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Habilitation thesis reviewer's report

Masaryk University	
Faculty of Science	
Field of study	**Mathematics - Geometry**
Applicant	**Mgr. Josef Šilhan, Ph.D.**
Unit	**Department of Mathematics and Statistics, Faculty of Science, Masaryk University**
Habilitation thesis	**Invariant quantization and differential symmetries on AHS structures**
Reviewer	Professor Andrew Waldron, Ph.D. (S.U.N.Y. Stony Brook), B.Sc. Hons (University of Melbourne)
Unit	Department of Mathematics, University of California, Davis

Reviewer's report

Before giving a detailed assessment of this Habilitation, it will likely be useful for the Thesis Committee to understand the standpoint from which I am basing the following opinions. At an American research university, there exists a promotion step from Assistant to Associate Professor similar to the Habilitation under review. As Full Professor, my role includes voting on many such promotions. Therefore it is reasonable to judge the present case on the basis of similar merit actions that have passed through our system. We adjudge our young faculty on their research, teaching and administrative achievements. Here, I will restrict my judgements only to the research merit of Mgr. Josef Silhan. His Habilitation package summarizes work appearing in six journal publications. In each case these have appeared in quality, professionally refereed journals. This level of research output would likely guarantee promotion in nearly all cases in our system. Indeed, coupled with positive opinions from independent letter writers, a promotion to a tenured position would be all but certain. Moreover, having spent quite some time working through these publications and the accompanying survey that Josef has written, I believe that this Habilitation easily meets the requirements for promotion.

Josef's work in the merit period has been in three main directions: parabolic geometries and almost hermitean structures, invariant quantization, and prolongation theory. Conformal geometries are the key example to which his and his collaboarator's methods apply. Broadly, his mathematical results impact differential geometry, representation theory and algebra. However, what is also rather impressive, is that his work has deep consequences for fundamental physics. As there there are two other referees who will focus on the mathematical consequences of his work, I will attempt to give some insight into importance of his work for fundamental physical models.

Symmetries of the Laplacian:

The Laplace equation, $\Delta f = 0$, has a startling number of applications across mathematics and physics. On pseudoRiemannian manifolds, solutions correspond to particle excitations.

Moreover, work instigated by the Vasiliev group in Moscow has shown that the symmetries of this equation in conformally flat spaces labels the space of gauge symmetries of interacting higher spin systems. Seminal work by Eastwood in 2005 classified the space of higher symmetries of this equation. A natural question is then to study how these symmetries generalize to the higher, conformally invariant Laplacian powers first constructed in generality by Graham, Jennes, Mason and Sparling. Josef's joint work with Rod Gover, constructs higher symmetries of these operators in conformally flat settings; as well as establishing their algebraic structure. This is a very nice result. For physical applications such as higher spin symmetries, their restriction to conformally flat goemetries is not a severe one, since this anyway seems to be the correct local model for the physical systems constructed so far. Nonetheless, in collaboration with Jean-Philippe and Fabian Radoux, Josef has also managed to classify second order symmetries of the generally curved Yamabe operator. These symmetries can be lifted to symmetries of natural particle motions in generally curved spaces, so again this is a result with likely important physical consequences.

Quantization:

Hamiltonian vector fields governing the evolution and symmetries of classical systems are, upon quantization, lifted to self-adjoint operators on Hilbert spaces. They then control the unitary evolution of wavefunctions and quantum symmetries. This lift is far from unique, and ultimately the correct choice is determined by experimental data. However, often one has theoretical information of the underlying geometry of a given a model. This can be used to constrain the space of possible quantizations. Almost Hermitean structures, and therefore physically pressing cases such as conformal or projective invariance are considerable interest here. In collaboration with Andreas Cap, Josef has constructed a map from the space of symbols (this encodes the leading derivative behavior of invariant differential operators) to maps from section spaces of certain bundles which respect the underlying almost Hermitean structure. Moreover, in a single author publication, he has attempted to perform a complete classification of conformal invariant quantization leading to maps from densities to densities. In particular, using modern tractor calculus machinery, he has an existence result for the case these maps avoid certain critical weights. Remembering that conformal densities are the correct objects to describe dimensionful geometric quantities, this is a very strong result. Again here there are indications that this could impact cutting edge physics research. For example, in the context of the AdS/CFT correspondence, which attempts to describe quantum field theories with conformal symmetries in terms of negative curved geometries whose metric obeys an Einstein condition. Densities and the invariant maps acting on these play a fundamental role. Josef's result provides an interesting link to underlying classical systems.

Prolongation and Tractor Calculus:

Systems of PDEs play a fundamental role for many physical systems. Often one is faced with higher order overdetermined systems. An important technique for handling these is to ``prolong" the set of equations to a first order system and then identify the underlying geometry. A classic example of this manuevre was the seminal work of Bailey, Eastwood and Gover which re-expressed the Einstein condition (describing cosmological geometries) in terms of a parallel condition for a scale tractor. The latter is a section of the so-called tractor bundle—a fundamental construction in conformal geometry. A similar picture extends to parallel sections of tractor bundles for almost Hermitean structures and associated invariant operators. In collaboration with Mattias Hammerl, Petr Somberg and Vladimir Soucek, Josef has studied to which extent this construction extends to generally curved cases and

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constructed corresponding prolongation connections. These are employed to write the associated parallel conditions. In particular, they focus on prolongations of first BGG operators. These have appeared as the gauge invariances of novel physical theories, for example partially massless models of higher spins propagating in cosmological spacetimes. This suggests the possibility that there might exist analogs of those models for general almost hermitean structures.

As is clear from the above synopsis of Josef's excellent research contributions, he works well both alone and in collaboration with leading international researchers. I have not had the opportunity to gauge the efficacy of his oral presentations, but his his written work (as evidence by this Habilitation and his published works) is of the highest quality.

Reviewer's questions for the habilitation thesis defence

There are some possibly very interesting relations of Josef's work to physical models, for example, have connections to symmetries of spinning particle models studied by van Holten, Gibbons and Rietdijk, as well as the Bologna group of Bastianelli been considered? Also, particle models in curved spaces could yield the underlying theories enjoying the higher symmetries studied in this Habiliation. Moreover their quantization (in the physical sense) could then be related to the quantization techniques pursued here. I would be interested in hearing any thoughts on such a relationship. Finally, I had difficulty understanding the last Equation on page 164 of the Habilitation (p162 of the Glob. Anal. Geom. article).

Conclusion

The habilitation thesis submitted by mgr. **Josef Šilhan, Ph.D.**, entitled "***Invariant quantization and differential symmetries on AHS structures***" ***meets*** the requirements applicable to habilitation theses in the field of **Mathematics - Geometry**.

In U.C. Davis on April 24, 2016

Andrew Waldron

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signature