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Riccati methods for half-linear differential equations

Habilitation Thesis

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Abstract

The form of the thesis is a commentary on a collection of eight scholarly papers devoted to studies of the second-order half-linear differential equations. We are interested mainly in the qualitative theory and asymptotic properties of studied equations, focusing on the applicability of the so-called modified Riccati technique. With its use, it is possible to investigate especially conditionally oscillatory equations, which occur at the threshold between oscillation and nonoscillation, and their perturbations. After the introductory part presenting the historical background and the methodology, the results providing concrete asymptotic formulas are commented. The next sections sum up our results in the form of various oscillation and nonoscillation criteria for ordinary and neutral half-linear equations. In the end, the numerical approach to finding approximate solutions of half-linear Euler-type equations, which makes use of the differential transformation method, is introduced. The thesis is concluded with a list of possible future directions of research.

Contents

Preface

Since establishing the foundations of calculus, differential equations are one of the most widely used tools for describing continuous real processes. The object of our interest - half-linear second-order differential equations - can be seen either as a generalization of ordinary linear second-order differential equations or as a scalar case of partial differential equations with p-Laplacian. The qualitative theory of half-linear equations has been deeply developed during the last decades and its research still continues.

The thesis is based on the papers [65, 66, 67, 109, 110, 111, 112, 114]. In the body of the thesis, their results are summarized and commented, and printouts of the full versions are enclosed at the end in the form of an attachment.

The work is organized as follows. In the first chapter, we provide a brief introduction to the topic in the historical context and present the main proving techniques used in our research. The second chapter is devoted to studies of asymptotic formulas for nonoscillatory solutions of the so-called conditionally oscillatory half-linear differential equations ([109, 110]). The third chapter brings a variety of oscillation results in the form of integral oscillation and nonoscillation criteria ([65, 66, 111]). An extension of some oscillation criteria to neutral half-linear differential equations is presented in the fourth chapter ([67]). In the last chapter, we change our perspective and turn our attention to numerical methods for finding approximate solutions of half-linear second-order differential equations, in particular to application of the differential transformation method to Euler type half-linear equations without delay and with proportional delay ([112, 114]).

I would like to express my thanks to my coauthors for their fruitful cooperation and for motivation.

My greatest gratitude goes in memoriam to Professor Ondřej Došlý for his kindness, willingness, patience, guidance, and inspiration.

Chapter 1

Introduction

The object of study of the thesis is mainly the half-linear second-order differential equation of the form

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \tag{1.1}$$

where $\Phi(x) := |x|^{p-1}\operatorname{sgn} x$, $p > 1$, and r, c are continuous functions, $r(t) > 0$. For $p = 2$ equation (1.1) reduces to the second-order linear Sturm-Liouville differential equation (also called second-order homogenous self-adjoint equation)

$$(r(t)x')' + c(t)x = 0. \tag{1.2}$$

From this point of view, the study of the properties of its generalization (1.1) is a natural direction of research.

On the other hand, half-linear equation (1.1) can be achieved as a transformation of some partial differential equations with the so-called p-Laplacian

$$\Delta_p u := \operatorname{div}(||\nabla u||^{p-2}\nabla u),$$

where for $u(x) = u(x_1, \ldots, x_N)$, $N \in \mathbb{N}$, the symbol ∇u stands for the Hamilton nabla operator and div denotes the usual divergence operator. If u is a radially symmetric function and $t = ||x||$, then the transformation $u(x) = y(t)$ reduces the partial differential p-Laplacian operator to the ordinary differential operator

$$\Delta_p u(x) = t^{1-N}(t^{1-N}\Phi(y'(t)))', \quad ' = \frac{d}{dt}.$$

In this connection half-linear equation (1.1) with $r(t) = 1$ can be considered as a class of differential equations with the one-dimensional p-Laplacian. Origins of p-Laplacian are described, for example, in the paper [4]. Accordingly, the history of p-Laplacian is closely linked to applications in the filtration of fluids through porous media and nonlinear non-Newtonian fluid dynamics. Another application can be found, for example, in [2]. Here p-Laplacian is used to model a non-homogenous diffusion to determine the height of a growing pile of noncohesive sand, where an ordinary differential equation arises in the limit case of "infinitely fast/slow" diffusion (see also [107]).

Furthermore, half-linear equation (1.1) appears as a special case of the quasilinear equation of the form

$$(r(t)\Phi_{p_1}(x'))' + c(t)\Phi_{p_2}(x) = 0, \quad \Phi_{p_1}(x) = |x|^{p_1-1}\operatorname{sgn} x,\ \Phi_{p_2}(x) = |x|^{p_2-1}\operatorname{sgn} x, \tag{1.3}$$

where $p_1, p_2 > 1$ and the coefficient functions r, c satisfy the same assumptions as in equation (1.1). If $p_1 = p_2$, then equation (1.3) reduces to equation (1.1). In this context, half-linear equation (1.1) can be understood as the borderline between the so-called super-half-linear and sub-half-linear equations, when $p_1 > p_2$ and $p_1 < p_2$, respectively. In the qualitative theory of quasilinear equation (1.3), half-linear equation (1.1) can be used for comparison purposes.

Finally, the quasilinear equation (1.3) can be generalized to the following second-order nonlinear equation of the form

$$(r(t)\Phi(x'))' + F(x,t) = 0,$$

where $F(x,t)$ is a general nonlinear function. Half-linear equation (1.1), as its particular case, can still play an important role in investigating even such general equations.

This chapter is devoted to an introductory treatise on the topic. The first section attempts to present the historical background and the general scientific context of the author's results. In the second section, some relevant notions and the proving technique common to a majority of the presented author's results are recalled.

1.1 Historical background

According to the monograph [39], which is considered as a primer of half-linear equations, the first results concerning the solution space of half-linear equations can be found in the papers [5, 6] by Bihari from the years 1957-58. Equation (1.1) firstly appeared probably in the year 1961 in [3], where the Riccati type transformation was introduced.

In Bihari's paper [7] equation (1.1) was for the first time referred to as “half-linear”. This designation comes from the fact that the solution space of (1.1) is only homogenous but it is not additive (i.e., it has got only half of the linearity properties). In the late seventies, Mirzov and Elbert published papers [104] and [55], respectively, which are considered as the pioneering works of the theory of half-linear equations. In [55] the half-linear Prüfer transformation was presented. With its use, for the intervals where $r(t) > 0$ and $r(t), c(t)$ are continuous, the existence and uniqueness theorem for an initial value problem can be proved. Hence solutions of (1.1) are extendable for arbitrarily large intervals.

In the following years, it was found out that the oscillation theory of half-linear equation (1.1) evinces many very similar aspects as that of linear equation (1.2). For example, the half-linear version of Sturmian theory extends from the linear theory almost verbatim. However, the missing additivity of the solution space of half-linear equations also causes some differences, as, for example, the unavailable Wronskian identity and Fredholm alternative. These missing tools have to be overcome by new approaches, which, surprisingly, sometimes lead to ways to obtain results that are new even in the linear case.

In the last decades of the 20th century and at the beginning of the new millennium, the qualitative theory of half-linear equations was thoroughly developed by many authors. A partial survey of the results was made in a chapter of the book [1] by Agarwal, Grace, and O'Regan in 2002. Three years later, in the year 2005, the monograph by Došlý and Řehák [39] was published, and since then, it has been widely used as a reference book by many authors. The book exhaustively sums up the basics of the half-linear theory, as well as many branches of relevant research up to the year of publication. After 2005, work on the research of half-linear equations has naturally continued. During this period, until today, hundreds of papers dealing with equation (1.1) and with equations or their systems generalizing it were published. Due to the content of this thesis, we consider mainly ordinary differential equations. Concerning partial differential equations related to half-linear equation (1.1), we refer, for example, to the group of authors around professor Drábek from Pilsen (see, for example [52, 53]).

In the rest of the section, we try to indicate the main directions that the study of equation (1.1) since 2005 went on. This attempt is not meant as an exhaustive overview, which exceeds the possibilities of this work. The aim is to show the main branches of the current research of half-linear equations, also that the topic is still actual and that it is being addressed by scientists all around the world.

The Japanese school of half-linear differential equations has played an important role in the research of half-linear equations throughout their whole history. The leading personality of the group working in the theory of half-linear equations in the observed period has been professor Sugie. Since 2005, he has collaborated on publications related to half-linear equations with many coauthors, for example, Yamaoka, Onitsuka, Yamaguchi, Kono, Matsumura, Hata, Wu, Ishibashi, and others. Sugie's interest has been not only in half-linear equations of the form (1.1), but also in other related fields such as nonlinear differential equations with p-Laplacian, delayed half-linear equations, nonautonomous half-linear differential systems and later also damped half-linear oscillators. From their papers, let us refer at least to some of those dealing with equation (1.1): [117, 118, 119, 122]. Very recently, the asymptotic properties of solutions of (1.1) were studied by Naito in [106]. Different groups of coauthors published their papers on half-linear differential systems and damped half-linear oscillators (Onitsuka, Soeda, Naito, Pasic, Tanaka, Enaka). The interest of Yoshida was in elliptic equations and inequalities with p-Laplacian. Yamaoka in [123] considered nonlinear perturbations of the half-linear Euler equation. Recently, a very new research direction has been introduced by Yamaoka and Fujimoto, who admit p to be a function $p(t)$ ([68, 69]).

Next, let us focus on the links between the two schools in Japan and in Brno. In addition to the collaboration between Yamaoka and Došlý, and a very recent common work of Fujimoto and Došlá, let us mention the position of professor Jaroš from Bratislava. In his publications with Brno coauthors ([34, 94]) he proved some oscillation and nonoscillation criteria for (1.1). In his own papers, he focused mainly on comparison and integral criteria for even-order half-linear equations ([85, 86, 87]), and he presented a reduction-of-order approach for higher-order equations ([88]). In cooperation with Kusano and Tanigawa, several papers were published not only for equation (1.1) ([91]) but also for half-linear differential systems ([89, 92]) and the fourth-order equation ([93]).

Another frequent coauthor of Kusano and Tanigawa has been Manojlović. Their cooperation resulted in papers on the existence of regularly varying solutions for retarded half-linear equations ([96, 100]), on criteria for the fourth-order half-linear equations ([97]) and asymptotics of solutions of (1.1) ([95]).

With respect to the author's scientific background and roots, we further focus on a more detailed description of the role of the Brno school. In the observed period, Brno has been one of the main research centers, where a group of authors around the leading personality of professor Došlý has continued in working on the theory of half-linear equations. Professor Došlý published many papers on equation (1.1) either on his own, with his Ph.D. students (e.g., Řezníčková, Fišnarová, Pátíková, Hasil, Veselý), or other coauthors (e.g., Lomtatidze, Ünal, Bognar, Özbekler, Cecchi, Marini, Yamaoka, Jaroš). The contribution of Došlý was in his innovative approach and many original ideas leading to new branches of the research field. To present some of the topics of his studies from this period in more detail, let us mention the following ones:

- various (non)oscillation criteria ([25, 35, 38, 41, 42, 46, 50, 37]),
- half-linear Euler type equation with its generalizations and perturbations ([19, 20, 30, 32, 40]),
- general conditionally oscillatory equations ([34, 47]),
- minimal solution of Riccati type equation ([8]),
- asymptotics of solutions ([9, 43, 48, 21]),
- equations with periodic coefficients and coefficients having mean values ([31, 33, 36, 49]),
- involving perturbation in both the terms of equation (1.1) ([23, 24, 26]),
- power comparison theorems ([28]),
- distinguishing the principal solution and its characterization ([15, 17]),
- higher-order half-linear differential equations ([44, 45]).

The most frequent methods used by Došlý include the Riccati technique and its modification, which is referred to as the modified Riccati technique, further comparing half-linear equations with linear equations, variational technique, and the Prüfer angle technique. The untimely decease of Došlý was a great loss for the entire scientific community working in the theory of half-linear differential equations.

The second author of the monograph [39] Řehák also continued in his work, especially in studies of asymptotics of solutions of half-linear equations in the frame of regularly varying functions. We refer to the survey paper [115], where a significant part of his previous work is summed up and where the classification of solutions in the sense of regular variation is finalized. Řehák extended oscillation and asymptotic theory also to half-linear difference and dynamic equations.

To the mental members of the Brno school also belongs the pair of coauthors Mařík and Fišnarová. Apart from their joint cooperation with Došlý ([27, 29]), let us mention their work on extending the Picone's identity ([59]), presenting a new class of constants for which Hille-Nehari type criteria hold ([61]), working with the modified Riccati technique and discussing the integral characterization of principal solution ([60, 62]). In the last years, their joint research has shifted to half-linear equations with delay and to neutral half-linear equations (see, for example [63, 64]). Mařík published several papers in fields linking the half-linear ordinary differential equations with partial differential equations with p-Laplacian (e.g., [102, 103]). Fišnarová has cooperated on a few papers also with Pátíková, who is the author of this thesis. Pátíková has worked mainly with perturbations of conditionally oscillatory and Euler-type equations ([65, 66, 111]), proved some asymptotic formulas for nonoscillatory solutions ([108, 109, 110]), considered neutral half-linear equations ([67]) and worked within the numerical approach on the application of the differential transformation method to half-linear Euler-type equations ([112]).

In the last years, a significant contribution to the topics studied in Brno has been made by the pair of coauthors Hasil and Veselý. They master and contribute mainly to techniques based on the generalized Prüfed transformation, averaging technique, and adapted Riccati transformation. Their interest has been in half-linear equations with periodic and almost periodic coefficients ([75, 76, 77, 78, 80, 83]) and coefficients having mean values ([73, 82]), they studied conditional oscillation of half-linear equations and of Euler-type equations ([18, 71, 79, 81]). Hasil and Veselý have extended some of their results also to half-linear equations on time scales and half-linear difference equations. Recently, they collaborate also with Ph.D. students (Juránek, Šišoláková, [72, 74]).

To complete the overview of the Brno school, let us also mention Šremr, who with his coauthors published results for systems of half-linear differential equations (see, for example [51]), Lomtatidze, whose interest was, for example, in boundary value problems, and the three collaborating authors Cecchi, Došlá and Marini. At the beginning of the regarded period, in the work of the three, they considered equation (1.1) and presented some results on the principal solution and asymptotic properties of solutions ([13, 14, 16]). Later, their attention has turned to more general nonlinear equations, for which half-linear equations are a special case.

In the last years, delayed, advanced, and neutral half-linear equations have attracted considerable attention. Apart from the authors already mentioned, let us present at least the strongest groups of coauthors from all over the world: Bohner, Grace, Jadlovská, Džurina, Li, Chatzarakis, Stavroulakis, Baculíková, Graef, Li, Zhang, Santra, Bazighifan (see, for example, the latest results and references therein [10, 11, 54, 116]).

Concerning other topics, which belong to the half-linear theory but which are less relevant to our work, let us continue with only a brief sketch of other branches. Half-linear equations also appear as a special class of some nonlinear/functional equations (see, for example, the works of Tiryaki). For the half-linear impulsive differential equations, see the papers of Zafer, Özbekler, and their coauthors. Tiryaki, Zafer, and Özbekler have also studied Lyapunov-type inequalities with the use in the theory of various types of half-linear equations (see also papers by Liu, Dhar, Kayar, Agarwal, Kong, Cakmak). Sahinez, Zafer, and Tiryaki also worked on elliptic equations with p-Laplacian.

Half-linear dynamic equations on time scales have been studied by a large number of authors (for example, Saker, Agarwal, Peterson, Zafer, Zhang, Bohner, Hassan, Li), as well as half-linear difference equations.

From the previous overview, it can be seen that the theory of half-linear equations is very rich, diverse, and deeply developed. The aim of this work is to present the contribution of the author to this theory. In particular, we have focused on the qualitative behaviour and asymptotical properties of solutions of (1.1) on a neighbourhood of infinity, i.e., for $t \in [T, \infty)$. We use the modified Riccati technique to obtain various types of oscillation and nonoscillation criteria, which are used mainly for equations at the boundary between oscillation and nonoscillation. Furthermore, we introduce the obtained asymptotic formulas of nonoscillatory solutions of certain equations, present our oscillatory criteria for neutral equations, and introduce the use of the differential transformation in the search for a numerical solution of the half-linear Euler equation.

1.2 Methods in qualitative theory of half-linear equations

In this section, we recall relevant preliminaries, remind the concept of principal solution, summarize some knowledge about conditionally oscillatory equations and Euler-type equations, and state the basis of the proving technique. Most of the preparatory work leads to the introduction and implementation of the so called modified Riccati technique, which plays key role in our research. Recall that the core idea of the modified Riccati technique was introduced by Došlý at the end of the last century and although in his older papers the technique was not named explicitly, its variations are present in many of his publications.

The classification of solutions of the object of our study - equation (1.1) - in terms of oscillation is used in the same manner as in the linear theory. If a solution is of one sign in some neighbourhood of infinity, it is called *nonoscillatory*. Otherwise, i.e., when a nontrivial solution has got infinitely many zeros tending to infinity, it is called *oscillatory*. Half-linear equation (1.1) can be classified as oscillatory if all its solutions are oscillatory and nonoscillatory if all of them are nonoscillatory. The fact that if equation (1.1) has one non(oscillatory) solution, then the equation preserves the same non(oscillation) property, is a direct consequence of the half-linear version of Sturm separation theorem (see, for example, [39, Theorem 1.2.3]). Hence, oscillatory and nonoscillatory solutions of (1.1) cannot coexist and oscillation or nonoscillation of equation (1.1) is implied by the form and properties of the coefficient functions $r(t)$ and $c(t)$.

If oscillatory behaviour, roughly speaking, depends only on a constant, we talk about the so-called conditional oscillation. More precisely, assuming in addition that $c(t)$ is positive and γ is a positive real parameter, the equation

$$(r(t)\Phi(x'))' + \gamma c(t)\Phi(x) = 0 \tag{1.4}$$

is called *conditionally oscillatory* if there exists a positive constant γ_0 such that equation (1.4) is oscillatory for $\gamma > \gamma_0$ and nonoscillatory for $\gamma < \gamma_0$. The number γ_0 is then called an *oscillation constant* of equation (1.4).

The main tools of proving techniques used in the half-linear theory make use of the Prüfer transformation or are implied by the so-called Roundabout theorem (see, for example, [39, Section 1.1.3] and [39, Theorem 1.2.7], respectively). The Roundabout theorem presents relationship between solvability of Riccati equation, positivity of the p-degree functional

$$F(y;a,b) = \int_a^b \left[r(t)|y'|^p - c(t)|y|^p\right] dt,$$

which makes the base of the so-called variational technique, and the fact that a solution has no zero on a particular interval. As its direct consequence for the interval $I = [T,\infty)$, one obtains the equivalence between nonoscillation of equation (1.1) and solvability of Riccati equation (1.6) on $[T,\infty)$. The use of this relationship is the basis of the so-called Riccati technique. Its modification, as described later, is the main means of our proofs.

Recall that Riccati transformation is widely used also in the theory of second-order linear equations (1.2) and the Riccati equation joined with equation (1.2) through the relation $w = r\frac{x'}{x}$ reads as

$$w' + c(t) + r^{-1}(t)w^2 = 0. \tag{1.5}$$

To extend the concept to half-linear equations (1.1), let x be a solution of (1.1) and q the so-called *conjugate number* to p for which

$$\frac{1}{p}+\frac{1}{q}=1.$$

Consider the half-linear Riccati transformation

$$w(t)=r(t)\Phi\left(\frac{x'}{x}\right).$$

Then a direct differentiation together with substituting from (1.1) gives

$$\begin{aligned} w' &= \frac{(r\Phi(x'))'\Phi(x)-(p-1)r\Phi(x')|x|^{p-2}x'}{\Phi^2(x)} = -c-(p-1)\frac{r|x'|^p}{|x|^p} \\ &= -c-(p-1)r^{1-q}|w|^q. \end{aligned}$$

Thus, w solves the Riccati-type differential equation

$$w'+c(t)+(p-1)r^{1-q}(t)|w|^q=0. \tag{1.6}$$

Observe that in the linear case for $p=2$ and $q=2$, equation (1.6) is reduced to (1.5).

Together with the Riccati technique, the method of comparing a pair of equations of the same form is often used. Similarly as in the oscillation theory of linear equations, the following Sturm comparison theorem holds. Consider the equation

$$(R(t)\Phi(y'))'+C(t)\Phi(y)=0, \tag{1.7}$$

where the functions R,C satisfy the same assumptions as r,c, respectively.

Theorem 1.1 ([39], Thm 1.2.4)**.** *Let $t_1<t_2$ be two consecutive zeros of a nontrivial solution x of* (1.1) *and suppose that*

$$C(t)\geq c(t), \quad r(t)\geq R(t)>0 \tag{1.8}$$

for $t\in[t_1,t_2]$. Then any solution of (1.7) *has a zero in (t_1,t_2) or it is a multiple of the solution x. The last possibility is excluded if one of the inequalities in* (1.8) *is strict on a set of positive measure.*

The concept of principal solution was extended to half-linear equations by Mirzov [105] (and independently by Elbert and Kusano [57], who used a different but equivalent approach). Given a nonoscillatory equation (1.1) and the associated Riccati equation (1.6), it was shown that among all solutions of (1.6) there exists a *minimal* one $\tilde{w}$, such that $w(t)>\tilde{w}(t)$ for any other solution w of (1.6) and for large t. The *principal solution* of (1.1) is then defined by

$$\tilde{x}=\exp\left\{\int^t r^{1-q}(s)\Phi^{-1}(\tilde{w}(s))\,ds\right\},$$

i.e., it is the one which is "produced" by the minimal solution $\tilde{w}$ of (1.6) with respect to the relation $\tilde{w}=r\Phi(\tilde{x}'/\tilde{x})$. Note that Φ^{-1} is the inverse operator of Φ.

In the theory of linear differential equations, the principal solution has its integral characterization. Its possible extension to half-linear equations was studied, for example, in [22], [17] and [60]. Generalizing the linear version, a half-linear equivalent of the integral characterization seems to be the condition

$$\int^{\infty} \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}} = \infty \tag{1.9}$$

and it was shown in [22] that it is under certain assumptions necessary or sufficient for principality of x, but a complete integral characterization of the principal solution applicable to general equation (1.1) has not been found.

Now, let us turn our attention back to conditional oscillation. A typical example of a conditionally oscillatory half-linear equation is the half-linear Euler equation and its extensions. Half-linear Euler equation

$$(\Phi(x'))' + \frac{\gamma}{t^p}\Phi(x) = 0 \tag{1.10}$$

has an oscillation constant

$$\gamma_p = \left(\frac{p-1}{p}\right)^p,$$

i.e., (1.10) is nonoscillatory for $\gamma < \gamma_p$ and oscillatory for $\gamma > \gamma_p$. If $\gamma = \gamma_p$, then (1.10) is nonoscillatory and it has got a pair of linearly independent nonoscillatory solutions. The principal one

$$h_1(t) = t^{\frac{p-1}{p}} \tag{1.11}$$

is known explicitly, and the second one is asymptotically equivalent to the function

$$h_2(t) = t^{\frac{p-1}{p}} \log^{\frac{2}{p}} t, \tag{1.12}$$

as $t \to \infty$, see [56]. Since the Euler equation in the critical case

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0 \tag{1.13}$$

is nonoscillatory, it is natural to ask if it can be perturbed with another term so that the perturbed equation remains nonoscillatory.

Up to this point, let us extend the the definition of conditionally oscillatory equations. Suppose that equation (1.1) is nonoscillatory and let $d(t)$ be a positive continuous function. The equation

$$(r(t)\Phi(x'))' + [c(t) + \mu d(t)]\Phi(x) = 0 \tag{1.14}$$

is called *conditionally oscillatory* if there exists a constant $\mu_0 > 0$ such that (1.14) is oscillatory for $\mu > \mu_0$ and nonoscillatory for $\mu < \mu_0$.

Searching for a perturbation of equation (1.13), so that the perturbed equation is conditionally oscillatory too, leads to the Riemann-Weber (sometimes called Euler-Weber) half-linear equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right)\Phi(x) = 0. \tag{1.15}$$

The oscillation constant of (1.15) is

$$\mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1},$$

i.e., equation (1.15) is oscillatory for $\mu > \mu_p$ and nonoscillatory for $\mu < \mu_p$, see [58]. The equation with the critical constant

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right)\Phi(x) = 0 \tag{1.16}$$

has a pair of solutions asymptotically close to the functions

$$h_1(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t, \quad h_2(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t \log^{\frac{2}{p}}(\log t), \tag{1.17}$$

as $t \to \infty$. Remark that h_1 is asymptotically close to the principal solution of (1.16).

The idea of looking for perturbations can be repeated and equations (1.10) and (1.15) can be further generalized. For this purpose, denote the iterated logarithm functions by

$$\log_1 t = \log t, \quad \log_k t = \log_{k-1}(\log t), \quad k \geq 2,$$

and their products by

$$\operatorname{Log}_j t = \prod_{k=1}^{j} \log_k t.$$

By a generalized Riemann-Weber half-linear equation or generalized Euler-type half-linear equation, we mean the equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n-1} \frac{\mu_p}{t^p \operatorname{Log}_j^2 t} + \frac{\mu}{t^p \operatorname{Log}_n^2 t}\right)\Phi(x) = 0, \tag{1.18}$$

where $n \geq 2$. The equation is conditionally oscillatory with the oscillation constant $\mu = \mu_p$ and naturally extends the Riemann-Weber equation (1.15). In the critical case $\mu = \mu_p$, the equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \operatorname{Log}_j^2 t}\right)\Phi(x) = 0 \tag{1.19}$$

is nonoscillatory and its solutions are asymptotically equivalent to the functions

$$h_1(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t, \quad h_2(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t, \tag{1.20}$$

for t tending to infinity, as was shown by Elbert and Schneider in [58]. Let us mention that the above-described process of finding further perturbations which remain conditionally oscillatory in the linear case was commented on, for example in [124, p. 133]. If $p = 2$, then also $q = 2$, the critical constants γ_p and μ_p become $\gamma_2 = \mu_2 = \frac{1}{4}$, and equation (1.19) reduces to

$$x'' + \left(\frac{1}{4t^2} + \sum_{j=1}^{n} \frac{1}{4t^2 \operatorname{Log}_j^2 t}\right)x = 0.$$

Let us now consider general half-linear equation (1.1) and assume that it is nonoscillatory. Let $h(t)$ be a positive solution of (1.1) such that $h'(t) \neq 0$ on some interval of the form $[T_0, \infty)$ and denote

$$R(t) := r(t)h^2(t)|h'(t)|^{p-2}, \quad G(t) := r(t)h(t)\Phi(h'(t)). \tag{1.21}$$

Remark that condition (1.9), which is related to the principality of a solution, can be expressed in the form $\int^\infty R^{-1}(t)dt = \infty$.

In [47] the authors showed that under the assumptions

$$\int^\infty \frac{dt}{R(t)} = \infty, \quad \liminf_{t\to\infty} |G(t)| > 0,$$

it is possible to construct a conditionally oscillatory equation seen as a perturbation of (1.1). It is of the form

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{\tilde{\mu}}{h^p(t)R(t)(\int^t R^{-1}(s)\,ds)^2}\right]\Phi(x) = 0 \tag{1.22}$$

and the critical oscillation constant of this equation is

$$\tilde{\mu}_0 = \frac{1}{2q}.$$

From this perspective, if we take the Euler equation with the critical constant (1.13) in place of the general equation (1.1) and use the principal solution of (1.10), i.e., the function $h(t) = t^{\frac{p-1}{p}}$, in the introduced construction, we arrive at the Riemann-Weber equation (1.15).

