot{x}(0), & \end{aligned}\right\} \tag{Π_m}$$

where $C_m : [0, T] \times \mathbb{R}^{4n} \times [0, 1] \multimap \mathbb{R}^n$ is defined by

$$C_m(t, x, y, u, v, \lambda) := \lambda\hat{C}(t, x, y, u, v) + \tau(x) \left(p(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|}.$$

Let us note that, according to their definitions, the mappings F_m and C_m are, for all $m \in \mathbb{N}$, upper-Carathéodory mappings.

Let us now prove that there exists $m_0 \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $m \geq m_0$, the problem (P_m) satisfies all assumptions of Proposition 3.1.

ad (P_i) Since $\hat{C}(t, x, y, u, v) \subset C(t, x, y, u, v)$, for all $(t, x, y, u, v) \in [0, T] \times \mathbb{R}^{4n}$, we have, for all $m \in \mathbb{N}$ and all $(t, x, y) \in [0, T] \times \mathbb{R}^{2n}$,

$$C_m(t, x, y, x, y, 1) \subset C(t, x, y, x, y) + \tau(x) \left(p(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|} \subset F_m(t, x, y).$$

Therefore, mappings F_m and C_m satisfy, for all $m \in \mathbb{N}$, condition (P_i) in Proposition 3.1.

ad (P_{ii}) Let us denote by P the Lipschitz constant of $\tau(x)\frac{\nabla V(x)}{|\nabla V(x)|}$. The mapping $C_m(t, \cdot, \cdot, r_1, r_2, \lambda)$ is, for all $t \in [0, T]$, $r_1 \in \overline{K}$, $r_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, Lipschitzian with Lipschitz function $L + l(t)$, where $l(t) := P \cdot (p(t)\chi_{\theta_m}(t) + \frac{1}{m})$. Since $p(\cdot) \in L^1([0, T], \mathbb{R})$ and $\mu(\theta_m) < \frac{1}{m}$, it is possible to find $\overline{m} \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $m \geq \overline{m}$, the integral $\int_0^T l(t)\,\mathrm{d}t$ is sufficiently small.

ad (P_{iii}), (P_{iv}) Conditions (P_{iii}) and (P_{iv}) follow immediately from the definition of C_m and from assumptions (iii) and (iv).

ad (P_v) For $\lambda = 0$, problem (Π_m) reduces to

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = \tau(x) \left(p(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|}, \quad & \text{for a.a. } t \in [0, T], \\ x(T) = Mx(0), \qquad \dot{x}(T) = N\dot{x}(0). & \end{aligned}\right\} \tag{20}$$

Let

$$\nu_m := \tilde{\Lambda} \left(\tilde{\Lambda}\tilde{\Gamma} + 1 \right) \int_0^T \left(p(s)\chi_{\theta_m}(s) + \frac{1}{m} \right) \mathrm{d}s$$

with $\tilde{\Lambda}$ and $\tilde{\Gamma}$ defined in (12). Since

$$\left| \tau(x) \left(p(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|} \right| \leq p(t)\chi_{\theta_m}(t) + \frac{1}{m}, \quad \text{for a.a. } t \in [0, T],$$

every solution y_m of (20) satisfies, according to Lemma 2.3,

$$|y_m(t)| \le \nu_m, \qquad |\dot{y}_m(t)| \le \nu_m, \quad \text{for all } t \in [0,T].$$

Since $\mu(\theta_m) < \frac{1}{m}$, it holds that $\nu_m \to 0$ as $m \to \infty$. Consequently, because $0 \in K$ and K is an open set, it is possible to find $\tilde{m}$ such that, for all $m \ge \tilde{m}$, every solution y_m of (20) satisfies $y_m(t) \in K$, for all $t \in [0,T]$.

ad (P_{vi}) Let x_m be a solution of (Π_m). It follows from Lemma 2.3 that

$$|x_m(t)| \le \eta_m, \quad |\dot{x}_m(t)| \le \eta_m, \quad \text{for all } t \in [0,T],$$

with

$$\eta_m = \tilde{\Lambda}\left(\tilde{\Lambda}\tilde{\Gamma}+1\right)\int_0^T \left(\alpha(s) + p(s)\chi_{\theta_m}(s) + \frac{1}{m}\right)\mathrm{d}s,$$

where $\tilde{\Lambda}$ and $\tilde{\Gamma}$ are defined in (12). Since $\mu(\theta_m) < \frac{1}{m}$, it is easy to see that

$$\int_0^T \left(p(s)\chi_{\theta_m}(s) + \frac{1}{m}\right)\mathrm{d}s \to 0,$$

implying $\eta_m \to \eta$ as $m \to \infty$. Therefore, it is possible to find $m^* \in \mathbb{N}$ such that $\eta_m < 2\eta$, for all $m \in \mathbb{N}$, $m \ge m^*$.

Let us now verify condition (P_{vi}), for all $m \in \mathbb{N},\ m \ge m^*$.

– At first, consider an arbitrary $t \in \theta_m$, $x \in N_{\frac{\delta}{2}}(\partial K) \cap N_h(\partial K) \cap \overline{K},\ \lambda \in (0,1)$, $v \in \mathbb{R}^n$ with $|v| \le 2\eta$ and $w_m \in C_m(t,x,v,x,v,\lambda) - A(t)v - B(t)x$. Then
$$w_m = w + \tau(x)\left(p(t)\chi_{\theta_m}(t) + \frac{1}{m}\right)\frac{\nabla V(x)}{|\nabla V(x)|}$$
with $w \in \lambda\hat{C}(t,x,v,x,v) - A(t)v - B(t)x$ and
$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_m\rangle \ge \langle \nabla V(x), w_m\rangle,$$
by means of condition (vi). Moreover,
$$\begin{aligned}\langle \nabla V(x), w_m\rangle &= \langle \nabla V(x), w\rangle + \tau(x)\left(p(t)\chi_{\theta_m}(t) + \frac{1}{m}\right)\cdot|\nabla V(x)| \\ &= \langle \nabla V(x), w\rangle + \left(p(t) + \frac{1}{m}\right)\cdot|\nabla V(x)| \ge \left(-|w| + p(t) + \frac{1}{m}\right)\cdot|\nabla V(x)| > 0,\end{aligned}$$
because $|\nabla V(x)| > 0$ and since, according to conditions (1_i), (iii), we have
$$|w| \le \alpha(t) + a(t)|v| + b(t)|x| \le \alpha(t) + 2a(t)\eta + b(t)\max_{k\in\overline{K}}|k| = p(t).$$

– Let $t \in (0,T)\setminus\theta_m,\ x \in \partial K,\ v \in \mathbb{R}^n$ with $|v| \le 2\eta$, $\lambda \in (0,1)$ and $w_m \in C_m(t,x,v,x,v,\lambda) - A(t)v - B(t)x$. Then $\chi_{\theta_m}(t) = 0$, $\tau(x) = 1$ and
$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_m\rangle \ge \langle \nabla V(x), w_m\rangle,$$
according to condition (vi). Moreover, there exists $w \in \lambda\hat{C}(t,x,v,x,v) - A(t)v - B(t)x$ such that
$$\langle \nabla V(x), w_m\rangle = \langle \nabla V(x), w\rangle + \frac{1}{m}\cdot|\nabla V(x)| > \frac{\gamma}{m},$$
by means of condition (v) and reasonings at the beginning of the proof.

According to the Scorza–Dragoni result and since $\hat{C} = \overline{C}$ on $([0,T]\setminus\theta_m)\times\overline{K}\times\overline{B_{2\eta}}\times\overline{K}\times\overline{B_{2\eta}}$, the mapping $\hat{C}$ is u.s.c. on $([0,T]\setminus\theta_m)\times\overline{K}\times\overline{B_{2\eta}}\times\overline{K}\times\overline{B_{2\eta}}$. By the same reason, $A,\ B$ are continuous on $[0,T]\setminus\theta_m$. Moreover, $V \in C^2(\mathbb{R}^n,\mathbb{R})$, and therefore, there exists $\kappa_m > 0$ such that
$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_m\rangle > 0,$$
for all $t \in (0,T)\setminus\theta_m,\ x \in N_{\kappa_m}(\partial K) \cap \overline{K} \cap N_h(\partial K)$, $v \in \mathbb{R}^n$ with $|v| \le 2\eta$, $\lambda \in (0,1)$ and $w_m \in C_m(t,x,v,x,v,\lambda) - A(t)v - B(t)x$.

Assumption (P_{vi}) is, therefore, satisfied with $\varepsilon = \min\left\{\frac{\delta}{2}, \kappa_m, h\right\}$.

ad (P_{vii}) Condition (P_{vii}) follows immediately from assumption (vii).

Therefore, we can apply Proposition 3.1 obtaining, for all $m \ge \max\{\overline{m}, m^*, \tilde{m}\}$, the existence of a solution x_m of the mth problem (P_m) such that $x_m(t) \in \overline{K}$, for each $t \in [0,T]$. Due to the continuation principle (see, e.g., [1, Theorem 1]) used for solving (P_m), x_m is indeed a solution of (Π_m), for $\lambda = 1$. Therefore, according to part ad (P_{vi}) of this proof, we obtain that $|\dot{x}_m(t)| \le 2\eta$, for all $m \ge m^*$ and $t \in [0,T]$ with η defined in (12). Hence, (1_i), assumption (iii) and the definition of τ imply $|\ddot{x}_m(t)| \le 4\eta a(t) + 2b(t)\max_{k\in\overline{K}}|k| + 2\alpha(t) + 1$. It is then possible to get $x \in C^1([0,T],\mathbb{R}^n)$ with absolutely continuous $\dot{x}$ and a subsequence (see, e.g., [13, Theorem 0.3.4]), again denoted as the sequence, such that $x_m \to x$, $\dot{x}_m \to \dot{x}$, uniformly in $[0,T]$, and $\ddot{x}_m \rightharpoonup \ddot{x}$, weakly in $L^1([0,T],\mathbb{R}^n)$, as $m \to \infty$. Thus, x satisfies the boundary conditions in (1). Put

$$\varphi_m(t) := \tau(x_m(t))\left(p(t)\chi_{\theta_m}(t) + \frac{1}{m}\right)\cdot\frac{\nabla V(x_m(t))}{|\nabla V(x_m(t))|}.$$

It is easy to see that

$$(\dot{x}_m(t), \ddot{x}_m(t) - \varphi_m(t))^T \in \left(0, \hat{C}(t, x_m(t), \dot{x}_m(t), x_m(t), \dot{x}_m(t))\right)^T - C(t)\,(x_m(t), \dot{x}_m(t))^T,$$

for a.a. $t \in [0, T]$, where the $2n \times 2n$ matrix function C was defined in (11). Since $|\varphi_m(t)| \le p(t)\chi_{\theta_m(t)} + \frac{1}{m}$, for a.a. $t \in [0, T]$, and $\varphi_m(t) \to 0$ as $m \to \infty$ in $[0, T]$, we have that $(\dot{x}_m(t), \ddot{x}_m(t) - \varphi_m(t)) \rightharpoonup (\dot{x}, \ddot{x})$, weakly in $L^1([0, T], \mathbb{R}^n)$, as $m \to \infty$. Therefore, a standard limiting argument (see [1, Proposition 2]) implies that x is a solution of problem (1). Finally, since $x_m(t) \in \overline{K}$, for all $m \in \mathbb{N}$ and $t \in [0, T]$, we obtain that also $x(t) \in \overline{K}$, for all $t \in [0, T]$, which completes the proof. □

As an application of Theorem 3.1, we conclude by the following simple illustrating example.

Example 3.1. Let us consider the vector dry friction b.v.p.

$$\left.\begin{aligned} &\ddot{x}(t) + a \cdot x(t) + b \cdot \operatorname{sgn} \dot{x}(t) = p(t, x(t)), \quad \text{for a.a. } t \in [0, T],\\ &x(T) = x(0), \qquad \dot{x}(T) = \dot{x}(0), \text{ or}\\ &x(T) = -x(0), \qquad \dot{x}(T) = -\dot{x}(0), \end{aligned}\right\} \tag{21}$$

where $a < 0$, $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\mathrm{T}}$, $\operatorname{sgn} \dot{x}(t) = (\operatorname{sgn} \dot{x}_1(t), \operatorname{sgn} \dot{x}_2(t), \dots, \operatorname{sgn} \dot{x}_n(t))^{\mathrm{T}}$ and $p : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function such that $|p(t, x)| \le p(t)$ with $p \in L^1([0, T], [0, \infty))$. Assume that $\langle p(t, x), x\rangle \ge 0$, for all $t \in (0, T)$ and $x \in \mathbb{R}^n$ with $|x| = R$, where $R > 0$ is a suitable number such that $b \in \left(\frac{aR}{n}, \frac{-aR}{n}\right)$.

Because of discontinuity in $\operatorname{sgn} y$, we can only consider Filippov solutions which can be identified (see, e.g., [8,16,19]) as Carathéodory solutions of

$$\left.\begin{aligned} &\ddot{x}(t) + a \cdot x(t) \in p(t, x(t)) - b \cdot \operatorname{Sgn} \dot{x}(t), \quad \text{for a.a. } t \in [0, T],\\ &x(T) = x(0), \qquad \dot{x}(T) = \dot{x}(0), \text{ or}\\ &x(T) = -x(0), \qquad \dot{x}(T) = -\dot{x}(0), \end{aligned}\right\} \tag{22}$$

where $\operatorname{Sgn} y = (\operatorname{Sgn} y_1, \operatorname{Sgn} y_2, \dots, \operatorname{Sgn} y_n)^{\mathrm{T}}$ and, for all $i \in \{1, 2, \dots, n\}$,

$$\operatorname{Sgn} y_i := \begin{cases} -1, & \text{for } y_i < 0,\\ [-1, 1], & \text{for } y_i = 0,\\ 1, & \text{for } y_i > 0. \end{cases}$$

We will show that, under the above assumptions, the dry friction b.v.p. (22) admits both periodic and anti-periodic solutions.

Defining $K := \{x \in \mathbb{R}^n \mid |x| < R\}$ and, for all $t \in [0, T]$ and $(x, y, u, v) \in \mathbb{R}^{4n}$, $C(t, x, y, u, v) := p(t, u) - b \cdot \operatorname{Sgn} v$, the mapping $C : [0, T] \times \mathbb{R}^{4n} \multimap \mathbb{R}^n$ obviously satisfies conditions (i)–(iii) from Theorem 3.1.

Moreover, both the associated homogeneous problems

$$\left.\begin{aligned} &\ddot{x}(t) + a \cdot x(t) = 0, \quad \text{for a.a. } t \in [0, T],\\ &x(T) = x(0), \qquad \dot{x}(T) = \dot{x}(0), \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} &\ddot{x}(t) + a \cdot x(t) = 0, \quad \text{for a.a. } t \in [0, T],\\ &x(T) = -x(0), \qquad \dot{x}(T) = -\dot{x}(0) \end{aligned}\right\} \tag{24}$$

have only the trivial solution, i.e. condition (iv) from Theorem 3.1 holds as well.

For verifying conditions (v)–(viii), let us define, for all $x \in \mathbb{R}^n$, $V(x) := \frac{1}{2}\left(|x|^2 - R^2\right)$. For all $x \in \mathbb{R}^n$, $\nabla V(x) = x$ and $HV(x) = I$.

Therefore, for all $\lambda \in (0, 1)$, $x = (x_1, \dots, x_n) \in \partial K$, $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ and $w \in \lambda \cdot p(t, x) - \lambda \cdot b \cdot \operatorname{Sgn} v - a \cdot x$,

$$\begin{aligned} \langle \nabla V(x), w\rangle &\ge \lambda\langle x, p(t, x)\rangle - \lambda\langle x, b \cdot \operatorname{Sgn} v\rangle - a\langle x, x\rangle\\ &\ge -|b|\,(|x_1| + \cdots + |x_n|) - a \cdot R^2 > -|b|nR - a \cdot R^2 > 0 \end{aligned}$$

since $b \in \left(\frac{aR}{n}, \frac{-aR}{n}\right)$ and $\langle x, p(t, x)\rangle \ge 0$, for all $t \in [0, T]$ and $x \in \mathbb{R}^n$ with $|x| = R$. Thus, condition (v) from Theorem 3.1 holds, too. Assumptions (vi) and (viii) follow immediately from the definition of the function V and the set K.

Condition (vii) is also satisfied, for both periodic and anti-periodic problems, because, for $M = N = I$, or, for $M = N = -I$,

$$\langle \nabla V(Mx), Nv\rangle \cdot \langle \nabla V(x), v\rangle = \langle x, v\rangle^2;$$

hence, when $x \in \partial K$ and $v \ne \mathbf{0}$, then $\langle x, v\rangle^2 > 0$ while $\langle \nabla V(Mx), Nv\rangle = \langle \nabla V(x), v\rangle = 0$ when $v = \mathbf{0}$.

All assumptions of Theorem 3.1 are so satisfied, by which, the dry friction problem (22) admits a periodic solution $x_1(\cdot)$ and an anti-periodic solution $x_2(\cdot)$ such that $|x_1(t)| \le R$ and $|x_2(t)| \le R$, for all $t \in [0, T]$. These solutions represent Filippov solutions of the original problem (21).

Remark 3.3. The sufficient conditions can be obviously slightly improved in the sense that $R > 0$ can be such that only $\langle x, p(t, x)\rangle > |b| \cdot n \cdot R + a \cdot R^2$, for $|x| = R$ and $t \in (0, T)$. Because of transparency, we preferred to present only a special, but much more explicit, case.

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A SCORZA–DRAGONI APPROACH TO DIRICHLET PROBLEM WITH AN UPPER-CARATHÉODORY RIGHT-HAND SIDE

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Abstract. In this paper, the existence and localization result will be proven for multivalued vector Dirichlet problem with an upper-Carathéodory right-hand side by using bound sets approach. Since Scorza–Dragoni type technique will be furthermore applied, the conditions for bounding functions can be required directly on the boundaries of bound sets and not at some vicinity of them.

1. Introduction

Boundary value problems (b.v.p.) for second-order differential inclusions have been studied for many years (see, e.g. [1], [5], [6], [9]–[11], [13]) since to their applications in several areas, such as physics, control theory or mathematical economics. In mentioned papers, various methods (like an upper and lower solutions technique, topological transformations, fixed point technique or tube solution method) were applied for obtaining the existence results. In this paper, except for the existence of a solution, also its localization is studied for multivalued vector Dirichlet problem. More concretely, let us consider the Dirichlet multivalued problem

$$\begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0, T], \\ x(T) = x(0) = \mathbf{0}, \end{cases} \tag{1.1}$$

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where $F\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping, together with the localization condition

$$x(t) \in K, \quad \text{for all } t \in [0,T], \tag{1.2}$$

where $K \subset \mathbb{R}^n$ is given open bounded set containing the null vector $\mathbf{0}$.

Let us note that the notion of a solution will be understood in the strong sense, i.e. by a *solution* of problem (1.1)–(1.2) we shall mean a function $x\colon [0,T] \to \mathbb{R}^n$ with absolutely continuous first derivative satisfying (1.1)–(1.2).

Dirichlet viability problem (1.1)–(1.2) was already studied in [7], [12]. In [7], the multivalued mapping F was globally u.s.c. and conditions for bounding functions were imposed directly on boundaries of bound sets, while in [12], F was an upper-Carathéodory multivalued mapping but the conditions concerning bounding functions were imposed at some vicinity of the boundaries of bounds sets. Since the Scorza–Dragoni type technique is applied in the present paper, conditions for bounding functions are imposed directly on boundaries of bound sets also in the case of upper-Carathéodory right-hand side. The obtained result is at the end of the paper illustrated by the vector dry friction problem.

2. Preliminaries

Let us start with notations we use in the paper. If (X,d) is a metric space and $A \subset X$, by $\overline{A}$, $\operatorname{Int} A$, and ∂A, we mean the *closure*, the *interior*, and the *boundary* of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \text{there exists } a \in A : d(x,a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighbourhood of the set A in X.

For a given compact real interval J, we denote by $C(J,\mathbb{R}^n)$ (by $C^1(J,\mathbb{R}^n)$) the set of all functions $x\colon J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $\mathrm{AC}^1(J,\mathbb{R}^n)$, we shall mean the set of all functions $x\colon J \to \mathbb{R}^n$ with absolutely continuous first derivatives on J.

We also need following definitions and notions from multivalued theory in the sequel. We say that F is a *multivalued mapping* from X to Y (written $F\colon X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given. A multivalued mapping $F\colon X \multimap Y$ is called *upper semi-continuous* (shortly, u.s.c.) if, for each open set $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

Let Y be a metric space and $(\Omega,\mathcal{U},\mu)$ be a *measurable space*, i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure μ on $\mathcal{U}$. A multivalued mapping $F\colon \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid F(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$. In the sequel, the symbol μ will exclusively denote the *Lebesgue measure* on $\mathbb{R}$.

We say that mapping $F\colon J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $F(\,\cdot\,,x)\colon J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $F(t,\cdot)\colon \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all $t \in J$, and the set $F(t,x)$ is compact and convex, for all $(t,x) \in J \times \mathbb{R}^m$.

In the proof of the main result, the following slight modification of Scorza–Dragoni type technique for multivalued mappings will be employed.

Proposition 2.1 (cf. e.g. [4, Proposition 8]). *Let $X \subset \mathbb{R}^m$ be compact and let $F\colon [a,b] \times X \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping. Then there exists a multivalued mapping $F_0\colon [a,b] \times X \multimap \mathbb{R}^n \cup \{\emptyset\}$ with compact, convex values and $F_0(t,x) \subset F(t,x)$, for all $(t,x) \in [a,b] \times X$, having the following properties*:

(a) *if $u,v\colon [a,b] \to \mathbb{R}^n$ are measurable functions with $v(t) \in F(t,u(t))$ on $[a,b]$, then $v(t) \in F_0(t,u(t))$ almost everywhere on $[a,b]$*;
(b) *for every $\varepsilon > 0$, there exists a closed $I_\varepsilon \subset [a,b]$ such that $\mu([a,b] \setminus I_\varepsilon) < \varepsilon$, $F_0(t,x) \neq \emptyset$, for all $(t,x) \in I_\varepsilon \times X$ and F_0 is u.s.c. on $I_\varepsilon \times X$.*

The proof of main result, Theorem 3.1 below, will be based (except from Proposition 2.1) also on the following proposition developed in [12]. Its proof was based on combination of bound sets approach with the continuation principle developed in [2]. The key point for application of the continuation principle lied in the fact that we assigned to the Dirichlet problem (1.1) the family of associated problems

$$\begin{cases} \ddot{x}(t) \in \lambda F(t,q(t),\dot{q}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = \mathbf{0}, \end{cases} \tag{2.1}$$

where $\lambda \in [0,1]$, and

$$q \in Q := \{q \in C^1([0,T],\mathbb{R}^n) \mid q(t) \in \overline{K} \text{ for all } t \in [0,T]\}. \tag{2.2}$$

Proposition 2.2 (cf. [12, Theorem 4.1 and Corollary 3.1]). *Let us consider the Dirichlet problem* (1.1)–(1.2) *where $F\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping. Moreover, assume that*

(a) *the closure $\overline{K}$ of the set K is a retract of $\mathbb{R}^n$,*
(b) *there exists a nonnegative, integrable function $\beta\colon [0,T] \to \mathbb{R}$ such that*
$$|F(t,q(t),\dot{q}(t))| \le \beta(t), \quad \text{a.e. in } [0,T],$$
for each $q \in Q$, where Q is defined by (2.2)
(c) *there exists a function $V \in C^2(\mathbb{R}^n,\mathbb{R})$ satisfying conditions*:
 (H1) $V|_{\partial K} = 0$,
 (H2) $V(x) \le 0$ *for all $x \in \overline{K}$,*
(d) *there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$, $\lambda \in (0,1]$ and $v \in \mathbb{R}^n$ with $|v| \le 2\int_0^T \beta(t)\,dt$, the following condition*:
$$\langle HV(x)v,v\rangle + \langle \nabla V(x),w\rangle > 0 \tag{2.3}$$
holds, for all $w \in \lambda F(t,x,v)$.

Then the Dirichlet viability problem (1.1)–(1.2) *has a solution.*

The function V satisfying conditions from Proposition 2.2 is called a (*Liapunov-like*) *bounding function.* Its existence guarantees that K is a *bound set* for the b.v.p.

$$\begin{cases} \ddot{x}(t) \in \lambda F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = \mathbf{0}, \end{cases} \tag{2.4}$$

for all $\lambda \in (0,1]$, i.e. ensures that there does not exist, for any $\lambda \in (0,1]$, a solution x of the b.v.p. (2.4) such that $x(t) \in \overline{K}$, for each $t \in [0,T]$, and $x(t_0) \in \partial K$, for some $t_0 \in [0,T]$.

3. Existence and localization result

Approximating the original problem by a sequence of problems satisfying conditions of Proposition 2.2 and applying the Scorza–Dragoni type result, we are already able to state the main result of the paper. The transversality condition imposed on the bounding function is now required only on the boundary ∂K of the set K, and not on the whole neighbourhood $\overline{K} \cap N_\varepsilon(\partial K)$, as in Proposition 2.2.

THEOREM 3.1. *Let us consider the Dirichlet viability problem* (1.1)–(1.2) *and assume that*

(a) *the closure $\overline{K}$ of the set K is a retract of $\mathbb{R}^n$,*
(b) *there exists a nonnegative, integrable function $\alpha\colon [0,T] \to \mathbb{R}$ such that*

$$|F(t, q(t), \dot{q}(t))| \le \alpha(t), \quad \textit{a.e. in } [0,T],$$

for each $q \in Q$, where Q is defined by formula (2.2),
(c) *there exists a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfying conditions* (H1), (H2),
(d) *for all $\lambda \in (0,1]$, $x \in \partial K$, $t \in (0,T)$, $v \in \mathbb{R}^n$ with $|v| \le 2\int_0^T (2\alpha(t)+1)\, dt$, and $w \in \lambda F(t,x,v)$, it holds that*

$$\langle \nabla V(x), w \rangle > 0, \tag{3.1}$$

(e) *there exists $h > 0$ such that $HV(x)$ is positive semi-definite, for all $x \in \overline{K} \cap N_h(\partial K)$.*

Then the Dirichlet viability problem (1.1)–(1.2) *has a solution.*

PROOF. At first, let us consider the family of associated problems (2.1) and let x be a solution of (2.1) for some $(q,\lambda) \in Q \times (0,1]$. Then it follows from the boundary conditions that there exists a point $\xi \in (0,T)$ such that $\dot{x}(\xi) = 0$. Therefore, according to condition (b),

$$|\dot{x}(0)| = |\dot{x}(\xi) - \dot{x}(0)| = \left| \int_0^\xi \ddot{x}(t)\, dt \right| \le \int_0^\xi |\ddot{x}(t)|\, dt \le \int_0^\xi \alpha(t)\, dt \le \int_0^T \alpha(t)\, dt.$$

Therefore, for almost all $t \in [0,T]$,

$$|\dot{x}(t)| \leq |\dot{x}(0)| + \int_0^t \alpha(s)\,ds \leq 2\int_0^T \alpha(s)\,ds.$$

Moreover, for almost all $t \in [0,T]$,

$$|x(t)| \leq |x(0)| + \int_0^t |\dot{x}(s)|\,ds \leq 2\int_0^T \int_0^T \alpha(s)\,ds\,du = 2T\int_0^T \alpha(s)\,ds.$$

Thus, x satisfies $|x(t)| \leq a$ and $|\dot{x}(t)| \leq b$, for almost all $t \in [0,T]$, where

$$a := 2T\int_0^T \alpha(s)\,ds \quad \text{and} \quad b := 2\int_0^T \alpha(s)\,ds. \tag{3.2}$$

It follows from condition (d) and from compactness of ∂K that there exists $\delta > 0$ such that $\nabla V(x) \neq 0$, for all $x \in N_\delta(\partial K)$. Moreover, there exists $\gamma > 0$ such that $|\nabla V(x)| \geq \gamma$, for all $x \in \partial K$.

Let us consider an open bounded set $K_0 \subset \mathbb{R}^n$ such that $\overline{K} \subset K_0$. Since F is an upper-Carathéodory mapping, we can apply a Scorza–Dragoni type result (cf. Proposition 2.1). Consequently, there exists a decreasing sequence $\{\theta_m\}$ of subsets of $[0,T]$ and a measurable mapping $\overline{F}\colon [0,T] \times \overline{K_0} \times \overline{B_{2b}} \multimap \mathbb{R}^n$ such that, for every $m \in \mathbb{N}$,

- $[0,T] \setminus \theta_m$ is compact and $\mu(\theta_m) < 1/m$,
- $\overline{F}(t,x,y) \subset F(t,x,y)$, for all $(t,x,y) \in [0,T] \times \overline{K_0} \times \overline{B_{2b}}$,
- $\overline{F}$ is u.s.c. on $([0,T] \setminus \theta_m) \times \overline{K_0} \times \overline{B_{2b}}$.

It is obvious that $\bigcap\limits_{m=1}^{\infty} \theta_m$ has zero Lebesque measure and that $\lim\limits_{m\to\infty} \chi_{\theta_m}(t) = 0$, for every $t \notin \bigcap\limits_{m=1}^{\infty} \theta_m$. Therefore, $\overline{F}$ is an upper-Carathéodory mapping.

Let us define the mapping $\widehat{F}\colon [0,T] \times \mathbb{R}^{2n} \multimap \mathbb{R}^n$ by the formula

$$\widehat{F}(t,x,y) := \begin{cases} \overline{F}(t,x,y), & \text{for } (t,x,y) \in [0,T] \times K_0 \times B_{2b}, \\ F(t,x,y), & \text{otherwise.} \end{cases}$$

Since K_0 is open and $\hat{F}(t,x,y) \subset F(t,x,y)$, for all $(t,x,y) \in [0,T] \times \mathbb{R}^{2n}$, the mapping $\widehat{F}$ is also an upper-Carathéodory mapping.

Let $\tau \in C^1(\mathbb{R}^n,[0,1])$ be such that $\tau \equiv 1$ on $N_{\delta/2}(\partial K)$ and $\tau \equiv 0$ on $\mathbb{R}^n \setminus N_\delta(\partial K)$ and let us consider (for all $m \in \mathbb{N}$) the m-th problem

$$\begin{cases} \ddot{x}(t) \in F_m(t,x(t),\dot{x}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = \mathbf{0}, \end{cases} \tag{P$_m$}$$

where an upper-Carathéodory mapping $F_m\colon [0,T] \times \mathbb{R}^{2n} \multimap \mathbb{R}^n$ is defined by

$$F_m(t,x,y) := F(t,x,y) + \tau(x)\left(\alpha(t)\chi_{\theta_m}(t) + \frac{1}{m}\right)\frac{\nabla V(x)}{|\nabla V(x)|}.$$

Moreover, let us consider the family of problems $(\Pi_{m,q,\lambda})$, associated to (P_m),

$$(\Pi_{m,q,\lambda}) \qquad \begin{cases} \ddot{x}(t) \in \lambda F_m(t, q(t), \dot{q}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = \mathbf{0}, \end{cases}$$

where $q \in Q$ and $\lambda \in [0,1]$.

Let us now prove that there exists $m_0 \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $m \geq m_0$, the problem (P_m) satisfies assumptions (b) and (d) from Proposition 2.2.

(b) Since for all $m \in \mathbb{N}$,

$$\left| \tau(x) \left(\alpha(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|} \right| \leq \alpha(t)\chi_{\theta_m}(t) + \frac{1}{m} \leq \alpha(t) + 1,$$

for almost all $t \in [0,T]$, it holds for all $m \in \mathbb{N}$, $q \in Q$ and almost all $t \in [0,T]$ that

$$|F_m(t, q(t), \dot{q}(t))| \leq \beta(t),$$

where $\beta(t) := 2\alpha(t) + 1$. Assumption (b) from Proposition 2.2 is therefore satisfied.

(d) If x_m be a solution of $(\Pi_{m,q,\lambda})$, then

$$|x_m(t)| \leq a_m, \quad |\dot{x}_m(t)| \leq b_m, \quad \text{for all } t \in [0,T],$$

where

$$a_m = 2T \int_0^T \left(\alpha(s) + \alpha(s)\chi_{\theta_m}(s) + \frac{1}{m} \right) ds,$$

and

$$b_m = 2 \int_0^T \left(\alpha(s) + \alpha(s)\chi_{\theta_m}(s) + \frac{1}{m} \right) ds.$$

Since $\mu(\theta_m) < 1/m$ and $1/m \to 0$ as $m \to \infty$, it is easy to see that

$$\int_0^T \left(\alpha(s)\chi_{\theta_m}(s) + \frac{1}{m} \right) ds \to 0,$$

implying $a_m \to a$ and $b_m \to b$ as $m \to \infty$, where a and b are defined by (3.2). Therefore, it is possible to find $m^* \in \mathbb{N}$ such that $a_m < 2a$ and $b_m < 2b$, for all $m \in \mathbb{N}$, $m \geq m^*$.

Let us now verify condition (d) for all $m \in \mathbb{N}$, $m \geq m^*$.

At first, consider an arbitrary $t \in \theta_m$, $x \in N_{\delta/2}(\partial K) \cap N_h(\partial K) \cap \overline{K}$, $\lambda \in (0,1]$, $v \in \mathbb{R}^n$ with $|v| \leq 2\int_0^T (2\alpha(t) + 1)\, dt$ and $w_m \in \lambda F_m(t,x,v)$. Then

$$w_m = w + \tau(x) \left(\alpha(t)\chi_{\theta_m}(t) + \frac{1}{m} \right) \frac{\nabla V(x)}{|\nabla V(x)|}$$

with $w \in \lambda \widehat{F}(t,x,v)$ and

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_m\rangle \geq \langle \nabla V(x), w_m\rangle,$$

by means of condition (e). Moreover,

$$\begin{aligned}\langle \nabla V(x), w_m\rangle &= \langle \nabla V(x), w\rangle + \tau(x)\bigg(\alpha(t)\chi_{\theta_m}(t) + \frac{1}{m}\bigg)\cdot |\nabla V(x)|\\ &= \langle \nabla V(x), w\rangle + \bigg(\alpha(t) + \frac{1}{m}\bigg)\cdot |\nabla V(x)|\\ &\geq \bigg(-|w| + \alpha(t) + \frac{1}{m}\bigg)\cdot |\nabla V(x)| > 0,\end{aligned}$$

because $|\nabla V(x)| > 0$, and since $|w| \leq \alpha(t)$.

Let $t \in (0,T)\setminus \theta_m$, $x \in \partial K$, $v \in \mathbb{R}^n$ with $|v| \leq 2\int_0^T (2\alpha(t)+1)\,dt$, $\lambda \in (0,1]$ and $w_m \in \lambda F_m(t,x,v)$. Then $\chi_{\theta_m}(t) = 0$, $\tau(x) = 1$ and

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_m\rangle \geq \langle \nabla V(x), w_m\rangle,$$

according to condition (e). Moreover, there exists $w \in \lambda\widehat{F}(t,x,v)$ such that

$$\langle \nabla V(x), w_m\rangle = \langle \nabla V(x), w\rangle + \frac{1}{m}\cdot |\nabla V(x)| > \frac{\gamma}{m},$$

by means of condition (d) and reasonings at the beginning of the proof.

According to the Scorza–Dragoni result and since $\widehat{F} = \overline{F}$ on $([0,T]\setminus\theta_m)\times\overline{K}\times\overline{B_{2b}}$, the mapping $\widehat{F}$ is u.s.c. on $([0,T]\setminus\theta_m)\times\overline{K}\times\overline{B_{2b}}$. Moreover, $V \in C^2(\mathbb{R}^n,\mathbb{R})$, and therefore, there exists $\kappa_m > 0$ such that

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_m\rangle > 0,$$

for all $t \in (0,T)\setminus\theta_m$, $x \in N_{\kappa_m}(\partial K)\cap\overline{K}\cap N_h(\partial K)$, $v \in \mathbb{R}^n$ with $|v| \leq 2\int_0^T (2\alpha(t) +1)\,dt$, $\lambda \in (0,1]$ and $w_m \in \lambda F_m(t,x,v)$.

Assumption (d) is, therefore, satisfied with $\varepsilon = \min\{\delta/2, \kappa_m, h\}$.

Thus, we can apply Proposition 2.2 obtaining, for all $m \geq m^*$, the existence of a solution x_m of the m-th problem (P_m) such that $x_m(t) \in \overline{K}$, for each $t \in [0,T]$. Due to the continuation principle (see [2]) used for solving (P_m), x_m is indeed a solution of $(\Pi_{m,q,\lambda})$, for $\lambda = 1$. Therefore, according to the previous part of this proof, we obtain that $|\dot{x}_m(t)| \leq 2b$, for all $m \geq m^*$ and $t \in [0,T]$, where b is defined by (3.2), and $|\ddot{x}_m(t)| \leq 2\alpha(t) + 1$. It is then possible to get $x \in C^1([0,T],\mathbb{R}^n)$ with absolutely continuous $\dot{x}$ and a subsequence (see e.g. [3, Theorem 0.3.4]), again denoted as the sequence, such that $x_m \to x$, $\dot{x}_m \to \dot{x}$, uniformly in $[0,T]$, and $\ddot{x}_m \rightharpoonup \ddot{x}$, weakly in $L^1([0,T],\mathbb{R}^n)$, as $m \to \infty$. Thus, x satisfies the boundary conditions in (1.1). Put

$$\varphi_m(t) := \tau(x_m(t))\bigg(\alpha(t)\chi_{\theta_m}(t) + \frac{1}{m}\bigg)\cdot\frac{\nabla V(x_m(t))}{|\nabla V(x_m(t))|}.$$

Since $|\varphi_m(t)| \leq \alpha(t)\chi_{\theta_m(t)} + 1/m$, for almost all $t \in [0,T]$, and $\varphi_m(t) \to 0$ as $m \to \infty$ in $[0,T]$, we have that $(\dot{x}_m(t), \ddot{x}_m(t) - \varphi_m(t)) \rightharpoonup (\dot{x}, \ddot{x})$, weakly in $L^1([0,T],\mathbb{R}^n)$, as $m \to \infty$. Therefore, a standard limiting argument implies that x is a solution of problem (1.1). Finally, since $x_m(t) \in \overline{K}$, for all $m \in \mathbb{N}$ and

$t \in [0,T]$, we obtain that also $x(t) \in \overline{K}$, for all $t \in [0,T]$, which completes the proof. $\square$

REMARK 3.2. As pointed out in the proof, the founded solution x of problem (1.1)–(1.2) is indeed a solution of (2.1), for $q = x$ and $\lambda = 1$. Therefore, due to the proof and the assumption (b), $|\dot{x}(t)| \le b$, for all $t \in [0,T]$, where $b := 2\int_0^T \alpha(s)\,ds$. It is so possible to enlarge the localization conditions and ensure the existence of solution of problem

$$\begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \end{cases}$$

$$\begin{aligned} x(t) &\in K, && \text{for all } t \in [0,T], \\ \dot{x}(t) &\in \overline{B_b}, && \text{for all } t \in [0,T]. \end{aligned}$$

As an application of Theorem 3.1, we conclude by the dry friction Dirichlet problem.

EXAMPLE 3.3. Let us consider the vector dry friction b.v.p.

$$\begin{cases} \ddot{x}(t) + a \cdot \operatorname{sgn} \dot{x}(t) = \varphi(t, x(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = \mathbf{0}, \end{cases} \tag{3.3}$$

where $a \in \mathbb{R}$, $x(t) = (x_1(t), \dots, x_n(t))^T$, $\operatorname{sgn} \dot{x}(t) = (\operatorname{sgn} \dot{x}_1(t), \dots, \operatorname{sgn} \dot{x}_n(t))^T$, and $\varphi\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function such that

$$|\varphi(t,x)| \le \beta(t)(1 + |x|) \quad \text{with } \beta \in L^1([0,T],[0,\infty)).$$

Because of discontinuity in $\operatorname{sgn} y$, we can only consider Filippov solutions which can be identified (see e.g. [1], [4], [8]) as Carathéodory solutions of

$$\begin{cases} \ddot{x}(t) \in \varphi(t, x(t)) - a \cdot \operatorname{Sgn} \dot{x}(t), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = \mathbf{0}, \end{cases} \tag{3.4}$$

where $\operatorname{Sgn} y = (\operatorname{Sgn} y_1, \dots, \operatorname{Sgn} y_n)^T$ and, for all $i \in \{1, \dots, n\}$,

$$\operatorname{Sgn} y_i := \begin{cases} -1, & \text{for } y_i < 0, \\ [-1,1], & \text{for } y_i = 0, \\ 1, & \text{for } y_i > 0. \end{cases}$$

If there exist $D > 0$ such that $\langle \varphi(t,x), x \rangle - a\sqrt{D}\sqrt{n} > 0$, for all $t \in (0,T)$ and $x \in \mathbb{R}^n$ with $|x| = D$, then the dry friction b.v.p. (3.4) admits, according to Theorem 3.1, a solution x such that $|x| < D$.

More concretely, for verifying conditions (a)–(e) from Theorem 3.1, let us define the set $K := \{x \in \mathbb{R}^n \mid |x| < D\}$, the bounding function $V(x) := (|x|^2 - D^2)/2$, and $\alpha(t) := \beta(t)(1 + D)$. Then, for all $x \in \mathbb{R}^n$, $\nabla V(x) = x$ and

$HV(x) = I$, and conditions (a), (c), (e) from Theorem 3.1 are obviously satisfied. Moreover, since for all $\lambda \in (0,1]$, $x \in \partial K$, $v \in \mathbb{R}^n$ and $w \in \lambda \cdot \varphi(t,x) - \lambda \cdot a \cdot \operatorname{Sgn} v$,

$$\langle \nabla V(x), w\rangle \geq \lambda(\langle x, \varphi(t,x)\rangle - a(|x_1| + \ldots + |x_n|)) \geq \lambda(\langle \varphi(t,x), x\rangle - a\sqrt{D}\sqrt{n}) > 0,$$

condition (d) from Theorem 3.1 holds, too.

All assumptions of Theorem 3.1 are so satisfied, by which, the dry friction problem (3.4) admits a solution x such that $|x| < D$. This solution represents the Filippov solution of the original problem (3.3).

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ON SECOND-ORDER BOUNDARY VALUE PROBLEMS IN BANACH SPACES: A BOUND SETS APPROACH

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Abstract. The existence and localization of strong (Carathéodory) solutions is obtained for a second-order Floquet problem in a Banach space. The combination of applied degree arguments and bounding (Liapunov-like) functions allows some solutions to escape from a given set. The problems concern both semilinear differential equations and inclusions. The main theorem for upper-Carathéodory inclusions is separately improved for Marchaud inclusions (i.e. for globally upper semicontinuous right-hand sides) in the form of corollary. Three illustrative examples are supplied.

1. Introduction

Let E be a Banach space (with the norm $\|\cdot\|$) satisfying the Radon–Nikodym property (e.g. reflexivity) and let us consider the Floquet boundary value problem (b.v.p.)

$$(1.1)\qquad \begin{cases} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0, T], \\ x(T) = Mx(0), \quad \dot{x}(T) = N\dot{x}(0). \end{cases}$$

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Throughout the paper, we assume (for the related definitions, see the next Section 2) that

($1_{\rm i}$) $A, B\colon [0,T] \to \mathcal{L}(E)$ are Bochner integrable, where $\mathcal{L}(E)$ stands for the Banach space of all linear, bounded transformations $L\colon E \to E$ endowed with the sup-norm,

($1_{\rm ii}$) $F\colon [0,T] \times E \times E \multimap E$ is an upper-Carathéodory multivalued mapping,

($1_{\rm iii}$) $M,\ N \in \mathcal{L}(E)$.

The notion of a solution will be understood in a strong (i.e. Carathéodory) sense. Namely, by a *solution* of problem (1.1), we mean a function $x\colon [0,T] \to E$ whose first derivative $\dot{x}(\,\cdot\,)$ is absolutely continuous and satisfies (1.1), for almost all $t \in [0,T]$.

Problems of this type can be related to those for *abstract nonlinear wave equations* in Hilbert spaces. For $t \in [0,T]$ and $\xi \in \Omega$, where Ω is a nonempty, bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, consider the functional evolution equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + \widetilde{B}u(t,\,\cdot\,) + \mathcal{B}||u(t,\,\cdot\,)||^{p-2}u = \varphi(t,u), \tag{1.2}$$

where $u = u(t,\xi)$, subject to boundary conditions

$$u(T,\,\cdot\,) = Mu(0,\,\cdot\,), \qquad \frac{\partial u(T,\,\cdot\,)}{\partial t} = N\frac{\partial u(0,\,\cdot\,)}{\partial t}. \tag{1.3}$$

Assume that $a \geq 0$, $\mathcal{B} \geq 0$, $p > 1$ are constants, $\widetilde{B}\colon L^2(\Omega) \to L^2(\Omega)$ is a linear operator and that $\varphi\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is sufficiently regular. The problem under consideration can be still restricted by a constraint:

$$u(t,\,\cdot\,) \in \overline{K} := \{e \in L^2(\Omega) \mid ||e|| \leq r\}, \quad t \in [0,T].$$

Taking $x(t) := u(t,\,\cdot\,)$ with $x \in AC^1([0,T], L^2(\Omega))$, $A(t) \equiv A := a$, $B(t)\colon L^2(\Omega) \to L^2(\Omega)$ defined by $x = u(t,\,\cdot\,) \to \widetilde{B}x$, $f\colon [0,T] \times L^2(\Omega) \to L^2(\Omega)$ defined by $(t,v) \to \varphi(t, v(\,\cdot\,))$, and $F(t,x,y) \equiv F(t,x) := -\mathcal{B}||x||^{p-2}x + f(t,x)$, the above problem can be rewritten into the form (1.1), possibly together with $x(t) \in \overline{K}, t \in [0,T]$, where $K \subset L^2(\Omega)$ is a nonempty, open, convex subset of $L^2(\Omega)$ containing 0.

If $\varphi(t,\,\cdot\,)$ is e.g. bounded, but discontinuous at finitely many points, then the Filippov regularization $\widetilde{\varphi}$ of $\varphi(t,\,\cdot\,)$ (cf. e.g. [6], [9]) can lead to a multivalued problem (1.1).

An interesting case occurs when $E = L^2(\Omega)$ and $\widetilde{B}u(t,\,\cdot\,) := -\Delta u(t,\,\cdot\,)$; equation (1.2) then becomes a hyperbolic equation (see e.g. [22, Chapter 5.2]). Since such a $\widetilde{B}$ is defined only on $W^{2,2}(\Omega) \subset E = L^2(\Omega)$, it does not satisfy condition ($1_{\rm i}$) and the related model can not be attached with the techniques developed in this work. Moreover, the Laplace operator is not bounded on $W^{2,2}(\Omega)$, as required in ($1_{\rm i}$). Indeed, the main purpose of the present paper is to prove the existence of a Carathéodory solution $x \in AC^1([0,T],E)$ to problem (1.1) in a given set Q. Section 6 also contains an applications of our results to the b.v.p. (1.2), (1.3), where $B \in \mathcal{L}(E)$.

Since the application of degree arguments will tendentiously allow some solutions of (1.1) to escape from Q, the crucial condition of the related continuation principle developed in Section 3 consists in guaranteeing the fixed point free boundary of Q w.r.t. an admissible homotopical bridge starting from (1.1) (see condition (e) in Proposition 3.1 below). This requirement will be verified by means of Liapunov-like bounding functions, i.e. via a bound sets technique (whence the title).

That is also why the whole Section 4 is devoted to this technique applied to Floquet problem (1.1) and in fact, as pointed out in remarks, to Floquet problems with general upper-Carathéodory differential inclusions (i.e. for A and B possibly equal to 0 in (1.1)). We distinguish two cases, namely when

(i) A, B are Bochner integrable transformations and F is an upper-Carathéodory mapping, and
(ii) A, B are continuous transformations and F is globally upper semicontinuous (i.e. a Marchaud mapping).

Unlike in the first case, the second one allows us to apply bounding functions which can be strictly localized on the boundaries of given bound sets.

The application of bounding functions to problems in abstract spaces was so far, to our best knowledge, exclusively related to first-order problems (see e.g. [5], [23], [24]). Moreover, guiding functions can only be (globally) applied in L^2-spaces or so, but not in general Banach spaces like here, as documented in [5] (see the related references therein). In this light, the bound sets approach to second-order problems in Banach spaces brings the main novelty of our paper.

Similarly as in finite-dimensional Euclidean spaces, the geometry concerning second-order problems, reflecting the behaviour of controlled trajectories, is much more sophisticated than for first-order problems. Moreover, to express desired transversality conditions in terms of bounding functions, it requires for second-order problems in Banach spaces to employ newly dual spaces, etc. On the other hand, the sufficient existence conditions are, in principle, better than those for equivalent first-order problems.

Although the main results formulated in Theorem 5.1 and Corollary 5.2 are rather abstract, they can be suitably applied for obtaining effective criteria of solvability of (1.1), as demonstrated especially by the third illustrative example supplied in Section 6.

Since the most important particular cases of the Floquet problem are related to a periodic problem ($M = N = \mathrm{id}$) and to an anti-periodic problem ($M = N = -\mathrm{id}$), the comparison of the obtained criteria with those of the other authors should preferably concern these two cases. However, since the methods applied by other authors in this field are significantly different from ours (see e.g. [1], [9], [12], [14], [7], [17], [18], [20], [21], [26]), we resigned to make such a comparison. If the localization of solutions, as the main advantage of our results, was guaranteed somewhere else, then it was almost exclusively done, in the frame of the viability theory, by means of various Nagumo-type (cone-type) conditions. Nevertheless, in the majority of quoted papers, the localization of solutions can be detected only with difficulties.

2. Preliminaries

Let E be a Banach space having the Radon–Nikodym property (see e.g. [19, pp. 694–695]) and $[0,T] \subset \mathbb{R}$ be a closed interval. By the symbol $L^1([0,T],E)$, we shall mean the set of all Bochner integrable functions $x\colon [0,T] \to E$. For the definition and properties, see e.g. [19, pp. 693–701]. The symbol $AC^1([0,T],E)$ will denote the set of functions $x\colon [0,T] \to E$ whose first derivative $\dot{x}(\,\cdot\,)$ is absolutely continuous. Then $\ddot{x} \in L^1([0,T],E)$ and the fundamental theorem of calculus (the Newton–Leibniz formula) holds (see e.g. [3, pp. 243–244], [19, pp. 695–696]). In the sequel, we shall always consider $AC^1([0,T],E)$ as a subspace of the Banach space $C^1([0,T],E)$. Given $C \subset E$ and $\varepsilon > 0$, the symbol $B(C,\varepsilon)$ will denote, as usually, the set $C + \varepsilon B$, where B is the open unit ball in E, i.e. $B = \{x \in E \mid ||x|| < 1\}$.

For each $L \in \mathcal{L}(E \times E)$, there exist unique $L_{ij} \in \mathcal{L}(E)$, $i,j = 1,2$, such that

$$L(x,y) = (L_{11}x + L_{12}y, L_{21}x + L_{22}y),$$

where $(x,y) \in E \times E$. For the sake of simplicity, we shall use the notation

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

We shall also need the following definitions and notions from multivalued analysis. Let X,Y be two metric spaces. We say that F is a *multivalued mapping* from X to Y (written $F\colon X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

A multivalued mapping $F\colon X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open subset $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

A multivalued mapping $F\colon X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x\in X} F(x)$ is contained in a compact subset of Y and it is called *quasi-compact* if it maps compact sets onto relatively compact sets.

The relationship between upper semicontinuous mappings and compact mappings with closed graphs is expressed by the following proposition (see, e.g. [15]).

Proposition 2.1. *Let X, Y be metric spaces and $F\colon X \multimap Y$ be a quasi-compact mapping with a closed graph. Then F is u.s.c.*

We say that a multivalued mapping $F\colon [0,T] \multimap Y$ with closed values is a *step multivalued mapping* if there exists a finite family of disjoint measurable subsets I_k, $k = 1, \dots, n$ such that $[0,T] = \bigcup I_k$ and F is constant, on every I_k. A multivalued mapping $F\colon [0,T] \multimap Y$ with closed values is called *strongly measurable* if there exists a sequence of step multivalued mappings $\{F_n\}$ such that $d_H(F_n(t), F(t)) \to 0$ as $n \to \infty$, for almost all $t \in [0,T]$, where d_H stands for the Hausdorff distance.

Let us note that if Y is a Banach space, then a strongly measurable mapping $F\colon [0,T] \multimap Y$ with compact values possesses a single-valued strongly measurable selection.

Let $J = [0,T]$ be a given compact interval. A multivalued mapping $F\colon J \times X \multimap Y$ is called an *upper-Carathéodory mapping* if the map $F(\,\cdot\,,x)\colon J \multimap Y$ is strongly measurable, for all $x \in X$, the map $F(t,\,\cdot\,)\colon X \multimap Y$ is u.s.c. for almost all $t \in J$, and the set $F(t,x)$ is compact and convex, for all $(t,x) \in J \times X$.

For more details concerning multivalued analysis, see e.g. [3], [De], [13], [15].

Definition 2.2. Let N be a partially ordered set, E be a Banach space and let $P(E)$ denote the family of all subsets of E. A function $\beta\colon P(E) \to N$ is called a *measure of non-compactness* (m.n.c.) in E if $\beta(\overline{\operatorname{co}\Omega}) = \beta(\Omega)$, for all $\Omega \in P(E)$, where $\overline{\operatorname{co}\Omega}$ denotes the closed convex hull of Ω.

A m.n.c. β is called:

(a) *monotone* if $\beta(\Omega_1) \le \beta(\Omega_2)$, for all $\Omega_1 \subset \Omega_2 \subset E$,
(b) *nonsingular* if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, for all $x \in E$ and $\Omega \subset E$,
(c) *invariant with respect to the union with compact sets* if $\beta(K \cup \Omega) = \beta(\Omega)$, for every relatively compact $K \subset E$ and every $\Omega \subset E$.
(d) *regular* when $\beta(\Omega) = 0$ if and only if Ω is relatively compact.

It is obvious that the m.n.c. which is invariant with respect to the union with compact sets is also nonsingular.

The typical example of an m.n.c. is the *Hausdorff measure of noncompactness* γ defined, for all $\Omega \subset E$ by

$$\gamma(\Omega) := \inf\biggl\{\varepsilon > 0 \biggm| \exists x_1, \dots, x_n \in E\colon \Omega \subset \bigcup_{i=1}^{n} B(\{x_i\}, \varepsilon)\biggr\}.$$

The Hausdorff measure of noncompactness is monotone and nonsingular. Moreover, if $L \in \mathcal{L}(E)$ and $\Omega \subset E$, then (see, e.g. [15])

$$\gamma(L\Omega) \le ||L||_{\mathcal{L}(E)}\gamma(\Omega). \tag{2.1}$$

Let $\{f_n\} \subset L([0,T],E)$ be such that $||f_n(t)|| \leq \alpha(t)$, $\gamma(\{f_n(t)\}) \leq c(t)$, for almost all $t \in [0,T]$, all $n \in \mathbb{N}$ and suitable $\alpha, c \in L([0,T],\mathbb{R})$, then (cf. [15])

$$\gamma\bigg(\bigg\{\int_0^T f_n(t)\,dt\bigg\}\bigg) \leq 2\int_0^T c(t), \quad \text{for a.a. } t \in [0,T]. \tag{2.2}$$

Moreover, for all subsets Ω of E (see e.g. [5]),

$$\gamma\bigg(\bigcup_{\lambda\in[0,1]} \lambda\Omega\bigg) \leq \gamma(\Omega). \tag{2.3}$$

Let us now introduce the function

$$\begin{aligned}\mu(\Omega) := \max_{\{w_n\}_n\subset\Omega}\bigg(&\sup_{t\in[0,T]}[\gamma(\{w_n(t)\}_n)+\gamma(\{\dot w_n(t)\}_n)],\\ &\operatorname{mod}_C(\{w_n\}_n)+\operatorname{mod}_C(\{\dot w_n\}_n)\bigg),\end{aligned} \tag{2.4}$$

defined on the bounded $\Omega \subset C^1([0,T],E)$, where the ordering is induced by the positive cone in $\mathbb{R}^2$ and where $\operatorname{mod}_C(\Omega)$ denotes the modulus of continuity of a subset $\Omega \subset C([0,T],E)$ ([1]). Such a μ is a m.n.c. in $C^1([0,T],E)$, as proven in the following lemma, where the properties of μ will be also discussed. We will use μ in order to solve problem (1.1) (cf. Theorem 5.1).

Lemma 2.3. *The function μ given by* (2.4) *defines an m.n.c. in $C^1([0,T],E)$; such a μ is monotone, invariant with respect to the union with compact sets and regular.*

Proof. At first, we show that μ is well-defined, i.e. that the maximum in (2.4) is reached. Indeed, let $\{x_n^{(m)}\}_n \subset \Omega$ and $\{y_n^{(m)}\}_n \subset \Omega$ be two sequences of denumerable sets such that, as $m \to \infty$,

$$\begin{aligned}\sup_{t\in[0,T]}[\gamma(\{x_n^{(m)}(t)\}_n)+\gamma(\{\dot x_n^{(m)}(t)\}_n)] &\to \sup_{\{w_n\}_n\subset\Omega}[\gamma(\{w_n(t)\}_n)+\gamma(\{\dot w_n(t)\}_n)],\\ \operatorname{mod}_C[(\{y_n^{(m)}\}_n)+(\{\dot y_n^{(m)}\}_n)] &\to \sup_{\{\dot w_n\}_n\subset\Omega}[\operatorname{mod}_C(\{w_n\}_n)+\operatorname{mod}_C(\{\dot w_n\}_n)].\end{aligned}$$

It is easy to see that the denumerable set

$$\{z_n\}_n := \bigg\{\bigg(\bigcup_{m=1}^\infty x_n^{(m)}, \bigcup_{m=1}^\infty y_n^{(m)}\bigg)\bigg\}_n$$

is such that

$$\mu(\Omega) = \bigg(\sup_{t\in[0,T]}[\gamma(\{z_n(t)\}_n)+\gamma(\{\dot z_n(t)\}_n)], \operatorname{mod}_C(\{z_n\}_n)+\operatorname{mod}_C(\{\dot z_n\}_n)\bigg).$$

([1]) The m.n.c. $\operatorname{mod}_C(\Omega)$ is a monotone, nonsingular and algebraically subadditive on $C([0,T],E)$ (cf. e.g. [15]) and it is equal to zero if and only if all the elements $x \in \Omega$ are equi-continuos.

Thus μ is well-defined. Observe that μ is also monotone, because if $\Omega_1 \subset \Omega_2 \subset C^1([0,T],E)$ are bounded, then the maximum for $\mu(\Omega_2)$ is taken on a larger set than for $\mu(\Omega_1)$, and so $\mu(\Omega_1) \leq \mu(\Omega_2)$. We now prove the equality $\mu(\overline{\text{co}}\,\Omega) = \mu(\Omega)$. By the monotonicity of μ, it is sufficient to prove that $\mu(\overline{\text{co}}\,\Omega) \leq \mu(\Omega)$. Let $\{y_n\}_n \subset (\overline{\text{co}}\,\Omega)$ be such that

$$\mu(\overline{\text{co}}\,\Omega) = \Big(\sup_{t\in[0,T]}[\gamma(\{y_n(t)\}_n) + \gamma(\{\dot{y}_n(t)\}_n)], \text{mod}_C(\{y_n\}_n) + \text{mod}_C(\{\dot{y}_n\}_n)\Big).$$

Hence, we can find $\{x_n\}_n$ such that $\{y_n\}_n \subset \overline{\text{co}}\,\{x_n\}_n$. According to the monotonicity of the Hausdorff m.n.c. and of $\text{mod}_C(\Omega)$, we obtain that

$$\begin{aligned}\gamma(\{y_n(t)\}_n) &\leq \gamma(\overline{\text{co}}\,\{x_n(t)\}_n) = \gamma(\{x_n(t)\}_n), \quad \text{for each } t \in [0,T],\\ \text{mod}_C(\{y_n\}_n) &\leq \text{mod}_C(\text{co}\,\{x_n\}_n) = \text{mod}_C(\{x_n\}_n),\end{aligned}$$

implying that $\mu(\{y_n\}_n) \leq \mu(\{x_n\}_n) \leq \mu(\Omega)$.

Now, we prove that μ is invariant with respect to the union with compact sets. Let $K \subset C^1([0,T],E)$ be relatively compact. Then, in view of monotonicity, $\mu(\Omega) \leq \mu(\Omega \cup K)$, for all bounded $\Omega \subset C^1([0,T],E)$. The reverse inequality $\mu(\Omega \cup K) \leq \mu(\Omega)$ can be proven as follows. Let $\{y_n\}_n \subset \Omega \cup K$ be a sequence where the maximum in the definition of $\mu(\Omega \cup K)$ is reached. Then

$$\gamma(\{y_n(t)\}) = \gamma((\{y_n\} \cap \Omega)(t) \cup (\{y_n\} \cap K)(t)) = \gamma((\{y_n\} \cap \Omega)(t)),$$

for all $t \in [0,T]$, and

$$\text{mod}_C(\{y_n\}) = \text{mod}_C((\{y_n\} \cap \Omega) \cup (\{y_n\} \cap K)) = \text{mod}_C(\{y_n\} \cap \Omega).$$

Put

$$\dot{\Omega} := \{x \in C([0,T],E) \mid \exists y \in \Omega\colon x(t) = \dot{y}(t),\ \text{for all } t \in [0,T]\}$$

and

$$\dot{K} := \{x \in C([0,T],E) \mid \exists y \in K\colon x(t) = \dot{y}(t),\ \text{for all } t \in [0,T]\}.$$

It is easy to see that both K and $\dot{K}$ are relatively compact in $C([0,T],E)$. Consequently,

$$\gamma(\{\dot{y}_n(t)\}) = \gamma((\{\dot{y}_n\} \cap \dot{\Omega})(t) \cup (\{\dot{y}_n\} \cap \dot{K})(t)) = \gamma((\{\dot{y}_n\} \cap \dot{\Omega})(t))$$

and

$$\text{mod}_C(\{\dot{y}_n\}) = \text{mod}_C((\{\dot{y}_n\} \cap \dot{\Omega}) \cup (\{\dot{y}_n\} \cap \dot{K})) = \text{mod}_C(\{\dot{y}_n\} \cap \dot{\Omega}).$$

Therefore,

$$\begin{aligned}\mu(\Omega \cup K) = \Big(&\sup_{t\in[0,T]}[\gamma((\{y_n\} \cap \Omega)(t)) + \gamma((\{\dot{y}_n\} \cap \dot{\Omega})(t))],\\ &\text{mod}_C(\{y_n\} \cap \Omega) + \text{mod}_C(\{\dot{y}_n\} \cap \dot{\Omega})\Big) \leq \mu(\Omega).\end{aligned}$$

Thus, the m.n.c. μ is invariant with respect to the union with compact sets, and so nonsingular as well.

It remains to show that μ is regular. If the set Ω is relatively compact, then each sequence $\{w_n\}_n \subset \Omega$ is also relatively compact. It implies that $\gamma(\{w_n(t)\}) = \gamma(\{\dot{w}_n(t)\}) = 0$, for every $t \in [0,T]$, and also that $\mathrm{mod}_C(\{w_n\}) = \mathrm{mod}_C(\{\dot{w}_n\}) = 0$. Hence, $\mu(\Omega) = (0,0)$.

On the other hand, if $\mu(\Omega) = (0,0)$, then $\gamma(\{w_n(t)\}) = \gamma(\{\dot{w}_n(t)\}) = \mathrm{mod}_C(\{w_n\}) = \mathrm{mod}_C(\{\dot{w}_n\}) = 0$, for each $t \in [0,T]$, and every $\{w_n\}_n \subset \Omega$. So, both $\{w_n\}_n$ and $\{\dot{w}_n\}_n$ are equi-continuous and, according to the regularity of the Hausdorff measure, the sets $\{w_n(t)\}_n$, $\{\dot{w}_n(t)\}_n$ are relatively compact, for every t. The well-known Arzelà–Ascoli lemma can be then applied to verify the relative compactness of $\{w_n\}_n$ which completes the proof. □

Definition 2.4. Let E be a Banach space and $X \subset E$. A multivalued mapping $F\colon X \multimap E$ with compact values is called *condensing with respect to an m.n.c.* β (shortly, β*-condensing*) if, for every $\Omega \subset X$ such that $\beta(F(\Omega)) \geq \beta(\Omega)$, it holds that Ω is relatively compact.

A family of mappings $G\colon X \times [0,1] \multimap E$ with compact values is called β-*condensing* if, for every $\Omega \subset X$ such that $\beta(G(\Omega \times [0,1])) \geq \beta(\Omega)$, it holds that Ω is relatively compact.

The following convergence result will be also employed.

Lemma 2.5 (cf. [3, Lemma III.1.30]). *Let E be a Banach space and assume that the sequence of absolutely continuous functions $x_k\colon [0,T] \to E$ satisfies the following conditions:*

(a) *the set $\{x_k(t) \mid k \in \mathbb{N}\}$ is relatively compact, for every $t \in [0,T]$,*
(b) *there exists $\alpha \in L^1([0,T],[0,\infty))$ such that*

$$||\dot{x}_k(t)|| \leq \alpha(t), \quad \textit{for a.a. } t \in [0,T] \textit{ and for all } k \in \mathbb{N},$$

(c) *the set $\{\dot{x}_k(t) \mid k \in \mathbb{N}\}$ is weakly relatively compact, for almost all $t \in [0,T]$.*

Then there exists a subsequence of $\{x_k\}$ (for the sake of simplicity denoted in the same way as the sequence) converging to an absolutely continuous function $x\colon [0,T] \to E$ in the following way:

(i) *$\{x_k\}$ converges uniformly to x, in $C([0,T],E)$,*
(ii) *$\{\dot{x}_k\}$ converges weakly in $L^1([0,T],E)$ to $\dot{x}$.*

The following lemma is well-known when the Banach spaces E_1 and E_2 coincide (see, e.g. [25, p. 88]). The present slight modification, for $E_1 \neq E_2$, was proved in [4].