The proving techniques used in our papers vary and utilize different specialized statements. Here we summarize the most frequent approaches. In studies of equations on the threshold between oscillation and nonoscillation (as, for example, Euler-type equations (1.15) and (1.18)), the classical Riccati technique is not efficient enough. It turned out that the following modification of the technique is suitable in that case. The early version of the modified Riccati transformation worked with two nonoscillatory equations, one of them was supposed to have a solution which is known exactly and the other equation was seen as a perturbation of the first one. To illustrate the idea, suppose that equation

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0, \tag{1.23}$$

is nonoscillatory, $h(t)$ be its positive solution such that $h'(t) \neq 0$ for large t. Within the perspective of perturbations, supposing $c(t) \geq \tilde{c}(t)$ for large t, equation (1.1) can be seen as a perturbation of (1.23), according to the rearrangement

$$c(t)\Phi(x) = \tilde{c}(t)\Phi(x) + (c(t) - \tilde{c}(t))\Phi(x).$$

Be x an eventually positive nonoscillatory solution (1.1). Denote $w = r\Phi(x'/x)$ and $w_h = r\Phi(h'/h)$. Then the transformation

$$v(t) = h^p(t)(w(t) - w_h(t))$$

applied to (1.6) leads to the modified Riccati equation

$$v'(t) + h^p(C(t) - c(t)) + (p-1)r^{1-q}(t)h^{-q}(t)H(v,G) = 0, \tag{1.24}$$

where

$$H(v,G) := |v+G|^q - q\Phi^{-1}(G)v - |G|^q \tag{1.25}$$

and G is defined by (1.21). The power of this approach is in the fact that the nonlinear function $H(v,G)$ can be under certain assumptions estimated by the function

$$K|G(t)|^{q-2}v^2(t), \quad K \in \mathbb{R}^+,$$

which is quadratic in v. The strength of such estimates is dependent on the knowledge of the properties of $H(v,G)$, which has gradually developed. The behaviour of $H(v,G)$ was considered, for example, in [23, 29, 58, 60]. The above described version of the modified Riccati equation was used in many papers by Došlý and his coauthors mainly at the beginning of the century and its derivation is clearly described, for example, in [46]. It is useful especially in the situation when the equation which is being perturbed is the Euler equation (1.13), since one of its solutions is known explicitly. However, it is insufficient to use it with more general equations such as (1.16) and (1.19). In 2010 Došlý and Fišnarová in [23] studied perturbations of both the coefficients r, c and although their modified transformation used again a known solution $h(t)$ of one of the two nonoscillatory equations, their calculation formed a base for a new situation. That is, when $h(t)$ is, instead of being a solution, only a function which is asymptotically close to it (in the formulation, there appears a general function $h(t)$, but the assumptions can ensure the closeness to a solution which is known only asymptotically). The generalized version of the modified Riccati equation, which extends the previous one, was firstly introduced in [60] by Fišnarová and Mařík.

Lemma 1.1 ([60], Lem 2.2)**.** *Let $h(t)$ be a differentiable function such that $h(t) \geq 0$ and $h'(t) \neq 0$ for large t, be $w_h = r\Phi(h'/h)$,*

$$l[x] := (r(t)\Phi(x'))' + c(t)\Phi(x),$$

and G, H defined by (1.21), (1.25)*, respectively. Put*

$$v := h^p(w - w_h),$$

where w is a continuously differentiable function. Then the following identity holds

$$h^p(t)(w' + c(t) + (p-1)r^{1-q}(t)|w|^q) = v'(t) + h(t)l[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(v,G). \tag{1.26}$$

In particular, if w is a solution of (1.6)*, then v is a solution of*

$$v'(t) + h(t)l[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(v,G) = 0 \tag{1.27}$$

and conversely, if v is a solution of (1.27)*, then $w = h^{-p}v + w_h$ is a solution of* (1.6).

With the use of the Roundabout theorem, we can say that Lemma 1.1 provides an equivalence between the nonoscillation of (1.1) and the solvability of the modified Riccati equation (1.27) on a neighbourhood of infinity. Remark that the transformation is sometimes presented in another equivalent form

$$v = h^p(w - w_h) = h^p w - G. \tag{1.28}$$

If h is a positive exact solution of (1.23), then

$$hl[h] = h[(r(t)\Phi(h'))' + c(t)\Phi(h)] = (c(t) - \tilde{c}(t))h^p$$

and this shows that (1.27) generalizes the prior approach in (1.24). The proof of (1.26) is straightforward. Indeed, according to the definitions of v, G and H (see (1.28), (1.21) and (1.25), respectively), we have (suppressing the argument t)

$$\begin{aligned}
v' &= (h^p w - G)' = ph^{p-1}h'w + h^p w' - ((r\Phi(h'))'h + r|h'|^p) \\
H(v,G) &= |v+G|^q - q\Phi^{-1}(G)v - |G|^q = |v+G|^q - q\Phi^{-1}(G)h^p w + (q-1)|G|^q \\
&= h^{pq}|w|^q - qr^{q-1}h^{q-1}h'h^p w + (q-1)r^q h^q |h'|^p \\
&= h^{pq}|w|^q - qr^{q-1}h^q h' w + (q-1)r^q h^q |h'|^p.
\end{aligned}$$

Now substitute to the right hand side of (1.26) and rearrange to the form

$$\begin{aligned}
RHS &= v'(t) + h(t)l[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(v,G) \\
&= ph^{p-1}h'w + h^p w' - ((r\Phi(h'))'h + r|h'|^p) + h(r\Phi(h'))' + ch^p \\
&+ (p-1)r^{1-q}h^{-q}(h^{pq}|w|^q - qr^{q-1}h^q h' w + (q-1)r^q h^q |h'|^p) \\
&= ph^{p-1}h'w + h^p w' - (r\Phi(h'))'h - r|h'|^p + h(r\Phi(h'))' + ch^p \\
&+ (p-1)r^{1-q}h^p|w|^q - ph^{p-1}h' + r|h'|^p \\
&= h^p w' + ch^p + (p-1)r^{1-q}h^p|w|^q,
\end{aligned}$$

which is equal to the left hand side of (1.26).

Under certain conditions, so that estimates of $H(v,G)$ can be done (usually v needs to tend to 0 as t goes to infinity), equation (1.27) is in a certain sense close to the first-order linear equation

$$u'(t) + h(t)l[h](t) + \frac{q}{2R(t)}u^2(t) = 0,$$

which is referred to as the *approximate Riccati equation.* In some situations, it is enough to work only with modified and approximate Riccati inequalities. The variant of the modified Riccati inequality for neutral half-linear equations is presented in Chapter 4.

Chapter 2

Asymptotic formulas of nonoscillatory solutions

The classification of the asymptotic behaviour of solutions is often made within the framework of regularly varying functions in the sense of Karamata (for an introduction, see [84]). Let us recall the relevant nomenclature (for more information, see, for example, the monograph [101]).

A measurable function $f : [a,\infty) \to (0,\infty)$ is called *regularly varying* (at infinity) of index ϑ (and we write $f \in RV(\vartheta)$) if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \quad \text{for every } \lambda > 0.$$

If $\vartheta = 0$, f is called *slowly varying*. The Representation Theorem (see, for example [115]) says that $f \in RV(\vartheta)$ if and only if it can be expressed in the form

$$f(t) = \varphi(t) t^{\vartheta} \exp\left\{ \int_a^t \frac{\psi(s)}{s}\, ds \right\},$$

where $t \geq a$ for some $a > 0$, φ and ψ are measurable functions such that $\lim_{t\to\infty} \varphi(t)$ is finite and positive and $\lim_{t\to\infty} \psi(t) = 0$.

Recall that the classification of solutions of half-linear equations in terms of regular variation was deeply described by Řehák, see [115] and references therein. Among other authors who considered half-linear equations in this context, mention, for example, Jaroš, Kusano, Marić, Manojlović, Tanigawa, see [90, 95, 98, 99]. In our contribution to the topic, we follow up on previous results and continue in finding the asymptotic formulas for solutions of equations at the threshold between oscillation and nonoscillation and their perturbations'.

2.1 Nonoscillatory solutions of conditionally oscillatory half-linear equations

In this section, we present asymptotic formulas for nonoscillatory solutions of conditionally oscillatory equation (1.22)

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{\tilde{\mu}}{h^p(t)R(t)(\int^t R^{-1}(s)\,ds)^2}\right]\Phi(x) = 0$$

in the cases when $\tilde{\mu} < \frac{1}{2q}$ and $\tilde{\mu} = \frac{1}{2q}$, as was proved in [109]. We also discuss a link to the special case the when the equation which is being perturbed is the Euler equation (1.13) and our asymptotic formulas generalize those of the Riemann-Weber equation (1.15).

The first statement deals with the subcritical case $\tilde{\mu} < \frac{1}{2q}$ and provides asymptotic formulas of a pair of linearly independent solutions.

Theorem 2.1 ([109], Thm 1)**.** *Suppose that* (1.1) *is nonoscillatory and possesses a positive solution $h(t)$ such that $h'(t) \neq 0$ for large t and let*

$$\int^\infty \frac{dt}{R(t)} = \infty \quad \text{and} \quad \liminf_{t\to\infty} |G(t)| > 0.$$

If $\tilde{\mu} < \frac{1}{2q}$, then conditionally oscillatory equation (1.22) *has a pair of solutions given by the asymptotic formula*

$$x_i = h(t)\left(\int^t R^{-1}(s)\,ds\right)^{(q-1)\lambda_i} L_i(t), \quad \text{as} \quad t\to\infty,$$

where λ_i are zeros of the quadratic equation

$$\frac{q}{2}\lambda^2 - \lambda + \tilde{\mu} = 0$$

and $L_i(t)$ are generalized normalized slowly varying functions of the form $L_i(t) = \exp\left\{\int^t \frac{\varepsilon_i(s)}{R(s)\int^s R^{-1}(\tau)\,d\tau}\,ds\right\}$ and $\varepsilon_i(t)\to 0$ for $t\to\infty$.

The second statement focuses on the critical case $\tilde{\mu} = \frac{1}{2q}$ and brings an asymptotic formula for a solution of the equation

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,ds)^2}\right]\Phi(x) = 0. \tag{2.1}$$

Theorem 2.2 ([109], Thm 2)**.** *Let the assumptions of the previous theorem be satisfied. Then equation* (2.1) *has a solution of the form*

$$x = h(t)\left(\int^t R^{-1}(s)\,ds\right)^{\frac{1}{p}} L(t), \tag{2.2}$$

where $L(t)$ is a generalized normalized slowly varying function of the form $L(t) = \exp\left\{\int^t \frac{\varepsilon(s)}{R(s)\int^s R^{-1}(\tau)\,d\tau}\,ds\right\}$ and $\varepsilon(t)\to 0$ for $t\to\infty$.

The proofs utilize the modified Riccati technique in combination with finding a fixed point of a certain integral operator with the use of the Schauder-Tychonoff fixed point theorem.

Consider the application of the theorems to the case when the equation which is being perturbed is the Euler equation with the critical constant (1.13). It means that $r(t) \equiv 1, c(t) = \gamma_p t^{-p}$ and $h(t) = t^{\frac{p-1}{p}}$. Then conditionally oscillatory equation (1.22) reduces to Riemann-Weber equation (1.15). Indeed, for $t \to \infty$ we have

$$R(t) = rh^2|h'|^{p-2} = \left(\frac{p-1}{p}\right)^{p-2} t, \quad \int^t R^{-1}(s)\, ds \sim \left(\frac{p}{p-1}\right)^{p-2} \log t$$

and hence

$$\frac{\tilde{\mu}}{h^p(t)R(t)(\int^t R^{-1}(s)\, ds)^2} \sim \frac{\left(\frac{p-1}{p}\right)^{p-2} \tilde{\mu}}{t^p \log^2 t}.$$

Observe that for $\tilde{\mu} = \frac{1}{2q}$ the constant in the last expression is

$$\left(\frac{p-1}{p}\right)^{p-2} \cdot \frac{1}{2q} = \mu_p.$$

Asymptotic formulas for $\tilde{\mu} < \frac{1}{2q}$ (and equivalently $\mu < \mu_p$ in (1.15)) then are

$$x_{1,2} \sim t^{\frac{p-1}{p}} \left(\left(\frac{p}{p-1}\right)^{p-2} \log t \right)^{\frac{2\tilde{\lambda}_{1,2}}{p}} = Const \cdot t^{\frac{p-1}{p}} \log t^{\frac{2\tilde{\lambda}_{1,2}}{p}} \quad \text{as } t \to \infty,$$

which coincides with the results of [108]. There the asymptotic formulas were found in the form

$$x_{1,2}(t) \sim t^{\frac{p-1}{p}} (\log t)^{\nu_{1,2}} \quad \text{as } t \to \infty,$$

where $\nu_{1,2} = \frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p} \lambda_{1,2}$ and $\lambda_{1,2}$ are roots of

$$\frac{\lambda^2}{4\mu_p} - \lambda + \mu = 0.$$

It is not hard to check that $\frac{2\tilde{\lambda}_{1,2}}{p} = \nu_{1,2}$, respectively, and so both the ways lead to the same result in a qualitative sense.

The second theorem can be applied to the Riemann-Weber equation in the critical case (1.16), i.e., to the equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right] \Phi(x) = 0.$$

The formula (2.2) then reduces to the formula asymptotically close to $h_1(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t$ as t tends to infinity (see (1.17), as was proved by Elbert and Schneider in [58]).

2.2 Subcritical half-linear Euler-type equation

The paper [110] is devoted to finding a nonscillatory solution of generalized Euler-type equation (1.18) with n terms in the sum (1.18), i.e.,

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n-1} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + \frac{\mu}{t^p \mathrm{Log}_n^2 t}\right)\Phi(x) = 0$$

in the subcritical case for $\mu \in (0, \mu_p)$. The found asymptotic formula can be seen as an extension of that type of result for nonoscillatory solutions of Riemann-Weber equation (1.15), which was proved in [108]. The main statement reads as follows.

Theorem 2.3 ([110], Thm 3.1)**.** *Equation* (1.18) *with* $\mu \in (0, \mu_p)$ *has a solution of the form*

$$x(t) = t^{\frac{p-1}{p}} \mathrm{Log}_{n-1}^{\frac{1}{p}} t \log_n^{\frac{2\lambda}{p}} t (1 + o(1)) \quad as \quad t \to \infty, \tag{2.3}$$

where $\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{\mu}{\mu_p}}$.

We showed that the solution (2.3) of equation (1.18) is a regularly varying function of index $\frac{p-1}{p}$ and the function $(1 + o(1))$ in its formula is a slowly varying function.

The proof is based on closeness of the modified Riccati equation (1.27) for equation (1.18)

$$v'(t) + \sum_{j=1}^{n-1} \frac{\mu_p}{t \,\mathrm{Log}_j^2 t} + \frac{\mu}{t \,\mathrm{Log}_n^2 t} + \frac{p-1}{t} H(v(t), 2\mu_p) = 0$$

and the so-called approximate modified Riccati equation

$$u'(t) + \sum_{j=1}^{n-1} \frac{\mu_p}{t \,\mathrm{Log}_j^2 t} + \frac{\mu}{t \,\mathrm{Log}_n^2 t} + \frac{1}{4\mu_p t} u^2(t) = 0. \tag{2.4}$$

It is possible to find the solution of (2.4), and it serves in constructing a suitable set, on which a certain integral operator is considered, and such that the Schauder-Tychonoff theorem can be used. The closeness of both the equations is shown with the use of estimates for the function $H(v, G)$ from [23].

Chapter 3

Oscillation and nonoscillation criteria

One of the most important tasks of the qualitative theory of differential equations is to provide the oscillation and nonoscillation criteria. With their use, one can decide, based on the form of the coefficient functions, whether an equation is oscillatory or nonoscillatory. In the first part of this chapter, we focus on extensions of Hille-Nehari type criteria for conditionally oscillatory half-linear equations. The latter part presents a generalization of an integral comparison theorem of Hille-Wintner type. Perturbations of conditionally oscillatory half-linear equations are considered.

3.1 Hille-Nehari type criteria

Throughout the section, we suppose that equation (1.1) is nonoscillatory and we study its perturbation in the form

$$(r(t)(\Phi(x'))' + (c(t) + d(t))\Phi(x) = 0. \tag{3.1}$$

Oscillation and nonoscillation criteria then say if $d(t)$ is large enough for (3.1) to become oscillatory or if it remains nonoscillatory too.

The typical feature of Hille-Nehari type oscillation and nonoscillation criteria for (3.1) is that they compare limits inferior and superior of certain integral expressions with concrete constants. These integral expressions are usually either of the form

$$\int_T^t R^{-1}(s)\,ds \int_t^\infty d(s)h^p(s)\,ds \quad \text{if } \int^\infty R^{-1}(t)\,dt = \infty \tag{A}$$

or

$$\int_t^\infty R^{-1}(s)\,ds \int_T^t d(s)h^p(s)\,ds \quad \text{if } \int^\infty R^{-1}(t)\,dt < \infty, \tag{B}$$

where h is a solution of equation (1.1) (or a function which is asymptotically close to a nonoscillatory solution of (1.1)). Recall that the function $R(t)$ is defined by (1.21).

3.1.1 Euler-type equation

Let us summarize the results we will follow up on. The perturbed half-linear Euler equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + d(t)\right)\Phi(x) = 0 \tag{3.2}$$

was considered in [35], where it was shown, under the notation

$$E(t) = \log t \int_t^\infty d(s)s^{p-1}\,ds,$$

that equation (3.2) is nonoscillatory if

$$\limsup_{t\to\infty} E(t) < \mu_p, \quad \liminf_{t\to\infty} E(t) > -3\mu_p$$

and oscillatory if

$$\liminf_{t\to\infty} E(t) > \mu_p.$$

The same couple of nonoscillation and oscillation criteria with the integral expression

$$E(t) = \frac{1}{\log t}\int_T^t d(s)s^{p-1}\log^2 s\,ds$$

was then proved by Došlý and Řezníčková in [40]. Let us point out that the relation between $E(t)$ and types (A) and (B) is given by taking $h(t) = t^{\frac{p-1}{p}}$ and $h(t) = t^{\frac{p-1}{p}}\log^{\frac{2}{p}} t$, respectively, which are (almost) solutions of Euler equation (1.13), see (1.11) and (1.12).

Now let us turn our attention to the perturbed Riemann-Weber half-linear equation with critical coefficients

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p\log^2 t} + d(t)\right)\Phi(x) = 0. \tag{3.3}$$

The criteria in terms of the integral expression

$$E(t) = \log(\log t)\int_t^\infty d(s)s^{p-1}\log s\,ds,$$

that complies with (A) taking $h(t) = t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t$, which is close to the principal solution of (1.16), were proved in [19] (the nonoscillation criterion) and [46] (the oscillation criterion). The case which corresponds to type (B) and to the function $h(t) = t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t \log^{\frac{2}{p}}(\log t)$, which is asymptotically close to the second solution of Riemann-Weber equation (1.16), see (1.17), remained open.

Finally, we come to the perturbations of general half-linear Euler-type equation with critical coefficients (1.19), i.e., to the equation

$$L_{RW}[x] := (\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^n \frac{\mu_p}{t^p \operatorname{Log}_j^2 t} + d(t)\right)\Phi(x) = 0. \tag{3.4}$$

Došlý in [20], with the use of the function $h(t) = t^{\frac{p-1}{p}}\operatorname{Log}_n^{\frac{1}{p}} t$ (i.e., $h_1(t)$ from (1.20)), proved the following statement.

Theorem 3.1 ([20], Thm 3.3)**.** *Suppose that the integral $\int^\infty d(t)t^{p-1}\operatorname{Log}_n t\,dt$ is convergent.*

(i) If

$$\limsup_{t\to\infty} \log_{n+1} t \int_t^\infty d(s)s^{p-1} \operatorname{Log}_n s\, ds < \mu_p,$$
$$\liminf_{t\to\infty} \log_{n+1} t \int_t^\infty d(s)s^{p-1} \operatorname{Log}_n s\, ds > -3\mu_p,$$

then (3.4) *is nonoscillatory.*

(ii) Suppose that there exists a constant $\gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2}$ *such that* $d(t)t^p \log^3 t \geq \gamma$ *for large* t *and*

$$\liminf_{t\to\infty} \log_{n+1} t \int_t^\infty d(s)s^{p-1} \operatorname{Log}_n s\, ds > \mu_p.$$

Then (3.4) *is oscillatory.*

If $n = 1$, then (3.4) reduces to (3.3) and the criteria from Theorem 3.1 reduce to those obtained in [19, 46].

In our paper [65] we continued further with complementing Theorem 3.1 (and the corresponding results of [19, 46] in the case $n = 1$). We utilized the second function $h_2 = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t$ from (1.20) and found a related couple of criteria for equation (3.4) formulated in terms of the expression

$$\frac{1}{\log_{n+1} t} \int^t d(s)s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s\, ds$$

which corresponds to type (B).

The results of [65] are presented in the following statements. First, we present the nonoscillation criterion. In comparison to previously mentioned nonoscillation criteria, instead of the constant μ_p on the right hand side of the inequality we have a class of constants, which enlarges the scope of the result.

Theorem 3.2 ([65], Thm 3.1)**.** *If*

$$\limsup_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t d(s)s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s\, ds < 2\mu_p(-\alpha + \sqrt{2\alpha}), \tag{3.5}$$
$$\liminf_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t d(s)s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s\, ds > 2\mu_p(-\alpha - \sqrt{2\alpha}) \tag{3.6}$$

for some $\alpha > 0$*, then equation* (3.4) *is nonoscillatory.*

Recall that if $\alpha = \frac{1}{2}$, then

$$2\mu_p(-\alpha + \sqrt{2\alpha}) = \mu_p, \quad 2\mu_p(-\alpha - \sqrt{2\alpha}) = -3\mu_p$$

and the constants from (3.5) and (3.6) in Theorem 3.2 reduce to the constants in Theorem 3.1, part (i).

The following statement is the oscillatory criterion which complements Theorem 3.2.

Theorem 3.3 ([65], Thm 3.3)**.** *Suppose that there exists a constant γ such that*

$$t^p \log^3 t \left(d(t) - \frac{\mu_p}{t^p \operatorname{Log}_{n+1}^2 t} \right) \geq \gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2}$$

for large t. If

$$\liminf_{t \to \infty} \frac{1}{\log_{n+1} t} \int_T^t d(s) s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s \, ds > \mu_p,$$

then (3.4) *is oscillatory.*

In the proofs we utilized general statements from [26] and [61]. There we employed the expansion of the term $h(t)L_{RW}[h](t)$ with the use of the function $h(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t$ and of the term $h(t)\tilde{L}_{RW}[h](t)$ with the function $h(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t$.

3.1.2 General conditionally oscillatory equation

Perturbations of general conditionally oscillatory equation with the critical coefficient (2.1) were studied firstly by Došlý and Ünal in [47]. One of its results (Theorem 5) provides a nonoscillatory Hille-Nehari type criterion for the equation

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{1}{2q h^p(t) R(t) \left(\int^t R^{-1}(s)\, ds \right)^2} + g(t) \right] \Phi(x) = 0 \tag{3.7}$$

and the integral expression

$$\log \left(\int^t R^{-1}(s)\, ds \right) \int_t^\infty g(s) h^p(s) \int^s R^{-1}(\tau)\, d\tau\, ds$$

is employed. Note that the result is of type (A). The crucial role in the proof of this criterion plays the fact that the asymptotic formulas for solutions of (2.1) are known. It was shown in [21, 47] (see also [109]) that equation (2.1) has a pair of linearly independent solutions that are asymptotically close (as $t \to \infty$) to the functions

$$\begin{aligned} x_1(t) &= h(t) \left(\int^t R^{-1}(s)\, ds \right)^{\frac{1}{p}}, \\ x_2(t) &= h(t) \left(\int^t R^{-1}(s)\, ds \right)^{\frac{1}{p}} \log^{\frac{2}{p}} \left(\int^t R^{-1}(s)\, ds \right). \end{aligned} \tag{3.8}$$

Our contribution to the topic was presented in the paper [66]. We improved the above-mentioned nonoscillation criterion for (3.7), formulated a relevant oscillation criterion for (3.7) and found a perturbation $g(t)$ in (3.7) such that the equation becomes conditionally oscillatory. We also formulated a version of nonoscillation and oscillation Hille–Nehari type criteria for equation (3.7) with the use of the asymptotic formula for the second solution of equation (2.1) (i.e., of type (B)).

The first theorem of [66] improves [47, Theorem 5]. The main difference is that we get by the condition

$$\lim_{t\to\infty} \log^2\big(\int^t R^{-1}(s)\,ds\big) R(t) G'(t) = 0$$

considered in [47] and that the inequalities hold for a set of constants.

Theorem 3.4 ([66], Thm 3.1)**.** *Suppose that h is a positive solution of* (1.1) *such that $h'(t) \neq 0$ for large t,*

$$\int^\infty R^{-1}(t)\,dt = \infty \quad \textit{and} \quad \liminf_{t\to\infty} |G(t)| > 0, \tag{3.9}$$

hold and the integral $\int^\infty g(t)h^p(t)\int^t R^{-1}(s)\,ds\,dt$ converges. If

$$\limsup_{t\to\infty} \log\left(\int^t R^{-1}(s)\,ds\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,d\tau\,ds < \frac{1}{q}(-\alpha+\sqrt{2\alpha}),$$
$$\liminf_{t\to\infty} \log\left(\int^t R^{-1}(s)\,ds\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,d\tau\,ds > \frac{1}{q}(-\alpha-\sqrt{2\alpha})$$

for some $\alpha > 0$, then (3.7) *is nonoscillatory.*

Note that the choice $\alpha = \frac{1}{2}$ reduces the right hand sides of the inequalities to the constants $\frac{1}{2q}$ and $-\frac{3}{2q}$, respectively, which appear in [47, Theorem 5].

Let us briefly describe the main characteristics of the other statements from [66] without all the details. In all of them, condition (3.9) is assumed. In [66, Thm 3.2], an oscillatory counterpart of Theorem 3.4 is presented. It summarizes the conditions, which together with the inequality

$$\liminf_{t\to\infty} \log\left(\int^t R^{-1}(s)\,ds\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,d\tau\,ds > \frac{1}{2q}$$

ensure that equation (3.7) is oscillatory. Note that the above mentioned results are of type (A).

Next results of [66] are oscillation and nonoscillation criteria of type (B), formulated in terms of the integral expression

$$\frac{1}{\log(\int^t R^{-1}(s)\,ds)}\int_T^t g(s)h^p(s)\Big(\int^s R^{-1}(\tau)\,d\tau\Big)\log^2\left(\int^s R^{-1}(\tau)\,d\tau\right)ds.$$

Finally, it is shown that for the perturbation $g(t)$ of the form

$$g(t) := \frac{\delta}{h^p(t)R(t)(\int^t R^{-1}(s)\,ds)^2\log^2(\int^t R^{-1}(s)\,ds)}, \quad \delta\in\mathbb{R}, \tag{3.10}$$

equation (3.7) becomes under certain conditions conditionally oscillatory. The precise formulation is presented in the following theorem.

Theorem 3.5 ([66], Thm 3.3)**.** *Suppose that h is a positive solution of* (1.1) *such that $h'(t) \neq 0$ for large t and* (3.9) *holds and consider the equation*

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{1}{h^p(t)R(t)(\int^t R^{-1}(s)\,ds)^2}\left(\frac{1}{2q} + \frac{\delta}{\log^2(\int^t R^{-1}(s)\,ds)}\right)\right]\Phi(x) = 0. \tag{3.11}$$

If $\delta \leq \frac{1}{2q}$, then (3.11) *is nonoscillatory. If $\delta > \frac{1}{2q}$ and there exists a constant β such that*

$$\frac{1}{R(t)\log^2(\int^s R^{-1}(\tau)\,d\tau)} \geq \frac{\beta|G'(t)|}{G^2(t)}, \quad \beta > \frac{p-2}{p}\operatorname{sgn} G'(t)$$

holds for large t, then (3.11) *is oscillatory.*

The proofs of the statements from the paper [66] refer to oscillation and nonoscillation criteria [61, Theorem 3.1 and 3.2], [26, Theorem 1 and 2]. The choice of the used helping function was x_1 and x_2 from (3.8) and $h = (\int^t R^{-1})^{\frac{1}{p}} \log^{\frac{1}{p}}(\int^t R^{-1})$, according to the conjecture that this function is asymptotically close to one of the solutions of (3.11) with $\delta = \frac{1}{2q}$.

3.2 Hille-Wintner type integral comparison theorems

Consider equation (1.1) together with the equation

$$L[x] := (r(t)(\Phi(x'))' + C(t)\Phi(x) = 0. \tag{3.12}$$

The Sturm comparison Theorem 1.1 compares the coefficient functions $c(t), C(t)$ pointwise. On the other hand, Hille-Wintner criteria compare integral expressions with these coefficient functions. The form of the criteria depends on convergence or divergence of the integral $\int^\infty r^{1-q}(t)\,dt$. If $\int^\infty r^{1-q}(t)\,dt = \infty$ and assuming $\int^\infty C(t)\,dt < \infty$, the Hille-Wintner theorem says that if (3.12) is nonoscillatory and

$$0 \leq \int_t^\infty c(s)\,ds \leq \int_t^\infty C(s)\,ds \quad \text{for large } t,$$

then (1.1) is nonoscillatory too. If $\int^\infty r^{1-q}(t)\,dt < \infty$, denote $\kappa(t) := \int_t^\infty r^{1-q}(s)\,ds$ and assume that $c(t) \geq 0$, $C(t) \geq 0$ for large t. Then if (3.12) is nonoscillatory and

$$\int_t^\infty c(s)\kappa^p(s)\,ds \leq \int_t^\infty C(s)\kappa^p(s)\,ds < \infty \quad \text{for large } t,$$

equation (1.1) is nonosillatory too (for both statements see [39, Section 2.3.1]).

Our work in this field utilizes adopting the perturbation approach, which leads to results applicable to equations on the threshold between oscillation and nonoscillation. Together with (1.1) and (3.12), consider the equation of the same form (1.23) which is supposed to be nonoscillatory and denote its left hand side by

$$\tilde{L}[x] := (r(t)(\Phi(x'))' + \tilde{c}(t)\Phi(x).$$

We look at equations (1.1) and (3.12) as if they were the perturbations of nonoscillatory equation (1.23).

The following theorem is the main result of the paper [37] and reads as follows.

Theorem 3.6 ([37], Thm 1). *Let $\int^\infty r^{1-q}(t)\,dt = \infty$. Suppose that equation* (1.23) *is nonoscillatory and possesses a positive principal solution h such that there exists a finite limit*

$$\lim_{t\to\infty} G(t) > 0$$

and

$$\int^\infty R^{-1}(t)\,dt = \infty.$$

Further suppose that $0 \le \int_t^\infty C(s)\,ds < \infty$ *and*

$$0 \le \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)\,ds \le \int_t^\infty (C(s) - \tilde{c}(s))h^p(s)\,ds < \infty,$$

all for large t. If equation (3.12) *is nonoscillatory, then* (1.1) *is also nonoscillatory.*

A direct application of the theorem leads to the corollary for the case when the equation which is being perturbed (1.23) is the Euler equation with the critical coefficient (1.13). We know its principal solution explicitly. Unfortunately, for Riemann-Weber equation (1.16) and Euler-type equation with n terms in the sum with the critical coefficients (1.19), only asymptotic formulas of the principal solutions are known. Hence, the theorem cannot be applied. The question, if the principal solution in Theorem 3.6 can be replaced by a function close to it, remained open for several years. Thanks to the fact that the modified Riccati technique has been deeper developed, the answer could have been found. In [111] we proved the following results.

Theorem 3.7 ([111], Thm 1). *Suppose that there exists a positive continuously differentiable function such that $h'(t) \neq 0$ for large t and the following conditions holds:*

$$\int_t^\infty R^{-1}(s)\,ds = \infty,$$

$$h(t)L[h](t) \ge 0, \qquad \int_t^\infty h(s)\tilde{L}[h](s)\,ds < \infty,$$

$$(\liminf_{t\to\infty} |G(t)| > 0 \quad \text{and} \quad \limsup_{t\to\infty} |G(t)| < \infty) \quad \text{or} \quad \lim_{t\to\infty} |G(t)| = \infty,$$

all for large t.

Let the inequality

$$-\int_t^\infty h(s)\tilde{L}[h](s)\,ds \le \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)\,ds \le \int_t^\infty (C(s) - \tilde{c}(s))h^p(s)\,ds < \infty$$

be satisfied. Then if equation (3.12) *is nonoscillatory, equation* (1.1) *is nonoscillatory too.*

As an immediate consequence of the previous theorem, we see that under the assumptions of Theorem 3.7 oscillation of equation (1.1) implies that of (3.12).

Next comes the consequence of Theorem 3.7 for the case where the nonoscillatory equation (1.23), which is being perturbed, is the Euler-type equation with the oscillation constant (1.19). Denote its left hand side by

$$L_{RW}[x] := (\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} \right) \Phi(x). \tag{3.13}$$

Recall that Došlý in [20] showed that for $h(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t$ and the operator defined in (3.13) we have

$$f(t) := h(t)L_{RW}[h](t) = \frac{\operatorname{Log}_n t}{t \log^3 t}\left[\frac{2\gamma_p p(2-p)}{3(p-1)^2} + o(1)\right] \quad \text{as } t \to \infty. \tag{3.14}$$

Corollary 3.1 ([111], Cor 2)**.** *Suppose that the condition*

$$L\left[t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t\right] \geq 0$$

holds for large t and take

$$\tilde{c}(t) = \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \operatorname{Log}_j^2 t}\right).$$

If the inequality

$$-\int_t^\infty f\,ds \leq \int_t^\infty (c-\tilde{c})s^{p-1} \operatorname{Log}_n(s)\,ds \leq \int_t^\infty (C-\tilde{c})s^{p-1} \operatorname{Log}_n(s)\,ds < \infty,$$

where $f(s)$ is defined by (3.14)*, is satisfied, and if equation* (3.12) *is nonoscillatory, then equation* (1.1) *is nonoscillatory too.*

Further, apply the results of Theorem 3.7 to the generalized Euler-type equation with $n+1$ terms in the sum as the testing equation (3.12) in order to obtain a Hille-Wintner type comparison criterion for the perturbed Euler-type equation (3.4). Then the inequality

$$-\int_t^\infty f(s)\,ds \leq \int_t^\infty d(s)s^{p-1} \operatorname{Log}_n(s)\,ds \leq \frac{\mu_p}{\log_{n+1}(t)},$$

where $f(t)$ is given by (3.14), implies nonoscillation of equation (3.4). Note that this consequence is in compliance with the Hille-Nehari type criterion, that was proved in [20], see the nonoscillatory part of Theorem 3.1.

Chapter 4

Neutral half-linear equations

In the last years, oscillation properties of delayed and neutral half-linear equations have attracted considerable attention. Our interest has been in looking for applications of a variant of the modified Riccati technique. This chapter is devoted to our results from [67] for the second-order half-linear neutral differential equation

$$\big(r(t)\Phi(z'(t))\big)' + c(t)\Phi(x(\tau(t))) = 0, \quad z(t) = x(t) + b(t)x(\sigma(t)), \tag{4.1}$$

where $t \geq t_0$ and $\Phi(x) = |x|^{p-2}x$, $p \in \mathbb{R}$, $p > 1$. We suppose that the coefficients of the equation satisfy the conditions $r \in C([t_0,\infty),\mathbb{R}^+)$, $b \in C^1([t_0,\infty),\mathbb{R}_0^+)$, $c \in C([t_0,\infty),\mathbb{R}_0^+)$, c is not identically equal to zero on any neighborhood of infinity and

$$b(t) \leq 1.$$

Concerning the deviating arguments, we assume that $\tau, \sigma \in C^1([t_0,\infty),\mathbb{R})$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \sigma(t) = \infty$ and

$$\tau' \geq 0, \quad \tau(t) \leq t, \quad \sigma(t) \leq t.$$

We also suppose that

$$\int^{\infty} r^{1-q}(t)\,dt = \infty,$$

where q denotes the conjugate number of p, i.e., $q = \frac{p}{p-1}$. The above setting and conditions are supposed to hold throughout the whole chapter.

Clearly, equation (4.1) can be seen as a generalization of equation (1.1) to which it is reduced for $\tau(t) = t$ and $b(t) = 0$. However, oscillation properties of both the equations significantly differ and equation (4.1) evinces more complex and complicated qualitative behavior.

By a solution of (4.1), one understands a differentiable function $x(t)$ ($x(t) \not\equiv 0$), such that $r(t)\Phi(z'(t))$ is differentiable and (4.1) holds for large t. Equation (4.1) is called *oscillatory* if it does not have a solution which is eventually positive or negative. Contrary to equation (1.1), oscillatory and nonoscillatory solutions can coexist and hence the existence of one oscillatory or nonoscillatory solution does not ensure that equation (4.1) is oscillatory or nonoscillatory, respectively.

One of the possible ways to prove oscillation criteria for (4.1) is via the Riccati technique and our contribution to the topic was to present some criteria which can be obtained by its modification (cf. Chapter 1.2). The (modified) Riccati method here allows to work only with Riccati type inequality, in contrast to Riccati equations (1.6) and (1.27) reated to equation (1.1). In more detail, suppose that equation (4.1) has an eventually positive solution $x(t)$. Take

$$w(t) = r(t)\Phi\left(\frac{z'(t)}{z(\tau(t))}\right).$$

A direct differentiation yields

$$w'(t) = \frac{(r(t)\Phi(z'(t)))'}{\Phi(z(\tau(t))} - (p-1)r(t)\tau'(t)\frac{\Phi(z'(t))z'(\tau(t))}{|z(\tau(t))|^p}$$

which, with the use of the inequality

$$\left(r(t)\Phi(z'(t))\right)' \leq -c(t)\Phi\left[z(\tau(t))(1-b(\tau(t))\right]$$

proved in [67, Lemma 1], gives

$$w'(t) \leq -c(t)\Phi(1-b(\tau(t))) - (p-1)r^{1-q}(t)\tau'(t)\frac{z'(\tau(t))}{z'(t)}|w(t)|^q.$$

Assuming that there exists a positive function $f(t)$ such that

$$\frac{z'(\tau(t))}{z'(t)} \geq f(t) > 0 \tag{4.2}$$

we obtain the Riccati type inequality of the form

$$w'(t) \leq -c(t)\Phi(1-b(\tau(t))) - (p-1)r^{1-q}(t)\tau'(t)f(t)|w(t)|^q.$$

To continue with the modification of the Riccati technique, let $h(t)$ be a positive differentiable function and denote

$$G(t) = r(t)h(\tau(t))\Phi\left(\frac{h'(\tau(t))}{f(t)}\right). \tag{4.3}$$

Using the modified Riccati transformation

$$v(t) = h^p(\tau(t))w(t) - G(t)$$

provides (see [67, Lemma 3]) the so-called modified Riccati inequality

$$v'(t) + C(t) + (p-1)r^{1-q}(t)\tau'(t)h^{-q}(\tau(t))f(t)H(v(t), G(t)) \leq 0,$$

where

$$C(t) = h(\tau(t))\left[\left(r(t)\Phi\left(\frac{h'(\tau(t))}{f(t)}\right)\right)' + c(t)\Phi(h(\tau(t)))\Phi(1-b(\tau(t)))\right] \tag{4.4}$$

and $H(v, G)$ is the function defined by (1.25).

4.1 Oscillation criteria

The first theorem of [67] can be seen as a result of the same character as, for example, [70, Theorem 5] or [120, Theorem 2.1]. In both the mentioned papers, a variant of Riccati transformation is used as the proving technique, whereas we use the modification of the Riccati technique as described in the previous section. The statement reads as follows.

Theorem 4.1 ([67], Thm 1)**.** *Let f be a positive function, G, H be defined by* (4.3), (1.25)*, respectively, h be a positive continuously differentiable function such that $h' \neq 0$ for large t and $C(t) \geq 0$ for large t. Moreover, let either*

$$\limsup_{t\to\infty} |G(t)| < \infty \tag{4.5}$$

and

$$\int^{\infty} \frac{f(t)\tau'(t)}{r^{q-1}(t)h^{q}(\tau(t))}\, dt = \infty \tag{4.6}$$

or

$$\lim_{t\to\infty} |G(t)| = \infty$$

and

$$\int^{\infty} \frac{\Phi(f(t))\tau'(t)}{r(t)h^{2}(\tau(t))|h'(\tau(t))|^{p-2}}\, dt = \infty.$$

If

$$\int^{\infty} C(t)\, dt = \infty, \tag{4.7}$$

then equation (4.1) *is either oscillatory or in every neighborhood of ∞ there exists t^* such that $\frac{z'(\tau(t^*))}{z'(t^*)f(t^*)} < 1$ for all solutions of* (4.1).

The main idea of the applicability of Theorem 4.1 lies in the fact that under our assumptions we know how to choose the function f. Indeed, according to [67, Lemma 1], for a positive solution $x(t)$ of equation (4.1), we have

$$\frac{z'(\tau(t))}{z'(t)} \geq \Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right). \tag{4.8}$$

This leads to the following consequence. Denote

$$R(t) = r(t)\frac{1}{\tau'(t)}h^{2}(\tau(t))|h'(\tau(t))|^{p-2}f^{1-p}(t). \tag{4.9}$$

Under the assumptions of the paper, according to (4.8), we can take $f(t) = \Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right)$ and the functions G, C and R (see (4.3), (4.4) and (4.9), respectively) get the following form:

$$\begin{aligned}
G_1(t) &= r(\tau(t))h(\tau(t))\Phi(h'(\tau(t))),\\
C_1(t) &= h(\tau(t))\left[(r(\tau(t))\Phi(h'(\tau(t))))' + c(t)\Phi(h(\tau(t)))\Phi(1-b(\tau(t)))\right],\\
R_1(t) &= r(\tau(t))\frac{1}{\tau'(t)}h^{2}(\tau(t))|h'(\tau(t))|^{p-2}.
\end{aligned}$$

In this particular case of the function f, we can formulate a version of Theorem 4.1 as follows.

Corollary 4.1 ([67], Cor 1)**.** *Let h be a positive continuously differentiable function such that $h' \neq 0$ for large t and $C_1(t) \geq 0$ for large t. Moreover, let either*

$$\limsup_{t\to\infty} |G_1(t)| < \infty \quad \text{and} \quad \int^{\infty} \frac{\tau'(t)}{r^{q-1}(\tau(t))h^q(\tau(t))}\, dt = \infty \tag{4.10}$$

or

$$\lim_{t\to\infty} |G_1(t)| = \infty \quad \text{and} \quad \int^{\infty} R_1^{-1}(t)\, dt = \infty.$$

If

$$\int^{\infty} C_1(t)\, dt = \infty, \tag{4.11}$$

then equation (4.1) *is oscillatory.*

The second result of [67] presents a Hille-Nehari type criterion which can be again applied to the same situation as in the preceding Corollary 4.1.

Theorem 4.2 ([67], Thm 2)**.** *Let h be a positive continuously differentiable function such that $h' \neq 0$ for large t and f be a positive function such that $C(t) \geq 0$ for large t,*

$$\int^{\infty} R^{-1}(t)\, dt = \infty, \quad \liminf_{t\to\infty} G(t) > 0$$

and

$$\limsup_{t\to\infty} G(t) < \infty \quad \text{or} \quad \lim_{t\to\infty} G(t) = \infty. \tag{4.12}$$

Suppose that $\int^{\infty} C(t)\, dt < \infty$. If

$$\liminf_{t\to\infty} \int^{t} R^{-1}(s)\, ds \int_{t}^{\infty} C(s)\, ds > \frac{1}{2q}, \tag{4.13}$$

then equation (4.1) *is either oscillatory or in every neighborhood of ∞ there exists t^* such that $\frac{z'(\tau(t^*))}{z'(t^*)f(t^*)} < 1$ for all solutions of* (4.1).

4.2 Application to neutral Euler-type equations

Consider the Euler-type equation

$$(\Phi(x(t) + b_0 x(\sigma(t)))')' + \frac{\gamma}{t^p}\Phi(x(\lambda t)) = 0, \tag{4.14}$$

where $\lambda \in (0,1)$, $\sigma(t) \leq t, \lim_{t\to\infty} \sigma(t) = \infty$, $b_0 \in [0,1)$. Equation (4.14) is of the form (4.1), where $r(t) = 1$, $c(t) = \frac{\gamma}{t^p}$, $\tau(t) = \lambda t$, $b(t) = b_0$. It is also a generalization of the Euler equation (1.10) to which it is reduced for $\tau(t) = t$ and $b_0 = 0$.

Take $h(t) = t^{\frac{p-1}{p}}$, then by a direct computation we have

$$G_1(t) = \left(\frac{p-1}{p}\right)^{p-1}, \quad R_1(t) = \left(\frac{p-1}{p}\right)^{p-2} t \tag{4.15}$$

and

$$\int^{\infty} \frac{\tau'(t)}{r^{q-1}(\tau(t))h^q(\tau(t))}\,dt = \int^{\infty} \frac{1}{t}\,dt \to \infty.$$

Hence condition (4.10) is satisfied. Furthermore, after some rearrangements, we have

$$\begin{aligned} C_1(t) &= (\lambda t)^{\frac{p-1}{p}} \left[\left(\left(\frac{p-1}{p} \right)^{p-1} (\lambda t)^{\frac{1-p}{p}} \right)' + c(t)\Phi(1-b_0) \left((\lambda t)^{\frac{p-1}{p}} \right)^{p-1} \right] \\ &= t^{-1} \left[- \left(\frac{p-1}{p} \right)^p + \gamma\Phi(1-b_0)\lambda^{p-1} \right]. \end{aligned} \tag{4.16}$$

Positivity of the expression $-\left(\frac{p-1}{p}\right)^p + \gamma\Phi(1-b_0)\lambda^{p-1}$ implies (4.11). Thus, by Corollary 4.1, equation (4.14) is oscillatory if

$$\gamma > \left(\frac{p-1}{p} \right)^p \frac{1}{\lambda^{p-1}\Phi(1-b_0)}. \tag{4.17}$$

This corresponds to the result known in the case $b_0 = 0$ (equations with delay, see [63]) and also to the case $b_0 = 0$ and $\lambda = 1$ (ordinary equations).

Grace et al. showed in [70] that under some additional assumptions, condition (4.8) can be strengthened. Similarly as in [125], they considered the sequence

$$g_0(\rho) := 1, \quad g_{n+1}(\rho) := e^{\rho g_n(\rho)}, \quad n = 0, 1, 2, \ldots, \tag{4.18}$$

where ρ is a positive constant. For $\rho \in (0, \frac{1}{e}]$, the sequence is increasing, bounded above and $\lim_{t\to\infty} g_n(\rho) = g(\rho) \in [1, e]$, where $g(\rho)$ is a real root of the equation

$$g(\rho) = e^{\rho g(\rho)}. \tag{4.19}$$

With the use of this sequence and the notation

$$\begin{aligned} \mathscr{Q}(t) &:= \Phi(1 - b(\tau(t)))c(t), \quad \mathscr{R}(t) := \int_{t_1}^{t} r^{1-q}(s)\,ds, \\ \tilde{\mathscr{R}}(t) &:= \mathscr{R}(t) + \frac{1}{p-1} \int_{t_1}^{t} \mathscr{R}(s)\Phi(\mathscr{R}(\tau(s)))\mathscr{Q}(s)\,ds \end{aligned}$$

for $t \geq t_1$, where t_1 is large enough, Grace et al. proved in [70, Lemma 4] that if $\tau(t)$ is strictly increasing, equation (4.1) has a positive solution $x(t)$ on $[t_0, \infty)$ and the condition

$$\int_{\tau(t)}^{t} \mathscr{Q}(s)\Phi(\tilde{\mathscr{R}}(s))\,ds \geq \rho \tag{4.20}$$

holds for some $\rho > 0$ and t large enough, then

$$\frac{z'(\tau(t))}{z'(t)} \geq \Phi^{-1}\left(\frac{g_n(\rho)r(t)}{r(\tau(t))} \right)$$

for every n and t large enough, where $g_n(\rho)$ is defined by (4.18).

Thanks to the preceding estimate, condition (4.17) can be even strengthened for a class of equations (4.14), which satisfy (4.20) with $\rho \in (0, \frac{1}{e})$. By a direct computation (or see [70]), one can show that for the considered Euler-type equation (4.14) we have

$$\rho = (1-b_0)^{p-1}\gamma\lambda^{p-1}\left(1+\frac{1}{p-1}\lambda^{p-1}(1-b_0)^{p-1}\gamma\right)^{p-1}\log\left(\frac{1}{p-1}\right).$$

If $\rho \in (0, \frac{1}{e})$, we can use in place of the positive function $f(t)$ from (4.2) the function $\Phi^{-1}\left(\frac{g(\rho)r(t)}{r(\tau(t))}\right)$, where $g(\rho)$ is defined by (4.19). Then functions G, C, R according to their definitions (4.3), (4.4) and (4.9), respectively, become

$$\begin{aligned}
G_2(t) &= r(\tau(t))h(\tau(t))\Phi(h'(\tau(t)))(g(\rho))^{-1},\\
C_2(t) &= h(\tau(t))\left[\left(r(\tau(t))\frac{\Phi(h'(\tau(t)))}{g(\rho)}\right)' + c(t)\Phi(h(\tau(t)))\Phi(1-b_0)\right],\\
R_2(t) &= r(\tau(t))\frac{1}{\tau'(t)}h^2(\tau(t))|h'(\tau(t))|^{p-2}(g(\rho))^{-1}.
\end{aligned}$$

These in our setting for equation (4.14) and again with $h(t) = t^{\frac{p-1}{p}}$ read as

$$G_2(t) = \left(\frac{p-1}{p}\right)^{p-1}(g(\rho))^{-1}, \quad R_1(t) = \left(\frac{p-1}{p}\right)^{p-2}(g(\rho))^{-1}t$$

and

$$C_2(t) = t^{-1}\left[-\left(\frac{p-1}{p}\right)^{p}(g(\rho))^{-1} + \gamma\Phi(1-b_0)\lambda^{p-1}\right].$$

Since conditions (4.5) and (4.6) hold and (4.7) is implied by positivity of the expression

$$-\left(\frac{p-1}{p}\right)^{p}(g(\rho))^{-1} + \gamma\Phi(1-b_0)\lambda^{p-1},$$

equation (4.14) is oscillatory, by Theorem 4.1, provided

$$\gamma > \left(\frac{p-1}{p}\right)^{p}\frac{1}{(g(\rho))\lambda^{p-1}\Phi(1-b_0)}.$$

This corresponds with the condition derived in [70].

For Euler equation (1.10) it is known that it oscillates if and only if $\gamma > \left(\frac{p-1}{p}\right)^{p}$ holds, which is the ordinary version of condition (4.17). The constant $\gamma_p = \left(\frac{p-1}{p}\right)^{p}$ is the critical constant between oscillation and nonoscillation of (4.14), and it is natural to study perturbations of the Euler-type equation with this critical constant and to find critical constants in the added terms. In the case of the delayed and neutral equations, there is not such a boundary between oscillation and nonoscillation of (4.14). However, based on the results known from the ordinary case, let us study the neutral version of the Euler-Weber type equation

$$(\Phi(x(t)+b_0x(\sigma(t)))')' + \left(\frac{\gamma_p}{\lambda^{p-1}\Phi(1-b_0)t^p} + \frac{\mu}{t^p\log^2 t}\right)\Phi(x(\lambda(t)) = 0. \tag{4.21}$$

For equation (4.21) we use Theorem 4.2 with $f(t) = \Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right)$. Similarly as previously, we take $h(t) = t^{\frac{p-1}{p}}$, for which (4.15) holds, and with the use of the relevant coefficient function $c(t)$ in (4.16) we have

$$C_1(t) = \frac{\mu}{t\log^2 t}\Phi(1-b_0)\lambda^{p-1}$$

and

$$\int_t^\infty C_1(s)\,ds = \int_t^\infty \frac{\mu\Phi(1-b_0)\lambda^{p-1}}{s\log^2 s}\,ds = \frac{\mu\Phi(1-b_0)\lambda^{p-1}}{\log t}.$$

Since $R_1^{-1} = \left(\frac{p}{p-1}\right)^{p-2}\frac{1}{t}$, condition (4.13) becomes

$$\left(\frac{p}{p-1}\right)^{p-2}\mu\Phi(1-b_0)\lambda^{p-1} > \frac{1}{2q}$$

which, by Theorem 4.2, implies that equation (4.21) is oscillatory if

$$\mu > \frac{1}{2\Phi(1-b_0)\lambda^{p-1}}\left(\frac{p-1}{p}\right)^{p-1}. \tag{4.22}$$

Note that in the ordinary case $\lambda = 1$ and $b_0 = 0$, the constant from (4.22) is critical, which means that (4.21) is oscillatory if and only if $\mu > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$.