Lemma 2.6. *Let $[0,T] \subset \mathbb{R}$ be a compact interval, let E_1, E_2 be Banach spaces and let $F\colon [0,T] \times E_1 \multimap E_2$ be a multivalued mapping satisfying the following conditions:*

(a) *$F(\,\cdot\,, x)$ has a strongly measurable selection, for every $x \in E_1$,*
(b) *$F(t, \cdot\,)$ is u.s.c., for a.a. $t \in [0,T]$,*
(c) *the set $F(t,x)$ is compact and convex, for all $(t,x) \in [0,T] \times E_1$.*

Assume in addition that, for every nonempty, bounded set $\Omega \subset E_1$, there exists $\nu = \nu(\Omega) \in L^1([0,T],(0,\infty))$ such that

$$||F(t,x)|| \leq \nu(t),$$

for almost all $t \in [0,T]$ and every $x \in \Omega$. Let us define the Nemytskiĭ operator $N_F\colon C([0,T],E_1) \multimap L^1([0,T],E_2)$ in the following way:

$$N_F(x) := \{f \in L^1([0,T],E_2) \mid f(t) \in F(t,x(t)),\ \text{a.e. on } [0,T]\},$$

for every $x \in C([0,T],E_1)$. Then, if sequences $\{x_k\} \subset C([0,T],E_1)$ and $\{f_k\} \subset L^1([0,T],E_2)$, $f_k \in N_F(x_k)$, $k \in \mathbb{N}$, are such that $x_k \to x$ in $C([0,T],E_1)$ and $f_k \to f$ weakly in $L^1([0,T],E_2)$, then $f \in N_F(x)$.

It will be also convenient to recall some basic facts concerning evolution equations. For a suitable introduction and more details, we refer, e.g. to [8], [16], [22].

Hence, let $C\colon [0,T] \to \mathcal{L}(E)$ be Bochner integrable and let $f \in L([0,T],E)$. Given $x_0 \in E$, consider the linear initial value problem

$$\dot{x}(t) = C(t)x(t) + f(t), \quad x(0) = x_0. \tag{2.5}$$

It is well-known (see, e.g. [8]) that, for the uniquely solvable problem (2.5), there exists the evolution operator $\{U(t,s)\}_{(t,s)\in\Delta}$, where $\Delta := \{(t,s)\colon 0 \leq s \leq t \leq T\}$, such that

$$||U(t,s)|| \leq \mathrm{e}^{\int_s^t ||C(\tau)||\,d\tau}, \quad \text{for all } (t,s) \in \Delta; \tag{2.6}$$

in addition, the unique solution $x(\,\cdot\,)$ of (2.5) is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)\,ds, \qquad t \in [0,T].$$

Given $D \in \mathcal{L}(E)$, the linear Floquet b.v.p.

$$\begin{cases} \dot{x}(t) = C(t)x(t) + f(t), \\ x(T) = Dx(0) \end{cases} \tag{2.7}$$

associated with the equation in (2.5), satisfies the following property.

LEMMA 2.7 (cf. [5]). *If the linear operator $D - U(T,0)$ is invertible, then (2.7) admits a unique solution given, for all $t \in [0,T]$, by*

$$x(t) = U(t,0)[D - U(T,0)]^{-1} \int_0^T U(T,\tau)f(\tau)\,d\tau + \int_0^t U(t,\tau)f(\tau)\,d\tau. \tag{2.8}$$

REMARK 2.8. Denoting

$$\Lambda := \mathrm{e}^{\int_0^T ||C(s)||\,ds}, \quad \Gamma := ||[D - U(T,0)]^{-1}||,$$

we obtain, in view of (2.6), (2.8) and the growth estimate imposed on $C(t)$, the following inequality for the solution $x(\,\cdot\,)$ of (2.7)

$$||x(t)|| \leq \Lambda(\Lambda\Gamma + 1) \int_0^T ||f(s)||\,ds.$$

Now, consider the second-order linear Floquet b.v.p.

$$\begin{cases} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = f(t), & \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0), \ \dot{x}(T) = N\dot{x}(0), \end{cases} \tag{2.9}$$

where A, B are Bochner integrable and $f \in L^1([0,T],E)$, and let

$$||(x,y)||_{E\times E} := \sqrt{||x||^2 + ||y||^2}, \quad \text{for all } x, y \in E.$$

Problem (2.9) is equivalent to the following first-order linear one

$$\begin{cases} \dot{\xi}(t) + C(t)\xi(t) = h(t), & \text{for a.a. } t \in [0,T], \\ \xi(T) = \widetilde{D}\xi(0), \end{cases} \tag{2.10}$$

where

$$\xi = (x,y) = (x,\dot{x}), \tag{2.11}$$

$$h(t) = (0, f(t)), \tag{2.12}$$

$$C(t)\colon E\times E \to E\times E, \quad (x,y) \mapsto (-y,\, B(t)x + A(t)y), \tag{2.13}$$

$$\widetilde{D}\colon E\times E \to E\times E, \quad (x,y) \mapsto (Mx,\, Ny). \tag{2.14}$$

Let us denote, for all $(t,s) \in [0,T]\times[0,T]$, by

$$U(t,s) := \begin{pmatrix} U_{11}(t,s) & U_{12}(t,s) \\ U_{21}(t,s) & U_{22}(t,s) \end{pmatrix}$$

the evolution operator associated with

$$\begin{cases} \dot{\xi}(t) + C(t)\xi(t) = h(t), & \text{for a.a. } t \in [0,T], \\ \xi(0) = \xi_0, \end{cases} \tag{2.15}$$

where ξ, h and C are defined by relations (2.11), (2.12) and (2.13), respectively, and $\xi_0 \in E \times E$. It is easy to see that $||C(t)|| \leq 1 + ||A(t)|| + ||B(t)||$ and, according to (2.6), we obtain

$$||U(t,s)|| \leq \mathrm{e}^{\int_0^T (1+||A(t)||+||B(t)||)\,dt}, \quad \text{for all } (t,s) \in \Delta.$$

Consequently, for all $i, j = 1, 2$,

$$||U_{ij}(t,s)|| \leq \mathrm{e}^{\int_0^T (1+||A(t)||+||B(t)||)\,dt}, \quad \text{for all } (t,s) \in \Delta. \tag{2.16}$$

Moreover, if we denote

$$[\widetilde{D} - U(T,0)]^{-1} := \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

and put

$$k := ||[\widetilde{D} - U(T,0)]^{-1}||, \tag{2.17}$$

then $||K_{ij}|| \leq k$, for $i, j = 1, 2$, and the solution $x(\,\cdot\,)$ of (2.9) and its derivative $\dot{x}(\,\cdot\,)$ take, for all $t \in [0,T]$, the forms

$$\begin{aligned} x(t) &= A_1(t) \int_0^T U_{12}(T,\tau) f(\tau)\,d\tau \\ &\quad + A_2(t) \int_0^T U_{22}(T,\tau) f(\tau)\,d\tau + \int_0^t U_{12}(t,\tau) f(\tau)\,d\tau, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \dot{x}(t) &= A_3(t) \int_0^T U_{12}(T,\tau) f(\tau)\,d\tau \\ &\quad + A_4(t) \int_0^T U_{22}(T,\tau) f(\tau)\,d\tau + \int_0^t U_{22}(t,\tau) f(\tau)\,d\tau, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} A_1(t) &:= U_{11}(t,0)K_{11} + U_{12}(t,0)K_{21}, \\ A_2(t) &:= U_{11}(t,0)K_{12} + U_{12}(t,0)K_{22}, \\ A_3(t) &:= U_{21}(t,0)K_{11} + U_{22}(t,0)K_{21}, \\ A_4(t) &:= U_{21}(t,0)K_{12} + U_{22}(t,0)K_{22}, \end{aligned}$$

for all $t \in [0,T]$. It holds that

$$||A_i(t)|| \leq 2k\mathrm{e}^{\int_0^T (1+||A(t)||+||B(t)||)\,dt}, \quad \text{for } i = 1,2,3,4 \text{ and } t \in [0,T]. \tag{2.20}$$

If there exists $\alpha \in L^1([0,T],[0,\infty))$ such that $||f(t)|| \leq \alpha(t)$, for almost all $t \in [0,T]$, then it immediately follows from Remark 2.8 that the following estimates hold for each solution $x(\,\cdot\,)$ of (2.9) and its derivative $\dot{x}(\,\cdot\,)$:

$$||x(t)|| \leq Z(Zk+1) \int_0^T \alpha(s)\,ds \quad \text{and} \quad ||\dot{x}(t)|| \leq Z(Zk+1) \int_0^T \alpha(s)\,ds,$$

where

$$Z := \mathrm{e}^{\int_0^T (||A(s)||+||B(s)||+1)\,ds} \tag{2.21}$$

with k defined in (2.17).

3. Continuation principle

In this section, consider the general multivalued b.v.p.

$$\begin{cases} \ddot{x}(t) \in \varphi(t, x(t), \dot{x}(t)) & \text{for a.a. } t \in J, \\ x \in S, \end{cases} \tag{3.1}$$

where $J = [0, T]$ is a given compact interval, $\varphi: J \times E \times E \multimap E$ is an upper-Carathéodory mapping. Furthermore, let $S \subset AC^1(J, E)$.

We also introduce the set $Q \subset AC^1(J, E)$ of candidate solutions of the b.v.p. (3.1) and associate to this problem a family of problems depending on two parameters $q \in Q$ and $\lambda \in [0, 1]$. The family of associated problems will be defined in such a way that if $\mathfrak{T}: Q \times [0, 1] \multimap AC^1(J, E)$ is its corresponding solution mapping, then all fixed points of the map $\mathfrak{T}(\,\cdot\,, 1)$ are solutions of (3.1) (see condition (3.2) below). In order to study the fixed point set of $\mathfrak{T}(\,\cdot\,, 1)$, a suitable topological degree technique will be employed.

Proposition 3.1. *Let us consider the b.v.p.* (3.1) *and let* $H: [0, T] \times E \times E \times E \times E \times [0, 1] \multimap E$ *be an upper-Carathéodory mapping such that*

$$H(t, c, d, c, d, 1) \subset \varphi(t, c, d), \quad \textit{for all } (t, c, d) \in [0, T] \times E \times E. \tag{3.2}$$

Moreover, assume that the following conditions hold:

(a) *There exist a closed set* $S_1 \subset S$ *and a closed, convex set* $Q \subset C^1([0, T], E)$ *with a non-empty interior* $\operatorname{Int} Q$ *such that each associated problem*

$$P(q,\lambda) \qquad \begin{cases} \ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda) & \textit{for a.a. } t \in [0, T], \\ x \in S_1, \end{cases}$$

where $q \in Q$ *and* $\lambda \in [0, 1]$, *has a non-empty, convex set of solutions* (*denoted by* $\mathfrak{T}(q, \lambda)$).

(b) *For every non-empty, bounded set* $\Omega \subset E \times E$, *there exists* $\nu_\Omega \in L^1([0, T], [0, \infty))$ *such that*

$$||H(t, x, y, q(t), \dot{q}(t), \lambda)|| \le \nu_\Omega(t),$$

for almost all $t \in [0, T]$ *and all* $(x, y) \in \Omega$, $q \in Q$ *and* $\lambda \in [0, 1]$.

(c) *The solution mapping* $\mathfrak{T}$ *is quasi-compact and* μ*-condensing with respect to a monotone and nonsingular measure of noncompactness* μ *defined on* $C^1([0, T], E)$.

(d) *For each* $q \in Q$*, the set of solutions of the problem* $P(q,0)$ *is a subset of* $\operatorname{Int} Q$*, i.e.* $\mathfrak{T}(q,0) \subset \operatorname{Int} Q$*, for all* $q \in Q$*.*

(e) *For each* $\lambda \in (0,1)$*, the solution mapping* $\mathfrak{T}(\,\cdot\,,\lambda)$ *has no fixed points on the boundary* ∂Q *of* Q*.*

Then the b.v.p. (3.1) *has a solution in* Q*.*

Proof. Let us observe that, according to condition (3.2), every fixed point of the solution mapping $\mathfrak{T}(\,\cdot\,,1)$ is a solution of the original problem (3.1) lying in Q. Thus, if the intersection $\operatorname{Fix}(\mathfrak{T}(\,\cdot\,,1)) \cap \partial Q$ is nonempty, then the b.v.p. (3.1) has a solution in Q and we are done. Otherwise, condition (e) can be reformulated (according to the above consideration and assumption (d)) as follows:

$$\operatorname{Fix}(\mathfrak{T}(\,\cdot\,,\lambda)) \cap \partial Q = \emptyset, \quad \text{for all } \lambda \in [0,1]. \tag{3.3}$$

Now, we will show that the solution mapping $\mathfrak{T}: Q \times [0,1] \multimap AC^1([0,T],E)$ is a u.s.c. mapping with compact values. Consequently, the properties of the solution mapping together with condition (3.3) will allow us to define the topological degree of $\mathfrak{T}$ and to prove that the b.v.p. (3.1) has a solution in Q.

At first, let us prove, by means of Lemmas 2.5 and 2.6, that the solution mapping $\mathfrak{T}$ has a closed graph $\Gamma_{\mathfrak{T}}$. For this purpose, let $\{q_k, \lambda_k, x_k\} \subset \Gamma_{\mathfrak{T}}$ be a sequence such that $(q_k, \lambda_k, x_k) \to (q_0, \lambda_0, x_0)$ in $C^1([0,T],E) \times \mathbb{R} \times C^1([0,T],E)$ as $k \to \infty$, where $q_0 \in Q$, $\lambda_0 \in [0,1]$ and $x_0 \in C^1([0,T],E)$. Since $\dot{x}_k(t) \to \dot{x}_0(t)$, the sequence $\{\dot{x}_k(t)\}_{k=1}^{\infty}$ is relatively compact, for all $t \in [0,T]$. Moreover, since $\{x_k, \dot{x}_k\}$ is uniformly convergent on $[0,T]$, there exists a constant $M > 0$ such that

$$\|x_k(t)\| \le M \quad \text{and} \quad \|\dot{x}_k(t)\| \le M, \quad \text{for all } t \in [0,T] \text{ and } k \in \mathbb{N}. \tag{3.4}$$

According to the estimates in (3.4) and condition (b), there exists $\nu \in L^1([0,T],[0,\infty))$ such that

$$\|H(x_k(t), \dot{x}_k(t), q_k(t), \dot{q}_k(t), \lambda_k)\| \le \nu(t),$$

for almost all $t \in [0,T]$ and all $k \in \mathbb{N}$. Therefore, $\|\ddot{x}_k(t)\| \le \nu(t)$, for almost all $t \in [0,T]$ and all $k \in \mathbb{N}$.

Now, let us show that, for almost all $t \in [0,T]$, $\{\ddot{x}_k(t)\}$ is relatively compact. For this purpose, let $t \in [0,T]$ be such that

$$\ddot{x}_k(t) \in H(t, x_k(t), \dot{x}_k(t), q_k(t), \dot{q}_k(t), \lambda_k), \quad \text{for all } k \in \mathbb{N}.$$

Since $H(t,\,\cdot\,)$ is u.s.c., for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$H(t,x,y,u,v,\lambda) \subset H(t, x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0) + \varepsilon B,$$

for all $(x,y,u,v,\lambda) \in E \times E \times E \times E \times [0,1]$ satisfying

$$\|(x,y,u,v,\lambda) - (x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0)\| < \delta.$$

The fact that $(q_k, \dot{q}_k, \lambda_k, x_k, \dot{x}_k) \to (q_0, \dot{q}_0, \lambda_0, x_0, \dot{x}_0)$ ensures the existence of $k_0 \in \mathbb{N}$ such that

$$H(t, x_k(t), \dot{x}_k(t), q_k(t), \dot{q}_k(t), \lambda_k) \subset H(t, x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0) + \varepsilon B,$$

for all $k \geq k_0$. Thus,

$$\{\ddot{x}_k(t)\}_{k=1}^{\infty} \subset \bigcup_{k=1}^{k_0} H(t, x_k(t), \dot{x}_k(t), q_k(t), \dot{q}_k(t), \lambda_k) \\ \cup H(t, x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0) + \varepsilon B.$$

Since H has compact values, the sequence $\{\ddot{x}_k(t)\}$ is relatively compact, for almost all $t \in [0,T]$.

The above reasonings imply that the sequence $\{\dot{x}_k\}$ satisfies all assumptions of Lemma 2.5. Thus, there exists a subsequence of $\{\dot{x}_k\}$, for the sake of simplicity denoted in the same way as the sequence, such that $\{\ddot{x}_k\}$ converges weakly to $\ddot{x}_0$ in $L^1([0,T],E)$.

If we set $y_k := \dot{x}_k$ and $z_k := (x_k, y_k)$, then $\dot{z}_k = (\dot{x}_k, \dot{y}_k) = (\dot{x}_k, \ddot{x}_k) \to (\dot{x}_0, \ddot{x}_0)$ weakly in $L^1([0,T],E)$. Let us now consider the system

$$\dot{z}_k(t) \in H^*(t, z_k(t), q_k(t), \dot{q}_k(t), \lambda_k), \quad \text{for a.a. } t \in [0,T],$$

where $H^*(t, z_k(t), q_k(t), \dot{q}_k(t), \lambda_k) = (y_k(t), H(t, z_k(t), q_k(t), \dot{q}_k(t), \lambda_k))$.

Applying Lemma 2.6, for $f_k := \dot{z}_k$, $f := (\dot{x}_0, \ddot{x}_0)$, $x_k := (z_k, q_k, \dot{q}_k, \lambda_k)$, it follows that

$$(\dot{x}_0(t), \ddot{x}_0(t)) \in H^*(t, x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0),$$

for almost all $t \in [0,T]$, i.e.

$$\ddot{x}_0(t) \in H(t, x_0(t), \dot{x}_0(t), q_0(t), \dot{q}_0(t), \lambda_0), \quad \text{for a.a. } t \in [0,T].$$

Moreover, since S_1 is closed, $x_0 \in S_1$, and so the solution mapping $\mathfrak{T}$ has a closed graph.

Thus, the set $\mathfrak{T}(q,\lambda)$ is closed, for all $(q,\lambda) \in Q \times [0,1]$, which (together with condition (c)) implies that $\mathfrak{T}$ has compact values. Furthermore, according to Proposition 2.1, $\mathfrak{T}$ is a u.s.c. mapping. Therefore, we can conclude that $\mathfrak{T}$ is a u.s.c. mapping with compact, convex values which is condensing on the closed set Q. This ensures that both the topological degree (see e.g. [15]) as well as the fixed point index (see e.g. [3]) can be defined on open sets with fixed point free boundaries. Moreover, both the degree and the index satisfy the standard properties. In particular, $\mathfrak{T}$ is an admissible homotopy according to (3.3) and the multivalued vector-fields $\phi_0(\,\cdot\,) := \mathrm{id} - \mathfrak{T}(\,\cdot\,,0)$, $\phi_1(\,\cdot\,) := \mathrm{id} - \mathfrak{T}(\,\cdot\,,1)$ are homotopic as well, and so $\deg_{C^1([0,T],E)}(\phi_1, Q) = \deg_{C^1([0,T],E)}(\phi_0, Q)$. Furthermore, since $\mathfrak{T}(Q \times \{0\}) \subset \operatorname{Int} Q$, the localization property of the degree ensures that $\deg_{C^1([0,T],E)}(\phi_0, Q) = \deg_Q(\phi_0, Q) = 1$. Therefore, the nonemptiness of $\mathrm{Fix}(\mathfrak{T}(\,\cdot\,,1))$ is ensured by the existence property of the degree which completes the proof. □

4. Bound sets technique

The continuation principle formulated in Proposition 3.1 requires, in particular, the existence of a suitable set $Q \subset AC^1(J,E)$ of candidate solutions. The set Q must satisfy the transversality condition (d), i.e. it must have fixed-point free boundary with respect to the solution mapping $\mathfrak{T}$. Since the direct verification of the transversality condition is usually a difficult task, we will devote this section to a bound sets technique which can be used for guaranteeing this condition. For this purpose, we will define the set Q as $Q = C^1([0,T],\overline{K})$, where K is nonempty and open in E and $\overline{K}$ denotes its closure.

Hence, let us consider the Floquet boundary value problem (1.1) and let $V\colon E \to \mathbb{R}$ be a C^1-function satisfying

(H1) $V|_{\partial K} = 0$,

(H2) $V(x) \le 0$, for all $x \in \overline{K}$.

DEFINITION 4.1. A nonempty open set $K \subset E$ is called a *bound set* for the b.v.p. (1.1) if every solution x of (1.1) such that $x(t) \in \overline{K}$, for each $t \in [0,T]$, does not satisfy $x(t^*) \in \partial K$, for any $t^* \in [0,T]$.

Let E' be the Banach space dual to E and let us denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the pairing (the duality relation) between E and E', i.e. for all $\Phi \in E'$ and $x \in E$, we put $\Phi(x) := \langle \Phi, x\rangle$.

PROPOSITION 4.2. *Let $K \subset E$ be an open set such that $0 \in K$. Moreover, let $M\partial K = \partial K$. Assume that the function $V \in C^1(E,\mathbb{R})$ has a locally Lipschitz Fréchet derivative $\dot{V}_x$ and satisfies conditions* (H1) *and* (H2). *Suppose, moreover, that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$ and $y \in E$, the following condition*

$$\limsup_{h \to 0^-} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + \langle \dot{V}_{x+hy}, w\rangle > 0 \tag{4.1}$$

holds, for all $w \in F(t,x,y) - A(t)y - B(t)x$, and that

$$\langle \dot{V}_{Mx}, Nz\rangle \cdot \langle \dot{V}_x, z\rangle > 0 \quad \text{or} \quad \langle \dot{V}_{Mx}, Nz\rangle = \langle \dot{V}_x, z\rangle = 0, \tag{4.2}$$

for all $x \in \partial K$ and $z \in E$. Then K is a bound set for the Floquet problem (1.1).

PROOF. Let $x\colon [0,T] \to \overline{K}$ be a solution of problem (1.1). We assume, by a contradiction, that there exists $t^* \in [0,T]$ such that $x(t^*) \in \partial K$. According to the boundary condition in (1.1) and in view of $M\partial K = \partial K$, we can take, without any loss of generality, $t^* \in (0,T]$.

Since $\dot{V}_x$ is locally Lipschitz, there exist a neighbourhood U of $x(t^*)$ and a constant $L > 0$ such that $\dot{V}|_U$ is Lipschitz with constant L. Let $\delta > 0$ be such that $x(t) \in U \cap B(\partial K, \varepsilon)$, for each $t \in [t^* - \delta, t^*]$.

In order to get the desired contradiction, let us define the function $g\colon [0,T] \to \mathbb{R}$ as the composition $g(t) := (V \circ x)(t)$. According to the regularity properties of x and V, $g \in C^1([0,T],\mathbb{R})$. Since $g(t^*) = 0$ and $g(t) \leq 0$, for all $t \in [0,T]$, t^* is a local maximum point for g. Therefore, $\dot{g}(t^*) \geq 0$ and $\dot{g}(t^*) = 0$, when $t^* \in (0,T)$. Moreover, there exists a point $t^{**} \in (t^* - \delta, t^*)$ such that $\dot{g}(t^{**}) \geq 0$.

According to boundary conditions, if $t^* = T$, then also $x(0) \in \partial K$ and

$$\dot{g}(0) = \langle \dot{V}_{x(0)}, \dot{x}(0) \rangle \leq 0.$$

Moreover, since $x(T) = Mx(0)$ and $\dot{x}(T) = N\dot{x}(0)$, we have

$$\dot{g}(T) = \langle \dot{V}_{x(T)}, \dot{x}(T) \rangle = \langle \dot{V}_{Mx(0)}, N\dot{x}(0) \rangle \geq 0.$$

Condition (4.2) then implies

$$\langle \dot{V}_{x(0)}, \dot{x}(0) \rangle = \langle V_{Mx(0)}, N\dot{x}(0) \rangle = 0$$

which is equivalent to $\dot{g}(0) = \dot{g}(T) = 0$.

Since $\dot{g}(t) = \langle V_{x(t)}, \dot{x}(t) \rangle$, where $\dot{V}_{x(t)}$ is locally Lipschitz and $\dot{x}(t)$ is absolutely continuous on $[t^* - \delta, t^*]$, $\ddot{g}(t)$ exists, for almost all $t \in [t^* - \delta, t^*]$. Consequently,

$$0 \geq -\dot{g}(t^{**}) = \dot{g}(t^*) - \dot{g}(t^{**}) = \int_{t^{**}}^{t^*} \ddot{g}(s)\, ds. \tag{4.3}$$

Let $t \in (t^{**}, t^*)$ be such that $\ddot{g}(t)$ and $\ddot{x}(t)$ exist. Then,

$$\lim_{h \to 0} \frac{\dot{x}(t+h) - \dot{x}(t)}{h} = \ddot{x}(t)$$

and, therefore, there exists a function $a(h)$, $a(h) \to 0$ as $h \to 0$ such that, for each h,

$$\dot{x}(t+h) = \dot{x}(t) + h[\ddot{x}(t) + a(h)].$$

Moreover, since $x \in C^1([0,T],E)$, there exists a function $b(h)$, $b(h) \to 0$ as $h \to 0$ such that, for each h,

$$x(t+h) = x(t) + h[\dot{x}(t) + b(h)].$$

Consequently, we obtain

$$\begin{aligned}
\ddot{g}(t) = \lim_{h \to 0} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} &= \limsup_{h \to 0^-} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} \\
&= \limsup_{h \to 0^-} \frac{\langle \dot{V}_{x(t+h)}, \dot{x}(t+h) \rangle - \langle V_{x(t)}, \dot{x}(t) \rangle}{h} \\
&= \limsup_{h \to 0^-} \frac{\langle \dot{V}_{x(t)+h[\dot{x}(t)+b(h)]}, \dot{x}(t) + h[\ddot{x}(t) + a(h)] \rangle - \langle \dot{V}_{x(t)}, \dot{x}(t) \rangle}{h}
\end{aligned}$$

$$\begin{aligned}
&\geq \limsup_{h\to 0^-} \frac{\langle \dot{V}_{x(t)+h\dot{x}(t)}, \dot{x}(t) + h[\ddot{x}(t)+a(h)]\rangle - \langle \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h} \\
&\quad - L\cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t)+a(h)]| \\
&= \limsup_{h\to 0^-} \frac{\langle \dot{V}_{x(t)+h\dot{x}(t)}, \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h} \\
&\quad - L\cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t)+a(h)]| + \langle \dot{V}_{x(t)+h\dot{x}(t)}, a(h)\rangle.
\end{aligned}$$

Since

$$\langle \dot{V}_{x(t)+h\dot{x}(t)}, a(h)\rangle - L\cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t)+a(h)]| \to 0 \quad \text{as } h \to 0,$$

$$\begin{aligned}
\ddot{g}(t) &\geq \limsup_{h\to 0^-} \frac{\langle \dot{V}_{x(t)+h\dot{x}(t)}, \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h} \\
&= \limsup_{h\to 0^-} \frac{\langle \dot{V}_{x(t)+h\dot{x}(t)} - \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h} + \langle \dot{V}_{x(t)+h\dot{x}(t)}, \ddot{x}(t)\rangle > 0,
\end{aligned}$$

according to assumption (4.1), it leads to a contradiction with (4.3). □

Remark 4.3. Observe that Proposition 4.2 holds, without any loss of generality, for the general upper-Carathéodory differential inclusion in (1.1), i.e. for $A = B \equiv 0$.

If the mapping $F(t,x,y) - A(t)y - B(t)x$ is globally u.s.c. in (t,x,y), then the transversality conditions can be localized directly on the boundary of K, as will be shown in the following propositions.

Proposition 4.4. *Let $K \subset E$ be a nonempty open set, $F\colon [0,T]\times E\times E \multimap E$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values and A and B be continuous. Assume that there exists a function $V \in C^1(E,\mathbb{R})$ with a locally Lipschitz Fréchet derivative $\dot{V}_x$ which satisfies conditions* (H1) *and* (H2). *Suppose moreover that, for all $x \in \partial K$, $t \in (0,T)$ and $y \in E$ with*

$$\langle \dot{V}_x, y\rangle = 0, \tag{4.4}$$

the following condition holds

$$\liminf_{h\to 0} \frac{\langle \dot{V}_{x+hy}, y + hw\rangle}{h} > 0, \tag{4.5}$$

for all $w \in F(t,x,y) - A(t)y - B(t)x$. Then all solutions $x\colon [0,T] \to \overline{K}$ of problem (1.1) *satisfy $x(t) \in K$, for every $t \in (0,T)$.*

Proof. Let $x\colon [0,T] \to \overline{K}$ be a solution of problem (1.1). We assume, by a contradiction, that there exists $t_0 \in (0,T)$ such that $x(t_0) \in \partial K$.

Let us define the function $g\colon [-t_0, T-t_0] \to (-\infty, 0]$ as the composition $g(h) := (V\circ x)(t_0+h)$. Then $g(0) = 0$ and $g(h) \leq 0$, for all $h \in [-t_0, T-t_0]$, i.e.

there is a local maximum for g at the point 0, and so $\dot{g}(0) = \langle \dot{V}_{x(t_0)}, \dot{x}(t_0)\rangle = 0$. Consequently, $v := \dot{x}(t_0)$ satisfies condition (4.4).

Since $\dot{V}_x$ is locally Lipschitz, there exist a neighborhood U of $x(t_0)$ and a constant $L > 0$ such that $\dot{V}|_U$ is Lipschitz with constant L.

Let $\{h_k\}_{k=1}^{\infty}$ be an arbitrary decreasing sequence of positive numbers such that $h_k \to 0^+$ as $k \to \infty$, $x(t_0 + h) \in U$, for all $h \in (0, h_1)$.

Since $g(0) = 0$ and $g(h) \le 0$, for all $h \in (0, h_k]$, there exists, for each $k \in \mathbb{N}$, $h_k^* \in (0, h_k)$ such that $\dot{g}(h_k^*) \le 0$.

Since $x \in C^1([0, T], E)$, for each $k \in \mathbb{N}$,

$$x(t_0 + h_k^*) = x(t_0) + h_k^*[\dot{x}(t_0) + b_k^*], \tag{4.6}$$

where $b_k^* \to 0$ as $k \to \infty$.

If we define, for each $t \in [0, T]$,

$$P(t, x(t), \dot{x}(t)) := -A(t)\dot{x}(t) - B(t)x(t) + F(t, x(t), \dot{x}(t)), \tag{4.7}$$

then (1.1) can be written in the form

$$\ddot{x}(t) \in P(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0, T].$$

Let

$$\zeta := \left\{ \frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*},\ k \in \mathbb{N} \right\}$$

and let $\varepsilon > 0$ be given. As a consequence of the regularity assumptions on F, A and B and of the continuity of both x and $\dot{x}$, there exists $\overline{\delta} = \overline{\delta}(\varepsilon) > 0$ such that, for each $t \in (0, T)$, $|t - t_0| \le \overline{\delta}$, it follows that

$$P(t, x(t), \dot{x}(t)) \subset P(t_0, x(t_0), \dot{x}(t_0)) + \varepsilon \overline{B}.$$

Subsequently, according to the Mean Value Theorem (see e.g. [6, Theorem 0.5.3]), there exists $k_\varepsilon \in \mathbb{N}$ such that, for each $k > k_\varepsilon$,

$$\frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*} = \frac{1}{h_k^*} \int_{t_0}^{t_0 + h_k^*} \ddot{x}(s)\, ds \in P(t_0, x(t_0), \dot{x}(t_0)) + \varepsilon \overline{B}.$$

Therefore,

$$\zeta \subset \left\{ \frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*},\ k = 1, \ldots, k(\varepsilon) \right\} \cup P(t_0, x(t_0), \dot{x}(t_0)) + \varepsilon \overline{B}.$$

Since P has compact values and ε is arbitrary, we obtain that ζ is a relatively compact set. Thus, there exist a subsequence, for the sake of simplicity denoted as the sequence, of $\{(\dot{x}(t_0 + h_k^*) - \dot{x}(t_0))/h_k^*\}$ and $w \in E$ such that

$$\frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*} \to w \tag{4.8}$$

as $k \to \infty$ implying, for the arbitrariness of $\varepsilon > 0$,

$$w \in P(t_0, x(t_0), \dot{x}(t_0)).$$

As a consequence of property (4.8), there exists a sequence $\{a_k^*\}_{k=1}^{\infty}$, $a_k^* \to 0$ as $k \to \infty$, such that

$$\dot{x}(t_0 + h_k^*) = \dot{x}(t_0) + h_k^*[w + a_k^*], \tag{4.9}$$

for each $k \in \mathbb{N}$. Since $h_k^* > 0$ and $\dot{g}(h_k^*) \leq 0$, in view of (4.6) and (4.9),

$$0 \geq \frac{\dot{g}(h_k^*)}{h_k^*} = \frac{\langle \dot{V}_{x(t_0+h_k^*)}, \dot{x}(t_0 + h_k^*)\rangle}{h_k^*} = \frac{\langle \dot{V}_{x(t_0)+h_k^*[\dot{x}(t_0)+b_k^*]}, \dot{x}(t_0) + h_k^*[w + a_k^*]\rangle}{h_k^*}.$$

Since $h_k^* \in (0, h_k) \subset (0, h_1)$, for all $k \in \mathbb{N}$, we have, according to (4.6), that $x(t_0) + h_k^*[\dot{x}(t_0) + b_k^*] \in U$, for each $k \in \mathbb{N}$. Since $b_k^* \to 0$ as $k \to \infty$, it is possible to find $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, it holds that $x(t_0) + \dot{x}(t_0)h_k^* \in U$. By means of the local Lipschitzianity of $\dot{V}$, for all $k \geq k_0$,

$$\begin{aligned}
0 &\geq \frac{\dot{g}(h_k^*)}{h_k^*}\\
&= \frac{\langle \dot{V}_{x(t_0)+h_k^*[\dot{x}(t_0)+b_k^*]} - \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)} + \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) + h_k^*[w + a_k^*]\rangle}{h_k^*}\\
&\geq \frac{\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) + h_k^*[w + a_k^*]\rangle}{h_k^*} - L \cdot |b_k^*| \cdot |\dot{x}(t_0) + h_k^*[w + a_k^*]|\\
&= \frac{\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) + h_k^* w\rangle}{h_k^*}\\
&\quad - L \cdot |b_k^*| \cdot |\dot{x}(t_0) + h_k^*[w + a_k^*]| + \langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, a_k^*\rangle.
\end{aligned}$$

Since $\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, a_k^*\rangle - L \cdot |b_k^*| \cdot |\dot{x}(t_0) + h_k^*[w + a_k^*]| \to 0$ as $k \to \infty$,

$$\liminf_{h \to 0^+} \frac{\langle \dot{V}_{x(t_0)+h\dot{x}(t_0)}, \dot{x}(t_0) + hw\rangle}{h} \leq 0. \tag{4.10}$$

If we consider, instead of the sequence $\{h_k\}_{k=1}^{\infty}$, an increasing sequence $\{\overline{h}_k\}_{k=1}^{\infty}$ of negative numbers such that $\overline{h}_k \to 0^-$ as $k \to \infty$, $x(t_0 + h) \in U$ for all $h \in (\overline{h}_1, 0)$, we are able to find, for each $k \in \mathbb{N}$, $\overline{h}_k^* \in (\overline{h}_k, 0)$ such that $\dot{g}(\overline{h}_k^*) \geq 0$. Therefore, using the same procedure as in the first part of the proof, we obtain, for $k \in \mathbb{N}$ sufficiently large, that

$$\begin{aligned}
0 \geq \frac{\dot{g}(\overline{h}_k^*)}{\overline{h}_k^*} &\geq \frac{\langle \dot{V}_{x(t_0)+\overline{h}_k^*\dot{x}(t_0)}, \dot{x}(t_0) + \overline{h}_k^*\overline{w}\rangle}{\overline{h}_k^*}\\
&\quad - L \cdot |\overline{b}_k^*| \cdot |\dot{x}(t_0) + \overline{h}_k^*[\overline{w} + \overline{a}_k^*]| + \langle \dot{V}_{x(t_0)+\overline{h}_k^*\dot{x}(t_0)}, \overline{a}_k^*\rangle,
\end{aligned}$$

where $\overline{a}_k^* \to 0$, $\overline{b}_k^* \to 0$ as $k \to \infty$ and $\overline{w} \in P(t_0, x(t_0), \dot{x}(t_0))$.

This means that $\langle \dot V_{x(t_0)+\overline{h}_k^*\dot x(t_0)}, \overline{a}_k^*\rangle - L\cdot|\overline{b}_k^*|\cdot|\dot x(t_0)+\overline{h}_k^*[\overline{w}+\overline{a}_k^*]| \to 0$ as $k\to\infty$ which implies

$$\liminf_{h\to 0^-} \frac{\langle \dot V_{x(t_0)+h\dot x(t_0)}, \dot x(t_0)+h\overline{w}\rangle}{h} \le 0. \tag{4.11}$$

Inequalities (4.10) and (4.11) are in a contradiction with condition (4.5), because $x(t_0)\in\partial K$, $\dot x(t_0)$ satisfies condition (4.4) and $w,\overline{w}\in P(t_0,x(t_0),\dot x(t_0))$. □

Remark 4.5. Observe that Proposition 4.4 holds, without any loss of generality, for the general second-order problem (3.1), i.e. for $A=B\equiv 0$.

Proposition 4.6. *Let $K\subset E$ be a nonempty open set, $F\colon[0,T]\times E\times E \multimap E$ be an upper semicontinuous mapping with nonempty, compact, convex values and A and B be continuous. Assume that there exists a function $V\in C^1(E,\mathbb{R})$ with a locally Lipschitz–Fréchet derivative $\dot V_x$ which satisfies conditions* (H1) *and* (H2). *Moreover, let M be invertible and such that*

$$M(\partial K)=\partial K. \tag{4.12}$$

Furthermore, assume that, for all $x\in\partial K$, $t\in(0,T)$ and $y\in E$ satisfying (4.4), *condition* (4.5) *holds, for all $w\in F(t,x,y)-A(t)y-B(t)x$. At last, assume that, for all $x\in\partial K$ and $y\in E$ with*

$$\langle \dot V_x, y\rangle \le 0 \le \langle \dot V_{Mx}, Ny\rangle, \tag{4.13}$$

at least one of the following conditions

$$\liminf_{h\to 0^+} \frac{\langle \dot V_{x+hy}, y+hw_1\rangle}{h} > 0 \tag{4.14}$$

or

$$\liminf_{h\to 0^-} \frac{\langle \dot V_{Mx+hNy}, Ny+hw_2\rangle}{h} > 0 \tag{4.15}$$

holds, for all $w_1\in F(0,x,y)-A(0)y-B(0)x$ or, for all $w_2\in F(T,Mx,Ny)-A(T)Ny-B(T)Mx$, respectively. Then K is a bound set for problem (1.1).

Proof. Applying Proposition 4.4, we only need to show that if $x\colon[0,T]\to\overline{K}$ is a solution of problem (1.1), then $x(0)\in K$ and $x(T)\in K$. As in the proof of Proposition 4.4, we argue by a contradiction. Since $x(0)\in\partial K$ if and only if $x(T)\in\partial K$ (according to condition (4.12) and the properties of M), we can take, without any loss of generality, a solution of (1.1) satisfying $x(0)\in\partial K$. Following the same reasoning as in the proof of Proposition 4.4, for $t_0=0$ we obtain

$$\langle \dot V_{x(0)}, \dot x(0)\rangle \le 0,$$

because $V(x(0))=0$ and $V(x(t))\le 0$ for all $t\in[0,T]$.

Moreover, since $V(x(T)) = 0$, it holds that

$$0 \le \langle \dot{V}_{x(T)}, \dot{x}(T)\rangle = \langle \dot{V}_{Mx(0)}, N\dot{x}(0)\rangle,$$

by virtue of the boundary conditions in (1.1). Therefore, $v := \dot{x}(0)$ satisfies condition (4.13).

Using the same procedure as in the proof of Proposition 4.4, for $t_0 = 0$, $h_k \to 0^+$ and for $t_0 = T$, $\overline{h}_k \to 0^-$, respectively, we obtain the existence of a sequence of positive numbers $\{h_k^*\}_{k=1}^\infty$, $h_k^* \in (0, h_k)$, of a sequence of negative numbers $\{\overline{h}_k^*\}_{k=1}^\infty$, $\overline{h}_k^* \in (\overline{h}_k, 0)$ and of points $w_0 \in P(0, x(0), \dot{x}(0)), w_T \in P(T, x(T), \dot{x}(T))$ (P is defined by formula (4.7)) such that

$$\frac{\dot{x}(h_k^*) - \dot{x}(0)}{h_k^*} \to w_0, \quad \text{as } k \to \infty,$$

$$\frac{\dot{x}(T + \overline{h}_k^*) - \dot{x}(T)}{\overline{h}_k^*} \to w_T, \quad \text{as } k \to \infty.$$

By the same arguments as in the previous proof, we get

$$\liminf_{h\to 0^+} \frac{\langle \dot{V}_{x(0)+h\dot{x}(0)}, \dot{x}(0) + hw_0\rangle}{h} \le 0, \tag{4.16}$$

$$\liminf_{h\to 0^-} \frac{\langle \dot{V}_{x(T)+h\dot{x}(T)}, \dot{x}(T) + hw_T\rangle}{h} \le 0. \tag{4.17}$$

Moreover, using the boundary conditions in (1.1), the inequality (4.17) can be written in the form

$$\liminf_{h\to 0^-} \frac{\langle \dot{V}_{Mx(0)+hN\dot{x}(0)}, N\dot{x}(0) + hw_T\rangle}{h} \le 0. \tag{4.18}$$

Inequalities (4.16) and (4.18) are in a contradiction with conditions (4.14) and (4.15) which completes the proof. □

Remark 4.7. Observe that Proposition 4.6 holds again, without any loss of generality, for the general upper-Carathéodory differential inclusion in (1.1), i.e. for $A = B \equiv 0$.

Definition 4.8. A C^1-function $V\colon E \to R$ with a locally Lipschitz–Fréchet derivative $\dot{V}$ which satisfies conditions (H1), (H2) and all assumptions in Propositions 4.2 or 4.6 is called a *bounding function* for problem (1.1).

5. Existence and localization results

Combining the continuation principle with the bound sets technique, we are ready to state the main result of the paper concerning the solvability and localization of a solution of the multivalued Floquet problem (1.1).

For this purpose, let us consider again the single-valued Floquet b.v.p. (2.9) which is equivalent to the first-order Floquet b.v.p. (2.10), provided ξ, $h(\,\cdot\,)$,

$C(\,\cdot\,)$ and $\widetilde{D}$ are defined by relations (2.11)–(2.14). Moreover, let $U(t,s)$ be the evolution operator associated with (2.15).

Theorem 5.1. *Consider the Floquet b.v.p.* (1.1). *Assume that conditions* (1_{i})–(1_{iii}) *are satisfied and that an open, convex set* $K \subset E$ *containing* 0 *exists such that* $M\partial K = \partial K$. *Furthermore, let the following conditions* (2_{i})–(2_{iv}) *be satisfied:*

- (2_{i}) $\widetilde{D} - U(T,0)$ *is invertible.*
- (2_{ii}) $\gamma(F(t,\Omega_1\times\Omega_2)) \le g(t)(\gamma(\Omega_1)+\gamma(\Omega_2))$, *for almost all* $t \in [0,T]$ *and each bounded* $\Omega_1,\Omega_2 \subset E$, *where* $g \in L^1([0,T],[0,\infty))$ *and* γ *is the Hausdorff measure of noncompactness in* E.
- (2_{iii}) *For every non-empty, bounded set* $\Omega \subset E\times E$, *there exists* $\nu_\Omega \in L^1([0,T],[0,\infty))$ *such that*

$$||F(t,x,y)|| \le \nu_\Omega(t),$$

for almost all $t \in [0,T]$ *and all* $(x,y) \in \Omega$.
- (2_{iv}) *The inequality*

$$4\mathrm{e}^{\int_0^T (1+||A(t)||+||B(t)||)\,dt}(4k\mathrm{e}^{\int_0^T (1+||A(t)||+||B(t)||)\,dt}+1)||g||_{L^1([0,T],[0,\infty))} < 1$$

holds, where k *is defined in* (2.17).

Finally, let there exist a function $V \in C^1(E,\mathbb{R})$ *with a locally Lipschitz Fréchet derivative* $\dot V$ *satisfying* (H1) *and* (H2), *jointly with condition* (4.2), *for all* $x \in \partial K$, $z \in E$ *and condition* (4.1), *for a suitable* $\varepsilon > 0$, *all* $x \in \overline{K} \cap B(\partial K,\varepsilon)$, $t \in (0,T)$, $y \in E$, $\lambda \in (0,1)$ *and* $w \in \lambda F(t,x,y) - A(t)y - B(t)x$. *Then the Floquet b.v.p.* (1.1) *admits a solution whose values are located in* $\overline{K}$.

Proof. Let us define the closed set $S = S_1$ by

$$S := \{x \in AC^1([0,T],E) : x(T) = Mx(0),\ \dot x(T) = N\dot x(0)\}$$

and let the set Q of candidate solutions be defined as $Q := C^1([0,T],\overline{K})$. Because of the convexity of K, the set Q is closed and convex.

For all $q \in Q$ and $\lambda \in [0,1]$, consider still the associated fully linearized problem

$$P(q,\lambda) \qquad \begin{cases} \ddot x(t) + A(t)\dot x(t) + B(t)x(t) \in \lambda F(t,q(t),\dot q(t)) & \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0),\ \dot x(T) = N\dot x(0), \end{cases}$$

and denote by $\mathfrak{T}$ a solution mapping which assigns to each $(q,\lambda) \in Q \times [0,1]$ the set of solutions of $P(q,\lambda)$. We will show that the family of the above b.v.p.s $P(q,\lambda)$ satisfies all assumptions of Proposition 3.1.

In this case, $\varphi(t,x,\dot x) = F(t,x,\dot x) - A(t)\dot x - B(t)x$ which, together with the definition of $P(q,\lambda)$, ensures the validity of (3.2).

(i) In order to verify condition (a) in Proposition 3.1, we need to show that, for each $(q,\lambda) \in Q \times [0,1]$, the problem $P(q,\lambda)$ is solvable with a convex set of solutions. So, let $(q,\lambda) \in Q \times [0,1]$ be arbitrary and let $f_q(\,\cdot\,)$ be a strongly measurable selection of $F(\,\cdot\,, q(\,\cdot\,), \dot q(\,\cdot\,))$. Then, according to $(2_{\rm i})$, Lemma 2.7 and the equivalence, stated in Section 2, between the b.v.p. (2.7) and (2.9), the single-valued Floquet problem

$$\begin{cases} \ddot x(t) + A(t)\dot x(t) + B(t)x(t) = \lambda f_q(t), & \text{for a.a. } t \in [0,T], \\ x(T) = Mx(0),\ \dot x(T) = N\dot x(0) \end{cases}$$

admits a unique solution which is one of solutions of $P(q,\lambda)$. Thus, the set of solutions of $P(q,\lambda)$ is nonempty. The convexity of the solution sets follows immediately from the property $(1_{\rm ii})$ and the fact that problems $P(q,\lambda)$ are fully linearized.

(ii) Assuming that $H\colon [0,T] \times E \times E \times E \times E \times [0,1] \multimap E$ is defined by $H(t,x,y,q,r,\lambda) := \lambda F(t,q,r) - A(t)x - B(t)y$, condition (b) in Proposition 3.1 is ensured directly by assumption $(2_{\rm iii})$.

(iii) Since the verification of condition (c) in Proposition 3.1 is technically the most complicated, it will be splitted into two parts: $({\rm iii}_1)$ the quasi-compactness of the solution operator $\mathfrak{T}$, $({\rm iii}_2)$ the condensity of $\mathfrak{T}$ w.r.t. the monotone and non-singular (cf. Lemma 2.3) m.n.c. μ defined by (2.4).

Ad $({\rm iii}_1)$. Let us firstly prove that the solution mapping $\mathfrak{T}$ is quasi-compact. Since $C^1([0,T],E)$ is a metric space, it is sufficient to prove the sequential quasi-compactness of $\mathfrak{T}$. Hence, let us consider the sequences $\{q_n\}, \{\lambda_n\}, q_n \in Q$, $\lambda_n \in [0,1]$, for all $n \in \mathbb{N}$, such that $q_n \to q$ in $C^1([0,T],E)$ and $\lambda_n \to \lambda$. Moreover, let $x_n \in \mathfrak{T}(q_n,\lambda_n)$, for all $n \in \mathbb{N}$. Then there exists, for all $n \in \mathbb{N}$, $f_n(\,\cdot\,) \in F(\,\cdot\,, q_n(\,\cdot\,), \dot q_n(\,\cdot\,))$ such that

$$\ddot x_n(t) + A(t)\dot x_n(t) + B(t)x_n(t) = \lambda_n f_n(t), \quad \text{for a.a. } t \in [0,T], \tag{5.1}$$

and that $x_n(T) = Mx_n(0)$, $\dot x_n(T) = N\dot x_n(0)$.

Since $q_n \to q$ and $\dot q_n \to \dot q$, there exists a bounded $\Omega \subset E \times E$ such that $(q_n(t), \dot q_n(t)) \in \Omega$, for all $t \in [0,T]$ and $n \in \mathbb{N}$. Therefore, there exists, according to condition $(2_{\rm iii})$, $\nu_\Omega \in L^1([0,T],[0,\infty))$ such that $||f_n(t)|| \le \nu_\Omega(t)$, for every $n \in \mathbb{N}$ and almost all $t \in [0,T]$. According to the arguments below Remark 2.8,

$$||x_n(t)|| \le J \quad \text{and} \quad ||\dot x_n(t)|| \le J, \quad \text{for a.a. } t \in [0,T],$$

where

$$J := Z(Zk+1)\int_0^T \nu_\Omega(s)\,ds$$

and k, Z are defined by relations (2.17) and (2.21). Consequently, for almost all $t \in [0,T]$, we have

$$|\ddot{x}_n(t)| \le ||A(t)||\,||\dot{x}_n(t)|| + ||B(t)||\,||x_n(t)|| \\ + ||f_n(t)|| \le (||A(t)|| + ||B(t)||) \cdot J + \nu_\Omega(t).$$

Thus, the sequences $\{x_n\}$ and $\{\dot{x}_n\}$ are bounded and $\{\ddot{x}_n\}$ is uniformly integrable.

The sequences $\{U_{ij}(t,s)f_n(s)\}$, $i,j \in \{1,2\}$, with $t \in (0,T]$, are uniformly integrable on $[0,t]$, because, according to (2.16),

$$||U_{ij}(t,s)f_n(s)|| \le Z\nu_\Omega(s), \tag{5.2}$$

for almost all $s \in [0,t]$ and all $n \in \mathbb{N}$.

Since the sequences $\{q_n\}$, $\{\dot{q}_n\}$ are converging, we obtain, in view of (2_{ii}),

$$\gamma(\{f_n(t)\}) \le g(t)(\gamma(\{q_n(t)\}) + \gamma(\{\dot{q}_n(t)\})) = 0, \quad \text{for a.a. } t \in [0,T],$$

which implies that $\{f_n(t)\}$ is relatively compact. For given $t \in (0,T]$, the sequences $\{U_{ij}(t,s)f_n(s)\}$, $i,j \in \{1,2\}$, are relatively compact as well, for a.a. $s \in [0,t]$, because, according to (2.1),

$$\gamma(\{U_{ij}(t,s)f_n(s)\}) \le ||U_{ij}(t,s)||\gamma(\{f_n(s)\}) = 0, \tag{5.3}$$

for all $i,j \in \{1,2\}$.

By means of (2.3) and (2.18),

$$\begin{aligned}\gamma(\{x_n(t)\}) \le{}& \gamma\bigg(\bigcup_{\lambda\in[0,1]} \lambda\bigg\{\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau \\ &+ A_1(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau + A_2(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\bigg\}\bigg) \\ \le{}& \gamma\bigg(\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau \\ &+ A_1(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau + A_2(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\bigg).\end{aligned}$$

By virtue of (2.1), (2.2), (5.2), (5.3) and the sub-additivity of γ, we finally arrive at

$$\begin{aligned}\gamma(\{x_n(t)\}) \le{}& \gamma\bigg(\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau\bigg) + ||A_1(t)||\gamma\bigg(\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau\bigg) \\ &+ ||A_2(t)||\gamma\bigg(\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\bigg) = 0.\end{aligned}$$

By similar reasonings, when using (2.19) instead of (2.18), we also get

$$\gamma(\{\dot{x}_n(t)\}) = 0$$

by which $\{x_n(t)\}$, $\{\dot{x}_n(t)\}$ are relatively compact, for almost all $t \in [0,T]$. Moreover, since x_n satisfies for all $n \in \mathbb{N}$ equation (5.1), $\{\ddot{x}_n(t)\}$ is relatively compact, for almost all $t \in [0,T]$. Thus, according to Lemma 2.5, there exist a subsequence of $\{\dot{x}_n\}$, for the sake of simplicity denoted in the same way as the sequence, and $x \in C^1([0,T],E)$ such that $\{\dot{x}_n\}$ converges to $\dot{x}$ in $C([0,T],E)$ and $\{\ddot{x}_n\}$ converges weakly to $\ddot{x}$ in $L^1([0,T],E)$. By similar arguments as in the proof of Proposition 3.1, we can obtain that $\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t) \in \lambda F(t,q(t),\dot{q}(t))$, for almost all $t \in [0,T]$. Since S is closed and $x_n \in S$, for all n, we deduce that x satisfies the boundary conditions in (1.1). This already implies the quasi-compactness of $\mathfrak{T}$.

Ad (iii$_2$). In order to show that $\mathfrak{T}$ is μ-condensing, where μ is defined by (2.4), we will prove that any bounded subset $\Theta \subset Q$ such that $\mu(\mathfrak{T}(\Theta\times[0,1])) \geq \mu(\Theta)$ is relatively compact. Let $\{x_n\}_n \subset \mathfrak{T}(\Theta\times[0,1])$ be a sequence such that

$$\begin{aligned}\mu(&\mathfrak{T}(\Theta\times[0,1]))\\ &= \Big(\sup_{t\in[0,T]}[\gamma(\{x_n(t)\}_n)+\gamma(\{\dot{x}_n(t)\}_n)], \operatorname{mod}_C(\{x_n\}_n)+\operatorname{mod}_C(\{\dot{x}_n\}_n)\Big).\end{aligned}$$

According to (2.18) and (2.19), we can find $\{q_n\}_n \subset \Theta$, $\{f_n\}_n$ satisfying $f_n(t) \in F(t,q_n(t),\dot{q}_n(t))$, for almost al $t \in [0,T]$, and $\{\lambda_n\}_n \subset [0,1]$ such that, for all $t \in [0,T]$,

$$\begin{aligned}x_n(t) = \lambda_n\bigg(&A_1(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau\\ &+A_2(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\bigg)+\lambda_n\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}\dot{x}_n(t) = \lambda_n\bigg(&A_3(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau\\ &+A_4(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\bigg)+\lambda_n\int_0^t U_{22}(t,\tau)f_n(\tau)\,d\tau.\end{aligned}\tag{5.5}$$

In view of (2$_{\text{ii}}$), we have, for all $t \in [0,T]$,

$$\begin{aligned}\gamma(\{f_n(t),n\in\mathbb{N}\}) &\leq g(t)(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot{q}_n(t),n\in\mathbb{N}\}))\\ &\leq g(t)\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot{q}_n(t),n\in\mathbb{N}\})).\end{aligned}$$

Since $\{q_n\}_n \subset \Theta$ and Θ is bounded in $C^1([0,T],E)$, by means of (2$_{\text{iii}}$), we get the existence of $\nu_\Theta \in L^1([0,T],[0,\infty))$ such that $|f_n(t)| \leq \nu_\Theta(t)$, for almost all $t \in [0,T]$ and all $n \in \mathbb{N}$. According to (2.16), this implies $|U_{i,j}(t)f_n(t)| \leq Z\nu_\Theta(t)$, for each $i,j = 1,2$, almost all $t \in [0,T]$ and all $n \in \mathbb{N}$. Moreover, by virtue of (2.1), for each $(t,\tau)\in\Delta$, we have (here, the notation $\{\,\cdot\,, n\in\mathbb{N}\}$ means the same as $\{\,\cdot\,\}_n$ before)

$$\begin{aligned}\gamma(\{U_{i,j}(t,\tau)f_n(\tau),n\in\mathbb{N}\})&\\ &\leq ||U_{ij}(t,\tau)||\gamma(\{f_n(\tau),n\in\mathbb{N}\})\leq Z\gamma(\{f_n(\tau),n\in\mathbb{N}\})\\ &\leq Zg(t)\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\})).\end{aligned}$$

Applying the property (2.2), for each $t\in[0,T]$, we so obtain

$$\begin{aligned}\gamma\bigg(\bigg\{\int_0^t U_{1,2}(t,\tau)f_n(\tau)\,d\tau,n\in\mathbb{N}\bigg\}\bigg)&\\ &\leq 2Z||g||_{L^1}\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\})).\end{aligned}$$

By a similar reasoning, we arrive, for $i=1,2$, at

$$\begin{aligned}\gamma\bigg(\bigg\{\int_0^T U_{i,2}(T,\tau)f_n(\tau)\,d\tau,n\in\mathbb{N}\bigg\}\bigg)&\\ &\leq 2Z||g||_{L^1}\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\})).\end{aligned}$$

Therefore, according to (2.20), (5.4), properties (2.1), (2.3) and the subadditivity of γ, for all $t\in[0,T]$, we have that

$$\begin{aligned}\gamma(\{x_n(t),n\in\mathbb{N}\})&\leq 2Z\|A_1(t)\|\|g\|_{L^1}\mathcal{S}+2Z\|A_2(t)\|\|g\|_{L^1}\mathcal{S}+2Z\|g\|_{L^1}\mathcal{S}\\ &=2Z\|g\|_{L^1}(4Zk+1)\mathcal{S},\end{aligned}$$

where $\mathcal{S}:=\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\}))$.

The same estimate can be obtained, at each $t\in[0,T]$, for $\gamma(\{\dot x_n(t),n\in\mathbb{N}\})$, when starting from condition (5.5). Subsequently,

$$\gamma(\{x_n(t),n\in\mathbb{N}\})+\gamma(\{\dot x_n(t),n\in\mathbb{N}\})\leq 4Z(4Zk+1)||g||_{L^1}\mathcal{S},$$

yielding

$$\sup_{t\in[0,T]}(\gamma(\{x_n(t),n\in\mathbb{N}\})+\gamma(\{\dot x_n(t),n\in\mathbb{N}\}))\leq 4Z(4Zk+1)||g||_{L^1}\mathcal{S}. \tag{5.6}$$

Since $\mu(\mathfrak{T}(\Theta\times[0,1]))\geq\mu(\Theta)$ and $\{q_n\}_n\subset\Theta$, we so get

$$\begin{aligned}\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\}))&\\ &\leq\sup_{t\in[0,T]}(\gamma(\{x_n(t),n\in\mathbb{N}\})+\gamma(\{\dot x_n(t),n\in\mathbb{N}\}))\end{aligned}$$

and, in view of (5.6) and $(2_{\rm iv})$, we have that

$$\sup_{t\in[0,T]}(\gamma(\{q_n(t),n\in\mathbb{N}\})+\gamma(\{\dot q_n(t),n\in\mathbb{N}\}))=0.$$

Inequality (5.6) implies that

$$\sup_{t\in[0,T]}(\gamma(\{x_n(t),n\in\mathbb{N}\})+\gamma(\{\dot{x}_n(t),n\in\mathbb{N}\}))=0. \tag{5.7}$$

Now, we show that both the sequences $\{x_n\}$ and $\{\dot{x}_n\}$ are equi-continuous. Let $\widetilde{\Theta}\subset E$ be such that $q(t)\in\widetilde{\Theta}$ and $\dot{q}(t)\in\widetilde{\Theta}$ for $q\in\Theta$ and $t\in[0,T]$. Hence, reasoning as in the formulas after condition (5.1), we can show that, for all $n\in\mathbb{N}$,

$$|x_n(t)|\le Z(Zk+1)\int_0^T \nu_{\widetilde{\Theta}}(s)\,ds,\qquad |\dot{x}_n(t)|\le Z(Zk+1)\int_0^T \nu_{\widetilde{\Theta}}(s)\,ds,$$

where Z is defined by (2.21) and $\nu_{\widetilde{\Theta}}\in L^1([0,T],[0,\infty))$ comes from ($2_{\rm iii}$). By the arguments as in the formulas below (5.1), we get that $\{\ddot{x}_n\}_n$ is uniformly integrable. It implies that $\{\dot{x}_n\}$ is equi-continuous. Since $\{\dot{x}_n\}_n$ is bounded, $\{x_n\}$ is also equi-continuous. Therefore,

$$\mathrm{mod}_C(\{x_n\})=\mathrm{mod}_C(\{\dot{x}_n\})=0.$$

In view of (5.7), we have obtained that

$$\mu(\mathfrak{T}(\Theta\times[0,1]))=(0,0).$$

Hence, also $\mu(\Theta)=(0,0)$ and since μ is regular, we have that Θ is relatively compact. Therefore, condition (c) in Proposition 3.1 holds.

(iv) For all $q\in Q$, the problem $P(q,0)$ has the trivial solution. According to Lemma 2.7 and the arguments below it, this is the only solution of $P(q,0)$, for all $q\in Q$. Since $0\in K$, condition *(iv)* in Proposition 3.1 is satisfied.

(v) Let $q_*\in Q$ be a solution of the b.v.p. $P(q_*,\lambda)$, for some $\lambda\in(0,1)$, i.e. a fixed point of the solution mapping $\mathfrak{T}$. In view of conditions (4.1), (4.2) (see Proposition 4.2), K is, for all $\lambda\in(0,1)$, a bound set for the problem

$$\begin{cases} \ddot{q}_*(t)+A(t)\dot{q}_*(t)+B(t)q_*(t)\in\lambda F(t,q_*(t),\dot{q}_*(t)), & \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0). \end{cases}$$

This implies that $q_*\notin\partial Q$ which ensures condition (e) in Proposition 3.1. □

If the mapping $F(t,x,y)-A(t)y-B(t)x$ is globally u.s.c. in (t,x,y), then we are able to improve Theorem 5.1, when just replacing the arguments in Proposition 3.1 by those in Propositions 4.4 and 4.6 (cf. condition (e) in Proposition 3.1), in the following way.

COROLLARY 5.2. *Let us consider the Floquet b.v.p.* (1.1)*, where* $F\colon[0,T]\times E\times E\multimap E$ *is an upper semicontinuous mapping with nonempty, compact, convex values and* A *and* B *are continuous. Moreover, let condition* ($1_{\rm iii}$) *hold and let there exist a nonempty, open, convex set* $K\subset E$ *containing* 0 *such that* $M\partial K=\partial K$*, where* M *is invertible.*

Furthermore, let there exist a function $V \in C^1(E,\mathbb{R})$ *with a locally Lipschitz Fréchet derivative* $\dot V$ *satisfying* (H1) *and* (H2)*. Moreover, let, for all* $x \in \partial K$, $t \in (0,T)$, $\lambda \in (0,1)$ *and* $y \in E$ *satisfying* (4.4)*, condition* (4.5) *hold, for all* $w \in \lambda F(t,x,y) - A(t)y - B(t)x$.

At last, suppose that, for all $x \in \partial K$, $\lambda \in (0,1)$ *and* $y \in E$ *satisfying (4.13) at least one of conditions* (4.14), (4.15) *holds, for all* $w_1 \in \lambda F(0,x,y) - A(0)y - B(0)x$ *or for all* $w_2 \in \lambda F(T,Mx,Ny) - A(T)Ny - B(T)Mx$*, respectively.*

If conditions (2_{i})–(2_{iv}) *from Theorem 5.1 are satisfied, then the Floquet b.v.p.* (1.1) *admits a solution whose values are located in* $\overline{K}$.

Remark 5.3. Observe that the rather technical inequality in condition (2_{iv}) can be trivially satisfied in finite-dimensional spaces or for compact maps F.

6. Illustrative examples

It is known (see e.g. [19, Example 1.2.41(b), Remark 3.12.13]) that if E is a Banach space and $V(x) = ||x||^2/2 - R$, then $V\colon E \to \mathbb{R}$ is a proper convex function and $\partial V = \{x^* \in E^* \mid \langle x^*, x\rangle = ||x||_E^2 = ||x^*||_{E^*}^2\}$, for all $x \in E$, where ∂V is the subdifferential of V. If, in particular, E is a Hilbert space, then $\partial V(x) = x$.

Moreover, if V is Gâteaux differentiable at $x \in E$, then $\partial V(x) = \{V'(x)\}$ (see e.g. [19, Theorem 1.2.37]). The same is all the better true, provided V is Fréchet differentiable which is, for all $x \in E\setminus\{0\}$, equivalent with E to be *locally uniformly smooth*, i.e.

$$\lim_{\tau\to 0^+} \frac{1}{\tau} \sup\{||x+\tau y|| + ||x-\tau y|| - 2||x|| \mid ||y|| = 1\} = 0$$

(see e.g. [10], [11]).

If E is *uniformly smooth*, i.e. if there exists the limit

$$\lim_{\tau\to 0} \frac{1}{\tau}(||x+\tau y|| - ||x||),$$

uniformly for $x, y \in S_E$, where $S_E := \{x \in E \mid ||x|| = 1\}$ is the unit sphere which is, according to the well-known Smuljan theorem, equivalent with E^* to be *uniformly convex*, i.e.

$$\inf\left\{1 - \frac{1}{2}||x^* + y^*||_{E^*} \mid x^*, y^* \in S_{E^*},\ ||x^* - y^*||_{E^*} = \varepsilon\right\} > 0,$$

for every $\varepsilon > 0$ (see e.g. [10], [11]), then E is obviously locally uniformly smooth as well. Moreover, E is also reflexive (see again e.g. [10], [11]).

Thus, if E is uniformly smooth, then $V(x) = ||x||^2/2 - R$ must be Fréchet differentiable, for all $x \in E$, and $\dot V_x = V'(x) = \{x^* \in X^* \mid \langle x^*, x\rangle = ||x||_E^2 = ||x^*||_{E^*}^2\}$, for $x \in E$. Observe that, despite the non-differentiability of $x \to ||x||$

at $x = 0$, the function V is entirely Fréchet differentiable in E (i.e. also at $x = 0$), because the square acts in its regularization. In fact, we have that

$$\begin{aligned}\lim_{h\to 0}\frac{\|V(h)-V(0)-\langle 0_E,h\rangle\|}{\|h\|} &= \lim_{h\to 0}\frac{|\|h\|^2/2-R+R-0|}{\|h\|}\\ &= \lim_{h\to 0}\frac{\|h\|^2/2}{\|h\|} = \lim_{h\to 0}\frac{1}{2}\|h\| = 0,\end{aligned}$$

where 0_E denotes the identically zero operator in E.

One can easily check that $\dot{V}_x$ is convex, i.e.

$$\dot{V}_{\lambda x_1+(1-\lambda)x_2} \le \lambda\dot{V}_{x_1} + (1-\lambda)\dot{V}_{x_2},$$

for all $x_1, x_2 \in E$ and $\lambda \in [0,1]$. We note that $\dot{V}_x$ is also locally Lipschitz continuous (see e.g. [19, Corollary 1.2.8]).

EXAMPLE 6.1. Let E be a uniformly smooth Banach space and consider problem (1.1). Assume that conditions $(1_{\rm i})$–$(1_{\rm iii})$ and $(2_{\rm i})$–$(2_{\rm iv})$ are satisfied. Putting $K := \{x \in E \mid ||x|| < \sqrt{2R}\}$, let $M\partial K = \partial K$; for instance, let $M = N = \mathrm{id}$, for a periodic problem, or $M = N = -\mathrm{id}$, for an anti-periodic problem.

Taking $V(x) = ||x||^2/2 - R$, where $R > 0$ is a given constant in the definition of K, in view of the above considerations, we have that the locally Lipschitz continuous derivative $\dot{V}_x$ satisfies

$$\dot{V}_x = \{x^* \in E^* \mid \langle x^*, x\rangle = ||x||_E^2 = ||x^*||_{E^*}^2\}, \quad \text{for } x \in E.$$

One can readily check that conditions (H1), (H2) trivially hold. Furthermore, condition (4.1) takes the form

$$\limsup_{h\to 0^-}\frac{\langle (x+hy)^* - x^*, y\rangle}{h} + \langle (x+hy)^*, w\rangle > 0, \tag{6.1}$$

for a suitable $\varepsilon > 0$, all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$, $y \in E$, $\lambda \in (0,1)$ and $w \in \lambda F(t,x,y) - A(t)y - B(t)x$.

If $M = N = \mathrm{id}$, then

$$\langle \dot{V}_{Mx}, Nz\rangle \cdot \langle \dot{V}_x, z\rangle = \langle \dot{V}_x, z\rangle \cdot \langle \dot{V}_x, z\rangle = \langle x^*, z\rangle^2 \ge 0$$

and, when $\langle x^*, z\rangle = 0$, then $\langle \dot{V}_{Mx}, Nz\rangle = \langle \dot{V}_x, z\rangle = 0$, by which condition (4.2) is satisfied.

It is easy to show that $\dot{V}_{-x} = -\dot{V}_x$, for all $x \in E$. Thus, if $M = N = -\mathrm{id}$, then

$$\langle \dot{V}_{Mx}, Nz\rangle \cdot \langle \dot{V}_x, z\rangle = -\langle \dot{V}_{-x}, z\rangle \cdot \langle \dot{V}_x, z\rangle = \langle \dot{V}_x, z\rangle \cdot \langle \dot{V}_x, z\rangle = \langle x^*, z\rangle^2 \ge 0;$$

so, as in the periodic case, when $\langle x^*, z\rangle = 0$, then $\langle \dot{V}_{-x}, -z\rangle = \langle \dot{V}_x, z\rangle = 0$. Hence, condition (4.2) is satisfied in the anti-periodic case as well.

In particular, if E is a Hilbert space, then condition (6.1) takes form,

$$\langle x, w\rangle + ||y||^2 > 0,$$

for a suitable $\varepsilon > 0$, all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$, $y \in E$, $\lambda \in (0,1)$ and $w \in \lambda F(t,x,y) - A(t)y - B(t)x$. Applying Theorem 5.1, problem (1.1) admits a solution whose values are located in $\overline{K}$.

For Marchaud inclusions, the application of Corollary 5.2 can be illustrated as follows.

Example 6.2. Let E be a uniformly smooth Banach space and consider problem (1.1), where this time F is an upper semicontinuous mapping and A, B are continuous. Assume that conditions (1_{iii}) and (2_{i})–(2_{iv}) are satisfied. Putting $K := \{x \in E \mid ||x|| < \sqrt{2R}\}$, let again $M\partial K = \partial K$.

For $V(x) = ||x||^2/2 - R$, conditions (H1), (H2) trivially hold, and with no change

$$\dot{V}_x = \{x^* \in X^* \mid \langle x^*, x\rangle = ||x||_E^2 = ||x^*||_{E^*}^2\}, \quad \text{for } x \in E.$$

Conditions (4.4) and (4.5) take the form: for all $x \in \partial K$ and $y \in E$ satisfying

$$\langle x^*, y\rangle = 0,$$

the following inequality holds

$$\liminf_{h\to 0} \frac{1}{h}\langle (x+hy)^*, y\rangle + \langle (x+hy)^*, w\rangle > 0,$$

for all $t \in (0,T)$, $\lambda \in (0,1)$ and $w \in \lambda F(t,x,y) - A(t)y - B(t)x$.

Furthermore, since for $M = N = \mathrm{id}$: $\langle \dot{V}_{Mx}, Ny\rangle = \langle \dot{V}_x, y\rangle = \langle x^*, y\rangle$, condition (4.13) is equivalent to

$$\langle x^*, y\rangle = 0, \quad \text{for all } x \in \partial K \text{ and } y \in E. \tag{6.2}$$

Since for $M = N = -\mathrm{id}$: $\langle \dot{V}_{Mx}, Ny\rangle = -\langle \dot{V}_{-x}, y\rangle = \langle \dot{V}_x, y\rangle$, condition (4.13) is also in this case equivalent to (6.2).

In view of

$$\frac{1}{h}\langle \dot{V}_{x+hy}, y + hw_1\rangle = \frac{1}{h}\langle (x+hy)^*, y\rangle + \langle (x+hy)^*, w_1\rangle,$$

condition (4.14) reads as

$$\liminf_{h\to 0^+} \frac{1}{h}\langle (x+hy)^*, y\rangle + \langle (x+hy)^*, w_1\rangle > 0,$$

for all $x \in \partial K$, $\lambda \in (0,1)$, $y \in E$ and $w_1 \in \lambda F(0,x,y) - A(0)y - B(0)x$.

Finally, in the case when $M = N = \mathrm{id}$ or $M = N = -\mathrm{id}$ condition (4.15) takes the respective forms

$$\liminf_{h\to 0^-} \frac{1}{h}\langle (x+hy)^*, y\rangle + \langle (x+hy)^*, w_2\rangle > 0,$$
$$\liminf_{h\to 0^-} \frac{1}{h}\langle (x+hy)^*, y\rangle - \langle (x+hy)^*, w_2\rangle > 0,$$

for all $x \in \partial K$, $y \in E$, $\lambda \in (0,1)$ and $w_2 \in \lambda F(T,x,y) - A(T)y - B(T)x$ or $w_2 \in \lambda F(T,-x,-y) + A(T)y + B(T)x$.

In particular, if E is a Hilbert space and if $M = N = \mathrm{id}$, then conditions (4.4), (4.5), (4.13)–(4.15) reduce to: for all $x \in \partial K$, $y \in E$, $t \in (0,T)$ and $\lambda \in (0,1)$ satisfying

$$\langle x, y\rangle = 0, \tag{6.3}$$

the inequalities

$$\langle x, w\rangle + ||y||^2 > 0, \qquad \max\{\langle x, w_1\rangle + ||y||^2, \langle x, w_2\rangle + ||y||^2) > 0$$

hold, for all $w \in \lambda F(t,x,y) - A(t)y - B(t)x$, $w_1 \in \lambda F(0,x,y) - A(0)y - B(0)x$ and all $w_2 \in \lambda F(T,x,y) - A(T)y - B(T)x$.

On the other hand, for $M = N = -\mathrm{id}$, i.e. for anti-periodic problems in Hilbert spaces, conditions (4.4), (4.5), (4.13)–(4.15) take the form: for all $x \in \partial K$, $y \in E$, $t \in (0,T)$ and $\lambda \in (0,1)$ satisfying (6.3) the inequalities

$$\langle x, w\rangle + ||y||^2 > 0, \qquad \max\{\langle x, w_1\rangle + ||y||^2, -\langle x, w_2\rangle + ||y||^2) > 0$$

hold, for all $w \in \lambda F(t,x,y) - A(t)y - B(t)x$, $w_1 \in \lambda F(0,x,y) - A(0)y - B(0)x$ and all $w_2 \in \lambda F(T,-x,-y) + A(T)y + B(T)x$.

Applying Corollary 5.2, problem (1.1) admits a solution whose values are located in $\overline{K}$.

Remark 6.3. Hilbert spaces are the best uniformly convex Banach spaces. Since they are self-adjoint, they are in particular reflexive and, according to the Smuljan theorem, uniformly smooth. That is also why illustrative examples in Hilbert spaces are, not only because of technically easy calculations, the most natural ones.

On the other hand, in uniformly smooth spaces which are not Hilbert, it depends on their concrete structure in order to express conditions in terms of the asterisque linear functionals, in Examples 6.1 and 6.2, explicitly.

Coming back to the stimulating example from introduction, we can now demonstrate how the main results apply to it.

Example 6.4. Consider again the problem in the Hilbert space $E := L^2(\Omega)$:

$$
(6.4) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} + a\dfrac{\partial u}{\partial t} + \widetilde{B}u(t,\,\cdot\,) + \mathcal{B}||u(t,\,\cdot\,)||^{p^*-2}u = \varphi(t,u), \\ u(T,\,\cdot\,) = Mu(0,\,\cdot\,), \quad \dfrac{\partial u(T,\,\cdot\,)}{\partial t} = N\dfrac{\partial u(0,\,\cdot\,)}{\partial t}, \end{cases}
$$

where, for the sake of simplicity, we put $\widetilde{B} := b < 0$, where b is a constant,

$$
p^* := p(x) = \begin{cases} p_0 \in [3,\infty) & \text{for } ||x|| \le 1, \\ p_1 \in (1,2] & \text{for } ||x|| > 1, \end{cases}
$$

and the other symbols have the same meaning as above.

Let the constraint be also the same:

$$
u(t,\,\cdot\,) \in \overline{K} := \{e \in L^2(\Omega) \mid ||e|| \le r\}, \quad t \in [0,T].
$$

where $r > 0$ is a given constant.

Rewriting this problem into the form of (1.1), let us verify successively all the related conditions, in order to apply Theorem 5.1 and Corollary 5.2. One can readily check that $K \subset E$ is a nonempty, open, convex set containing 0 and that, for $M = N = \mathrm{id}$ or for $M = N = -\mathrm{id}$, the equality $M\partial K = \partial K$ trivially holds. Moreover, conditions (1_{i})–(1_{iii}), or their analogies in Corollary 5.2, are easily satisfied, provided $f\colon [0,T] \times E \to E$ is Carathéodory or continuous.

For $a \ge 0$, $b < 0$, the spectrum $\sigma(U(T,0))$ of the evolution operator U, associated with the homogeneous equation

$$
\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0, \quad t \in [0,T],
$$

can be calculated as

$$
\sigma(U(T,0)) = \sigma(\mathrm{e}^{CT}).
$$

Moreover, it can be shown that

$$
\sigma(\mathrm{e}^{CT}) = \{\mathrm{e}^{\lambda_1 T}, \mathrm{e}^{\lambda_2 T}\},
$$

where $0 < \mathrm{e}^{\lambda_1 T} = \mathrm{e}^{(T/2)(a-\sqrt{a^2-4b})} < 1$, $\mathrm{e}^{\lambda_2 T} = \mathrm{e}^{(T/2)(a+\sqrt{a^2-4b})} > 1$.

Thus, the spectrum $\sigma(U(T,0))$ does not intersect the unit cycle which is, at least for $M = N = \mathrm{id}$ and $M = N = -\mathrm{id}$, equivalent with the invertibility of the operator $\widetilde{D} - U(T,0) = (M,N) - U(T,0)$, provided $\widetilde{D} - U(T,0)$ is still surjective (cf. [8], [19]).

Since the homogeneous equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$ has constant coefficients, we can compute the linear operator e^{CT} and it is not difficult to show that it takes the following form

$$
\mathrm{e}^{CT} = \begin{pmatrix} c_1\mathrm{id}_E & c_2\mathrm{id}_E \\ c_3\mathrm{id}_E & c_4\mathrm{id}_E \end{pmatrix}, \quad c_1,c_2,c_3,c_4 \in \mathbb{R},
$$

implying that the 2×2 real matrix

$$\widehat{C} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

has the same eigenvalues as the linear operator e^{CT}. Moreover, $\pm\mathrm{id} - \mathrm{e}^{CT}$ is surjective if and only if $\pm\mathrm{id}_{\mathbb{R}^2\times\mathbb{R}^2} - \widehat{C}$ is so. Since, in our case, $\widetilde{D} - U(T,0) = \pm\mathrm{id}_{E\times E} - \mathrm{e}^{CT}$, one can check that, for $a \geq 0$, $b < 0$, we have

$$\begin{aligned} \det(\mathrm{id}_{\mathbb{R}^2\times\mathbb{R}^2} - \widehat{C}) &= 1 - (\mathrm{e}^{\lambda_1 T} + \mathrm{e}^{\lambda_2 T}) + \mathrm{e}^{\lambda_1 T}\mathrm{e}^{\lambda_2 T} < 0, \\ \det(-\mathrm{id}_{\mathbb{R}^2\times\mathbb{R}^2} - \widehat{C}) &= 1 + (\mathrm{e}^{\lambda_1 T} + \mathrm{e}^{\lambda_2 T}) + \mathrm{e}^{\lambda_1 T}\mathrm{e}^{\lambda_2 T} > 2, \end{aligned}$$

which guarantees the surjectivity of $\widetilde{D} - U(T,0)$. Indeed, the function $\det(\lambda\mathrm{id} - \mathrm{e}^{CT}) = \lambda^2 - (\mathrm{e}^{\lambda_1 T} + \mathrm{e}^{\lambda_2 T})\lambda + \mathrm{e}^{\lambda_1 T}\mathrm{e}^{\lambda_2 T}$ is obviously strictly convex in λ with two zero points $\mathrm{e}^{\lambda_1 T}$, $\mathrm{e}^{\lambda_2 T}$ and one minimum at $(\mathrm{e}^{\lambda_1 T} + \mathrm{e}^{\lambda_2 T})/2 \in (\mathrm{e}^{\lambda_1 T}, \mathrm{e}^{\lambda_2 T})$.

Since, for $||x|| > 0$, we obtain the estimate (cf. [19, p. 263])

$$\begin{aligned} ||(\mathcal{B}||x||^{p^*-2}x)'|| &= \mathcal{B}||(p^* - 2)||x||^{p^*-3}\frac{x}{||x||} + ||x||^{p^*-2}|| \\ &\leq \mathcal{B}(|p^* - 2| \cdot ||x||^{p^*-3} + ||x||^{p^*-2}) \leq \mathcal{B}\max(p_0 - 1, -p_1 + 3), \end{aligned}$$

the mapping $x \to \mathcal{B}||x||^{p^*-2}x$ is Lipschitz with the constant $L := \mathcal{B}\max(p_0 - 1, -p_1 + 3) \geq \mathcal{B}$. If

$$\mathcal{B} < \frac{1}{\max(p_0 - 1, -p_1 + 3)} \quad (\leq 1),$$

then it is a contraction with the coefficient $L < 1$, and so condensing. Thus, condition (2_{ii}) reduces into $\gamma(f(t,\Omega)) \leq g(t)\gamma(\Omega)$, for almost all $t \in [0,T]$ and each bounded $\Omega \subset E$, where $g \in L^1([0,T],[0,\infty))$. Obviously, if f is compact or contractive in x, then (2_{ii}) trivially holds.

Let us have e.g. a growth estimate for f:

$$||f(t,x)|| \leq c_0(t) + c_1(t)||x||^m, \quad \text{for all } x \in E,$$

where $m \geq 0$, $c_0, c_1 \in L^1([0,T],[0,\infty))$ are suitable functions. Then, in view of the inequalities ($||x||^{p^*-2} \leq 1$)

$$||f(t,x) - \mathcal{B}||x||^{p^*-2}x|| \leq ||f(t,x)|| + \mathcal{B}||x|| \leq c_0(t) + c_1(t)||x||^m + \mathcal{B}||x||,$$

it is enough to take

$$\nu_\Omega(t) := c_0(t) + \omega^m c_1(t) + \omega\mathcal{B}, \quad \text{where } \omega := \sup_{x\in\Omega}||x||,$$

in order (2_{iii}) to be satisfied.

Condition (2_{iv}) simplifies into the inequality

$$4\mathrm{e}^{T(1+a-b)}(4k\mathrm{e}^{T(1+a-b)} + 1)||g||_{L^1([0,T],[0,\infty))} < 1, \tag{6.5}$$

where k was defined in (2.17). This inequality is satisfied, $||g||_{L^1([0,T],[0,\infty))}$ is sufficiently small and g is related only to f.

Now, defining $V(x) := ||x||^2/2 - r^2/2$, conditions (H1) and (H2) are trivially satisfied. Moreover, conditions (4.1) and (4.5), (4.14) yield the inequality $(-||x||^{p^*-2} \geq -1)$

$$\begin{aligned} (6.6)\quad & \langle x, \lambda f(t,x) - \lambda\mathcal{B}||x||^{p^*-2}x - ay - bx\rangle + ||y||^2 \\ & \geq -b||x||^2 + ||y||^2 - a\langle x,y\rangle + \lambda(\langle x, f(t,x)\rangle - \mathcal{B}||x||^2\rangle) > 0, \end{aligned}$$

for all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $y \in E$, $t \in (0,T)$, $\lambda \in (0,1)$, and for all $x \in \partial K$ (i.e. $||x|| = r > 0$), $y \in E$, $t \in [0,T)$, $\lambda \in (0,1)$, respectively.

If $a^2 \leq -4b$, $b < 0$, then

$$\begin{aligned} -b||x||^2 - a\langle x,y\rangle + ||y||^2 &\geq -b||x||^2 - a\langle x,y\rangle - \frac{a^2}{4b}||y||^2 \\ &\geq -b||x||^2 - a||x||\,||y|| - \frac{a^2}{4b}||y||^2 \\ &= \bigg(\sqrt{|b|}\;||x|| - \frac{a}{2\sqrt{|b|}}||y||\bigg)^2 \geq 0, \end{aligned}$$

and if $a^2 < -4b$, $||x|| > 0$, then we get $-b||x||^2 - a\langle x,y\rangle + ||y||^2 > 0$. Thus, if

$$(6.7)\qquad \langle x, f(t,x)\rangle \geq \mathcal{B}||x||^2$$

holds, where $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$ or, where $x \in \partial K$, $t \in [0,T)$, then for $a^2 < -4b$, the inequality (6.6) holds, on the respective sets. In the latter case, in view of condition (4.4), it is enough to take only $a^2 \leq -4b$, $b < 0$.

Otherwise, condition (6.7) can be obviously replaced by

$$(6.8)\qquad (d - \mathcal{B})||x||^2 + \langle x, f(t,x)\rangle \geq 0,$$

provided $d > 0$ is a constant such that $a^2 < -4(b+d)$ or $a^2 \leq -4(b+d)$, respectively.

By the similar arguments, for $M = N = \mathrm{id}$, condition (4.15) can be (in view of (4.13)) satisfied, provided $a^2 \leq -4b, b < 0$ and $\langle x, f(T,x)\rangle \geq \mathcal{B}r^2$, where $x \in \partial K$, or if there exists a constant $d > 0$ such that $a^2 \leq -4(b+d)$ and $(d-\mathcal{B})r^2 + \langle x, f(T,x)\rangle \geq 0$, for $x \in \partial K$.

For $M = N = -\mathrm{id}$, condition (4.15) can be (in view of (4.13)) satisfied, provided $a^2 \leq -4b, b < 0$ and $-\langle x, f(T,-x)\rangle \geq \mathcal{B}r^2$, where $x \in \partial K$, or if there exists a constant $d > 0$ such that $a^2 \leq -4(b+d)$ and $(d-\mathcal{B})r^2 - \langle x, f(T,-x)\rangle \geq 0$, for $x \in \partial K$.

Summing up, for $M = N = \mathrm{id}$ or for $M = N = -\mathrm{id}$ together with $f(t,-x) \equiv -f(t,x)$, where $f \in C([0,T] \times E, E)$, conditions (4.5), (4.14), (4.15) are (in view of (4.4), (4.13)) satisfied, provided $a^2 \leq -4b, b < 0$ and condition (6.7) holds, for $x \in \partial K$, $t \in [0,T]$. If there exists $d > 0$ such that $a^2 \leq -4(b+d)$, then condition (6.7) can be replaced by (6.8), for $x \in \partial K$, $t \in [0,T]$. If f is Carathéodory, then it need not be odd (for $M = N = -\mathrm{id}$), but conditions (6.7) or (6.8) should hold, for $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$. Moreover, the class of Floquet boundary conditions with $M\partial K = \partial K$ can be larger than two particular cases above.

If, in particular, $a = 0$ and $b < 0$, then the only condition

$$\langle x, f(t,x)\rangle \geq (b + \mathcal{B})||x||^2$$

is sufficient (instead of (6.7) or (6.8)), on the respective sets.

REMARK 6.5. Observe that, if $r \leq 1$ in the bound set $K_1 := \{e \in L^2(\Omega) \mid ||e|| < r\}$, then also the original problem with $p \in [3, \infty)$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + a\dfrac{\partial u}{\partial t} + \widetilde{B}u(t,\,\cdot\,) + \mathcal{B}||u(t,\,\cdot\,)||^{p-2}u = \varphi(t,u), \\ x(T,\,\cdot\,) = Mx(0,\,\cdot\,), \quad \dfrac{\partial x(T,\,\cdot\,)}{\partial t} = N\dfrac{\partial x(0,\,\cdot\,)}{\partial t}, \end{cases} \tag{6.9}$$

admits, according to Theorem 5.1 and Corollary 5.2, the same solution $x(t) := u(t,\,\cdot\,) \in \overline{K}_1$, $t \in [0,T]$, as for (6.4), because $p^* = p_0 := p$, where $||x|| \leq 1$.

More precisely, problem (6.9), where $M = N = \mathrm{id}$ or $M = N = -\mathrm{id}$ together with $\varphi(t,-u) \equiv -\varphi(t,u)$, admits a (strong) solution $x(t) := u(t,\cdot)$ such that $x(t) \in \overline{K}_1$, $t \in [0,T]$, provided

(a) $a \geq 0$, $b < 0, 0 \leq \mathcal{B} < 1/(p-1)$, where $p \in [3,\infty)$,

(b) φ is Carathéodory (resp. continuous) and such that

$$|\varphi(t,\xi)| \leq \frac{c_0(t)}{\sqrt{|\Omega|+1}} + \frac{c_1(t)}{\sqrt{|\Omega|+1}}|\xi|^{2m}, \quad t \in [0,T],\ \xi \in \Omega,$$

where c_0, c_1 are suitable integrable coefficients

($\Rightarrow$ f is Carathéodory (resp. continuous) and such that $||f(t,x)|| \leq c_0(t) + c_1(t)||x||^m$, for all $x \in E$),

(c) $\varphi(t,\xi)$ is Lipschitz in ξ with a constant L (independent of t) such that

$$4\mathrm{e}^{T(1+a-b)}(4k\mathrm{e}^{T(1+a-b)} + 1)LT < 1 \quad (\text{cf. } (6.5)) \tag{6.10}$$

($\Rightarrow$ f satisfies the γ-regularity condition, namely $\gamma(f(t,\widetilde{\Omega})) \leq L\gamma(\widetilde{\Omega})$, for almost all $t \in [0,T]$ and each bounded $\widetilde{\Omega} \subset E$, with $g(t) := L$ satisfying (6.5),

(d) condition (6.8) holds on the set $(0,T) \times \overline{K}_1 \cap B(\partial K, \varepsilon)$ (resp. on $[0,T] \times \partial K_1$), where $d \geq 0$ is a suitable constant such that $a^2 < -4(b+d)$ (resp. $a^2 \leq -4b(b+d)$).

REMARK 6.6. It would be nice to express condition (d), as conditions (a)–(c), for function φ. Thus, for instance, the related equality $\sqrt{\int_\Omega x^2(\xi)\,d\xi} = r$ would, however, lead to the inequality

$$z\varphi(t,z) \geq (\mathcal{B} - d)z^2$$

required, for all $(t,z)\in[0,T]\times\mathbb{R}$. In this way, the information concerning the localization of solutions would be lost.

Remark 6.7. The most technical requirement (in nontrivial situations) is so the inequality (6.10) in condition (c). Nevertheless, the quotient

$$k := \|[\widetilde{D}-U(T,0)]^{-1}\| = \|[\pm\mathrm{id}-\mathrm{e}^{CT}]^{-1}\|_{E\times E}$$

in can be calculated as

$$k = k_0^{-1}\left\|\begin{matrix} \pm 1+\dfrac{\lambda_1\mathrm{e}^{\lambda_1 T}-\lambda_2\mathrm{e}^{\lambda_2 T}}{\lambda_2-\lambda_1} & \dfrac{\mathrm{e}^{\lambda_2 T}-\mathrm{e}^{\lambda_1 T}}{\lambda_2-\lambda_1} \\ \dfrac{\lambda_1\lambda_2(\mathrm{e}^{\lambda_1 T}-\mathrm{e}^{\lambda_2 T})}{\lambda_2-\lambda_1} & \pm 1+\dfrac{\lambda_1\mathrm{e}^{\lambda_2 T}-\lambda_2\mathrm{e}^{\lambda_1 T}}{\lambda_2-\lambda_1}\end{matrix}\right\|_{\mathbb{R}^2\times\mathbb{R}^2},$$

where

$$k_0^{-1} = [1\mp(\mathrm{e}^{\lambda_1 T}+\mathrm{e}^{\lambda_2 T})+\mathrm{e}^{\lambda_1 T+\lambda_2 T}]^{-1},$$
$$\lambda_1 = \frac{-a-\sqrt{a^2-4b}}{2},\quad \lambda_2 = \frac{-a+\sqrt{a^2-4b}}{2}.$$

For instance, for $a=0$, $b=-1$, we get $k\le(1+\mathrm{e}^T)/(2+\mathrm{e}^T+\mathrm{e}^{-T})<1$; condition (6.10) can be then satisfied, when e.g. $L\le 1/T(16\mathrm{e}^{4T}+4\mathrm{e}^{2T})$.

7. Concluding remarks

Assuming, for $M=N=\mathrm{id}$, that $A(t)\equiv A(t+T)$ and $B(t)\equiv B(t+T)$ or, for $M=N=-\mathrm{id}$, that $A(t)\equiv -A(t+T)$ and $B(t)\equiv -B(t+T)$, the solutions of the homogeneous problem

$$\begin{cases}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)=0, & \text{for a.a. } t\in[0,T],\\ x(T)=\pm x(0),\quad \dot{x}(T)=\pm\dot{x}(0),\end{cases}$$

where the sign plus in the b.v.p. refers to the case when A and B are T-periodic, while minus refers to the case when A and B are anti-periodic in $[0,T]$, can be obviously prolonged onto $(-\infty,\infty)$ in a T-periodic or a $2T$-periodic way, respectively.

Let the spectrum $\sigma(U(T,0))$ of $U(T,0)$ (or $\sigma(U(2T,0))$ of $U(2T,0)$) not intersect the unit circle, and so contain components lying in the interior or the exterior or in both of the unit circle.

In this context, $U(T,0)$ is called the *monodromy operator*. If $U(T,0)$ has a logarithm, that is if there is an operator S such that $U(T,0)=\mathrm{e}^S$, then its Floquet representation takes the form (cf. [8, Chapter V.1])

$$U(t,0)=R(t)\mathrm{e}^{-tT^{-1}\ln U(T,0)}\quad (\text{or } U(t,0)=R(t)\mathrm{e}^{-t(2T)^{-1}\ln U(2T,0)}),$$

where $R(t)\equiv R(t+T)$ (or $R(t)\equiv R(t+2T)$) is a suitable operator.

The condition imposed on the spectrum is equivalent (see e.g. [8, Theorem 2.1]) with the *regular exponential dichotomy* of the homogenous equation

$$\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = 0 \tag{7.1}$$

which implies that the above T-periodic or $2T$-periodic prolongations either would tend to 0 or diverge to ∞, in the norm. Consider the inhomogeneous equation

$$\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = f(t),$$

where $f \in L^1([0,T],E)$ is essentially bounded and such that $f(t) \equiv \pm f(t+T)$. It admits a unique entirely bounded solution

$$x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)\,ds$$

whose first derivative

$$\dot{x}(t) = \int_{-\infty}^{\infty} \frac{\partial G(t,s)}{\partial t} f(s)\,ds$$

is entirely bounded as well. The symbol G means the *principal Green function* of (7.1) (see e.g. [8, Theorem IV.3.2]).

Since the spectral condition is, by the definition, also equivalent (cf. e.g. [19]) with (2_{i}), the bounded solution $x(\cdot)$ and its derivative $\dot{x}(\cdot)$ must be, according to Lemma 2.7, T-periodic ($2T$-periodic). If E is reflexive, then the T-periodicity or $2T$-periodicity of $x(\,\cdot\,)$ and $\dot{x}(\,\cdot\,)$ alternatively follows already from their boundedness on the half-line (see e.g. [16, Theorem II.114C]).

Thus, if f is essentially bounded, then for the solvability of T-periodic or $2T$-periodic problems, by means of the principal Green functions, condition (2_{i}) can be replaced by the spectral requirement on $U(T,0)$ or $U(2T,0)$, as indicated above.

Theorem 5.1 and Corollary 5.2 deal only with the localization of solutions, but not with their first derivatives. This is, however, not a disadvantage, because otherwise additional requirements occur. In such a case, it is more convenient to consider the equivalent first-order problems (see [5]).

The parameter set Q of candidate solutions was taken everywhere as $Q := C^1([0,T],\overline{K})$, but it is without any loss of generality to take it as $Q := AC^1([0,T],\overline{K})$. On the other hand, if Q is only taken as $Q := C([0,T],\overline{K})$, then the solution derivatives can behave still in a more liberal way. Nevertheless, it would be practically very delicate to employ this theoretical possibility.

Unlike in finite-dimensional spaces (cf. [4]), the localization of solution values in a nonconvex bound set K is always a difficult task because of a cumbersome application of degree arguments (cf. [2], [3, Chapter II.11]). Bound sets of the type $K_0 := \{w \in W^{2,2}(\Omega) \mid ||w|| < r \text{ and } \mathrm{Tr}(w) = 0 \text{ on } \partial\Omega\}$, where Ω is a nonempty, bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, are convex but not open in $W^{2,2}(\Omega)$, and so not suitable for applications, too.

Moreover, in finite-dimensional spaces the diagonalization argument can be applied to guarantee sequentially entirely bounded solutions in given sets by means of results on compact intervals (see e.g. [3, Proposition III.1.37]). On the other hand, the compactness requirements in infinite-dimensional spaces (see e.g. [3, Proposition III.1.36]) allow us to employ e.g. appropriate results for Cauchy (initial value) problems, but not those obtained e.g. for periodic or anti-periodic problems. For first-order problems, this was solved in a sequential way (using the diagonalization arguments) in [5] and directly in [2]. Second-order problems, where e.g. some solutions should be entirely bounded and localized in a given set, but not necessarily their derivatives, will be treated by ourselves elsewhere.

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A Scorza–Dragoni approach to second-order boundary value problems in abstract spaces

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Abstract: The existence and localization of strong (Carathéodory) solutions is proved for a second-order Floquet problem in a Banach space. The result is obtained by combining a continuation principle together with a bounding (Liapunov-like) functions approach. The application of the Scorza–Dragoni type technique allows us to use strictly localized transversality conditions.

Keywords: Second-order Floquet problem, Scorza–Dragoni type results, bounding functions, solutions in a given set, evolution equations, condensing multivalued operators.

1. Introduction

The main aim of this paper is to present a theorem concerning the existence and localization of solutions to second-order Floquet boundary value problems for upper-Carathéodory differential inclusions in Banach spaces. For some related references, see e.g. [6,7] and those quoted in [3]. The novelty consists in the application of strictly localized Liapunov-type bounding functions guaranteeing the transversality behaviour of trajectories on bound sets, i.e. the fixed points free property required in the applied degree arguments.

The first-order problems were considered in [6,7]. The same second-order problem was already studied by ourselves via a bound sets approach in [3]. The conditions concerning bounding functions were not however imposed directly on the boundaries of bound sets like here, but at some vicinity of them. On the other hand, such a strict localization, allowed by means of the Scorza–Dragoni type technique developed in [15], demands a higher regularity of applied bounding functions which brings here some obstructions. Nevertheless, our result is new even in a single-valued case of equations.

Hence, let E be a separable Banach space (with the norm $\|\cdot\|$) satisfying the Radon–Nikodym property (e.g. reflexivity) and let us consider the Floquet boundary value problem (b.v.p.)

$$\left.\begin{array}{l}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in F(t,x(t),\dot{x}(t)), \quad \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0).\end{array}\right\} \tag{1}$$

Throughout the paper, we assume (for the related definitions, see the next Section 2) that

(1_i) $A, B\colon [0,T]\to\mathcal{L}(E)$ are Bochner integrable, where $\mathcal{L}(E)$ stands for the Banach space of all linear, bounded transformations $L\colon E\to E$ endowed with the sup-norm,
(1_{ii}) $F\colon [0,T]\times E\times E\multimap E$ is an upper-Carathéodory multivalued mapping,
(1_{iii}) $M,\ N\in\mathcal{L}(E)$ with M non-singular.

Let us note that in the entire paper, all derivatives will be always understood in the sense of Fréchet, and by the measurability, we mean the one with respect to the Lebesque σ-algebra in $[0,T]$ and the Borel σ-algebra in E.

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The notion of a solution will be understood in a strong (i.e. Carathéodory) sense. Namely, by a *solution* of problem (1), we mean a function $x\colon [0,T] \to E$ whose first derivative $\dot{x}(\cdot)$ is absolutely continuous and satisfies (1), for almost all $t \in [0,T]$.

The solution of the b.v.p. (1) will be obtained as the limit of a sequence of solutions of approximating problems that we construct by means of a Scorza–Dragoni type result developed in [15]. The approximating problems will be treated by means of the continuation principle developed in [3].

For the main result (Theorem 1) in Section 3, we collect all necessary technicalities and applied tools in the next Section 2. Concluding remarks in Section 4 concern an illustrative example of the application of Theorem 1. Since the applied bounding function V takes the form $V(x) := \frac{1}{2}\left(\|x\|^2 - r\right)$ and since one condition in Theorem 1 deals with $V \in C^2(E,\mathbb{R})$, we only restrict ourselves there to Hilbert spaces, where $\ddot{V}(x) \equiv Id$. In particular, we take $E := L^2(\Omega)$, where Ω is a suitable nonempty, bounded domain in $\mathbb{R}^n$.

2. Preliminaries

Let E be a Banach space having the Radon-Nikodym property (see e.g. [13, pp. 694–695]) and $[0,T] \subset \mathbb{R}$ be a closed interval. By the symbol $L^1([0,T],E)$, we shall mean the set of all Bochner integrable functions $x\colon [0,T] \to E$. For the definition and properties, see e.g. [13, pp. 693–701]. The symbol $AC^1([0,T],E)$ will denote the set of functions $x\colon [0,T] \to E$ whose first derivative $\dot{x}(\cdot)$ is absolutely continuous. Then $\ddot{x} \in L^1([0,T],E)$ and the fundamental theorem of calculus (the Newton–Leibniz formula) holds (see e.g. [1, pp. 243–244], [13, pp. 695–696]). In the sequel, we shall always consider $AC^1([0,T],E)$ as a subspace of the Banach space $C^1([0,T],E)$.

Given $C \subset E$ and $\varepsilon > 0$, the symbol $B(C,\varepsilon)$ will denote, as usually, the set $C + \varepsilon B$, where B is the open unit ball in E, i.e. $B = \{x \in E \mid ||x|| < 1\}$. In what follows, the symbol μ will denote the Lebesque measure on $\mathbb{R}$.

For each $L \in \mathcal{L}(E \times E)$, there exist unique $L_{ij} \in \mathcal{L}(E)$, $i,j = 1,2$, such that

$$L(x,y) = (L_{11}x + L_{12}y,\ L_{21}x + L_{22}y)\,,$$

where $(x,y) \in E \times E$. For the sake of simplicity, we shall use the notation

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

Let E' be the Banach space dual to E and let us denote by $\langle\cdot,\cdot\rangle$ the pairing (the duality relation) between E and E', i.e., for all $\Phi \in E'$ and $x \in E$, we put $\Phi(x) := \langle\Phi, x\rangle$.

We shall also need the following definitions and notions from multivalued analysis. Let X, Y be two metric spaces. We say that F is a *multivalued mapping* from X to Y (written $F\colon X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, the subset of $X \times Y$, defined by $\Gamma_F := \{(x,y) \in X \times Y \mid y \in F(x)\}$.

A multivalued mapping $F\colon X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open subset $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

Let $J \subset \mathbf{R}$ be a compact interval. A mapping $F\colon J \multimap Y$ with closed values, where Y is a separable metric space, is called *measurable* if, for each open subset $U \subset Y$, the set $\{t \in J \mid F(t) \subset U\}$ belongs to a σ-algebra of subsets of J.

If $F\colon J \multimap Y$ is compact-valued and $Y = E$ is a separable Banach space, then the notion of measurability coincides with those of strong measurability (cf. e.g. [11, Theorem 1.3.1]) as well as of weak measurability (cf. e.g. [1, Proposition I.3.45.4]). For the definitions and more details, see e.g. [1,10,11].

A multivalued mapping $F\colon X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x\in X} F(x)$ is contained in a compact subset of Y and it is called *quasi-compact* if it maps compact sets onto relatively compact sets.

The relationship between upper semicontinuous mappings and quasi-compact mappings with closed graphs is expressed by the following proposition (see, e.g., [11]).

Proposition 1. *Let X, Y be metric spaces and $F\colon X \multimap Y$ be a quasi-compact mapping with a closed graph. Then F is u.s.c.*

Let $J = [0,T]$ be a given compact interval. A multivalued mapping $F\colon J \times X \multimap Y$, where Y is a separable Banach space, is called an *upper-Carathéodory mapping* if the map $F(\cdot,x)\colon J \multimap Y$ is measurable, for all $x \in X$, the map $F(t,\cdot)\colon X \multimap Y$ is u.s.c., for almost all $t \in J$, and the set $F(t,x)$ is compact and convex, for all $(t,x) \in J \times X$.

The technique that will be used for proving the existence and localization result consists in constructing a sequence of approximating problems. This construction will be made on the basis of the Scorza–Dragoni type result in [15] (cf. [5]).

Definition 1. *An upper-Carathéodory mapping $F\colon [0,T] \times X \times X \multimap X$ is said to have the Scorza–Dragoni property if there exists a multivalued mapping $F_0\colon [0,T] \times X \times X \multimap X \cup \{\emptyset\}$ with compact, convex values having the following properties:*

(i) $F_0(t,x,y) \subset F(t,x,y)$, *for all* $(t,x,y) \in [0,T] \times X \times X$,
(ii) if $u, v\colon [0,T] \to X$ *are measurable functions with* $v(t) \in F(t,u(t),\dot{u}(t))$, *for a.a.* $t \in [0,T]$, *then also* $v(t) \in F_0(t,u(t),\dot{u}(t))$, *for a.a.* $t \in [0,T]$,
(iii) for every $\varepsilon > 0$, *there exists a closed* $I_\varepsilon \subset [0,T]$ *such that* $\mu([0,T] \setminus I_\varepsilon) < \varepsilon$, $F_0(t,x,y) \neq \emptyset$, *for all* $(t,x,y) \in I_\varepsilon \times X \times X$, *and* F_0 *is u.s.c. on* $I_\varepsilon \times X \times X$.

The following two propositions are crucial in our investigation. The first one is almost a direct consequence of the main result in [15] (cf. [5] and [7, Theorem 2.1]); precisely, the quoted results deal with a multivalued map $F\colon [0,T] \times X \multimap X$, but it is straightforward to see that they are still valid in this case, where F is defined on $[0,T] \times X \times X$. The second one allows us to construct a sequence of approximating problems of (1).

Proposition 2. *Let* X *be a separable Banach space and* $F\colon [0,T] \times X \times X \multimap X$ *be an upper-Carathéodory mapping. If* F *is globally measurable or quasi-compact, then* F *has the Scorza–Dragoni property.*

Proposition 3. (cf. [7, Theorem 2.2]) *Let* X *be a Banach space and* $K \subset X$ *a nonempty, open, convex, bounded set such that* $0 \in K$. *Moreover, let* $\varepsilon > 0$ *and* $V\colon X \to \mathbb{R}$ *be a Fréchet differentiable function with* $\dot{V}$ *Lipschitzian in* $\overline{B(\partial K, \varepsilon)}$ *satisfying*

$(H1)$ $V|_{\partial K} = 0$,
$(H2)$ $V(x) \leq 0$, *for all* $x \in \overline{K}$,
$(H3)$ $\|\dot{V}(x)\| \geq \delta$, *for all* $x \in \partial K$, *where* $\delta > 0$ *is given.*

Then there exists a bounded Lipschitzian function $\phi\colon \overline{B(\partial K, \varepsilon)} \to X$ *such that* $\langle \dot{V}_x, \phi(x) \rangle = 1$, *for every* $x \in \overline{B(\partial K, \varepsilon)}$

Example 1. Let us note that the function $x \to \phi(x)||\dot{V}_x||$, where ϕ and $\dot{V}_x$ occur in Proposition 3, is Lipschitzian and bounded in $\overline{B(\partial K, \varepsilon)}$. The symbol $\dot{V}_x$ denotes as usually the first Fréchet derivative of V at x.

For more details concerning multivalued analysis, see e.g. [1,10,11].

Definition 2. *Let* N *be a partially ordered set,* E *be a Banach space and let* $P(E)$ *denote the family of all subsets of* E. *A function* $\beta\colon P(E) \to N$ *is called a measure of non-compactness (m.n.c.) in* E *if* $\beta(\overline{co\ \Omega}) = \beta(\Omega)$, *for all* $\Omega \in P(E)$, *where* $\overline{co\ \Omega}$ *denotes the closed convex hull of* Ω.

A m.n.c. β *is called:*

(i) monotone if $\beta(\Omega_1) \leq \beta(\Omega_2)$, *for all* $\Omega_1 \subset \Omega_2 \subset E$,
(ii) nonsingular if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, *for all* $x \in E$ *and* $\Omega \subset E$,
(iii) invariant with respect to the union with compact sets if $\beta(K \cup \Omega) = \beta(\Omega)$, *for every relatively compact* $K \subset E$ *and every* $\Omega \subset E$,
(iv) regular when $\beta(\Omega) = 0$ *if and only if* Ω *is relatively compact.*

It is obvious that the m.n.c. which is invariant with respect to the union with compact sets is also nonsingular.

The typical example of an m.n.c. is the *Hausdorff measure of noncompactness* γ defined, for all $\Omega \subset E$ by

$$\gamma(\Omega) := \inf\{\varepsilon > 0 \mid \exists x_1, \ldots, x_n \in E\colon \Omega \subset \cup_{i=1}^n B(\{x_i\}, \varepsilon)\}.$$

The Hausdorff m.n.c. is monotone, invariant with respect to the union with compact sets and regular. Moreover, if $L \in \mathcal{L}(E)$ and $\Omega \subset E$, then (see, e.g., [11])

$$\gamma(L\Omega) \leq \|L\|_{\mathcal{L}(E)} \gamma(\Omega). \tag{2}$$

Let $\{f_n\} \subset L([0,T], E)$ be such that $||f_n(t)|| \leq \alpha(t)$, $\gamma(\{f_n(t)\}) \leq c(t)$, for a.a. $t \in [0,T]$, all $n \in \mathbb{N}$ and suitable $\alpha, c \in L([0,T], \mathbb{R})$, then (cf. [11])

$$\gamma\left(\left\{\int_0^T f_n(t)\, dt\right\}\right) \leq \int_0^T c(t)\, dt. \tag{3}$$

Moreover, for all subsets Ω of E (see e.g. [4]),

$$\gamma(\cup_{\lambda \in [0,1]} \lambda \Omega) = \gamma(\Omega).$$

Let us now introduce the function

$$\mu(\Omega) := \max_{\{w_n\}_n \subset \Omega} \Big(\sup_{t \in [0,T]} [\gamma(\{w_n(t)\}_n) + \gamma(\{\dot{w}_n(t)\}_n)], \mathrm{mod}_C(\{w_n\}_n) + \mathrm{mod}_C(\{\dot{w}_n\}_n)\Big), \tag{4}$$

defined on the bounded $\Omega \subset C^1([0,T],E)$, where the ordering is induced by the positive cone in $\mathbb{R}^2$ and where $\mathrm{mod}_C(\Omega)$ denotes the modulus of continuity of a subset $\Omega \subset C([0,T],E)$.[1] It was proved in [3] that the function μ given by (4) is an m.n.c. in $C^1([0,T],E)$ that is monotone, invariant with respect to the union with compact sets and regular.

Definition 3. *Let E be a Banach space and $X \subset E$. A multivalued mapping $F\colon X \multimap E$ with compact values is called condensing with respect to an m.n.c. β (shortly, β-condensing) if, for every $\Omega \subset X$ such that $\beta(F(\Omega)) \geq \beta(\Omega)$, it holds that Ω is relatively compact.*

A family of mappings $G\colon X \times [0,1] \multimap E$ with compact values is called β-condensing if, for every $\Omega \subset X$ such that $\beta(G(\Omega \times [0,1])) \geq \beta(\Omega)$, it holds that Ω is relatively compact.

It will be also convenient to recall some basic facts concerning evolution equations. For a suitable introduction and more details, we refer, e.g., to [8,12,16].

Hence, let $C\colon [0,T] \to \mathcal{L}(E)$ be Bochner integrable and let $f \in L([0,T],E)$. Given $x_0 \in E$, consider the linear initial value problem

$$\dot{x}(t) = C(t)x(t) + f(t), \quad x(0) = x_0. \tag{5}$$

It is well-known (see, e.g., [8]) that, for the uniquely solvable problem (5), there exists the evolution operator

$$\{U(t,s)\}_{(t,s)\in\Delta},$$

where $\Delta := \{(t,s)\colon 0 \leq s \leq t \leq T\}$, such that

$$U(t,s) \in \mathcal{L}(E) \quad \text{and} \quad \|U(t,s)\| \leq \mathrm{e}^{\int_s^t \|C(\tau)\|\,d\tau}, \quad \text{for all } (t,s) \in \Delta; \tag{6}$$

in addition, the unique solution $x(\cdot)$ of (5) is given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)\,ds, \qquad t \in [0,T].$$

Given $D \in \mathcal{L}(E)$, the linear Floquet b.v.p.

$$\left.\begin{aligned} &\dot{x}(t) = C(t)x(t) + f(t), \\ &x(T) = Dx(0), \end{aligned}\right\} \tag{7}$$

associated with the equation in (5), satisfies the following property.

Lemma 1. (cf. [4]) *If the linear operator $D - U(T,0)$ is invertible, then (7) admits a unique solution given, for all $t \in [0,T]$, by*

$$x(t) = U(t,0)\Big[D - U(T,0)\Big]^{-1} \int_0^T U(T,\tau)f(\tau)\,d\tau + \int_0^t U(t,\tau)f(\tau)\,d\tau. \tag{8}$$

Example 2. Denoting

$$\Lambda := \mathrm{e}^{\int_0^T \|C(s)\|\,ds}, \quad \Gamma := \|[D - U(T,0)]^{-1}\|,$$

we obtain, in view of (6), (8) and the growth estimate imposed on $C(t)$, the following inequality for the solution $x(\cdot)$ of (7):

$$\|x(t)\| \leq \Lambda\,(\Lambda\Gamma + 1) \int_0^T \|f(s)\|\,ds. \tag{9}$$

Now, consider the second-order linear Floquet b.v.p.

$$\left.\begin{aligned} &\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = f(t), \quad \text{for a.a. } t \in [0,T], \\ &x(T) = Mx(0),\ \dot{x}(T) = N\dot{x}(0), \end{aligned}\right\}, \tag{10}$$

where A, B are Bochner integrable and $f \in L^1([0,T],E)$, and let

$$\|(x,y)\|_{E\times E} := \sqrt{\|x\|^2 + \|y\|^2}, \quad \text{for all } x,y \in E.$$

[1] The m.n.c. $\mathrm{mod}_C(\Omega)$ is a monotone, nonsingular and algebraically subadditive on $C([0,T],E)$ (cf. e.g. [11]) and it is equal to zero if and only if all the elements $x \in \Omega$ are equi-continuos.

Problem (10) is equivalent to the following first-order linear one

$$\left.\begin{array}{l}\dot{\xi}(t) + C(t)\xi(t) = h(t), \quad \text{for a.a. } t \in [0,T],\\ \xi(T) = \tilde{D}\xi(0),\end{array}\right\} \tag{11}$$

where

$$\xi = (x, y) = (x, \dot{x}), \tag{12}$$

$$h(t) = (0, f(t)), \tag{13}$$

$$C(t)\colon E \times E \to E \times E, \quad (x, y) \longmapsto (-y, B(t)x + A(t)y) \tag{14}$$

and

$$\tilde{D}\colon E \times E \to E \times E, \quad (x, y) \longmapsto (Mx, Ny)\,. \tag{15}$$

Let us denote, for all $(t, s) \in [0,T] \times [0,T]$, by

$$U(t,s) := \begin{pmatrix} U_{11}(t,s) & U_{12}(t,s) \\ U_{21}(t,s) & U_{22}(t,s) \end{pmatrix}$$

the evolution operator associated with

$$\left.\begin{array}{l}\dot{\xi}(t) + C(t)\xi(t) = h(t), \quad \text{for a.a. } t \in [0,T],\\ \xi(0) = \xi_0,\end{array}\right\} \tag{16}$$

where ξ, h and C are defined by relations (12), (13) and (14), respectively, and $\xi_0 \in E \times E$. It is easy to see that $\|C(t)\| \leq 1 + \|A(t)\| + \|B(t)\|$ and, according to (6), we obtain

$$\|U(t,s)\| \leq \mathrm{e}^{\int_0^T (1+\|A(t)\|+\|B(t)\|)\,dt}, \quad \text{for all } (t,s) \in \Delta.$$

Consequently, for all $i, j = 1, 2$,

$$\|U_{ij}(t,s)\| \leq \mathrm{e}^{\int_0^T (1+\|A(t)\|+\|B(t)\|)\,dt}, \quad \text{for all } (t,s) \in \Delta. \tag{17}$$

Moreover, if we assume that $\tilde{D} - U(T,0)$ is invertible, denote

$$[\tilde{D} - U(T,0)]^{-1} := \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

and put

$$k := \|[\tilde{D} - U(T,0)]^{-1}\|, \tag{18}$$

then $\|K_{ij}\| \leq k$, for $i, j = 1, 2$, and the solution $x(\cdot)$ of (10) and its derivative $\dot{x}(\cdot)$ take, for all $t \in [0,T]$, the forms

$$x(t) = A_1(t)\int_0^T U_{12}(T,\tau)f(\tau)\,d\tau + A_2(t)\int_0^T U_{22}(T,\tau)f(\tau)\,d\tau + \int_0^t U_{12}(t,\tau)f(\tau)\,d\tau, \tag{19}$$

and

$$\dot{x}(t) = A_3(t)\int_0^T U_{12}(T,\tau)f(\tau)\,d\tau + A_4(t)\int_0^T U_{22}(T,\tau)f(\tau)\,d\tau + \int_0^t U_{22}(t,\tau)f(\tau)\,d\tau, \tag{20}$$

where

$$\begin{aligned}
A_1(t) &:= U_{11}(t,0)K_{11} + U_{12}(t,0)K_{21},\\
A_2(t) &:= U_{11}(t,0)K_{12} + U_{12}(t,0)K_{22},\\
A_3(t) &:= U_{21}(t,0)K_{11} + U_{22}(t,0)K_{21},\\
A_4(t) &:= U_{21}(t,0)K_{12} + U_{22}(t,0)K_{22},
\end{aligned}$$

for all $t \in [0,T]$. It holds that

$$\|A_i(t)\| \leq 2k\mathrm{e}^{\int_0^T (1+\|A(t)\|+\|B(t)\|)\,dt}, \quad \text{for } i = 1,2,3,4 \text{ and } t \in [0,T]. \tag{21}$$

If there exists $\alpha \in L^1([0,T],[0,\infty))$ such that $\|f(t)\| \le \alpha(t)$, for a.a. $t \in [0,T]$, then it immediately follows from Remark 2 that the following estimates hold for each solution $x(\cdot)$ of (10) and its derivative $\dot{x}(\cdot)$:

$$\|x(t)\| \le Z\,(4Zk+1)\int_0^T \alpha(s)\,ds$$

and

$$\|\dot{x}(t)\| \le Z\,(4Zk+1)\int_0^T \alpha(s)\,ds,$$

where

$$Z := \mathrm{e}^{\int_0^T(\|A(s)\|+\|B(s)\|+1)\,ds} \tag{22}$$

with k defined in (18).

The proof of the main result (cf. Theorem 1 below) will be based on the following slight modification of the continuation principle developed in [3]. Since the proof of this modified version differs from the one in [3] only slightly in technical details, we omit it here.

Proposition 4. *Let us consider the b.v.p.*

$$\left.\begin{array}{l}\ddot{x}(t) \in \varphi(t,x(t),\dot{x}(t)), \text{ for a.a. } t \in [0,T],\\ x \in S,\end{array}\right\} \tag{23}$$

where $\varphi\colon [0,T]\times E\times E \multimap E$ is an upper-Carathéodory mapping and $S \subset AC^1([0,T],E)$. Let $H\colon [0,T]\times E\times E\times E\times E\times[0,1] \multimap E$ be an upper-Carathéodory mapping such that

$$H(t,c,d,c,d,1) \subset \varphi(t,c,d), \text{ for all } (t,c,d) \in [0,T]\times E\times E.$$

Moreover, assume that the following conditions hold:

(i) There exist a closed set $S_1 \subset S$ and a closed, convex set $Q \subset C^1([0,T],E)$ with a non-empty interior $Int\,Q$ such that each associated problem

$$\left.\begin{array}{ll}\ddot{x}(t) \in H(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda), & \text{for a.a. } t \in [0,T],\\ x \in S_1,\end{array}\right\} \tag{$P(q,\lambda)$}$$

where $q \in Q$ and $\lambda \in [0,1]$, has a non-empty, convex set of solutions (denoted by $\mathfrak{T}(q,\lambda)$).

(ii) For every non-empty, bounded set $\Omega \subset E\times E\times E\times E$, there exists $\nu_\Omega \in L^1([0,T],[0,\infty))$ such that

$$\|H(t,x,y,u,v,\lambda)\| \le \nu_\Omega(t),$$

for a.a. $t \in [0,T]$ and all $(x,y,u,v) \in \Omega$ and $\lambda \in [0,1]$.

(iii) The solution mapping $\mathfrak{T}$ is quasi-compact and μ-condensing with respect to a monotone and nonsingular m.n.c. μ defined on $C^1([0,T],E)$.

(iv) For each $q \in Q$, the set of solutions of the problem $P(q,0)$ is a subset of $Int\,Q$, i.e. $\mathfrak{T}(q,0) \subset Int\,Q$, for all $q \in Q$.

(v) For each $\lambda \in (0,1)$, the solution mapping $\mathfrak{T}(\cdot,\lambda)$ has no fixed points on the boundary ∂Q of Q.

Then the b.v.p. (23) has a solution in Q.

3. Main result

Combining the foregoing continuation principle with the Scorza–Dragoni type technique (cf. Proposition 2), we are ready to state the main result of the paper concerning the solvability and localization of a solution of the multivalued Floquet problem (1).

For this purpose, let us consider again the single-valued Floquet b.v.p. (10) which is equivalent to the first-order Floquet b.v.p. (11), provided ξ, $h(\cdot)$, $C(\cdot)$ and $\tilde{D}$ are defined by relations (12)–(15). Moreover, let $U(t,s)$ be the evolution operator associated with (16).

Theorem 1. *Consider the Floquet b.v.p. (1), under conditions (1_i)–(1_{iii}), and suppose that F has the Scorza–Dragoni property. Assume that an open, convex, bounded set $K \subset E$ containing 0 exists such that $M\partial K = \partial K$. Furthermore, let the following conditions (2_i)–(2_{iv}) be satisfied:*

(2_i) *$\tilde{D} - U(T,0)$ is invertible.*
(2_{ii}) *$\gamma\left(F(t,\Omega_1\times\Omega_2)\right) \le g(t)\left(\gamma(\Omega_1)+\gamma(\Omega_2)\right)$, for a.a. $t\in[0,T]$ and each bounded $\Omega_1,\Omega_2\subset E$, where $g\in L^1([0,T],[0,\infty))$ and γ is the Hausdorff m.n.c. in E.*
(2_{iii}) *For every non-empty, bounded $\Omega\subset E$, there exists $\nu_\Omega\in L^1([0,T],[0,\infty))$ such that*

$$\|F(t,x,y)\| \le \nu_\Omega(t), \tag{24}$$

for a.a. $t\in[0,T]$ and all $(x,y)\in\Omega\times E$.
(2_{iv})*The inequality*

$$2e^{\int_0^T(1+\|A(t)\|+\|B(t)\|)\,dt}\left(4ke^{\int_0^T(1+\|A(t)\|+\|B(t)\|)\,dt}+1\right)\|g\|_{L^1([0,T],[0,\infty))}<1$$

holds, where k is defined in (18).

Furthermore, let there exist $\varepsilon>0$ and a function $V\in C^2(E,\mathbb{R})$, i.e. a twice continuously differentiable function in the sense of Fréchet, satisfying (H1)–(H3) with Fréchet derivative $\dot{V}$ Lipschitzian in $\overline{B(\partial K,\varepsilon)}$.[2] *Moreover, let there exist $h>0$ such that*

$$\left\langle \ddot{V}_x(v),v\right\rangle \ge 0, for all x\in B(\partial K,h),\ v\in E, \tag{25}$$

where $\ddot{V}_x(v)$ denotes the second Fréchet derivative of V at x in the direction $(v,v)\in E\times E$. Finally, let

$$\langle \dot{V}_x,w\rangle > 0, \tag{26}$$

and

$$\left\langle \dot{V}_{Mx},Nv\right\rangle\cdot\left\langle \dot{V}_x,v\right\rangle>0,\ \text{or}\ \left\langle \dot{V}_{Mx},Nv\right\rangle=\left\langle \dot{V}_x,v\right\rangle=0, \tag{27}$$

and for all $x\in\partial K$, $t\in(0,T)$, $v\in E$, $\lambda\in(0,1)$ and $w\in\lambda F(t,x,v)-A(t)v-B(t)x$.
Then the Floquet b.v.p. (1) admits a solution whose values are located in $\overline{K}$.

Proof. Since the proof of this result is rather technical, it will be divided into several steps. At first, let us define the sequence of approximating problems. For this purpose, let us consider a continuous function $\tau: E\to[0,1]$ such that $\tau(x)=0$, for all $x\in E\setminus B(\partial K,\varepsilon)$, and $\tau(x)=1$, for all $x\in\overline{B(\partial K,\frac{\varepsilon}{2})}$. According to Proposition 3 (see also Remark 1), the function $\hat{\phi}: E\to E$, where

$$\hat{\phi}(x)=\begin{cases}\tau(x)\cdot\phi(x)\cdot\|\dot{V}_x\|, & \text{for all } x\in\overline{B(\partial K,\varepsilon)},\\ 0, & \text{for all } x\in E\setminus\overline{B(\partial K,\varepsilon)},\end{cases}$$

is well-defined, continuous and bounded. Since $(t,y)\to A(t)y$ and $(t,x)\to B(t)x$ are Carathéodory maps, on $[0,T]\times E$, they are also almost-continuous (cf. [14]). Therefore, the mapping $(t,x,y)\multimap -A(t)y-B(t)x+F(t,x,y)$ has the Scorza–Dragoni property. So, we are able to find a decreasing sequence $\{J_m\}$ of subsets of $[0,T]$ and a mapping $F_0:[0,T]\times E\times E\multimap E\cup\{\emptyset\}$ such that, for all $m\in\mathbb{N}$,

– $\mu(J_m)<\frac{1}{m}$,
– $[0,T]\setminus J_m$ is closed,
– $(t,x,y)\multimap -A(t)y-B(t)x+F_0(t,x,y)$ is u.s.c. on $[0,T]\setminus J_m\times E\times E$,
– $\nu_{\overline{K}}(t)$ is continuous in $[0,T]\setminus J_m$.

If we put $J=\cap_{m=1}^\infty J_m$, then $\mu(J)=0$, $F_0(t,x,y)\neq\emptyset$, for all $t\in[0,T]\setminus J$ and the mapping $(t,x,y)\multimap -A(t)y-B(t)x+F_0(t,x,y)$ is u.s.c. on $[0,T]\setminus J\times E\times E$.

For each $m\in\mathbb{N}$, let us define the mapping $F_m:[0,T]\times E\times E\multimap E$ with compact, convex values by the formula

$$F_m(t,x,y):=\begin{cases}F_0(t,x,y)-p(t)\left(\chi_{J_m}(t)+\frac{1}{m}\right)\hat{\phi}(x), & \text{for all } (t,x,y)\in[0,T]\setminus J\times E\times E,\\ -p(t)\left(\chi_{J_m}(t)+\frac{1}{m}\right)\hat{\phi}(x), & \text{for all } (t,x,y)\in J\times E\times E,\end{cases}$$

where

$$p(t)=: -\nu_{\overline{K}}(t)-\|A(t)\|Z(4Zk+1)\|\nu_{\overline{K}}\|_{L^1([0,T],[0,\infty))}-\|B(t)\|\left(\|\partial K\|+\frac{\varepsilon}{2}\right). \tag{28}$$

[2] Since a C^2-function V has only a locally Lipschitzian Fréchet derivative $\dot{V}$ (cf. e.g. [13]), we had to assume explicitly the global Lipschitzianity of $\dot{V}$ in a noncompact set $\overline{B(\partial K,\varepsilon)}$.

with k and Z defined by (18) and (22), respectively.

Let us consider the b.v.p.

$$\left.\begin{array}{ll}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in F_m(t,x(t),\dot{x}(t)), & \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0).\end{array}\right\} \tag{P_m}$$

Now, let us verify the solvability of problems (P_m). Let $m\in\mathbb{N}$ be fixed. Since F_0 is globally u.s.c. on $[0,T]\setminus J\times E\times E$, $F_m(\cdot,x,y)$ is measurable, for each $(x,y)\in E\times E$, and, due to the continuity of $\hat{\phi}$, $F_m(t,\cdot,\cdot)$ is u.s.c., for all $t\in[0,T]\setminus J$. Therefore, F_m is an upper-Carathéodory mapping. Moreover, let us define the upper-Carathéodory mapping $H_m\colon[0,T]\times E\times E\times E\times E\times[0,1]\multimap E$ by the formula

$$\begin{aligned}&H_m(t,x,y,u,v,\lambda)\equiv H_m(t,u,v,\lambda)\\ &:=\begin{cases}\lambda F_0(t,u,v)-p(t)\left(\chi_{J_m}(t)+\frac{1}{m}\right)\hat{\phi}(u), & \text{for all } (t,x,y,u,v,\lambda)\in[0,T]\setminus J\times E^4\times[0,1],\\ -p(t)\left(\chi_{J_m}(t)+\frac{1}{m}\right)\hat{\phi}(u), & \text{for all } (t,x,y,u,v,\lambda)\in J\times E^4\times[0,1].\end{cases}\end{aligned}$$

Let us show that, when $m\in\mathbb{N}$ is sufficiently large, all assumptions of Proposition 4 (for $\varphi(t,x,\dot{x}):=F_m(t,x,\dot{x})-A(t)\dot{x}-B(t)x$) are satisfied.

For this purpose, let us define the closed set $S=S_1$ by

$$S:=\{x\in AC^1([0,T],E)\colon x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0)\}$$

and let the set Q of candidate solutions be defined as $Q:=C^1([0,T],\overline{K})$. Because of the convexity of K, the set Q is closed and convex.