Finally, let us consider the perturbation of the Euler-Weber equation (4.21) with $\mu = \mu_p$. Take $h(t) = t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t$ and consider equation (4.1) with $r(t) = 1$, $\tau(t) = \lambda t$, $b(t) = 0$. Observe that

$$h'(t) = \frac{p-1}{p}t^{-\frac{1}{p}}\log^{\frac{1}{p}} t\left[1 + \frac{1}{(p-1)\log t}\right].$$

By direct computation, we see that

$$G_1(t) = \left(\frac{p-1}{p}\right)^{p-1}\log(\lambda t)(1+o(1)),$$
$$R_1(t) = \left(\frac{p-1}{p}\right)^{p-2}t\log(\lambda t)(1+o(1))$$

as $t\to\infty$. With the use of the power expansion formula

$$(1+x)^s = 1 + sx + \frac{s(s-1)}{2}x^2 + \frac{s(s-1)(s-2)}{6}x^3 + o(x^3) \quad \text{as} \quad x\to 0$$

one can show that for

$$c(t) = \frac{\gamma_p}{\lambda^{p-1}t^p} + \frac{1}{2}\left(\frac{p}{p-1}\right)^{p-1}\frac{1}{\lambda^{p-1}t^p\log^2(\lambda t)} + \frac{\mu}{t^p\log^2(\lambda t)\log^2(\log(\lambda t))} \tag{4.23}$$

we have

$$C_1(t) = \frac{\mu\lambda^{p-1}}{t\log(\lambda t)\log^2(\log(\lambda t))}(1+o(1)) \quad \text{as } t\to\infty.$$

Because

$$\int R_1^{-1}(t)\,dt \sim \left(\frac{p-1}{p}\right)^{2-p} \log(\log(\lambda t))$$

and

$$\int C_1(t)\,dt \sim -\frac{\mu\lambda^{p-1}}{\log(\log(\lambda t))},$$

as $t \to \infty$, condition (4.13) becomes

$$\left(\frac{p-1}{p}\right)^{2-p} \mu\lambda^{p-1} > \frac{1}{2q}.$$

Since $\frac{1}{q} = \frac{p-1}{p}$, the considered perturbed Euler-Weber equation (4.1) with $r(t) = 1$, $\tau(t) = \lambda t$, $b(t) = 0$ and with $c(t)$ given by (4.23) is oscillatory if

$$\mu > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \frac{1}{\lambda^{p-1}}.$$

One can see that the constant is reduced to μ_p in the undelayed case $\lambda = 1$.

Chapter 5

Numerical approach

Within the numerical approach to finding approximate solutions of differential equations, we focus on a semi-analytical method based on the differential transformation. Contrary to other numerical methods, where the result is a set of points and approximate function values, we arrive at an approximate polynomial.

In [114] we proposed an algorithm using the differential transformation, which is convenient for finding numerical solutions to initial value problems for functional differential equations. Our focus was on retarded equations with delays which in general are functions of the independent variable. The main idea of the algorithm is that a delayed differential equation is turned into an ordinary differential equation using the method of steps, and subsequently, the ordinary differential equation is transformed into a recurrence relation in one variable using the differential transformation. The approximate numerical solution has the form of a Taylor polynomial whose coefficients are determined by solving the recurrence relation. In the paper [112], the algorithm is used for finding approximate numerical solutions to second-order half-linear Euler equation with and without delay, and recurrence equations for Taylor series coefficients of solutions to initial value problems are derived. Here we recall the basics of the differential method in such a depth, which is necessary for understanding how the application to the Euler equations is made, and we present the process, how the approximate solutions for half-linear Euler equations in the form of Taylor's polynomial can be found.

5.1 Differential transformation method

Differential transformation of a real function $u(t)$ at a point $t_0 \in \mathbb{R}$ is

$$\mathscr{D}\{u(t)\}[t_0] = \{U(k)[t_0]\}_{k=0}^{\infty}. \tag{5.1}$$

where $U(k)[t_0]$, i.e., the $k-$th component of the differential transformation of the function $u(t)$ at t_0, $k \in \mathbb{N}_0$, is defined as

$$U(k)[t_0] = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}, \tag{5.2}$$

provided that the original function $u(t)$ is analytic in a neighbourhood of t_0. Inverse differential transformation of $\{U(k)[t_0]\}_{k=0}^{\infty}$ is defined as

$$u(t) = \mathscr{D}^{-1}\Big\{\{U(k)[t_0]\}_{k=0}^{\infty}\Big\}[t_0] = \sum_{k=0}^{\infty} U(k)[t_0](t-t_0)^k. \tag{5.3}$$

In applications, the function $u(t)$ is expressed by a finite sum

$$u(t) = \sum_{k=0}^{N} U(k)[t_0](t-t_0)^k.$$

Now we present the formulas for the differential transformation of such functions, which we need for transforming the half-linear Euler equation. They can be found, for example, in [114]:

Lemma 5.1. *Assume that $F(k)[t_0]$ and $U(k)[t_0]$ are differential transformations of functions $f(t)$ and $u(t)$, respectively.*

$$\text{If } f(t) = \frac{d^n u(t)}{dt^n}, \text{ then } F(k)[t_0] = \frac{(k+n)!}{k!} U(k+n)[t_0]. \tag{5.4}$$

$$\text{If } f(t) = t^r, r \in \mathbb{R}, \text{ then } F(k)[t_0] = \binom{r}{k} t_0^{r-k} \text{ for all } t \text{ such that } |t-t_0| < |t_0|, \tag{5.5}$$

$$\text{where } \binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!} = \frac{(r)_k}{k!}, (r)_k \text{ being the Pochhammer symbol.}$$

$$\text{If } f(t) = g(t)h(t), \text{ then } F(k)[t_0] = \sum_{l=0}^{k} G(l)[t_0]H(k-l)[t_0]. \tag{5.6}$$

Differential transformation of components containing nonlinear terms can be obtained using the algorithm described in [113]:

Lemma 5.2. *Let g and f be real functions analytic near t_0 and $g(t_0)$ respectively, and let h be the composition $h(t) = (f \circ g)(t) = f(g(t))$. Denote $\mathscr{D}\{g(t)\}[t_0] = \{G(k)\}_{k=0}^{\infty}$, $\mathscr{D}\{f(t)\}[g(t_0)] = \{F(k)\}_{k=0}^{\infty}$ and $\mathscr{D}\{(f \circ g)(t)\}[t_0] = \{H(k)\}_{k=0}^{\infty}$ the differential transformations of functions g, f and h at t_0, $g(t_0)$ and t_0 respectively. Then the numbers $H(k)$ in the sequence $\{H(k)\}_{k=0}^{\infty}$ satisfy the relations $H(0) = F(0)$ and*

$$H(k) = \sum_{l=1}^{k} F(l) \cdot \hat{B}_{k,l}\big(G(1), \dots, G(k-l+1)\big) \text{ for } k \geq 1, \tag{5.7}$$

where $\hat{B}_{k,l}(\hat{x}_1, \dots, \hat{x}_{k-l+1})$ are the partial ordinary Bell polynomials.

Here, according to [113], the partial ordinary Bell polynomials $\hat{B}_{k,l}(\hat{x}_1, \dots, \hat{x}_{k-l+1})$, $l = 1, 2, \dots$, $k \geq l$, satisfy the recurrence relation

$$\hat{B}_{k,l}(\hat{x}_1, \dots, \hat{x}_{k-l+1}) = \sum_{i=1}^{k-l+1} \frac{i \cdot l}{k} \hat{x}_i \hat{B}_{k-i,l-1}(\hat{x}_1, \dots, \hat{x}_{k-i-l+2}),$$

where $\hat{B}_{0,0} = 1$ and $\hat{B}_{k,0} = 0$ for $k \geq 1$.

Finally, we need the following result which is proved in [114]. If $\mathscr{D}\{f(t)\}[t_0] = \{F(k)[t_0]\}_{k=0}^{\infty}$ is the differential transformation of the function $f(t)$ at t_0. Then the components $F(k)[t_1]$ of the differential transformation $\mathscr{D}\{f(t)\}[t_1] = \{F(k)[t_1]\}_{k=0}^{\infty}$ of $f(t)$ at $t_1 > t_0$ may be expressed as

$$F(k)[t_1] = \sum_{j=0}^{\infty} \binom{k+j}{j} (t_1 - t_0)^j F(k+j)[t_0], \quad k \geq 0. \tag{5.8}$$

5.2 Application of the differential transformation to half-linear Euler equation

In the paper [112], the differential transformation method is applied to half-linear Euler equation in the form

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}} x^{\alpha}(t) = 0, \tag{5.9}$$

as well as to the second-order half-linear Euler equation with a proportional delay

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}} x^{\alpha}(\lambda t) = 0,$$

where $\alpha > 0$ is a quotient of two odd positive numbers and $\lambda \in (0,1)$. Note that $\alpha = p-1$ and equation (5.9) is a special case of equation (1.10).

Consider the initial value problem for half-linear Euler equation

$$((x')^{\alpha})' + \frac{\gamma}{t^{\alpha+1}} x^{\alpha} = 0, \quad x(t_0) = a, \quad x'(t_0) = b. \tag{5.10}$$

To apply the differential transformation of (5.10) at t_0, we use Lemma 5.1 and successively obtain:

$$\begin{aligned}
\mathscr{D}\{x(t)\} &\overset{(5.1)}{=} \{X(k)\}_{k=0}^{\infty}, \quad X(0) \overset{(5.2)}{=} a,\ X(1) \overset{(5.2)}{=} b, \\
\mathscr{D}\{x'(t)\} &\overset{(5.4)}{=} \{(k+1)X(k+1)\}_{k=0}^{\infty}, \\
\mathscr{D}\{(x'(t))^{\alpha}\} &\overset{(5.7)}{=} \left\{ \sum_{l=1}^{k} \binom{\alpha}{l} (X(1))^{\alpha-l} \hat{B}_{k,l}(2X(2), 3X(3), \dots) \right\}_{k=1}^{\infty} =: \{H_1(k)\}_{k=1}^{\infty}, \\
H_1(0) &= \binom{\alpha}{0} (X(1))^{\alpha}, \\
\mathscr{D}\{((x'(t))^{\alpha})'\} &\overset{(5.4)}{=} \{(k+1)H_1(k+1)\}_{k=0}^{\infty}, \\
\mathscr{D}\{(x(t))^{\alpha}\} &\overset{(5.7)}{=} \left\{ \sum_{l=1}^{k} \binom{\alpha}{l} (X(0))^{\alpha-l} \hat{B}_{k,l}(X(1), X(2), \dots) \right\}_{k=1}^{\infty} =: \{H_2(k)\}_{k=1}^{\infty}, \\
H_2(0) &= \binom{\alpha}{0} (X(0))^{\alpha}, \\
\mathscr{D}\left\{ \frac{1}{t^{\alpha+1}} \right\} &\overset{(5.5)}{=} \left\{ \binom{-\alpha-1}{k} t_0^{-\alpha-1-k} \right\}_{k=0}^{\infty} =: \{F_1(k)\}_{k=0}^{\infty}.
\end{aligned}$$

Hence equation (5.10) transformed at $t = t_0$, with use of the rule for transforming product (5.6), reads as

$$(k+1)H_1(k+1) + \gamma \sum_{l=0}^{k} F_1(l)H_2(k-l) = 0, \quad k \geq 0. \tag{5.11}$$

This is a recurrence relation, to which it is possible to substitute the initial conditions and then compute successively further members of the sequence X. In the end, the inverse transformation (5.3) is applied.

Example 5.1. As a concrete example, take Euler equation (5.10) with $\alpha = 3$. We know that if $\gamma = \gamma_3 = \left(\frac{3}{4}\right)^4$, then $t^{3/4}$ is a solution of the initial value problem

$$((x')^3)' + \frac{\left(\frac{3}{4}\right)^4}{t^4} x^3 = 0, \quad x(1) = 1, \quad x'(1) = \frac{3}{4}. \tag{5.12}$$

Let us illustrate the process of finding the Taylor series of the solution on this problem (5.12). Transformed equation (5.11) expands for (5.12) with $\alpha = p-1 = 3, t_0 = 1, \gamma = \left(\frac{3}{4}\right)^4$ to

$$\begin{aligned} 0 = {} & (k+1) \sum_{l=1}^{k+1} \binom{3}{l} (X(1))^{3-l} \hat{B}_{k+1,l}(2X(2), \ldots) \\ & + \gamma \sum_{l=0}^{k-1} \binom{-4}{l} \sum_{j=1}^{k-l} \binom{3}{j} (X(0))^{3-j} \hat{B}_{k-l,j}(X(1), \ldots). + \gamma \binom{-4}{k} \binom{3}{0} (X(0))^3 \end{aligned}$$

The initial conditions imply $X(0) = 1$ and $X(1) = \frac{3}{4}$. Substituting for $k = 0, 1, 2$ provides the results

$$X(2) = -\frac{3}{32}, \quad X(3) = \frac{5}{128}, \quad X(4) = -\frac{45}{2048}.$$

The precise calculations can be found in [112]. We see, that using the differential transformation algorithm we get the Taylor series coefficients of the exact solution. Indeed, Taylor series expansion of $t^{3/4}$ at $t_0 = 1$ is

$$1 + \frac{3}{4}(t-1) - \frac{3}{32}(t-1)^2 + \frac{5}{128}(t-1)^3 - \frac{45}{2048}(t-1)^4 + O((t-1)^5), \quad |t-1| \leq 1.$$

A different approach to find recurrence relations for obtaining the coefficients of a Taylor series of the solution is used in the Parker-Sochacki method (see, for example [12, 121]). The main idea is to transform the studied ordinary differential equation into a polynomial system, if possible. The polynomial system is then transformed, and recurrence relations are found. Such a method can be applied to a wide class of ordinary differential equations. However, the description of the process of finding the polynomial form is not simple, and the polynomial system might not be unique. A-priori error estimates presented in [121] can be applied especially to this polynomial form of the equation. Within this context, choosing the transformations $y_1 = x(t), y_2 = x'(t), y_3 = t^{-4}, y_4 = t, y_5 = 1/y_2$, the initial value problem (5.12) can be rewritten in the polynomial form, as is shown in [112]. However, we find our approach more elegant less demanding with respect to the number of needed calculations. Note that the role of Theorem 5.2, which enables us to transform compound functions, is crucial within our algorithm.

Now let us focus on the half-linear Euler equation with proportional delay. We show how the process of combining the differential transformation with the method of steps, as described in [114], is applied. Consider the initial value problem (for $\lambda \in (0,1)$)

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}}(x(\lambda t))^{\alpha} = 0, \quad x(t_0) = a, \quad x'(t_0) = b, \tag{5.13}$$

with the initial function

$$\phi(t) = a + b(t - t_0), \quad t \in (0, t_0] = I_0. \tag{5.14}$$

Denote $t_i = \frac{t_0}{\lambda^i}$ and $I_i = [t_{i-1}, t_i]$, $i \geq 1$. If $t \in I_i$ then λt lies in I_{i-1}.

For $t \in I_1 = [t_0, t_1]$ we denote the solution of initial value problem (5.13)+(5.14) as $x_1(t)$ and its differential transformation at t_0 by $X_1(k)[t_0]$, $k \geq 0$. Since λt for $t \in I_1$ falls into I_0, we substitute for $x(\lambda t)$ the initial function (5.14) and rewrite equation (5.13) into the form

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}}(a + b(\lambda t - t_0))^{\alpha} = 0, \quad x(t_0) = a, \quad x'(t_0) = b. \tag{5.15}$$

Because

$$a + b(\lambda t - t_0) = a + bt_0(\lambda - 1) + b\lambda(t - t_0),$$

we have

$$\mathscr{D}\{a + b(\lambda t - t_0)\}[t_0] = \{a + bt_0(\lambda - 1), b\lambda, 0, 0, \ldots\},$$

$$\begin{aligned}
\mathscr{D}\{(a + b(\lambda t - t_0))^{\alpha}\}(k)[t_0] &\overset{(5.7)}{=} \sum_{l=1}^{k} \binom{\alpha}{l} (a + bt_0(\lambda - 1))^{\alpha - l} \hat{B}_{k,l}(b\lambda, 0, 0, \ldots) \\
&= \binom{\alpha}{k} (a + bt_0(\lambda - 1))^{\alpha - k} (b\lambda)^k, k \geq 1, \\
\mathscr{D}\{(a + b(\lambda t - t_0))^{\alpha}\}(0)[t_0] &= (a + bt_0(\lambda - 1))^{\alpha}.
\end{aligned}$$

Equation (5.15) transformed at t_0 with the use of the rule for transforming the product (5.6) reads as

$$\begin{aligned}
&(k+1) \sum_{l=1}^{k+1} \binom{\alpha}{l} (X_1(1)[t_0])^{\alpha - l} \hat{B}_{k+1,l}(2X_1(2)[t_0], 3X_1(3)[t_0] \ldots) \\
&\quad + \gamma \sum_{l=0}^{k} \binom{-\alpha - 1}{l} (t_0)^{-\alpha - 1 - l} \binom{\alpha}{k - l} (a + bt_0(\lambda - 1))^{\alpha - k + l} (b\lambda)^{k - l} = 0
\end{aligned} \tag{5.16}$$

and the initial conditions transform, according to (5.2), to

$$X_1(0)[t_0] = a, \quad X_1(1)[t_0] = b.$$

Substituting for $k = 0, 1, \ldots$ into (5.16) provides recurrent equations, from which one can successively calculate $X_1(k)[t_0]$ for $k \geq 2$. Resulting solution on interval I_1 then is

$$x_1(t) = \sum_{k=0}^{\infty} X_1(k)[t_0](t - t_0)^k.$$

In the applications and programming, this series (and every other) will be truncated. The solution x_1 will be just approximate.

Notice that with a different (general) initial function ϕ, equation (5.16) would be of the form:

$$(k+1)\sum_{l=1}^{k+1}\binom{\alpha}{l}(X_1(1)[t_0])^{\alpha-l}\hat{B}_{k+1,l}(2X_1(2)[t_0],3X_1(3)[t_0]\dots)$$
$$+\gamma\sum_{l=0}^{k}\binom{-\alpha-1}{l}(t_0)^{-\alpha-1-l}\mathscr{D}\{\phi(\lambda t)\}(k-l)[t_0]=0.$$

Now let us move to the second step, take $t\in I_2=[t_1,t_2]$, denote $x_2(t)$ approximate solution on I_2 and $X_2(k)[t_1]$, $k\geq 0$, its differential transformation with the center at t_1. Because λt for $t\in I_2$ lies in I_1, we substitute for $x(\lambda t)$ the function $x_1(\lambda t)$ and rewrite equation (5.13) into the form

$$((x'(t))^{\alpha})'+\frac{\gamma}{t^{\alpha+1}}(x_1(\lambda t))^{\alpha}=0,\quad x(t_1)=x_1(t_1),\quad x'(t_1)=x_1'(t_1). \tag{5.17}$$

Since $\lambda t=\lambda t_1+\lambda(t-t_1)$, we have

$$\mathscr{D}\{\lambda t\}=\{\lambda t_1,\lambda,0,0,\dots\}$$

and because $\lambda t_1=t_0$, we have with the use of (5.7)

$$\mathscr{D}\{x_1(\lambda t)\}(k)[t_1]=\sum_{l=1}^{k}X_1(l)[\lambda t_1]\hat{B}_{k,l}(\lambda,0,0,\dots)=X_1(k)[t_0]\lambda^k=:G(k)[t_1],k\geq 1,$$
$$\mathscr{D}\{x_1(\lambda t)\}(0)[t_1]=X_1(0)[t_0]=a=:G(0)[t_1].$$

Furthermore, again according to the way how to transform a compound function (5.7),

$$\mathscr{D}\{(x_1(\lambda t))^{\alpha}\}(k)[t_1]=\sum_{l=0}^{k}\binom{\alpha}{l}(G(0)[t_1])^{\alpha-l}\hat{B}_{k,l}(G(1)[t_1],G(2)[t_1],\dots),k\geq 0.$$

Equation (5.17) transformed at t_1 reads as

$$\begin{aligned}0=(k+1)&\sum_{l=1}^{k+1}\binom{\alpha}{l}(X_2(1)[t_1])^{\alpha-l}\hat{B}_{k+1,l}(2X_2(2)[t_1],3X_2(3)[t_1]\dots)\\&+\gamma\sum_{l=0}^{k}\binom{-\alpha-1}{l}(t_1)^{-\alpha-1-l}\sum_{j=0}^{k-l}\binom{\alpha}{j}(G(0)[t_1])^{\alpha-j}\hat{B}_{k-l,j}(G(1)[t_1],G(2)[t_1],\dots)\end{aligned} \tag{5.18}$$

and the initial conditions are (according to (5.8))

$$X_2(0)[t_1]=X_1(0)[t_1]=\sum_{k=0}^{\infty}X_1(k)[t_0](t_1-t_0)^k$$
$$X_2(1)[t_1]=X_1(1)[t_1]=\sum_{k=0}^{\infty}(k+1)X_1(k+1)[t_0](t_1-t_0)^k.$$

The approximate solution for $t \in I_2$ then is

$$x_2(t) = \sum_{k=0}^{\infty} X_2(k)[t_1](t-t_1)^k.$$

Further steps for $t \in I_i$, $i \geq 3$ employ again recurrent equation (5.18), only with shifted indexes (e.g., X_2 becomes X_i and t_1 becomes t_{i-1}). The process of the i-th step follows the path of the second step.

The paper [112] presents numerical results for a concrete example of the initial value problem (5.13)+(5.14) together with their comparison with results obtained by Matlab function ddesd. In particular, the following problem is considered

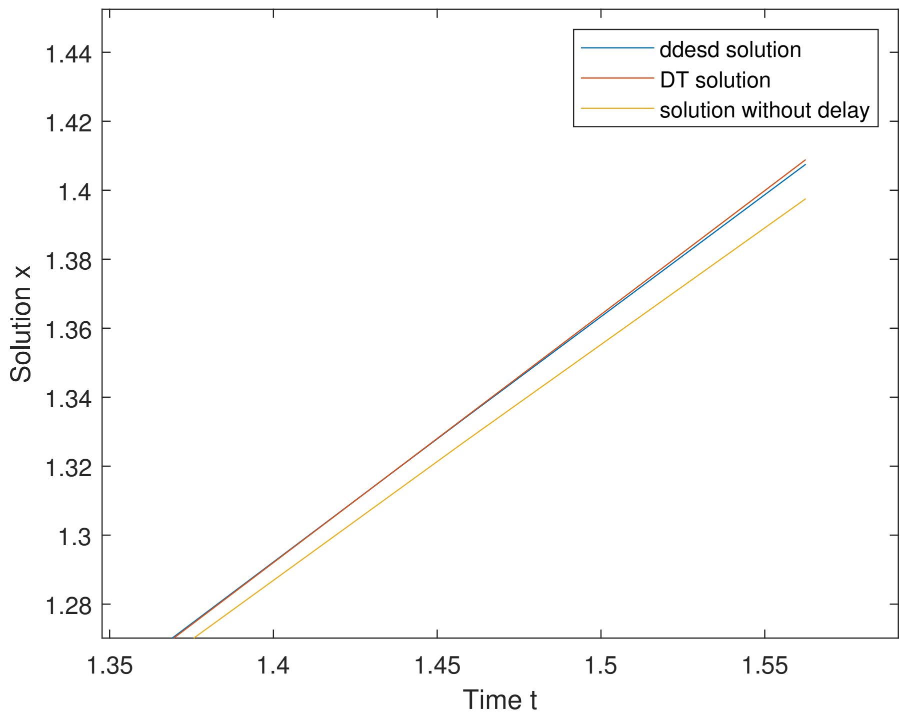
$$((x'(t))^3)' + \left(\frac{3}{4}\right)^4 \frac{1}{t^4}(x(0.8t))^3 = 0, \quad x(1) = 1, \quad x'(1) = \frac{3}{4}, \tag{5.19}$$

with the initial function

$$\phi(t) = 1 + \frac{3}{4}(t-1), \quad t \in (0,1]. \tag{5.20}$$

The first step of the algorithm takes place on the interval $[t_0, t_1] = [1, \frac{10}{8}]$ and the second step on the interval $[t_1, t_2] = [\frac{10}{8}, \frac{100}{64}]$ (because $t_i = \frac{t_0}{\lambda^i}$).

The results are illustrated in the figure. There we see, how the approximate solutions of the delayed problem (5.19)+(5.20) tend to rise more at the end of the interval $[t_0, t_2]$ compared to the original undelayed solution of the initial value problem (5.12). We can observe, and it is discussed in more depth above the table of numerical results in [112], that our algorithm provides an approximate solution that is very close to the solution calculated by the built-in Matlab function ddesd designed for solving delay differential equations.



Conclusion

We conclude the thesis with some remarks presenting further possible research directions and open problems and with a brief summary.

Remark 5.1. Further possible work in a search for asymptotic formulas of nonoscillatory solutions is straightforward and concerns the generalized Euler-type equation in the subcritical case (2.2). The formula for its second nonoscillatory solution linearly independent to (2.3) is still to be found.

Another potential direction would be turning the attention to formulas for oscillatory solutions. Some of the formulas for oscillatory solutions of (1.22) in the supercritical case, where proposed by Došlý in [21], but as far as we know, this area remains unexplored.

Remark 5.2. Based on the results of Section 3.2 for perturbations of general conditionally equations and their comparison with those for the perturbed Euler-type equation in Section 3.1, we suppose that we can study perturbations of equation (3.11) with $\delta = \frac{1}{2q}$ and find a perturbation such that the obtained perturbed equation is conditionally oscillatory. More generally, we conjecture that the equation with an arbitrary number of iterated logarithmic terms

$$(\Phi(x'))' + \left(c(t) + \sum_{j=0}^{n} \frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,ds)^2 \operatorname{Log}_j^2(\int^t R^{-1}(s)\,ds)} \right) \Phi(x) = 0 \quad (5.21)$$

is, under certain assumptions, conditionally oscillatory (here $\operatorname{Log}_0 t := 1$). This would generalize the result of [58] concerning generalized Euler-type equation (1.19) and give us the possibility to generalize the oscillation and nonoscillation criteria of the paper [66] to the case when we study perturbations of equation (5.21), similarly as in [20, 65], where perturbations of generalized Euler-type equation (1.19) are studied. Another task would be to find the asymptotics of nonoscillatory solutions of equation (5.21).

Remark 5.3. Interesting problems arise from the approach of Fujimoto and Yamaoka ([68, 69]), who started to consider second-order half-linear equations with function $p(t)$ instead of the number p in the operator Φ. A natural question from our perspective is to what extent the Riccati technique can be applied and if there could exist a variant of the modified Riccati transformation which would be of use.

Remark 5.4. The qualitative theory of delayed and neutral half-linear equations is still a rather new research field where many branches deserve exploration. Compared to ordinary half-linear equations, one of the disadvantages is that, within the Riccati technique, only Riccati type inequalities can be applied. This fact leads to the bigger number of oscillation criteria, which are currently proved, whereas nonoscillation criteria are harder to achieve.

Remark 5.5. The applicability of the differential transform method for finding the numerical solutions of differential equations is limited by the proposed calculating algorithms in concrete programs like Matlab. To overcome the limits of extending the solution to longer intervals, a different approach to implementing the code could help. One of the possible directions to overcome the effect of division by a number close to zero might be the concept of automatic differentiation. Another task also is to find the recurrent equations providing Taylor's polynomial of solutions of more general types, like half-linear Riemann-Weber equation and further generalizations of the half-linear Euler equation.

The focus of the thesis has been mainly on the usage of the modified Riccati technique for half-linear second-order Euler-type and generally conditionally oscillatory equations to obtain oscillation and nonoscillation criteria and to prove asymptotic formulas of nonoscillatory solutions. Our results are applicable especially for equations at the narrow boundary between oscillation and nonoscillation, for example, for half-linear Euler equation, Riemann-Weber equation, and generalized Euler-type equation. Results of our studies of half-linear neutral equations and of numerical approach within the application of the differential transform method are presented too. We have also pointed out further possible extensions of the considered research fields.