For all $q\in Q$ and $\lambda\in[0,1]$, consider still the associated fully linearized problem

$$\left.\begin{array}{ll}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in H_m(t,q(t),\dot{q}(t),\lambda), & \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0),\end{array}\right\} \tag{$P_m(q,\lambda)$}$$

and denote by $\mathfrak{T}_m$ the solution mapping which assigns to each $(q,\lambda)\in Q\times[0,1]$ the set of solutions of $P_m(q,\lambda)$.

ad (i) In order to verify condition (i) in Proposition 4, we need to show that, for each $(q,\lambda)\in Q\times[0,1]$, the problem $P_m(q,\lambda)$ is solvable with a convex set of solutions. So, let $(q,\lambda)\in Q\times[0,1]$ be arbitrary and let $f_q(\cdot)$ be a measurable selection of $H_m(\cdot,q(\cdot),\dot{q}(\cdot),\lambda)$. Then, according to (2_i), Lemma 1 and the equivalence, stated in Section 2, between the b.v.p. (10) and (11), the single-valued Floquet problem

$$\left.\begin{array}{ll}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)=f_q(t), & \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0)\end{array}\right\}$$

admits a unique solution which is one of solutions of $P_m(q,\lambda)$. Thus, the set of solutions of $P_m(q,\lambda)$ is nonempty. The convexity of the solution sets follows immediately from the definition of H_m and the fact that problems $P_m(q,\lambda)$ are fully linearized.

ad (ii) Let $\Omega\subset E\times E\times E\times E$ be bounded. Then, there exists a bounded $\Omega_1\subset E$ such that $\Omega\subset\Omega_1\times\Omega_1\times\Omega_1\times\Omega_1$ and, according to (2_{iii}) and the definition of H_m, there exists $\hat{J}\subset[0,T]$ with $\mu\left(\hat{J}\right)=0$ such that, for all $t\in[0,T]\setminus\left(J\cup\hat{J}\right)$, $(x,y,u,v)\in\Omega$ and $\lambda\in[0,1]$,

$$\|H_m(t,u,v,\lambda)-A(t)y-B(t)x\|\leq\nu_{\Omega_1}(t)+2p(t)\cdot\max_{x\in B(\partial K,\varepsilon)}\|\hat{\phi}(x)\|+\|A(t)\|\cdot\|y\|+\|B(t)\|\cdot\|x\|.$$

Therefore, the mapping $H_m(t,q(t),\dot{q}(t),\lambda)-A(t)\dot{x}(t)-B(t)x(t)$ satisfies condition (ii) from Proposition 4.

ad (iii) Since the verification of condition (iii) in Proposition 4 is technically the most complicated, it will be split into two parts: (iii_1) the quasi-compactness of the solution operator $\mathfrak{T}_m$, (iii_2) the condensity of $\mathfrak{T}_m$ w.r.t. the monotone and non-singular m.n.c. μ defined by (4).

ad (iii_1) Let us firstly prove that the solution mapping $\mathfrak{T}_m$ is quasi-compact. Since $C^1([0,T],E)$ is a complete metric space, it is sufficient to prove the sequential quasi-compactness of $\mathfrak{T}_m$. Hence, let us consider the sequences $\{q_n\},\{\lambda_n\}, q_n\in Q,\lambda_n\in[0,1]$, for all $n\in\mathbb{N}$, such that $q_n\to q$ in $C^1([0,T],E)$ and $\lambda_n\to\lambda$. Moreover, let $x_n\in\mathfrak{T}_m(q_n,\lambda_n)$, for all $n\in\mathbb{N}$. Then there exists, for all $n\in\mathbb{N}$, $k_n(\cdot)\in F_0(\cdot,q_n(\cdot),\dot{q}_n(\cdot))$ such that

$$\ddot{x}_n(t)+A(t)\dot{x}_n(t)+B(t)x_n(t)=\lambda_nk_n(t)-p(t)\left(\chi_{J_m}(t)+\frac{1}{m}\right)\hat{\phi}(q_n(t)),\quad \text{for a.a. } t\in[0,T], \tag{29}$$

and that $x_n(T) = Mx_n(0),\ \dot{x}_n(T) = N\dot{x}_n(0)$.

According to condition (2_{iii}) and the definition of Q, $\|k_n(t)\| \le \nu_{\overline{K}}(t)$, for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$. According to formula (19),

$$x_n(t) = A_1(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau + A_2(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau + \int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau, \tag{30}$$

where

$$f_n(t) = \lambda_n k_n(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(q_n(t)). \tag{31}$$

Therefore, for all $t \in [0,T]$ and $n \in \mathbb{N}$,

$$\|x_n(t)\| \le Z(4Zk+1)\hat{C},$$

where k, Z are defined by relations (18), (22) and

$$\hat{C} := \left[\|\nu_{\overline{K}}\|_{L^1([0,T],[0,\infty))} + 2\cdot \max_{x\in B(\partial K,\varepsilon)}\|\hat{\phi}(x)\| \cdot \|p\|_{L^1([0,T],[0,\infty))}\right]. \tag{32}$$

This implies that the sequence $\{x_n\}$ is bounded.

Moreover, since

$$\dot{x}_n(t) = A_3(t)\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau + A_4(t)\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau + \int_0^t U_{22}(t,\tau)f_n(\tau)\,d\tau, \tag{33}$$

where $f_n(t)$ is defined by formula (31), we can obtain, by the similar arguments, that $\|\dot{x}_n(t)\| \le Z(4Zk+1)\hat{C}$ for all $t \in [0,T]$ and $n \in \mathbb{N}$.

Consequently, for a.a. $t \in [0,T]$, we have

$$\|\ddot{x}_n(t)\| \le \|A(t)\|\cdot\|\dot{x}_n(t)\| + \|B(t)\|\cdot\|x_n(t)\| + \|f_n(t)\|$$

$$\le (\|A(t)\| + \|B(t)\|)\cdot Z(4Zk+1)\hat{C} + \nu_{\overline{K}}(t) + 2\cdot \max_{x\in B(\partial K,\varepsilon)}\|\hat{\phi}(x)\|\cdot p(t).$$

Thus, $\{\ddot{x}_n\}$ is uniformly integrable.

For each $t \in [0,T]$, the properties of the Hausdorff m.n.c. yield

$$\gamma(\{f_n(t)\}_n) \le \gamma\left(\{\lambda_n k_n(t)\}_n\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\hat{\phi}(q_n(t))\}_n\right)$$

$$\le \gamma\left(\cup_{\lambda\in[0,1]}\{\lambda k_n(t)\}_n\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K,\varepsilon)}\}\right)$$

$$= \gamma\left(\{k_n(t)\}_n\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K,\varepsilon)}\}\right).$$

Therefore, according to condition (2_{ii}), for a.a. $t \in [0,T]$,

$$\gamma(\{f_n(t)\}_n) \le g(t)\left(\gamma\left(\{q_n(t)\}_n\right) + \gamma\left(\{\dot{q}_n(t)\}_n\right)\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K,\varepsilon)}\}\right)$$

$$\le g(t)\sup_{t\in[0,T]}\left(\gamma\left(\{q_n(t)\}_n\right) + \gamma\left(\{\dot{q}_n(t)\}_n\right)\right) + p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\{\phi(q_n(t))\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K,\varepsilon)}\}\right).$$

Since the function $x \to \phi(x)\|\dot{V}_x\|$ is Lipschitzian on $\overline{B(\partial K,\varepsilon)}$ with some Lipschitz constant $\hat{L} > 0$ (see Remark 1), we get that

$$\gamma(\{f_n(t)\}_n) \le \left(g(t) + \hat{L}p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\right)\sup_{t\in[0,T]}\left(\gamma\left(\{q_n(t)\}_n\right) + \gamma\left(\{\dot{q}_n(t)\}_n\right)\right).$$

Since $q_n \to q$ and $\dot{q}_n \to \dot{q}$ in $C([0,T],E)$, we get that, for a.a. $t \in [0,T]$, $\gamma(\{q_n(t)\}_n) = \gamma(\{\dot{q}_n(t)\}_n) = 0$, which implies that $\gamma(\{f_n(t)\}_n) = 0$, for a.a. $t \in [0,T]$.

For a given $t \in (0,T]$, the sequences $\{U_{ij}(t,s)f_n(s)\}$, $i,j \in \{1,2\}$, are relatively compact as well, for a.a. $s \in [0,t]$, because, according to (2),

$$\gamma(\{U_{ij}(t,s)f_n(s)\}_n) \leq \|U_{ij}(t,s)\|\gamma(\{f_n(s)\}_n) = 0, \tag{34}$$

for all $i,j \in \{1,2\}$.

Moreover, according to (17) and (22),

$$\|U_{ij}(t,s)f_n(s)\| \leq Z\left(\nu_{\overline{K}}(s) + 2\cdot \max_{x\in B(\partial K,\varepsilon)}\|\hat{\phi}(x)\|\cdot p(s)\right), \tag{35}$$

for a.a. $s \in [0,t]$ and all $n \in \mathbb{N}$.

By virtue of (2), (3), (34), (35) and the sub-additivity of γ, we finally arrive at

$$\gamma(\{x_n(t)\}_n) \leq \gamma\left(\left\{\int_0^t U_{12}(t,\tau)f_n(\tau)\,d\tau\right\}_n\right) + \|A_1(t)\|\cdot\gamma\left(\left\{\int_0^T U_{12}(T,\tau)f_n(\tau)\,d\tau\right\}_n\right)$$

$$+\|A_2(t)\|\cdot\gamma\left(\left\{\int_0^T U_{22}(T,\tau)f_n(\tau)\,d\tau\right\}_n\right) = 0.$$

By similar reasonings, when using (20) instead of (19), we also get

$$\gamma(\{\dot{x}_n(t)\}_n) = 0$$

by which $\{x_n(t)\}$, $\{\dot{x}_n(t)\}$ are relatively compact, for a.a. $t \in [0,T]$. Moreover, since x_n satisfies for all $n \in \mathbb{N}$ equation (29), $\{\ddot{x}_n(t)\}$ is relatively compact, for a.a. $t \in [0,T]$. Thus, according to [1, Lemma III.1.30], there exist a subsequence of $\{\dot{x}_n\}$, for the sake of simplicity denoted in the same way as the sequence, and $x \in C^1([0,T],E)$ such that $\{\dot{x}_n\}$ converges to $\dot{x}$ in $C([0,T],E)$ and $\{\ddot{x}_n\}$ converges weakly to $\ddot{x}$ in $L^1([0,T],E)$. According to the classical closure results (cf. e.g. [11, Lemma 5.1.1]), $x \in \mathfrak{T}_m(q,\lambda)$, which implies the quasi-compactness of $\mathfrak{T}_m$.

ad (iii_2) In order to show that, for $m \in \mathbb{N}$ sufficiently large, $\mathfrak{T}_m$ is μ-condensing with respect to the m.n.c. μ defined by (4), let us consider a bounded subset $\Theta \subset Q$ such that $\mu(\mathfrak{T}_m(\Theta\times[0,1])) \geq \mu(\Theta)$. Let $\{x_n\} \subset \mathfrak{T}_m(\Theta\times[0,1])$ be a sequence such that

$$\mu(\mathfrak{T}_m(\Theta\times[0,1])) = \left(\sup_{t\in[0,T]}[\gamma(\{x_n(t)\}_n)+\gamma(\{\dot{x}_n(t)\}_n)], \mathrm{mod}_C(\{x_n\}_n)+\mathrm{mod}_C(\{\dot{x}_n\}_n)\right).$$

According to (19) and (20), we can find $\{q_n\} \subset \Theta$, $\{\lambda_n\} \subset [0,1]$ and $\{k_n\}$ satisfying $k_n(t) \in F_0(t,q_n(t),\dot{q}_n(t))$, for a.a. $t \in [0,T]$, such that, for all $t \in [0,T]$, $x_n(t)$ and $\dot{x}_n(t)$ are defined by formulas (30) and (33), respectively, where $f_n(t)$ is defined by formula (31).

By the similar reasonings as in the part **ad** (iii_1), we can obtain that

$$\gamma(\{f_n(t)\}_n) \leq \left(g(t)+\hat{L}p(t)(\chi_{J_m}(t)+\frac{1}{m})\right)\sup_{t\in[0,T]}(\gamma(\{q_n(t)\}_n)+\gamma(\{\dot{q}_n(t)\}_n)),$$

for a.a. $t \in [0,T]$.

Let us put

$$\mathcal{S} := \sup_{t\in[0,T]}(\gamma(\{q_n(t)\}_n)+\gamma(\{\dot{q}_n(t)\}_n)),$$

fix $\tau \in [0,T]$ and let $i,j = 1,2$. Then, according to (17) and (22), we have that, for all $n \in \mathbb{N}$,

$$\|U_{ij}(\tau,t)f_n(t)\| \leq \|U_{ij}(\tau,t)\|\cdot\|f_n(t)\| \leq Z\left(\|k_n(t)\|+2\cdot\max_{x\in B(\partial K,\varepsilon)}\|\hat{\phi}(x)\|\cdot p(t)\right), \quad fora.a.t \in [0,\tau].$$

Since $k_n(t) \in F_0(t,q_n(t),\dot{q}_n(t))$, for a.a. $t \in [0,T]$, and $q_n \in \Theta$, for all $n \in \mathbb{N}$, where Θ is a bounded subset of $C^1([0,T],E)$, there exists $\Omega \subset \overline{K}$ such that $q_n(t) \in \Omega$, for all $n \in \mathbb{N}$ and $t \in [0,T]$. Hence, it follows from condition (2_{iii}) that

$$\|U_{ij}(\tau,t)f_n(t)\| \leq Z\left(\nu_\Omega(t)+2\cdot p(t)\cdot\max_{x\in B(\partial K,\varepsilon)}\|\hat{\phi}(x)\|\right), \quad fora.a.t \in [0,\tau].$$

As a consequence of (17), (22) and property (2), we also have that

$\gamma\left(\{U_{ij}(\tau,t)f_n(t)\}_n\right) \le Z\gamma\left(\{f_n(t)\}_n\right), \quad for a.a. t \in [0,\tau].$

Therefore, we can use (3) in order to show that

$$\gamma\left(\left\{\int_0^T U_{ij}(T,t)f_n(t)\,dt\right\}_n\right) \le Z\mathcal{S}\int_0^T \left(g(t)+\hat{L}p(t)\left(\chi_{J_m}(t)+\frac{1}{m}\right)\right)dt, \quad ij=1,2,$$

and also

$$\gamma\left(\left\{\int_0^t U_{i2}(t,\tau)f_n(\tau)\,d\tau\right\}_n\right) \le Z\mathcal{S}\int_0^t \left(g(\tau)+\hat{L}p(\tau)\left(\chi_{J_m}(\tau)+\frac{1}{m}\right)\right)d\tau, \quad i=1,2.$$

Consequently, according to (2), (21), (30) and the subadditivity of γ, we have that, for a.a. $t\in[0,T]$,

$$\gamma\left(\{x_n(t)\}_n\right) \le Z\mathcal{S}\left(\|A_1(t)\|+\|A_2(t)\|+1\right)\int_0^T \left(g(t)+\hat{L}p(t)\left(\chi_{J_m}(t)+\frac{1}{m}\right)\right)dt$$

$$\le Z\mathcal{S}\left(4kZ+1\right)\left(\|g\|_{L^1([0,T],[0,\infty))}+\hat{L}\left(\|p\|_{L^1(J_m)}+\frac{1}{m}\|p\|_{L^1([0,T],[0,\infty))}\right)\right).$$

The same estimate can be obtained for $\gamma\left(\{\dot{x}_n(t)\}_n\right)$, when starting from condition (33). Subsequently,

$$\gamma\left(\{x_n(t)\}_n\right)+\gamma\left(\{\dot{x}_n(t)\}_n\right) \le 2Z\mathcal{S}\left(4kZ+1\right)\left(\|g\|_{L^1([0,T],[0,\infty))}+\hat{L}\left(\|p\|_{L^1(J_m)}+\frac{1}{m}\|p\|_{L^1([0,T],[0,\infty))}\right)\right).$$

Since we assume that $\mu\left(\mathfrak{T}_m(\Theta\times[0,1])\right)\ge\mu(\Theta)$ and $\{q_n\}_n\subset\Theta$, we get

$$\mathcal{S}=\sup_{t\in[0,T]}\left(\gamma\left(\{q_n(t)\}_n\right)+\gamma\left(\{\dot{q}_n(t)\}_n\right)\right)\le\sup_{t\in[0,T]}\left(\gamma\left(\{x_n(t)\}_n\right)+\gamma\left(\{\dot{x}_n(t)\}_n\right)\right)$$

$$\le 2Z(4Zk+1)\left(\|g\|_{L^1([0,T],[0,\infty))}+\hat{L}\left(\|p\|_{L^1(J_m)}+\frac{1}{m}\|p\|_{L^1([0,T],[0,\infty))}\right)\right)\mathcal{S}.$$

Since we have, according to (2_{iv}), that $2Z(4kZ+1)\|g\|_{L^1([0,T],[0,\infty))}<1$, we can choose $m_0\in\mathbb{N}$ such that, for all $m\in\mathbb{N}$, $m\ge m_0$, it holds that

$$2Z(4kZ+1)\left(\|g\|_{L^1([0,T],[0,\infty))}+\hat{L}\left(\|p\|_{L^1(J_m)}+\frac{1}{m}\|p\|_{L^1([0,T],[0,\infty))}\right)\right)<1.$$

Therefore, we get, for sufficiently large $m\in\mathbb{N}$, the contradiction $\mathcal{S}<\mathcal{S}$ which ensures the validity of condition (iii) in Proposition 4.

ad (iv) For all $q\in Q$, the set $\mathfrak{T}_m(q,0)$ coincides with the unique solution x_m of the linear system

$$\left.\begin{array}{l}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)=-p(t)\left(\chi_{J_m}(t)+\frac{1}{m}\right)\hat{\phi}(q(t)),\ \text{for a.a. } t\in[0,T],\\ x(T)=Mx(0),\ \dot{x}(T)=N\dot{x}(0).\end{array}\right\}$$

According to (19) and (20), for all $t\in[0,T]$,

$$x_m(t)=A_1(t)\int_0^T U_{12}(T,\tau)\varphi_m(\tau)\,d\tau+A_2(t)\int_0^T U_{22}(T,\tau)\varphi_m(\tau)\,d\tau+\int_0^t U_{12}(t,\tau)\varphi_m(\tau)\,d\tau,$$

and

$$\dot{x}_m(t)=A_3(t)\int_0^T U_{12}(T,\tau)\varphi_m(\tau)\,d\tau+A_4(t)\int_0^T U_{22}(T,\tau)\varphi_m(\tau)\,d\tau+\int_0^t U_{22}(t,\tau)\varphi_m(\tau)\,d\tau,$$

where $\varphi_m(t):=-p(t)\left(\chi_{J_m}(t)+\frac{1}{m}\right)\hat{\phi}(q_m(t))$.

Since

$$\|\varphi_m\|_{L^1([0,T],[0,\infty))}\le\max_{x\in B(\partial K,\varepsilon)}\|\hat{\phi}(x)\|\cdot\left(\|p\|_{L^1(J_m,[0,\infty))}+\frac{\|p\|_{L^1([0,T],[0,\infty))}}{m}\right),$$

we have that, for all $t \in [0,T]$,

$$\|x_m(t)\| \leq Z \cdot (4Zk+1) \cdot \max_{x \in B(\partial K, \varepsilon)} \|\hat{\phi}(x)\| \cdot \left(\|p\|_{L^1(J_m,[0,\infty))} + \frac{\|p\|_{L^1([0,T],[0,\infty))}}{m} \right), \tag{36}$$

where k, Z are defined by relations (18), (22).

Let us now consider $r > 0$ such that $rB \subset K$. Then, it follows from (36) that we are able to find $m_0 \in \mathbb{N}$ such that, for all $m \in \mathbb{N},\ m \geq m_0$, and $t \in [0,T]$, $\|x_m\| \leq r$. Therefore, for all $m \in \mathbb{N},\ m \geq m_0$, $\mathfrak{T}_m(q,0) \subset Int\ Q$, for all $q \in Q$, which ensures the validity of condition (iv) in Proposition 4.

ad (v) Let $m \in \mathbb{N}$ be fixed and let us show that each (P_m) satisfies the transversality condition (v) in Proposition 4. We reason by a contradiction, and assume the existence of $\lambda \in (0,1)$ and $q \in \partial Q$ such that $q \in \mathfrak{T}_m(q,\lambda)$. According to the definition of the solution operator $\mathfrak{T}_m$, there is $f_0 \in L^1([0,T],E)$ with $f_0(t) \in F_0(t,q(t),\dot{q}(t))$, for a.a. $t \in [0,T] \setminus J$, satisfying

$$\ddot{q}(t) + A(t)\dot{q}(t) + B(t)q(t) = \lambda f_0(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(q(t)), \quad for a.a. t \in [0,T] \setminus J. \tag{37}$$

Since, moreover, $\mu(J) = 0$, condition (37) is indeed valid for a.a. $t \in [0,T]$.

Since $q \in \partial Q$, there exists $t_0 \in [0,T]$ satisfying $q(t_0) \in \partial K$. If we further assume that $t_0 = 0$, then $q(T) = Mq(0) \in M\partial K = \partial K$. With no loss of generality we can then take $t_0 \in (0,T]$. According to condition (H3), $\|\dot{V}_{q(t_0)}\| \geq \delta$. Furthermore, since $t \longmapsto \|\dot{V}_{q(t)}\|$ is continuous, there is $h_0 > 0$ such that $q(t) \in B\left(\partial K, \min\{h, \frac{\varepsilon}{2}\}\right)$ and $\|\dot{V}_{q(t)}\| \geq \frac{\delta}{2}$, for all $t \in [t_0 - h_0, t_0]$. Since J_m is open in $[0,T]$, if, in addition, $t_0 \in J_m$, we can take h_0 in such a way that $[t_0 - h_0, t_0] \subset J_m$.

Consider now the function $g: [0,T] \to \mathbb{R}$ defined by $g(t) = V(q(t))$.

According to the regularity conditions imposed on V and q, we have that $g \in C^1([0,T],\mathbb{R})$ and $\dot{g}(t) = \langle \dot{V}_{q(t)},\ \dot{q}(t)\rangle$, for all $t \in [0,T]$. Since, moreover, $V \in C^2(E,\mathbb{R})$ and $\dot{q}$ is absolutely continuous on $[0,T]$, we obtain that also $\dot{g}$ is absolutely continuous, implying that $\ddot{g}(t)$ exists, for a.a. $t \in [t_0 - h_0, t_0]$.

Since $g(t) \leq 0$, for all $t \in [0,T]$ with $g(t_0) = 0$, t_0 is a local maximum point. Hence, $\dot{g}(t_0) \geq 0$ and $\dot{g}(t_0) = 0$, whenever $t_0 \in (0,T)$. Consider now the special case when $t_0 = T$. Since $q(0) = M^{-1}q(T)$, according to the properties of M, we have that $q(0) \in \partial K$, and thus $\dot{g}(0) = \langle \dot{V}_{q(0)},\ \dot{q}(0)\rangle \leq 0$. Note, moreover, that $\dot{q}(T) = N\dot{q}(0)$. Consequently, we have that $\langle \dot{V}_{Mq(0)}, N\dot{q}(0)\rangle \cdot \langle \dot{V}_{q(0)}, \dot{q}(0)\rangle = \dot{g}(T) \cdot \dot{g}(0) \leq 0$ and according to (27) we obtain that

$$\dot{g}(0) = \left\langle \dot{V}_{q(0)}, \dot{q}(0) \right\rangle = \dot{g}(T) = \left\langle \dot{V}_{q(T)}, \dot{q}(T) \right\rangle = 0.$$

Let $t \in [t_0 - h_0, t_0]$ be such that both $\ddot{q}(t)$ and $\ddot{x}(t)$ exist. Then

$$\ddot{g}(t) = \lim_{h \to 0} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} = \lim_{h \to 0} \frac{\langle \dot{V}_{q(t+h)}, \dot{q}(t+h)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h}.$$

According to the regularity of q, there exist two functions $a(h)$ and $b(h)$ from $[-t, T-t]$ to E with $a(h) \to 0$ and $b(h) \to 0$ when $h \to 0$ such that

$$\dot{q}(t+h) = \dot{q}(t) + h\left[\ddot{q}(t) + a(h)\right], \quad q(t+h) = q(t) + h\left[\dot{q}(t) + b(h)\right].$$

Consequently,

$$\ddot{g}(t) = \lim_{h \to 0} \frac{\langle \dot{V}_{q(t+h)}, \dot{q}(t) + h\left[\ddot{q}(t) + a(h)\right]\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h}$$

$$= \lim_{h \to 0} \frac{\langle \dot{V}_{q(t+h)}, \dot{q}(t)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h} + \frac{\langle \dot{V}_{q(t+h)}, h\left[a(h)\right]\rangle}{h} + \langle \dot{V}_{q(t+h)}, \ddot{q}(t)\rangle.$$

Since $h \longmapsto \|\dot{V}_{q(t+h)}\|$ is continuous, it is bounded, for $t \in [-t, T-t]$, and therefore

$$\left| \frac{\langle \dot{V}_{q(t+h)}, h\left[a(h)\right]\rangle}{h} \right| \leq \|\dot{V}_{q(t+h)}\| \|a(h)\| \to 0, \quad h \to 0.$$

Thus, we obtain that

$$\ddot{g}(t) = \lim_{h \to 0} \frac{\langle \dot{V}_{q(t+h)}, \dot{q}(t)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h} + \langle \dot{V}_{q(t+h)}, \ddot{q}(t)\rangle$$

$$= \lim_{h\to 0} \frac{\langle \dot{V}_{q(t)+h[\dot{q}(t)+b(h)]}, \dot{q}(t)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h} + \langle \dot{V}_{q(t+h)}, \ddot{q}(t)\rangle.$$

According to the regularity condition imposed on V, there exists $O(h) \in E'$ with

$$\frac{\|O(h)\|}{h} \to 0 \quad for h \to 0$$

such that

$$\dot{V}_{q(t)+h[\dot{q}(t)+b(h)]} = \dot{V}_{q(t)} + \ddot{V}_{q(t)}\,(h\dot{q}(t) + hb(h)) + O(h)$$

implying

$$\frac{\langle \dot{V}_{q(t)+h[\dot{q}(t)+b(h)]}, \dot{q}(t)\rangle - \langle \dot{V}_{q(t)}, \dot{q}(t)\rangle}{h} = \frac{\langle \ddot{V}_{q(t)}(\dot{h}q(t)), \dot{q}(t)\rangle}{h} + \frac{\langle \ddot{V}_{q(t)}(hb(h)), \dot{q}(t)\rangle}{h} + \frac{\langle O(h), \dot{q}(t)\rangle}{h}$$

$$= \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle + \langle \ddot{V}_{q(t)}(b(h)), \dot{q}(t)\rangle + \frac{\langle O(h), \dot{q}(t)\rangle}{h}.$$

Therefore,

$$\ddot{g}(t) = \lim_{h\to 0} \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle + \langle \ddot{V}_{q(t)}(b(h)), \dot{q}(t)\rangle + \langle \dot{V}_{q(t+h)}, \ddot{q}(t)\rangle + \frac{\langle O(h), \dot{q}(t)\rangle}{h}$$

$$= \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle + \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle. \tag{38}$$

Let us now consider the case when $t_0 \in J_m$. According to the properties of g, it is possible to find $\hat{t}_0 \in (t_0 - h_0, t_0)$ such that $\dot{g}(\hat{t}_0) \geq 0$. Therefore, we obtain that

$$0 \geq -\dot{g}(\hat{t}_0) = \dot{g}(t_0) - \dot{g}(\hat{t}_0) = \int_{\hat{t}_0}^{t_0} \ddot{g}(t)\, dt.$$

According to (25) and (38), we have that

$$0 \geq -\dot{g}(\hat{t}_0) = \int_{\hat{t}_0}^{t_0} \ddot{g}(t)\, dt = \int_{\hat{t}_0}^{t_0} \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle + \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle\, dt \geq \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle\, dt$$

$$= \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \left(1 + \frac{1}{m}\right) p(t)\hat{\phi}(q(t))\rangle\, dt$$

$$= \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \left(1 + \frac{1}{m}\right) p(t)\tau(q(t))\|\dot{V}_{q(t)}\|\phi(q(t))\rangle\, dt.$$

Since $q(t) \in B(\partial K, \frac{\epsilon}{2})$, for all $t \in [\hat{t}_0, t_0]$, $\tau(q(t)) = 1$ and, according to Proposition 3, $\langle \dot{V}_{q(t)}, \phi(q(t))\rangle = 1$. Therefore, we obtain that

$$0 \geq -\dot{g}(\hat{t}_0) \geq \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \left(1 + \frac{1}{m}\right) p(t)\tau(q(t))\|\dot{V}_{q(t)}\|\phi(q(t))\rangle\, dt$$

$$= \int_{\hat{t}_0}^{t_0} \left(\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t)\rangle - \left(1 + \frac{1}{m}\right) p(t)\|\dot{V}_{q(t)}\| \right)\, dt$$

$$\geq \int_{\hat{t}_0}^{t_0} \|\dot{V}_{q(t)}\| \left(\kappa(t) - \left(1 + \frac{1}{m}\right) p(t) \right)\, dt,$$

where

$$\kappa(t) := -\nu_{\overline{K}}(t) - \|A(t)\|Z(4Zk+1)\|\nu_{\overline{K}}\|_{L^1([0,T],[0,\infty))} - \|B(t)\| \left(\|\partial K\| + \frac{\varepsilon}{2} \right).$$

According to the definition of p, we have that the last integral is strictly positive, so we get the contradictory conclusion $0 \geq -\dot{g}(\hat{t}_0) > 0$. It implies that $t_0 \notin J_m$.

Therefore, let us study the case when $t_0 \in [0,T] \setminus J_m$. If we are able to get a contradiction also when $t_0 \in [0,T] \setminus J_m$, then $q \in \mathfrak{T}_m(\lambda, q)$ with $q \in \partial Q$ is not possible, and so problem (P_m) satisfies the required tranversality condition.

Let $w_0 \in F(t_0, q(t_0), \dot{q}(t_0))$. According to Proposition 3, and since $t_0 \notin J_m$, we have that

$$\langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0) - p(t_0)(\chi_{J_m}(t_0) + \frac{1}{m})\hat{\phi}(q(t_0))\rangle$$

$$= \langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0) - \frac{p(t_0)}{m}\hat{\phi}(q(t_0))\rangle$$

$$= \langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0)\rangle - \frac{p(t_0)}{m}\|\dot{V}_{q(t_0)}\|.$$

Therefore, as a consequence of (26), the negativity of p and condition $(H3)$, we have that

$$\left\langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0) - \frac{p(t_0)}{m}\hat{\phi}(q(t_0))\right\rangle \geq -\frac{p(t_0)}{m}\|\dot{V}_{q(t_0)}\| \geq -\frac{\delta p(t_0)}{m} > 0,$$

for all $w_0 \in F(t_0, q(t_0), \dot{q}(t_0))$. The multivalued map F is compact-valued and the map $\dot{V}_{q(t_0)}: E \to \mathbb{R}$ is continuous. Thus, we can find $\sigma > 0$ such that

$$\left\langle \dot{V}_{q(t_0)}, \lambda w_0 - A(t_0)\dot{q}(t_0) - B(t_0)q(t_0) - \frac{p(t_0)}{m}\hat{\phi}(q(t_0))\right\rangle \geq 2\sigma,$$

for all $w_0 \in F(t_0, q(t_0), \dot{q}(t_0))$.

In $[0,T] \setminus J_m$, the multivalued map

$$t \multimap \lambda F_0(t, q(t), \dot{q}(t)) - A(t)\dot{q}(t) - B(t)q(t) - \frac{p(t)}{m}\hat{\phi}(q(t))$$

is u.s.c. and, therefore, $\Phi: [0,T] \setminus J_m \multimap \mathbb{R}$ defined by

$$t \multimap \left\{\langle \dot{V}_{q(t)}, \lambda w - A(t)\dot{q}(t) - B(t)q(t) - \frac{p(t)}{m}\hat{\phi}(q(t))\rangle, : \ w \in F_0(t, q(t), \dot{q}(t))\right\}$$

is u.s.c. Thus, we can find $\tilde{h}_0 \leq h_0$ such that $\Phi(t) \in [\sigma, +\infty)$, for all $t \in [t_0 - \tilde{h}_0, t_0] \setminus J_m$.

Since $g(t_0 - \tilde{h}_0) \leq 0$, also in $[t_0 - \tilde{h}_0, t_0]$, we can find $\tilde{t}_0$ with $\dot{g}(\tilde{t}_0) \geq 0$. Now, we reason as before and get

$$0 \geq -\dot{g}(\tilde{t}_0) = \dot{g}(t_0) - \dot{g}(\tilde{t}_0) = \int_{\tilde{t}_0}^{t_0} \ddot{g}(t)\, dt$$

$$= \int_{\tilde{t}_0}^{t_0} \langle \ddot{V}_{q(t)}(\dot{q}(t)), \dot{q}(t)\rangle\, dt + \int_{\tilde{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle\, dt \geq \int_{\hat{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \ddot{q}(t)\rangle\, dt$$

$$= \int_{\tilde{t}_0}^{t_0} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - p(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(q(t))\rangle\, dt$$

$$= \int_{[\tilde{t}_0, t_0] \setminus J_m} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \frac{p(t)}{m}\hat{\phi}(q(t))\rangle\, dt$$

$$+ \int_{[\tilde{t}_0, t_0] \cap J_m} \langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - p(t)(1 + \frac{1}{m})\hat{\phi}(q(t))\rangle\, dt.$$

Since the multivalued map $\Phi(t)$ is u.s.c. and since $t_0 \notin J_m$, we have that

$$\int_{[\tilde{t}_0, t_0] \setminus J_m} \left\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - \frac{p(t)}{m}\hat{\phi}(q(t))\right\rangle dt \geq \sigma \int_{[\tilde{t}_0, t_0] \setminus J_m} > 0.$$

Otherwise, from the definition of p and by a similar reasoning as before, we obtain that

$$\int_{[\tilde{t}_0, t_0] \cap J_m} \left\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - p(t)\left(1 + \frac{1}{m}\right)\hat{\phi}(q(t))\right\rangle dt$$

$$= \int_{[\tilde{t}_0,t_0]\cap J_m} \left\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t) - p(t)\left(1+\frac{1}{m}\right)\|\dot{V}_{q(t)}\|\phi(q(t))\right\rangle dt$$

$$= \int_{[\tilde{t}_0,t_0]\cap J_m} \left(\left\langle \dot{V}_{q(t)}, \lambda f_0(t) - A(t)\dot{q}(t) - B(t)q(t)\right\rangle - p(t)\left(1+\frac{1}{m}\right)\|\dot{V}_{q(t)}\|\right) dt$$

$$\geq \int_{[\tilde{t}_0,t_0]\cap J_m} \|\dot{V}_{q(t)}\| \left(-\nu_{\overline{K}}(t) - \|A(t)\|Z(4Zk+1)\|\nu_{\overline{K}}\|_{L^1([0,T],[0,\infty))}\right) dt$$

$$- \int_{[\tilde{t}_0,t_0]\cap J_m} \|\dot{V}_{q(t)}\| \left(\|B(t)\|\left(\|\partial K\| + \frac{\varepsilon}{2}\right) + \left(1+\frac{1}{m}\right)p(t)\,dt\right) > 0$$

In the case when $t_0 \in [0,T] \setminus J_m$, we obtain the contradictory conclusion $0 \geq -\dot{g}(\tilde{t}_0) > 0$ as well, and the tranversality condition (v) in Proposition 4 is so verified.

Summing up, we have proved that there exists $m_0 \in \mathbb{N}$ such that every problem (P_m), where $m \geq m_0$, satisfies all the assumptions of Proposition 4. This implies that every such (P_m) admits a solution, denoted by x_m, with $x_m(t) \in \overline{K}$, for all $t \in [0,T]$. Consequently, there exists a sequence $\{k_m\}_m$ in $L^1([0,T],E)$ satisfying

$$\ddot{x}_m(t) + A(t)\dot{x}_m(t) + B(t)x_m(t) = k_m(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x_m(t)) \tag{39}$$

and also $k_m(t) \in F(t, x_m(t), \dot{x}_m(t))$, for a.a. $t \in [0,T]$ and every $m \geq m_0$. Moreover, according to (2_{ii}), we obtain that $\|k_m(t)\| \leq \nu_{\overline{K}}(t)$, for a.a. $t \in [0,T]$ and every $m \geq m_0$. Therefore, reasoning as in **ad** (iii_1), we have that $\|\dot{x}_m(t)\| \leq Z(4Zk+1)\hat{C}$ with $\hat{C}$ defined by (32). We can then apply (2_{ii}) and get

$$\gamma\left(\{k_m(t)\}_m\right) \leq g(t)\left[\gamma\left(\{x_m(t)\}_m\right) + \gamma\left(\{\dot{x}_m(t)\}_m\right)\right], \quad for a.a. t \in [0,T]. \tag{40}$$

Let us put $\hat{\mathcal{S}} := \gamma\left(\{x_m(t)\}_m\right) + \gamma\left(\{\dot{x}_m(t)\}_m\right)$ and let $\{f_m\} \subset L^1([0,T],E)$ be defined by $f_m(t) := k_m(t) - p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x_m(t))$, for a.a. $t \in [0,T]$. When $t \notin J$, there is $\hat{m} = \hat{m}(t) \geq m_0$ such that $t \notin J_m$, for all $m \geq \hat{m}$. If we further apply the subadditivity of the Hausdorff m.n.c., we obtain

$$\gamma\left(\{f_m(t)\}_m\right) \leq \gamma\left(\{k_m(t)\}_m\right) + \gamma\left(\{-p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x_m(t))\}_m\right)$$

$$\leq \gamma\left(\{k_m(t)\}_m\right) + \gamma\left(\{-p(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}(x_m(t)),\ m = m_0, \ldots, \hat{m}(t)-1\}_m\right)$$

$$+\gamma\left(\{-\frac{p(t)}{m}\hat{\phi}(x_m(t)),\, m \geq \hat{m}(t)\}_m\right) = \gamma\left(\{k_m(t)\}_m\right) + \gamma\left(\{-\frac{p(t)}{m}\hat{\phi}(x_m(t)),\, m \geq \hat{m}(t)\}_m\right).$$

Since $\hat{\phi}$ is bounded, we obtain that

$$\frac{p(t)}{m}\hat{\phi}(x_m(t)) \to 0, \quad m \to \infty$$

implying that $\gamma\left(\{f_m(t)\}_m\right) \leq \gamma\left(\{k_m(t)\}_m\right)$, for a.a. $t \in [0,T]$. According to (40), we have that $\gamma\left(\{f_m(t)\}_m\right) \leq \hat{\mathcal{S}}g(t)$, for a.a. $t \in [0,T]$. Reasoning as in **ad**(iii_1), it is also possible to show that

$$\gamma\left(\{x_m(t)\}_m\right) \leq Z(4Zk+1)\hat{\mathcal{S}}\|g\|_{L^1([0,T],[0,\infty))},$$

and the same estimate is valid for $\gamma\left(\{\dot{x}_m(t)\}_m\right)$. Consequently, according (2_{iii}), we obtain that

$$\hat{\mathcal{S}} = \gamma\left(\{x_m(t)\}_m\right) + \gamma\left(\{\dot{x}_m(t)\}_m\right) \leq 2Z(4Zk+1)\hat{\mathcal{S}}\|g\|_{L^1([0,T],[0,\infty))} < \hat{\mathcal{S}},$$

implying that $\hat{\mathcal{S}} = 0$. Hence, $\gamma\left(\{x_m(t)\}_m\right) = \gamma\left(\{\dot{x}_m(t)\}_m\right) = 0$, for every $t \notin J$. Thus, also $\gamma\left(\{f_m(t)\}_m\right) = 0$. According to (39), we then obtain that $\gamma\left(\{\ddot{x}_m(t)\}_m\right) = 0$, for a.a. $t \in [0,T]$. Therefore, a classical convergence result (see e.g. [1, Lemma III.1.30])) assures the existence of a subsequence, denoted as the sequence, and of a function $x \in AC^1([0,T],E)$ such that $x_m \to x$ and $\dot{x}_m \to \dot{x}$ in $C([0,T],E)$ and also $\ddot{x}_m \rightharpoonup x$ in $L^1([0,T],E)$, when $m \to \infty$. Finally, a classical closure result (see e.g. [11, Lemma 5.1.1]) guarantees that x is a solution of (1) satisfying $x(t) \in \overline{K}$, for all $t \in [0,T]$, and the proof is so complete.

4. Concluding remarks

Observe that in a Hilbert space E, for $V(x) := \frac{1}{2}\left(\|x\|^2 - r\right)$, we have that (cf. [3], [13]) $\partial V(x) = \{\dot{V}(x)\} = x$, i.e. we obtain that $\ddot{V}(x) \equiv Id$. In particular $V \in C^2(E, \mathbb{R})$, as required in Theorem 1. On the other hand, if $\|\cdot\|^2$ (i.e. also $V(\cdot)$) is twice Fréchet differentiable at 0 in a Banach space $(E, \|\cdot\|)$, then E is isomorphic to a Hilbert space (see e.g. [9, p. 180]).

As pointed out in [3], problems of type (1) can be related to those for *abstract nonlinear wave equations* in Hilbert spaces $E := L^2(\Omega)$. Hence, for $t \in [0,T]$ and $\xi \in \Omega$, where Ω is a nonempty, bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, consider the functional evolution equation

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu(t,\cdot) + \mathcal{B}\|u(t,\cdot)\|^{p-2}u = \varphi(t,u), \tag{41}$$

where $u = u(t,\xi)$, subject to boundary conditions

$$u(T,\cdot) = Mu(0,\cdot), \quad \frac{\partial u(T,\cdot)}{\partial t} = N\frac{\partial u(0,\cdot)}{\partial t}. \tag{42}$$

Assume that $a \geq 0, b < 0, \mathcal{B} \geq 0, p \in [3,\infty)$ are constants and that $\varphi\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ is sufficiently regular. The problem under consideration can be still restricted by a constraint $u(t,\cdot) \in \overline{K_1}$, where

$$K_1 := \{e \in L^2(\Omega) \mid \|e\| < 1\},\ t \in [0,T].$$

Taking $x(t) := u(t,\cdot)$ with $x \in AC^1([0,T], L^2(\Omega))$, $A(t) \equiv A := a$, $B(t) \equiv B := b$, $f\colon [0,T]\times L^2(\Omega) \to L^2(\Omega)$ defined by $(t,v) \to \varphi(t,v(\cdot))$, and $F(t,x,y) \equiv F(t,x) := -\mathcal{B}\|x\|^{p-2}x + f(t,x)$, the above problem can be rewritten into the form (1), possibly together with $x(t) \in \overline{K_1}, t \in [0,T]$.

In view of the above arguments, all illustrative examples in [3], related to $V(x) := \frac{1}{2}\|x\|^2 - R$ acting in Hilbert spaces, can be improved by means of Theorem 1 in the sense that all relations holding for $(t,x) \in (0,T)\times\overline{K_1}\cap B(\partial K_1,\varepsilon)$ can be strictly localized to $(0,T)\times\partial K_1$. More concretely, problem (41), (42), where $M = N = Id$ or $M = N = -Id$ together with $\varphi(t,-u) \equiv -\varphi(t,u)$, admits in this way a (strong) solution $x(t) := u(t,\cdot)$ such that $x(t) \in \overline{K_1}$, $t \in [0,T]$, provided (for more details, see [3])

(i) $a \geq 0,\ b < 0, 0 \leq \mathcal{B} < \frac{1}{p-1}$, where $p \in [3,\infty)$,

(ii) φ is Carathéodory (resp. continuous) and such that

$$|\varphi(t,\xi)| \leq \frac{c_0(t)}{\sqrt{|\Omega|+1}} + \frac{c_1(t)}{\sqrt{|\Omega|+1}}|\xi|^{2m}, \qquad t \in [0,T],\ \xi \in \Omega,$$

where c_0, c_1 are suitable integrable coefficients

($\Rightarrow f$ is Carathéodory and such that $\|f(t,x)\| \leq c_0(t) + c_1(t)\|x\|^m$, for all $x \in L^2(\Omega)$),

(iii) $\varphi(t,\xi)$ is Lipschitzian in ξ with a constant L (independent of t) such that (k will be specified below)

$$4\mathrm{e}^{T(1+a-b)}\left(4k\mathrm{e}^{T(1+a-b)} + 1\right)LT < 1 \tag{43}$$

($\Rightarrow f$ satisfies the $\gamma-$regularity condition, namely $\gamma\left(f(t,\tilde{\Omega})\right) \leq L\gamma(\tilde{\Omega})$, for a.a. $t \in [0,T]$ and each bounded $\tilde{\Omega} \subset E$, with $g(t) := L$ satisfying the inequality

$$4\mathrm{e}^{T(1+a-b)}\left(4k\mathrm{e}^{T(1+a-b)} + 1\right)\|g\|_{L^1([0,T],[0,\infty))} < 1),$$

(iv) condition

$$(d-\mathcal{B})\|x\|^2 + \langle x, f(t,x)\rangle \geq 0,$$

holds on the set $[0,T]\times\partial K_1$, where $d \geq 0$ is a suitable constant such that $a^2 \leq -4b(b+d)$.

It would be nice to express condition (iv), as conditions (i)–(iii), for function φ. For instance, the related equality $\sqrt{\int_\Omega x^2(\xi)\, d\xi} = r$ would then, however, lead to the inequality

$$z\varphi(t,z) \geq (\mathcal{B}-d)z^2$$

required, for all $(t,z) \in [0,T]\times\mathbb{R}$. In this way, the information concerning the localization of solutions would be lost.

The most technical requirement (in nontrivial situations) is so the inequality (43) in condition (iii). Nevertheless, the quotient in (43)

$$k := \|[\tilde{D} - U(T,0)]^{-1}\| = \|[\pm Id - e^{CT}]^{-1}\|_{E\times E}$$

can be calculated as

$$k = k_0^{-1}\left\|\begin{matrix} \pm 1 + \frac{\lambda_1 e^{\lambda_1 T} - \lambda_2 e^{\lambda_2 T}}{\lambda_2 - \lambda_1}, & \frac{e^{\lambda_2 T} - e^{\lambda_1 T}}{\lambda_2 - \lambda_1} \\ \frac{\lambda_1\lambda_2(e^{\lambda_1 T} - e^{\lambda_2 T})}{\lambda_2 - \lambda_1}, & \pm 1 + \frac{\lambda_1 e^{\lambda_2 T} - \lambda_2 e^{\lambda_1 T}}{\lambda_2 - \lambda_1}\end{matrix}\right\|_{\mathbb{R}^2\times\mathbb{R}^2},$$

where

$$k_0^{-1} = \left[1 \mp (e^{\lambda_1 T} + e^{\lambda_2 T}) + e^{\lambda_1 T + \lambda_2 T}\right]^{-1}, \quad \lambda_1 = \frac{-a - \sqrt{a^2 - 4b}}{2}, \quad \lambda_2 = \frac{-a + \sqrt{a^2 - 4b}}{2}.$$

For instance, for $a = 0, b = -1$, we get $k \leq \frac{1+e^T}{2+e^T+e^{-T}} < 1$; condition (43) can be then satisfied, when e.g. $L \leq \frac{1}{T(16e^{4T}+4e^{2T})}$.

After all, since the usage of bounding function $V(x) := \frac{1}{2}\|x\|^2 - R$ is the most standard one, the illustrative example demonstrates that, in view of the above arguments, the practical application of Theorem 1 reduces to separable Hilbert spaces.

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Dirichlet problem in Banach spaces: the bound sets approach

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Abstract

The existence and localization result is obtained for a multivalued Dirichlet problem in a Banach space. The upper-Carathéodory and Marchaud right-hand sides are treated separately because in the latter case, the transversality conditions derived by means of bounding functions can be strictly localized on the boundaries of bound sets.

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Keywords: Dirichlet problem; bounding functions; solutions in a given set; condensing multivalued operators

1 Introduction

The Dirichlet problem and its special case with homogeneous boundary conditions, usually called the Picard problem, belong to the most frequently studied boundary value problems. A lot of results concerning the standard problem for scalar second-order ordinary differential equations were generalized in various directions.

In Euclidean spaces, besides many extensions to vector equations, vector inclusions were under consideration, *e.g.*, in [1–4]. In abstract spaces, usually in Banach and Hilbert spaces, equations, *e.g.*, in [5–11] and inclusions, *e.g.*, in [9, 12, 13] were treated.

Sadovskii's or Darbo's fixed point theorems, jointly with the usage of a measure of noncompactness, were applied in [5, 8, 9, 11]. Kakutani's or Ky Fan's fixed point theorems were applied with the upper and lower solutions technique in [9] and with a measure of noncompactness in [13]. On the other hand, continuation principles were employed in [2, 4, 7].

The main aim of our present paper is an extension of the finite-dimensional results in [2, 4] into infinite-dimensional ones. We were also stimulated by the work of Jean Mawhin in [7], where degree arguments were applied to the Dirichlet problem in a Hilbert space probably for the first time, and in [14], where a bound sets approach was systematically developed. Hence, besides these two approaches, our extension consists in the consideration of differential inclusions in rather general Banach spaces and the usage of a measure of noncompactness. Similar results were already obtained in an analogous way by ourselves for Floquet problems in [15–18].

Besides the existence, the localization of solutions will be obtained in our main theorems (see Theorem 5.1 and Theorem 5.2). Unlike in [10], where the solutions belong to a positively invariant set, in our paper, some trajectories can escape from the prescribed set of candidate solutions. Moreover, the associated bound set need not be compact as in

[10]. Similarly, the main difference between our results and those in [9, 13] consists in the application of a continuation principle jointly with a bound sets approach, which allows us to check fixed point free boundaries of given bound sets. This, in particular, means that, unlike in [9, 13], some trajectories can again escape from the prescribed set of candidate solutions in a transversal way.

Let E be a Banach space (with the norm $\|\cdot\|$) satisfying the Radon-Nikodym property (*e.g.*, reflexivity) and let us consider the Dirichlet boundary value problem (b.v.p.)

$$\left.\begin{array}{ll} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)) & \text{for a.a. } t \in [0, T], \\ x(0) = x(T) = 0, & \end{array}\right\} \tag{1}$$

where $F\colon [0, T] \times E \times E \multimap E$ is an upper-Carathéodory mapping or a globally upper semicontinuous mapping with compact, convex values (for the related definitions, see Section 2).

The main purpose of the present paper is to prove the existence of a Carathéodory solution $x \in AC^1([0, T], E)$ to problem (1) in a given set Q. This will be achieved by means of a suitable continuation principle. The crucial condition of the continuation principle described in Section 3 consists in guaranteeing the fixed point free boundary of Q w.r.t. an admissible homotopical bridge starting from (1) (see condition (v) in Proposition 3.1 below). This requirement will be verified by means of Lyapunov-like bounding functions, *i.e.*, via a bound sets technique. That is also why the whole Section 4 is devoted to this technique applied to Dirichlet problem (1). We will distinguish two cases, namely when F is an upper-Carathéodory mapping and when F is globally upper semicontinuous (*i.e.*, a Marchaud mapping). Unlike in the first case, the second one allows us to apply bounding functions which can be strictly localized at the boundaries of given bound sets.

2 Preliminaries

Let E be a Banach space having the *Radon-Nikodym property* (see, *e.g.*, [19, pp.694-695]), *i.e.*, if for every finite measure space $(\mathcal{M}, \Sigma, \mu)$ and every vector measure $m\colon \Sigma \to E$ of bounded variation, which is absolutely continuous w.r.t. μ, we can find a Bochner integrable function $f\colon \mathcal{M} \to E$ such that

$$m(C) = \int_C f(v)\, d\mu$$

for each $C \in \Sigma$. Let $[0, T] \subset \mathbb{R}$ be a closed interval. By the symbol $L^1([0, T], E)$, we will mean the set of all Bochner integrable functions $x\colon [0, T] \to E$. For the definition and properties, see, *e.g.*, [19, pp.693-701].

The symbol $AC^1([0, T], E)$ will denote the set of functions $x\colon [0, T] \to E$ whose first derivative $\dot{x}(\cdot)$ is absolutely continuous. Then $\ddot{x} \in L^1([0, T], E)$ and the fundamental theorem of calculus (the Newton-Leibniz formula) holds (see, *e.g.*, [15, pp.243-244], [19, pp.695-696]). In the sequel, we will always consider $AC^1([0, T], E)$ as a subspace of the Banach space $C^1([0, T], E)$.

Given $C \subset E$ and $\varepsilon > 0$, the symbol $B(C, \varepsilon)$ will denote, as usually, the set $C + \varepsilon B$, where B is the open unit ball in E, *i.e.*, $B = \{x \in E \mid \|x\| < 1\}$.

We will also need the following definitions and notions from multivalued analysis. Let X, Y be two metric spaces. We say that F is a *multivalued mapping* from X to Y (written

$F\colon X \multimap Y$) if for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

A multivalued mapping $F\colon X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if for each open subset $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

A multivalued mapping $F\colon X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x\in X} F(x)$ is contained in a compact subset of Y; it is called *quasi-compact* if it maps compact sets onto relatively compact sets; and *completely continuous* if it maps bounded sets onto relatively compact sets.

We say that a multivalued mapping $F\colon [0,T] \multimap Y$ with closed values is a *step multivalued mapping* if there exists a finite family of disjoint measurable subsets I_k, $k = 1,\dots,n$ such that $[0,T] = \bigcup I_k$ and F is constant on every I_k. A multivalued mapping $F\colon [0,T] \multimap Y$ with closed values is called *strongly measurable* if there exists a sequence of step multivalued mappings $\{F_n\}$ such that $d_H(F_n(t), F(t)) \to 0$ as $n \to \infty$ for a.a. $t \in [0,T]$, where d_H stands for the Hausdorff distance.

It is well known that if Y is a Banach space, then a strongly measurable mapping $F\colon [0,T] \multimap Y$ with compact values possesses a single-valued strongly measurable selection (see, *e.g.*, [12, 20]).

A multivalued mapping $F\colon [0,T] \times X \multimap Y$ is called an *upper-Carathéodory mapping* if the map $F(\cdot,x)\colon [0,T] \multimap Y$ is strongly measurable for all $x \in X$, the map $F(t,\cdot)\colon X \multimap Y$ is u.s.c. for almost all $t \in [0,T]$ and the set $F(t,x)$ is compact and convex for all $(t,x) \in [0,T] \times X$.

Let us note that if X, Y are Banach spaces, then an upper-Carathéodory mapping $F\colon [0,T] \times X \multimap Y$ is weakly superpositionally measurable, *i.e.*, that for each continuous $g\colon [0,T] \to X$, the composition $F(\cdot, g(\cdot))\colon [0,T] \multimap Y$ possesses a single-valued measurable selection (see, *e.g.*, [12, 20]).

A multivalued mapping $F\colon [0,T] \times X \times X \multimap Y$ is called *Lipschitzian* in $(x,y) \in X \times X$ if there exists a constant $L > 0$ such that

$$d_H\bigl(F(t,x_1,y_1), F(t,x_2,y_2)\bigr) \le L\bigl(\|x_1 - x_2\| + \|y_1 - y_2\|\bigr)$$

for a.a. $t \in [0,T]$ and for all $x_1, x_2, y_1, y_2 \in X$.

For more details concerning multivalued analysis, see, *e.g.*, [12, 15, 20, 21].

In the sequel, the measure of noncompactness will also be employed.

Definition 2.1 Let N be a partially ordered set, E be a Banach space and let $P(E)$ denote the family of all nonempty subsets of E. A function $\beta\colon P(E) \to N$ is called a *measure of noncompactness* (m.n.c.) in E if $\beta(\overline{co\Omega}) = \beta(\Omega)$ for all $\Omega \in P(E)$, where $\overline{co\Omega}$ denotes the closed convex hull of Ω.

An m.n.c. β is called:

(i) *monotone* if $\beta(\Omega_1) \le \beta(\Omega_2)$ for all $\Omega_1 \subset \Omega_2 \subset E$,

(ii) *nonsingular* if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$ for all $x \in E$ and $\Omega \subset E$,

(iii) *invariant with respect to the union with compact sets* if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact $K \subset E$ and every $\Omega \subset E$,

(iv) *regular* when $\beta(\Omega) = 0$ if and only if Ω is relatively compact,
(v) *algebraically semi-additive* if $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$ for all $\Omega_1, \Omega_2 \subset E$.

Definition 2.2 An m.n.c. β with values in a cone of a Banach space has the *semi-homogeneity* property if $\beta(t\Omega) = |t|\beta(\Omega)$ for all $t \in \mathbb{R}$ and all $\Omega \subset E$.

It is obvious that an m.n.c. which is invariant with respect to the union with compact sets is also nonsingular.

The typical example of an m.n.c. is the *Hausdorff measure of noncompactness* γ defined, for all $\Omega \subset E$, by

$$\gamma(\Omega) := \inf\left\{\varepsilon > 0 \;\middle|\; \exists x_1, \ldots, x_n \in E \colon \Omega \subset \bigcup_{i=1}^{n} B\bigl(\{x_i\}, \varepsilon\bigr)\right\}.$$

The Hausdorff measure of noncompactness is monotone, nonsingular, algebraically semi-additive and has the semi-homogeneity property.

Let $\{f_n\} \subset L^1([0,T],E)$ be such that $\|f_n(t)\| \leq \alpha(t)$, $\gamma(\{f_n(t)\}) \leq c(t)$ for a.a. $t \in [0,T]$, all $n \in \mathbb{N}$ and suitable $\alpha, c \in L^1([0,T],\mathbb{R})$, then (*cf.* [20])

$$\gamma\left(\left\{\int_0^T f_n(t)\,dt\right\}\right) \leq 2\int_0^T c(t)\,dt \quad \text{for a.a. } t \in [0,T]. \tag{2}$$

Moreover, for all subsets Ω of E (see, *e.g.*, [18]),

$$\gamma\left(\bigcup_{\lambda\in[0,1]} \lambda\Omega\right) \leq \gamma(\Omega). \tag{3}$$

Let us now introduce the function

$$\begin{aligned}\mu(\Omega) := \max_{\{w_n\}_n\subset\Omega}\Bigl(&\sup_{t\in[0,T]}\bigl[\gamma\bigl(\{w_n(t)\}_n\bigr) + \gamma\bigl(\{\dot{w}_n(t)\}_n\bigr)\bigr],\\ &\operatorname{mod}_C\bigl(\{w_n\}_n\bigr) + \operatorname{mod}_C\bigl(\{\dot{w}_n\}_n\bigr)\Bigr),\end{aligned} \tag{4}$$

defined on the bounded set $\Omega \subset C^1([0,T],E)$, where the ordering is induced by the positive cone in $\mathbb{R}^2$ and where $\operatorname{mod}_C(\Omega)$ denotes the modulus of continuity of a subset $\Omega \subset C([0,T],E)$.[a] Such a μ is an m.n.c. in $C^1([0,T],E)$, as shown in the following lemma (proven in [16]), where the properties of μ will be also discussed.

Lemma 2.1 *The function μ given by* (4) *defines an m.n.c. in $C^1([0,T],E)$; such an m.n.c. μ is monotone, invariant with respect to the union with compact sets and regular.*

The m.n.c. μ defined by (4) will be used in order to solve problem (1) (*cf.* Theorem 5.1).

Definition 2.3 Let E be a Banach space and $X \subset E$. A multivalued mapping $F \colon X \multimap E$ with compact values is called *condensing with respect to an m.n.c. β* (shortly, *β-condensing*) if for every $\Omega \subset X$ such that $\beta(F(\Omega)) \geq \beta(\Omega)$, it holds that Ω is relatively compact.

A family of mappings $G \colon X \times [0,1] \multimap E$ with compact values is called *β-condensing* if for every $\Omega \subset X$ such that $\beta(G(\Omega \times [0,1])) \geq \beta(\Omega)$, it holds that Ω is relatively compact.

The following convergence result will be also employed.

Lemma 2.2 (*cf.* [15, Lemma III.1.30]) *Let E be a Banach space and assume that the sequence of absolutely continuous functions $x_k\colon [0,T] \to E$ satisfies the following conditions*:

(i) *the set $\{x_k(t) | k \in \mathbb{N}\}$ is relatively compact for every $t \in [0,T]$,*

(ii) *there exists $\alpha \in L^1([0,T],[0,\infty))$ such that $\|\dot{x}_k(t)\| \le \alpha(t)$ for a.a. $t \in [0,T]$ and for all $k \in \mathbb{N}$,*

(iii) *the set $\{\dot{x}_k(t) | k \in \mathbb{N}\}$ is weakly relatively compact for a.a. $t \in [0,T]$.*

Then there exists a subsequence of $\{x_k\}$ (for the sake of simplicity denoted in the same way as the sequence) converging to an absolutely continuous function $x\colon [0,T] \to E$ in the following way:

1. *$\{x_k\}$ converges uniformly to x in $C([0,T],E)$,*
2. *$\{\dot{x}_k\}$ converges weakly in $L^1([0,T],E)$ to $\dot{x}$.*

The following lemma is well known when the Banach spaces E_1 and E_2 coincide (see, *e.g.*, [22, p.88]). The present slight modification for $E_1 \neq E_2$ was proved in [23].

Lemma 2.3 *Let $[0,T] \subset \mathbb{R}$ be a compact interval, let E_1, E_2 be Banach spaces and let $F\colon [0,T] \times E_1 \multimap E_2$ be a multivalued mapping satisfying the following conditions*:

(i) *$F(\cdot,x)$ has a strongly measurable selection for every $x \in E_1$,*

(ii) *$F(t,\cdot)$ is u.s.c. for a.a. $t \in [0,T]$,*

(iii) *the set $F(t,x)$ is compact and convex for all $(t,x) \in [0,T] \times E_1$.*

Assume in addition that for every nonempty, bounded set $\Omega \subset E_1$, there exists $\nu = \nu(\Omega) \in L^1([0,T],(0,\infty))$ such that

$$\|F(t,x)\| \le \nu(t)$$

for a.a. $t \in [0,T]$ and every $x \in \Omega$. Let us define the Nemytskiǐ operator $N_F\colon C([0,T],E_1) \multimap L^1([0,T],E_2)$ in the following way: $N_F(x) := \{f \in L^1([0,T],E_2) \mid f(t) \in F(t,x(t))$, a.e. on $[0,T]\}$ for every $x \in C([0,T],E_1)$. Then, if sequences $\{x_k\} \subset C([0,T],E_1)$ and $\{f_k\} \subset L^1([0,T],E_2)$, $f_k \in N_F(x_k)$, $k \in \mathbb{N}$, are such that $x_k \to x$ in $C([0,T],E_1)$ and $f_k \to f$ weakly in $L^1([0,T],E_2)$, then $f \in N_F(x)$.

3 Continuation principle

The proof of the main result (*cf.* Theorem 5.1 below) will be based on the combination of a bound sets technique together with the following continuation principle developed in [16].

Proposition 3.1 *Let us consider the general multivalued b.v.p.*

$$\left.\begin{aligned} &\ddot{x}(t) \in \varphi(t,x(t),\dot{x}(t)) \quad \text{for a.a. } t \in [0,T], \\ &x \in S, \end{aligned}\right\} \tag{5}$$

where $\varphi\colon [0,T] \times E \times E \multimap E$ is an upper-Carathéodory mapping and $S \subset AC^1([0,T],E)$. Let $H\colon [0,T] \times E \times E \times E \times E \times [0,1] \multimap E$ be an upper-Carathéodory mapping such that

$$H(t,c,d,c,d,1) \subset \varphi(t,c,d) \tag{6}$$

for all $(t,c,d) \in [0,T] \times E \times E$. *Moreover, assume that the following conditions hold*:

(i) *There exist a closed set* $S_1 \subset S$ *and a closed, convex set* $Q \subset C^1([0,T],E)$ *with a nonempty interior* $\operatorname{Int} Q$ *such that each associated problem*

$$\left.\begin{array}{ll} \ddot{x}(t) \in H(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda), & \text{for a.a. } t \in [0,T], \\ x \in S_1, & \end{array}\right\} \qquad P(q,\lambda)$$

where $q \in Q$ *and* $\lambda \in [0,1]$, *has a nonempty, convex set of solutions* (*denoted by* $\mathfrak{T}(q,\lambda)$).

(ii) *For every nonempty, bounded set* $\Omega \subset E \times E$, *there exists* $\nu_\Omega \in L^1([0,T],[0,\infty))$ *such that*

$$\big\|H\big(t,x,y,q(t),\dot{q}(t),\lambda\big)\big\| \le \nu_\Omega(t)$$

for a.a. $t \in [0,T]$ *and all* $(x,y) \in \Omega$, $q \in Q$ *and* $\lambda \in [0,1]$.

(iii) *The solution mapping* $\mathfrak{T}$ *is quasi-compact and* μ-*condensing with respect to a monotone and nonsingular m.n.c.* μ *defined on* $C^1([0,T],E)$.

(iv) *For each* $q \in Q$, *the set of solutions of the problem* $P(q,0)$ *is a subset of* $\operatorname{Int} Q$, *i.e.*, $\mathfrak{T}(q,0) \subset \operatorname{Int} Q$ *for all* $q \in Q$.

(v) *For each* $\lambda \in (0,1)$, *the solution mapping* $\mathfrak{T}(\cdot,\lambda)$ *has no fixed points on the boundary* ∂Q *of* Q.

Then the b.v.p. (5) *has a solution in* Q.