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ASYMPTOTIC FORMULAS FOR NONOSCILLATORY SOLUTIONS OF CONDITIONALLY OSCILLATORY HALF-LINEAR EQUATIONS

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ABSTRACT. We establish asymptotic formulas for nonoscillatory solutions of a special conditionally oscillatory half-linear second order differential equation, which is seen as a perturbation of a general nonoscillatory half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1,$$

where r, c are continuous functions and $r(t) > 0$.

1. Introduction

In this paper we investigate asymptotic properties of nonoscillatory solutions of a special conditionally oscillatory half-linear second order differential equation, which was constructed in [3] as a perturbation of a general half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{1}$$

where $t \in [t_0, \infty)$, r, c are continuous functions and $r(t) > 0$. In the case $p = 2$, equation (1) reduces to the linear Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0 \tag{2}$$

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and it is well known that the linear oscillation theory of (2) can be naturally extended also to half-linear equation (1). In particular, (1) is called *oscillatory* if its every nontrivial solution has infinitely many zeros tending to infinity and *nonoscillatory* otherwise.

In the whole paper we suppose that (1) is nonoscillatory. Let $d(t)$ be a positive continuous function, we say that the equation

$$(r(t)\Phi(x'))' + [c(t) + \mu d(t)]\Phi(x) = 0 \tag{3}$$

is *conditionally oscillatory* if there exists a constant $\mu_0 > 0$ such that (3) is oscillatory for $\mu > \mu_0$ and nonoscillatory for $\mu < \mu_0$.

Let $h(t)$ be a positive solution of nonoscillatory equation (1) such that $h'(t) \neq 0$ on some interval of the form $[T_0, \infty)$ and denote

$$R(t) := r(t)h^2(t)|h'(t)|^{p-2}, \quad G(t) := r(t)h(t)\Phi(h'(t)). \tag{4}$$

Under the asumptions

$$\int^{\infty} \frac{\mathrm{d}t}{R(t)} = \infty, \quad \liminf_{t\to\infty} |G(t)| > 0,$$

the authors of [3] constructed a conditionally oscillatory equation seen as a perturbation of (1) in the form

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{\mu}{h^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\right]\Phi(x) = 0. \tag{5}$$

The critical oscillation constant of this equation is $\mu_0 = \frac{1}{2q}$, where q is the conjugate number to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. In [3] it is also shown that (5) has for this constant $\mu = \mu_0$ a solution with the asymptotic formula

$$x(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}\left(1 + O\left(\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{-1}\right)\right) \quad \text{as} \quad t \to \infty. \tag{6}$$

The aim of this paper is to give more precise asymptotic formulas in terms of slowly and regularly varying functions in the case where the constant μ is less than or equal to $\frac{1}{2q}$.

The "perturbation approach", when the studied equation is regarded as a perturbation of another half-linear equation, has been also used in [7]. Here, the asymptotics of nonoscillatory solutions of

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) + \tilde{c}(t)\Phi(x) = 0, \tag{7}$$

where $\gamma_p = \left(\frac{p-1}{p}\right)^p$, was established under the assumption

$$\lim_{t\to\infty} \log t \int_t^\infty \tilde{c}(s)s^{p-1}\,\mathrm{d}s \in \left(-\infty, \tfrac{1}{2}\left(\tfrac{p-1}{p}\right)^{p-1}\right].$$

Equation (7) has been seen as a perturbation of the half-linear Euler type equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0 \tag{8}$$

with the critical constant γ_p.

In this paper, we apply the perturbation principle combined with the so called Riccati technique to get our asymptotical results for (5) with $\mu \le \frac{1}{2q}$.

2. Preliminaries

As in the linear oscillation theory, the nonoscillation of equation (1) is equivalent to the solvability of a Riccati type equation (for details see [2]). In particular, if x is an eventually positive or negative solution of the nonoscillatory equation (1) on some interval of the form $[T_0, \infty)$, then $w(t) = r(t)\Phi\left(\frac{x'}{x}\right)$ solves the Riccati type equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0. \tag{9}$$

Conversely, having a solution $w(t)$ of (9) for $t \in [T_0, \infty)$, the corresponding solution of (1) can be expressed as

$$x(t) = C\exp\left\{\int^t r^{1-q}(s)\Phi^{-1}(w)\,\mathrm{d}s\right\},$$

where Φ^{-1} is the inverse function of Φ and C a constant.

Using the concept of perturbations it appears useful to deal with the so called modified (or generalized) Riccati equation. Let h be a positive solution of (1) and $w_h(t) = r(t)\Phi\left(\frac{h'}{h}\right)$ be the corresponding solution of the Riccati equation (9). Let us consider another nonoscillatory equation

$$(r(t)\Phi(x'))' + C(t)\Phi(x) = 0 \tag{10}$$

and let $w(t)$ be a solution of the Riccati equation associated with (10). Then $v(t) = (w(t) - w_h(t))h^p(t)$ solves the modified Riccati equation

$$v' + (C(t) - c(t))h^p + pr^{1-q}h^p P(\Phi^{-1}(w_h), w) = 0, \tag{11}$$

where

$$P(u,v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0,$$

with the equality $P(u,v) = 0$ if and only if $v = \Phi(u)$. Equation (11), in this form, was derived e.g. in [1]. We deal with this equation in a slightly different, but still equivalent, form

$$v' + (C(t) - c(t))h^p + (p-1)r^{1-q}h^{-q}|G|^q F\left(\frac{v}{G}\right) = 0, \tag{12}$$

where $G(t)$ is defined by (4) and

$$F(u) = |u+1|^q - qu - 1. \tag{13}$$

Regularly and slowly varying functions in the sense of Karamata (see [4], [5] and the references therein) have an important role in half-linear theory, see e.g. [6]. Let us recall their nomenclature.

Let a continuously differentiable function $J(t)\colon [T_0, \infty) \to (0, \infty)$ be such that

$$J'(t) > 0 \quad \text{for } t \geq T_0, \qquad \lim_{t\to\infty} J(t) = \infty$$

and let $g(t), \varepsilon(t)$ be some measurable functions satisfying

$$\lim_{t\to\infty} g(t) = g \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \varepsilon(t) = \varrho \in \mathbb{R}.$$

According to the terminology of the above mentioned papers a positive measurable function $f(t)$, such that $f \circ J^{-1}$ is defined for all large t, and which can be expressed in the form

$$f(t) = g(t)\exp\left\{\int_{t_0}^{t} \frac{J'(s)\varepsilon(s)}{J(s)}\,\mathrm{d}s\right\}, \qquad t \geq T_0$$

for some $T_0 > t_0$, is called *generalized regularly varying function of index* ϱ *with respect to* J (the notation $f \in RV_J(\varrho)$ is then used). If $g(t) \equiv g$ (a constant), $f(t)$ is called *normalized regularly varying function.* For $\varrho = 0$ the terminology *(normalized) slowly varying function* is used.

Let us only remark that in [6] this terminology is introduced for $J(t)$ defined on the interval $[0,\infty)$ but since the point is in describing the asymptotic behavior, the interval of existence of the function $f(t)$ is sufficient even for $J(t)$.

3. Main results

In this section we establish asymptotic formulas for nonoscillatory solutions of conditionally oscillatory equation (5) in the cases $\mu < \frac{1}{2q}$ and $\mu = \frac{1}{2q}$, respectively.

Theorem 1. *Suppose that* (1) *is nonoscillatory and posseses a positive solution $h(t)$ such that $h'(t) \neq 0$ for large t and let*

$$\int^{\infty} \frac{\mathrm{d}t}{R(t)} = \infty, \tag{14}$$

and

$$\liminf_{t\to\infty} |G(t)| > 0. \tag{15}$$

If $\mu < \frac{1}{2q}$, then the conditionally oscillatory equation (5) *has a pair of solutions given by the asymptotic formula*

$$x_i = h(t)\left(\int^{t} R^{-1}(s)\,\mathrm{d}s\right)^{(q-1)\lambda_i} L_i(t),$$

where λ_i are zeros of the quadratic equation

$$\frac{q}{2}\lambda^2 - \lambda + \mu = 0 \tag{16}$$

and $L_i(t)$ are generalized normalized slowly varying functions of the form $L_i(t) = \exp\left\{\int^{t} \frac{\varepsilon_i(s)}{R(s)\int^{s} R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}$ and $\varepsilon_i(t) \to 0$ for $t \to \infty$.

P r o o f. We are looking for solutions of the modified Riccati equation associated with (5), which reads as

$$v'(t) + \frac{\mu}{R(t)(\int^{t} R^{-1}(s)\,\mathrm{d}s)^2} + (p-1)r^{1-q}(t)h^{-q}(t)|G(t)|^q F\left(\frac{v(t)}{G(t)}\right) = 0, \tag{17}$$

where G is defined in (4) and F in (13).

Assumptions (14) and (15) imply the convergence of the integral $\int^{\infty} r^{1-q}(t)h^{-q}(t)|G(t)|^q F\left(\frac{v(t)}{G(t)}\right) \mathrm{d}t$, from which follows (see [3]) that $v(t) \to 0$ and $\frac{v(t)}{G(t)} \to 0$ for $t \to \infty$.

Let $C_0[T, \infty)$ be the set of all continuous functions on the interval $[T, \infty)$ (concrete T will be specified later) which converge to zero for $t \to \infty$ and let us consider a set of functions

$$V = \{\omega \in C_0[T, \infty) : \ |\omega(t)| < \varepsilon, \ \ t \geq T\},$$

where $\varepsilon > 0$ is so small that

$$\frac{1}{\sqrt{1-2q\mu}}(q+1)\,\varepsilon < \frac{1}{2}. \tag{18}$$

Let us also observe that the fact

$$\frac{1}{\sqrt{1-2q\mu}}\left(\frac{q}{2}+1\right)\varepsilon \leq 1 \tag{19}$$

is implied.

Let us denote the roots of the quadratic equation (16) for $\mu < \frac{1}{2q}$ as

$$\lambda_1 = \frac{1-\sqrt{1-2q\mu}}{q}, \quad \lambda_2 = \frac{1+\sqrt{1-2q\mu}}{q}.$$

We assume that two solutions of the modified Riccati equation (17) are in the form

$$v_i(z,t) := \frac{\lambda_i + z(t)}{\int^t R^{-1}(s)\,\mathrm{d}s},$$

for $t \in [T, \infty)$ and $z \in V$, $i = 1, 2$. Substituting this function and its derivative into (17), we have

$$z'(t) + \frac{\mu - \lambda_i - z(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} + (p-1)r^{1-q}(t)h^{-q}(t)|G(t)|^q F\left(\frac{v_i(z,t)}{G(t)}\right)\int^t R^{-1}(s)\,\mathrm{d}s = 0$$

which can be rewritten as

$$z'(t) \quad + \quad \frac{(-1+\lambda_i q)z(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \frac{1}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}E_i(z,t) = 0, \tag{20}$$

where

$$E_i(z,t) := \mu - \lambda_i - \lambda_i q z(t) + (p-1)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2 G^2(t) F\left(\frac{v_i(z,t)}{G(t)}\right).$$

This means that looking for solutions v_i of the modified Riccati equation (17) is equivalent to looking for solutions z_i of the equation (20). In the next we shall show that two solutions of (20) can be found through the Banach fixed-point theorem used onto suitable integral operators.

Firstly, let us turn our attention to the behavior of the function $F(u)$, which plays an important role in estimating of certain needed terms.

Studying the behavior of $F(u)$ and $F'(u)$ for u in a neighbourhood of 0, we have

$$\begin{aligned} F(u) &= \frac{F''(0)}{2}u^2 + \frac{F'''(\zeta)}{6}u^3 \\ &= \frac{q(q-1)}{2}u^2 + \frac{q(q-1)(q-2)}{6}|1+\zeta|^{q-3}\operatorname{sgn}(1+\zeta)u^3, \end{aligned}$$

where ζ is between 0 and u. For $|u| < \frac{1}{2}$ and hence also $|\zeta| < \frac{1}{2}$ there exists a positive constant M_q such that

$$\left|\frac{q(q-2)}{6}|1+\zeta|^{q-3}\operatorname{sgn}(1+\zeta)\right| \le M_q \qquad \text{for} \quad q>1.$$

Therefore

$$\left|F(u) - \frac{q(q-1)}{2}u^2\right| \le (q-1)M_q|u|^3. \tag{21}$$

Similarly,

$$F'(u) = F''(0)u + \frac{F'''(\zeta')}{2}u^2 = q(q-1)u + \frac{q(q-1)(q-2)}{2}|1+\zeta'|^{q-3}\operatorname{sgn}(1+\zeta')u^2,$$

where ζ' is between 0 and u. Again, considering $|\zeta'| < \frac{1}{2}$ we have

$$|F'(u) - q(q-1)u| \le 3(q-1)M_q|u|^2. \tag{22}$$

Now, let us denote $J(t) := \int^{t} R^{-1}(s)\,\mathrm{d}s$, then the estimate of the function $E_i(z,t)$ for $z \in V$, reads as

$$
\begin{aligned}
|E_i(z,t)| = \Big| & \mu - \lambda_i - \lambda_i q z(t) + \frac{q}{2} J^2(t) v_i^2(z,t) \\
& + (p-1) J^2(t) G^2(t) F\left(\frac{v_i(z,t)}{G(t)}\right) - \frac{q}{2} J^2(t) v_i^2(z,t) \Big| \\
\le & \left|\mu - \lambda_i - \lambda_i q z(t) + \frac{q}{2}(\lambda_i + z(t))^2\right| \\
& + \left|(p-1) J^2(t) G^2(t) \left[F\left(\frac{v_i(z,t)}{G(t)}\right) - \frac{q(q-1)}{2}\left(\frac{v_i(z,t)}{G(t)}\right)^2\right]\right| \\
\le & \frac{q}{2}|z(t)|^2 + \frac{M_q |\lambda_i + z(t)|^3}{|G(t)||J(t)|} \\
\le & \frac{q}{2}|z(t)|^2 + \frac{K M_q |\lambda_i + z(t)|^3}{|J(t)|},
\end{aligned}
$$

where (21) was used for $u = \frac{v_i}{G}$ and $K := \sup\limits_{t \ge T} \frac{1}{|G(t)|}$ is a finite constant for T sufficiently large because of (15). According to (14) there exists T_1 such that the last term in the previous inequality is less than ε^2 and therefore

$$
|E_i(z,t)| \le \frac{q}{2}\varepsilon^2 + \varepsilon^2 \le \varepsilon^2\left(\frac{q}{2} + 1\right) \tag{23}
$$

for $t \ge T_1$.

Furthermore, for $z_1, z_2 \in V$ we have

$$
\begin{aligned}
& |E_i(z_1,t) - E_i(z_2,t)| \\
= & \left| -\lambda_i q (z_1 - z_2) + (p-1) J^2(t) G^2(t) \left[F\left(\frac{v_i(z_1,t)}{G(t)}\right) - F\left(\frac{v_i(z_2,t)}{G(t)}\right)\right]\right|,
\end{aligned}
$$

which, by the mean value theorem with a suitable $z(t) \in V$, becomes

$$
\begin{aligned}
= \Big| & -\lambda_i q (z_1 - z_2) + q J(t) v_i(z,t)(z_1 - z_2) \\
& + (p-1) J(t) G(t) F'\left(\frac{v_i(z,t)}{G(t)}\right)(z_1 - z_2) - q J(t) v_i(z,t)(z_1 - z_2) \Big| \\
\le \Big| & -\lambda_i q + q J(t) v_i(z,t) \\
& + (p-1) J(t) G(t) \left(F'\left(\frac{v_i(z,t)}{G(t)}\right) - q(q-1)\frac{v_i(z,t)}{G(t)}\right) \Big| \cdot \|z_1 - z_2\|
\end{aligned}
$$

$$\leq \left(q|z(t)| + \left|\frac{3KM_q(\lambda_i + z(t))^2}{J(t)}\right|\right) \cdot \|z_1 - z_2\|,$$

where (22) was used. Similarly as in the previous estimate, there exists T_2 such that the middle term in the last row of the inequality is less than ε and hence

$$|E_i(z_1, t) - E_i(z_2, t)| \leq \varepsilon(q + 1) \cdot \|z_1 - z_2\| \tag{24}$$

for $t \in [T_2, \infty)$.

Now, let us consider the pair of functions

$$r_i(t) := \exp\left\{\int^t \frac{-1 + \lambda_i q}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}, \qquad i = 1, 2.$$

Then equation (20) is equivalent to

$$(r_i(t)z(t))' + r_i(t)\frac{1}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}E_i(z, t) = 0. \tag{25}$$

For $i = 1$, we have a function

$$r_1(t) = \exp\left\{\int^t \frac{-1 + \lambda_1 q}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\} = \exp\left\{\int^t \frac{-\sqrt{1 - 2q\mu}}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}$$

and it is easy to see that $r_1(t) \to 0$ for $t \to \infty$.

Finally, let us define an integral operator F_1 on the set of functions V by

$$(F_1 z)(t) = \frac{1}{r_1(t)}\int_t^\infty \frac{r_1(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}E_1(z, s)\,\mathrm{d}s.$$

We observe that

$$\int_t^\infty \frac{r_1(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s = \frac{r_1(t)}{\sqrt{1 - 2q\mu}}.$$

Taking $T = \max\{T_1, T_2\}$, by (23) and (18) we have

$$\begin{aligned} |(F_1 z)(t)| &\leq \frac{1}{r_1(t)}\int_t^\infty \frac{r_1(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}|E_1(z, s)|\,\mathrm{d}s \\ &\leq \frac{1}{\sqrt{1 - 2q\mu}}\left(\frac{q}{2} + 1\right)\varepsilon^2 \leq \varepsilon, \end{aligned}$$

which means that F_1 maps the set V into itself, and by (24) and (19) we see that

$$|(F_1z_1)(t) - (F_1z_2)(t)| \leq \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{R(s) \int^s R^{-1}(\tau)\,\mathrm{d}\tau} |E_1(z_1, s) - E_1(z_2, s)|\,\mathrm{d}s$$
$$\leq \|z_1 - z_2\| \frac{1}{\sqrt{1-2q\mu}} \varepsilon(q+1) < \frac{1}{2}\|z_1 - z_2\|,$$

which implies that F_1 is a contraction. Using the Banach fixed-point theorem we can find a function $z_1(t)$, that satisfies $z_1 = F_1z_1$. That means that $z_1(t)$ is a solution of (25) and also of (20) and $v_1(t) = \frac{\lambda_1 + z_1(t)}{\int^t R^{-1}\,\mathrm{d}s}$ is a solution of (17).

For $i = 2$ we have

$$r_2(t) = \exp\left\{\int^t \frac{-1 + \lambda_2 q}{R(s) \int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\} = \exp\left\{\int^t \frac{\sqrt{1-2q\mu}}{R(s) \int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}$$

and we define an integral operator F_2 by

$$(F_2z)(t) = -\frac{1}{r_2(t)} \int^t \frac{r_2(s)}{R(s) \int^s R^{-1}(\tau)\,\mathrm{d}\tau} E_2(z, s)\,\mathrm{d}s.$$

Since

$$\int^t \frac{r_2(s)}{R(s) \int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s = \frac{r_2(t) - c}{\sqrt{1-2q\mu}},$$

where c is a positive suitable constant, the inequality

$$\frac{1}{r_2(t)} \int^t \frac{r_2(s)}{R(s) \int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s \leq \frac{1}{\sqrt{1-2q\mu}}$$

holds for t sufficiently large, as $r_2(t) \to \infty$ for $t \to \infty$. Taking $T = \max\{T_1, T_2\}$, the estimates for the operator F_2 are the same as in the previous case and we can find a fixed point $z_2(t)$ satisfying $F_2z_2 = z_2$. Thus $z_2(t)$ solves (25) and $v_2(t) = \frac{\lambda_2 + z_2(t)}{\int^t R^{-1}\,\mathrm{d}s}$ solves the modified Riccati equation (17).

Expressing the solutions of the standard Riccati equation for (5) corresponding to the solutions $v_i(z_i, t)$ of the modified Riccati equation, we have

$$\begin{aligned} w_i(t) &= h^{-p}(t)v_i(z_i,t) + w_h(t) = w_h(t)\left(1 + \frac{v_i(z_i,t)}{h^p(t)w_h(t)}\right) \\ &= w_h(t)\left(1 + \frac{\lambda_i + z_i(t)}{h^p(t)w_h(t)\int^t R^{-1}\,\mathrm{d}s}\right) = w_h(t)\left(1 + \frac{\lambda_i + z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right). \end{aligned}$$

Since solutions of (5) are given by the formula $x(t) = \exp\left\{\int^t r^{1-q}(s)\Phi^{-1}(w)\,\mathrm{d}s\right\}$, we need to express

$$\begin{aligned} & r^{1-q}(t)\Phi^{-1}(w_i) \\ &= \frac{h'(t)}{h(t)}\left(1 + \frac{\lambda_i + z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right)^{q-1} \\ &= \frac{h'(t)}{h(t)}\left(1 + (q-1)\frac{\lambda_i + z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s} + o\left(\frac{\lambda_i + z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right)\right) \\ &= \frac{h'(t)}{h(t)} + \frac{(q-1)\lambda_i}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \frac{(q-1)z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} + o\left(\frac{\lambda_i + z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right). \end{aligned}$$

Because

$$\begin{aligned} o\left(\frac{\lambda_i + z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right) &= \frac{R(t)\int^t R^{-1}(s)\,\mathrm{d}s\; o\left(\frac{\lambda_i + z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} \\ &= \frac{o(\lambda_i + z_i(t))}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} \end{aligned}$$

holds for large t, the pair of solutions of (5) for $i = 1, 2$ is in the form

$$\begin{aligned} x_i(t) = \exp\Bigg\{ & \log h(t) + \log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{(q-1)\lambda_i} \\ & + \int^t \frac{(q-1)z_i(s) + o(\lambda_i + z_i(s))}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\Bigg\}. \end{aligned}$$

As $z_i \in V$ and hence $z_i(t) \to 0$ for $t \to \infty$, the statement of the theorem holds for $\varepsilon_i(t) = (q-1)z_i(t) + o(\lambda_i + z_i(t))$. $\square$

Now let us present the asymptotic formula in case $\mu = \frac{1}{2q}$, which gives an improved version of (6).

Theorem 2. *Let the assumptions of the previous theorem be satisfied and let $\mu = \frac{1}{2q}$. Then equation* (5) *has a solution of the form*

$$x = h(t)\left(\int^{t} R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} L(t), \tag{26}$$

where $L(t)$ is a generalized normalized slowly varying function of the form $L(t) = \exp\left\{\int^{t} \frac{\varepsilon(s)}{R(s)\int^{s} R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}$ and $\varepsilon(t) \to 0$ for $t \to \infty$.

Proof. For $\mu = \frac{1}{2q}$ the quadratic equation (16) has a double root $\lambda = \frac{1}{q}$. We assume the solution of modified Riccati equation to be in the form (for $z \in V$)

$$v(z,t) = \frac{\frac{1}{q} + z(t)}{\int^{t} R^{-1}(s)\,\mathrm{d}s},$$

which gives, after substituting into the modified Riccati equation (17) for $\mu = \frac{1}{2q}$,

$$z'(t) + \frac{-z(t) - \frac{1}{2q}}{R(t)\int^{t} R^{-1}(s)\,\mathrm{d}s} + (p-1)r^{1-q}(t)h^{-q}(t)|G(t)|^{q} F\left(\frac{v(z,t)}{G(t)}\right)\int^{t} R^{-1}(s)\,\mathrm{d}s = 0.$$

Let us denote

$$E(z,t) = -z(t) - \frac{1}{2q} + (p-1)\left(\int^{t} R^{-1}(s)\,\mathrm{d}s\right)^{2} G^{2}(t) F\left(\frac{v(z,t)}{G(t)}\right)$$

and let us consider an integral operator F_3

$$(F_3 z)(t) = \int_t^{\infty} \frac{1}{R(s)\int^{s} R^{-1}(\tau)\,\mathrm{d}\tau} E(z,s)\,\mathrm{d}s$$

on a set of continuous functions

$$V = \{\omega \in C_0[T,\infty) :\ |\omega(t)| < \varepsilon,\ t \geq T\},$$

where T and ε are to be established similarly as in the proof of the previous theorem. Then the solution of modifed Riccati equation and also the solution of the studied equation can be found in almost the same manner as for the previous statement. $\square$

Remark 1. If $r(t) \equiv 1, c(t) = \gamma_p t^{-p}$ and $h(t) = t^{\frac{p-1}{p}}$ then the conditionally oscillatory equation (5) with $\mu = \frac{1}{2q}$, seen as a perturbation of the Euler equation (8), becomes the Euler-Weber (or alternatively Riemann-Weber) half-linear differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right] \Phi(x) = 0$$

with the so-called critical coefficient $\mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$. The asymptotic formula (26) then reduces to the formula given in [7, Theorem 2].

Remark 2. For the Euler-Weber half linear equation also the asymptotic formula for its second linearly independent solution is known (see [8]). An open question remains whether the second linearly independent solution of (5) with $\mu = \frac{1}{2q}$ could be found in a similar form

$$x_2(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}\left(\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\right)^{\frac{2}{p}} L_2(t),$$

where

$$L_2(t) = \exp\left\{\int^t \frac{\varepsilon_2(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau \log\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)}\,\mathrm{d}s\right\}$$

and $\varepsilon_2(t) \to 0$ for $t \to \infty$.

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SPECIAL ISSUE PAPER

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Nonoscillatory solutions of half-linear Euler-type equation with n terms

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We consider the half-linear Euler-type equation with n terms

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n-1} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + \frac{\mu}{t^p \mathrm{Log}_n^2 t}\right)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\mathrm{sgn}x$$

in the subcritical case when $0<\mu<\mu_p$ and $p>1$. The solutions of this nonoscillatory equation cannot be found in an explicit form and can be studied only asymptotically. In this paper, with the use of the perturbation principle, modified Riccati technique, and the fixed point theorem, we establish an asymptotic formula for one of its solutions.

KEYWORDS
asymptotic formulas, Euler equation, half-linear differential equation, nonoscillatory solution, perturbation

MSC CLASSIFICATION
34C10

1 | INTRODUCTION

The aim of the paper is to find an asymptotic formula for a nonoscillatory solution of half-linear Euler-type differential equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n-1} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + \frac{\mu}{t^p \mathrm{Log}_n^2 t}\right)\Phi(x) = 0 \tag{1}$$

for $0<\mu<\mu_p$, $t\in[T,\infty)$. The operator Φ is defined as $\Phi(x):=|x|^{p-1}\mathrm{sgn}x$, $p>1$, $n \in \mathbb{N}$, γ_p and μ_p are the constants

$$\gamma_p := \left(\frac{p-1}{p}\right)^p, \quad \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$$

and $\mathrm{Log}_j t$ are products of iterated logarithmic functions:

$$\mathrm{Log}_j t := \prod_{k=1}^{j} \log_k t, \quad \log_1 t := \log t, \quad \log_k t := \log_{k-1}(\log t), \quad k \geq 2.$$

The studied Equation (1) is a special case of a general half-linear second-order differential equation

$$L[x] := (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \tag{2}$$

where $x=x(t)$, functions $r(t),c(t)$ are continuous, and $r(t)$ is positive on the interval of consideration. The solution space of (2) is linear (but not additive) and contains either oscillatory or nonoscillatory solutions. Within the studies of Equation (2) in some neighborhood of infinity, ie, $t\in[T,\infty)$ for some T, its solution can be classified as oscillatory, if it has got infinitely many zeros tending to infinity, and nonoscillatory otherwise. Oscillatory and nonoscillatory solutions cannot coexist; hence, half-linear equations are said to be oscillatory or nonoscillatory according to behavior of their all solutions (for more information, see the basic literature[1] summing up the results for half-linear equations up to the year 2005).