The proof of the continuation principle is based on the fact that the family $P(q,\lambda)$ of problems depending on two parameters $q \in Q$ and $\lambda \in [0,1]$ is associated to the original b.v.p. (5). This family is defined in such a way that if $\mathfrak{T}\colon Q \times [0,1] \multimap AC^1([0,T],E)$ is its corresponding solution mapping, then all fixed points of the map $\mathfrak{T}(\cdot,1)$ are solutions of (5) (see condition (6)).

4 Bound sets technique

The continuation principle formulated in Proposition 3.1 requires, in particular, the existence of a suitable set $Q \subset AC^1([0,T],E)$ of candidate solutions. The set Q should satisfy the transversality condition (v), *i.e.*, it should have a fixed-point free boundary with respect to the solution mapping $\mathfrak{T}$. Since the direct verification of the transversality condition is usually a difficult task, we will devote this section to a bound sets technique which can be used for guaranteeing such a condition. For this purpose, we will define the set Q as $Q := C^1([0,T],\overline{K})$, where K is nonempty and open in E and $\overline{K}$ denotes its closure.

Hence, let us consider the Dirichlet boundary value problem (1) and let $V\colon E \to \mathbb{R}$ be a C^1-function satisfying

(H1) $V|_{\partial K} = 0$,

(H2) $V(x) \le 0$ for all $x \in \overline{K}$.

Definition 4.1 A nonempty open set $K \subset E$ is called a *bound set* for the b.v.p. (1) if every solution x of (1) such that $x(t) \in \overline{K}$ for each $t \in [0,T]$ does not satisfy $x(t^*) \in \partial K$ for any $t^* \in [0,T]$.

Let E' be the Banach space dual to E and let us denote by $\langle\cdot,\cdot\rangle$ the pairing (the duality relation) between E and E', *i.e.*, for all $\Phi \in E'$ and $x \in E$, we put $\Phi(x) := \langle\Phi, x\rangle$. The proof of the following proposition is quite analogous to the finite-dimensional case considered in [4]. Nevertheless, for the sake of completeness, we present it here, too.

Proposition 4.1 *Let $K \subset E$ be an open set such that $0 \in K$ and $F\colon [0,T] \times E \times E \multimap E$ be an upper-Carathéodory mapping. Assume that the function $V \in C^1(E,\mathbb{R})$ has a locally Lipschitzian Fréchet derivative $\dot{V}_x$ and satisfies conditions* (H1) *and* (H2). *Suppose, moreover, that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$ and $y \in E$, at least one of the following conditions*:

$$\limsup_{h\to 0^-} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + \langle \dot{V}_x, w\rangle > 0, \tag{7}$$

$$\limsup_{h\to 0^+} \frac{\langle \dot{V}_{x+hy} - \dot{V}_x, y\rangle}{h} + \langle \dot{V}_x, w\rangle > 0 \tag{8}$$

holds for all $w \in F(t,x,y)$. Then K is a bound set for the Dirichlet problem (1).

Proof Let $x\colon [0,T] \to \overline{K}$ be a solution of problem (1). We assume, by a contradiction, that there exists $t^* \in [0,T]$ such that $x(t^*) \in \partial K$. The point t^* must lie in $(0,T)$ according to the Dirichlet boundary conditions and the fact that $0 \in K$.

Since $\dot{V}_x$ is locally Lipschitzian, there exist a neighborhood U of $x(t^*)$ and a constant $L > 0$ such that $\dot{V}|_U$ is Lipschitzian with a constant L. Let $\delta > 0$ be such that $x(t) \in U \cap B(\partial K, \varepsilon)$ for each $t \in [t^* - \delta, t^* + \delta]$.

In order to get the desired contradiction, let us define the function $g\colon [0,T] \to \mathbb{R}$ as the composition $g(t) := (V \circ x)(t)$. According to the regularity properties of x and V, $g \in C^1([0,T],\mathbb{R})$. Since $g(t^*) = 0$ and $g(t) \le 0$ for all $t \in [0,T]$, t^* is a local maximum point for g. Therefore, $\dot{g}(t^*) = 0$. Moreover, there exist points $t^{**} \in (t^* - \delta, t^*)$, $t^{***} \in (t^*, t^* + \delta)$ such that $\dot{g}(t^{**}) \ge 0$ and $\dot{g}(t^{***}) \le 0$.

Since $\dot{g}(t) = \langle \dot{V}_{x(t)}, \dot{x}(t)\rangle$, where $\dot{V}_{x(t)}$ is locally Lipschitzian and $\dot{x}(t)$ is absolutely continuous on $[t^* - \delta, t^*]$, $\ddot{g}(t)$ exists for a.a. $t \in [t^* - \delta, t^* + \delta]$. Consequently,

$$0 \ge -\dot{g}\bigl(t^{**}\bigr) = \dot{g}\bigl(t^*\bigr) - \dot{g}\bigl(t^{**}\bigr) = \int_{t^{**}}^{t^*} \ddot{g}(s)\,ds \tag{9}$$

and

$$0 \ge \dot{g}\bigl(t^{***}\bigr) = \dot{g}\bigl(t^{***}\bigr) - \dot{g}\bigl(t^*\bigr) = \int_{t^*}^{t^{***}} \ddot{g}(s)\,ds. \tag{10}$$

At first, let us assume that condition (7) holds and let $t \in (t^{**}, t^*)$ be such that $\ddot{g}(t)$ and $\ddot{x}(t)$ exist. Then

$$\lim_{h\to 0} \frac{\dot{x}(t+h) - \dot{x}(t)}{h} = \ddot{x}(t),$$

and so there exists a function $a(h), a(h) \to 0$ as $h \to 0$, such that for each h,

$$\dot{x}(t+h) = \dot{x}(t) + h\bigl[\ddot{x}(t) + a(h)\bigr].$$

Moreover, since $x \in C^1([0,T],E)$, there exists a function $b(h)$, $b(h) \to 0$ as $h \to 0$, such that for each h,

$$x(t+h) = x(t) + h\big[\dot{x}(t) + b(h)\big].$$

Consequently, we obtain

$$\begin{aligned}
\ddot{g}(t) &= \lim_{h\to 0}\frac{\dot{g}(t+h)-\dot{g}(t)}{h} = \limsup_{h\to 0^-}\frac{\dot{g}(t+h)-\dot{g}(t)}{h}\\
&= \limsup_{h\to 0^-}\frac{\langle \dot{V}_{x(t+h)}, \dot{x}(t+h)\rangle - \langle \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h}\\
&= \limsup_{h\to 0^-}\frac{\langle \dot{V}_{x(t)+h[\dot{x}(t)+b(h)]}, \dot{x}(t)+h[\ddot{x}(t)+a(h)]\rangle - \langle \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h}\\
&\geq \limsup_{h\to 0^-}\frac{\langle \dot{V}_{x(t)+h\dot{x}(t)}, \dot{x}(t)+h[\ddot{x}(t)+a(h)]\rangle - \langle \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h}\\
&\quad - L\cdot\big|b(h)\big|\cdot\big\|\dot{x}(t)+h\big[\ddot{x}(t)+a(h)\big]\big\|\\
&= \limsup_{h\to 0^-}\frac{\langle \dot{V}_{x(t)+h\dot{x}(t)}, \dot{x}(t)+h\ddot{x}(t)\rangle - \langle \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h}\\
&\quad - L\cdot\big|b(h)\big|\cdot\big\|\dot{x}(t)+h\big[\ddot{x}(t)+a(h)\big]\big\| + \big\langle \dot{V}_{x(t)+h\dot{x}(t)}, a(h)\big\rangle.
\end{aligned}$$

Since $\langle \dot{V}_{x(t)+h\dot{x}(t)}, a(h)\rangle - L\cdot|b(h)|\cdot\|\dot{x}(t)+h[\ddot{x}(t)+a(h)]\| \to 0$ as $h \to 0$,

$$\begin{aligned}
\ddot{g}(t) &\geq \limsup_{h\to 0^-}\frac{\langle \dot{V}_{x(t)+h\dot{x}(t)}, \dot{x}(t)+h\ddot{x}(t)\rangle - \langle \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h}\\
&= \limsup_{h\to 0^-}\frac{\langle \dot{V}_{x(t)+h\dot{x}(t)} - \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h} + \big\langle \dot{V}_{x(t)+h\dot{x}(t)}, \ddot{x}(t)\big\rangle.
\end{aligned}$$

Moreover, for every $x, w \in E$ and $h \in \mathbb{R}$, we have that

$$\langle \dot{V}_{x+hy}, w\rangle = \langle \dot{V}_x, w\rangle + \big[\langle \dot{V}_{x+hy}, w\rangle - \langle \dot{V}_x, w\rangle\big].$$

According to the Lipschitzianity of $\dot{V}$, when $|h|$ is sufficiently small, we have that

$$\begin{aligned}
\big|\langle \dot{V}_{x+hy}, w\rangle - \langle \dot{V}_x, w\rangle\big| &= \big|\langle \dot{V}_{x+hy} - \dot{V}_x, w\rangle\big|\\
&\leq \|V_{x+hy} - \dot{V}_x\|\cdot\|w\| \leq L|h|\cdot\|y\|\cdot\|w\|,
\end{aligned}$$

where L denotes the local Lipschitz constant of $\dot{V}$ in a neighborhood of x. It implies that

$$\lim_{h\to 0}\langle \dot{V}_{x+hy}, w\rangle - \langle \dot{V}_x, w\rangle = 0$$

and then

$$\lim_{h\to 0}\langle \dot{V}_{x+hy}, w\rangle = \langle \dot{V}_x, w\rangle.$$

Therefore,

$$\ddot{g}(t) \geq \limsup_{h\to 0^-} \frac{\langle \dot{V}_{x(t)+h\dot{x}(t)} - \dot{V}_{x(t)}, \dot{x}(t)\rangle}{h} + \langle \dot{V}_{x(t)}, \ddot{x}(t)\rangle > 0,$$

according to assumption (7), it leads to a contradiction with inequality (9).

Secondly, let us assume that condition (8) holds and let $s \in (t^*, t^{***})$ be such that $\ddot{g}(s)$ and $\ddot{x}(s)$ exist. Then it is possible to show, using the same procedure as before, that according to assumption (8),

$$\ddot{g}(s) \geq \limsup_{h\to 0^+} \frac{\langle \dot{V}_{x(s)+h\dot{x}(s)} - \dot{V}_{x(s)}, \dot{x}(s)\rangle}{h} + \langle \dot{V}_{x(s)}, \ddot{x}(s)\rangle > 0,$$

which leads to a contradiction with inequality (10).

Therefore, we get the contradiction in case that at least one of conditions (7), (8) holds which completes the proof. □

If the mapping $F(t,x,y)$ is globally u.s.c. in (t,x,y), then the transversality conditions can be localized directly on the boundary of K, as will be shown in the following proposition, whose proof is again quite analogous to the finite-dimensional case considered in [2].

Proposition 4.2 *Let $K \subset E$ be a nonempty open set such that $0 \in K$ and $F\colon [0,T] \times E \times E \multimap E$ be an upper semicontinuous multivalued mapping with compact, convex values. Assume that there exists a function $V \in C^1(E,\mathbb{R})$ with a locally Lipschitzian Fréchet derivative $\dot{V}_x$ which satisfies conditions* (H1) *and* (H2). *Suppose, moreover, that for all $x \in \partial K$, $t \in (0,T)$ and $y \in E$ with*

$$\langle \dot{V}_x, y\rangle = 0, \tag{11}$$

the following condition holds:

$$\liminf_{h\to 0} \frac{\langle \dot{V}_{x+hy}, y\rangle}{h} + \langle \dot{V}_x, w\rangle > 0 \tag{12}$$

for all $w \in F(t,x,y)$. Then K is a bound set for problem (1).

Proof Let $x\colon [0,T] \to \overline{K}$ be a solution of problem (1). We assume, by a contradiction, that there exists $t_0 \in [0,T]$ such that $x(t_0) \in \partial K$. Since $0 \in K$ and x satisfies Dirichlet boundary conditions, $t_0 \in (0,T)$.

Let us define the function $g : [-t_0, T - t_0] \to (-\infty, 0]$ as the composition $g(h) := (V \circ x)(t_0 + h)$. Then $g(0) = 0$ and $g(h) \leq 0$ for all $h \in [-t_0, T - t_0]$, *i.e.*, there is a local maximum for g at the point 0, and so $\dot{g}(0) = \langle \dot{V}_{x(t_0)}, \dot{x}(t_0)\rangle = 0$. Consequently, $v := \dot{x}(t_0)$ satisfies condition (11).

Since $\dot{V}_x$ is locally Lipschitzian, there exist a neighborhood U of $x(t_0)$ and a constant $L > 0$ such that $\dot{V}|_U$ is Lipschitzian with a constant L.

Let $\{h_k\}_{k=1}^{\infty}$ be an arbitrary decreasing sequence of positive numbers such that $h_k \to 0^+$ as $k \to \infty$, $x(t_0 + h) \in U$ for all $h \in (0, h_1)$.

Since $g(0) = 0$ and $g(h) \leq 0$ for all $h \in (0, h_k]$, there exists, for each $k \in \mathbb{N}$, $h_k^* \in (0, h_k)$ such that $\dot{g}(h_k^*) \leq 0$.

Since $x \in C^1([0,T], E)$, for each $k \in \mathbb{N}$,

$$x\bigl(t_0 + h_k^*\bigr) = x(t_0) + h_k^*\bigl[\dot{x}(t_0) + b_k^*\bigr], \tag{13}$$

where $b_k^* \to 0$ as $k \to \infty$.

Let

$$\zeta := \left\{ \frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*}, k \in \mathbb{N} \right\}$$

and let $\varepsilon > 0$ be given. As a consequence of the regularity assumptions imposed on F and of the continuity of both x and $\dot{x}$, there exists $\overline{\delta} = \overline{\delta}(\varepsilon) > 0$ such that for each $t \in (0, T)$, $|t - t_0| \leq \overline{\delta}$, it follows that

$$F\bigl(t, x(t), \dot{x}(t)\bigr) \subset F\bigl(t_0, x(t_0), \dot{x}(t_0)\bigr) + \varepsilon\overline{B}.$$

Subsequently, according to the mean-value theorem (see, *e.g.*, [24, Theorem 0.5.3]), there exists $k_\varepsilon \in \mathbb{N}$ such that for each $k > k_\varepsilon$,

$$\frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*} = \frac{1}{h_k^*} \int_{t_0}^{t_0 + h_k^*} \ddot{x}(s)\, ds \in F\bigl(t_0, x(t_0), \dot{x}(t_0)\bigr) + \varepsilon\overline{B}.$$

Therefore,

$$\zeta \subset \left\{ \frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*}, k = 1, 2, \ldots, k(\varepsilon) \right\} \cup F\bigl(t_0, x(t_0), \dot{x}(t_0)\bigr) + \varepsilon\overline{B}.$$

Since F has compact values and ε is arbitrary, we obtain that ζ is a relatively compact set. Thus, there exist a subsequence, for the sake of simplicity denoted as the sequence, of $\{\frac{\dot{x}(t_0+h_k^*)-\dot{x}(t_0)}{h_k^*}\}$ and $w \in E$ such that

$$\frac{\dot{x}(t_0 + h_k^*) - \dot{x}(t_0)}{h_k^*} \to w \tag{14}$$

as $k \to \infty$ implying, for the arbitrariness of $\varepsilon > 0$,

$$w \in F\bigl(t_0, x(t_0), \dot{x}(t_0)\bigr).$$

As a consequence of the property (14), there exists a sequence $\{a_k^*\}_{k=1}^{\infty}$, $a_k^* \to 0$ as $k \to \infty$, such that

$$\dot{x}\bigl(t_0 + h_k^*\bigr) = \dot{x}(t_0) + h_k^*\bigl[w + a_k^*\bigr] \tag{15}$$

for each $k \in \mathbb{N}$. Since $h_k^* > 0$ and $\dot{g}(h_k^*) \leq 0$, in view of (13) and (15),

$$0 \geq \frac{\dot{g}(h_k^*)}{h_k^*} = \frac{\langle \dot{V}_{x(t_0+h_k^*)}, \dot{x}(t_0 + h_k^*)\rangle}{h_k^*} = \frac{\langle \dot{V}_{x(t_0)+h_k^*[\dot{x}(t_0)+b_k^*]}, \dot{x}(t_0) + h_k^*[w + a_k^*]\rangle}{h_k^*}.$$

Since $h_k^* \in (0, h_k) \subset (0, h_1)$ for all $k \in \mathbb{N}$, we have, according to (13), that $x(t_0) + h_k^*[\dot{x}(t_0) + b_k^*] \in U$ for each $k \in \mathbb{N}$. Since $b_k^* \to 0$ as $k \to \infty$, it is possible to find $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, it holds that $x(t_0) + \dot{x}(t_0)h_k^* \in U$. By means of the local Lipschitzianity of $\dot{V}$, for all $k \geq k_0$,

$$\begin{aligned}
0 \geq \frac{\dot{g}(h_k^*)}{h_k^*} &= \frac{\langle \dot{V}_{x(t_0)+h_k^*[\dot{x}(t_0)+b_k^*]} - \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)} + \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) + h_k^*[w + a_k^*] \rangle}{h_k^*} \\
&\geq \frac{\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) + h_k^*[w + a_k^*] \rangle}{h_k^*} - L \cdot |b_k^*| \cdot \| \dot{x}(t_0) + h_k^*[w + a_k^*] \| \\
&= \frac{\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) + h_k^* w \rangle}{h_k^*} - L \cdot |b_k^*| \cdot \| \dot{x}(t_0) + h_k^*[w + a_k^*] \| + \langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, a_k^* \rangle \\
&= \frac{\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) \rangle}{h_k^*} + \langle \dot{V}_{x(t_0)}, w \rangle - L \cdot |b_k^*| \cdot \| \dot{x}(t_0) + h_k^*[w + a_k^*] \| \\
&\quad + \langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, a_k^* \rangle.
\end{aligned}$$

Since $\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, a_k^* \rangle - L \cdot |b_k^*| \cdot \| \dot{x}(t_0) + h_k^*[w + a_k^*] \| \to 0$ as $k \to \infty$,

$$\liminf_{h \to 0^+} \frac{\langle \dot{V}_{x(t_0)+h_k^*\dot{x}(t_0)}, \dot{x}(t_0) \rangle}{h_k^*} + \langle \dot{V}_{x(t_0)}, w \rangle \leq 0. \tag{16}$$

If we consider, instead of the sequence $\{h_k\}_{k=1}^{\infty}$, an increasing sequence $\{\bar{h}_k\}_{k=1}^{\infty}$ of negative numbers such that $\bar{h}_k \to 0^-$ as $k \to \infty$, $x(t_0 + h) \in U$ for all $h \in (\bar{h}_1, 0)$, we are able to find, for each $k \in \mathbb{N}$, $\bar{h}_k^* \in (\bar{h}_k, 0)$ such that $\dot{g}(\bar{h}_k^*) \geq 0$. Therefore, using the same procedure as in the first part of the proof, we obtain, for $k \in \mathbb{N}$ sufficiently large, that

$$\begin{aligned}
0 \geq \frac{\dot{g}(\bar{h}_k^*)}{\bar{h}_k^*} \geq{} & \frac{\langle \dot{V}_{x(t_0)+\bar{h}_k^*\dot{x}(t_0)}, \dot{x}(t_0) \rangle}{\bar{h}_k^*} \\
& + \langle \dot{V}_{x(t_0)}, \overline{w} \rangle - L \cdot |\bar{b}_k^*| \cdot \| \dot{x}(t_0) + \bar{h}_k^*[\overline{w} + \bar{a}_k^*] \| + \langle \dot{V}_{x(t_0)+\bar{h}_k^*\dot{x}(t_0)}, \bar{a}_k^* \rangle,
\end{aligned}$$

where $\bar{a}_k^* \to 0$, $\bar{b}_k^* \to 0$ as $k \to \infty$ and $\overline{w} \in F(t_0, x(t_0), \dot{x}(t_0))$.

This means that $\langle \dot{V}_{x(t_0)+\bar{h}_k^*\dot{x}(t_0)}, \bar{a}_k^* \rangle - L \cdot |\bar{b}_k^*| \cdot \| \dot{x}(t_0) + \bar{h}_k^*[\overline{w} + \bar{a}_k^*] \| \to 0$ as $k \to \infty$, which implies

$$\liminf_{h \to 0^-} \frac{\langle \dot{V}_{x(t_0)+h\dot{x}(t_0)}, \dot{x}(t_0) \rangle}{h} + \langle \dot{V}_{x(t_0)}, \overline{w} \rangle \leq 0. \tag{17}$$

Inequalities (16) and (17) are in a contradiction with condition (12), because $x(t_0) \in \partial K$, $\dot{x}(t_0)$ satisfies condition (11) and $w, \overline{w} \in F(t_0, x(t_0), \dot{x}(t_0))$. □

Remark 4.1 One can readily check that for $V \in C^2(E, \mathbb{R})$, inequalities (7) and (8), as well as (12), become

$$\langle \ddot{V}_x(y), y \rangle + \langle \dot{V}_x, w \rangle > 0,$$

with t, x, y, w as in Proposition 4.1 or in Proposition 4.2.

The typical case occurs when $E = H$ is a Hilbert space, $\langle\, ,\, \rangle$ denotes the scalar product and

$$V(x) := \frac{1}{2}\left(\|x\|^2 - R^2\right) = \frac{1}{2}\left(\langle x, x\rangle - R^2\right)$$

for some $R > 0$. In this case, $V \in C^2(H, \mathbb{R})$ and it is not difficult to see that conditions (7) and (8), as well as (12), become

$$\langle y, y\rangle + \langle x, w\rangle > 0$$

with t, x, y and w as in Proposition 4.1 or in Proposition 4.2, where $K := \{x \in H \mid \|x\| < R\}$.

Definition 4.2 A C^1-function $V\colon E \to R$ with a locally Lipschitzian Fréchet derivative $\dot{V}$ which satisfies conditions (H1), (H2) and all assumptions in Proposition 4.1 or Proposition 4.2 is called a *bounding function* for problem (1).

5 Existence and localization results

Combining the continuation principle with the bound sets technique, we are ready to state the main result of the paper concerning the solvability and localization of a solution of the multivalued Dirichlet problem (1).

Theorem 5.1 *Consider the Dirichlet b.v.p.* (1), *where* $F\colon [0,T] \times E \times E \multimap E$ *is an upper-Carathéodory multivalued mapping. Assume that* $K \subset E$ *is an open, convex set containing* 0. *Furthermore, let the following conditions be satisfied*:

(5_i) $\gamma(F(t, \Omega_1 \times \Omega_2)) \le g(t)(\gamma(\Omega_1) + \gamma(\Omega_2))$ *for a.a.* $t \in [0,T]$ *and each bounded* $\Omega_1, \Omega_2 \subset E$, *where* $g \in L^1([0,T],[0,\infty))$ *and* γ *is the Hausdorff measure of noncompactness in* E.

(5_{ii}) *For every nonempty, bounded set* $\Omega \subset E \times E$, *there exists* $\nu_\Omega \in L^1([0,T],[0,\infty))$ *such that*

$$\left\|F(t,x,y)\right\| \le \nu_\Omega(t) \tag{18}$$

for a.a. $t \in [0,T]$ *and all* $(x,y) \in \Omega$,

(5_{iii}) $(T+4)\|g\|_{L^1([0,T],[0,\infty))} < 2$.

Finally, let there exist a function $V \in C^1(E,\mathbb{R})$ *with a locally Lipschitzian Fréchet derivative* $\dot{V}$ *satisfying conditions* (H1), (H2), *and at least one of conditions* (7), (8) *for a suitable* $\varepsilon > 0$, *all* $x \in \overline{K} \cap B(\partial K, \varepsilon)$, $t \in (0,T)$, $y \in E$, $\lambda \in (0,1)$ *and* $w \in \lambda F(t,x,y)$. *Then the Dirichlet b.v.p.* (1) *admits a solution whose values are located in* $\overline{K}$.

Proof Let us define the closed set $S = S_1$ by

$$S := \left\{x \in AC^1\left([0,T],E\right) \colon x(T) = x(0) = 0\right\}$$

and let the set Q of candidate solutions be defined as $Q := C^1([0,T],\overline{K})$. Because of the convexity of K, the set Q is closed and convex.

For all $q \in Q$ and $\lambda \in [0,1]$, consider still the associated fully linearized problem

$$\left.\begin{array}{ll} \ddot{x}(t) \in \lambda F(t, q(t), \dot{q}(t)) & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \end{array}\right\} \qquad P(q,\lambda)$$

and denote by $\mathfrak{T}$ a solution mapping which assigns to each $(q,\lambda) \in Q \times [0,1]$ the set of solutions of $P(q,\lambda)$. We will show that the family of the above b.v.p.s $P(q,\lambda)$ satisfies all assumptions of Proposition 3.1.

In this case, $\varphi(t,x,\dot{x}) = F(t,x,\dot{x})$ which, together with the definition of $P(q,\lambda)$, ensures the validity of (6).

ad (i) In order to verify condition (i) in Proposition 3.1, we need to show that for each $(q,\lambda) \in Q \times [0,1]$, the problem $P(q,\lambda)$ is solvable with a convex set of solutions. So, let $(q,\lambda) \in Q \times [0,1]$ be arbitrary and let $f_q(\cdot)$ be a strongly measurable selection of $F(\cdot, q(\cdot), \dot{q}(\cdot))$. The homogeneous problem corresponding to b.v.p. $P(q,\lambda)$,

$$\left.\begin{array}{ll} \ddot{x}(t) = 0 & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \end{array}\right\} \qquad (19)$$

has only the trivial solution, and therefore the single-valued Dirichlet problem

$$\left.\begin{array}{ll} \ddot{x}(t) = \lambda f_q(t) & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0 & \end{array}\right\}$$

admits a unique solution $x_{q,\lambda}(\cdot)$ which is one of solutions of $P(q,\lambda)$. This is given, for a.a. $t \in [0,T]$, by $x_{q,\lambda}(t) = \int_0^T G(t,s)\lambda f_q(s)\,ds$, where G is the Green function associated to the homogeneous problem (19). The Green function G and its partial derivative $\frac{\partial}{\partial t}G$ are defined by (*cf.*, *e.g.*, [12, pp.170-171])

$$G(t,s) = \begin{cases} \frac{(s-T)t}{T} & \text{for all } 0 \le t \le s \le T, \\ \frac{(t-T)s}{T} & \text{for all } 0 \le s \le t \le T, \end{cases}$$

$$\frac{\partial}{\partial t}G(t,s) = \begin{cases} \frac{(s-T)}{T} & \text{for all } 0 \le t \le s \le T, \\ \frac{s}{T} & \text{for all } 0 \le s \le t \le T. \end{cases}$$

Thus, the set of solutions of $P(q,\lambda)$ is nonempty. The convexity of the solution sets follows immediately from the properties of a mapping F and the fact that problems $P(q,\lambda)$ are fully linearized.

ad (ii) Assuming that $H\colon [0,T] \times E \times E \times E \times E \times [0,1] \multimap E$ is defined by $H(t,x,y,q,r,\lambda) := \lambda F(t,q,r)$, condition (ii) in Proposition 3.1 is ensured directly by assumption (5_{ii}).

ad (iii) Since the verification of condition (iii) in Proposition 3.1 is technically the most complicated, it will be subdivided into two parts: (iii_1) the quasi-compactness of the solution operator $\mathfrak{T}$, (iii_2) the condensity of $\mathfrak{T}$ w.r.t. the monotone and nonsingular (*cf.* Lemma 2.1) m.n.c. μ defined by (4).

ad (iii_1) Let us firstly prove that the solution mapping $\mathfrak{T}$ is quasi-compact. Since $C^1([0,T],E)$ is a metric space, it is sufficient to prove the sequential quasi-compactness of $\mathfrak{T}$. Hence, let us consider the sequences $\{q_n\}, \{\lambda_n\}, q_n \in Q, \lambda_n \in [0,1]$ for all $n \in \mathbb{N}$ such that $q_n \to q$ in $C^1([0,T],E)$ and $\lambda_n \to \lambda$. Moreover, let $x_n \in \mathfrak{T}(q_n,\lambda_n)$ for all $n \in \mathbb{N}$. Then there exists, for all $n \in \mathbb{N}$, $f_n(\cdot) \in F(\cdot, q_n(\cdot), \dot{q}_n(\cdot))$ such that

$$\ddot{x}_n(t) = \lambda_n f_n(t) \quad \text{for a.a. } t \in [0,T], \tag{20}$$

and that $x_n(T) = x_n(0) = 0$.

Since $q_n \to q$ and $\dot{q}_n \to \dot{q}$ in $C([0,T],E)$, there exists a bounded $\Omega \subset E \times E$ such that $(q_n(t), \dot{q}_n(t)) \in \Omega$ for all $t \in [0,T]$ and $n \in \mathbb{N}$. Therefore, there exists, according to condition (5_{ii}), $\nu_\Omega \in L^1([0,T],[0,\infty))$ such that $\|f_n(t)\| \le \nu_\Omega(t)$ for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$.

Moreover, for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$,

$$x_n(t) = \lambda_n \int_0^T G(t,s) f_n(s)\, ds$$

and

$$\dot{x}_n(t) = \lambda_n \int_0^T \frac{\partial}{\partial t} G(t,s) f_n(s)\, ds.$$

Thus, x_n satisfies, for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$, $\|x_n(t)\| \le a$ and $\|\dot{x}_n(t)\| \le b$, where

$$a := \frac{T}{4} \int_0^T \nu_\Omega(s)\, ds$$

and

$$b := \int_0^T \nu_\Omega(s)\, ds.$$

Furthermore, for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$, we have

$$\left\|\ddot{x}_n(t)\right\| \le \nu_\Omega(t).$$

Hence, the sequences $\{x_n\}$ and $\{\dot{x}_n\}$ are bounded and $\{\ddot{x}_n\}$ is uniformly integrable.

Since the sequences $\{q_n\}$, $\{\dot{q}_n\}$ are converging, we obtain, in view of (5_i),

$$\gamma\left(\left\{f_n(t)\right\}\right) \le g(t)\left(\gamma\left(\left\{q_n(t)\right\}\right) + \gamma\left(\left\{\dot{q}_n(t)\right\}\right)\right) = 0$$

for a.a. $t \in [0,T]$, which implies that $\{f_n(t)\}$ is relatively compact.

For all $(t,s) \in [0,T] \times [0,T]$, the sequence $\{G(t,s)f_n(s)\}$ is relatively compact as well since, according to the semi-homogeneity of the Hausdorff m.n.c.,

$$\gamma\left(\left\{G(t,s)f_n(s)\right\}\right) \le \left|G(t,s)\right| \gamma\left(\left\{f_n(s)\right\}\right) = 0 \quad \text{for all } (t,s) \in [0,T] \times [0,T]. \tag{21}$$

Moreover, by means of (2), (3), (21) and the semi-homogeneity of the Hausdorff m.n.c.,

$$\gamma\left(\left\{x_n(t)\right\}\right) \le \gamma\left(\bigcup_{\lambda \in [0,1]} \lambda \left\{\int_0^T G(t,s) f_n(s)\, ds\right\}\right) \le \gamma\left(\left\{\int_0^T G(t,s) f_n(s)\, ds\right\}\right) = 0.$$

By similar reasonings, we can also get

$$\gamma\big(\{\dot{x}_n(t)\}\big) = 0,$$

by which $\{x_n(t)\}$, $\{\dot{x}_n(t)\}$ are relatively compact for a.a. $t \in [0,T]$. Moreover, since x_n satisfies for all $n \in \mathbb{N}$ equation (20), $\{\ddot{x}_n(t)\}$ is relatively compact for a.a. $t \in [0,T]$. Thus, according to Lemma 2.2, there exist a subsequence of $\{\dot{x}_n\}$, for the sake of simplicity denoted in the same way as the sequence, and $x \in C^1([0,T],E)$ such that $\{\dot{x}_n\}$ converges to $\dot{x}$ in $C([0,T],E)$ and $\{\ddot{x}_n\}$ converges weakly to $\ddot{x}$ in $L^1([0,T],E)$. Therefore, the mapping $\mathfrak{T}$ is quasi-compact.

ad (iii$_2$) In order to show that $\mathfrak{T}$ is μ-condensing, where μ is defined by (4), we will prove that any bounded subset $\Theta \subset Q$ such that $\mu(\mathfrak{T}(\Theta \times [0,1])) \geq \mu(\Theta)$ is relatively compact. Let $\{x_n\}_n \subset \mathfrak{T}(\Theta \times [0,1])$ be a sequence such that

$$\begin{aligned}&\mu\big(\mathfrak{T}(\Theta \times [0,1])\big)\\&\quad= \Big(\sup_{t\in[0,T]}\big[\gamma\big(\{x_n(t)\}_n\big) + \gamma\big(\{\dot{x}_n(t)\}_n\big)\big], \operatorname{mod}_C\big(\{x_n\}_n\big) + \operatorname{mod}_C\big(\{\dot{x}_n\}_n\big)\Big).\end{aligned}$$

Then we can find $\{q_n\}_n \subset \Theta$, $\{f_n\}_n$ satisfying $f_n(t) \in F(t, q_n(t), \dot{q}_n(t))$ for a.a. $t \in [0,T]$ and $\{\lambda_n\}_n \subset [0,1]$ such that for all $t \in [0,T]$,

$$x_n(t) = \lambda_n \int_0^T G(t,s) f_n(s)\, ds \tag{22}$$

and

$$\dot{x}_n(t) = \lambda_n \int_0^T \frac{\partial}{\partial t} G(t,s) f_n(s)\, ds. \tag{23}$$

In view of (5$_i$), we have, for all $t \in [0,T]$,

$$\begin{aligned}&\gamma\big(\{f_n(t), n \in \mathbb{N}\}\big)\\&\quad\leq g(t)\big(\gamma\big(\{q_n(t), n \in \mathbb{N}\}\big) + \gamma\big(\{\dot{q}_n(t), n \in \mathbb{N}\}\big)\big)\\&\quad\leq g(t) \sup_{t\in[0,T]}\big(\gamma\big(\{q_n(t), n \in \mathbb{N}\}\big) + \gamma\big(\{\dot{q}_n(t), n \in \mathbb{N}\}\big)\big).\end{aligned}$$

Since $\{q_n\}_n \subset \Theta$ and Θ is bounded in $C^1([0,T],E)$, by means of (5$_{ii}$), we get the existence of $\nu_\Theta \in L^1([0,T],[0,\infty))$ such that $\|f_n(t)\| \leq \nu_\Theta(t)$ for a.a. $t \in [0,T]$ and all $n \in \mathbb{N}$. This implies $\|G(t,s)f_n(t)\| \leq |G(t,s)|\nu_\Theta(t)$ for a.a. $t,s \in [0,T]$ and all $n \in \mathbb{N}$.

Moreover, by virtue of the semi-homogeneity of the Hausdorff m.n.c., for all $(t,s) \in [0,T] \times [0,T]$, we have

$$\begin{aligned}&\gamma\big(\{G(t,s)f_n(s), n \in \mathbb{N}\}\big)\\&\quad\leq \big|G(t,s)\big|\gamma\big(\{f_n(s), n \in \mathbb{N}\}\big) \leq \frac{T}{4}\gamma\big(\{f_n(s), n \in \mathbb{N}\}\big)\\&\quad\leq \frac{T}{4} g(t) \sup_{t\in[0,T]}\big(\gamma\big(\{q_n(t), n \in \mathbb{N}\}\big) + \gamma\big(\{\dot{q}_n(t), n \in \mathbb{N}\}\big)\big).\end{aligned}$$

According to (2), (3) and (22), we so obtain for each $t \in [0,T]$,

$$\begin{aligned}\gamma\left(\left\{x_n(t), n \in \mathbb{N}\right\}\right) &\leq \gamma\left(\left\{\int_0^T G(t,s) f_n(s)\, ds, n \in \mathbb{N}\right\}\right) \\ &\leq 2\frac{T}{4}\|g\|_{L^1} \sup_{t\in[0,T]}\left(\gamma\left(\left\{q_n(t), n \in \mathbb{N}\right\}\right) + \gamma\left(\left\{\dot{q}_n(t), n \in \mathbb{N}\right\}\right)\right) \\ &= \frac{T}{2}\|g\|_{L^1}\mathcal{S},\end{aligned}$$

where

$$\mathcal{S} := \sup_{t\in[0,T]}\left(\gamma\left(\left\{q_n(t), n \in \mathbb{N}\right\}\right) + \gamma\left(\left\{\dot{q}_n(t), n \in \mathbb{N}\right\}\right)\right).$$

By the similar reasonings, we can obtain that for each $t \in [0,T]$,

$$\gamma\left(\left\{\dot{x}_n(t), n \in \mathbb{N}\right\}\right) \leq 2\|g\|_{L^1}\mathcal{S},$$

when starting from condition (23). Subsequently,

$$\gamma\left(\left\{x_n(t), n \in \mathbb{N}\right\}\right) + \gamma\left(\left\{\dot{x}_n(t), n \in \mathbb{N}\right\}\right) \leq \frac{T+4}{2}\|g\|_{L^1}\mathcal{S},$$

yielding

$$\sup_{t\in[0,T]}\left(\gamma\left(\left\{x_n(t), n \in \mathbb{N}\right\}\right) + \gamma\left(\left\{\dot{x}_n(t), n \in \mathbb{N}\right\}\right)\right) \leq \frac{T+4}{2}\|g\|_{L^1}\mathcal{S}. \tag{24}$$

Since $\mu(\mathfrak{T}(\Theta \times [0,1])) \geq \mu(\Theta)$ and $\{q_n\}_n \subset \Theta$, we so get

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\gamma\left(\left\{q_n(t), n \in \mathbb{N}\right\}\right) + \gamma\left(\left\{\dot{q}_n(t), n \in \mathbb{N}\right\}\right)\right) \\ &\quad\leq \sup_{t\in[0,T]}\left(\gamma\left(\left\{x_n(t), n \in \mathbb{N}\right\}\right) + \gamma\left(\left\{\dot{x}_n(t), n \in \mathbb{N}\right\}\right)\right)\end{aligned}$$

and, in view of (24) and (5_{iii}), we have that

$$\sup_{t\in[0,T]}\left(\gamma\left(\left\{q_n(t), n \in \mathbb{N}\right\}\right) + \gamma\left(\left\{\dot{q}_n(t), n \in \mathbb{N}\right\}\right)\right) = 0.$$

Inequality (24) implies that

$$\sup_{t\in[0,T]}\left(\gamma\left(\left\{x_n(t), n \in \mathbb{N}\right\}\right) + \gamma\left(\left\{\dot{x}_n(t), n \in \mathbb{N}\right\}\right)\right) = 0. \tag{25}$$

Now, we show that both the sequences $\{x_n\}$ and $\{\dot{x}_n\}$ are equi-continuous. Let $\tilde{\Theta} \subset E$ be such that $q_n(t) \in \tilde{\Theta}$ and $\dot{q}_n(t) \in \tilde{\Theta}$ for all $n \in \mathbb{N}$ and $t \in [0,T]$. Thus, we get that $\|\ddot{x}_n(t)\| = \lambda_n\|f_n(t)\| \leq \nu_{\tilde{\Theta}}(t)$, where $\nu_{\tilde{\Theta}} \in L^1([0,T],[0,\infty))$ comes from (5_{ii}), and so $\{\ddot{x}_n\}_n$ is uniformly integrable. This implies that $\{\dot{x}_n\}_n$ is equi-continuous. Moreover, according to (23), we obtain that

$$\left\|\dot{x}_n(t)\right\| \leq \int_0^T \nu_{\tilde{\Theta}}(s)\, ds$$

for all $n \in \mathbb{N}$ and $t \in [0,T]$, implying that $\{\dot{x}_n\}_n$ is bounded; consequently, also $\{x_n\}_n$ is equi-continuous. Therefore,

$$\mathrm{mod}_C(\{x_n\}) = \mathrm{mod}_C(\{\dot{x}_n\}) = 0.$$

In view of (25), we have so obtained that

$$\mu(\mathfrak{T}(\Theta \times [0,1])) = (0,0).$$

Hence, also $\mu(\Theta) = (0,0)$ and since μ is regular, we have that Θ is relatively compact. Therefore, condition (iii) in Proposition 3.1 holds.

ad (iv) For all $q \in Q$, the problem $P(q,0)$ has only the trivial solution. Since $0 \in K$, condition (iv) in Proposition 3.1 is satisfied.

ad (v) Let $q_* \in Q$ be a solution of the b.v.p. $P(q_*,\lambda)$ for some $\lambda \in (0,1)$, *i.e.*, a fixed point of the solution mapping $\mathfrak{T}$. In view of conditions (7), (8) (see Proposition 4.1), K is, for all $\lambda \in (0,1)$, a bound set for the problem

$$\left.\begin{array}{ll} \ddot{q}_*(t) \in \lambda F(t,q_*(t),\dot{q}_*(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0. & \end{array}\right\}$$

This implies that $q_* \notin \partial Q$, which ensures condition (v) in Proposition 3.1.

□

If the mapping $F(t,x,y)$ is globally u.s.c. in (t,x,y) (*i.e.*, a Marchaud map), then we are able to improve Theorem 5.1 in the following way.

Theorem 5.2 *Consider the Dirichlet b.v.p.* (1), *where* $F\colon [0,T] \times E \times E \multimap E$ *is an upper semicontinuous mapping with compact, convex values. Assume that* $K \subset E$ *is an open, convex set containing* 0. *Moreover, let conditions* (5_i), (5_{ii}), (5_{iii}) *from Theorem* 5.1 *be satisfied.*

Furthermore, let there exist a function $V \in C^1(E,\mathbb{R})$ *with a locally Lipschitz Frechét derivative* $\dot{V}$ *satisfying* (H1) *and* (H2). *Moreover, let, for all* $x \in \partial K$, $t \in (0,T)$, $\lambda \in (0,1)$ *and* $y \in E$ *satisfying* (11), *condition* (12) *hold for all* $w \in \lambda F(t,x,y)$. *Then the Dirichlet b.v.p.* (1) *admits a solution whose values are located in* $\overline{K}$.

Proof The verification is quite analogous as in Theorem 5.1 when just replacing the usage of Proposition 4.1 by Proposition 4.2. □

6 Illustrative example

Example 6.1 Let $E = H$ be a Hilbert space and let us consider the Dirichlet b.v.p.

$$\left.\begin{array}{ll} \ddot{x}(t) \in F_1(t,x(t),\dot{x}(t)) + F_2(t,x(t),\dot{x}(t)), & \text{for a.a. } t \in [0,T], \\ x(0) = x(T) = 0, & \end{array}\right\} \tag{26}$$

where

(i) $F_1\colon [0,T] \times H \times H \multimap H$ is an upper-Carathéodory multivalued mapping and $F_1(t,\cdot,\cdot)\colon H \times H \multimap H$ is completely continuous for a.a. $t \in [0,T]$ such that

$$\|F_1(t,x,y)\| \leq \nu_1(t,D_0,D_1) \in L^1([0,T],[0,\infty))$$

for a.a. $t \in [0,T]$ and all $x, y \in H$ with $\|x\| \le D_0$, $\|y\| \le D_1$,

(ii) $F_2\colon [0,T] \times H \times H \multimap H$ is a Carathéodory multivalued mapping such that

$$\|F_2(t,0,0)\| \le \nu_2(t) \in L^1([0,T],[0,\infty)) \quad \text{for a.a. } t \in [0,T],$$

and $F_2(t,\cdot,\cdot)\colon H \times H \multimap H$ is Lipschitzian for a.a. $t \in [0,T]$ with the Lipschitz constant

$$L < \frac{2}{T(T+4)}.$$

Moreover, suppose that

(iii) there exist $R > 0$ and $\varepsilon > 0$ such that, for all $x \in H$ with $R - \varepsilon < \|x\| \le R$, $t \in (0,T)$, $y \in H$, $\lambda \in (0,1)$ and $w \in \lambda(F_1(t,x,y) + F_2(t,x,y))$, we have

$$\langle y, y\rangle + \langle x, w\rangle > 0.$$

Then the Dirichlet problem (26) admits, according to Theorem 5.1, a solution $x(\cdot)$ such that $\|x(t)\| \le R$ for all $t \in [0,T]$.

Indeed. The properties of F_2 guarantee that F_2 satisfies the inequality (*cf.*, *e.g.*, [20])

$$\gamma\big(F_2(t,\Omega_1 \times \Omega_2)\big) \le L\big(\gamma(\Omega_1) + \gamma(\Omega_2)\big) \tag{27}$$

for a.a. $t \in [0,T]$ and every bounded $\Omega_1, \Omega_2 \subset H$, where γ stands for the Hausdorff measure of noncompactness in H.

Since $F_1(t,\cdot,\cdot)$ is completely continuous and thanks to the algebraic semi-additivity of γ, inequality (27) can be rewritten into

$$\gamma\big(F_1(t,\Omega_1 \times \Omega_2) + F_2(t,\Omega_1 \times \Omega_2)\big) \le L\big(\gamma(\Omega_1) + \gamma(\Omega_2)\big)$$

for a.a. $t \in [0,T]$ and every bounded $\Omega_1, \Omega_2 \subset H$, *i.e.*, ($5_i$), for $g := L < \frac{2}{T(T+4)}$ (*cf.* (5_{iii})).

Moreover, according to the Lipschitzianity of F_2, the following inequalities take place:

$$d_H\big(F_2(t,x,y),0\big) \le d_H\big(F_2(t,x,y),F_2(t,0,0)\big) + d_H\big(F_2(t,0,0),0\big) \le L\big(\|x\| + \|y\|\big) + \nu_2(t)$$

for a.a. $t \in [0,T]$ and all $x, y \in H$.

Thus, for $\|x\| \le D_0$, $\|y\| \le D_1$, we arrive at

$$\big\|F_1(t,x,y) + F_2(t,x,y)\big\| \le L(D_0 + D_1) + \nu_1(t,D_0,D_1) + \nu_2(t) := \nu_\Omega(t) \in L^1\big([0,T],[0,\infty)\big),$$

i.e., (18) in (5_{ii}).

Finally, in view of Remark 4.1, we can define the bounding function $V \in C^2(H,\mathbb{R})$ by the formula

$$V(x) := \frac{1}{2}\big(\langle x, x\rangle - R^2\big)$$

and the bound set K as $K := \{x \in H \mid \|x\| < R\}$ in order to get a claim.

Remark 6.1 Consider again (26) in a Hilbert space H, but let this time F_1, F_2 be globally u.s.c. mappings with compact, convex values ($\Rightarrow F_2([0,T],0,0)$ is compact (*cf.*, *e.g.*, [15, Proposition I.3.20]) and, in particular, bounded) such that

(i) $F_1(t,\cdot,\cdot)\colon H\times H \multimap H$ is a completely continuous mapping for a.a. $t\in[0,T]$ such that

$$\|F_1(t,x,y)\| \le \nu_1(t,D_0,D_1) \in L^1\bigl([0,T],[0,\infty)\bigr)$$

for a.a. $t\in[0,T]$ and all $x,y\in H$ with $\|x\|\le D_0$, $\|y\|\le D_1$.

(ii) $F_2(t,\cdot,\cdot)\colon H\times H \multimap H$ is a Lipschitzian mapping for a.a. $t\in[0,T]$ with the Lipschitz constant

$$L < \frac{2}{T(T+4)}.$$

(iii^{usc}) There exists $R>0$ such that, for all $x\in H$ with $\|x\|=R$, $t\in(0,T)$, $y\in H$ satisfying $\langle x,y\rangle = 0$, $\lambda\in(0,1)$ and $w\in\lambda(F_1(t,x,y)+F_2(t,x,y))$, we have

$$\langle y,y\rangle + \langle x,w\rangle > 0.$$

Applying now Theorem 5.2, by the analogous arguments as in Example 6.1, the Dirichlet problem (26) admits a solution $x(\cdot)$ such that $\|x(t)\|\le R$ for all $t\in[0,T]$.

Remark 6.2 Since the solution derivative $\dot{x}(\cdot)$ takes the form

$$\dot{x}(t) \in \int_0^T \frac{\partial}{\partial t}G(t,s)\bigl[F_1\bigl(s,x(s),\dot{x}(s)\bigr) + F_2\bigl(s,x(s),\dot{x}(s)\bigr)\bigr]\,ds,$$

where

$$\frac{\partial}{\partial t}G(t,s) = \begin{cases} \frac{(s-T)}{T} & \text{for all } 0\le t\le s\le T,\\ \frac{s}{T} & \text{for all } 0\le s\le t\le T,\end{cases}$$

and so $|\frac{\partial}{\partial t}G(t,s)|\le 1$ for all $t,s\in[0,T]$, we obtain (under the above assumptions) the implicit inequality

$$D_1 \le \frac{1}{1-LT}\left[\int_0^T \nu_1(t,R,D_1)\,dt + \int_0^T \nu_2(t)\,dt + LRT\right] \quad \text{for all } t\in[0,T],$$

for $D_1 := \max_{t\in[0,T]}\|\dot{x}(t)\|$.

Thus, for $F_1(t,x,y)\equiv F_1(t,x)$, we have $\nu_1(t,R,D_1)\equiv\nu_1(t,R)$, and subsequently

$$\|\dot{x}(t)\| \le \frac{1}{1-LT}\left[\int_0^T \nu_1(t,R)\,dt + \int_0^T \nu_2(t)\,dt + LRT\right] \quad \text{for all } t\in[0,T].$$

Similarly, if $F_1\colon[0,T]\times H\times H\mapsto H$ is compact, then

$$\int_0^T \nu_1(t,R,D_1)\,dt \le C_1 T$$

holds with a suitable constant $C_1 \geq \|F_1(t,x,y)\|$, and the following estimate holds:

$$\|\dot{x}(t)\| \leq \frac{1}{1-LT}\left[C_1 T + LRT + \int_0^T \nu_2(t)\,dt\right] \quad \text{for all } t \in [0,T].$$

Because of the Dirichlet boundary conditions $x(0) = x(T) = 0$ for $H = R$, there exists a zero point $t_0 \in [0,T]$ of $\dot{x}(\cdot)$, *i.e.*, $\dot{x}(t_0) = 0$, by which the same estimates can be also obtained without an explicit usage of the Green function above. Otherwise, it is not so easy to obtain such estimates, because Rolle's theorem fails in general.

For obtaining the estimation of the solution derivative $\dot{x}(\cdot)$ in a Hilbert space H, one can also apply, under natural assumptions, the p-Nagumo condition derived in [7].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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Endnote
[a] The m.n.c. $\mathrm{mod}_C(\Omega)$ is monotone, nonsingular and algebraically subadditive on $C([0,T],E)$ (*cf.*, *e.g.*, [20]) and it is equal to zero if and only if all the elements $x \in \Omega$ are equi-continuous.

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Scorza-Dragoni approach to Dirichlet problem in Banach spaces

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Abstract

Hartman-type conditions are presented for the solvability of a multivalued Dirichlet problem in a Banach space by means of topological degree arguments, bounding functions, and a Scorza-Dragoni approximation technique. The required transversality conditions are strictly localized on the boundaries of given bound sets. The main existence and localization result is applied to a partial integro-differential equation involving possible discontinuities in state variables. Two illustrative examples are supplied. The comparison with classical single-valued results in this field is also made.

MSC: 34A60; 34B15; 47H04

Keywords: Dirichlet problem; Scorza-Dragoni-type technique; strictly localized bounding functions; solutions in a given set; condensing multivalued operators

1 Introduction

In this paper, we will establish sufficient conditions for the existence and localization of strong solutions to a multivalued Dirichlet problem in a Banach space via degree arguments combined with a bound sets technique. More precisely, Hartman-type conditions (*cf.* [1]), *i.e.* sign conditions w.r.t. the first state variable and growth conditions w.r.t. the second state variable, will be presented, provided the right-hand side is a multivalued upper-Carathéodory mapping which is γ-regular w.r.t. the Hausdorff measure of non-compactness γ.

The main aim will be two-fold: (i) strict localization of sign conditions on the boundaries of bound sets by means of a technique originated by Scorza-Dragoni [2], and (ii) the application of the obtained abstract result (see Theorem 3.1 below) to an integro-differential equation involving possible discontinuities in a state variable. The first aim allows us, under some additional restrictions, to extend our earlier results obtained for globally upper semicontinuous right-hand sides and partly improve those for upper-Carathéodory right-hand sides (see [3]). As we shall see, the latter aim justifies such an abstract setting, because the problem can be transformed into the form of a differential inclusion in a Hilbert L^2-space. Roughly speaking, problems of this type naturally require such an abstract setting. In order to understand in a deeper way what we did and why, let us briefly recall classical results in this field and some of their extensions.

Hence, consider firstly the Dirichlet problem in the simplest vector form:

$$\left.\begin{array}{ll}\ddot{x}(t) = f(t,x(t),\dot{x}(t)), & t \in [0,1],\\ x(1) = x(0) = 0, & \end{array}\right\} \tag{1}$$

where $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is, for the sake of simplicity allowing the comparison of the related results, a continuous function.

The first existence results, for a bounded f in (1), are due to Scorza-Dragoni [4, 5]. Let us note that his name in the title is nevertheless related to the technique developed in [2] rather than to the existence results in [4, 5].

It is well known (see *e.g.* [3, 6–13]) that the problem (1) is solvable on various levels of generality provided:

($\mathrm{i_{sign}}$) $\exists R > 0$ such that $\langle f(t,x,y), x\rangle > 0$, for $(t,x,y) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$ with $\|x\| = R$,

($\mathrm{ii_{growth}}$) $\exists C_1 \geq 0, C_2 \geq 0$ such that $C_1R < 1$ and $\|f(t,x,y)\| \leq C_1\|y\|^2 + C_2$, for $(t,x,y) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$ with $\|x\| \leq R$.

Let us note that the existence of the same constant $R > 0$ in ($\mathrm{i_{sign}}$) and ($\mathrm{ii_{growth}}$) can be assumed either explicitly as in [6, 7, 9, 11, 13] or it follows from the assumptions as those in [8, 10, 12].

(1) Hartmann [9] (*cf.* also [1]) generalized both conditions as follows:

($\mathrm{i_H}$) $\exists R > 0$ such that $\langle f(t,x,y), x\rangle + \|y\|^2 > 0$, for $t \in [0,1]$ and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\|x\| = R$ and $\langle x, y\rangle = 0$,

($\mathrm{i_H}$) the well-known Bernstein-Nagumo-Hartman condition (for its definition and more details, see *e.g.* [1, 14]).

Let us note that the strict inequality in ($\mathrm{i_H}$) can be replaced by a non-strict one (see *e.g.* [1, Chapter XII,II,5], [11, Corollary 6.2]).

(2) Lasota and Yorke [10] improved condition ($\mathrm{i_{sign}}$) with suitable constants $K_1 \geq 0$ and $K_2 > 0$ in the following way:

($\mathrm{i_{LY}}$) $\langle f(t,x,y), x\rangle + \|y\|^2 \geq -K_1(1 + \|x\| + \langle x,y\rangle) + K_2\|y\|$,

but for $t \in [0,1]$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, and replaced ($\mathrm{ii_{growth}}$) by the Bernstein-Nagumo-Hartman condition.

Since ($\mathrm{i_{LY}}$) implies (*cf.* [10]) the existence of a constant $K \geq 0$ such that

$$\bigl\langle f(t,x,y), x\bigr\rangle + \|y\|^2 \geq -K\bigl(1 + \|x\| + \bigl|\langle x,y\rangle\bigr|\bigr),$$

for $(t,x,y) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$, the sign condition ($\mathrm{i_{LY}}$) is obviously more liberal than ($\mathrm{i_{sign}}$) as well as than ($\mathrm{i_H}$), on the intersection of their domains.

If $K_1 > 0$ in ($\mathrm{i_{LY}}$), then constant K_2 can be even equal to zero, *i.e.* $K_2 = 0$, in ($\mathrm{i_{LY}}$) (see *e.g.* [7, Corollary V.26 on p.74]). Moreover, the related Bernstein-Nagumo-Hartman condition can only hold for x in a suitable convex, closed, bounded subset of $\mathbb{R}^n$ (see again *e.g.* [7]).

(3) Following the ideas of Mawhin in [7, 11, 12], Amster and Haddad [6] demonstrated that an open, bounded subset of $\mathbb{R}^n$, say $D \subset \mathbb{R}^n$, need not be convex, provided it has a C^2-boundary ∂D such that condition ($\mathrm{i_H}$) can be generalized as follows:

($\mathrm{i_{AH}}$) $\langle f(t,x,y), n_x\rangle \geq I_x(y)$, $(t,x,y) \in [0,1] \times T\partial D \times \mathbb{R}^n$, with $\langle n_x, y\rangle = 0$,

where n_x is the outer-pointing normal unit vector field, $T\partial D$ denotes the tangent vector bundle and $I_x(y)$ stands for the second fundamental form of the hypersurface.

Since for the ball $D := B(0,R)$, $R > 0$, we can have

$$I_x(y) = -\frac{\|y\|^2}{R} \quad \text{and} \quad n_x = \frac{x}{R},$$

condition (i_{AH}) is obviously more general than the original Hartman condition (i_{H}).

Nevertheless, the growth condition takes there only the form ($\mathrm{ii}_{\mathrm{growth}}$), namely with $\|x\| \le R$ replaced by $x \in \overline{D}$, where R denotes, this time, the radius of D.

For a convex, open, bounded subset $D \subset \mathbb{R}^n$, the particular case of (i_{AH}) can read as follows:

($\mathrm{i}_{\mathrm{conv}}$) $\langle f(t,x,y), n_x \rangle > 0$, for $(t,x,y) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$ with $x \in \partial D$ and $\langle n_x, y \rangle = 0$,

which is another well-known generalization of ($\mathrm{i}_{\mathrm{sign}}$).

(4) In a Hilbert space H, for a completely continuous mapping f, Mawhin [12] has shown that, for real constants a, b, c such that $a + b < 1$, condition ($\mathrm{i}_{\mathrm{sign}}$) can be replaced in particular by

(i_{M}) $\langle f(t,x,y), x \rangle \ge -(a\|x\|^2 + b\|x\|\|y\| + c\|x\|)$, $(t,x,y) \in [0,1] \times H \times H$,

and ($\mathrm{ii}_{\mathrm{growth}}$) by an appropriate version of the Bernstein-Nagumo-Hartman condition.

(5) In a Banach space E, Schmitt and Thompson [13] improved, for a completely continuous mapping f, condition ($\mathrm{i}_{\mathrm{conv}}$) in the sense that the strict inequality in ($\mathrm{i}_{\mathrm{conv}}$) can be replaced by a non-strict one. More concretely, if there exists a convex, open, bounded subset $D \subset E$ of E with $0 \in D$ such that

(i_{ST}) $\langle f(t,x,y), n_x \rangle \ge 0$, for $(t,x,y) \in [0,1] \times E \times E$, with $x \in \partial D$ and $\langle n_x, y \rangle = 0$,

where $\langle \cdot, \cdot \rangle$ denotes this time the pairing between E and its dual E', jointly with the appropriate Bernstein-Nagumo-Hartman condition, then the problem (1) admits a solution whose values are located in $\overline{D}$ (see [13, Theorem 4.1]).

In the Carathéodory case of $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ in (1), for instance, the strict inequality in condition ($\mathrm{i}_{\mathrm{sign}}$) can be replaced, according to [8, Theorem 6.1], by a non-strict one and the constants C_1, C_2 can be replaced without the requirement $C_1 R < 1$, but globally in $[0,1] \times \mathbb{R}^n \times \mathbb{R}^n$, by functions $c_1(t,x)$, $c_2(t,x)$ which are bounded on bounded sets. Moreover, system (1) can be additively perturbed, for the same goal, by another Carathéodory function which is sublinear in both states variables x and y.

On the other hand, the Carathéodory case brings about some obstructions in a strict localization of sign conditions on the boundaries of bound sets (see *e.g.* [3, 15]). The same is also true for other boundary value problems (for Floquet problems, see *e.g.* [16–18]). Therefore, there naturally exist some extensions of classical results in this way. Further extensions concern problems in abstract spaces, functional problems, multivalued problems, *etc.* For the panorama of results in abstract spaces, see *e.g.* [19], where multivalued problems are also considered.

Nevertheless, let us note that in abstract spaces, it is extremely difficult (if not impossible) to avoid the convexity of given bound sets, provided the degree arguments are applied for non-compact maps (for more details, see [20]).

In this light, we would like to modify in the present paper the Hartman-type conditions ($\mathrm{i}_{\mathrm{sign}}$), ($\mathrm{ii}_{\mathrm{growth}}$) at least in the following way:

- the given space E to be Banach (or, more practically, Hilbert),
- the right-hand side to be a multivalued upper-Carathéodory mapping F which is γ-regular w.r.t. $(x, y) \in E \times E$ and either globally measurable or globally quasi-compact,
- the inequality in $(\mathrm{i}_{\mathrm{sign}})$ to hold w.r.t. x strictly on the boundary ∂D of a convex, bounded subset $D \subset E$ (or, more practically, of the ball $B(0, R) \subset E$),
- condition $(\mathrm{ii}_{\mathrm{growth}})$ to be replaced by a suitable growth condition which would allow us reasonable applications (the usage of the Bernstein-Nagumo-Hartman-type condition will be employed in this context by ourselves elsewhere).

Hence, let E be a separable Banach space (with the norm $\|\cdot\|$) satisfying the Radon-Nikodym property (*e.g.* reflexivity, see *e.g.* [21, pp.694-695]) and let us consider the Dirichlet boundary value problem (b.v.p.)

$$\left.\begin{aligned} &\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0, T], \\ &x(T) = x(0) = 0, \end{aligned}\right\} \tag{2}$$

where $F : [0, T] \times E \times E \multimap E$ is an upper-Carathéodory multivalued mapping.

Let us note that in the entire paper all derivatives will be always understood in the sense of Fréchet and, by the measurability, we mean the one with respect to the Lebesgue σ-algebra in $[0, T]$ and the Borel σ-algebra in E.

The notion of a solution will be understood in a strong (*i.e.* Carathéodory) sense. Namely, by a *solution* of problem (2) we mean a function $x : [0, T] \to E$ whose first derivative $\dot{x}(\cdot)$ is absolutely continuous and satisfies (2), for almost all $t \in [0, T]$.

The solution of the b.v.p. (2) will be obtained as the limit of a sequence of solutions of approximating problems that we construct by means of a Scorza-Dragoni-type result developed in [22]. The approximating problems will be treated by means of the continuation principle developed in [19].

2 Preliminaries

Let E be as above and $[0, T] \subset \mathbb{R}$ be a closed interval. By the symbol $L^1([0, T], E)$, we shall mean the set of all Bochner integrable functions $x : [0, T] \to E$. For the definition and properties of Bochner integrals, see *e.g.* [21, pp.693-701]. The symbol $AC^1([0, T], E)$ will be reserved for the set of functions $x : [0, T] \to E$ whose first derivative $\dot{x}(\cdot)$ is absolutely continuous. Then $\ddot{x} \in L^1([0, T], E)$ and the fundamental theorem of calculus (the Newton-Leibniz formula) holds (see *e.g.* [21, pp.695-696], [23, pp.243-244]). In the sequel, we shall always consider $AC^1([0, T], E)$ as a subspace of the Banach space $C^1([0, T], E)$ and by the symbol $\mathcal{L}(E)$ we shall mean the Banach space of all linear, bounded transformations $L : E \to E$ endowed with the sup-norm.

Given $C \subset E$ and $\varepsilon > 0$, the symbol $B(C, \varepsilon)$ will denote, as usually, the set $C + \varepsilon B$, where B is the open unit ball in E centered at 0, *i.e.* $B = \{x \in E \mid \|x\| < 1\}$. In what follows, the symbol μ will denote the Lebesgue measure on $\mathbb{R}$.

Let E' be the Banach space dual to E and let us denote by $\langle \cdot, \cdot \rangle$ the pairing (the duality relation) between E and E', *i.e.*, for all $\Phi \in E'$ and $x \in E$, we put $\Phi(x) =: \langle \Phi, x \rangle$.

We recall also the Pettis measurability theorem which will be used in Section 4 and which we state here in the form of proposition.

Proposition 2.1 [24, p.278] *Let (X, Σ) be a measure space, E be a separable Banach space. Then $f : X \to E$ is measurable if and only if for every $e \in E'$ the function $e \circ f : X \to \mathbb{R}$ is measurable with respect to Σ and the Borel σ-algebra in $\mathbb{R}$.*

We shall also need the following definitions and notions from multivalued analysis. Let X, Y be two metric spaces. We say that F is a *multivalued mapping* from X to Y (written $F : X \multimap Y$) if, for every $x \in X$, a non-empty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, the subset of $X \times Y$, defined by $\Gamma_F := \{(x, y) \in X \times Y \mid y \in F(x)\}$.

A multivalued mapping $F : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open subset $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

Let $J \subset \mathbb{R}$ be a compact interval. A mapping $F : J \multimap Y$, where Y is a separable metric space, is called *measurable* if, for each open subset $U \subset Y$, the set $\{t \in J \mid F(t) \subset U\}$ belongs to a σ-algebra of subsets of J.

A multivalued mapping $F : X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x \in X} F(x)$ is contained in a compact subset of Y and it is called *quasi-compact* if it maps compact sets onto relatively compact sets.

Let $J \subset \mathbb{R}$ be a given compact interval. A multivalued mapping $F : J \times X \multimap Y$, where Y is a separable Banach space, is called an *upper-Carathéodory mapping* if the map $F(\cdot, x) : J \multimap Y$ is measurable, for all $x \in X$, the map $F(t, \cdot) : X \multimap Y$ is u.s.c., for almost all $t \in J$, and the set $F(t, x)$ is compact and convex, for all $(t, x) \in J \times X$.

The technique that will be used for proving the existence and localization result consists in constructing a sequence of approximating problems. This construction will be made on the basis of the Scorza-Dragoni-type result developed in [22] (*cf.* also [25]).

For more details concerning multivalued analysis, see *e.g.* [23, 26, 27].

Definition 2.1 An upper-Carathéodory mapping $F : [0, T] \times X \times X \multimap X$ is said to have the *Scorza-Dragoni property* if there exists a multivalued mapping $F_0 : [0, T] \times X \times X \multimap X \cup \{\emptyset\}$ with compact, convex values having the following properties:

(i) $F_0(t, x, y) \subset F(t, x, y)$, for all $(t, x, y) \in [0, T] \times X \times X$,

(ii) if $u, v : [0, T] \to X$ are measurable functions with $v(t) \in F(t, u(t), \dot{u}(t))$, for a.a. $t \in [0, T]$, then also $v(t) \in F_0(t, u(t), \dot{u}(t))$, for a.a. $t \in [0, T]$,

(iii) for every $\varepsilon > 0$, there exists a closed $I_\varepsilon \subset [0, T]$ such that $\mu([0, T] \setminus I_\varepsilon) < \varepsilon$, $F_0(t, x, y) \neq \emptyset$, for all $(t, x, y) \in I_\varepsilon \times X \times X$, and F_0 is u.s.c. on $I_\varepsilon \times X \times X$.

The following two propositions are crucial in our investigation. The first one is almost a direct consequence of the main result in [22] (*cf.* [25] and [16, Proposition 2]). The second one allows us to construct a sequence of approximating problems of (2).

Proposition 2.2 *Let E be a separable Banach space and $F : [0, T] \times E \times E \multimap E$ be an upper-Carathéodory mapping. If F is globally measurable or quasi-compact, then F has the Scorza-Dragoni property.*

Proposition 2.3 (*cf.* [18, Theorem 2.2]) *Let E be a Banach space and $K \subset E$ a non-empty, open, convex, bounded set such that $0 \in K$. Moreover, let $\varepsilon > 0$ and $V : E \to \mathbb{R}$ be a Fréchet differentiable function with $\dot{V}$ Lipschitzian in $\overline{B(\partial K, \varepsilon)}$ satisfying*

(H1) $V|_{\partial K} = 0$,

(H2) $V(x) \leq 0$, *for all* $x \in \overline{K}$,

(H3) $\|\dot{V}(x)\| \geq \delta$, *for all* $x \in \partial K$, *where* $\delta > 0$ *is given.*

Then there exist $k \in (0, \varepsilon]$ *and a bounded Lipschitzian function* $\phi : \overline{B(\partial K, k)} \to E$ *such that* $\langle \dot{V}_x, \phi(x) \rangle = 1$, *for every* $x \in \overline{B(\partial K, k)}$.

Remark 2.1 Let us note that the function $x \to \phi(x)\|\dot{V}_x\|$, where ϕ and $\dot{V}_x$ are the same as in Proposition 2.3, is Lipschitzian and bounded in $\overline{B(\partial K, k)}$. The symbol $\dot{V}_x$ denotes as usually the first Fréchet derivative of V at x.

Example 2.1 If V satisfies all the assumptions of Proposition 2.3, then it is easy to prove the existence of $\sigma \in (0, \varepsilon]$ such that $\|\dot{V}_x\| \geq \frac{\delta}{2}$, for all $x \in \overline{B(\partial K, \sigma)}$. Consequently, when E is an arbitrary Hilbert space, we can define $\phi : \overline{B(\partial K, \sigma)} \to E$ by the formula

$$\phi(x) := \frac{\nabla V(x)}{\|\nabla V(x)\|^2}$$

which satisfies all the properties mentioned in Proposition 2.3.

Definition 2.2 Let N be a partially ordered set, E be a Banach space and let $P(E)$ denote the family of all non-empty bounded subsets of E. A function $\beta : P(E) \to N$ is called a *measure of non-compactness* (m.n.c.) in E if $\beta(\overline{\text{co}\,\Omega}) = \beta(\Omega)$, for all $\Omega \in P(E)$, where $\overline{\text{co}\,\Omega}$ denotes the closed convex hull of Ω.

A m.n.c. β is called:

(i) *monotone* if $\beta(\Omega_1) \leq \beta(\Omega_2)$, for all $\Omega_1 \subset \Omega_2 \subset E$,
(ii) *non-singular* if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, for all $x \in E$ and $\Omega \subset E$.

If N is a cone in a Banach space, then a m.n.c. β is called:

(iii) *semi-homogeneous* if $\beta(t\Omega) = |t|\beta(\Omega)$, for every $t \in \mathbb{R}$ and every $\Omega \subset E$,
(iv) *regular* when $\beta(\Omega) = 0$ if and only if Ω is relatively compact,
(v) *algebraically subadditive* if $\gamma(\Omega_1 + \Omega_2) \leq \gamma(\Omega_1) + \gamma(\Omega_2)$, for all $\Omega_1, \Omega_2 \subset E$.

The typical example of an m.n.c. is the *Hausdorff measure of non-compactness* γ defined, for all $\Omega \subset E$ by

$$\gamma(\Omega) := \inf\left\{\varepsilon > 0 : \exists n \geq 1\ \exists x_1, \ldots, x_n \in E : \Omega \subset \bigcup_{i=1}^{n} B\big(\{x_i\}, \varepsilon\big)\right\}.$$

The Hausdorff m.n.c. is monotone, non-singular, semi-homogeneous and regular. Moreover, if $M \in \mathcal{L}(E)$ and $\Omega \subset E$, then (see, *e.g.*, [27])

$$\gamma(M\Omega) \leq \|M\|_{\mathcal{L}(E)}\gamma(\Omega). \tag{3}$$

Let E be a separable Banach space and $\{f_n\}_n \subset L^1([0,T], E)$ be such that $\|f_n(t)\| \leq \alpha(t)$, $\gamma(\{f_n(t)\}_n) \leq c(t)$, for a.a. $t \in [0,T]$, all $n \in \mathbb{N}$ and suitable $\alpha, c \in L^1([0,T], \mathbb{R})$, then (*cf.* [27])

$$\gamma\left(\left\{\int_0^T f_n(t)\,dt\right\}_n\right) \leq \int_0^T c(t)\,dt. \tag{4}$$

Moreover, if $h : E \multimap E$ is L-Lipschitzian, then

$$\gamma\big(h(\Omega)\big) \leq L\gamma(\Omega), \tag{5}$$

for all bounded $\Omega \subset E$.

Furthermore, for all subsets Ω of E (see *e.g.* [17]),

$$\gamma\Big(\bigcup_{\lambda\in[0,1]} \lambda\Omega\Big) = \gamma(\Omega). \tag{6}$$

Let us now introduce the function

$$\alpha(\Omega) := \max_{\{w_n\}_n\subset\Omega}\Big(\sup_{t\in[0,T]}\big[\gamma\big(\{w_n(t)\}_n\big) + \gamma\big(\{\dot{w}_n(t)\}_n\big)\big],$$
$$\operatorname{mod}_C\big(\{w_n\}_n\big) + \operatorname{mod}_C\big(\{\dot{w}_n\}_n\big)\Big), \tag{7}$$

defined on the bounded $\Omega \subset C^1([0,T],E)$, where the ordering is induced by the positive cone in $\mathbb{R}^2$ and where $\operatorname{mod}_C(\Omega)$ denotes the modulus of continuity of a subset $\Omega \subset C([0,T],E)$.[a] It was proved in [19] that the function α given by (7) is an m.n.c. in $C^1([0,T],E)$ that is monotone, non-singular and regular.

Definition 2.3 Let E be a Banach space and $X \subset E$. A multivalued mapping $F: X \multimap E$ with compact values is called *condensing with respect to an m.n.c.* β (shortly, β*-condensing*) if, for every bounded $\Omega \subset X$ such that $\beta(F(\Omega)) \geq \beta(\Omega)$, we see that Ω is relatively compact.

A family of mappings $G: X \times [0,1] \multimap E$ with compact values is called β*-condensing* if, for every bounded $\Omega \subset X$ such that $\beta(G(\Omega \times [0,1])) \geq \beta(\Omega)$, we see that Ω is relatively compact.

The proof of the main result (*cf.* Theorem 3.1 below) will be based on the following slight modification of the continuation principle developed in [19]. Since the proof of this modified version differs from the one in [19] only slightly in technical details, we omit it here.

Proposition 2.4 *Let us consider the b.v.p.*

$$\left.\begin{aligned} &\ddot{x}(t) \in \varphi(t,x(t),\dot{x}(t)), \quad \text{for a.a. } t \in [0,T],\\ &x \in S, \end{aligned}\right\} \tag{8}$$

where $\varphi: [0,T] \times E \times E \multimap E$ *is an upper-Carathéodory mapping and* $S \subset AC^1([0,T],E)$. *Let* $H: [0,T] \times E \times E \times E \times E \times [0,1] \multimap E$ *be an upper-Carathéodory mapping such that*

$$H(t,c,d,c,d,1) \subset \varphi(t,c,d), \quad \text{for all } (t,c,d) \in [0,T] \times E \times E.$$

Moreover, assume that the following conditions hold:

(i) *There exist a closed set* $S_1 \subset S$ *and a closed, convex set* $Q \subset C^1([0,T],E)$ *with a non-empty interior* Int Q *such that each associated problem*

$$P(q,\lambda) \quad \left.\begin{aligned} &\ddot{x}(t) \in H(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda), \quad \text{for a.a. } t \in [0,T],\\ &x \in S_1, \end{aligned}\right\}$$

where $q \in Q$ *and* $\lambda \in [0,1]$, *has a non-empty, convex set of solutions* (*denoted by* $\mathfrak{T}(q,\lambda)$).

(ii) *For every non-empty, bounded set* $\Omega \subset E \times E \times E \times E$, *there exists* $\nu_\Omega \in L^1([0,T],[0,\infty))$ *such that*

$$\|H(t,x,y,u,v,\lambda)\| \le \nu_\Omega(t),$$

for a.a. $t \in [0,T]$ *and all* $(x,y,u,v) \in \Omega$ *and* $\lambda \in [0,1]$.

(iii) *The solution mapping* $\mathfrak{T}$ *is quasi-compact and* μ*-condensing with respect to a monotone and non-singular m.n.c.* μ *defined on* $C^1([0,T],E)$.

(iv) *For each* $q \in Q$, *the set of solutions of problem* $P(q,0)$ *is a subset of* $\operatorname{Int} Q$, *i.e.* $\mathfrak{T}(q,0) \subset \operatorname{Int} Q$, *for all* $q \in Q$.

(v) *For each* $\lambda \in (0,1)$, *the solution mapping* $\mathfrak{T}(\cdot,\lambda)$ *has no fixed points on the boundary* ∂Q *of* Q.

Then the b.v.p. (8) *has a solution in* Q.

3 Main result

Combining the foregoing continuation principle with the Scorza-Dragoni-type technique (*cf.* Proposition 2.2), we are ready to state the main result of the paper concerning the solvability and localization of a solution of the multivalued Dirichlet problem (2).

Theorem 3.1 *Consider the Dirichlet b.v.p.* (2). *Suppose that* $F : [0,T] \times E \times E \multimap E$ *is an upper-Carathéodory mapping which is either globally measurable or quasi-compact. Furthermore, let* $K \subset E$ *be a non-empty, open, convex, bounded subset containing* 0 *of a separable Banach space* E *satisfying the Radon-Nikodym property. Let the following conditions* (2_{i})-(2_{iii}) *be satisfied*:

(2_{i}) $\gamma(F(t,\Omega_1 \times \Omega_2)) \le g(t)(\gamma(\Omega_1) + \gamma(\Omega_2))$, *for a.a.* $t \in [0,T]$ *and each* $\Omega_1 \subset \overline{K}$, *and each bounded* $\Omega_2 \subset E$, *where* $g \in L^1([0,T],[0,\infty))$ *and* γ *is the Hausdorff m.n.c. in* E.

(2_{ii}) *For every non-empty, bounded* $\Omega \subset E$, *there exists* $\nu_\Omega \in L^1([0,T],[0,\infty))$ *such that*

$$\|F(t,x,y)\| \le \nu_\Omega(t), \tag{9}$$

for a.a. $t \in [0,T]$ *and all* $(x,y) \in \Omega \times E$.

(2_{iii})

$$(T+4)\|g\|_{L^1([0,T],\mathbb{R})} < 4.$$

Furthermore, let there exist $\varepsilon > 0$ *and a function* $V \in C^2(E,\mathbb{R})$, *i.e. a twice continuously differentiable function in the sense of Fréchet, satisfying* (H1)-(H3) (*cf. Proposition* 2.3) *with Fréchet derivative* $\dot{V}$ *Lipschitzian in* $\overline{B(\partial K,\varepsilon)}$.[b] *Let there still exist* $h > 0$ *such that*

$$\langle \ddot{V}_x(v), v\rangle \ge 0, \quad \text{for all } x \in B(\partial K,h), v \in E, \tag{10}$$

where $\ddot{V}_x(v)$ *denotes the second Fréchet derivative of* V *at* x *in the direction* $(v,v) \in E \times E$. *Finally, let*

$$\langle \dot{V}_x, w\rangle > 0, \tag{11}$$

for a.a. $t \in (0,T)$ *and all* $x \in \partial K$, $v \in E$, *and* $w \in F(t,x,v)$.

Then the Dirichlet b.v.p. (2) *admits a solution whose values are located in* $\overline{K}$. *If, moreover,* $0 \notin F(t,0,0)$, *for a.a.* $t \in [0,T]$, *then the obtained solution is non-trivial.*

Proof Since the proof of this result is rather technical, it will be divided into several steps. At first, let us define the sequence of approximating problems. For this purpose, let k be as in Proposition 2.3 and consider a continuous function $\tau : E \to [0,1]$ such that $\tau(x) = 0$, for all $x \in E \setminus B(\partial K, k)$, and $\tau(x) = 1$, for all $x \in \overline{B(\partial K, \frac{k}{2})}$. According to Proposition 2.3 (see also Remark 2.1), the function $\hat{\phi} : E \to E$, where

$$\hat{\phi}(x) = \begin{cases} \tau(x) \cdot \phi(x) \cdot \|\dot{V}_x\|, & \text{for all } x \in \overline{B(\partial K, k)}, \\ 0, & \text{for all } x \in E \setminus \overline{B(\partial K, k)}, \end{cases}$$

is well defined, continuous and bounded.

Since the mapping $(t,x,y) \multimap F(t,x,y)$ has, according to Proposition 2.2, the Scorza-Dragoni property, we are able to find a decreasing sequence $\{J_m\}_m$ of subsets of $[0,T]$ and a mapping $F_0 : [0,T] \times E \times E \multimap E \cup \{\emptyset\}$ with compact, convex values such that, for all $m \in \mathbb{N}$,

- $\mu(J_m) < \frac{1}{m}$,
- $[0,T] \setminus J_m$ is closed,
- $(t,x,y) \multimap F_0(t,x,y)$ is u.s.c. on $[0,T] \setminus J_m \times E \times E$,
- $\nu_{\overline{K}}$ is continuous in $[0,T] \setminus J_m$ (*cf. e.g.* [2]).

If we put $J = \bigcap_{m=1}^{\infty} J_m$, then $\mu(J) = 0$, $F_0(t,x,y) \neq \emptyset$, for all $t \in [0,T] \setminus J$, the mapping $(t,x,y) \multimap F_0(t,x,y)$ is u.s.c. on $[0,T] \setminus J \times E \times E$ and $\nu_{\overline{K}}$ is continuous in $[0,T] \setminus J$.

For each $m \in \mathbb{N}$, let us define the mapping $F_m : [0,T] \times E \times E \multimap E$ with compact, convex values by the formula

$$F_m(t,x,y) := \begin{cases} F_0(t,x,y) + \nu_{\overline{K}}(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(x), & \text{for all } (t,x,y) \in [0,T] \setminus J \times E \times E, \\ \nu_{\overline{K}}(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(x), & \text{for all } (t,x,y) \in J \times E \times E. \end{cases}$$

Let us consider the b.v.p.

$$(P_m) \qquad \left. \begin{array}{l} \ddot{x}(t) \in F_m(t,x(t),\dot{x}(t)), \quad \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0. \end{array} \right\}$$

Now, let us verify the solvability of problems (P_m). Let $m \in \mathbb{N}$ be fixed. Since F_0 is globally u.s.c. on $[0,T] \setminus J \times E \times E$, $F_m(\cdot,x,y)$ is measurable, for each $(x,y) \in E \times E$, and, due to the continuity of $\hat{\phi}$, $F_m(t,\cdot,\cdot)$ is u.s.c., for all $t \in [0,T] \setminus J$. Therefore, F_m is an upper-Carathéodory mapping. Moreover, let us define the upper-Carathéodory mapping $H_m : [0,T] \times E \times E \times E \times E \times [0,1] \multimap E$ by the formula

$$\begin{aligned} & H_m(t,x,y,u,v,\lambda) \\ & \quad \equiv H_m(t,u,v,\lambda) \\ & \quad := \begin{cases} \lambda F_0(t,u,v) + \nu_{\overline{K}}(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(u), & \text{for all } (t,x,y,u,v,\lambda) \in [0,T] \setminus J \\ & \qquad \times E^4 \times [0,1], \\ \nu_{\overline{K}}(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(u), & \text{for all } (t,x,y,u,v,\lambda) \in J \times E^4 \times [0,1]. \end{cases} \end{aligned}$$

Let us show that, when $m \in \mathbb{N}$ is sufficiently large, all assumptions of Proposition 2.4 (for $\varphi(t,x,\dot{x}) := F_m(t,x,\dot{x})$) are satisfied.

For this purpose, let us define the closed set $S = S_1$ by

$$S := \{x \in AC^1([0,T],E) : x(T) = x(0) = 0\}$$

and let the set Q of candidate solutions be defined as $Q := C^1([0,T],\overline{K})$. Because of the convexity of K, the set Q is closed and convex.

For all $q \in Q$ and $\lambda \in [0,1]$, consider still the associated fully linearized problem

$$P_m(q,\lambda) \quad \left.\begin{array}{ll} \ddot{x}(t) \in H_m(t,q(t),\dot{q}(t),\lambda), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \end{array}\right\}$$

and denote by $\mathfrak{T}_m$ the solution mapping which assigns to each $(q,\lambda) \in Q \times [0,1]$ the set of solutions of $P_m(q,\lambda)$.

ad (i) In order to verify condition (i) in Proposition 2.4, we need to show that, for each $(q,\lambda) \in Q \times [0,1]$, the problem $P_m(q,\lambda)$ is solvable with a convex set of solutions. So, let $(q,\lambda) \in Q \times [0,1]$ be arbitrary and let $f_{q,\lambda}(\cdot)$ be a measurable selection of $H_m(\cdot,q(\cdot),\dot{q}(\cdot),\lambda)$, which surely exists (see, *e.g.*, [27, Theorem 1.3.5]). According to (2_{ii}) and the definition of H_m, it is also easy to see that $f_{q,\lambda} \in L^1([0,T],E)$. The homogeneous problem corresponding to b.v.p. $P_m(q,\lambda)$,

$$\left.\begin{array}{ll} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \end{array}\right\} \tag{12}$$

has only the trivial solution, and therefore the single-valued Dirichlet problem

$$\left.\begin{array}{ll} \ddot{x}(t) = f_{q,\lambda}(t), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0 & \end{array}\right\}$$

admits a unique solution $x_{q,\lambda}(\cdot)$ which is one of solutions of $P_m(q,\lambda)$. This is given, for a.a. $t \in [0,T]$, by $x_{q,\lambda}(t) = \int_0^T G(t,s) f_{q,\lambda}(s)\,ds$, where G is the Green function associated to the homogeneous problem (12). The Green function G and its partial derivative $\frac{\partial}{\partial t}G$ are defined by (*cf. e.g.* [28, pp.170-171])

$$G(t,s) = \begin{cases} \frac{(s-T)t}{T}, & \text{for all } 0 \le t \le s \le T, \\ \frac{(t-T)s}{T}, & \text{for all } 0 \le s \le t \le T, \end{cases}$$

$$\frac{\partial}{\partial t}G(t,s) = \begin{cases} \frac{(s-T)}{T}, & \text{for all } 0 \le t < s \le T, \\ \frac{s}{T}, & \text{for all } 0 \le s < t \le T. \end{cases}$$

Thus, the set of solutions of $P_m(q,\lambda)$ is non-empty. The convexity of the solution sets follows immediately from the definition of H_m and the fact that problems $P_m(q,\lambda)$ are fully linearized.

ad (ii) Let $\Omega \subset E \times E \times E \times E$ be bounded. Then, there exists a bounded $\Omega_1 \subset E$ such that $\Omega \subset \Omega_1 \times \Omega_1 \times \Omega_1 \times \Omega_1$ and, according to (2_{ii}) and the definition of H_m, there exists

$\hat{J} \subset [0,T]$ with $\mu(\hat{J}) = 0$ such that, for all $t \in [0,T] \setminus (J \cup \hat{J})$, $(x,y,u,v) \in \Omega$ and $\lambda \in [0,1]$,

$$\|H_m(t,u,v,\lambda)\| \le \nu_{\Omega_1}(t) + 2\nu_{\overline{K}}(t) \cdot \max_{x \in \overline{B(\partial K,k)}} \|\hat{\phi}(x)\|.$$

Therefore, the mapping $H_m(t,q(t),\dot{q}(t),\lambda)$ satisfies condition (ii) from Proposition 2.4.

ad (iii) Since the verification of condition (iii) in Proposition 2.4 is technically the most complicated, it will be split into two parts: (iii$_1$) the quasi-compactness of the solution operator $\mathfrak{T}_m$, (iii$_2$) the condensity of $\mathfrak{T}_m$ w.r.t. the monotone and non-singular m.n.c. α defined by (7).

ad (iii$_1$) Let us firstly prove that the solution mapping $\mathfrak{T}_m$ is quasi-compact. Since $C^1([0,T],E)$ is a complete metric space, it is sufficient to prove the sequential quasi-compactness of $\mathfrak{T}_m$. Hence, let us consider the sequences $\{q_n\}$, $\{\lambda_n\}$, $q_n \in Q$, $\lambda_n \in [0,1]$, for all $n \in \mathbb{N}$, such that $q_n \to q$ in $C^1([0,T],E)$ and $\lambda_n \to \lambda$. Moreover, let $x_n \in \mathfrak{T}_m(q_n,\lambda_n)$, for all $n \in \mathbb{N}$. Then there exists, for all $n \in \mathbb{N}$, $k_n(\cdot) \in F_0(\cdot,q_n(\cdot),\dot{q}_n(\cdot))$ such that

$$\ddot{x}_n(t) = f_n(t), \quad \text{for a.a. } t \in [0,T], \tag{13}$$

where

$$f_n(t) = \lambda_n k_n(t) + \nu_{\overline{K}}(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\hat{\phi}\big(q_n(t)\big), \tag{14}$$

and that

$$x_n(T) = x_n(0) = 0.$$

Since $q_n \to q$ and $\dot{q}_n \to \dot{q}$ in $C([0,T],E)$, there exists a bounded $\Omega \times \Omega \subset E \times E$ such that $(q_n(t),\dot{q}_n(t)) \in \Omega \times \Omega$, for all $t \in [0,T]$ and $n \in \mathbb{N}$. Therefore, there exists, according to condition (2$_{\text{ii}}$), $\nu_\Omega \in L^1([0,T],[0,\infty))$ such that $\|f_n(t)\| \le \varpi(t)$, for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$, where $\varpi(t) := \nu_\Omega(t) + 2\nu_{\overline{K}}(t) \cdot \max_{x \in \overline{B(\partial K,\varepsilon)}} \|\hat{\phi}(x)\|$.

Moreover, for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$,

$$x_n(t) = \int_0^T G(t,s) f_n(s)\,ds \tag{15}$$

and

$$\dot{x}_n(t) = \int_0^T \frac{\partial}{\partial t} G(t,s) f_n(s)\,ds. \tag{16}$$

Thus, x_n satisfies, for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$, $\|x_n(t)\| \le a$ and $\|\dot{x}_n(t)\| \le b$, where

$$a := \frac{T}{4}\int_0^T \varpi(s)\,ds \quad \text{and} \quad b := \int_0^T \varpi(s)\,ds.$$

Furthermore, for every $n \in \mathbb{N}$ and a.a. $t \in [0,T]$, we have

$$\|\ddot{x}_n(t)\| \le \varpi(t).$$

Hence, the sequences $\{x_n\}$ and $\{\dot{x}_n\}$ are bounded and $\{\ddot{x}_n\}$ is uniformly integrable.

For each $t \in [0, T]$, the properties of the Hausdorff m.n.c. yield

$$\begin{aligned}\gamma\left(\left\{f_n(t)\right\}_n\right) &\le \gamma\left(\left\{\lambda_n k_n(t)\right\}_n\right) + \nu_{\overline{K}}(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\left\{\hat{\phi}\left(q_n(t)\right)\right\}_n\right)\\ &\le \gamma\left(\left\{k_n(t)\right\}_n\right) + \nu_{\overline{K}}(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\\ &\quad \times \gamma\left(\left\{\phi\left(q_n(t)\right)\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K, \varepsilon)}\right\}\right).\end{aligned}$$

Since $q_n(t) \in \overline{K}$, for all $t \in [0, T]$ and all $n \in \mathbb{N}$, it follows from condition (2_{i}) that, for a.a. $t \in [0, T]$,

$$\begin{aligned}\gamma\left(\left\{f_n(t)\right\}_n\right) &\le g(t)\left(\gamma\left(\left\{q_n(t)\right\}_n\right) + \gamma\left(\left\{\dot{q}_n(t)\right\}_n\right)\right)\\ &\quad + \nu_{\overline{K}}(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\left\{\phi\left(q_n(t)\right)\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K, \varepsilon)}\right\}\right)\\ &\le g(t) \sup_{t\in[0,T]}\left(\gamma\left(\left\{q_n(t)\right\}_n\right) + \gamma\left(\left\{\dot{q}_n(t)\right\}_n\right)\right)\\ &\quad + \nu_{\overline{K}}(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\gamma\left(\left\{\phi\left(q_n(t)\right)\|\dot{V}_{q_n(t)}\| : q_n(t) \in \overline{B(\partial K, \varepsilon)}\right\}\right).\end{aligned}$$

Since the function $x \to \phi(x)\|\dot{V}_x\|$ is Lipschitzian on $\overline{B(\partial K, \varepsilon)}$ with some Lipschitz constant $\hat{L} > 0$ (see Remark 2.1), we get

$$\gamma\left(\left\{f_n(t)\right\}_n\right) \le \left(g(t) + \hat{L}\nu_{\overline{K}}(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\right) \sup_{t\in[0,T]}\left(\gamma\left(\left\{q_n(t)\right\}_n\right) + \gamma\left(\left\{\dot{q}_n(t)\right\}_n\right)\right). \tag{17}$$

Since $q_n \to q$ and $\dot{q}_n \to \dot{q}$ in $C([0, T], E)$, we get, for all $t \in [0, T]$, $\gamma(\{q_n(t)\}_n) = \gamma(\{\dot{q}_n(t)\}_n) = 0$, which implies that $\gamma(\{f_n(t)\}_n) = 0$, for all $t \in [0, T]$.

For all $(t, s) \in [0, T] \times [0, T]$, the sequence $\{G(t, s)f_n(s)\}$ is relatively compact as well since, according to the semi-homogeneity of the Hausdorff m.n.c.,

$$\gamma\left(\left\{G(t, s)f_n(s)\right\}\right) \le \left|G(t, s)\right|\gamma\left(\left\{f_n(s)\right\}\right) = 0, \quad \text{for all } (t, s) \in [0, T] \times [0, T]. \tag{18}$$

Moreover, by means of (4) and (18),

$$\gamma\left(\left\{x_n(t)\right\}\right) = \gamma\left(\left\{\int_0^T G(t, s)f_n(s)\, ds\right\}\right) = 0, \quad \text{for all } t \in [0, T].$$

By similar reasoning, we also get

$$\gamma\left(\left\{\dot{x}_n(t)\right\}\right) = 0, \quad \text{for all } t \in [0, T],$$

by which $\{x_n(t)\}$, $\{\dot{x}_n(t)\}$ are relatively compact, for all $t \in [0, T]$.

Moreover, since x_n satisfies for all $n \in \mathbb{N}$ (13), $\{\ddot{x}_n(t)\}$ is relatively compact, for a.a. $t \in [0, T]$. Thus, according to [23, Lemma III.1.30], there exist a subsequence of $\{\dot{x}_n\}$, for the sake of simplicity denoted in the same way as the sequence, and $x \in C^1([0, T], E)$ such that $\{\dot{x}_n\}$ converges to $\dot{x}$ in $C([0, T], E)$ and $\{\ddot{x}_n\}$ converges weakly to $\ddot{x}$ in $L^1([0, T], E)$. According to the classical closure results (*cf. e.g.* [27, Lemma 5.1.1]), $x \in \mathfrak{T}_m(q,\lambda)$, which implies the quasi-compactness of $\mathfrak{T}_m$.

ad (iii$_2$) In order to show that, for $m \in \mathbb{N}$ sufficiently large, $\mathfrak{T}_m$ is α-condensing with respect to the m.n.c. α defined by (7), let us consider a bounded subset $\Theta \subset Q$ such that $\alpha(\mathfrak{T}_m(\Theta \times [0,1])) \geq \alpha(\Theta)$. Let $\{x_n\} \subset \mathfrak{T}_m(\Theta \times [0,1])$ be a sequence such that

$$\begin{aligned}&\alpha\bigl(\mathfrak{T}_m\bigl(\Theta \times [0,1]\bigr)\bigr)\\&\quad=\Bigl(\sup_{t\in[0,T]}\bigl[\gamma\bigl(\bigl\{x_n(t)\bigr\}_n\bigr)+\gamma\bigl(\bigl\{\dot{x}_n(t)\bigr\}_n\bigr)\bigr],\mathrm{mod}_C\bigl(\{x_n\}_n\bigr)+\mathrm{mod}_C\bigl(\{\dot{x}_n\}_n\bigr)\Bigr).\end{aligned}$$

At first, let us show that the set $\mathfrak{T}_m(\Theta \times [0,1])$ is bounded. If $x \in \mathfrak{T}_m(\Theta \times [0,1])$, then there exist $q \in \Theta$, $\lambda \in [0,1]$ and $k(\cdot) \in F_0(\cdot, q(\cdot), \dot{q}(\cdot))$ such that

$$x(t)=\int_0^T G(t,s)f(s)\,ds, \qquad \dot{x}(t)=\int_0^T \frac{\partial G(t,s)}{\partial t}f(s)\,ds, \quad \text{for all } t\in[0,T],$$

with $f(t) = \lambda k(t) + \nu_{\overline{K}}(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(q(t))$, for a.a. $t \in [0,T]$.

Since Θ is bounded, there exists $\Omega \subset E$ such that $q(t) \in \Omega$, for all $q \in \Theta$ and all $t \in [0,T]$. Hence, according to (2$_{\text{ii}}$), there exists $\nu_\Omega \in L^1([0,T])$ such that $\|k(t)\| \leq \nu_\Omega(t)$, for a.a. $t \in [0,T]$. Consequently

$$\begin{aligned}\bigl\|x(t)\bigr\|_E&\leq \max_{(t,s)\in[0,1]\times[0,1]}\bigl|G(t,s)\bigr|\biggl[\int_0^T \nu_\Omega(s)\,ds+2\max_{x\in\overline{B(\partial K,k)}}\bigl\|\hat{\phi}(x)\bigr\|\int_0^T \nu_{\overline{K}}(t)\biggr]\\&\leq \frac{T}{4}\|\nu_\Omega\|+2\max_{x\in\overline{B(\partial K,k)}}\bigl\|\hat{\phi}(x)\bigr\|\cdot\|\nu_{\overline{K}}\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\bigl\|\dot{x}(t)\bigr\|_E&\leq \max_{(t,s)\in[0,1]\times[0,1]}\biggl|\frac{\partial G(t,s)}{\partial}\biggr|\biggl[\int_0^T \bigl\|k(s)\bigr\|\,ds+2\max_{x\in\overline{B(\partial K,k)}}\bigl\|\hat{\phi}(x)\bigr\|\int_0^T \nu_{\overline{K}}(t)\biggr]\\&\leq \|\nu_\Omega\|+2\max_{x\in\overline{B(\partial K,k)}}\bigl\|\hat{\phi}(x)\bigr\|\cdot\|\nu_{\overline{K}}\|.\end{aligned}$$

Thus, the set $\mathfrak{T}_m(\Theta \times [0,1])$ is bounded.

Moreover, we can find $\{q_n\} \subset \Theta$, $\{\lambda_n\} \subset [0,1]$ and $\{k_n\}$ satisfying, for a.a. $t \in [0,T]$, $k_n(t) \in F_0(t, q_n(t), \dot{q}_n(t))$, such that, for all $t \in [0,T]$, $x_n(t)$ and $\dot{x}_n(t)$ are defined by (15) and (16), respectively, where $f_n(t)$ is defined by (14).

By similar reasoning as in the part *ad* (iii$_1$), we obtain

$$\gamma\bigl(\bigl\{f_n(t)\bigr\}_n\bigr)\leq\biggl(g(t)+\hat{L}\nu_{\overline{K}}(t)\biggl(\chi_{J_m}(t)+\frac{1}{m}\biggr)\biggr)\sup_{t\in[0,T]}\bigl(\gamma\bigl(\bigl\{q_n(t)\bigr\}_n\bigr)+\gamma\bigl(\bigl\{\dot{q}_n(t)\bigr\}_n\bigr)\bigr),$$

for a.a. $t \in [0,T]$, and that

$$\bigl\|f_n(t)\bigr\|\leq\bigl\|k_n(t)\bigr\|+2\cdot\max_{x\in\overline{B(\partial K,\varepsilon)}}\bigl\|\hat{\phi}(x)\bigr\|\cdot\nu_{\overline{K}}(t), \quad \text{for a.a. } t\in[0,T] \text{ and all } n\in\mathbb{N}.$$

Since $k_n(t) \in F_0(t, q_n(t), \dot{q}_n(t))$, for a.a. $t \in [0,T]$, and $q_n \in \Theta$, for all $n \in \mathbb{N}$, where Θ is a bounded subset of $C^1([0,T],E)$, there exists $\Omega \subset \overline{K}$ such that $q_n(t) \in \Omega$, for all $n \in \mathbb{N}$ and

$t \in [0,T]$. Hence, it follows from condition (2_{ii}) that

$$\|f_n(t)\| \leq \nu_\Omega(t) + 2 \cdot \nu_{\overline{K}}(t) \cdot \max_{x \in \overline{B(\partial K, \varepsilon)}} \|\hat{\phi}(x)\|, \quad \text{for a.a. } t \in [0,T]. \tag{19}$$

This implies $\|G(t,s)f_n(t)\| \leq |G(t,s)|(\nu_\Omega(t) + 2 \cdot \nu_{\overline{K}}(t) \cdot \max_{x \in \overline{B(\partial K, \varepsilon)}} \|\hat{\phi}(x)\|)$, for a.a. $t,s \in [0,T]$ and all $n \in \mathbb{N}$.

Moreover, by virtue of the semi-homogeneity of the Hausdorff m.n.c., for all $(t,s) \in [0,T] \times [0,T]$, we have

$$\begin{aligned}\gamma\left(\left\{G(t,s)f_n(s)\right\}_n\right) &\leq \left|G(t,s)\right|\gamma\left(\left\{f_n(s)\right\}_n\right) \leq \frac{T}{4}\gamma\left(\left\{f_n(s)\right\}_n\right) \\ &\leq \frac{T}{4}\left(g(t) + \hat{L}\nu_{\overline{K}}(t)\left(\chi_{J_m}(t) + \frac{1}{m}\right)\right) \\ &\quad \times \sup_{t \in [0,T]}\left(\gamma\left(\left\{q_n(t)\right\}_n\right) + \gamma\left(\left\{\dot{q}_n(t)\right\}_n\right)\right).\end{aligned}$$

Let us denote

$$\mathcal{S} := \sup_{t \in [0,T]}\left(\gamma\left(\left\{q_n(t)\right\}_n\right) + \gamma\left(\left\{\dot{q}_n(t)\right\}_n\right)\right)$$

and

$$\mathcal{S}^* := \sup_{t \in [0,T]}\left(\gamma\left(\left\{x_n(t)\right\}_n\right) + \gamma\left(\left\{\dot{x}_n(t)\right\}_n\right)\right).$$

According to (4) and (15) we thus obtain for each $t \in [0,T]$,

$$\begin{aligned}\gamma\left(\left\{x_n(t)\right\}_n\right) &= \gamma\left(\left\{\int_0^T G(t,s)f_n(s)\,ds\right\}_n\right) \\ &\leq \frac{T}{4}\left(\|g\|_{L^1} + \hat{L}\left(\|\nu_{\overline{K}}\|_{L^1(J_m)} + \frac{1}{m}\|\nu_{\overline{K}}\|_{L^1}\right)\right)\mathcal{S}.\end{aligned}$$

By similar reasonings, we can see that, for each $t \in [0,T]$,

$$\gamma\left(\left\{\dot{x}_n(t)\right\}_n\right) \leq \left(\|g\|_{L^1} + \hat{L}\left(\|\nu_{\overline{K}}\|_{L^1(J_m)} + \frac{1}{m}\|\nu_{\overline{K}}\|_{L^1}\right)\right)\mathcal{S},$$

when starting from condition (16). Subsequently,

$$\mathcal{S}^* \leq \frac{T+4}{4}\left(\|g\|_{L^1} + \hat{L}\left(\|\nu_{\overline{K}}\|_{L^1(J_m)} + \frac{1}{m}\|\nu_{\overline{K}}\|_{L^1}\right)\right)\mathcal{S}. \tag{20}$$

Since we assume that $\alpha(\mathfrak{T}_m(\Theta \times [0,1])) \geq \alpha(\Theta)$ and $\{q_n\}_n \subset \Theta$, we get

$$\mathcal{S} \leq \mathcal{S}^* \leq \frac{T+4}{4}\left(\|g\|_{L^1} + \hat{L}\left(\|\nu_{\overline{K}}\|_{L^1(J_m)} + \frac{1}{m}\|\nu_{\overline{K}}\|_{L^1}\right)\right)\mathcal{S}.$$

Since we have, according to (2_{iii}), $\frac{T+4}{4}\|g\|_{L^1} < 1$, we can choose $m_0 \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $m \geq m_0$, we have

$$\frac{T+4}{4}\left(\|g\|_{L^1} + \hat{L}\left(\|\nu_{\overline{K}}\|_{L^1(J_m)} + \frac{1}{m}\|\nu_{\overline{K}}\|_{L^1}\right)\right) < 1.$$

Therefore, we get, for sufficiently large $m \in \mathbb{N}$, the contradiction $\mathcal{S} < \mathcal{S}$ which ensures the validity of condition (iii) in Proposition 2.4.

ad (iv) For all $q \in Q$, the set $\mathfrak{T}_m(q,0)$ coincides with the unique solution x_m of the linear system

$$\left.\begin{array}{ll}\ddot{x}(t) = \nu_{\overline{K}}(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(q(t)), & \text{for a.a. } t \in [0,T],\\ x(T) = x(0) = 0. & \end{array}\right\}$$

According to (15) and (16), for all $t \in [0,T]$,

$$x_m(t) = \int_0^T G(t,s)\varphi_m(s)\,ds$$

and

$$\dot{x}_m(t) = \int_0^T \frac{\partial}{\partial t}G(t,s)\varphi_m(s)\,ds,$$

where $\varphi_m(t) := \nu_{\overline{K}}(t)(\chi_{J_m}(t) + \frac{1}{m})\hat{\phi}(q(t))$.

Since

$$\|\varphi_m\|_{L^1} \le \max_{x \in \overline{B(\partial K,\varepsilon)}} \|\hat{\phi}(x)\| \cdot \left(\|\nu_{\overline{K}}\|_{L^1(J_m)} + \frac{\|\nu_{\overline{K}}\|_{L^1}}{m}\right),$$

we have, for all $t \in [0,T]$,

$$\|x_m(t)\| \le \frac{T^2}{4} \cdot \max_{x \in \overline{B(\partial K,\varepsilon)}} \|\hat{\phi}(x)\| \cdot \left(\|\nu_{\overline{K}}\|_{L^1(J_m)} + \frac{\|\nu_{\overline{K}}\|_{L^1}}{m}\right). \tag{21}$$

Let us now consider $r > 0$ such that $rB \subset K$. Then it follows from (21) that we are able to find $m_0 \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$, $m \ge m_0$, and $t \in [0,T]$, $\|x_m\| \le r$. Therefore, for all $m \in \mathbb{N}$, $m \ge m_0$, $\mathfrak{T}_m(q,0) \subset \operatorname{Int} Q$, for all $q \in Q$, which ensures the validity of condition (iv) in Proposition 2.4.

ad (v) The validity of the transversality condition (v) in Proposition 2.4 can be proven quite analogously as in [16] (see pp.40-43 in [16]) with the following differences:

- due to the Dirichlet boundary conditions, t_0 belongs to the open interval $(0,T)$,
- since $A(t) = B(t) = 0$, we have $p(t) = -\nu_{\overline{K}}(t)$.

In this way, we can prove that there exists $m_0 \in \mathbb{N}$ such that every problem (P_m), where $m \ge m_0$, satisfies all the assumptions of Proposition 2.4. This implies that every such (P_m) admits a solution, denoted by x_m, with $x_m(t) \in \overline{K}$, for all $t \in [0,T]$. By similar arguments as in [16], but with the expression $Z(4Zk+1)$ replaced by $\frac{T}{4}$, according to condition (2_{ii}), we can obtain the result that there exists a subsequence, denoted as the sequence, and a function $x \in AC^1([0,T],E)$ such that $x_m \to x$ and $\dot{x}_m \to \dot{x}$ in $C([0,T],E)$ and also $\ddot{x}_m \rightharpoonup x$ in $L^1([0,T],E)$, when $m \to \infty$. Thus, a classical closure result (see *e.g.* [27, Lemma 5.1.1]) guarantees that x is a solution of (2) satisfying $x(t) \in \overline{K}$, for all $t \in [0,T]$, and the sketch of proof is so complete. □

The case when $F = F_1 + F_2$, with $F_1(t,\cdot,\cdot)$ to be completely continuous and $F_2(t,\cdot,\cdot)$ to be Lipschitzian, for a.a. $t \in [0,T]$, represents the most classical example of a map which is

γ-regular w.r.t. the Hausdorff measure of non-compactness γ. The following corollary of Theorem 3.1 can be proved quite analogously as in [3, Example 6.1 and Remark 6.1].

Corollary 3.1 *Let $E = H$ be a separable Hilbert space and let us consider the Dirichlet b.v.p.:*

$$\left.\begin{array}{l} \ddot{x}(t) \in F_1(t, x(t), \dot{x}(t)) + F_2(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0, T], \\ x(0) = x(T) = 0, \end{array}\right\} \tag{22}$$

where

(i) *$F_1 : [0, T] \times H \times H \multimap H$ is an upper-Carathéodory, globally measurable, multivalued mapping and $F_1(t, \cdot, \cdot) : H \times H \multimap H$ is completely continuous, for a.a. $t \in [0, T]$, such that*

$$\|F_1(t, x, y)\| \le \nu_1(t, D_0),$$

for a.a. $t \in [0, T]$, all $x \in H$ with $\|x\| \le D_0$, where $D_0 > 0$ is an arbitrary constant, $\nu_1 \in L^1([0, T], [0, \infty))$, and all $y \in H$,

(ii) *$F_2 : [0, T] \times H \times H \multimap H$ is a Carathéodory multivalued mapping such that*

$$\|F_2(t, 0, 0)\| \le \nu_2(t), \quad \text{for a.a. } t \in [0, T],$$

where $\nu_2 \in L^1([0, T], [0, \infty))$, and $F_2(t, \cdot, \cdot) : H \times H \multimap H$ is Lipschitzian, for a.a. $t \in [0, T]$, with the Lipschitz constant

$$L < \frac{4}{T(T + 4)}.$$

Moreover, suppose that

(iii) *there exists $R > 0$ such that, for all $x \in H$ with $\|x\| = R$, $t \in (0, T)$, $y \in H$ and $w \in F_1(t, x, y) + F_2(t, x, y)$, we have*

$$\langle x, w\rangle > 0.$$

Then the Dirichlet problem (22) *admits, according to Theorem* 3.1, *a solution $x(\cdot)$ such that $\|x(t)\| \le R$, for all $t \in [0, T]$.*

Remark 3.1 For $F_2(t, x, y) \equiv 0$, the completely continuous mapping $F_1(t, x, y)$ allows us to make a comparison with classical single-valued results recalled in the Introduction. Unfortunately, our F_1 in (i) (see also (2_{ii}) in Theorem 3.1) is the only mapping which is (unlike in [3, Example 6.1 and Remark 6.1], where under some additional restrictions quite liberal growth restrictions were permitted) globally bounded w.r.t. $y \in H$. Furthermore, our sign condition in (iii) is also (unlike again in [3, Example 6.1 and Remark 6.1], where under some additional restrictions the Hartman-type condition like ($\mathrm{i_H}$) in the Introduction was employed) the most restrictive among their analogies in [6–13]. On the other hand, because of multivalued upper-Carathéodory maps $F_1 + F_2$ in a Hilbert space which are γ-regular, our result has still, as far as we know, no analogy at all.

4 Illustrative examples

The first illustrative example of the application of Theorem 3.1 concerns the integro-differential equation

$$\begin{aligned} & u_{tt}(t,x) + \varphi\bigl(t,x,u_t(t,x)\bigr) \\ & \quad = b(t)u(t,x) + \int_{\mathbb{R}} k(x,y)u(t,y)\,dy + p\biggl(\int_{\mathbb{R}} \psi(x)u(t,x)\,dx\biggr) f\bigl(u(t,x)\bigr), \\ & \quad t \in [0,T], x \in \mathbb{R}, \end{aligned} \tag{23}$$

involving discontinuities in a state variable. In this equation, the non-local diffusion term $\int_{\mathbb{R}} k(x,y)u(t,y)\,dy$ replaces the classical diffusion behavior given by $u_{xx}(t,x)$. In dispersal models such an integral term takes into account the long-distance interactions between individuals (see *e.g.* [29]). Moreover, when φ is linear in u_t, (23) can be considered as an alternative version of the classical telegraph equation (see *e.g.* [30] and the references therein), where the classical diffusivity is replaced by the present non-local diffusivity.

Telegraph equations appear in many fields such as modeling of an anomalous diffusion, a wave propagation phenomenon, sub-diffusive systems or modeling of a pulsate blood flow in arteries (see *e.g.* [31, 32]).

For the sake of simplicity, we will discuss here only the case when φ is globally bounded w.r.t. u_t. On the other hand, for non-strictly localized transversality conditions as in [3], for instance, a suitable linear growth estimate w.r.t. u_t can be permitted.

Example 4.1 Let us consider the integro-differential equation (23) with $\varphi : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $b : [0,T] \to \mathbb{R}$, $k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}$ and $p : \mathbb{R} \to \mathbb{R}$. We assume that

(a) φ is Carathéodory, *i.e.* $\varphi(\cdot,x,y)$ is measurable, for all $x,y \in \mathbb{R}$, and $\varphi(t,\cdot,\cdot)$ is continuous, for a.a. $t \in [0,T]$; $\varphi(t,x,\cdot)$ is $L(t)$-Lipschitzian with $L \in L^1([0,T])$; $|\varphi(t,x,y)| \le \varphi_1(t)\varphi_2(x)$, for a.a. $t \in [0,T]$ and all $x,y \in \mathbb{R}$, where $\varphi_1 \in L^1([0,T])$ and $\varphi_2 \in L^2(\mathbb{R})$; $\varphi(t,x,0) \neq 0$, for all a.a. $t \in [0,T]$ and all $x \in \mathbb{R}$,

(b) $b \in L^1([0,T])$ and satisfies $b(t) \ge b_0 > 1$, for a.a. $t \in [0,T]$,

(c) $k \in L^2(\mathbb{R} \times \mathbb{R})$ with $\|k\|_{L^2(\mathbb{R}\times\mathbb{R})} = 1$,

(d) $p(r) \ge 0$, for all $r \in \mathbb{R}$; and there can exist $r_1 < r_2 < \cdots < r_k$ such that $p(\cdot)$ is continuous, for $r \neq r_i$, and $p(\cdot)$ has discontinuities at r_i, for $i = 1,\ldots,k$, with $p(r_i^{\mp}) := \lim_{r\to r_i^{\mp}} p(r) \in \mathbb{R}$,

(e) f is L-Lipschitzian; $L > 0$; $f(0) = 0$; and $xf(x) > 0$, for all $x \neq 0$,

(f) $\psi \in L^2(\mathbb{R})$ with $\|\psi\|_{L^2(\mathbb{R})} = 1$.

Since the function p can have some discontinuities, a solution of (23) satisfying the Dirichlet conditions

$$u(0,x) = u(T,x) = 0, \quad \text{for all } x \in \mathbb{R}, \tag{24}$$

will be appropriately interpreted in the sense of Filippov. More precisely, let us define $P : \mathbb{R} \multimap \mathbb{R}$ by the formula

$$P(r) := \begin{cases} p(r) & \text{if } r \neq r_i, \\ [\min\{p(r_i), p(r_i^-), p(r_i^+)\}, \max\{p(r_i), p(r_i^-), p(r_i^+)\}] & \text{if } r = r_i, i = 1,2,\ldots,k. \end{cases}$$

A function $u(t,x)$ is said to be a *solution* of (23), (24) if $u(t,\cdot) \in L^2(R)$, for all $t \in [0,T]$, the map $[0,T] \to L^2(R)$ defined by $t \to u(t,\cdot)$ is C^1 if it is a solution of the inclusion

$$\begin{aligned} &u_{tt}(t,x) + \varphi\bigl(t,x,u_t(t,x)\bigr) \\ &\quad \in \int_{\mathbb{R}} k(x,y)u(t,y)\,dy + b(t)u(t,x) + P\biggl(\int_{\mathbb{R}} \psi(x)u(t,x)\,dx\biggr) f\bigl(u(t,x)\bigr) \end{aligned} \tag{25}$$

and if it satisfies (24).

If we further assume the existence of $R > 0$ such that

$$R > \frac{\varphi_1(t)}{b_0 - 1}\|\varphi_2\|_{L^2(\mathbb{R})}, \quad \text{for a.a. } t \in [0,T], \tag{26}$$

and that

$$\bigl\|L(t) + b(t)\bigr\|_{L^1([0,T])} + (1 + mL)T < \frac{4}{T+4}, \tag{27}$$

where

$$m := \max_{r\in[-R,R]} \max\bigl\{p(r), p\bigl(r^-\bigr), p\bigl(r^+\bigr)\bigr\}, \tag{28}$$

then the problem (23), (24) has a solution, in the sense of Filippov, satisfying $\|u(t,\cdot)\|_{L^2(\mathbb{R})} \le R$, for a.a. $t \in [0,T]$.

In fact, problem (25), (24) can be transformed into the abstract setting

$$\begin{cases} \ddot{y}(t) \in F(t, y(t), \dot{y}(t)), & t \in [0,T], \\ y(T) = y(0) = 0, \end{cases} \tag{29}$$

where $y(t) := u(t,\cdot) \in L^2(\mathbb{R})$, for all $t \in [0,T]$, and $F : [0,T] \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \multimap L^2(\mathbb{R})$ is defined by

$$F(t,y,w) := -\hat{\varphi}(t,w) + b(t)y + K(y) + \hat{F}(y),$$

where $\hat{\varphi} : [0,T] \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(t,y) \mapsto (x \mapsto \varphi(t,x,y(x)))$, $K : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $w \mapsto (x \mapsto \int_{\mathbb{R}} k(x,y)w(y)\,dy)$, $\hat{f} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $y \mapsto (x \mapsto f(y(x)))$ and $\hat{F} : L^2(\mathbb{R}) \multimap L^2(\mathbb{R})$, $y \multimap \{p\hat{f}(y) : p \in P(\int_{\mathbb{R}} \psi(x)y(x)\,dx)\}$.

Let us now examine the properties of F. According to (a), $\hat{\varphi}$ is well defined. Given $y \in L^2(\mathbb{R})$, let us show that $\hat{\varphi}(\cdot,y)$ is measurable. For this purpose, let Ψ be an arbitrary element in the dual space $(L^2(\mathbb{R}))'$ of $L^2(\mathbb{R})$. Hence, there exists $\psi \in L^2(\mathbb{R})$ such that $\Psi(z) = \int_{\mathbb{R}} \psi(x)z(x)\,dx$, for all $z \in L^2(\mathbb{R})$, and consequently the composition $\Psi \circ \hat{\varphi}(\cdot,y) : [0,T] \to \mathbb{R}$ is such that $t \to \int_{\mathbb{R}} \psi(x)\varphi(t,x,y(x))\,dx$. Since φ is Carathéodory, it is globally measurable, and so the mapping $(t,x) \to \psi(x)\varphi(t,x,y(x))$ is globally measurable as well. This implies that, according to the Fubini Theorem, the mapping $\Psi \circ \hat{\varphi}(\cdot,y)$ is measurable, too. Finally, since Ψ was arbitrary, according to the Pettis Theorem (see Proposition 2.1), $\hat{\varphi}(\cdot,y)$ is measurable.

Furthermore, let us show that $\hat{F}$ is u.s.c. For this purpose, let $y_0 \in L^2(\mathbb{R})$ be fixed.

(i) If $r_0 := \int_{\mathbb{R}} \psi(x)y_0(x)\,dx \neq r_i$, $i = 1,2,\dots,k$, then it is possible to find $\delta > 0$ such that $\hat{F} : B(y_0,\delta) \to L^2(\mathbb{R})$ is single-valued, *i.e.* $\hat{F}(y) = p(r)\hat{f}(y)$, $r := \int_{\mathbb{R}} \psi(x)y(x)\,dx \in [r_0 - \delta, r_0 + \delta]$, for all $y \in B(y_0,\delta)$ and $r_i \notin [r_0 - \delta, r_0 + \delta]$, for $i = 1,2,\dots,k$. Since p is continuous in $[r_0 - \delta, r_0 + \delta]$ and $\hat{f}$ is Lipschitzian, $\hat{F}$ is continuous in $B(y_0,\delta)$.

(ii) Let $r_0 = r_j$, for some $j \in \{i = 1,2,\dots,k\}$ and let $U \subset L^2(\mathbb{R})$ be open and such that $\hat{F}(y_0) \subset U$. Moreover, let $\sigma > 0$ be such that $r := \int_{\mathbb{R}} \psi(x)y(x)\,dx \neq r_i$, $i \neq j$, for any $y \in B(y_0,\sigma)$. This implies that $\hat{F}(y)$ is equal either to $p(r)\hat{f}(y)$ or to $P(r_j)\hat{f}(y)$, for all $y \in B(y_0,\sigma)$. If $r < r_j$ is such that $\hat{F}(y) = p(r)\hat{f}(y)$, then

$$\begin{aligned}\|\hat{F}(y) - p(r_j^-)\hat{f}(y_0)\|_{L^2(\mathbb{R})} &= \|p(r)\hat{f}(y) - p(r_j^-)\hat{f}(y_0)\|_{L^2(\mathbb{R})} \\ &\leq |p(r) - p(r_j^-)| \cdot \|\hat{f}(y)\| + p(r_j^-) \cdot \|\hat{f}(y) - \hat{f}(y_0)\|,\end{aligned}$$

which implies that it is possible to find $\sigma_1 > 0$ such that $F(y) \subset U$, for all $y \in B(y_0,\sigma_1)$. Similarly, we would obtain the same when assuming $r > r_j$.

If $\hat{F}(y) = P(r_j)\hat{f}(y)$ then, for every $p \in P(r_j)$,

$$\|p\hat{f}(y) - p\hat{f}(y_0)\|_{L^2(\mathbb{R})} = |p| \cdot \|\hat{f}(y) - \hat{f}(y_0)\| \leq m\|\hat{f}(y) - \hat{f}(y_0)\|,$$

which implies that also in this case it is possible to find $\sigma_2 > 0$ such that $F(y) \subset U$, for all $y \in B(y_0,\sigma_2)$.

Moreover, according to (a) and (c), $\hat{\varphi}$ is a Carathéodory mapping such that $\hat{\varphi}(t,\cdot)$ is $L(t)$-Lipschitzian, for all $t \in [0,T]$, and K is well defined and 1-Lipschitzian. It can also be shown that, according to (d) and (e), $\hat{F}$ has compact and convex values. Therefore, the mapping F is globally measurable, and so has the Scorza-Dragoni property (*cf.* Proposition 2.2).

Let us now verify particular assumptions of Theorem 3.1.

Let $\Omega_1 \subset \{y \in L^2(\mathbb{R}) \mid \|y\|_{L^2(\mathbb{R})} \leq R\}$. Then, according to (f),

$$\int_{\mathbb{R}} \psi(x)y(x)\,dx \in [-R,R],$$

for all $y \in \Omega_1$. Hence,

$$\begin{aligned}\hat{F}(\Omega_1) &= \left\{p\hat{f}(y) : p \in P\left(\int_{\mathbb{R}} \psi(x)y(x)\,dx\right), y \in \Omega_1\right\} \subset \{p\hat{f}(\Omega_1) : p \in [0,m]\} \\ &= \{m \cdot \alpha \cdot \hat{f}(\Omega_1) : \alpha \in [0,1]\},\end{aligned}$$

where m is defined by (28).

Thus,

$$\gamma(\hat{F}(\Omega_1)) \leq m\gamma(\{\alpha \cdot \hat{f}(\Omega_1) : \alpha \in [0,1]\}) \leq m \cdot L \cdot \gamma(\Omega_1),$$

according to the Lipschitzianity of $\hat{f}$ and property (6). For a.a. $t \in [0,T]$ and all $\Omega_2 \subset L^2(\mathbb{R})$, we have

$$\begin{aligned}\gamma(F(t,\Omega_1 \times \Omega_2)) &\leq \gamma(\hat{\varphi}(t,\Omega_2)) + \gamma(b(t)\Omega_1) + \gamma(K(\Omega_1)) + \gamma(\hat{F}(\Omega_1)) \\ &\leq L(t)\gamma(\Omega_2) + b(t)\gamma(\Omega_1) + \gamma(\Omega_1) + m \cdot L \cdot \gamma(\Omega_1),\end{aligned}$$

and so condition (2_{i}) is satisfied with $g(t) = L(t) + b(t) + 1 + m \cdot L$. The obtained form of $g(t)$ together with assumption (27) directly guarantee the condition (2_{iii}). It can also be easily shown that properties of F ensure the validity of condition (2_{ii}).

In order to verify conditions imposed on a bounding function, let us define $V : L^2(\mathbb{R}) \to \mathbb{R}$, $\alpha \to \frac{1}{2}(\|\alpha\|^2_{L^2(\mathbb{R})} - R^2)$. The function $V \in C^2(L^2(\mathbb{R}), \mathbb{R})$ with $\dot{V}_x : h \to \langle x, h\rangle$ obviously satisfies (10), so it is only necessary to check condition (11). Thus, let $\alpha \in L^2(\mathbb{R})$, $\|\alpha\|_{L^2(\mathbb{R})} = R$, $t \in (0, T)$, $v \in L^2(\mathbb{R})$ and $z \in F(t, \alpha, v)$. Then there exists $p^* \in P(\int_{\mathbb{R}} \psi(x)\alpha(x)\,dx)$ such that

$$z = -\hat{\varphi}(t, v) + b(t)\alpha + K(\alpha) + p^*\hat{f}(\alpha).$$

Moreover, since $p^* \geq 0$ and $\int_{\mathbb{R}} \alpha(x)f(\alpha(x))\,dx \geq 0$, we see that

$$p^* \int_{\mathbb{R}} \alpha(x)f\bigl(\alpha(x)\bigr)\,dx \geq 0, \tag{30}$$

and since

$$\left|\int_{\mathbb{R}} \alpha(x) \int_{\mathbb{R}} k(x, y)\alpha(y)\,dy\,dx\right| \leq R \int_{\mathbb{R}} \bigl|\alpha(x)\bigr| \cdot \bigl\|k(x, y)\bigr\|_{L^2(\mathbb{R})}\,dx \leq R^2,$$

we see that

$$\int_{\mathbb{R}} \alpha(x) \int_{\mathbb{R}} k(x, y)\alpha(y)\,dy\,dx \geq -R^2. \tag{31}$$

The properties (a)-(f) together with the well-known Hölder inequality then yield

$$\begin{aligned}
\langle \dot{V}_\alpha, z\rangle &= \langle \alpha, z\rangle \\
&= -\int_{\mathbb{R}} \alpha(x)\varphi\bigl(t, x, v(x)\bigr)\,dx + b(t)\int_{\mathbb{R}} \alpha^2(x)\,dx \\
&\quad + \int_{\mathbb{R}} \alpha(x) \int_{\mathbb{R}} k(x, y)\alpha(y)\,dy\,dx + p^* \int_{\mathbb{R}} \alpha(x)f\bigl(\alpha(x)\bigr)\,dx \\
&\geq -R\varphi_1(t)\|\varphi_2\|_{L^2(\mathbb{R})} + b_0R^2 - R^2 > 0,
\end{aligned}$$

in view of condition (26), (30), and (31).

Hence, the Dirichlet problem (29) admits, according to Theorem 3.1, a solution y satisfying $\|y(t)\|_{L^2(\mathbb{R})} \leq R$, for a.a. $t \in (0, T)$. If $u(t, x) := y(t)(x)$, then u is a solution of (24), (25) which is the Filippov solution of the original problem (23), (24).

Finally, we can sum up the above result in the form of the following theorem.

Theorem 4.1 *Let the assumptions* (a)-(f) *be satisfied. If still conditions* (26), (27) *hold, then the problem* (23), (24) *admits a non-trivial solution u in the sense of Fillippov such that* $\|u(t, \cdot)\|_{L^2(\mathbb{R})} \leq R$.

Remark 4.1 In [13, Example 5.2], the following formally simpler integro-differential equation in $\mathbb{R}$:

$$u_{tt}(t, x) = \int_0^1 \tilde{k}(x, y)u(t, y)\,dy, \quad t \in (0, 1), x \in [0, 1],$$

with non-homogeneous Dirichlet conditions

$$u(0,x) = u_0(x), \qquad u(1,x) = u_1(x), \quad x \in [0,1], u_0, u_1 \in L^2\big([0,1]\big),$$

was solved provided $\tilde{k} : [0,1] \times [0,1] \to (0,\infty)$ is a positive kernel of the Hilbert-Schmidt-type and the norms $\|u_0\|_{L^2([0,1])}$ and $\|u_1\|_{L^2([0,1])}$ are finite.

After the homogenization of boundary conditions, the Dirichlet problem takes the form

$$u_{tt}(t,x) = \tilde{\varphi}(t,x) + \int_0^1 \tilde{k}(x,y)u(t,y)\,dy, \quad t \in (0,1), x \in [0,1],$$
$$u(0,x) = u(1,x) = 0,$$

where $\tilde{\varphi}(t,x) := \int_0^1 \tilde{k}(x,y)\{[u_1(y) - u_0(y)]t + u_0(y)\}\,dy$.

Thus, it can be naturally extended onto the infinite strip $[0,1] \times \mathbb{R}$, into the form (23), (24), where

$$\varphi(t,x,y) \equiv \varphi(t,x) := \begin{cases} -\tilde{\varphi}(t,x) & \text{if } (t,x) \in (0,1) \times [0,1], \\ 0 & \text{otherwise}, \end{cases}$$

$$k(t,z) := \begin{cases} \tilde{k}(x,z) & \text{if } (x,z) \in [0,1] \times [0,1], \\ 0 & \text{otherwise}, \end{cases}$$

and $b(t) \equiv 0$, $p(r) \equiv 0$ or $f(s) \equiv 0$.

The result in [13, Example 5.2] cannot be, however, deduced from Theorem 3.1, because condition (b) in Example 4.1 cannot be satisfied in this way.

On the other hand, the linear term with coefficient b could not be implemented in their equation, because it is not completely continuous in (36) below, as required in [13].

In view of the arguments in Remark 4.1, we can conclude by the second illustrative example.

Example 4.2 Consider the following non-homogeneous Dirichlet problem in $\mathbb{R}$:

$$\left.\begin{aligned} &u_{tt}(t,x) = b(t)u(t,x) + \textstyle\int_0^1 \tilde{k}(x,y)u(t,y)\,dy, \quad t \in (0,1), x \in [0,1], \\ &u(0,x) = u_0(x), \qquad u(1,x) = u_1(x), \quad x \in [0,1], u_0, u_1 \in L^2([0,1]), \end{aligned}\right\} \tag{32}$$

where $\tilde{k} : [0,1] \times [0,1] \to (0,\infty)$ is a positive kernel of the Hilbert-Schmidt-type such that

$$k_0 := \|\tilde{k}\|_{L^2([0,1]\times[0,1])} < \infty$$

and $b \in L^1((0,1))$ is such that $b(t) \geq b_0 > 0$, for a.a. $t \in (0,1)$.

Furthermore, let there exist a constant $L < \frac{4}{5}$ such that

$$\operatorname*{ess\,sup}_{t\in(0,1)} b(t) \leq L. \tag{33}$$

The properties of u_0 and u_1 guarantee that there exists $B \geq 0$ such that

$$\|u_0 - u_1\|_{L^2([0,1])} + \|u_0\|_{L^2([0,1])} \leq B. \tag{34}$$

We will show that, under (33) and (34), problem (32) is solvable, in the abstract setting, by means of Corollary 3.1.