Describing asymptotic properties of solutions is one of the main tasks of qualitative theory of differential equations. Asymptotic behavior of solutions is often being classified with the use of the theory of regularly varying functions in the sense of Karamata. Let us recall the following notation (for more information, see, for example, the monograph[2]).

A measurable function $f:[a,\infty)\to(0,\infty)$ is called regularly varying (at infinity) of index ϑ (and we write $f\in RV(\vartheta)$) if

$$\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)}=\lambda^{\vartheta}\quad\text{for every}\quad\lambda>0.$$

If $\vartheta=0$, f is called slowly varying. The representation theorem (see, for example, Řehák[3]) says that $f\in RV(\vartheta)$ if and only if it can be expressed in the form

$$f(t)=\varphi(t)t^{\vartheta}\exp\left\{\int_a^t\frac{\psi(s)}{s}\mathrm{d}s\right\},$$

where $t\geq a$ for some $a>0$, φ and ψ are measurable functions such that $\lim_{t\to\infty}\varphi(t)$ is finite and positive and $\lim_{t\to\infty}\psi(t)=0$.

Half-linear equations have been studied in the framework of regularly varying functions, for example, by the group of authors Jaroš, Kusano, Manojlović, Marić, Tanigawa, and Řehák, see previous works[3-6] and references therein. Considering the latest papers that sum up, improve, and extend previous results in this field, Řehák[3] provided an exhaustive overview of asymptotic formulas for (normalized) regularly varying solutions of (2) in the case when r is positive and c negative on $[T,\infty)$. Furthermore, Kusano and Manojlović[5] established asymptotic formulas for nonoscillatory solutions of (2) depending on the rate of decay toward zero of the positive function

$$Q_C=t^{p-1}\int_t^{\infty}c(s)\mathrm{d}s-C$$

as $t\to\infty$, where $C<\dfrac{\gamma_p}{p-1}$. However, the results are inapplicable to our Equation (1), because the constant C in this case does not satisfy strict inequality, but equality. The reason is that Equation (1) lies on the threshold between oscillation and nonoscillation. For such equations, it was shown, for example, in Pátíková[7] that it can be useful to regard the studied equation as a perturbation of a nonoscillatory equation with less terms.

Equation (1) can be seen as a perturbation and in some sense also a generalization of the half-linear Euler equation

$$(\Phi(x'))'+\frac{\gamma}{t^p}\Phi(x)=0. \tag{3}$$

The critical coefficient $\gamma=\gamma_p$ is its oscillation constant: Replaced by a greater constant, the Euler equation becomes oscillatory, and for γ_p and smaller constants, it is nonoscillatory. In view of this property, one can say that the Euler equation is conditionally oscillatory. Equation (3) in the case $\gamma=\gamma_p$ has a pair of linearly independent nonoscillatory solutions (see, for example, Došlý and Řehák[1], chapter 1.4.2)

$$x_1(t)=t^{\frac{p-1}{p}},\quad x_2(t)=t^{\frac{p-1}{p}}\log^{\frac{2}{p}}(t)(1+o(1))\quad\text{as}\quad t\to\infty.$$

If $\gamma<\gamma_p$, then (3) has a pair of solutions

$$x_{1,2}=t^{\lambda_{1,2}^{q-1}},$$

where $\lambda_{1,2}$ are zeros of $|\lambda|^q-\lambda+\dfrac{\gamma}{p-1}=0$, as can be seen by a direct substitution.

The Euler equation (3) with the critical constant can be perturbed so that the resulting equation is again conditionally oscillatory. Such a one term perturbation leads to the half-linear Riemann-Weber equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right)\Phi(x) = 0, \tag{4}$$

whose oscillation constant is $\mu = \mu_p$. In this critical case for $\mu = \mu_p$, Equation (4) is nonoscillatory and possesses a pair of linearly independent solutions (see Elbert and Schneider[8])

$$x_1(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t(1 + o(1)), \quad x_2(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t \log^{\frac{2}{p}} (\log t)(1 + o(1)) \quad \text{as} \quad t \to \infty.$$

In the subcritical case for $0 < \mu < \mu_p$, the asymptotic formulas of a pair of linearly independent solutions were found in Pátíková[7], namely,

$$x_{1,2}(t) = t^{\frac{p-1}{p}} (\log t)^{\nu_{1,2}} (1 + o(1)) \quad \text{as} \quad t \to \infty,$$

where $\nu_{1,2} = \frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p} \lambda_{1,2}$ and $\lambda_{1,2}$ are zeros of $\frac{\lambda^2}{4\mu_p} - \lambda + \mu = 0$. These formulas can be obtained also as a special case of more general results proved in Došlý[9] (a link between these two approaches can be found in Pátíková[10]).

Finally, the Euler-type equation (also called the generalized Riemann-Weber equation) with an arbitrary number of perturbation terms (1) was studied in Elbert and Schneider,[8] and it was shown that its oscillation constant is again $\mu = \mu_p$ and that the equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \operatorname{Log}_j^2 t}\right)\Phi(x) = 0 \tag{5}$$

has a pair of linearly independent solutions

$$x_1(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t(1 + o(1)), \quad x_2(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t(1 + o(1)) \quad \text{as} \quad t \to \infty.$$

Notice that all the above solutions are regularly varying of certain indexes and the functions of the form $(1+o(1))$ are slowly varying functions (see other studies[7,11]).

In this paper, we consider the subcritical case of (1) when $0<\mu<\mu_p$ and reveal asymptotics of one of its solutions. Our motivation comes among others from Pátíková,[10] where the asymptotic formulas for solutions of (1) were proposed in the case $n=2$. Let us point out that also further generalizations of (1) are a subject of recent studies, see Došlý[9] and Hasil[12] and references given therein.

2 | PRELIMINARIES

An important role in the proof of the main result is played by the so-called Riccati technique, which is based on the following facts (see, for example, Došlý and Řehák[1]). If $x(t) \neq 0$ is a solution of (2) on I, then $w = r(t)\Phi\left(\frac{x'(t)}{x(t)}\right)$ is a solution of the half-linear Riccati equation

$$w(t) + c(t) + (p-1)r^{1-q}(t)|w(t)|^q = 0, \quad q = \frac{p}{p-1}. \tag{6}$$

Within the perturbation approach, the Riccati equation (6) is insufficient for our purposes. The way how to employ the idea of perturbation in the Riccati equation is in making its following modification. Denote for a positive differentiable function $h(t)$

$$R(t) = r(t)h^2(t)|h'(t)|^{p-2}, \quad G(t) = r(t)h(t)\Phi(h'(t)). \tag{7}$$

Then $v(t) = h^p(t)w(t) - G(t)$ is a solution of the so-called modified Riccati equation (see, for example, Došlý and Fišnarová[13])

$$v'(t) + h(t)L[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(v, G) = 0, \tag{8}$$

where

$$H(v,\, G) = |v+G|^q - q\Phi^{-1}(G)v - |G|^q.$$

According to Došlý,[9] Equation (8) is in some sense close to the so-called approximate Riccati equation

$$u'(t) + h(t)L[h](t) + \frac{q}{2R(t)}u^2(t) = 0, \tag{9}$$

where the nonlinearity is only quadratic. The estimates of how close to each other are the solutions of (8) and (9) can be done with the use of the calculations from the proof of theorem 2 in Došlý and Ünal[14] (see also Došlý[9], lemma 1), which we formulate here in the case of our interest for $r(t)=1$, $h(t) = t^{\frac{p-1}{p}}$, $G=2\mu_p=\Gamma_p$, $R = \left(\frac{p-1}{p}\right)^{p-2} t$.

Lemma 2.1. *If* $\varepsilon\in(0,1)$ *and*

$$b(\varepsilon) = \begin{cases} \left|\dfrac{q(q-2)}{6}\right|(1+\varepsilon)^{q-3} \text{ for } & q \geq 3, \\ \left|\dfrac{q(q-2)}{6}\right|(1-\varepsilon)^{q-3} \text{ for } & q < 3, \end{cases}$$

then for $|v/2\mu_p|<\varepsilon$,

$$\left|\frac{(p-1)}{t}H(v,\,\Gamma_p) - \frac{1}{4\mu_p t}v^2\right| \leq \frac{Lb(\varepsilon)}{t}|v|^3, \tag{10}$$

where $L = \left(\frac{p-1}{p}\right)^{3-2p}$.

3 | MAIN RESULT

The main result of our paper providing an asymptotic formula of a solution of (1) reads as follows.

Theorem 3.1. *Equation (1) with* $\mu\in(0,\mu_p)$ *has a solution of the form*

$$x(t) = t^{\frac{p-1}{p}}\mathrm{Log}_{n-1}^{\frac{1}{p}} t \log_n^{\frac{2\lambda}{p}} t(1+o(1)) \quad \text{as} \quad t\to\infty, \tag{11}$$

where $\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{\mu}{\mu_p}}$.

Proof We consider Equation (1) to be a perturbation of the Euler equation (3), which is the reason for choosing its solution as the function h in the modified Riccati equation (8). The modified Riccati equation (8) for (1) with

$$h(t) = t^{\frac{p-1}{p}},\; G(t) = \Gamma_p = 2\mu_p \tag{12}$$

reads as

$$v'(t) + \sum_{j=1}^{n-1}\frac{\mu_p}{t\mathrm{Log}_j^2 t} + \frac{\mu}{t\mathrm{Log}_n^2 t} + \frac{p-1}{t}H(v(t),\,\Gamma_p) = 0, \tag{13}$$

and the approximate Riccati equation (9) is the in the form

$$u'(t) + \sum_{j=1}^{n-1}\frac{\mu_p}{t\mathrm{Log}_j^2 t} + \frac{\mu}{t\mathrm{Log}_n^2 t} + \frac{1}{4\mu_p t}u^2(t) = 0. \tag{14}$$

One can see that, with the use of the substitution $u = 4\mu_p t\frac{y'}{y}$, this is the classical Riccati equation of the linear second-order equation

$$(4\mu_p t y'(t))' + \left(\sum_{j=1}^{n-1}\frac{\mu_p}{t\mathrm{Log}_j^2 t} + \frac{\mu}{t\mathrm{Log}_n^2 t}\right)y(t) = 0,$$

which can be rewritten, using the notation $\tau = \frac{\mu}{\mu_p}$, as

$$(ty'(t))' + \left(\sum_{j=1}^{n-1} \frac{1}{4t\mathrm{Log}_j^2 t} + \frac{\tau}{4t\mathrm{Log}_n^2 t}\right) y(t) = 0. \tag{15}$$

The transformation of variable $\frac{1}{t}\mathrm{d}t = \mathrm{d}s$, $t = e^s$, $s = \log t$ leads to the equation

$$y''(s) + \left(\frac{1}{4s^2} + \sum_{j=1}^{n-2} \frac{1}{4s^2\mathrm{Log}_j^2 s} + \frac{\tau}{4s^2\mathrm{Log}_{n-1}^2 s}\right) y(s) = 0.$$

First, we use the change of variable and substitution $s = e^{u_1}$, $y(s) = \sqrt{s}z_1(u_1)$, and if $n>2$, we continue in the same manner with $u_i = e^{u_{i+1}}$, $z_i(u_i) = \sqrt{u_i}z_{i+1}(u_{i+1})$ for $i=2,...,n-1$ to obtain the equation

$$z''_{n-1}(u_{n-1}) + \frac{\tau}{4u_{n-1}^2} z_{n-1}(u_{n-1}) = 0.$$

This equation has a couple of solutions $z_{n-1}(u_{n-1}) = u_{n-1}^{\lambda_{1,2}}$, where $\lambda_{1,2}$ are two (real) zeros of the quadratic equation

$$\lambda^2 - \lambda + \frac{\tau}{4} = \lambda^2 - \lambda + \frac{\mu}{4\mu_p} = 0. \tag{16}$$

Backward transformations result into the couple of solutions of (15)

$$y_{1,2}(t) = \sqrt{\mathrm{Log}_{n-1} t}\,(\log_n t)^{\lambda_{1,2}}$$

and solutions of (14) are

$$\begin{aligned} u_{1,2} = 4\mu_p t\frac{y'}{y} &= 4\mu_p t\left(\frac{\mathrm{Log}'_{n-1} t}{2\mathrm{Log}_{n-1} t} + \frac{\lambda_{1,2}\log'_n t}{\log_n t}\right) \\ &= 2\mu_p \sum_{i=1}^{n-1} \frac{1}{\mathrm{Log}_i t} + 4\mu_p\lambda_{1,2}\frac{1}{\mathrm{Log}_n t}. \end{aligned} \tag{17}$$

Observe that for $\mu\in(0,\mu_p)$, the zeros $\lambda_{1,2}$ of (16) are in the interval (0,1). The bigger one $\lambda_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{\mu}{\mu_p}}$ lies in the interval $(\frac{1}{2}, 1)$.Now, let us introduce the function

$$\varphi(t) = \int \frac{u_1^3(s)}{s}\mathrm{d}s \tag{18}$$

and the set of functions

$$\mathcal{V} = \{v \in C[T^*, \infty),\ |v(t) - u_1(t)| \le K\varphi(t)\},$$

where T^* and K will be specified later. Since $\varphi(t)=o(u_1(t))$ as $t\to\infty$ and $u_1(t)$ is positive for large t, there exists T_0 such that $u_1(t)-K\varphi(t)>0$ for $t>T_0$. x To find a solution of the modified Riccati equation (13), we construct the integral operator

$$\mathcal{F}(v)(t) = \int_t^\infty \left(\sum_{j=1}^{n-1} \frac{\mu_p}{s\mathrm{Log}_j^2 s} + \frac{\mu}{s\mathrm{Log}_n^2 s}\right) ds + \int_t^\infty \frac{p-1}{s} H(v, \Gamma_p)\mathrm{d}s \tag{19}$$

and show that it has a fixed point on the set $\mathcal{V}$.

First, we show that the integral $\int^\infty \frac{p-1}{t} H(v, \Gamma_p)\mathrm{d}t$ converges. Be T_1 such that the estimate (10) holds. For $v \in \mathcal{V}$ and $t>T_0$, we have $0<v(t)\le u_1(t)+K\varphi(t)$, and since the function H is increasing in v for $v>0$, we observe (using Lemma 2.1 and suppressing arguments) that for $t > \max\{T_0, T_1\}$

$$\begin{aligned}\int^{\infty}\frac{p-1}{t}H(v,\Gamma_p)\mathrm{d}t &\leq \int^{\infty}\left|\frac{p-1}{t}H(v,\Gamma_p)-\frac{1}{4\mu_p t}v^2\right|\mathrm{d}t+\int^{\infty}\frac{1}{4\mu_p t}v^2\mathrm{d}t\\ &\leq Lb(\varepsilon)\int^{\infty}\frac{v^3}{t}\mathrm{d}t+\int^{\infty}\frac{v^2}{4\mu_p t}\mathrm{d}t\\ &\leq Lb(\varepsilon)\int^{\infty}\frac{(u_1+K\varphi)^3}{t}\mathrm{d}t+\int^{\infty}\frac{(u_1+K\varphi)^2}{4\mu_p t}\mathrm{d}t<\infty,\end{aligned}$$

since $\varphi(t)=o(u_1(t))$ as $t\to\infty$ and $\int^{\infty}\frac{u_1^2}{t}\mathrm{d}t<\infty$.

Next, we show that for suitably chosen constants K and T^*, the operator $\mathcal{F}$ maps $\mathcal{V}$ into itself. Using (19) together with (14), Lemma 2.1, and estimates of v, we have for t large enough

$$\begin{aligned}|\mathcal{F}(v)(t)-u_1(t)| &= \left|\int_t^{\infty}\frac{p-1}{s}H(v,\Gamma_p)\mathrm{d}s-\frac{1}{4\mu_p}\int_t^{\infty}\frac{u_1^2}{s}\mathrm{d}s\right|\\ &\leq \left|\int_t^{\infty}\frac{p-1}{s}H(v,\Gamma_p)\mathrm{d}s-\frac{1}{4\mu_p}\int_t^{\infty}\frac{v^2}{s}\mathrm{d}s\right|+\frac{1}{4\mu_p}\int_t^{\infty}\frac{|v^2-u_1^2|}{s}\mathrm{d}s\\ &\leq Lb(\varepsilon)\int_t^{\infty}\frac{v^3}{s}\mathrm{d}s+\frac{1}{4\mu_p}\int_t^{\infty}\frac{(u_1+v)|u_1-v|}{s}\mathrm{d}s\\ &\leq Lb(\varepsilon)\int_t^{\infty}\frac{(u_1+K\varphi)^3}{s}\mathrm{d}s+\frac{1}{4\mu_p}\int_t^{\infty}\frac{(2u_1+K\varphi)K\varphi}{s}\mathrm{d}s.\end{aligned}$$

The first integral satisfies

$$\begin{aligned}\int_t^{\infty}\frac{(u_1+K\varphi)^3}{s}\mathrm{d}s &= \int_t^{\infty}\frac{u_1^3}{s}\mathrm{d}s+\int_t^{\infty}\frac{3Ku_1^2\varphi+3K^2u_1\varphi^2+K^3\varphi^3}{s}\mathrm{d}s\\ &\leq \varphi(t)+o(\varphi(t))=\varphi(t)(1+o(1))\quad\text{as}\quad t\to\infty.\end{aligned}$$

Next, we have

$$\int_t^{\infty}\frac{K^2\varphi^2}{s}\mathrm{d}s=o(\varphi(t))\quad\text{as}\quad t\to\infty.$$

Finally, we show that

$$\frac{1}{4\mu_p}\int_t^{\infty}\frac{2u_1K\varphi}{s}\mathrm{d}s\leq K\lambda_1\varphi(t). \tag{20}$$

Up to this point, first, we show that

$$\frac{u_1^2(t)}{t}\leq -4\mu_p\lambda_1u_1'(t) \tag{21}$$

for t large enough. Indeed, using (17) and some arrangements, we arrive at the inequality

$$0\leq(2\lambda_1-1)\sum_{i=1}^{n-1}\frac{1}{\mathrm{Log}_i^2t}+2(\lambda_1-1)\sum_{1\leq i<j\leq n-1}\frac{1}{\mathrm{Log}_it\mathrm{Log}_jt}+\frac{4\lambda_1(\lambda_1-1)^n}{\mathrm{Log}_n}\sum_{i=1}^{n-1}\frac{1}{\mathrm{Log}_it}. \tag{22}$$

Observe that $\lambda_1-1\in(-\frac{1}{2},0)$, $\lambda_1(\lambda_1-1)=-\frac{\mu}{4\mu_p}\in(0,-\frac{1}{4})$ and $2\lambda_1-1\in(0,1)$. Since $\sum_{1\leq i<j<n-1}\frac{1}{\mathrm{Log}_it\mathrm{Log}_jt}=o\left(\sum_{i=1}^{n-1}\frac{1}{\mathrm{Log}_i^2t}\right)$ and $\sum_{i=1}^{n-1}\frac{1}{\mathrm{Log}_it\mathrm{Log}_nt}=o\left(\sum_{i=1}^{n-1}\frac{1}{\mathrm{Log}_i^2t}\right)$, there exists T_2 such that inequality (22) and also (21) is satisfied for $t>T_2$. Now, inequality (21) implies

$$\int_t^{\infty}\frac{u_1^3(s)}{s}\mathrm{d}s\leq 2\mu_p\lambda_1\int_t^{\infty}(-2u_1(s)u_1'(s))\mathrm{d}s,$$

which is equivalent to

$$\varphi(t) \leq 2\mu_p\lambda_1 u_1^2(t).$$

Multiplying by $\frac{u_1}{t}$ leads to the integral inequality

$$\int_t^\infty \frac{u_1(s)\varphi(s)}{s}\mathrm{d}s \leq 2\mu_p\lambda_1\int_t^\infty \frac{u_1^3(s)}{s}\mathrm{d}s$$

and with the use of the definition of φ (18), one can see that (20) holds for $t>T_2$.
In total, we have

$$|\mathcal{F}(v)(t)-u_1(t)| \leq (Lb(\varepsilon)+K\lambda_1+o(1))\varphi(t).$$

Let T_3 be so large that the term o(1) is less than or equal to 1 for $t>T_3$. Then for $T^* = \max\{T_0, T_1, T_2, T_3\}$ and for $K \geq \frac{Lb(\varepsilon)+1}{1-\lambda_1}$ (such K exists since $\lambda_1 \in (\frac{1}{2}, 1)$), we obtain

$$|\mathcal{F}(v)(t)-u_1(t)| \leq K\varphi(t).$$

Hence, $\mathcal{F}$ maps $\mathcal{V}$ into itself. All the other assumptions of the Schauder-Tichonoff fixed point theorem are satisfied too: $\mathcal{F}(\mathcal{V})$ is bounded, and since the derivatives $\mathcal{F}'(v)(t)$ are bounded on compact subintervals of $[T^*,\infty)$, the operator $\mathcal{F}(\mathcal{V})$ is also equicontinuous. Hence, the operator $\mathcal{F}(v)(t)$ has a fixed point

$$\mathcal{F}(v)(t) = v(t)$$

on $\mathcal{V}$, for which $v(t)=u_1+O(\varphi(t))$. The function $v(t)$ now generates a solution of the half-linear Riccati equation (6) $w(t) = \Phi\left(\frac{x'}{x}\right) = h^{-p}(v+G)$, from which one can express a solution of (1) in the form $x(t) = exp\{\int^t \Phi^{-1}(w(s))\mathrm{d}s\}$. In more detail, according to (12) and (17),

$$w(t) = \Phi\left(\frac{x'(t)}{x(t)}\right) = 2\mu_p t^{p-1}\left(1 + t\frac{\mathrm{Log}'_{n-1}t}{\mathrm{Log}_{n-1}t} + 2t\lambda_1\frac{\log'_n t}{\log_n t} + O(\varphi(t))\right).$$

Applying the inverse function Φ^{-1}, we have

$$\frac{x'(t)}{x(t)} = \frac{p-1}{p}\frac{1}{t}\left(1 + t\frac{\mathrm{Log}'_{n-1}t}{\mathrm{Log}_{n-1}t} + 2t\lambda_1\frac{\log'_n t}{\log_n t} + O(\varphi(t))\right)^{q-1},$$

and using the power expansion formula, we arrive at

$$\begin{aligned}\frac{x'(t)}{x(t)} &= \frac{p-1}{p}\frac{1}{t}\left(1 + (q-1)t\frac{\mathrm{Log}'_{n-1}t}{\mathrm{Log}_{n-1}t} + 2t\lambda_1(q-1)\frac{\log'_n t}{\log_n t} + O(\varphi(t))\right)\\ &= \frac{p-1}{p}\frac{1}{t} + \frac{1}{p}\frac{\mathrm{Log}'_{n-1}t}{\mathrm{Log}_{n-1}t} + \frac{2\lambda_1}{p}\frac{\log'_n t}{\log_n t} + O\left(\frac{\varphi(t)}{t}\right).\end{aligned}$$

Finally, integrating gives (11).

4 | CONCLUDING REMARKS

1. Within the framework of regularly varying functions, we can say that the solution (11) is a regularly varying function of index $\frac{p-1}{p}$ (see the properties of indexes described, for example, in the appendix of Řehák[3]) and the function $(1+o(1))$ in its formula is a slowly varying function.
2. The so-called approximate Riccati equation (14) has two linearly independent solutions u_1,u_2 described by (17). To get the solution of (13) and consequently of (1), we used u_1—the one with the larger zero of (16). The natural question arises whether also u_2 with the smaller zero of (16) could be used to find the second linearly independent solution of (1). This remains as an open problem.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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Perturbed generalized half-linear Riemann–Weber equation – further oscillation results

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Abstract. We establish new oscillation and nonoscillation criteria for the perturbed generalized Riemann–Weber half-linear equation with critical coefficients

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + \tilde{c}(t)\right)\Phi(x) = 0$$

in terms of the expression

$$\frac{1}{\log_{n+1} t}\int^t \tilde{c}(s)s^{p-1}\,\mathrm{Log}_{\,n}s\,\log_{n+1}^2 s\,\mathrm{d}s.$$

The obtained criteria complement results of [O. Došlý, *Electron. J. Qual. Theory Differ. Equ., Proc. 10'th Coll. Qualitative Theory of Diff. Equ.* **2016**, No. 10, 1–14].

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1 Introduction

Consider the half-linear differential equation of the form

$$L[x] := (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \qquad p > 1, \tag{1.1}$$

where r, c are continuous functions, $r(t) > 0$ and $t \in [T, \infty)$ for some $T \in \mathbb{R}$. The terminology *half-linear* comes from the fact that the solution space of (1.1) is homogenous, but generally not additive for $p \neq 2$. In the special case $p = 2$ this equation reduces to the linear Sturm–Liouville differential equation

$$(r(t)x')' + c(t)x = 0. \tag{1.2}$$

In this paper we deal with oscillatory properties of equations of the form (1.1). It is well known that the classical linear Sturmian theory of (1.2) can be naturally extended also to (1.1), see [8].

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In particular, (1.1) is called *oscillatory* if all of its solutions are oscillatory, i.e., it has infinitely many zeros tending to infinity. In the opposite case all solutions of (1.1) are nonoscillatory, i.e., they are eventually positive or negative and (1.1) is said to be *nonoscillatory*. Let us emphasize that oscillatory and nonoscillatory solutions of (1.1) cannot coexist.

If we suppose that (1.1) is nonoscillatory, one can study the influence of the perturbation $\tilde{c}$ on the oscillatory behavior of the equation of the form

$$(r(t)(\Phi(x'))' + (c(t) + \tilde{c}(t))\Phi(x) = 0. \tag{1.3}$$

The concrete (non)oscillation criteria measure the positiveness of the function $\tilde{c}$ (generally of arbitrary sign). If $\tilde{c}$ is "sufficiently positive" then the perturbed equation (1.3) becomes oscillatory, if $\tilde{c}$ is negative or "not too much positive", then (1.3) remains nonoscillatory. This approach is sometimes referred to as the perturbation principle and leads, e.g., to the Hille–Nehari type (non)oscillation criteria for (1.3) which compare limits inferior and superior of certain integral expressions with concrete constants. These integral expressions are usually either of the form

$$\int_T^t R^{-1}(s)\,\mathrm{d}s \int_t^\infty \tilde{c}(s)h^p(s)\,\mathrm{d}s \quad \text{if} \int^\infty R^{-1}(t)\,\mathrm{d}t = \infty \tag{1.4}$$

or

$$\int_t^\infty R^{-1}(s)\,\mathrm{d}s \int_T^t \tilde{c}(s)h^p(s)\,\mathrm{d}s \quad \text{if} \int^\infty R^{-1}(t)\,\mathrm{d}t < \infty, \tag{1.5}$$

where h is a solution of (1.1) (or a function which is asymptotically close to a solution of (1.1)) and $R = rh^2|h'|^{p-2}$. Criteria of this type can be found in [1–3, 5–7, 9, 10, 13], see also the references therein. Note that the divergence or convergence of the integral $\int^\infty R^{-1}(t)\,\mathrm{d}t$ is closely connected with the so called principality of the solution h of (1.1), see [4, 8] for details.