Problem (32) can be homogenized as follows:

$$\left.\begin{aligned}&\hat{u}_{tt}(t,x) = \varphi(t,x) + b(t)\hat{u}(t,x) + \textstyle\int_0^1 \tilde{k}(x,y)\hat{u}(t,y)\,dy, \quad t \in (0,1), x \in [0,1],\\ &\hat{u}(0,x) = \hat{u}(1,x) = 0,\end{aligned}\right\} \tag{35}$$

where

$$\varphi(t,x) := b(t)\bigl\{\bigl[u_1(x) - u_0(x)\bigr]t + u_0(x)\bigr\} + w(t,x), \quad t \in (0,1), x \in [0,1],$$

with $w(t,x) := \int_0^1 \tilde{k}(x,y)\{[u_1(y) - u_0(y)]t + u_0(y)\}\,dy$.

Since the Hilbert-Schmidt operator

$$\int_0^1 \overline{k}(y)(\cdot)\,dy : L^2\bigl([0,1]\bigr) \to L^2\bigl([0,1]\bigr),$$

where $\overline{k}(y)(\cdot) := \tilde{k}(x,y)$ is well known to be completely continuous (*cf.* [13, Example 5.2]) and $b(t)(\cdot) : L^2([0,1]) \to L^2([0,1])$ is, according to (33), L-Lipschitzian with $L < \frac{4}{5}$, conditions (i), (ii) in Corollary 3.1 can be easily satisfied, for $\overline{u}(t) := u(t,x)$, $\overline{u} \in L^2([0,1])$,

$$F_1(t,\overline{u},\overline{v}) \equiv F_1(t,\overline{u}) := \overline{\varphi}(t) + f(\overline{u}),$$

where $\overline{\varphi}(t) := b(t)\{[\overline{u}_1 - \overline{u}_0]t + \overline{u}_0\} + \overline{w}(t)$, $\overline{w}(t) := w(t,x)$,

$$f(\overline{u}) := \int_0^1 \overline{k}(y)\overline{u}(y)\,dy$$

and

$$F_2(t,\overline{u},\overline{v}) \equiv F_2(t,\overline{u}) := b(t)\overline{u}.$$

In this setting, problem (35) takes the abstract form as (22), namely

$$\left.\begin{aligned}&\ddot{\overline{u}}(t) = f(\overline{u}(t)) + b(t)\overline{u}(t) + \overline{\varphi}(t), \quad \text{for a.a. } t \in (0,1),\\ &\overline{u}(0) = \overline{u}(1) = 0.\end{aligned}\right\} \tag{36}$$

Since $\langle f(\overline{u}), \overline{u}\rangle \geq 0$ holds, for all $\overline{u} \in L^2([0,1])$ (see [13, Example 5.2]) one can check that the strict inequality in (iii) in Corollary 3.1 can be easily satisfied, for (32), whenever

$$R > \frac{B(L + k_0)}{b_0}. \tag{37}$$

Hence, applying Corollary 3.1, problem (36) admits a solution, say $\hat{u}(\cdot)$, such that

$$\|\hat{u}\|_{L^2([0,1])} \leq R,$$

where R satisfies (37), and subsequently the same is true for (32), *i.e.*

$$\max_{t\in[0,1]} \left\| \hat{u}(t,\cdot) \right\|_{L^2([0,1])} \leq R, \tag{38}$$

as claimed.

After all, we can sum up the sufficient conditions for the existence of a solution $\hat{u}$ of (32) satisfying (38) as follows:

- $\tilde{k}$ is a positive kernel of the Hilbert-Schmidt operator with the finite norm

$$k_0 := \|\tilde{k}\|_{L^2([0,1]\times[0,1])} < \infty,$$

- there exists $b_0 > 0$, $L < \frac{4}{5}$: $b_0 \leq b(t) \leq L$, for a.a. $t \in (0,1)$,
- condition (37) holds.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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Endnotes

[a] The m.n.c. $\mathrm{mod}_C(\Omega)$ is a monotone, non-singular and algebraically subadditive on $C([0,T],E)$ (*cf. e.g.* [27]) and it is equal to zero if and only if all the elements $x \in \Omega$ are equi-continuous.

[b] Since a C^2-function V has only a locally Lipschitzian Fréchet derivative $\dot{V}$ (*cf. e.g.* [21]), we had to assume explicitly the global Lipschitzianity of $\dot{V}$ in a non-compact set $\overline{B(\partial K, \varepsilon)}$.

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BOUNDING FUNCTION APPROACH FOR IMPULSIVE DIRICHLET PROBLEMS WITH UPPER-CARATHÉODORY RIGHT-HAND SIDE

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Abstract. In this article, we prove the existence and localization of solutions for a vector impulsive Dirichlet problem with multivalued upper-Carathéodory right-hand side. The result is obtained by combining the continuation principle with a bound sets technique. The main theorem is illustrated by an application to the forced pendulum equation with viscous damping term and dry friction coefficient.

1. Introduction

Given an upper-Carathéodory multivalued mapping $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$, we consider the multivalued vector Dirichlet problem

$$\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0,T], \tag{1.1}$$

$$x(T) = x(0) = 0. \tag{1.2}$$

Moreover, let a finite number of points $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, and real $n \times n$ matrices A_i, B_i, $i = 1, \dots, p$, be given.

In this article, we study the solvability of the boundary-value problem (1.1)-(1.2), in the presence of the impulse conditions

$$x(t_i^+) = A_i x(t_i), \quad i = 1, \dots, p, \tag{1.3}$$

$$\dot{x}(t_i^+) = B_i \dot{x}(t_i), \quad i = 1, \dots, p, \tag{1.4}$$

where $\lim_{t\to a^+} x(t) = x(a^+)$.

By a solution of (1.1)-(1.4) we mean a function $x \in PAC^1([0,T], \mathbb{R}^n)$ (see Section 2 for the definition) satisfying (1.1)–(1.4).

Boundary value problems with impulses have attracted lots of interest because of their applications in many areas such as: aircraft control, drug administration, biotechnology and population dynamics, where processes are characterized by the fact that the model parameters are subject to short term perturbations in time. For instance, in the treatment of some diseases, impulses may correspond to administration of a drug treatment; in environmental sciences, impulses may correspond to seasonal changes or harvesting; in economics, impulses may correspond to abrupt

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changes of prices. Impulsive differential equations and inclusions are adequate apparatus for modeling such processes and phenomena. The theory of single valued impulsive problems is widely developed and presents in many cases direct analogies with the results for problems without impulses (see, e.g., [11, 12, 24, 30]). The theory dealing with multivalued impulsive problems arises e.g. from single valued problems with discontinuous right-hand sides, problems with inaccurately known right-hand sides or from control theory. This field has not been so deeply studied and the results have been obtained in particular for the first-order problems and using fixed point theorems or upper and lower-solutions methods; for the overview of known results, we recommend the monographs [13, 21] and the references therein. Few results were obtained for Dirichlet impulsive problems using topological or variational approaches in cases when right-hand sides do not dependent on the first derivative or when the impulses depend only on the first derivative (see [1, 15, 16, 18, 25, 29]).

In this paper, not only the existence but also the localization of solutions for the impulsive multivalued Dirichlet problem (1.1)-(1.4) are obtained by means of bound sets technique. The bound sets approach was introduced in the single valued case by Gaines and Mawhin [20] for obtaining the existence of solutions of first and second order differential equations. This technique was applied for multivalued Dirichlet, Floquet or two-point problems without impulses in [4]-[9], [28, 32]. The existence and localization result presented in Theorem 4.1 below will be obtained by combining the bound sets approach with the continuation principle developed in Section 2.

This article is organized as follows. In the second section, we recall suitable definitions and statements which will be used in the sequel. Section 3 is devoted to the study of bound sets and Liapunov-like bounding functions for impulsive Dirichlet problems. At first, we consider C^1-bounding functions with locally Lipschitzian gradients. Consequently, it is shown how conditions ensuring the existence of bound set become in case of C^2-bounding functions. In Section 4, the bound sets approach is combined with the continuation principle and an existence and localization result is obtained in this way for the impulsive Dirichlet problem (1.1)-(1.4). Section 5 deals with an application to the forced pendulum equation with viscous damping term and dry friction coefficient.

2. Preliminaries

We start with the notation used in this article. Let (X, d) be a metric space and $A \subset X$. By $\overline{A}$, $\operatorname{int} A$ and ∂A, we mean the closure, interior and boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X : \exists a \in A : d(x,a) < \varepsilon\}$, hence $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a continuous function $r : X \to A$ satisfying $r(x) = x$ for every $x \in A$; this function is called a retraction.

For a given compact real interval J, we denote by $C(J, \mathbb{R}^n)$ (by $C^1(J, \mathbb{R}^n)$) the set of all functions $x : J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $AC^1(J, \mathbb{R}^n)$, we denote the set of functions $x : J \to \mathbb{R}^n$ with absolutely continuous first derivatives on J. In the sequel, the norm of a real $n \times n$ matrix will be denoted by $\|\cdot\|$ and the norm in $L^1(J, \mathbb{R})$ by the symbol $\|\cdot\|_1$.

Let $PAC^1([0,T],\mathbb{R}^n)$ be the space of functions $x:[0,T]\to\mathbb{R}^n$ such that

$$x(t)=\begin{cases} x_{[0]}(t), & \text{for } t\in[0,t_1],\\ x_{[1]}(t), & \text{for } t\in(t_1,t_2],\\ \dots \\ x_{[p]}(t), & \text{for } t\in(t_p,T],\end{cases}$$

where $x_{[0]}\in AC^1([0,t_1],\mathbb{R}^n)$, $x_{[i]}\in AC^1((t_i,t_{i+1}],\mathbb{R}^n)$, $x(t_i^+)=\lim_{t\to t_i^+}x(t)\in\mathbb{R}$ and $\dot{x}(t_i^+)=\lim_{t\to t_i^+}\dot{x}(t)\in\mathbb{R}$, for $i=1,\dots,p$. The space $PAC^1([0,T],\mathbb{R}^n)$ equipped with the norm

$$\|x\|_E:=\sup_{t\in[0,T]}|x(t)|+\sup_{t\in[0,T]}|\dot{x}(t)|, \tag{2.1}$$

is denoted by $(E,\|\cdot\|_E)$. In a similar way, we can define the spaces $PC([0,T],\mathbb{R}^n)$ and $PC^1([0,T],\mathbb{R}^n)$ as the spaces of functions $x:[0,T]\to\mathbb{R}^n$ satisfying the previous definition with $x_{[0]}\in C([0,t_1],\mathbb{R}^n)$, $x_{[i]}\in C((t_i,t_{i+1}],\mathbb{R}^n)$, and with $x_{[0]}\in C^1([0,t_1],\mathbb{R}^n)$, $x_{[i]}\in C^1((t_i,t_{i+1}],\mathbb{R}^n)$, for $i=1,\dots,p$, respectively. The space $PC^1([0,T],\mathbb{R}^n)$ with the norm defined in (2.1) is a Banach space (see [27, page 128]). A compactness result for subsets of $PC^1([0,T],\mathbb{R}^n)$ will be needed. So we recall that a family $\mathcal{F}\subset PC([0,T],\mathbb{R}^n)$ is left equicontinuous (see [27]) if for every $\epsilon>0$ and $x\in[0,T]$ there exists $\delta>0$ such that, for every $f\in\mathcal{F}$,

$$|f(x)-f(y)|<\epsilon,\quad \text{for all } y\in(x-\delta,x]$$

and

$$|f(x^+)-f(y)|<\epsilon,\ \text{for all }\ y\in(x,x+\delta).$$

In the sequel, we use a generalized Ascoli-Arzelà theorem whose prove is given in [27, Theorem 2], in a slightly different case, i.e. when the real valued functions are discontinuous from the left and are just continuous in each interval $[t_i,t_{i+1})$.

Proposition 2.1. *A family $\mathcal{F}\subset PC^1([0,T],\mathbb{R}^n)$ is compact if and only if it is bounded, left equicontinuous and the set $\{f':f\in\mathcal{F}\}$ is left equicontinuous.*

We also need the following definitions and notion for multivalued mappings. We say that F is a *multivalued mapping* from X to Y (written $F:X\multimap Y$), if, for every $x\in X$, a nonempty subset $F(x)$ of Y is given. We associate to F its graph Γ_F, i.e. the subset of $X\times Y$ defined by

$$\Gamma_F:=\{(x,y)\in X\times Y \mid y\in F(x)\}.$$

The single valued function $f:X\to Y$ is called a *selection* of F if $\Gamma_f\subset\Gamma_F$, i.e. if $f(x)\in F(x)$, for every $x\in X$.

A multivalued mapping $F:X\multimap Y$ is called *upper semi-continuous* (abbreviated, u.s.c.) if, for each open set $U\subset Y$, the set $\{x\in X:F(x)\subset U\}$ is open in X. A multivalued mapping $F:X\multimap Y$ is called *compact* if the set $F(X)=\cup_{x\in X}F(x)$ is contained in a compact subset of Y. Let us note that every u.s.c. mapping with closed values has a closed graph and that every compact multivalued mapping with closed graph is u.s.c.

Let Y be a metric space and $(\Omega,\mathcal{U},\mu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure μ on $\mathcal{U}$. A multivalued mapping $F:\Omega\multimap Y$ is called *measurable* if $\{\omega\in\Omega:F(\omega)\subset V\}\in\mathcal{U}$, for each open set $V\subset Y$.

We say that the mapping $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $F(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $F(t, \cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for a.a. $t \in J$, and the set $F(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

We shall use the following selection result, which was proved in [14, Proposition 6] in a quite general setting for a continuous function q. Its proof can be easily extended to piecewise continuous functions, so we omit it here.

Proposition 2.2. *Let $J \subset \mathbb{R}$ be a compact interval and $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that for every $r > 0$ there exists an integrable function $\mu_r : J \to [0, \infty)$ satisfying $|y| \le \mu_r(t)$, for every $(t, x) \in J \times \mathbb{R}^m$, with $|x| \le r$, and every $y \in F(t, x)$. Then the composition $F(t, q(t))$ admits, for every $q \in PC(J, \mathbb{R}^m)$, a measurable selection.*

Let $X \cap Y \neq \emptyset$ and $F : X \multimap Y$. We say that a point $x \in X \cap Y$ is a *fixed point* of F if $x \in F(x)$. The set of all fixed points of F is denoted by $Fix(F)$, i.e.

$$Fix(F) := \{x \in X : x \in F(x)\}.$$

The following proposition will be applied for obtaining the existence of solutions to boundary value problems. It follows from a result in [2, 3].

Proposition 2.3. *Let X be a retract of a Banach space Y, and let $\mathfrak{T} : X \times [0, 1] \multimap Y$ be a compact u.s.c. mapping with convex values such that $\mathfrak{T}(X, 0) \subset X$ and that $Fix(\mathfrak{T}(x, \lambda)) \cap \partial X = \emptyset$, for every $\lambda \in [0, 1)$. Then $\mathfrak{T}(\cdot, 1)$ has a fixed point.*

We also need the following modification of the continuation principle developed in [10] for problems on arbitrary, possibly non-compact, intervals. The differences between the presented result and the one in [10] consist in replacement of the non-compact interval by the compact one which simplify the last, so called transversality condition, and in replacement of the space $AC^1_{loc}([0, T], \mathbb{R}^n)$ by the space E defined above. For the completeness, the proof of this modified result is given here.

Proposition 2.4. *Let us consider the boundary-value problem*

$$\begin{gathered} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0, T], \\ x \in S, \end{gathered} \tag{2.2}$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of E. Let $H : [0, T] \times \mathbb{R}^{4n} \times [0, 1] \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that

$$H(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in [0, T] \times \mathbb{R}^{2n}. \tag{2.3}$$

Assume that

(i) *there exists a retract Q of $PC^1([0, T], \mathbb{R}^n)$, with $Q \setminus \partial Q \neq \emptyset$, and a closed subset S_1 of S such that the associated problem*

$$\begin{gathered} \ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in [0, T], \\ x \in S_1 \end{gathered} \tag{2.4}$$

has, for each $(q, \lambda) \in Q \times [0, 1]$, a non-empty and convex set of solutions $\mathfrak{T}(q, \lambda)$;

(ii) *there exists a nonnegative, integrable function $\alpha : [0, T] \to \mathbb{R}$ such that*

$$|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)| \le \alpha(t)(1 + |x(t)| + |\dot{x}(t)|),$$

for a.a. $t \in [0, T]$, and for any $(q, \lambda, x) \in \Gamma_{\mathfrak{T}}$;

(iii) $\mathfrak{T}(Q \times \{0\}) \subset Q$;
(iv) *there exist constants* $M_0 \geq 0$, $M_1 \geq 0$ *such that* $|x(0)| \leq M_0$ *and* $|\dot{x}(0)| \leq M_1$, *for all* $x \in \mathfrak{T}(Q \times [0,1])$;
(v) *the solution map* $\mathfrak{T}(\cdot, \lambda)$ *has no fixed points on the boundary* ∂Q *of* Q, *for every* $\lambda \in [0,1)$.

Then (2.2) *has a solution in* $S_1 \cap Q$.

Proof. Let us apply Proposition 2.3, where $X = Q$ is a retract of the Banach space $Y = PC^1([0,T], \mathbb{R}^n)$. First of all, notice that if there exists $q \in \partial Q$ such that $\mathfrak{T}(q,1) = q$, then the result is proven. Otherwise, we get that $\mathfrak{T}(Q \times [0,1]) \cap \partial Q = \emptyset$, according to assumption (v). Moreover, it follows from conditions (i) and (iii), that $\mathfrak{T}$ has convex values and that $\mathfrak{T}(Q, 0) \subset Q$.

Let us now show that $\mathfrak{T}$ has a closed graph. Let $\{(q_k, \lambda_k, x_k)\} \subset \Gamma_{\mathfrak{T}}$ such that $(q_k, \lambda_k, x_k) \to (q, \lambda, x)$, $(q, \lambda) \in Q \times [0,1]$ be arbitrary. Then, since $x_k \in S_1$, $x_k \to x$ and S_1 is closed, it holds that $x \in S_1$. Moreover, x_k is a solution of (2.4), and so, according to Proposition 2.2, we get the existence of $h_k \in H(\cdot, x_k(\cdot), \dot{x}_k(\cdot), q_k(\cdot), \dot{q}_k(\cdot), \lambda_k)$ such that $\dot{x}_k(t_{i+1}) - \dot{x}_k(t) = \int_t^{t_{i+1}} h_k(s) ds$, for every $t \in (t_i, t_{i+1}]$ and $i = 0, \dots, p$. The convergence of $\{x_k\}$ implies its boundedness in $PC^1([0,T], \mathbb{R}^n)$, and therefore, we get from (ii) that $|h_k(t)| \leq \alpha(t)(1 + M)$, for some $M > 0$, every $k \in \mathbb{N}$ and a.a. $t \in [0,T]$. This implies that $\{h_k\}$ is bounded in $L^1([0,T], \mathbb{R}^n)$, and so it has a weakly convergent subsequence, for the sake of simplicity still denoted as the sequence, which converges to a function h. In particular, $\int_t^{t_{i+1}} h_k(s)\, ds \to \int_t^{t_{i+1}} h(s)\, ds$, for every $t \in (t_i, t_{i+1}]$ and $i = 0, \dots, p$. Hence,

$$\dot{x}(t_{i+1}) - \dot{x}(t) = \lim_{k \to \infty} [\dot{x}_k(t_{i+1}) - \dot{x}_k(t)] = \lim_{k \to \infty} \int_t^{t_{i+1}} h_k(s)\, ds = \int_t^{t_{i+1}} h(s)\, ds,$$

for $t \in (t_i, t_{i+1}]$ and $i = 0, \dots, p$. Therefore, there exists $\ddot{x}(t) = h(t)$, for a.a. $t \in [0,T]$. It remains to prove that $h \in H(\cdot, x(\cdot), \dot{x}(\cdot), q(\cdot), \dot{q}(\cdot), \lambda)$. Since H is upper-Carathéodory, there exists, for every $\epsilon > 0$ and a.a. $t \in [0,T]$, a positive number δ such that, if $|(c,d,e,f,g) - (q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda)| \leq \delta$, then

$$H(t,c,d,e,f,g) \subset H(t, q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda) + B_0^{\epsilon}.$$

Recalling that the convergence in $PC^1([0,T], \mathbb{R}^n)$ of q_k to q and x_k to x implies the pointwise convergence of both sequences and of the sequences of their derivatives to the same limits, we get that, for every $t \in [0,T]$ and $\delta > 0$, there exists $\overline{k}$ such that, for $k \geq \overline{k}$, $|(q_k(t), \dot{q}_k(t), x_k(t), \dot{x}_k(t), \lambda_k) - (q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda)| \leq \delta$. Therefore, for every $\epsilon > 0$ and a.a. $t \in [0,T]$, there exists $\overline{k}$ such that, if $k \geq \overline{k}$, then

$$h_k(t) \in H(t, q_k(t), \dot{q}_k(t), x_k(t), \dot{x}_k(t), \lambda_k) \subset H(t, q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda) + B_0^{\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, we get that $h(t) \in H(t, q(t), \dot{q}(t), x(t), \dot{x}(t), \lambda)$, for a.a. $t \in [0,T]$, i.e. that $\mathfrak{T}$ has a closed graph. Recalling that a compact mapping with closed graph is u.s.c. and has compact values, it remains only to prove that $\mathfrak{T}$ is compact. According to Proposition 2.1, we need to prove that $\mathfrak{T}(Q \times [0,1])$ is bounded, left equicontinuous, and has left equicontinuous set of derivatives.

Let $x \in \mathfrak{T}(q, \lambda)$. Then there exists $h \in H(\cdot, x(\cdot), \dot{x}(\cdot), q(\cdot), \dot{q}(\cdot), \lambda)$ such that, for every $\bar{t}, \tilde{t} \in (t_i, t_{i+1}]$, with $\bar{t} > \tilde{t}$, and $i = 0, \dots, p$,

$$\dot{x}(\bar{t}) = \dot{x}(\tilde{t}) + \int_{\tilde{t}}^{\bar{t}} h(s)\, ds, \tag{2.5}$$

and consequently, according to Fubini's theorem,

$$\begin{aligned} x(\bar{t}) &= x(\tilde{t}) + \dot{x}(\tilde{t})(\bar{t}-\tilde{t}) + \int_{\tilde{t}}^{\bar{t}} \int_{\tilde{t}}^{r} h(s)\,ds\,dr \\ &= x(\tilde{t}) + \dot{x}(\tilde{t})(\bar{t}-\tilde{t}) + \int_{\tilde{t}}^{\bar{t}} (\bar{t}-s)h(s)\,ds. \end{aligned} \tag{2.6}$$

According to (ii) and (iv), for every $t \in [0, t_1]$, it holds that

$$|x(t)| + |\dot{x}(t)| \le M_0 + M_1(t_1+1) + (t_1+1)\int_0^t \alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,ds.$$

Therefore, if we denote by $\beta_1 := M_0 + M_1(t_1+1) + (t_1+1)\int_0^{t_1}\alpha(s)ds$, we obtain by Gronwall's lemma that

$$|x(t)| + |\dot{x}(t)| \le \beta_1 + \beta_1(t_1+1)\int_0^{t_1} \alpha(s) e^{(t_1+1)\int_s^{t_1}\alpha(r)\,dr}\,ds := C_1.$$

Take now $t \in (t_1, t_2]$. Reasoning as above we obtain

$$\begin{aligned} &|x(t)| + |\dot{x}(t)| \\ &\le |x(t_1^+)| + |\dot{x}(t_1^+)|(t_2+1) + (t_2 - t_1 + 1)\int_{t_1}^t \alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,ds \\ &\le \|A_1\|\cdot|x(t_1)| + \|B_1\|\cdot|\dot{x}(t_1)|(t_2+1) \\ &\quad + (t_2-t_1+1)\int_{t_1}^t \alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,ds \\ &\le \max\{\|A_1\|, \|B_1\|(t_2+1)\}C_1 + (t_2-t_1+1)\int_{t_1}^t \alpha(s)(1+|x(s)|+|\dot{x}(s)|)\,ds. \end{aligned}$$

Hence, denoted by $\beta_2 := \max\{\|A_1\|, \|B_1\|(t_2+1)\}C_1 + (t_2-t_1+1)\int_{t_1}^{t_2}\alpha(s)\,ds$, we obtain that

$$|x(t)| + |\dot{x}(t)| \le \beta_2 + \beta_2(t_2-t_1+1)\int_{t_1}^{t_2} \alpha(s) e^{(t_2-t_1+1)\int_s^{t_2}\alpha(r)dr}\,ds := C_2.$$

Iterating we obtain the existence of $D > 0$ such that $|x(t)| + |\dot{x}(t)| \le D$, for every $t \in [0, T]$, i.e. we obtain that $\mathfrak{T}(Q \times [0,1])$ is bounded in $PC^1([0,T], \mathbb{R}^n)$.

Moreover, it follows from (2.5) and (2.6) that, that for every $\bar{t}, \tilde{t} \in (t_i, t_{i+1}]$ with $\bar{t} > \tilde{t}$ and $i = 0, \dots, p$,

$$|\dot{x}(\bar{t}) - \dot{x}(\tilde{t})| = \Big|\int_{\tilde{t}}^{\bar{t}} h(s)ds\Big| \le (1+D)\int_{\tilde{t}}^{\bar{t}} \alpha(s)ds,$$

$$|x(\bar{t}) - x(\tilde{t})| \le D|\bar{t}-\tilde{t}| + (1+D)\int_{\tilde{t}}^{\bar{t}} (\bar{t}-s)\alpha(s)ds.$$

Thus, if $t \ne t_1, \dots, t_p$, one can take δ sufficiently small such that $(t-\delta, t+\delta) \cap \{t_1, \dots, t_p\} = \emptyset$ and conclude (from the absolute continuity of the Lebesgue integral) that the functions x and $\dot{x}$ are equicontinuous at t. The left equicontinuity can be deduced similarly for $t \in \{t_1, \dots, t_p\}$.

So, we have proved that $\mathfrak{T}(Q \times [0,1))$ is compact, and hence, it follows from Proposition 2.3, that there exists a fixed point of $\mathfrak{T}(\cdot, 1)$ in $S_1 \cap Q$. $\square$

The continuation principle described in Proposition 2.4 requires in particular that any of corresponding problems does not have solutions tangent to the boundary of a given set Q of candidate solutions. In Section 4, we will ensure that the candidate solutions are not tangent to the boundary of Q by means of Hartman-type conditions (see Section 3) and by means of the following result based on Nagumo conditions (see [31, Lemma 2.1] and [23, Lemma 5.1]).

Proposition 2.5. *Let $\psi : [0, +\infty) \to [0, +\infty)$ be a continuous and non-decreasing function, with*

$$\lim_{s\to\infty} \frac{s^2}{\psi(s)} ds = \infty, \tag{2.7}$$

and let R be a positive constant. Then there exists a positive constant

$$B = \psi^{-1}(\psi(2R) + 2R) \tag{2.8}$$

such that if $x \in PC^1([0, T], \mathbb{R}^n)$ is such that $|\ddot{x}(t)| \le \psi(|\dot{x}(t)|)$, for a.a. $t \in [0, T]$, and $|x(t)| \le R$, for every $t \in [0, T]$, then it holds that $|\dot{x}(t)| \le B$, for every $t \in [0, T]$.

Let us note that the previous result is classically given for C^2-functions. However, it is easy to prove (see, e.g., [7]) that the statement holds also for piecewise continuously differentiable functions.

3. Bound sets theory for impulsive Dirichlet problems

The direct verification of transversality condition (v) in Proposition 2.4 is quite complicated. Therefore, we now introduce a Liapunov-like function V, usually called *bounding function*, which can guarantee this condition.

Let $K \subset \mathbb{R}^n$ be a nonempty, open set with $0 \in K$ and let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying

(H1) $V|_{\partial K} = 0$,
(H2) $V(x) \le 0$, for all $x \in \overline{K}$.

Definition 3.1. A set K is called a *bound set for the impulsive Dirichlet problem* (1.1)-(1.4) if every solution x of (1.1)-(1.4) such that $x(t) \in \overline{K}$, for each $t \in [0, T]$, does not satisfy $x(t^*) \in \partial K$, for any $t^* \in [0, T]$.

Remark 3.2. Note that the existence of a bound set K for problem (1.1)-(1.4) does not guarantee the existence of a solution for (1.1) -(1.4). It only ensures that if there would exist a solution laying in $\overline{K}$, then this solution would not touch the boundary of K at any point, i.e. it would lay in $\operatorname{int} K$.

At first, the sufficient conditions for the existence of a bound set for the impulsive Dirichlet problem (1.1)-(1.4) in the general case will be shown in Proposition 3.3 below. Afterwards, the regularity assumptions on the bounding function V will be made more strict and the practically applicable version of Proposition 3.3 will be obtained (see Corollary 3.5 below).

Proposition 3.3. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$ and $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping. Let a finite number of points $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, be given and let A_i, B_i, $i = 1, \ldots, p$, be real $n \times n$ matrices such that $A_i \partial K = \partial K$, for all $i = 1, \ldots, p$.*

Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$, with ∇V locally Lipschitzian, satisfying conditions (H1) *and* (H2)*. Suppose, moreover, that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$ and $v \in \mathbb{R}^n$, the condition*

$$\limsup_{h \to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{3.1}$$

holds for all $w \in F(t,x,v)$, and that

$$\langle \nabla V(A_i x), B_i v\rangle \cdot \langle \nabla V(x), v\rangle > 0, \tag{3.2}$$

for all $i = 1, \dots, p$, $x \in \partial K$ and $v \in \mathbb{R}^n$ with $\langle \nabla V(x), v\rangle \neq 0$.

Then K is a bound set for the impulsive Dirichlet problem (1.1)-(1.4)*.*

Proof. We assume, by a contradiction, that K is not a bound set for the Dirichlet problem (1.1)-(1.4), i.e. that there exist a solution $x : [0,T] \to \overline{K}$ of (1.1)-(1.4) and $t^* \in [0,T]$ such that $x(t^*) \in \partial K$. The point t^* must lay in $(0,T)$, according to the boundary condition (1.2) and the fact that $0 \in K$.

Let us define a function $g : [0,T] \to \mathbb{R}$ by the formula $g(t) := V(x(t))$. According to the properties of x and V, $g \in PC^1([0,T], \mathbb{R})$ and $g(t) \le 0$ for all t. Since $g(t^*) = 0$, the point t^* is a local maximum point for g. Therefore, if $t^* \notin \{t_1, \dots, t_p\}$, $\dot g(t^*) = 0$. Let us now prove that $\dot g(t^*) = 0$ also when $t^* = t_{i+1}$, for some $i = 0, \dots, p-1$. By a contradiction, suppose that

$$0 < \dot g(t_{i+1}) = \langle \nabla V(x(t_{i+1})), \dot x(t_{i+1})\rangle. \tag{3.3}$$

Notice that also $A_{i+1}x(t_{i+1}) \in \partial K$, and hence $g(t_{i+1}^+) = g(A_{i+1}x(t_{i+1})) = 0$. According to condition (3.2), there exist two functions $a(h)$ and $b(h)$, with $a(h) \to 0, b(h) \to 0$ when $h \to 0$, such that

$$\begin{aligned}
\dot g(t_{i+1}^+) &= \lim_{h\to 0^+} \frac{V(x(t_{i+1}+h)) - V(x(t_{i+1}^+))}{h}\\
&= \lim_{h\to 0^+} \frac{V(x(t_{i+1}^+) + \dot x(t_{i+1}^+)h + a(h)h) - V(x(t_{i+1}^+))}{h}\\
&= \lim_{h\to 0^+} \frac{\langle \nabla V(x(t_{i+1}^+), \dot x(t_{i+1}^+) + a(h)\rangle h + b(h)h}{h}\\
&= \langle \nabla V(x(t_{i+1}^+)), \dot x(t_{i+1}^+)\rangle\\
&= \langle \nabla V(A_{i+1}x(t_{i+1})), B_{i+1}\dot x(t_{i+1})\rangle > 0.
\end{aligned}$$

Thus, for $t > t_{i+1}$ sufficiently close to t_{i+1}, we get that $0 \ge g(t) > g(t_{i+1}^+) = 0$, a contradiction. Therefore, $\dot g(t^*) = 0$ also in the case when $t^* = t_{i+1}$.

Since ∇V is locally Lipschitzian, there exist a bounded set $U \subset \mathbb{R}^n$ with $x(t^*) \in U$ and a constant $L > 0$ such that $\nabla V|_U$ is Lipschitzian with constant L. The continuity of x in $(t_i, t_{i+1}]$ then yields the existence of $\delta > 0$, $\delta < t^* - t_i$, such that $x(t) \in U \cap N_\varepsilon(\partial K)$, for each $t \in [t^* - \delta, t^*]$. Since $\dot g(t) = \langle \nabla V(x(t)), \dot x(t)\rangle$, where $\nabla V(x(t))$ is locally Lipschitzian and $\dot x(t)$ is absolutely continuous on $[t^* - \delta, t^*]$, there exists $\ddot g \in L^1([t^* - \delta, t^*], \mathbb{R})$. Moreover, there exists a point $t^{**} \in (t^* - \delta, t^*)$, such that $\dot g(t^{**}) \ge 0$, because t^* is a local maximum point. Consequently,

$$0 \ge -\dot g(t^{**}) = \dot g(t^*) - \dot g(t^{**}) = \int_{t^{**}}^{t^*} \ddot g(s)\, ds. \tag{3.4}$$

Let $t \in (t^{**}, t^*)$ be such that $\ddot{g}(t)$ and $\ddot{x}(t)$ exist. Then there exist two functions $a(h)$ and $b(h)$, with $a(h) \to 0$, $b(h) \to 0$ when $h \to 0$, such that, for each h,

$$\dot{x}(t+h) = \dot{x}(t) + h[\ddot{x}(t) + a(h)], \tag{3.5}$$

$$x(t+h) = x(t) + h[\dot{x}(t) + b(h)]. \tag{3.6}$$

Consequently,

$$\begin{aligned}
&\ddot{g}(t) \\
&= \lim_{h\to 0} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} = \limsup_{h\to 0^-} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} \\
&= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t+h)), \dot{x}(t+h)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h[\dot{x}(t) + b(h)]), \dot{x}(t) + h[\ddot{x}(t) + a(h)]\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&\geq \limsup_{h\to 0^-} \Big[\frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h[\ddot{x}(t) + a(h)]\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&\quad - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]|\Big] \\
&= \limsup_{h\to 0^-} \Big[\frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} \\
&\quad - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| + \langle \nabla V(x(t) + h\dot{x}(t)), a(h)\rangle\Big].
\end{aligned}$$

Since $\langle \nabla V(x(t) + h\dot{x}(t)), a(h)\rangle - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| \to 0$ as $h \to 0$ and since assumption (3.1) holds,

$$\ddot{g}(t) \geq \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} > 0,$$

which leads to a contradiction with inequality (3.4). $\square$

Definition 3.4. A function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying (H1), (H2), (3.1), and (3.2) is called a *bounding function* for (1.1)-(1.4).

When the bounding function V is of class C^2, condition (3.1) can be rewritten in terms of gradients and Hessian matrices.

Corollary 3.5. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$ and $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping. Let a finite number of points $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, be given and let A_i, B_i, $i = 1, \dots, p$, be real $n \times n$ matrices such that $A_i \partial K = \partial K$, for all $i = 1, \dots, p$.*

Assume that there exists a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ satisfying conditions (H1), (H2), *and* (3.2). *Moreover, assume that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0, T)$ and $v \in \mathbb{R}^n$, the condition*

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0 \tag{3.7}$$

holds for all $w \in F(t, x, v)$. Then K is a bound set for problem (1.1)-(1.4).

Proof. The statement follows immediately from the fact that if $V \in C^2(\mathbb{R}^n, \mathbb{R})$, then, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0, T)$, $v \in \mathbb{R}^n$ and $w \in F(t, x, v)$, there exists

$$\lim_{h\to 0} \frac{\langle \nabla V(x + hv), v + hw\rangle - \langle \nabla V(x), v\rangle}{h} = \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle.$$

□

Remark 3.6. In conditions (3.1), (3.2) and (3.7), the element v plays the role of the first derivative of the solution x. If $x(t) \in \overline{K}$, for every $t \in J$, then, according to Proposition 2.5 and the fact that $R = \max\{|c| : c \in \overline{K}\} \in \mathbb{R}$, it holds that $|\dot{x}(t)| \leq B$, for every $t \in J$, where B is defined by (2.8). Hence, it is sufficient to require conditions (3.1), (3.2) and (3.7) in Proposition 3.3 and Corollary 3.5 only for all $v \in \mathbb{R}^n$ with $|v| \leq B$ and not for all $v \in \mathbb{R}^n$.

4. Existence and localization results for Dirichlet problems

In this section,we study (1.1)-(1.4) by combining the continuation principle in Proposition 2.4 with bound sets results developed in the previous section. After rewriting (1.1)-(1.4) in the abstract form (2.2), we will be able to verify all conditions in Proposition 2.4.

Theorem 4.1. *Let $K \subset \mathbb{R}^n$ be a nonempty, open, bounded and convex set with $0 \in K$ and let us consider (1.1)-(1.4), where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, and A_i, B_i, $i = 1, \dots, p$, are real $n \times n$ matrices with $A_i \partial K = \partial K$, for all $i = 1, \dots, p$. Moreover, assume that*

(i) *there exists a function $\beta : [0,\infty) \to [0,\infty)$ continuous and non-decreasing satisfying*
$$\lim_{s\to\infty} \frac{s^2}{\beta(s)} ds = \infty \tag{4.1}$$
such that
$$|F(t,c,d)| \leq \beta(|d|), \tag{4.2}$$
for a.a. $t \in [0,T]$ and every $c, d \in \mathbb{R}^n$ with $|c| \leq R := \max\{|x| : x \in \overline{K}\}$;

(ii) *the problem*
$$\begin{gathered} \ddot{x}(t) = 0, \quad \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i), \quad i = 1, \dots, p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), \quad i = 1, \dots, p, \end{gathered} \tag{4.3}$$
has only the trivial solution;

(iii) *there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$, with ∇V locally Lipschitzian, satisfying conditions (H1) and (H2);*

(iv) *there exists $\varepsilon > 0$ such that, for all $\lambda \in (0,1), x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$, and $v \in \mathbb{R}^n$, with $|v| \leq \phi^{-1}(\phi(2R) + 2R)$, the condition*
$$\limsup_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{4.4}$$
holds for all $w \in \lambda F(t,x,v)$;

(v) *for all $i = 1, \dots, p$, $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \leq \phi^{-1}(\phi(2R) + 2R)$ and $\langle \nabla V(x), v\rangle \neq 0$, it holds*
$$\langle \nabla V(A_i x), B_i v\rangle \cdot \langle \nabla V(x), v\rangle > 0.$$

Then (1.1)-(1.4) has a solution $x(\cdot)$ such that $x(t) \in K$, for all $t \in [0,T]$.

Proof. For every $c \in \overline{K}$, it holds that $|c| \le R$. According to Proposition 2.5, for every $x \in PC^1([0,T],\mathbb{R}^n)$ with $|\ddot{x}(t)| \le \beta(|\dot{x}(t)|)$, for a.a. $t \in [0,T]$, and $x(t) \in \overline{K}$, for every $t \in [0,T]$, it holds $|\dot{x}(t)| \le B$, for every $t \in [0,T]$, with B defined by

$$B = \beta^{-1}(\beta(2R) + 2R).$$

Define

$$Q := \{q \in PC^1([0,T],\mathbb{R}^n) : q(t) \in \overline{K},\ |\dot{q}(t)| \le 2B, \quad \text{for all } t \in [0,T]\}, \tag{4.5}$$

$S = S_1 = Q$ and $H(t,c,d,e,f,\lambda) = \lambda F(t,e,f)$. Thus the associated problem (2.4) is the fully linearized problem

$$\begin{gathered}\ddot{x}(t) \in \lambda F(t,q(t),\dot{q}(t)), \quad \text{for a.a. } t \in [0,T],\\ x(T) = x(0) = 0,\\ x(t_i^+) = A_i x(t_i),\ i = 1,\dots,p,\\ \dot{x}(t_i^+) = B_i \dot{x}(t_i),\ i = 1,\dots,p.\end{gathered} \tag{4.6}$$

For each $(q,\lambda) \in Q \times [0,1]$, let $\mathfrak{T}(q,\lambda)$ be the solution set of (4.6). Now we check that all the assumptions of Proposition 2.4 are satisfied.

Since the closure of a convex set is still a convex set, it follows that Q is convex, and hence a retract of $PC^1([0,T],\mathbb{R}^n)$.

Condition (ii) follows from assumption (i) and the fact that

$$\begin{aligned}|H(t,x(t),\dot{x}(t),q(t),\dot{q}(t),\lambda| &= \lambda|F(t,q(t),\dot{q}(t))| \le \beta(|\dot{q}(t)|) \le \beta(2B)\\ &\le \beta(2B)(1 + |x(t)| + |\dot{x}(t)|),\end{aligned}$$

for every $\lambda \in [0,1], q \in Q, x \in \mathfrak{T}(q,\lambda)$. In particular $|F(t,e,f)| \le \beta(r)$ for every $(t,e,f) \in J \times \mathbb{R}^{2n}$ with $|f| \le r$.

Let $q \in Q$ and let f_q be a measurable selection of $F(\cdot,q(\cdot),\dot{q}(\cdot))$, whose existence is guaranteed applying Proposition 2.2 with $\mu_r(t) \equiv \beta(r)$. Then, for any $\lambda \in [0,1]$, λf_q is a measurable selection of $\lambda F(\cdot,q(\cdot),\dot{q}(\cdot))$. Let us consider the corresponding single valued linear problem with linear impulses

$$\begin{gathered}\ddot{x}(t) = \lambda f_q(t),\ \text{for a.a. } t \in [0,T],\\ x(T) = x(0) = 0,\\ x(t_i^+) = A_i x(t_i),\ i = 1,\dots,p,\\ \dot{x}(t_i^+) = B_i \dot{x}(t_i),\ i = 1,\dots,p.\end{gathered} \tag{4.7}$$

First of all, let us prove that problem (4.7) has a unique solution $x_{\lambda f_q}$. If we denote

$$C := \begin{cases} B_1(T - t_1) & \text{if } p = 1\\ \prod_{l=1}^p B_l(T - t_p) + \prod_{k=1}^p A_k t_1 & \\ + \sum_{j=2}^p \prod_{k=j}^p A_k \prod_{l=1}^{j-1} B_l(t_j - t_{j-1}) & \text{if } p \ge 2,\end{cases} \tag{4.8}$$

it is easy to prove that the initial problem

$$\begin{gathered}\ddot{x}(t) = 0, \quad \text{for a.a. } t \in [0,T],\\ x(0) = 0,\\ x(t_i^+) = A_i x(t_i), \quad i = 1,\dots,p,\\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), \quad i = 1,\dots,p\end{gathered}$$

has infinitely many solutions,

$$x_0(t)=\begin{cases}\dot x_0(0)t & \text{if } t\in[0,t_1],\\ B_1\dot x_0(0)(t-t_1) & \text{if } t\in(t_1,t_2]\\ \Big[\prod_{l=1}^i B_l(t-t_i)+\prod_{k=1}^i A_k t_1\\ +\sum_{j=2}^i\prod_{k=j}^i A_k\prod_{l=1}^{j-1}B_l(t_j-t_{j-1})\Big]\dot x_0(0)\\ \qquad \text{if } t\in(t_i,t_{i+1}],\ 2\le i\le p+1\end{cases}$$

with $\dot x_0(0)\in\mathbb{R}^n$. Since $x_0(T)=0$ if and only if $C\dot x_0(0)=0$, condition (ii) holds if and only if C is regular. Then, for every $\lambda\in[0,1]$, $q\in Q$ and every measurable selection f_q of $F(\cdot,q(\cdot)\dot q(\cdot))$, (4.7) has a unique solution,

$$x_{\lambda f_q}(t)=\begin{cases}\dot x_{\lambda f_q}(0)t+\int_0^t(t-\tau)f_q(\tau)d\tau & \text{if } t\in[0,t_1],\\ B_1\dot x_{\lambda f_q}(0)(t-t_1)+\int_{t_1}^t(t-\tau)f_q(\tau)d\tau\\ +B_1(t-t_1)\int_0^{t_1}f_q(\tau)d\tau & \text{if } t\in(t_1,t_2]\\ \prod_{l=1}^i B_l\dot x_{\lambda f_q}(0)(t-t_i)+\int_{t_i}^t(t-\tau)f_q(\tau)d\tau\\ +\sum_{r=1}^i\prod_{l=r}^i B_l(t-t_i)\int_{t_{r-1}}^{t_r}f_q(\tau)d\tau+\prod_{k=1}^i A_k\dot x_{\lambda f_q}(0)t_1\\ +\prod_{k=1}^i A_k\int_0^{t_1}(t_1-\tau)f_q(\tau)d\tau\\ +\sum_{j=2}^i\prod_{k=j}^i A_k\Big[\prod_{l=1}^{j-1}B_l\dot x_{\lambda f_q}(0)(t_j-t_{j-1})\\ +\int_{t_{j-1}}^{t_j}(t_j-\tau)f_q(\tau)d\tau\\ +\sum_{r=1}^{k-1}\prod_{l=r}^{k-1}B_l(t_j-t_{j-1})\int_{t_{r-1}}^{t_r}f_q(\tau)d\tau\Big]\\ \qquad \text{if } t\in(t_i,t_{i+1}],\ 2\le i\le p+1\end{cases}$$

with

$$\dot x_{\lambda f_q}(0)=-C^{-1}\Big(\int_{t_1}^T(T-\tau)f_q(\tau)d\tau+B_1(T-t_1)\int_0^{t_1}f_q(\tau)d\tau\Big) \tag{4.9}$$

if $p=1$, and

$$\begin{aligned}\dot x_{\lambda f_q}(0)=-C^{-1}\Big(&\int_{t_p}^T(T-\tau)f_q(\tau)d\tau+\sum_{r=1}^p\prod_{l=r}^p B_l(T-t_p)\int_{t_{r-1}}^{t_r}f_q(\tau)d\tau\\ &+\prod_{k=1}^p A_k\int_0^{t_1}(t_1-\tau)f_q(\tau)d\tau+\sum_{j=2}^p\prod_{k=j}^p A_k\Big[\int_{t_{j-1}}^{t_j}(t_j-\tau)f_q(\tau)d\tau\\ &+\sum_{r=1}^{k-1}\prod_{l=r}^{k-1}B_l(t_j-t_{j-1})\int_{t_{r-1}}^{t_r}f_q(\tau)d\tau\Big]\Big)\end{aligned} \tag{4.10}$$

if $p\ge 2$. Therefore

$$\mathfrak{T}(q,\lambda)=\{x_{\lambda f_q}: f_q \text{ is a selection of } F(\cdot,q(\cdot),\dot q(\cdot))\}\neq\emptyset.$$

Given $x_1,x_2\in\mathfrak{T}(q,\lambda)$, there exist measurable selections f_q^1,f_q^2 of $F(\cdot,q(\cdot),\dot q(\cdot))$ such that $x_1=x_{\lambda f_q^1}$ and $x_2=x_{\lambda f_q^2}$. Since the right-hand side F has convex values, it holds that, for any $c\in[0,1]$, $cf_q^1+(1-c)f_q^2$ is a measurable selection of $F(\cdot,q(\cdot),\dot q(\cdot))$ as well. The linearity of both the equation and of the impulses yields that $cx_1+(1-c)x_2=x_{cf_q^1+(1-c)f_q^2}$, i.e. that the set of solutions of problem

(4.6) is convex, for each $(q,\lambda) \in Q \times [0,1]$. Therefore, assumptions (i) and (ii) in Proposition 2.4 are satisfied.

Condition (iii) follows immediately from the fact that $0 \in K$ and that, for $\lambda = 0$, the associated problem has only the trivial solution, see assumption (ii).

Let $x_{\lambda f_q}$ be the solution of (4.7). Then $|x_{\lambda f_q}(0)| = 0$. Moreover, according to assumption (i) and formulas (4.9) and (4.10),

$$|\dot{x}_{\lambda f_q}(0)| \le \|C^{-1}\| \Big[\beta(2B)\frac{1}{2}T^2 + T^2\|B_1\|\beta(2B)\Big] = T^2\|C^{-1}\| \cdot \beta(2B)\Big[\frac{1}{2} + \|B_1\|\Big]$$

if $p = 1$ and

$$\begin{aligned} |\dot{x}_{\lambda f_q}(0)| \le \|C^{-1}\| \Big[&\frac{1}{2}T^2\beta(2B) + T^2\prod_{l=1}^{p}\|B_l\| \cdot \beta(2B) \\ &+ \frac{1}{2}T^2\prod_{k=1}^{p}\|A_k\|\beta(2B) + T^2\prod_{l=1}^{p}\|B_l\|\prod_{k=1}^{p}\|A_k\| \cdot \beta(2B)\Big] \\ = T^2\|C^{-1}\| \cdot \beta(2B)\Big[&\frac{1}{2} + \prod_{l=1}^{p}\|B_l\| \\ &+ \prod_{k=1}^{p}\|A_k\| + \prod_{l=1}^{p}\|B_l\|\prod_{k=1}^{p}\|A_k\|\Big] \end{aligned}$$

if $p \ge 2$. Therefore there exists a constant M_1 such that $|\dot{x}(0)| \le M_1$, for all solutions x of (4.6). Hence, condition (iv) in Proposition 2.4 is satisfied.

Let us assume that $q_* \in Q$ is, for some $\lambda \in [0,1)$, a fixed point of the solution mapping $\mathfrak{T}(\cdot,\lambda)$. We will show now that q_* can not lay in ∂Q.

At first, let us investigate the case when $\lambda = 0$. Then (4.6) transforms into (4.3) which has only the trivial solution. Therefore, for $\lambda = 0$, it holds that $q_* \equiv 0$ which lays in $Int\ Q$. Hence, if $\lambda = 0$, condition (v) in Proposition 2.4 is satisfied.

Secondly, let us assume that $\lambda \in (0,1)$. If q_* belongs to ∂Q, then there exists $t_0 \in [0,T]$ such that $q_*(t_0) \in \partial K$ or $|\dot{q}_*(t_0)| = 2B$. Since, for a.a. $t \in [0,T]$, we have

$$|\ddot{q}_*(t)| = \lambda|F(t,q_*(t),\dot{q}_*(t))| \le \beta(|\dot{q}_*(t)|)$$

and $|q_*(t)| \le R$, for every $t \in [0,T]$, Proposition 2.5 implies that $|\dot{q}_*(t)| \le B < 2B$, for every $t \in [0,T]$. Hence, $q(t_0) \in \partial K$, which is impossible, since, according to Remark 3.6, hypotheses (iii), (iv) and (v) guarantee that K is a bound set for (4.6), i.e. that $q_*(t) \in K$, for all $t \in [0,T]$. Thus $q_* \in Int\ Q$.

Therefore, condition (v) from Proposition 2.4 is satisfied, for all $\lambda \in [0,1]$, which completes the proof. $\square$

Remark 4.2. An easy example of impulses conditions guaranteeing assumption (ii) in Theorem 4.1 are the antiperiodic impulses, i.e. $A_i = B_i = -I$, for every $i = 1,\dots,p$. It follows from the proof of Theorem 4.1 that for the fulfilment of assumption (ii), it is sufficient to prove the regularity of the matrix C defined in (4.8). For $p = 1$, $C = (t_1 - T)I$ which is obviously regular. Let us show that C is regular also when $p \ge 2$. If p is even, then $\prod_{k=j}^{p}(-I)\prod_{l=1}^{j-1}(-I) = \prod_{l=1}^{p}(-I) = I$. Hence

$$C = [T - t_p + t_1 + \sum_{j=2}^{p}(t_j - t_{j-1})]I = TI$$

which is regular. It can be shown that a similar reasoning holds also in the case when p is odd.

Remark 4.3. When V is of class C^2, then, according to Corollary 3.5, condition (iv) in Theorem 4.1 is equivalent to requiring that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$, and $v \in \mathbb{R}^n$, with $|v| \le \phi^{-1}(\phi(2R) + 2R)$,

$$\langle HV(x)v, v\rangle + \lambda\langle \nabla V(x), w\rangle > 0, \quad \text{for every } \lambda \in (0,1) \text{ and } w \in F(t,x,v). \tag{4.11}$$

Since the function $g(\lambda) = \lambda\langle \nabla V(x), w\rangle$ is monotone, (4.11) is then equivalent to the following two conditions

$$\langle HV(x)v, v\rangle \ge 0 \quad \text{and} \quad \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle \ge 0 \tag{4.12}$$

that do not depend on λ.

5. Application to the forced pendulum equation

Let us consider the forced (mathematical) pendulum equation with viscous damping and dry friction terms

$$\ddot{x} + e\dot{x} + b\sin x + f\operatorname{sgn}\dot{x} = h(t), \quad \text{for a.a. } t \in [0,\pi], \tag{5.1}$$

with antiperiodic impulses and Dirichlet boundary conditions

$$x(t_i^+) = -x(t_i), \quad i = 1,\dots,p, \tag{5.2}$$

$$\dot{x}(t_i^+) = -\dot{x}(t_i), \quad i = 1,\dots,p, \tag{5.3}$$

$$x(0) = x(\pi) = 0, \tag{5.4}$$

where e, b and f are real constants and $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = \pi$, $p \in \mathbb{N}$. The function $h : [0,\pi] \to \mathbb{R}$ plays the role of the forcing term and we assume that $h \in L^\infty([0,\pi],\mathbb{R})$.

The study of the pendulum equation (i.e. the case $b > 0$, $e = f = 0$) dates back to a century ago (see [22]), when it was shown that it is worth to consider Dirichlet boundary conditions since the symmetry of the equation implies that such solutions are related to periodic solutions. The mathematical pendulum equation (i.e. the case $b < 0$, $e = f = 0$) was considered for the first time in [19]. More recently, the pendulum equation was generalized introducing a non-zero viscous damping coefficient e or a non-zero friction coefficient f (see [5, 26] for more details about this topic). Let us mention also the paper [17], where an impulse problem is considered in the case $e = f = 0$.

Because the function $\operatorname{sgn} y$ is discontinuous at $y = 0$, we should consider Filippov solutions of (5.1) which can be identified as Carathéodory solutions of the inclusion

$$\ddot{x} + e\dot{x} + b\sin x \in h(t) - f\operatorname{Sgn}\dot{x}, \tag{5.5}$$

where

$$\operatorname{Sgn} y := \begin{cases} -1, & \text{for } y < 0,\\ [-1,1], & \text{for } y = 0,\\ 1, & \text{for } y > 0. \end{cases}$$

Let us now consider the Dirichlet multivalued problem (5.5), (5.4) with impulse conditions (5.2), (5.3) and let us check that all the assumptions of Theorem 4.1 are satisfied.

To verify condition (i), let us define the continuous and non-decreasing function

$$\beta(d) = \|h\|_\infty + |e||d| + |b| + |f|, \quad \text{for all } d \in \mathbb{R}.$$

The function β obviously satisfies (4.1) and $F(t,c,d) = h(t) - ed - b\sin c - f\operatorname{Sgn} d$ satisfies (4.2), for all $t \in [0,\pi]$ and all $c, d \in \mathbb{R}$.

Assumption (ii) holds as well since, according to Remark 4.2, the associated homogeneous problem has only the trivial solution.

For verifying condition (iii), consider the nonempty, open, bounded, convex and symmetric neighbourhood of the origin $K = (-k,k)$ with $k \in (0,\frac{\pi}{2}]$ which will be specified later and the C^2-function $V(x) = \frac{1}{2}(x^2 - k^2)$ that trivially satisfies conditions (H1) and (H2).

To check condition (4.4) (which takes in our case the form (4.12), according to Corollary 3.5 and Remark 4.3), since $\langle HV(x)v, v\rangle = v^2$ is obviously non-negative, it is sufficient to verify that

$$\begin{aligned} &v^2 + x\big(h(t) - ev - b\sin x - f\operatorname{Sgn} v\big) \\ &= v^2 - exv + xh(t) - bx\sin x - fx\operatorname{Sgn} v \subset (0,\infty), \end{aligned} \tag{5.6}$$

for every $t \in (0,\pi), v \in \mathbb{R}$ and $x \in \mathbb{R}$ with $k - \varepsilon \le |x| \le k$.

(1) If $x = k$, then (5.6) becomes

$$v^2 - ekv + kh(t) - bk\sin k - fk\operatorname{Sgn} v \subset (0,\infty), \tag{5.7}$$

for every $t \in (0,\pi)$ and $v \in \mathbb{R}$. Since $k > 0$,

$$kh(t) \ge k \inf_{t\in(0,\pi)} h(t), \text{ for all } t \in (0,\pi),$$

and so condition (5.7) holds if

$$v^2 - ekv + k\inf_{t\in(0,\pi)} h(t) - bk\sin k - fk\operatorname{Sgn} v \subset (0,\infty), \forall v \in \mathbb{R}. \tag{5.8}$$

(a) If $v = 0$, then (5.8) takes the form

$$k\inf_{t\in(0,\pi)} h(t) - bk\sin k - fks > 0$$

for every $s \in [-1,1]$. This is equivalent to

$$\inf_{t\in(0,\pi)} h(t) > b\sin k + |f|, \tag{5.9}$$

since $\max_{s\in[-1,1]} fs = |f|$.

(b) If $v > 0$, then (5.8) takes the form

$$v^2 - ekv + k\inf_{t\in(0,\pi)} h(t) - bk\sin k - fk > 0. \tag{5.10}$$

If we define the function $g : [0,\infty) \to \mathbb{R}$ by $g(v) = v^2 - ekv + k\inf_{t\in(0,\pi)} h(t) - bk\sin k - fk$, then $g(0) > 0$, according to (5.9), and the minimum of g is achieved at the point $\bar{v} = \frac{ek}{2}$. Therefore, the inequality (5.10) holds if (5.9) is satisfied in case of $e \le 0$ and if

$$\inf_{t\in(0,\pi)} h(t) > \frac{e^2k}{4} + b\sin k + f, \quad \text{for } e > 0.$$

Summing up, inequality (5.10) holds if

$$\inf_{t\in(0,\pi)} h(t) > \frac{e^2k}{4} + b\sin k + f.$$

(c) If $v < 0$, then (5.8) takes the form

$$v^2 - ekv + k \inf_{t\in(0,\pi)} h(t) - bk\sin k + fk > 0. \tag{5.11}$$

In the same way as before, it is possible to obtain that (5.11) holds if

$$\inf_{t\in(0,\pi)} h(t) > \frac{e^2k}{4} + b\sin k - f, \quad \text{for } e < 0,$$
$$\inf_{t\in(0,\pi)} h(t) > b\sin k - f, \text{ for } e \geq 0.$$

Summing up, (5.6) holds, for $x = k$, if

$$\inf_{t\in(0,\pi)} h(t) > \frac{e^2k}{4} + b\sin k + |f|. \tag{5.12}$$

(2) If $x = -k$, then (5.6) becomes

$$v^2 + ekv - kh(t) - bk\sin k + fk \operatorname{Sgn} v \subset (0,\infty), \quad \text{for every } t \in (0,\pi) \text{ and } v \in \mathbb{R}$$

and analogously as in the case $x = k$, we obtain that (5.6) holds for $x = -k$ if

$$\sup_{t\in(0,\pi)} h(t) < -\frac{e^2k}{4} - b\sin k - |f|. \tag{5.13}$$

Therefore, (5.6) holds, for all $t \in (0,\pi), v \in \mathbb{R}$ and $x \in \mathbb{R}$ with $k - \varepsilon \leq |x| \leq k$, for some $\varepsilon > 0$ sufficiently small, (due to the continuity and the inequalities (5.12) and (5.13)) if

$$\frac{e^2k}{4} + b\sin k + |f| < \inf_{t\in(0,\pi)} h(t) \leq \sup_{t\in(0,\pi)} h(t) < -\frac{e^2k}{4} - b\sin k - |f|,$$

which, in particular, implies that $\frac{e^2k}{4} + b\sin k + |f| < 0$.

Since $\nabla V(x) = \dot{V}(x) = x$ and $HV(x) = \ddot{V}(x) = 1$, for all $x \in \mathbb{R}$, condition (v) trivially holds.

In conclusion, assuming that $k \in (0,\pi/2]$ is such that

$$\frac{e^2k}{4} + b\sin k + |f| < 0 \tag{5.14}$$

and that

$$|h(t)| < -\frac{e^2k}{4} - b\sin k - |f|, \text{ for all } t \in (0,\pi),$$

then all the assumptions of Theorem 4.1 are satisfied, and problem (5.1) admits a solution laying in $[-k,k]$. We stress that such solution is not trivial, according to the presence of the forcing term. Notice moreover that condition (5.14) is consistent, since it never holds for small k, and therefore (5.6) is not satisfied in the whole corresponding set $\overline{K}$ but only in some neighborhood of its boundary, as required.

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On the impulsive Dirichlet problem for second-order differential inclusions

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Abstract. Solutions in a given set of an impulsive Dirichlet boundary value problem are investigated for second-order differential inclusions. The method used for obtaining the existence and the localization of a solution is based on the combination of a fixed point index technique developed by ourselves earlier with a bound sets approach and Scorza-Dragoni type result. Since the related bounding (Liapunov-like) functions are strictly localized on the boundaries of parameter sets of candidate solutions, some trajectories are allowed to escape from these sets.

Keywords: impulsive Dirichlet problem, differential inclusions, topological methods, bounding functions, Scorza-Dragoni technique.

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1 Introduction

Let us consider the Dirichlet boundary value problem

$$\begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0, T], \\ x(T) = x(0) = 0, \end{cases} \tag{1.1}$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping.

Moreover, let a finite number of points $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T,\ p \in \mathbb{N}$, and real $n \times n$ matrices A_i, B_i, $i = 1, \dots, p$, be given. In the paper, the solvability of the Dirichlet b.v.p. (1.1) will be investigated in the presence of the following impulse conditions

$$x(t_i^+) = A_i x(t_i), \qquad i = 1, \dots, p, \tag{1.2}$$

$$\dot{x}(t_i^+) = B_i \dot{x}(t_i), \qquad i = 1, \dots, p, \tag{1.3}$$

where the notation $\lim_{t \to a^+} x(t) = x(a^+)$ is used.

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By a *solution* of problem (1.1)–(1.3) we shall mean a function $x \in PAC^1([0,T],\mathbb{R}^n)$ (see Section 2 for the definition) satisfying (1.1), for almost all $t \in [0,T]$, and fulfilling the conditions (1.2) and (1.3).

Boundary value problems with impulses have been widely studied because of their applications in areas, where the parameters are subject to certain perturbations in time. For instance, in the treatment of some diseases, impulses may correspond to administration of a drug treatment or in environmental sciences, they can describe the seasonal changes or harvesting.

While the theory of single valued impulsive problems is deeply examined (see, e.g. [9,10,22]), the theory dealing with multivalued impulsive problems has not been studied so much yet (for the overview of known results see, e.g., the monographs [11,19] and the references therein). However, it is worth to study also the multivalued case, since the multivalued problems come e.g. from single valued problems with discontinuous right-hand sides, or from control theory.

The most of mentioned results dealing with impulsive problems have been obtained using fixed point theorems, upper and lower-solutions methods, or using topological and variational approaches.

In this paper, the existence and the localization of a solution for the impulsive Dirichlet b.v.p. (1.1)–(1.3) will be studied using a continuation principle. On this purpose, it will be necessary to embed the original problem into a family of problems and to ensure that the boundary of a prescribed set of candidate solutions is fixed point free, i.e. to verify so called transversality condition. This condition can be guaranteed by a bound sets technique that was described by Gaines and Mawhin in [17] for single valued problems without impulses. Recently, in [25], a bound sets technique for the multivalued impulsive b.v.p. using non strictly localized bounding (Liapunov-like) functions has been developed. Such a non-strict localization of bounding functions makes parameter sets of candidate solutions "only" positively invariant.

In this paper, the conditions imposed on the bounding function will be strictly localized on the boundary of the set of candidate solutions, which eliminates this unpleasant handicap. Both the possible cases will be discussed – problems with an upper semicontinuous r.h.s. and also problems with an upper-Carathéodory r.h.s. More concretely, in Theorem 4.3 below, the upper semicontinuous case is considered and the transversality condition is obtained reasoning pointwise via a C^1-bounding function with a locally Lipschitzian gradient. In Theorem 5.2, the upper-Carathéodory case and a C^2-bounding function will be considered and the reasoning will be based on a Scorza-Dragoni approximation technique. In fact, even if the first kind of regularity of the r.h.s. is a special case of the second one, in the first case the stronger regularity will allow to use C^1-bounding functions, while in the second case, C^2-bounding functions will be needed. Moreover, even when using C^2-bounding functions, the more regularity of the r.h.s. allows to obtain the result under weaker conditions. Let us note that a similar approach was employed for problems with upper semicontinous r.h.s. without impulses e.g. in [3,6] and for problems with upper-Carathéodory r.h.s. without impulses e.g. in [4,24].

This paper is organized as follows. In the second section, we recall suitable definitions and statements which will be used in the sequel. Section 3 is devoted to the study of bound sets and Liapunov-like bounding functions for impulsive Dirichlet problems with an upper semicontinuous r.h.s. At first, C^1-bounding functions with locally Lipschitzian gradients are considered. Consequently, it is shown how conditions ensuring the existence of bound set change in case of C^2-bounding functions. In Section 4, the bound sets approach is combined with a continuation principle and the existence and localization result is obtained in this way for the impulsive Dirichlet problem (1.1)–(1.3). Section 5 deals with the existence and localization of a solution of the Dirichlet impulsive problem in case when the r.h.s. is an upper-Carathéodory mapping. In Section 6, the obtained result is applied to an illustrative example.