Let us summarize the known results concerning the above mentioned criteria which apply to perturbations of the Euler and Rieman–Weber type equations. Denote

$$\gamma_p := \left(\frac{p-1}{p}\right)^p, \qquad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

An example of a nonoscillatory equation of the form (1.1) is the half-linear Euler type equation with the critical coefficient γ_p (called also the oscillation constant)

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \tag{1.6}$$

whose principal solution is $h_1(t) = t^{\frac{p-1}{p}}$ and the second one (linearly independent of h_1) is asymptotically equivalent to $h_2(t) = t^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$, see [11]. Note that the criticality of γ_p in (1.6) means that if we replace γ_p in (1.6) by another constant γ, then (1.6) is oscillatory for $\gamma > \gamma_p$ and nonoscillatory for $\gamma < \gamma_p$. It was shown in [7] that the perturbed Euler type equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \tilde{c}(t)\right)\Phi(x) = 0 \tag{1.7}$$

is nonoscillatory if

$$\limsup_{t\to\infty} E(t) < \mu_p, \qquad \liminf_{t\to\infty} E(t) > -3\mu_p$$

and oscillatory if

$$\liminf_{t\to\infty} E(t) > \mu_p,$$

where $E(t) = \log t \int_t^\infty \tilde{c}(s)s^{p-1}\,\mathrm{d}s$. Došlý and Řezníčková [9] proved the same couple of nonoscillation and oscillation criteria with $E(t) = \frac{1}{\log t}\int_T^t \tilde{c}(s)s^{p-1}\log^2 s\,\mathrm{d}s$. Compare both cases of $E(t)$ with (1.4) and (1.5) taking $h(t) = h_1(t)$ and $h(t) = h_2(t)$, respectively.

Further natural step was to find similar statements also for perturbations of the Riemann–Weber (sometimes called Euler–Weber) half-linear equation with critical coefficients

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p\log^2 t}\right)\Phi(x) = 0. \tag{1.8}$$

This equation has a pair of solutions asymptotically close to the functions $h_1(t) = t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t$ and $h_2(t) = t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t\log^{\frac{2}{p}}(\log t)$ and if we replace the constant μ_p in (1.8) by a different constant μ, then (1.8) is oscillatory for $\mu > \mu_p$ and nonoscillatory for $\mu < \mu_p$, see [12]. The (non)oscillation criteria for the perturbed equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p\log^2 t} + \tilde{c}(t)\right)\Phi(x) = 0 \tag{1.9}$$

were formulated in terms of

$$E(t) = \log(\log t)\int_t^\infty \tilde{c}(s)s^{p-1}\log s\,\mathrm{d}s,$$

which complies with (1.4) taking $h(t) = h_1(t)$. The relevant nonoscillation criterion for (1.9) was proved in [2] and oscillatory criterion in [10]. The case which corresponds to (1.5) and to the second function h_2 remained open.

Recently, the criteria from [2, 10] were generalized in [3] to perturbations of the following generalized Riemann–Weber half-linear equation with critical coefficients

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n}\frac{\mu_p}{t^p\mathrm{Log}_j^2 t}\right)\Phi(x) = 0, \tag{1.10}$$

where $n \in \mathbb{N}$ and

$$\log_1 t = \log t, \qquad \log_k t = \log_{k-1}(\log t), \quad k \geq 2, \qquad \mathrm{Log}_j t = \Pi_{k=1}^j \log_k t.$$

Elbert and Schneider in [12] derived the asymptotic formulas for the two linearly independent nonoscillatory solutions of (1.10). These solutions are asymptotically equivalent to the functions

$$h_1(t) = t^{\frac{p-1}{p}}\mathrm{Log}_n^{\frac{1}{p}} t, \qquad h_2(t) = t^{\frac{p-1}{p}}\mathrm{Log}_n^{\frac{1}{p}} t\log_{n+1}^{\frac{2}{p}} t. \tag{1.11}$$

Došlý in [3] studied the equation

$$L_{RW}[x] := (\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n}\frac{\mu_p}{t^p\mathrm{Log}_j^2 t} + \tilde{c}(t)\right)\Phi(x) = 0 \tag{1.12}$$

and proved the following statement.

Theorem A. *Suppose that the integral $\int^\infty \tilde{c}(t)t^{p-1}\operatorname{Log}_n t\,\mathrm{d}t$ is convergent.*

(i) If

$$\limsup_{t\to\infty} \log_{n+1} t \int_t^\infty \tilde{c}(s)s^{p-1}\operatorname{Log}_n s\,\mathrm{d}s < \mu_p,$$
$$\liminf_{t\to\infty} \log_{n+1} t \int_t^\infty \tilde{c}(s)s^{p-1}\operatorname{Log}_n s\,\mathrm{d}s > -3\mu_p,$$

then (1.12) is nonoscillatory.

(ii) Suppose that there exists a constant $\gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2}$ such that $\tilde{c}(t)t^p \log^3 t \geq \gamma$ for large t and

$$\liminf_{t\to\infty} \log_{n+1} t \int_t^\infty \tilde{c}(s)s^{p-1}\operatorname{Log}_n s\,\mathrm{d}s > \mu_p.$$

Then (1.12) is oscillatory.

Observe that the integral expression from Theorem A relates to (1.4) with $h(t) = h_1(t)$ from (1.11). If $n = 1$, then (1.12) reduces to (1.9) and the criteria from Theorem A reduce to that obtained in [2, 10].

The aim of this paper is to complement Theorem A (and also the corresponding results of [2, 10] in case $n = 1$). We utilize the second function h_2 from (1.11) and find a related couple of criteria for equation (1.12) formulated in terms of the expression

$$\frac{1}{\log_{n+1} t}\int^t \tilde{c}(s)s^{p-1}\operatorname{Log}_n s \log_{n+1}^2 s\,\mathrm{d}s$$

which corresponds to (1.5).

2 Auxiliary statements

In this section we present the known statements which will be used in the proofs of our main results in the next section. Denote

$$R(t) := r(t)h^2(t)|h'(t)|^{p-2}, \qquad G(t) := r(t)h(t)\Phi(h'(t)) \tag{2.1}$$

and recall that $q = \frac{p}{p-1}$ is the so called conjugate number of p.

The following statement comes from [13].

Theorem B. *Let h be a function such that $h(t) > 0$ and $h'(t) \neq 0$, both for large t. Suppose that the following conditions hold:*

$$\int^\infty R^{-1}(t)\ \mathrm{d}t < \infty, \qquad \lim_{t\to\infty} G(t)\int_t^\infty R^{-1}(s)\ \mathrm{d}s = \infty. \tag{2.2}$$

If

$$\limsup_{t\to\infty} \int_t^\infty R^{-1}(s)\ \mathrm{d}s \int_T^t h(s)L[h](s)\ \mathrm{d}s < \frac{1}{q}(-\alpha + \sqrt{2\alpha}), \tag{2.3}$$

$$\liminf_{t\to\infty} \int_t^\infty R^{-1}(s)\ \mathrm{d}s \int_T^t h(s)L[h](s)\ \mathrm{d}s > \frac{1}{q}(-\alpha - \sqrt{2\alpha}) \tag{2.4}$$

for some $\alpha > 0$, then (1.1) is nonoscillatory.

The following theorem was proved in [6].

Theorem C. *Let h be a positive continuously differentiable function satisfying the following conditions:*

$$h(t)L(h)(t) \geq 0 \quad \text{for large } t, \qquad \int^{\infty} h(t)L(h)(t)\, \mathrm{d}t = \infty, \tag{2.5}$$

$$\int^{\infty} R^{-1}(t)\, \mathrm{d}t = \infty \quad \text{and} \quad \lim_{t\to\infty} G(t) = \infty. \tag{2.6}$$

Then (1.1) is oscillatory.

In the following lemma we summarize some technical facts which are either evident or were derived in [3].

Lemma 2.1. *For $n \geq 2$ and large t we have*

$$\operatorname{Log}_n t > \cdots > \operatorname{Log}_1 t = \log t > \cdots > \log_n t$$

and

$$(\log_n t)' = \frac{1}{t \operatorname{Log}_{n-1} t}, \qquad (\operatorname{Log}_n t)' = \frac{\operatorname{Log}_n t}{t} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t}.$$

Moreover, for $h(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t$ and the operator defined in (1.12) we have

$$h'(t) = \frac{p-1}{p} t^{-\frac{1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t}\right)$$

and

$$h(t)L_{RW}[h](t) = \frac{\operatorname{Log}_n t}{t \log^3 t} \left[\frac{2\gamma_p p(2-p)}{3(p-1)^2} + \tilde{c}(t) t^p \log^3 t + o(1)\right] \quad \text{as } t \to \infty. \tag{2.7}$$

3 Main results

Our main result concerning nonoscillation of (1.12) reads as follows.

Theorem 3.1. *If*

$$\limsup_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t \tilde{c}(s) s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s\, \mathrm{d}s < 2\mu_p(-\alpha + \sqrt{2\alpha}), \tag{3.1}$$

$$\liminf_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t \tilde{c}(s) s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s\, \mathrm{d}s > 2\mu_p(-\alpha - \sqrt{2\alpha}) \tag{3.2}$$

for some $\alpha > 0$, then equation (1.12) is nonoscillatory.

Proof. We prove the statement with the use of the function $h(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t$ in Theorem B. By a direct differentiation (and using Lemma 2.1) we have

$$\begin{aligned} h'(t) &= \frac{p-1}{p} t^{-\frac{1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t + \frac{1}{p} t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}-1} t \frac{\operatorname{Log}_n t}{t} \left(\frac{1}{\log t} + \cdots + \frac{1}{\operatorname{Log}_n t}\right) \log_{n+1}^{\frac{2}{p}} t \\ &\quad + \frac{2}{p} t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}-1} t \frac{1}{t \operatorname{Log}_n t} \\ &= \frac{p-1}{p} t^{-\frac{1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right). \end{aligned}$$

Denote $\Gamma_p = \left(\frac{p-1}{p}\right)^{p-1}$. Then

$$\Phi(h') = \Gamma_p t^{-1+\frac{1}{p}} \operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-1}.$$

By a direct differentiation (and using Lemma 2.1 again) we obtain

$$\begin{aligned}
(\Phi(h'))' = &-\gamma_p t^{-2+\frac{1}{p}} \operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-1} \\
&+ \gamma_p t^{-2+\frac{1}{p}} \operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-1} \\
&+ 2\gamma_p t^{-2+\frac{1}{p}} \operatorname{Log}_n^{-\frac{1}{p}} t \log_{n+1}^{1-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-1} \\
&+ \Gamma_p (p-1) t^{-1+\frac{1}{p}} \operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-2} \\
&\times \frac{-1}{(p-1)t} \left[\frac{1}{\log^2 t} + \frac{1}{\operatorname{Log}_2 t}\left(\frac{1}{\log t} + \frac{1}{\operatorname{Log}_2 t}\right) + \cdots + \frac{1}{\operatorname{Log}_n t} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t}\right. \\
&\qquad \left. + \frac{2}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}\right].
\end{aligned}$$

Observe that the expression in the square brackets can be rearranged as follows:

$$\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} + \frac{2}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}.$$

Hence

$$\begin{aligned}
(\Phi(h'))' = &\, t^{-2+\frac{1}{p}} \operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-2} \\
&\times \left\{-\gamma_p \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)\right. \\
&\quad + \gamma_p \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right) \\
&\quad + 2\gamma_p \frac{1}{\operatorname{Log}_{n+1} t} \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right) \\
&\quad \left. - \Gamma_p \left[\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} + \frac{2}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}\right]\right\}.
\end{aligned}$$

Denote by $A(t)$ the expression in the curly brackets. By a direct computation with using the fact that

$$\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t}\right)^2 = \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + \sum_{1 \le i < j \le n} \frac{2}{\operatorname{Log}_i t \operatorname{Log}_j t}$$

we obtain

$$\begin{aligned} A(t) = & -\gamma_p + \gamma_p \frac{p-2}{p-1} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + 2\gamma_p \frac{p-2}{p-1} \frac{1}{\operatorname{Log}_{n+1} t} - \gamma_p \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} \\ & + \gamma_p \frac{2-p}{p-1} \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} + 2\gamma_p \frac{2-p}{p-1} \frac{1}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}. \end{aligned} \tag{3.3}$$

Next, denote

$$B(t) := \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-2}.$$

Using the power expansion

$$(1+x)^s = 1 + sx + \frac{s(s-1)}{2}x^2 + \frac{s(s-1)(s-2)}{6}x^3 + o(x^3) \quad \text{as } x \to 0,\ s \in \mathbb{R},$$

we obtain

$$\begin{aligned} B(t) = 1 &+ \frac{p-2}{p-1}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \frac{2}{\operatorname{Log}_{n+1} t}\right) + \frac{(p-2)(p-3)}{2(p-1)^2}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \frac{2}{\operatorname{Log}_{n+1} t}\right)^2 \\ &+ \frac{(p-2)(p-3)(p-4)}{6(p-1)^3}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \frac{2}{\operatorname{Log}_{n+1} t}\right)^3 + o(\log^{-3} t), \end{aligned}$$

as $t \to \infty$.

Next observe that if at least one of the indices i, j, k is greater than one, then

$$\frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t \operatorname{Log}_k t} = o\left(\log^{-3} t\right) \quad \text{as } t \to \infty.$$

Hence we can write $B(t)$ in the form

$$\begin{aligned} B(t) = 1 &+ \frac{p-2}{p-1}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \frac{2}{\operatorname{Log}_{n+1} t}\right) \\ &+ \frac{(p-2)(p-3)}{2(p-1)^2}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + \sum_{1 \le i < j \le n} \frac{2}{\operatorname{Log}_i t \operatorname{Log}_j t} + \frac{4}{\operatorname{Log}_{n+1} t}\sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}\right) \\ &+ \frac{(p-2)(p-3)(p-4)}{6(p-1)^3}\frac{1}{\log^3 t} + o(\log^{-3} t) \end{aligned} \tag{3.4}$$

as $t \to \infty$.

From (3.3) and (3.4), we obtain

$$\begin{aligned} & A(t) \cdot B(t) \\ & \quad = -\gamma_p + \gamma_p \frac{p-2}{p-1} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + 2\gamma_p \frac{p-2}{p-1} \frac{1}{\operatorname{Log}_{n+1} t} - \gamma_p \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} \\ & \qquad + \gamma_p \frac{2-p}{p-1} \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} + 2\gamma_p \frac{2-p}{p-1} \frac{1}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t} \\ & \qquad - \gamma_p \frac{p-2}{p-1} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \gamma_p \left(\frac{p-2}{p-1}\right)^2 \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + 2\gamma_p \left(\frac{p-2}{p-1}\right)^2 \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} \end{aligned}$$

$$
\begin{aligned}
&+2\gamma_p\left(\frac{p-2}{p-1}\right)^2\frac{1}{\operatorname{Log}_{n+1}t}\sum_{i=1}^{n}\frac{1}{\operatorname{Log}_i t}-\gamma_p\frac{p-2}{p-1}\frac{1}{\log^3 t}\\
&-2\gamma_p\frac{p-2}{p-1}\frac{1}{\operatorname{Log}_{n+1}t}+2\gamma_p\left(\frac{p-2}{p-1}\right)^2\frac{1}{\operatorname{Log}_{n+1}t}\sum_{i=1}^{n}\frac{1}{\operatorname{Log}_i t}+4\gamma_p\left(\frac{p-2}{p-1}\right)^2\frac{1}{\operatorname{Log}^2_{n+1}t}\\
&-\gamma_p\frac{(p-2)(p-3)}{2(p-1)^2}\sum_{i=1}^{n}\frac{1}{\operatorname{Log}^2_i t}+\gamma_p\frac{(p-2)^2(p-3)}{2(p-1)^3}\frac{1}{\log^3 t}\\
&-\gamma_p\frac{(p-2)(p-3)}{(p-1)^2}\sum_{1\le i<j\le n}\frac{1}{\operatorname{Log}_i t\operatorname{Log}_j t}-2\gamma_p\frac{(p-2)(p-3)}{(p-1)^2}\frac{1}{\operatorname{Log}_{n+1}t}\sum_{i=1}^{n+1}\frac{1}{\operatorname{Log}_i t}\\
&-\gamma_p\frac{(p-2)(p-3)(p-4)}{6(p-1)^3}\frac{1}{\log^3 t}\ +o(\log^{-3}t)\\
=&-\gamma_p-\mu_p\sum_{i=1}^{n}\frac{1}{\operatorname{Log}^2_i t}-\frac{2\gamma_p p(p-2)}{3(p-1)^2}\frac{1}{\log^3 t}+o(\log^{-3}t)
\end{aligned}
$$

as $t\to\infty$. Summarizing the above computations, we have

$$
(\Phi(h'))'=t^{-2+\frac{1}{p}}\operatorname{Log}_n^{1-\frac{1}{p}}t\log_{n+1}^{2-\frac{2}{p}}t\left(-\gamma_p-\mu_p\sum_{i=1}^{n}\frac{1}{\operatorname{Log}^2_i t}-\frac{2\gamma_p p(p-2)}{3(p-1)^2}\frac{1}{\log^3 t}+o(\log^{-3}t)\right)
$$

as $t\to\infty$. Consequently, for the operator L_{RW} defined in (1.12) we have

$$
\begin{aligned}
hL_{RW}[h]&=h(\Phi(h'))'+h^p\left(\frac{\gamma_p}{t^p}+\sum_{j=1}^{n}\frac{\mu_p}{t^p\operatorname{Log}^2_j t}+\tilde{c}(t)\right)\\
&=\frac{\operatorname{Log}_n t\log^2_{n+1}t}{t}\left(-\gamma_p-\mu_p\sum_{i=1}^{n}\frac{1}{\operatorname{Log}^2_i t}-\frac{2\gamma_p p(p-2)}{3(p-1)^2}\frac{1}{\log^3 t}+o(\log^{-3}t)\right)\\
&\quad+t^{p-1}\operatorname{Log}_n t\log^2_{n+1}t\left(\frac{\gamma_p}{t^p}+\sum_{j=1}^{n}\frac{\mu_p}{t^p\operatorname{Log}^2_j t}+\tilde{c}(t)\right)\\
&=\frac{\operatorname{Log}_n t\log^2_{n+1}t}{t\log^3 t}\left(-\frac{2\gamma_p p(p-2)}{3(p-1)^2}+o(1)\right)+\tilde{c}(t)t^{p-1}\operatorname{Log}_n t\log^2_{n+1}t
\end{aligned}
\tag{3.5}
$$

as $t\to\infty$. In order to check conditions (2.2), express $R(t)$ and $G(t)$ from (2.1):

$$
\begin{aligned}
R(t)&=h^2(t)|h'(t)|^{p-2}\\
&=\left(\frac{p-1}{p}\right)^{p-2}t\operatorname{Log}_n t\log^2_{n+1}t\left(1+\sum_{i=1}^{n}\frac{1}{(p-1)\operatorname{Log}_i t}+\frac{2}{(p-1)\operatorname{Log}_{n+1}t}\right)^{p-2}\\
&=\left(\frac{p-1}{p}\right)^{p-2}t\operatorname{Log}_n t\log^2_{n+1}t(1+o(1))
\end{aligned}
$$

and

$$
\begin{aligned}
G(t)&=h(t)\Phi(h'(t))\\
&=\left(\frac{p-1}{p}\right)^{p-1}\operatorname{Log}_n t\log^2_{n+1}t\left(1+\sum_{i=1}^{n}\frac{1}{(p-1)\operatorname{Log}_i t}+\frac{2}{(p-1)\operatorname{Log}_{n+1}t}\right)^{p-1}\\
&=\left(\frac{p-1}{p}\right)^{p-1}\operatorname{Log}_n t\log^2_{n+1}t(1+o(1))
\end{aligned}
$$

as $t \to \infty$. Since $\int^\infty R^{-1}(s)\,\mathrm{d}s < \infty$, the first condition in (2.2) is satisfied. The second condition in (2.2) is also fulfilled, since

$$\int_t^\infty \frac{1}{R(s)}\,\mathrm{d}s = \left(\frac{p}{p-1}\right)^{p-2} \frac{1}{\log_{n+1} t}(1+o(1)) \tag{3.6}$$

and hence

$$G(t)\int_t^\infty \frac{1}{R(s)}\,\mathrm{d}s = \frac{p-1}{p} \operatorname{Log}_n t \log_{n+1} t(1+o(1)) \to \infty$$

as $t \to \infty$.

Finally, we show that conditions (2.3) and (2.4) hold. To this end, let $\varepsilon \in (0,1)$. Then

$$\lim_{t\to\infty} \frac{\operatorname{Log}_n t \log_{n+1}^2 t}{\log^{1+\varepsilon} t} < \lim_{t\to\infty} \frac{\log_2^{n+1} t}{\log^{\varepsilon} t} = \lim_{t\to\infty} \frac{(n+1)!}{\varepsilon^{n+1}\log^{\varepsilon} t} = 0$$

and hence

$$\lim_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t \frac{\operatorname{Log}_n s \log_{n+1}^2 s}{s\log^3 s}\,\mathrm{d}s < \lim_{t\to\infty} \frac{1}{\log_{n+1} t}\int_T^t \frac{1}{s\log^{2-\varepsilon} s}\,\mathrm{d}s = 0. \tag{3.7}$$

From (3.5) and (3.6) we obtain

$$\begin{aligned}
&\left(\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right)\left(\int_T^t h(s)L_{RW}[h](s)\,\mathrm{d}s\right) \\
&\quad = \left(\frac{p}{p-1}\right)^{p-2} \frac{1}{\log_{n+1} t}(1+o(1)) \\
&\qquad \times \int_T^t \frac{\operatorname{Log}_n s \log_{n+1}^2 s}{s\log^3 s}\left(-\frac{2\gamma_p p(p-2)}{3(p-1)^2} + o(1)\right) + \tilde{c}(s)s^{p-1}\operatorname{Log}_n s \log_{n+1}^2 s\,\mathrm{d}s
\end{aligned}$$

as $t \to \infty$. Conditions (3.1) and (3.2) together with (3.7) imply

$$\begin{aligned}
&\limsup_{t\to\infty}\left(\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right)\left(\int_T^t h(s)L_{RW}[h](s)\,\mathrm{d}s\right) \\
&\quad = \left(\frac{p}{p-1}\right)^{p-2} \limsup_{t\to\infty} \frac{1}{\log_{n+1} t}\int_T^t \tilde{c}(s)s^{p-1}\operatorname{Log}_n s \log_{n+1}^2 s\,\mathrm{d}s < \frac{1}{q}(-\alpha+\sqrt{2\alpha})
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{t\to\infty}\left(\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right)\left(\int_T^t h(s)L_{RW}[h](s)\,\mathrm{d}s\right) \\
&\quad = \left(\frac{p}{p-1}\right)^{p-2} \liminf_{t\to\infty} \frac{1}{\log_{n+1} t}\int_T^t \tilde{c}(s)s^{p-1}\operatorname{Log}_n s \log_{n+1}^2 s\,\mathrm{d}s > \frac{1}{q}(-\alpha-\sqrt{2\alpha}).
\end{aligned}$$

All assumptions of Theorem B are true, which finishes the proof. □

To obtain the oscillatory counterpart of Theorem 3.1, we first prove the following criterion for the equation

$$\tilde{L}_{RW}[x] := (\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n+1}\frac{\mu_p}{t^p \operatorname{Log}_j^2 t} + d(t)\right)\Phi(x) = 0, \tag{3.8}$$

which is in fact equation (1.12) shifted from n to $n+1$. The reason why we formulate this criterion rather for (3.8) than for (1.12) is only technical.

Theorem 3.2. *Suppose that there exists a constant* γ *such that*

$$d(t)t^p \log^3 t \geq \gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2} \tag{3.9}$$

for large t. *If*

$$\int^\infty d(t)t^{p-1} \operatorname{Log}_{n+1} t \, dt = \infty, \tag{3.10}$$

then equation (3.8) *is oscillatory.*

Proof. Take $h(t) = t^{\frac{p-1}{p}} \operatorname{Log}_{n+1}^{\frac{1}{p}} t$. According to Lemma 2.1 (with n replaced by $n+1$)

$$h'(t) = \frac{p-1}{p} t^{-\frac{1}{p}} \operatorname{Log}_{n+1}^{\frac{1}{p}} t(1+o(1)) \quad \text{as } t \to \infty.$$

Hence, by (2.1)

$$R(t) = h^2(t)|h'(t)|^{p-2} = \left(\frac{p-1}{p}\right)^{p-2} t \operatorname{Log}_{n+1} t(1+o(1)) \quad \text{as } t \to \infty$$

and consequently

$$\int^t R^{-1}(s) \, ds = \left(\frac{p-1}{p}\right)^{2-p} \log_{n+2} t(1+o(1)) \to \infty \quad \text{as } t \to \infty.$$

Further,

$$G(t) = h(t)\Phi(h'(t)) = \left(\frac{p-1}{p}\right)^{p-1} \operatorname{Log}_{n+1} t(1+o(1)) \to \infty \quad \text{as } t \to \infty.$$

Rewriting (2.7) for the operator from (3.8) we have

$$\begin{aligned} h(t)\tilde{L}_{RW}[h](t) &= \frac{\operatorname{Log}_{n+1} t}{t \log^3 t} \left[\frac{2\gamma_p p(2-p)}{3(p-1)^2} + d(t)t^p \log^3 t + o(1)\right] \\ &= \left[\frac{2\gamma_p p(2-p)}{3(p-1)^2} + o(1)\right] \frac{\operatorname{Log}_{n+1} t}{t \log^3 t} + d(t)t^{p-1} \operatorname{Log}_{n+1} t \end{aligned}$$

as $t \to \infty$. Because the integral $\int^\infty \frac{\operatorname{Log}_{n+1} t}{t \log^3 t} \, dt$ is convergent, condition (3.10) implies

$$\int^\infty h\tilde{L}_{RW}[h](t) \, dt = \infty.$$

Thanks to condition (3.9) we have also $h\tilde{L}_{RW}[h](t) \geq 0$ for large t. This means that equation (3.8) is oscillatory by Theorem C. □

The following statement is the oscillatory criterion which complements Theorem 3.1.