2 Some preliminaries

Let us recall at first some geometric notions of subsets of metric spaces. If (X, d) is an arbitrary metric space and $A \subset X$, by $\mathrm{Int}(A), \overline{A}$ and ∂A we mean the interior, the closure and the boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x,a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X.

For a given compact real interval J, we denote by $C(J, \mathbb{R}^n)$ (by $C^1(J, \mathbb{R}^n)$) the set of all functions $x : J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $AC^1(J, \mathbb{R}^n)$, we shall mean the set of all functions $x : J \to \mathbb{R}^n$ with absolutely continuous first derivatives on J.

Let $PAC^1([0, T], \mathbb{R}^n)$ be the space of all functions $x : [0, T] \to \mathbb{R}^n$ such that

$$x(t) = \begin{cases} x_{[0]}(t), & \text{for } t \in [0, t_1], \\ x_{[1]}(t), & \text{for } t \in (t_1, t_2], \\ \quad\vdots & \\ x_{[p]}(t), & \text{for } t \in (t_p, T], \end{cases}$$

where $x_{[0]} \in AC^1([0, t_1], \mathbb{R}^n), x_{[i]} \in AC^1((t_i, t_{i+1}], \mathbb{R}^n), x(t_i^+) = \lim_{t \to t_i^+} x(t) \in \mathbb{R}$ and $\dot{x}(t_i^+) = \lim_{t \to t_i^+} \dot{x}(t) \in \mathbb{R}$, for every $i = 1, \dots, p$. The space $PAC^1([0, T], \mathbb{R}^n)$ is a normed space with the norm

$$\|x\| := \sup_{t \in [0,T]} |x(t)| + \sup_{t \in [0,T]} |\dot{x}(t)|. \tag{2.1}$$

In a similar way, we can define the spaces $PC([0, T], \mathbb{R}^n)$ and $PC^1([0, T], \mathbb{R}^n)$ as the spaces of functions $x : [0, T] \to \mathbb{R}^n$ satisfying the previous definition with $x_{[0]} \in C([0, t_1], \mathbb{R}^n), x_{[i]} \in C((t_i, t_{i+1}], \mathbb{R}^n)$ or with $x_{[0]} \in C^1([0, t_1], \mathbb{R}^n)$, $x_{[i]} \in C^1((t_i, t_{i+1}], \mathbb{R}^n)$, for every $i = 1, \dots, p$, respectively. The space $PC^1([0, T], \mathbb{R}^n)$ with the norm defined by (2.1) is a Banach space (see [23, page 128]).

A subset $A \subset X$ is called a *retract* of a metric space X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$. We say that a space X is an *absolute retract* (*AR*-space) if, for each space Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over Y. If f is extendable only over some neighborhood of A, for each closed $A \subset Y$ and each continuous mapping $f : A \to X$, then X is called an *absolute neighborhood retract* (*ANR*-space). Let us note that X is an *ANR*-space if and only if it is a retract of an open subset of a normed space and that X is an *AR*-space if and only if it is a retract of some normed space (see, e.g. [2]). Conversely, if X is a retract (of an open subset) of a convex set in a Banach space, then it is an *AR*-space (*ANR*-space). So, the space $C^1(J, \mathbb{R}^n)$, where $J \subset \mathbb{R}$ is a compact interval, is an *AR*-space as well as its convex subsets or retracts, while its open subsets are *ANR*-spaces.

A nonempty set $A \subset X$ is called an R_δ*-set* if there exists a decreasing sequence $\{A_n\}_{n=1}^\infty$ of compact AR-spaces such that

$$A = \bigcap_{n=1}^{\infty} A_n.$$

The following hierarchy holds for nonempty subsets of a metric space:

$$\text{compact+convex} \subset \text{compact } AR\text{-space} \subset R_\delta\text{-set}, \tag{2.2}$$

and all the above inclusions are proper. For more details concerning the theory of retracts, see [14].

We also employ the following definitions from the multivalued analysis in the sequel. Let X and Y be arbitrary metric spaces. We say that φ is a multivalued mapping from X to Y (written $\varphi : X \multimap Y$) if, for every $x \in X$, a nonempty subset $\varphi(x)$ of Y is prescribed. We associate with F its *graph* Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

Let us mention also some basic notions concerning multivalued mappings. A multivalued mapping $\varphi : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \subset U\}$ is open in X.

Let $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper semicontinuous multimap and let, for all $r > 0$, exist an integrable function $\mu_r : J \to [0, \infty)$ such that $|y| \le \mu_r(t)$, for every $(t, x) \in J \times \mathbb{R}^m$, with $|x| \le r$, and every $y \in F(t, x)$. Then if we consider the composition of F with a function $q \in PC^1([0, T], \mathbb{R}^n)$, the corresponding superposition multioperator $\mathcal{P}_F(q)$ given by

$$\mathcal{P}_F(q) = \{f \in L^1([0, T]; \mathbb{R}^m) \ : \ f(t) \in F(t, q(t)) \text{ a.a. } t \in [0, T]\},$$

is well defined and nonempty (see [12, Proposition 6]).

Let Y be a metric space and $(\Omega, \mathcal{U}, \nu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $\varphi : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$. Obviously, every u.s.c. mapping is measurable.

We say that mapping $\varphi : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $\varphi(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $\varphi(t, \cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all $t \in J$, and the set $\varphi(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

If $X \cap Y \neq \emptyset$ and $\varphi : X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed point* of φ if $x \in \varphi(x)$. The set of all fixed points of φ is denoted by $\text{Fix}(\varphi)$, i.e.

$$\text{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}.$$

For more information and details concerning multivalued analysis, see, e.g., [2, 8, 18, 21].

The continuation principle which will be applied in the paper requires in particular the transformation of the studied problem into a suitable family of associated problems which does not have solutions tangent to the boundary of a given set Q of candidate solutions. This will be ensured by means of Hartman-type conditions (see Section 3) and by means of the following result based on Nagumo conditions (see [27, Lemma 2.1] and [20, Lemma 5.1]).

Proposition 2.1. *Let $\psi : [0, +\infty) \to [0, +\infty)$ be a continuous and increasing function, with*

$$\lim_{s\to\infty} \frac{s^2}{\psi(s)} ds = \infty, \tag{2.3}$$

and let R be a positive constant. Then there exists a positive constant

$$B = \psi^{-1}(\psi(2R) + 2R) \tag{2.4}$$

such that if $x \in PC^1([0, T], \mathbb{R}^n)$ is such that $|\ddot{x}(t)| \leq \psi(|\dot{x}(t)|)$, for a.a. $t \in [0, T]$, and $|x(t)| \leq R$, for every $t \in [0, T]$, then it holds that $|\dot{x}(t)| \leq B$, for every $t \in [0, T]$.

Let us note that the previous result is classically given for C^2-functions. However, it is easy to prove (see, e.g., [5]) that the statement holds also for piecewise continuously differentiable functions.

For obtaining the existence and localization result for the case of upper-Carathéodory r.h.s., we will need the following Scorza-Dragoni type result for multivalued maps (see [15, Proposition 5.1]).

Proposition 2.2. *Let $X \subset \mathbb{R}^m$ be compact and let $F : [a, b] \times X \multimap \mathbb{R}^n$ be an upper-Carathéodory map. Then there exists a multivalued mapping $F_0 : [a, b] \times X \multimap \mathbb{R}^n \cup \{\emptyset\}$ with compact, convex values and $F_0(t, x) \subset F(t, x)$, for all $(t, x) \in [a, b] \times X$, having the following properties:*

(i) if $u : [a, b] \to \mathbb{R}^m$, $v : [a, b] \to \mathbb{R}^n$ are measurable functions with $v(t) \in F(t, u(t))$, on $[a, b]$, then $v(t) \in F_0(t, u(t))$, a.e. on $[a, b]$;

(ii) for every $\epsilon > 0$, there exists a closed $I_\epsilon \subset [a, b]$ such that $\nu([a, b] \setminus I_\epsilon) < \epsilon$, $F_0(t, x) \neq \emptyset$, for all $(t, x) \in I_\epsilon \times X$ and F_0 is u.s.c. on $I_\epsilon \times X$.

3 Bound sets for Dirichlet problems with upper semicontinuous r.h.s.

In this section, we consider an u.s.c. multimap F and we are interested in introducing a Liapunov-like function V, usually called a *bounding function*, verifying suitable transversality conditions which assure that there does not exist a solution of the b.v.p. lying in a closed set $\overline{K}$ and touching the boundary ∂K of K at some point.

Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$ and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$(H1)$ $V|_{\partial K} = 0$,

$(H2)$ $V(x) \leq 0$, for all $x \in \overline{K}$.

Definition 3.1. A nonempty open set $K \subset \mathbb{R}^n$ is called a *bound set* for problem (1.1)–(1.3) if there does not exist a solution x of (1.1)–(1.3) such that $x(t) \in \overline{K}$, for each $t \in [0, T]$, and $x(t_0) \in \partial K$, for some $t_0 \in [0, T]$.

Firstly, we show sufficient conditions for the existence of a bound set for the second-order impulsive Dirichlet problem (1.1)–(1.3) in the case of a smooth bounding function V with a locally Lipschitzian gradient.

Proposition 3.2. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$, $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies conditions $(H1)$ and $(H2)$. Suppose moreover that, for all $x \in \partial K$, $t \in (0,T) \setminus \{t_1, \ldots, t_p\}$ and $v \in \mathbb{R}^n$ with*

$$\langle \nabla V(x), v \rangle = 0, \tag{3.1}$$

the following condition holds

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} > 0, \tag{3.2}$$

for all $w \in F(t,x,v)$. Then all solutions $x : [0,T] \to \overline{K}$ of problem (1.1) satisfy $x(t) \in K$, for every $t \in [0,T] \setminus \{t_1, \ldots, t_p\}$.

Proof. Let $x : [0,T] \to \overline{K}$ be a solution of problem (1.1). We assume by a contradiction that there exists $\bar{t} \in [0,T] \setminus \{t_1, \ldots, t_p\}$ such that $x(\bar{t}) \in \partial K$. Since $x(0) = x(T) = 0 \in K$, it must be $\bar{t} \in (0,T)$.

Let us define the function g in the following way $g(h) := V(x(\bar{t} + h))$. Then $g(0) = 0$ and there exists $\alpha > 0$ such that $g(h) \leq 0$, for all $h \in [-\alpha, \alpha]$, i.e., there is a local maximum for g at the point 0, and $g \in C^1([-\alpha, \alpha], \mathbb{R}^n)$, so $\dot{g}(0) = \langle \nabla V(x(\bar{t})), \dot{x}(\bar{t}) \rangle = 0$. Consequently, $x := x(\bar{t}), v := \dot{x}(\bar{t})$ satisfy condition (3.1).

Since ∇V is locally Lipschitzian, there exist an open set $U \subset \mathbb{R}^n$, with $x(\bar{t}) \in U$, and a constant $L > 0$ such that $\nabla V|_U$ is Lipschitzian with constant L. We can assume, without loss of generality, that $x(\bar{t} + h) \in U$ for all $h \in [-\alpha, \alpha]$.

Since $g(0) = 0$ and $g(h) \leq 0$, for all $h \in [-\alpha, 0)$, there exists an increasing sequence of negative numbers $\{h_k\}_{k=1}^{\infty}$such that $h_1 > -\alpha, h_k \to 0^-$ as $k \to \infty$, and $\dot{g}(h_k) \geq 0$, for each $k \in \mathbb{N}$. Since $x \in C^1([-\alpha, 0], \mathbb{R}^n)$, it holds, for each $k \in \mathbb{N}$, that

$$x(\bar{t} + h_k) = x(\bar{t}) + h_k[\dot{x}(\bar{t}) + b_k], \tag{3.3}$$

where $b_k \to 0$ as $k \to \infty$.

Since $x([-\alpha, 0])$ and $\dot{x}([-\alpha, 0])$ are compact sets and F is globally upper semicontinuous with compact values, $F(\cdot, x(\cdot), \dot{x}(\cdot))$ must be bounded on $[-\alpha, 0]$, by which $\dot{x}$ is Lipschitzian on $[-\alpha, 0]$. Thus, there exists a constant λ such that, for all $k \in \mathbb{N}$,

$$\left| \frac{\dot{x}(\bar{t} + h_k) - \dot{x}(\bar{t})}{h_k} \right| \leq \lambda,$$

i.e. the sequence $\{\frac{\dot{x}(\bar{t}+h_k) - \dot{x}(\bar{t})}{h_k}\}_{k=1}^{\infty}$ is bounded. Therefore, there exist a subsequence, for the sake of simplicity denoted as the sequence, of $\{\frac{\dot{x}(\bar{t}+h_k) - \dot{x}(\bar{t})}{h_k}\}$ and $w \in \mathbb{R}^n$ such that

$$\frac{\dot{x}(\bar{t} + h_k) - \dot{x}(\bar{t})}{h_k} \to w \tag{3.4}$$

as $k \to \infty$.

Let $\varepsilon > 0$ be given. Then, as a consequence of the regularity assumptions on F and of the continuity of both x and $\dot{x}$ on $[-\alpha, 0]$, there exists $\overline{\delta} = \overline{\delta}(\varepsilon) > 0$ such that, for each $h \in [-\alpha, 0]$, $h \geq -\overline{\delta}$, it follows that

$$F(\bar{t} + h, x(\bar{t} + h), \dot{x}(\bar{t} + h)) \subset F(\bar{t}, x(\bar{t}), \dot{x}(\bar{t})) + \varepsilon \overline{B}_n,$$

where B_n denotes the unit open ball in $\mathbb{R}^n$ centered at the origin. Subsequently, since F is convex valued, according to the Mean-Value Theorem (See [8], Theorem 0.5.3), there exists $k_\varepsilon \in \mathbb{N}$ such that, for each $k \geq k_\varepsilon$,

$$\frac{\dot{x}(\bar{t}+h_k)-\dot{x}(\bar{t})}{h_k} = \frac{1}{-h_k}\int_{\bar{t}+h_k}^{\bar{t}} \ddot{x}(s)\, ds \in F(\bar{t}, x(\bar{t}), \dot{x}(\bar{t})) + \varepsilon \overline{B}_n.$$

Since F has compact values and $\varepsilon > 0$ is arbitrary,

$$w \in F(\bar{t}, x(\bar{t}), \dot{x}(\bar{t})).$$

As a consequence of property (3.4) , there exists a sequence $\{a_k\}_{k=1}^{\infty}$, $a_k \to 0$ as $k \to \infty$, such that

$$\dot{x}(\bar{t}+h_k) = \dot{x}(\bar{t}) + h_k[w + a_k], \tag{3.5}$$

for each $k \in \mathbb{N}$. Since $h_k < 0$ and $\dot{g}(h_k) \geq 0$, in view of (3.3) and (3.5),

$$\begin{aligned} 0 \geq \frac{\dot{g}(h_k)}{h_k} &= \frac{\langle \nabla V(x(\bar{t}+h_k)), \dot{x}(\bar{t}+h_k)\rangle}{h_k} \\ &= \frac{\langle \nabla V(x(\bar{t}) + h_k[\dot{x}(\bar{t})+b_k]), \dot{x}(\bar{t}) + h_k[w+a_k]\rangle}{h_k}. \end{aligned}$$

Since $b_k \to 0$ when $k \to +\infty$, it is possible to find $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, it holds that $x(\bar{t}) + \dot{x}(\bar{t})h_k \in U$, because U is open. By means of the local Lipschitzianity of ∇V, for all $k \geq k_0$,

$$\begin{aligned} 0 \geq \frac{\dot{g}(h_k)}{h_k} &\geq \frac{\langle \nabla V(x(\bar{t}) + h_k\dot{x}(\bar{t})), \dot{x}(\bar{t}) + h_k[w+a_k]\rangle}{h_k} - L \cdot |b_k| \cdot |\dot{x}(\bar{t}) + h_k[w+a_k]| \\ &= \frac{\langle \nabla V(x(\bar{t}) + h_k\dot{x}(\bar{t})), \dot{x}(\bar{t}) + h_k w\rangle}{h_k} - L \cdot |b_k| \cdot |\dot{x}(\bar{t}) + h_k[w+a_k]| + \langle \nabla V(x(\bar{t}) + h_k\dot{x}(\bar{t})), a_k\rangle. \end{aligned}$$

Since $\langle \nabla V(x(\bar{t}) + h_k\dot{x}(\bar{t})), a_k\rangle - L \cdot |b_k| \cdot |\dot{x}(\bar{t}) + h_k[w+a_k]| \to 0$ as $k \to \infty$,

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x(\bar{t}) + h\dot{x}(\bar{t})), \dot{x}(\bar{t}) + hw\rangle}{h} \leq 0 \tag{3.6}$$

in contradiction with (3.2). Thus $x(t) \in K$ for every $t \in [0,T] \setminus \{t_1, \ldots, t_p\}$. □

Remark 3.3. It is obvious that condition (3.2) in Proposition 3.2 can be replaced by the following assumption: suppose that, for all $x \in \partial K$, $t \in (0,T) \setminus \{t_1, \ldots, t_p\}$ and $v \in \mathbb{R}^n$ satisfying (3.1) the following condition holds

$$\liminf_{h \to 0^+} \frac{\langle \nabla V(x + hv), v + hw\rangle}{h} > 0, \tag{3.7}$$

for all $w \in F(t, x, v)$.

Now, let us focus our attention also to the impulsive points $t_1, \ldots, t_p$.

Theorem 3.4. *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$, $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies* *conditions* $(H1)$ *and* $(H2)$. *Furthermore, assume that* A_i, B_i, $i = 1, \dots, p$, *are real* $n \times n$ *matrices such that* A_i, $i = 1, \dots, p$, *satisfy*

$$A_i(\partial K) = \partial K, \quad \textit{for all } i = 1, \dots, p. \tag{3.8}$$

Moreover, let, for all $x \in \partial K$, $t \in (0, T) \setminus \{t_1, \dots t_p\}$ *and* $v \in \mathbb{R}^n$ *satisfying* (3.1), *condition* (3.2) *holds, for all* $w \in F(t, x, v)$.

At last, suppose that, for all $x \in \partial K$ *and* $v \in \mathbb{R}^n$ *with*

$$\langle \nabla V(A_i x), B_i v\rangle \le 0 \le \langle \nabla V(x), v\rangle, \quad \textit{for some } i = 1, \dots, p, \tag{3.9}$$

the following condition

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw\rangle}{h} > 0 \tag{3.10}$$

holds, for all $w \in F(t_i, x, v)$. *Then* K *is a bound set for the impulsive Dirichlet problem* (1.1)–(1.3).

Proof. Applying Proposition 3.2, we only need to show that if $x : [0, T] \to \overline{K}$ is a solution of problem (1.1), then $x(t_i) \in K$, for all $i = 1, \dots, p$. As in the proof of Proposition 3.2, we argue by a contradiction, i.e. we assume that there exists $i \in \{1, \dots, p\}$ such that $x(t_i) \in \partial K$. Following the same reasoning as in the proof of Proposition 3.2, for $\bar{t} = t_i$, we obtain

$$\langle \nabla V(x(t_i)), \dot{x}(t_i)\rangle \ge 0,$$

because $V(x(t_i)) = 0$ and $V(x(t)) \le 0$, for all $t \in [0, T]$.

Moreover, according to the condition (3.8), $V(A_i(x(t_i))) = 0$ as well, and so we can apply the same reasoning to the function $\tilde{g}(h) = V(x(t_i + h))$ for $h > 0$ and $\tilde{g}(0) = V(x(t_i^+))$. Since $x \in PC^1([0, T], \mathbb{R}^n)$, also $\tilde{g} \in C^1([0, \alpha], \mathbb{R})$ and $\tilde{g}(h) \le 0$ for $h > 0$ and $\tilde{g}(0) = 0$ imply $\dot{\tilde{g}}(0) \le 0$, i.e.

$$0 \ge \langle \nabla V(A_i(x(t_i))), B_i \dot{x}(t_i)\rangle.$$

Therefore, $x := x(t_i), v := \dot{x}(t_i)$ satisfy condition (3.9).

Using the same procedure as in the proof of Proposition 3.2, for $\bar{t} = t_i$, we obtain the existence of a sequence of negative numbers $\{h_k\}_{k=1}^{\infty}$ and of point $w \in F(t_i, x(t_i), \dot{x}(t_i))$ such that

$$\frac{\dot{x}(t_i + h_k) - \dot{x}(t_i)}{h_k} \to w \quad \text{as } k \to \infty.$$

By the same arguments as in the previous proof, we get

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x(t_i) + h\dot{x}(t_i)), \dot{x}(t_i) + hw\rangle}{h} \le 0. \tag{3.11}$$

Inequality (3.11) is in a contradiction with condition (3.10), which completes the proof. □

Remark 3.5. If condition (3.10) holds, for some $x \in \partial K$, $v \in \mathbb{R}^n$ satisfying (3.9) and $w \in F(t_i, x, v)$, then, according to the continuity of ∇V,

$$\langle \nabla V(x), v\rangle = 0. \tag{3.12}$$

Indeed

$$\liminf_{h \to 0^-} \frac{\langle \nabla V(x + hv), v + hw\rangle}{h} = \liminf_{h \to 0^-} \left[\frac{\langle \nabla V(x + hv), v\rangle}{h} + \langle \nabla V(x + hv), w\rangle\right]$$

which, since $\langle \nabla V(x), v\rangle \ge 0$, can be positive only if (3.12) holds.

Definition 3.6. A function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying all assumptions of Theorem 3.4 is called a *bounding function* for the set K relative to (1.1)–(1.3).

For our main result concerning the existence and localization of a solution of the Dirichlet b.v.p., we need to ensure that no solution of given b.v.p lies on the boundary ∂Q of a parameter set Q of candidate solutions. In the following section, it will be shown that if the set Q is defined as follows

$$Q := \{q \in PC^1([0,T], \mathbb{R}^n) \mid q(t) \in \overline{K}, \ \text{ for all } t \in [0,T]\} \tag{3.13}$$

and if all assumptions of Theorem 3.4 are satisfied, then solutions of the b.v.p. (1.1)–(1.3) behave as indicated.

Proposition 3.7. *Let $K \subset \mathbb{R}^n$ be a nonempty open bounded set with $0 \in K$, let $Q \subset PC^1([0,T], \mathbb{R}^n)$ be defined by the formula (3.13) and let $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ with a locally Lipschitzian gradient ∇V which satisfies conditions $(H1)$ and $(H2)$. Moreover, assume that A_i, B_i, $i = 1, \dots, p$, are real $n \times n$ matrices such that A_i, $i = 1, \dots, p$, satisfy (3.8).*

Furthermore, suppose that, for all $x \in \partial K$, $t \in (0,T) \setminus \{t_1, \dots, t_p\}$ and $v \in \mathbb{R}^n$ satisfying (3.1), condition (3.2) holds, for all $w \in F(t,x,v)$, and that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ satisfying (3.9), the condition (3.10) holds, for all $w \in F(t_i,x,v)$. Then problem (1.1)–(1.3) has no solution on ∂Q.

Proof. One can readily check that if $x \in \partial Q$, then there exists a point $t_x \in [0,T]$ such that $x(t_x) \in \partial K$. But then, according to Theorem 3.4, x cannot be a solution of (1.1)–(1.3). □

Let us now consider the particular case when the bounding function V is of class C^2. Then conditions (3.2) and (3.10) can be rewritten in terms of gradients and Hessian matrices and the following result can be directly obtained.

Corollary 3.8. *Let $K \subset \mathbb{R}^n$ be a nonempty open bounded set with $0 \in K$, let $Q \subset PC^1([0,T], \mathbb{R}^n)$ be defined by the formula (3.13) and let $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with nonempty, compact, convex values. Assume that there exists a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$ which satisfies conditions $(H1)$ and $(H2)$. Moreover, assume that A_i, B_i, $i = 1, \dots, p$, are real $n \times n$ matrices such that A_i, $i = 1, \dots, p$, satisfy (3.8).*

Furthermore, suppose that, for all $x \in \partial K$ and $v \in \mathbb{R}^n$ the following holds:

$$\textit{if } \langle \nabla V(x), v \rangle = 0, \quad \textit{then } \langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0, \tag{3.14}$$

for all $t \in (0,T) \setminus \{t_1, \dots, t_p\}$ and $w \in F(t,x,v)$, and fixed $i = 1, \dots, n$

$$\textit{if } \langle \nabla V(A_i x), B_i v \rangle \le 0 \le \langle \nabla V(x), v \rangle \quad \textit{then } \langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle > 0, \tag{3.15}$$

for all $w \in F(t_i,x,v)$. Then problem (1.1)–(1.3) has no solution on ∂Q.

Proof. The statement of Corollary 3.8 follows immediately from Remark 3.5 and the fact that if $V \in C^2(\mathbb{R}^n, \mathbb{R})$, then, for all $x \in \partial K$, $t \in (0,T)$, $v \in \mathbb{R}^n$ and $w \in F(t,x,v)$, there exists

$$\begin{aligned} \lim_{h \to 0} \frac{\langle \nabla V(x + hv), v + hw \rangle}{h} &= \lim_{h \to 0} \frac{\langle \nabla V(x + hv), v + hw \rangle - \langle \nabla V(x), v \rangle}{h} \\ &= \langle HV(x)v, v \rangle + \langle \nabla V(x), w \rangle. \end{aligned}$$

□

Remark 3.9. In conditions (3.2), (3.10), (3.14) and (3.15), the element v plays the role of the first derivative of the solution x. If x is a solution of (1.1)–(1.3) such that $x(t) \in \overline{K}$, for every $t \in [0, T]$, and there exists a continuous increasing function $\psi : [0, \infty) \to [0, \infty)$ satisfying condition (2.3) and such that

$$|F(t, c, d)| \leq \psi(|d|), \tag{3.16}$$

for a.a. $t \in [0, T]$ and every $c, d \in \mathbb{R}^n$ with $|c| \leq R := \max\{|x| : x \in \overline{K}\}$, then, according to Proposition 2.1, it holds that $|\dot{x}(t)| \leq B$, for every $t \in [0, T]$, where B is defined by (2.4). Hence, it is sufficient to require conditions (3.2), (3.10), (3.14) and (3.15) in Proposition 3.2, Theorem 3.4 and Corollary 3.8 only for all $v \in \mathbb{R}^n$ with $|v| \leq B$ and not for all $v \in \mathbb{R}^n$.

4 Existence and localization result for the impulsive Dirichlet problem with upper semi-continuous r.h.s.

In order to obtain the main existence theorem, the bound sets technique described in the previous section will be combined with the topological method which was developed by ourselves in [25] for the impulsive boundary value problems. The version of the continuation principle for problems without impulses can be found e.g. in [7].

Proposition 4.1 ([25, Proposition 2.4])**.** *Let us consider the b.v.p.*

$$\begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), & \text{for a.a. } t \in [0, T], \\ x \in S, \end{cases} \tag{4.1}$$

where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of $PC^1([0, T], \mathbb{R}^n)$. Let $H : [0, T] \times \mathbb{R}^{4n} \times [0, 1] \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that

$$H(t, c, d, c, d, 1) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in [0, T] \times \mathbb{R}^{2n}. \tag{4.2}$$

Assume that

(*i*) *there exists a retract Q of $PC^1([0, T], \mathbb{R}^n)$, with $Q \setminus \partial Q \neq \emptyset$, and a closed subset S_1 of S such that the associated problem*

$$\begin{cases} \ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), & \text{for a.a. } t \in [0, T], \\ x \in S_1 \end{cases} \tag{4.3}$$

has, for each $(q, \lambda) \in Q \times [0, 1]$, a non-empty and convex set of solutions $\mathfrak{T}(q, \lambda)$;

(*ii*) *there exists a nonnegative, integrable function $\alpha : [0, T] \to \mathbb{R}$ such that*

$$|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)| \leq \alpha(t)(1 + |x(t)| + |\dot{x}(t)|), \quad \text{for a.a. } t \in [0, T],$$

for any $(q, \lambda, x) \in \Gamma_{\mathfrak{T}}$;

(*iii*) *$\mathfrak{T}(Q \times \{0\}) \subset Q$;*

(*iv*) *there exist constants $M_0 \geq 0$, $M_1 \geq 0$ such that $|x(0)| \leq M_0$ and $|\dot{x}(0)| \leq M_1$, for all $x \in \mathfrak{T}(Q \times [0, 1])$;*

(*v*) *the solution map $\mathfrak{T}(\cdot, \lambda)$ has no fixed points on the boundary ∂Q of Q, for every $\lambda \in [0, 1)$.*

Then the b.v.p. (4.1) *has a solution in* $S_1 \cap Q$.

Remark 4.2. The condition that Q is a retract of $PC^1([0,T],\mathbb{R}^n)$ in Proposition 4.1 can be replaced by the assumption that Q is an absolute neighborhood retract and $\operatorname{ind}(\mathfrak{T}(\cdot,0),Q,Q) \neq 0$ (see, e.g., [2]). It is therefore possible to assume alternatively that Q is a retract of a convex subset of $PC^1([0,T],\mathbb{R}^n)$ or of an open subset of $PC^1([0,T],\mathbb{R}^n)$ together with $\operatorname{ind}(\mathfrak{T}(\cdot,0),Q,Q)\neq 0$.

The solvability of (1.1) will be now proved, on the basis of Proposition 4.1. Defining namely the set Q of candidate solutions by the formula (3.13), we are able to verify, for each $(q,\lambda) \in Q \times [0,1)$, the transversality condition (v) in Proposition 4.1.

Theorem 4.3. *Let* $K \subset \mathbb{R}^n$ *be a nonempty, open, bounded and convex set with* $0 \in K$ *and let us consider the impulsive Dirichlet problem* (1.1)–(1.3), *where* $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper semicontinuous multivalued mapping,* $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T, p \in \mathbb{N}$, *and* A_i, B_i, $i = 1,\ldots,p$, *are real* $n \times n$ *matrices with* $A_i\partial K = \partial K$, *for all* $i = 1,\ldots,p$. *Moreover, assume that*

(i) *there exists a function* $\beta : [0,\infty) \to [0,\infty)$ *continuous and increasing satisfying*

$$\lim_{s\to\infty} \frac{s^2}{\beta(s)} ds = \infty$$

such that

$$|F(t,c,d)| \leq \beta(|d|),$$

for a.a. $t \in [0,T]$ *and every* $c,d \in \mathbb{R}^n$ *with* $|c| \leq R := \max\{|x| : x \in \overline{K}\}$;

(ii) *the problem*

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\ldots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\ldots,p, \end{cases} \tag{4.4}$$

has only the trivial solution;

(iii) *there exists a function* $V \in C^1(\mathbb{R}^n,\mathbb{R})$, *with* ∇V *locally Lipschitzian, satisfying conditions* $(H1)$ *and* $(H2)$;

(iv) *for all* $x \in \partial K$ *and* $v \in \mathbb{R}^n$ *with* $|v| \leq \beta^{-1}(\beta(2R) + 2R)$, *the inequality*

$$\liminf_{h\to 0^-} \frac{\langle \nabla V(x+hv), v + h\lambda w\rangle}{h} > 0$$

holds, for all $t \in (0,T) \setminus \{t_1,\ldots,t_p\}, \lambda \in (0,1)$ *and* $w \in F(t,x,v)$ *if* $\langle \nabla V(x), v\rangle = 0$ *and for all* $\lambda \in (0,1), w \in F(t_i,x,v)$ *if* $\langle \nabla V(A_i x), B_i v\rangle \leq 0 \leq \langle \nabla V(x), v\rangle$.

Then the Dirichlet problem (1.1)–(1.3) *has a solution* $x(\cdot)$ *such that* $x(t) \in \overline{K}$, *for all* $t \in [0,T]$.

Proof. Define

$$B = \beta^{-1}(\beta(2R) + 2R),$$

$$S = S_1 = Q := \{q \in PC^1([0,T],\mathbb{R}^n) \mid q(t) \in \overline{K},\ |\dot{q}(t)| \leq 2B,\ \text{for all } t \in [0,T]\}$$

and $H(t,c,d,e,f,\lambda) = \lambda F(t,e,f)$. Thus the associated problem (4.3) is the fully linearized problem

$$\begin{cases} \ddot{x}(t) \in \lambda F(t,q(t),\dot{q}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p. \end{cases} \tag{4.5}$$

For each $(q,\lambda) \in Q \times [0,1]$, let $\mathfrak{T}(q,\lambda)$ be the solution set of (4.5). We will check now that all the assumptions of Proposition 4.1 are satisfied.

Since the closure of a convex set is still a convex set, it follows that Q is convex, and hence a retract of $PC^1([0,T],\mathbb{R}^n)$. Moreover,

$$\operatorname{Int} Q = \{q \in PC^1([0,T],\mathbb{R}^n) \mid q(t) \in K,\ |\dot{q}(t)| < 2B,\ \text{for all } t \in [0,T]\} \neq \emptyset,$$

since K is nonempty.

Notice now that, for every $t \in [0,T], c, d \in \mathbb{R}^n$, the inequality

$$|H(t,e,f,c,d,\lambda)| = \lambda |F(t,e,f)| \leq \beta(|f|) \tag{4.6}$$

holds. Hence, denoting $z = (c,d,e,f,\lambda) \in \mathbb{R}^{4n+1}$, since $|f| \leq |z|$, when $|z| \leq r$, the monotonicity of β implies that $|H(t,c,d,e,f,\lambda)| \leq \beta(r)$, which ensures, for every $q \in Q$, the existence of $f_q \in \mathcal{P}_F(q)$. Given $q \in Q$, $\lambda \in [0,1]$, and a L^1-selection $f_q(\cdot)$ of $F(\cdot,q(\cdot),\dot{q}(\cdot))$, let us consider the corresponding single valued linear problem with linear impulses

$$\begin{cases} \ddot{x}(t) = \lambda f_q(t), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p. \end{cases} \tag{4.7}$$

Clearly, for all $q \in Q$ and $\lambda \in [0,1]$,

$$\mathfrak{T}(q,\lambda) = \{x_{\lambda f_q} \in PC^1([0,T],\mathbb{R}^n) : x_{\lambda f_q} \text{ is a solution of (4.7), for some } f_q \in \mathcal{P}_F(q)\}.$$

Using the notation

$$C := \begin{cases} B_1(T-t_1) + A_1 t_1 & \text{if } p = 1 \\ \displaystyle\prod_{l=1}^{p} B_l (T-t_p) + \prod_{k=1}^{p} A_k t_1 + \sum_{j=2}^{p} \prod_{k=j}^{p} A_k \prod_{l=1}^{j-1} B_l (t_j - t_{j-1}) & \text{if } p \geq 2, \end{cases} \tag{4.8}$$

it is easy to prove that the initial problem

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(0) = 0, \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p \end{cases}$$

has infinitely many solutions given by

$$x_0(t) = \begin{cases} \dot{x}_0(0)t & \text{if } t \in [0,t_1], \\ B_1 \dot{x}_0(0)(t-t_1) + A_1 \dot{x}_0(0) t_1 & \text{if } t \in (t_1,t_2] \\ \left[\displaystyle\prod_{l=1}^{i} B_l (t-t_i) + \prod_{k=1}^{i} A_k t_1 + \sum_{j=2}^{i} \prod_{k=j}^{i} A_k \prod_{l=1}^{j-1} B_l (t_j - t_{j-1})\right] \dot{x}_0(0) & \text{if } t \in (t_i,t_{i+1}],\ 2 \leq i \leq p \end{cases}$$

with $\dot{x}_0(0) \in \mathbb{R}^n$. Since $x_0(T) = 0$ if and only if $C\dot{x}_0(0) = 0$, assumption (ii) holds if and only if C is regular. Then (4.7) has a unique solution given by

$$x_{\lambda f_q}(t) = \begin{cases} \dot{x}_{\lambda f_q}(0)t + \lambda \int_0^t (t-\tau) f_q(\tau)d\tau \quad \text{if } t \in [0, t_1], \\ B_1 \dot{x}_{\lambda f_q}(0)(t-t_1) + \lambda \int_{t_1}^t (t-\tau) f_q(\tau)d\tau + B_1(t-t_1)\lambda \int_0^{t_1} f_q(\tau)d\tau + A_1 \dot{x}_{\lambda f_q}(0)t_1 \\ \qquad + A_1 \lambda \int_0^{t_1} (t_1 - \tau) f_q(\tau)d\tau \quad \text{if } t \in (t_1, t_2], \\ \prod_{l=1}^{i} B_l \dot{x}_{\lambda f_q}(0)(t-t_i) + \lambda \int_{t_i}^t (t-\tau) f_q(\tau)d\tau + \sum_{r=1}^{i} \prod_{l=r}^{i} B_l (t-t_i)\lambda \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \\ \qquad + \prod_{k=1}^{i} A_k \dot{x}_{\lambda f_q}(0)t_1 + \prod_{k=1}^{i} A_k \lambda \int_0^{t_1} (t_1-\tau) f_q(\tau)d\tau \\ \qquad + \sum_{j=2}^{i} \prod_{k=j}^{i} A_k \Bigg[\prod_{l=1}^{j-1} B_l \dot{x}_{\lambda f_q}(0)(t_j - t_{j-1}) + \lambda \int_{t_{j-1}}^{t_j} (t_j - \tau) f_q(\tau)d\tau \\ \qquad + \sum_{r=1}^{k-1} \prod_{l=r}^{k-1} B_l (t_j - t_{j-1}) \lambda \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \Bigg] \quad \text{if } t \in (t_i, t_{i+1}],\ 2 \le i \le p \end{cases}$$

with

$$\dot{x}_{\lambda f_q}(0) = -C^{-1}\left(\lambda \int_{t_1}^T (T-\tau) f_q(\tau)d\tau + B_1(T-t_1)\lambda \int_0^{t_1} f_q(\tau)d\tau + A_1\lambda \int_0^{t_1} (t_1-\tau) f_q(\tau)d\tau \right) \tag{4.9}$$

if $p = 1$ and

$$\begin{aligned} \dot{x}_{\lambda f_q}(0) = -C^{-1}\Bigg(& \lambda \int_{t_p}^T (T-\tau) f_q(\tau)d\tau + \sum_{r=1}^{p} \prod_{l=r}^{p} B_l (T-t_p)\lambda \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \\ & + \prod_{k=1}^{p} A_k \lambda \int_0^{t_1} (t_1-\tau) f_q(\tau)d\tau \\ & + \sum_{j=2}^{p} \prod_{k=j}^{p} A_k \Bigg[\lambda \int_{t_{j-1}}^{t_j} (t_j-\tau) f_q(\tau)d\tau + \sum_{r=1}^{k-1} \prod_{l=r}^{k-1} B_l (t_j - t_{j-1}) \lambda \int_{t_{r-1}}^{t_r} f_q(\tau)d\tau \Bigg] \Bigg) \end{aligned} \tag{4.10}$$

if $p \ge 2$. Therefore $\mathfrak{T}(q, \lambda) \ne \emptyset$. Moreover, given $x_1, x_2 \in \mathfrak{T}(q, \lambda)$, there exist f_q^1, f_q^2 such that $x_1 = x_{\lambda f_q^1}$ and $x_2 = x_{\lambda f_q^2}$. Since the right-hand side F has convex values, it holds that, for any $c \in [0, 1]$ and $t \in [0, T]$, $cf_q^1(t) + (1-c)f_q^2(t) \in F(t, q(t), \dot{q}(t))$ as well. The linearity of both the equation and of the impulses yields that $cx_1 + (1-c)x_2 = x_{cf_q^1 + (1-c)f_q^2}$, i.e. that the set of solutions of problem (4.5) is, for each $(q, \lambda) \in Q \times [0, 1]$, convex. Hence assumption (i) of Proposition 4.1 is satisfied.

Moreover, from (4.6), we obtain that, for every $\lambda \in [0, 1], q \in Q, x \in \mathfrak{T}(q, \lambda)$,

$$|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda| \le \beta(|\dot{q}(t)|) \le \beta(2B) \le \beta(2B)(1 + |x(t)| + |\dot{x}(t)|), \tag{4.11}$$

thus also assumption (ii) of the same proposition holds.

The fulfillment of condition (iii) in Proposition 4.1 follows from the fact that, for $\lambda = 0$, problems (4.7) and (4.4) coincide and the latter one has only the trivial solution. Hence, $\mathfrak{T}(q, 0) = 0 \in \operatorname{Int} Q$, because $0 \in K$.

For every $\lambda \in [0,1]$, $q \in Q$ and every solution $x_{\lambda f_q}$ of (4.7), $|x_{\lambda f_q}(0)| = 0$. Moreover, according to assumption (i) and formulas (4.9) and (4.10),

$$
\begin{aligned}
|\dot{x}_{\lambda f_q}(0)| &\leq \|C^{-1}\| \left[\beta(2B)\frac{1}{2}T^2 + T^2\|B_1\|\beta(2B) + \frac{1}{2}T^2\|A_1\|\beta(2B)\right] \\
&= T^2\|C^{-1}\| \cdot \beta(2B)\left[\frac{1}{2} + \|B_1\| + \frac{1}{2}\|A_1\|\right]
\end{aligned}
$$

if $p = 1$ and

$$
\begin{aligned}
|\dot{x}_{\lambda f_q}(0)| \leq \|C^{-1}\| \Bigg[&\frac{1}{2}T^2\beta(2B) + T^2\prod_{l=1}^{p}\|B_l\| \cdot \beta(2B) \\
&+ T^2\prod_{k=1}^{p}\|A_k\|\beta(2B) + T^2\prod_{l=1}^{p}\|B_l\|\prod_{k=1}^{p}\|A_k\| \cdot \beta(2B)\Bigg] \\
= T^2\|C^{-1}\| \cdot \beta(2B)&\left[\frac{1}{2} + \prod_{l=1}^{p}\|B_l\| + \prod_{k=1}^{p}\|A_k\| + \prod_{l=1}^{p}\|B_l\|\prod_{k=1}^{p}\|A_k\|\right]
\end{aligned}
$$

if $p \geq 2$. Therefore there exists a constant M_1 such that $|\dot{x}(0)| \leq M_1$, for all solutions x of (4.5). Hence, condition (iv) in Proposition 4.1 is satisfied as well.

At last, let us assume that $q_* \in Q$ is, for some $\lambda \in [0,1)$, a fixed point of the solution mapping $\mathfrak{T}(\cdot,\lambda)$. We will show now that q_* can not lay in ∂Q. We already proved this property if $\lambda = 0$, thus we can assume that $\lambda \in (0,1)$. From (4.11), we have, for a.a. $t \in [0,T]$, that

$$
|\ddot{q}_*(t)| = \lambda|F(t,q_*(t),\dot{q}_*(t))| \leq \beta(|\dot{q}_*(t)|).
$$

Therefore, since $|q_*(t)| \leq R$, for every $t \in [0,T]$, Proposition 2.1 implies that $|\dot{q}_*(t)| \leq B < 2B$, for every $t \in [0,T]$. Moreover, according to Theorem 3.4 and Remark 3.9, hypotheses (iii) and (iv) guarantee that $q_*(t) \in K$, for all $t \in [0,T]$. Thus $q_* \in \operatorname{Int} Q$, which implies that condition (v) from Proposition 4.1 is satisfied, for all $\lambda \in [0,1)$, and the proof is completed. □

Remark 4.4. An easy example of impulses conditions guaranteeing assumption (ii) in Theorem 4.3 are the antiperiodic impulses, i.e. $A_i = B_i = -I$, for every $i = 1,\ldots,p$. In this case, the matrix $C = (-1)^pTI$ (see [25]) and it is clearly regular. If $p = 1$ condition (ii) holds also e.g. for $A_1 = -I$ and $B_1 = I$ provided $T \neq 2t_1$.

5 Existence and localization result for the impulsive Dirichlet problem with upper-Carathéodory r.h.s.

In this section, we will study the impulsive Dirichlet b.v.p. (1.1)–(1.3) with an upper-Carathéodory r.h.s. and we will develop the bounding functions method with the strictly localized bounding functions also in this more general case. The technique which will be applied for obtaining the final result consists in replacing the original problem by the sequence of problems with non-strict localized bounding functions which satisfy all the assumptions of the following result developed by ourselves recently in [25].

Proposition 5.1 ([25, Theorem 4.1 and Remark 4.3]). *Let $K \subset \mathbb{R}^n$ be a nonempty, open, bounded and convex set with $0 \in K$ and let us consider the impulsive Dirichlet problem (1.1)–(1.3), where $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, and A_i, B_i, $i = 1,\ldots,p$, are real $n \times n$ matrices with $A_i\partial K = \partial K$, for all $i = 1,\ldots,p$. Moreover, assume that* (*i*) *there exists a function* $\varphi : [0, \infty) \to [0, \infty)$ *continuous and increasing satisfying*

$$\lim_{s\to\infty} \frac{s^2}{\varphi(s)} ds = \infty \tag{5.1}$$

such that

$$|F(t, c, d)| \le \varphi(|d|), \tag{5.2}$$

for a.a. $t \in [0, T]$ *and every* $c, d \in \mathbb{R}^n$ *with* $|c| \le R := \max\{|x| : x \in \overline{K}\}$;

(*ii*) *the problem*

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1, \ldots, p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1, \ldots, p, \end{cases} \tag{5.3}$$

has only the trivial solution;

(*iii*) *there exists a function* $V \in C^1(\mathbb{R}^n, \mathbb{R})$, *with* ∇V *locally Lipschitzian, satisfying conditions* $(H1)$ *and* $(H2)$;

(*iv*) *there exists* $\varepsilon > 0$ *such that, for all* $\lambda \in (0, 1)$, $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0, T)$, *and* $v \in \mathbb{R}^n$, *with* $|v| \le \varphi^{-1}(\varphi(2R) + 2R)$, *the following condition*

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0 \tag{5.4}$$

holds, for all $w \in \lambda F(t, x, v)$;

(*v*) *for all* $i = 1, \ldots, p$, $x \in \partial K$ *and* $v \in \mathbb{R}^n$, *with* $|v| \le \varphi^{-1}(\varphi(2R) + 2R)$ *and* $\langle \nabla V(x), v\rangle \ne 0$, *it holds that*

$$\langle \nabla V(A_i x), B_i v\rangle \cdot \langle \nabla V(x), v\rangle > 0.$$

Then the Dirichlet problem (1.1)–(1.3) *has a solution* $x(\cdot)$ *such that* $x(t) \in \overline{K}$, *for all* $t \in [0, T]$.

Approximating the original problem by a sequence of problems satisfying conditions of Proposition 5.1 and applying the Scorza-Dragoni type result (Proposition 2.2), we are already able to state the second main result of the paper. The transversality condition is now required only on the boundary ∂K of the set K and not on the whole neighborhood $\overline{K} \cap N_\varepsilon(\partial K)$, as in Proposition 5.1.

Theorem 5.2. *Let* $K \subset \mathbb{R}^n$ *be a nonempty, open, bounded and convex set with* $0 \in K$ *and let us consider the impulsive Dirichlet problem* (1.1)–(1.3), *where* $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper Carathéodory multivalued mapping,* $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T, p \in \mathbb{N}$, *and* A_i, B_i, $i = 1, \ldots, p$, *are real* $n \times n$ *matrices with* $A_i \partial K = \partial K$, *for all* $i = 1, \ldots, p$. *Moreover, assume that*

(*i*) *there exists a function* $\beta : [0, \infty) \to [0, \infty)$ *continuous and increasing satisfying*

$$\lim_{s\to\infty} \frac{s^2}{\beta(s)} ds = \infty$$

such that

$$|F(t, c, d)| \le \beta(|d|),$$

for a.a. $t \in [0, T]$ *and every* $c, d \in \mathbb{R}^n$ *with* $|c| \le R := \max\{|x| : x \in \overline{K}\}$;

(ii) *the problem*

$$\begin{cases} \ddot{x}(t) = 0, & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p, \end{cases} \tag{5.5}$$

has only the trivial solution;

(iii) *there exists $h > 0$ and a function $V \in C^2(\mathbb{R}^n, \mathbb{R})$, with $HV(x)$ positive semidefinite in $N_h(\partial K)$, satisfying conditions $(H1), (H2)$;*

(iv) *for all $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \le \beta^{-1}(\beta(2R) + 2R)$, the inequality*

$$\langle \nabla V(x), w \rangle > 0$$

holds for all $t \in (0,T)$ and $w \in F(t,x,v)$;

(v) *for all $i = 1,\dots,p$, $x \in \partial K$ and $v \in \mathbb{R}^n$, with $|v| \le \beta^{-1}(\beta(2R) + 2R)$ and $\langle \nabla V(x), v \rangle \neq 0$, it holds that*

$$\langle \nabla V(A_i x), B_i v \rangle \cdot \langle \nabla V(x), v \rangle > 0.$$

Then the Dirichlet problem (1.1)–(1.3) has a solution $x(\cdot)$ such that $x(t) \in \overline{K}$, for all $t \in [0,T]$.

Proof. Since $V \in C^2(\mathbb{R}^n, \mathbb{R})$, the function $x \to |\nabla V(x)|$ is continuous on the compact set ∂K, and hence there exists $k > 0$ such that $|\nabla V(x)| > 0$ for every $x \in N_k(\partial K)$. Define $\delta = \min\{h, k\}$. According to Urysohn's Lemma, there exists a function $\mu \in C(\mathbb{R}^n, [0,1])$ such that $\mu \equiv 1 \in N_{\frac{\delta}{2}}(\partial K)$ and $\mu \equiv 0 \in \mathbb{R}^n \setminus N_\delta(\partial K)$. Take a sequence of positive numbers $\{\epsilon_m\}$ decreasing to zero, an open and bounded set G, with $\overline{K} \subset G$, and $L > \beta^{-1}(\beta(2R) + 2R)$. According to Proposition 2.2 there exist a monotone decreasing sequence $\{\theta_m\}$ of open subsets of $[0,T]$ and a measurable multimap $F_0 : [0,T] \times \overline{G} \times \{v \in \mathbb{R}^n : |v| \le L\} \multimap \mathbb{R}^n$ such that $\nu(\theta_m) \le \epsilon_m, F_0(t,x,v) \subset F(t,x,v)$ and F_0 is u.s.c. on $([0,T] \setminus \theta_m) \times \overline{G} \times \{v \in \mathbb{R}^n : |v| \le L\}$ for every $m \in \mathbb{N}$. Trivially $\nu(\cap_{m=1}^\infty \theta_m) = 0$ and $\lim_{m\to\infty} \chi_{\theta_m}(t) = 0$ for every $t \notin \cap_{m=1}^\infty \theta_m$.

Define, for each $m \in \mathbb{N}$, $(t,x,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$F_m(t,x,v) = \begin{cases} F_0(t,x,v) + 2\mu(x)\beta(|v|)\chi_{\theta_m}(t)\frac{\nabla V(x)}{|\nabla V(x)|} & \text{if } x \in G \text{ and } |v| < L \\ F(t,x,v) + 2\mu(x)\beta(|v|)\chi_{\theta_m}(t)\frac{\nabla V(x)}{|\nabla V(x)|}, & \text{otherwise.} \end{cases}$$

Since $\delta \le k$, we have that $\mu(x) = 0$ for $x \in \mathbb{R}^n \setminus N_\delta(\partial K)$ and $\nabla V(x) \neq 0$ in $N_\delta(\partial K)$, hence it follows that F_m is well defined. Since μ and β are continuous, V is of class C^2, G is open, $F_0(t,x,v) \subset F(t,x,v)$, and F is an upper-Carathéodory map, F_m is a Carathéodory map as well.

Let us now prove that problem

$$\begin{cases} \ddot{x}(t) \in F_m(t,x(t),\dot{x}(t)), & \text{for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, & \\ x(t_i^+) = A_i x(t_i), & i = 1,\dots,p, \\ \dot{x}(t_i^+) = B_i \dot{x}(t_i), & i = 1,\dots,p. \end{cases} \tag{5.6}$$

satisfies the assumptions of Proposition 5.1.

First of all notice that, since $0 \le \mu(x) \le 1, 0 \le \chi_{\theta_m}(t) \le 1$, for every $x \in \mathbb{R}^n, t \in [0,T]$, it holds, according to (i),

$$|F_m(t,c,d)| \le |F(t,c,d)| + 2\beta(|d|) \le 3\beta(|d|),$$

for every $(t,c,d) \in t \times \mathbb{R}^n \times \mathbb{R}^n$ with $|c| \le R$. Thus condition (i) of Proposition 5.1 is satisfied by the continuous increasing function $\varphi = 3\beta$, since it clearly holds that

$$\lim_{s\to\infty} \frac{s^2}{\varphi(s)} = \frac{1}{3} \lim_{s\to\infty} \frac{s^2}{\beta(s)} = \infty.$$

Moreover, conditions (ii) and (iii) imply the analogous conditions in Proposition 5.1.

Let us now observe that, since $\varphi(s) = 3\beta(s)$, then $\varphi^{-1}(s) = \beta^{-1}(\frac{s}{3})$, which is an increasing function, as inverse of an increasing function. Hence

$$\varphi^{-1}(\varphi(2R) + 2R) = \beta^{-1}\left(\frac{3\beta(2R) + 2R}{3}\right) = \beta^{-1}\left(\beta(2R) + \frac{2}{3}R\right) \le \beta^{-1}(\beta(2R) + 2R).$$

Therefore, condition (v) implies the analogous condition of Proposition 5.1. Moreover, for every $\lambda \in (0,1), x \in \overline{K} \cap N_{\frac{\delta}{2}}(\partial K)$, $t \in (0,T)$, and $v \in \mathbb{R}^n$, with $|v| \le \varphi^{-1}(\varphi(2R) + 2R), w_1 \in \lambda F_m(t,x,v)$,

$$\begin{aligned}\langle HV(x)v, v\rangle + \langle \nabla V(x), w_1\rangle &= \langle HV(x)v, v\rangle + \lambda[\langle \nabla V(x), w\rangle + 2\mu(x)\beta(|v|)\chi_{\theta_m}(t)|\nabla V(x)|] \\ &= \langle HV(x)v, v\rangle + \lambda[\langle \nabla V(x), w\rangle + 2\beta(|v|)\chi_{\theta_m}(t)|\nabla V(x)|].\end{aligned}$$

Then, if $t \in [0,T] \setminus \theta_m$

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_1\rangle = \langle HV(x)v, v\rangle + \lambda\langle \nabla V(x), w\rangle \ge \lambda\langle \nabla V(x), w\rangle,$$

with $w \in F_0(t,x,v)$, because $\overline{K} \cap N_{\frac{\delta}{2}}(\partial K) \subset \overline{K} \subset G$ and $\varphi^{-1}(\varphi(2R) + 2R) \le \beta^{-1}(\beta(2R) + 2R) < L$. Since V is of class C^2, F_0 is u.s.c. on the compact set $([0,T] \setminus \theta_m) \times \partial K \times \{v \in \mathbb{R}^n : |v| \le \varphi^{-1}(\varphi(2R) + 2R)\}$, and F_0 is compact valued, condition (iv) implies that there exists $k_1 > 0$ such that

$$\langle \nabla V(x), w\rangle > 0$$

for every $t \in [0,T] \setminus \theta_m, x \in \overline{K} \cap N_{k_1}(\partial K), v \in \mathbb{R}^n : |v| \le \varphi^{-1}(\varphi(2R) + 2R), w \in F_0(t,x,v)$. Hence,

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w_1\rangle \ge \lambda\langle \nabla V(x), w\rangle > 0,$$

for all $\lambda \in (0,1), t \in [0,T] \setminus \theta_m, x \in \overline{K} \cap N_{k_1}(\partial K), v \in \mathbb{R}^n : |v| \le \varphi^{-1}(\varphi(2R) + 2R), w_1 \in \lambda F_m(t,x,v)$.

On the other hand, if $t \in \theta_m$, since $x \in N_{\frac{\delta}{2}}(\partial K)$ and $h \ge \delta$,

$$\begin{aligned}\langle HV(x)v, v\rangle + \langle \nabla V(x), w_1\rangle &\ge \lambda[\langle \nabla V(x), w\rangle + 2\beta(|v|)|\nabla V(x)|] \\ &\ge \lambda[-|w| + 2\beta(|v|)]|\nabla V(x)| \ge \lambda\beta(|v|)|\nabla V(x)| > 0.\end{aligned}$$

Condition (iv) in Proposition 5.1 follows taking $\epsilon = \min\{k_1, \frac{\delta}{2}\}$.

Applying Proposition 5.1 we obtain that, for every $m \in \mathbb{N}$, there exists a solution x_m of (5.6) such that $x_m(t) \in \overline{K}$ and $|\dot{x}_m(t)| \le \varphi^{-1}(\varphi(2R) + 2R)$, for every $t \in [0,T]$. Hence $|\ddot{x}_m(t)| \le \varphi(2R) + 2R$ for every $t \in [0,T]$. The Ascoli–Arzelà theorem implies that $\{x_m\} \to x$ uniformly in $C^1([0,T],\mathbb{R}^n)$ and $\ddot{x}_m \to \ddot{x}$ weakly in $L^1([0,T],\mathbb{R}^n)$. Thus $x(t) \in \overline{K}$, $|\dot{x}(t)| \leq \varphi^{-1}(\varphi(2R)+2R)$ for every $t \in [0,T]$, and x satisfies (1.2)–(1.3). Moreover, since $\nu(\cap_{n=1}^{\infty}\theta_m) = 0$,

$$\lim_{m\to\infty} 2\mu(x_m(t))\beta(|\dot{x}_m(t)|)\chi_{\theta_m}(t)\frac{\nabla V(x_m(t))}{|\nabla V(x_m(t))|} = 0,$$

for a.a. $t \in [0,T]$. Consequently, a standard limiting argument (see e.g. [28, Theorem 3.1.2]) implies that x is a solution of

$$\ddot{x}(t) = F_0(t,x(t),\dot{x}(t))$$

and, since $F_0(t,x(t),\dot{x}(t)) \subset F(t,x(t),\dot{x}(t))$, a solution of the problem (1.1)–(1.3). □

Remark 5.3. Both Theorems 4.3 and 5.2 give an existence result for an impulsive Dirichlet boundary value problem with a strictly localized bounding function respectively for u.s.c. and upper-Carathéodory multimap. However Theorem 5.2 does not represent an extension of Theorem 4.3, since the first one deals with a C^2-bounding function, while the second one is related to a C^1-bounding function and can not be easily extended to the Carathéodory case.

In the case when the multivalued mapping F is u.s.c. and the bounding function V is of class C^2, i.e. when it is possible to apply both theorems, conditions of Theorem 4.3 are weaker than assumptions of Theorem 5.2. In fact, in this case, according to Corollary 3.8, condition (iv) of the first theorem reads as

$$\langle HV(x)v,v\rangle + \lambda\langle\nabla V(x),w\rangle > 0$$

for every $x \in \partial K, \lambda \in (0,1)$, $v \in \mathbb{R}^n$, with $|v| \leq \beta^{-1}(\beta(2R)+2R)$, and for every $t \in [0,T] \setminus \{t_1,\dots,t_p\}, w \in F(t,x,v)$ if $\langle\nabla V(x),v\rangle \neq 0$, or for every $w \in F(t_i,x,v)$ if $\langle\nabla V(A_i x),B_i v\rangle \leq 0 \leq \langle\nabla V(x),v\rangle$, which are implied by assumptions (iii) and (iv) of the second theorem.

6 An application of the main result

As an application of Theorem 5.2, let us consider the second-order inclusion

$$\ddot{x}(t) \in a(t)\dot{x}(t) + h(t,x(t))), \quad \text{for a.a. } t \in [0,T], \tag{6.1}$$

together with antiperiodic impulses and Dirichlet boundary conditions

$$x(t_i^+) = -x(t_i), \qquad i = 1,\dots,p, \tag{6.2}$$
$$\dot{x}(t_i^+) = -\dot{x}(t_i), \qquad i = 1,\dots,p, \tag{6.3}$$
$$x(0) = x(T) = 0, \tag{6.4}$$

where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$. Assume that $a \in L^\infty([0,T],\mathbb{R})$, with $\|a\|_\infty > 0$, and $h : [0,T] \times \mathbb{R} \multimap \mathbb{R}$ is an upper-Carathédory multivalued mapping with

$$|h(t,y)| \leq \alpha(t)g(y)$$

for some $\alpha \in L^\infty([0,T],\mathbb{R})$, $g \in C(\mathbb{R},\mathbb{R})$.

When h is a function, the impulsive Dirichlet boundary value problem associated to the single valued equation $\ddot{x}(t) = a(t)\dot{x}(t) + h(t,x(t))$ represents a generalization of a wide class of equations which are widely studied in literature (see, e.g., [1,13,16,26,29]) for its several applications (including biological phenomena involving thresholds, models describing population dynamics or inspection processes in operations research). Much more rare are the results concerning the multivalued case which can be e.g. used for modelling optimal control problems in economics.

We will show now that, under very general conditions, the Dirichlet multivalued problem (6.1), (6.4) together with impulse conditions (6.2), (6.3) satisfies all the assumptions of Theorem 5.2. On this purpose, let us consider the nonempty, open, bounded, convex and symmetric neighbourhood of the origin $K = (-k, k)$, with k to be specified later, and the C^2-function $V(x) = \frac{1}{2}(x^2 - k^2)$ that trivially satisfies conditions $(H1)$ and $(H2)$.

In order to verify condition (i), let us define the continuous and increasing function

$$\beta(d) = \|a\|_\infty d + \|\alpha\|_\infty \overline{g}, \quad \text{for all } d \in [0, +\infty),$$

where $\overline{g} = \max_{|x|\le k} |g(x)|$. The function β obviously satisfies (5.1) and $F(t, c, d) := a(t)d + h(t, c)$ satisfies (3.16), for all $t \in [0, T]$ and all c, $d \in \mathbb{R}$, with $|c| \le k$.

Assumption (ii) holds as well since, according to Remark 4.4, the associated homogeneous problem has only the trivial solution.

Condition (iii) follows from the fact that $\dot{V}(x) = x$ and $\ddot{V}(x) = 1$, for every $x \in \mathbb{R}$.

Notice moreover that, whenever $xv \neq 0$, then $(-x)(-v)xv = x^2v^2 > 0$, hence also condition (v) holds.

Finally, since $\beta^{-1}(d) = \frac{1}{\|a\|_\infty}(d - \|\alpha\|_\infty \overline{g})$, we easily get that

$$\beta^{-1}(\beta(2k) + 2k) = 2k\left(1 + \frac{1}{\|a\|_\infty}\right).$$

Thus condition (iv) reads as

$$a(t)xv + xw > 0 \tag{6.5}$$

for every $t \in [0, T]$, x with $|x| = k$, v with $|v| \le 2k(1 + \frac{1}{\|a\|_\infty})$ and $w \in h(t, x)$. Taking $x = k$ we then get $w > -a(t)v$, for every $w \in h(t, k)$. Since the previous condition must hold both for positive and negative values of v, $h(t, k)$ must take only positive values and the transversality condition is satisfied if

$$w > \|a\|_\infty 2k\left(1 + \frac{1}{\|a\|_\infty}\right) = 2k(\|a\|_\infty + 1) \qquad \forall\, w \in h(t, k).$$

Similarly,taking $x = -k$ we get that (6.5) is equivalent to $w < -a(t)v$, for every $w \in h(t, -k)$ which is satisfied only if w is negative. A sufficient condition then becomes

$$w < -2k(\|a\|_\infty + 1) \qquad \forall\, w \in h(t, -k).$$

Thus condition (iv) holds if there exists $k > 0$ such that for every $w_1 \in h(t, k), w_2 \in h(t, -k)$,

$$w_1 > 2k(\|a\|_\infty + 1) \quad \text{and} \quad w_2 < -2k(\|a\|_\infty + 1). \tag{6.6}$$

The previous result can be stated in the form of the following theorem.

Theorem 6.1. *Assume that $a \in L^\infty([0, T], \mathbb{R})$, with $\|a\|_\infty > 0$, $h : [0, T] \times \mathbb{R} \multimap \mathbb{R}$ is an upper-Carathédory multivalued mapping with*

$$|h(t, y)| \le \alpha(t)g(y),$$

for some $\alpha \in L^\infty([0, T], \mathbb{R})$, $g \in C(\mathbb{R}, \mathbb{R})$. Moreover, assume that there exists $k > 0$ such that, for every $t \in [0, T]$, and $w \in h(t, k)$,

$$w > 2k(\|a\|_\infty + 1)$$

and that, for every $t \in [0,T]$, and $w \in h(t,-k)$,

$$w < -2k(\|a\|_\infty + 1).$$

Then problem (6.1)–(6.4) *has a solution $x(\cdot)$ such that $|x(t)| \leq k$, for every $t \in [0,T]$.*

Remark 6.2. Suppose that, in (6.1), $h(t,x) = \gamma(t) + \alpha(t)f(x)$, where f is an odd semicontinuous multimap and $\alpha, \gamma \in L^\infty([0,T],\mathbb{R})$. Then (6.6) is equivalent to require the existence of $k > 0$ such that, for every $t \in [0,T]$,

$$\alpha(t)f(k) > 2k(|a\|_\infty + 1) - \gamma(t).$$

If $\alpha(t) \geq \overline{\alpha} > 0$, for every $t \in [0,T]$, then (6.6) is equivalent to

$$\overline{\alpha}f(k) > 2k(\|a\|_\infty + 1) - \|\gamma^-\|_\infty,$$

where $\gamma^-(t) = \min\{0, \gamma(t)\}$, which holds, e.g., if f is superlinear at infinity, which is true in many applications. The superlinearity of f at infinity is a sufficient condition also if $\alpha(t) \leq -\overline{\alpha} < 0$, for every $t \in [0,T]$. Notice that the obtained solution is a nonzero function whenever γ is a nonzero function.

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