Theorem 3.3. *Suppose that there exists a constant* γ *such that*

$$t^p \log^3 t \left(\tilde{c}(t) - \frac{\mu_p}{t^p \operatorname{Log}_{n+1}^2 t}\right) \geq \gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2} \tag{3.11}$$

for large t. *If*

$$\liminf_{t \to \infty} \frac{1}{\log_{n+1} t} \int_T^t \tilde{c}(s)s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s \, ds > \mu_p, \tag{3.12}$$

then (1.12) *is oscillatory.*

Proof. Let us rewrite (1.12) into the form

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n+1} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + \left(\tilde{c}(t) - \frac{\mu_p}{t^p \,\mathrm{Log}_{\,n+1}^{\,2} t}\right)\right) \Phi(x) = 0,$$

so (1.12) is seen as a perturbation of the generalized Riemann–Weber equation with the critical coefficients and with $n+1$ elements in the sum. We apply Theorem 3.2 with the perturbation term $d(t) = \tilde{c}(t) - \frac{\mu_p}{t^p \,\mathrm{Log}_{n+1}^2 t}$. Then (3.9) is guaranteed by (3.11). With respect to (3.12) there exist $\varepsilon > 0$ and $\tilde{T} > T$ such that

$$\liminf_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t \tilde{c}(s) s^{p-1} \,\mathrm{Log}_{\,n+1} s \log_{n+1} s \;\mathrm{d}s > \mu_p + \varepsilon$$

and also

$$\int_T^t \tilde{c}(s) s^{p-1} \,\mathrm{Log}_{\,n+1} s \log_{n+1} s \;\mathrm{d}s > (\mu_p + \varepsilon) \log_{n+1} t$$

for $t > \tilde{T}$. Let $b > \tilde{T}$. With the use of integration by parts and the above inequality, we have

$$\begin{aligned}
&\int_T^b \left(\tilde{c}(t) - \frac{\mu_p}{t^p \,\mathrm{Log}_{\,n+1}^{\,2} t}\right) t^{p-1} \,\mathrm{Log}_{\,n+1} t \;\mathrm{d}t \\
&\quad = \int_T^b \tilde{c}(t) t^{p-1} \,\mathrm{Log}_{\,n+1} t \;\mathrm{d}t - \int_T^b \frac{\mu_p}{t \,\mathrm{Log}_{\,n+1} t} \;\mathrm{d}t \\
&\quad = \int_T^b \frac{1}{\log_{n+1} t} \tilde{c}(t) t^{p-1} \,\mathrm{Log}_{\,n+1} t \log_{n+1} t \;\mathrm{d}t - \mu_p \left[\log_{n+2} t\right]_T^b \\
&\quad = \left[\frac{1}{\log_{n+1} t} \int_T^t \tilde{c}(s) s^{p-1} \,\mathrm{Log}_{\,n+1} s \log_{n+1} s \;\mathrm{d}s\right]_T^b + \int_T^{\tilde{T}} \frac{\int_T^t \tilde{c}(s) s^{p-1} \,\mathrm{Log}_{\,n+1} s \log_{n+1} s \;\mathrm{d}s}{t \,\mathrm{Log}_{\,n} t \log_{n+1}^2 t} \;\mathrm{d}t \\
&\qquad + \int_{\tilde{T}}^b \frac{\int_T^t \tilde{c}(s) s^{p-1} \,\mathrm{Log}_{\,n+1} s \log_{n+1} s \;\mathrm{d}s}{t \,\mathrm{Log}_{\,n} \log_{n+1}^2 t} \;\mathrm{d}t - \mu_p \left[\log_{n+2} t\right]_T^b \\
&\quad \geq \frac{1}{\log_{n+1} b} \int_T^b \tilde{c}(t) t^{p-1} \,\mathrm{Log}_{\,n+1} t \log_{n+1} t \;\mathrm{d}t + K_1 + \int_{\tilde{T}}^b \frac{\mu_p + \varepsilon}{t \,\mathrm{Log}_{\,n+1} t} \;\mathrm{d}t - \mu_p \left[\log_{n+2} t\right]_T^b \\
&\quad \geq \mu_p + \varepsilon + K_1 + (\mu_p + \varepsilon) \left[\log_{n+2} t\right]_{\tilde{T}}^b - \mu_p \left[\log_{n+2} t\right]_T^b \\
&\quad = \mu_p + \varepsilon + K_1 + \varepsilon \log_{n+2} b - K_2 \to \infty
\end{aligned}$$

as $b \to \infty$, where

$$K_1 = \int_T^{\tilde{T}} \frac{\int_T^t \tilde{c}(s) s^{p-1} \,\mathrm{Log}_{\,n+1} s \log_{n+1} s \;\mathrm{d}s}{t \,\mathrm{Log}_{\,n} t \log_{n+1}^2 t} \;\mathrm{d}t, \qquad K_2 = (\mu_p + \varepsilon) \log_{n+2} \tilde{T} - \mu_p \log_{n+2} T.$$

Hence condition (3.10) is satisfied and (1.12) is oscillatory according to Theorem 3.2. □

Remark 3.4. If $\alpha = \frac{1}{2}$ in Theorem 3.1, then

$$2\mu_p(-\alpha + \sqrt{2\alpha}) = \mu_p, \qquad 2\mu_p(-\alpha - \sqrt{2\alpha}) = -3\mu_p$$

and the constants from (3.1) and (3.2) in Theorem 3.1 reduce to the constants in Theorem A, part (i). The generalization for $\alpha \neq \frac{1}{2}$ is due to Theorem B. Note also that the constants in the nonoscillatory part of Theorem A could be generalized in the same way by utilizing [13, Theorem 3.2] in the proof of Theorem A.

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Hille–Nehari type criteria and conditionally oscillatory half-linear differential equations

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Abstract. We study perturbations of the generalized conditionally oscillatory half-linear equation of the Riemann–Weber type. We formulate new oscillation and nonoscillation criteria for this equation and find a perturbation such that the perturbed Riemann–Weber type equation is conditionally oscillatory.

Keywords: half-linear differential equation, generalized Riemann–Weber equation, non(oscillation) criteria, perturbation principle.

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1 Introduction

In the paper we study oscillatory properties of the half-linear equation

$$L[x] := (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{1.1}$$

where the coefficients r, c are continuous functions, $r(t) > 0$ on the interval under consideration, which is a neighbourhood of infinity. In the special case when $p = 2$ this equation becomes the linear Sturm–Liouville equation. If $p \neq 2$, equation (1.1) is called half-linear since it has one half of the properties that characterize linearity: the solution space is homogeneous, but is generally not additive. Despite the missing additivity, the classical linear Sturmian theory has been extended to half-linear equations. We refer to the book [8] for the overview of the methods and results concerning half-linear equations up to year 2005. Concerning the recent results on half-linear differential equtions, see, e.g., [14–20] and the references therein.

Recall that equation (1.1) is said to be *oscillatory* if all its solutions are oscillatory, i.e., all the solutions have infinitely many zeros tending to infinity. In the opposite case equation (1.1) is said to be *nonoscillatory*. Note also that oscillatory and nonoscillatory solutions of (1.1) cannot coexist and this means that this equation is nonoscillatory if all solutions have constant sign eventually.

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Throughout this paper we suppose that equation (1.1) is nonoscillatory and we study the influence of perturbations of the coefficient c on the oscillatory behaviour of equation (1.1), i.e., we study equations of the form

$$(r(t)\Phi(x'))' + (c(t) + \tilde{c}(t))\Phi(x) = 0. \tag{1.2}$$

It is known from the Sturmian theory, that if the perturbation $\tilde{c}$ is "sufficiently positive", the equation becomes oscillatory, if it is "not too much positive", the equation remains nonoscillatory. If we find a positive function d and a constant λ_0 such that the equation

$$(r(t)\Phi(x'))' + (c(t) + \lambda d(t))\Phi(x) = 0 \tag{1.3}$$

is nonoscillatory for $\lambda < \lambda_0$ and oscillatory for $\lambda > \lambda_0$, we say that equation (1.3) is *conditionally oscillatory* with the *oscillation constant* λ_0. Examples of conditionally oscillatory equations (written below with the oscillation constants) are the Euler type equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p \tag{1.4}$$

and the perturbed Euler type equations, such as the Riemann–Weber type equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right)\Phi(x) = 0, \quad \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} \tag{1.5}$$

or equations with arbitrary number of perturbation terms of the form

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \operatorname{Log}_j^2 t}\right)\Phi(x) = 0, \tag{1.6}$$

where $n \in \mathbb{N}$, $\log_1 t = \log t$, $\log_k t = \log_{k-1}(\log t)$, $k \geq 2$, $\operatorname{Log}_j t = \prod_{k=1}^{j} \log_k t$. All these equations are nonoscillatory also in the critical case with the oscillation constants. The appropriate results concerning the Euler type equation and its perturbations in the coefficient $\frac{\gamma_p}{t^p}$ including the asymptotic formulas for nonoscillatory solutions of these equations can be found in the paper of Elbert and Schneider [11]. Note that the result of Elbert and Schneider has been generalized to the case when also perturbations in the term with derivative are allowed and also to the case of equations with non-constant coefficients, see, e.g., [4,6,7,14] and the references therein.

In this paper we study perturbations of general nonoscillatory equation (1.1). We suppose that h is a solution of (1.1) such that $h(t) > 0$ and $h'(t) \neq 0$, for $t \geq t_0$, where t_0 is a real number from the interval of consideration of (1.1). Moreover, we suppose that

$$\int^{\infty} R^{-1}(t)\,\mathrm{d}t = \infty \quad \text{and} \quad \liminf_{t\to\infty} |G(t)| > 0, \tag{1.7}$$

where

$$R(t) = r(t)h^2(t)|h'(t)|^{p-2}, \quad G(t) = r(t)h(t)\Phi(h'(t)). \tag{1.8}$$

Note that we follow the notation used in [10] and wherever we consider the integral $\int^{\infty} R^{-1}(t)\,\mathrm{d}t$, its lower limit is omitted, as it can be a constant greater or equal to t_0 such that all relevant conditions hold.

The motivation for our research comes from paper [10]. In that paper, under assumptions (1.7), it is shown that if $d(t) = \big(h^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2\big)^{-1}$ in (1.3), then (1.3) is conditionally oscillatory equation and the oscillation constant is $\frac{1}{2q}$, where q is a conjugate number to p, i.e., $\frac{1}{p}+\frac{1}{q}=1$. It is also shown that the equation in the critical case with the oscillation constant $\frac{1}{2q}$

$$\hat{L}[x] := (r(t)\Phi(x'))' + \left[c(t) + \frac{1}{2qh^p(t)R(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)^2}\right]\Phi(x) = 0 \tag{1.9}$$

is nonoscillatory and the asymptotic formula for one of solutions of (1.9) is established. Consequently, the perturbed equation

$$\tilde{L}[x] := (r(t)\Phi(x'))' + \left[c(t) + \frac{1}{2qh^p(t)R(t)\big(\int^t R^{-1}(s)\ \mathrm{d}s\big)^2} + g(t)\right]\Phi(x) = 0 \tag{1.10}$$

is studied. In particular, a nonoscillation criterion of the Hille–Nehari type for (1.10), where limits inferior and superior of the expression

$$\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

are compared with certain constants, is proved, see [10, Theorem 5]. The crucial role in the proof of this criterion plays the fact that the asymptotic formula for a solution of (1.9) is known.

The aim of our paper is to improve the above mentioned nonoscillation criterion for (1.10), to formulate a relevant oscillation criterion for (1.10) and to find a perturbation g in (1.10) such that (1.10) becomes conditionally oscillatory. We also formulate a version of a nonoscillatory Hille–Nehari type criterion for (1.10) in the case when we handle the asymptotic formula for the second solution of (1.9), which has been found recently in [3].

The paper is organized as follows. In the next section we formulate auxiliary results and technical lemmas which are important in our proofs. The main results, oscillation and nonoscillation criteria for (1.10), are presented in Section 3. The last section is devoted to remarks.

2 Auxiliary results

The proofs of our main results are based on the following theorems which can be found in [5] and [12]. For a positive and differentiable function $\tilde{x}$ denote

$$\tilde{R}(t) := r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2}, \quad \tilde{G}(t) := r(t)\tilde{x}(t)\Phi(\tilde{x}'(t)). \tag{2.1}$$

Theorem A ([12, Theorem 3.2])**.** *Let $\tilde{x}$ be a function such that $\tilde{x}(t) > 0$ and $\tilde{x}'(t) \neq 0$, both for large t. Suppose that*

$$\int^\infty \tilde{x}(t)L[\tilde{x}](t)\,\mathrm{d}t \quad \textit{is convergent and}$$

$$\lim_{t\to\infty} |\tilde{G}(t)|\int_T^t \frac{\mathrm{d}s}{\tilde{R}(s)} = \infty, \tag{2.2}$$

where $T \in \mathbb{R}$ is sufficiently large. If

$$\limsup_{t\to\infty} \int_T^t \tilde{R}^{-1}(s)\ \mathrm{d}s \int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\ \mathrm{d}s < \frac{1}{q}(-\alpha + \sqrt{2\alpha}), \tag{2.3}$$

$$\liminf_{t\to\infty} \int_T^t \tilde{R}^{-1}(s)\ \mathrm{d}s \int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\ \mathrm{d}s > \frac{1}{q}(-\alpha - \sqrt{2\alpha}) \tag{2.4}$$

for some $\alpha > 0$, then (1.1) is nonoscillatory.

Theorem B ([5, Theorem 1])**.** *Let $\tilde{x}$ be a continuously differentiable function satisfying conditions*

$$\tilde{x}(t)L[\tilde{x}](t) \ge 0 \quad \textit{for large } t, \qquad \int^\infty \tilde{x}(t)L[\tilde{x}](t)\,dt < \infty, \tag{2.5}$$

$$\int^\infty \frac{dt}{\tilde{R}(t)} = \infty \quad \textit{and} \quad \lim_{t\to\infty} \tilde{G}(t) = \infty. \tag{2.6}$$

If

$$\liminf_{t\to\infty} \int_T^t \tilde{R}^{-1}(s)\,\mathrm{d}s \int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\,\mathrm{d}s > \frac{1}{2q}, \tag{2.7}$$

where $T \in \mathbb{R}$ is sufficiently large, then (1.1) is oscillatory.

Theorem C ([12, Theorem 3.1])**.** *Let $\tilde{x}$ be a function such that $\tilde{x}(t) > 0$ and $\tilde{x}'(t) \neq 0$, both for large t. Suppose that the following conditions hold:*

$$\int^\infty \tilde{R}^{-1}(t)\ \mathrm{d}t < \infty, \quad \lim_{t\to\infty} |\tilde{G}(t)| \int_t^\infty \tilde{R}^{-1}(s)\ \mathrm{d}s = \infty. \tag{2.8}$$

If

$$\limsup_{t\to\infty} \int_t^\infty \tilde{R}^{-1}(s)\ \mathrm{d}s \int_T^t \tilde{x}(s)L[\tilde{x}](s)\ \mathrm{d}s < \frac{1}{q}(-\alpha + \sqrt{2\alpha}), \tag{2.9}$$

$$\liminf_{t\to\infty} \int_t^\infty \tilde{R}^{-1}(s)\ \mathrm{d}s \int_T^t \tilde{x}(s)L[\tilde{x}](s)\ \mathrm{d}s > \frac{1}{q}(-\alpha - \sqrt{2\alpha}) \tag{2.10}$$

for some $\alpha > 0$, $T \in \mathbb{R}$ sufficiently large, then (1.1) is nonoscillatory.

Theorem D ([5, Theorem 2])**.** *Let $\tilde{x}$ be a positive continuously differentiable function satisfying the following conditions:*

$$\tilde{x}(t)L[\tilde{x}](t) \ge 0 \ \textit{ for large } t, \qquad \int^\infty \tilde{x}(t)L[\tilde{x}](t)\ \mathrm{d}t = \infty, \tag{2.11}$$

$$\int^\infty \tilde{R}^{-1}(t)\ \mathrm{d}t = \infty \quad \textit{and} \quad \lim_{t\to\infty} \tilde{G}(t) = \infty. \tag{2.12}$$

Then (1.1) is oscillatory.

In the next lemma we collect some technical facts which are frequently used in the proofs of our main results.

Lemma 2.1. *Suppose that conditions (1.7) hold.*

(i) Let $j \in \mathbb{Z}$ be arbitrary and $k, l \in \mathbb{Z}$ be such that $k > 0$, $l \ge 0$. Then

$$\lim_{t\to\infty} \frac{\log^j(\int^t R^{-1}(s)\,\mathrm{d}s)}{G^l(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^k} = 0. \tag{2.13}$$

(ii) The integrals

$$\int_T^\infty \frac{G'(t)\log^j(\int^t R^{-1}(s)\,\mathrm{d}s)}{G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s}, \quad \int_T^\infty \frac{\log^j(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2} \tag{2.14}$$

are convergent for arbitrary $j \in \mathbb{Z}$, $T \in \mathbb{R}$ sufficiently large.

Proof. (i) Assumptions (1.7) imply that there exists a constant K such that $\frac{1}{|G(t)|} \leq K$ for sufficiently large t and

$$\lim_{t\to\infty} \frac{\log^j(\int^t R^{-1}(s)\,\mathrm{d}s)}{(\int^t R^{-1}(s)\,\mathrm{d}s)^k} = 0 \tag{2.15}$$

which can be shown by L'Hospital's Rule as follows. If $j \leq 0$, then (2.15) is evident. If $j > 0$, then we apply L'Hospital's Rule j times to obtain

$$\lim_{t\to\infty} \frac{\log^j(\int^t R^{-1}(s)\,\mathrm{d}s)}{(\int^t R^{-1}(s)\,\mathrm{d}s)^k} = \frac{j!}{k^j}\lim_{t\to\infty} \frac{1}{(\int^t R^{-1}(s)\,\mathrm{d}s)^k} = 0.$$

Therefore also (2.13) holds.

(ii) The integrals are convergent by the comparison test for improper integrals. The first integral in (2.14) is convergent, because the integral

$$\int_T^\infty \frac{G'(t)}{G^2(t)}\,\mathrm{d}t = \frac{1}{G(T)} - \lim_{t\to\infty}\frac{1}{G(t)}$$

is convergent and

$$\lim_{t\to\infty} \frac{\log^j(\int^t R^{-1}(s)\,\mathrm{d}s)}{\int^t R^{-1}(s)\,\mathrm{d}s} = 0.$$

Concerning the second integral in (2.14) we show that the integral

$$\int_T^\infty \frac{\log^j(\int^t R^{-1}(s)\,\mathrm{d}s)}{R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\,\mathrm{d}t \tag{2.16}$$

is convergent. If $j = 0$, the convergence follows immediately from (1.7), since in this case

$$\int_T^\infty \frac{1}{R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\,\mathrm{d}t = \frac{1}{\int^T R^{-1}(s)\,\mathrm{d}s} - \lim_{t\to\infty}\frac{1}{\int^t R^{-1}(s)\,\mathrm{d}s}.$$

By induction, suppose that integral in (2.16) is convergent for a positive integer j and consider the case $j+1$. Using integration by parts we obtain

$$\begin{aligned}\int_T^\infty \frac{\log^{j+1}(\int^t R^{-1}(s)\,\mathrm{d}s)}{R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\,\mathrm{d}t &= \frac{\log^{j+1}(\int^T R^{-1}(s)\,\mathrm{d}s)}{\int^T R^{-1}(s)\,\mathrm{d}s} - \lim_{t\to\infty}\frac{\log^{j+1}(\int^t R^{-1}(s)\,\mathrm{d}s)}{\int^t R^{-1}(s)\,\mathrm{d}s}\\ &\quad + (j+1)\int_T^\infty \frac{\log^j(\int^t R^{-1}(s)\,\mathrm{d}s)}{R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\,\mathrm{d}t.\end{aligned}$$

This implies the convergence of the integral in (2.16) for any positive integer j. If j is negative, the convergence is evident. The convergence of the second integral in (2.14) follows then from the fact that $\frac{1}{|G(t)|}$ is bounded for large t. $\square$

In the last part of this section we evaluate $\tilde{x}\hat{L}[\tilde{x}]$ from (1.9) for some particular functions $\tilde{x}$. The first of the following statements comes from [10]. The identity (2.17) follows from [10, Theorem 3], where we use the fact that $h'/h = G/R$ and fix the constant in the leading term. The convergence of the corresponding integral is shown in the proof of [10, Theorem 4]. It follows also from Lemma 2.1.

Lemma 2.2. *Let h be a positive solution of* (1.1) *such that $h'(t) \neq 0$ for large t and* (1.7) *holds. Set $\tilde{x}(t) := h(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)^{\frac{1}{p}}$. Then*

$$\begin{aligned}\tilde{x}(t)\hat{L}[\tilde{x}](t) &= \frac{(p-2)(1-p)G'(t)}{2p^2G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s}(1+o(1)) \\ &\quad + \frac{2(1-p)(p-2)}{3p^2G(t)R(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)^2}(1+o(1))\end{aligned} \tag{2.17}$$

as $t \to \infty$ and the integral

$$\int^\infty \tilde{x}(t)\hat{L}[\tilde{x}](t)\,\mathrm{d}t$$

converges.

In the proofs of the following two statements we use the notation

$$\varphi(t) := \int^t R^{-1}(s)\,\mathrm{d}s. \tag{2.18}$$

Lemma 2.3. *Let h be a positive solution of* (1.1) *such that $h'(t) \neq 0$ for large t and* (1.7) *holds. Set $\tilde{x}(t) := h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \log^{\frac{1}{p}}\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)$. Then*

$$\begin{aligned}&\tilde{x}(t)\hat{L}[\tilde{x}](t) + \frac{1}{2qR(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)\log\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)} \\ &\quad = \frac{(p-2)(1-p)G'(t)\log\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)}{2p^2G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s}(1+o(1)) \\ &\qquad + \frac{2(1-p)(p-2)\log\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)}{3p^2G(t)R(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)^2}(1+o(1))\end{aligned} \tag{2.19}$$

as $t \to \infty$.

Proof. We use notation (2.18). By a direct computation and using the fact that $hG = h'R$ we obtain

$$\begin{aligned}\tilde{x}'(t) &= h'(t)\varphi^{\frac{1}{p}}(t)\log^{\frac{1}{p}}\varphi(t) + \frac{1}{p}h(t)R^{-1}(t)\varphi^{\frac{1}{p}-1}(t)\log^{\frac{1}{p}}\varphi(t) \\ &\quad + \frac{1}{p}h(t)R^{-1}(t)\varphi^{\frac{1}{p}-1}(t)\log^{\frac{1}{p}-1}\varphi(t) \\ &= h'(t)\varphi^{\frac{1}{p}}(t)\log^{\frac{1}{p}}\varphi(t)\left[1 + \frac{h(t)}{ph'(t)R(t)\varphi(t)} + \frac{h(t)}{ph'(t)R(t)\varphi(t)\log\varphi(t)}\right] \\ &= h'(t)\varphi^{\frac{1}{p}}(t)\log^{\frac{1}{p}}\varphi(t)\left[1 + \frac{1}{pG(t)\varphi(t)} + \frac{1}{pG(t)\varphi(t)\log\varphi(t)}\right].\end{aligned} \tag{2.20}$$

Let us denote

$$A(t) := 1 + \frac{1}{pG(t)\varphi(t)} + \frac{1}{pG(t)\varphi(t)\log\varphi(t)}.$$

Then

$$r(t)\Phi(\tilde{x}'(t)) = r(t)\Phi(h'(t))\varphi^{\frac{p-1}{p}}(t)\big(\log\varphi(t)\big)^{\frac{p-1}{p}}A^{p-1}(t)$$

and hence,

$$\begin{aligned}\big(r(t)\Phi(\tilde{x}'(t))\big)' = {} & \big(r(t)\Phi(h'(t))\big)'\varphi^{\frac{p-1}{p}}(t)\big(\log\varphi(t)\big)^{\frac{p-1}{p}}A^{p-1}(t)\\ & + \frac{p-1}{p}r(t)\Phi(h'(t))R^{-1}(t)\varphi^{\frac{-1}{p}}(t)(\log\varphi(t))^{\frac{p-1}{p}}A^{p-1}(t)\\ & + \frac{p-1}{p}r(t)\Phi(h'(t))R^{-1}(t)\varphi^{\frac{-1}{p}}(t)\big(\log\varphi(t)\big)^{-\frac{1}{p}}A^{p-1}(t)\\ & + (p-1)r(t)\Phi(h'(t))\varphi^{\frac{p-1}{p}}(t)\big(\log\varphi(t)\big)^{\frac{p-1}{p}}A^{p-2}(t)A'(t).\end{aligned}$$

Consequently,

$$\tilde{x}(t)\big(r(t)\Phi(\tilde{x}'(t))\big)' = h(t)A^{p-2}(t)B(t),$$

where

$$\begin{aligned}B(t) := {} & \big(r(t)\Phi(h'(t))\big)'\varphi(t)\log\varphi(t)A(t) + \frac{p-1}{p}r(t)\Phi(h'(t))R^{-1}(t)\log\varphi(t)A(t)\\ & + \frac{p-1}{p}r(t)\Phi(h'(t))R^{-1}(t)A(t) + (p-1)r(t)\Phi(h'(t))\varphi(t)\log\varphi(t)A'(t).\end{aligned}$$

Next, for the derivative of $A(t)$ we have

$$\begin{aligned}A'(t) = {} & -\frac{G'(t)\varphi(t) + G(t)R^{-1}(t)}{pG^2(t)\varphi^2(t)}\\ & -\frac{G'(t)\varphi(t)\log\varphi(t) + G(t)R^{-1}(t)\log\varphi(t) + G(t)R^{-1}(t)}{pG^2(t)\varphi^2(t)\log^2\varphi(t)}\\ = {} & -\frac{G'(t)}{pG^2(t)\varphi(t)} - \frac{1}{pG(t)R(t)\varphi^2(t)} - \frac{G'(t)}{pG^2(t)\varphi(t)\log\varphi(t)}\\ & -\frac{1}{pG(t)R(t)\varphi^2(t)\log\varphi(t)} - \frac{1}{pG(t)R(t)\varphi^2(t)\log^2\varphi(t)},\end{aligned}$$

hence, substituting formulas for $A(t)$ and $A'(t)$ in $B(t)$, we obtain

$$\begin{aligned}B(t) = {} & (r\Phi(h'(t)))'\varphi(t)\log\varphi(t) + \frac{(r(t)\Phi(h'(t)))'\log\varphi(t)}{pG(t)} + \frac{(r(t)\Phi(h'(t)))'}{pG(t)}\\ & + \frac{(p-1)r(t)\Phi(h'(t))\log\varphi(t)}{pR(t)} - \frac{(p-1)^2r(t)\Phi(h'(t))\log\varphi(t)}{p^2G(t)R(t)\varphi(t)}\\ & - \frac{(p-1)(p-2)r(t)\Phi(h'(t))}{p^2G(t)R(t)\varphi(t)} + \frac{(p-1)r\Phi(h')}{pR(t)} - \frac{(p-1)^2r(t)\Phi(h'(t))}{p^2G(t)R(t)\varphi(t)\log\varphi(t)}\\ & - \frac{(p-1)r(t)\Phi(h'(t))G'(t)\log\varphi(t)}{pG^2(t)} - \frac{(p-1)r(t)\Phi(h'(t))G'(t)}{pG^2(t)}.\end{aligned}$$

Using the fact that $G' = h(r\Phi(h'))' + h'r\Phi(h')$ and $hG = h'R$, we simplify the previous formula as follows

$$\begin{aligned} B(t) &= (r(t)\big(\Phi(h'(t))\big)'\varphi(t)\log\varphi(t) + \frac{(2-p)(r(t)\big(\Phi(h'(t))\big)'\log\varphi(t)}{pG(t)} \\ &\quad + \frac{(2-p)(r(t)\big(\Phi(h'(t))\big)'}{pG(t)} - \frac{(p-1)^2 r(t)\Phi(h'(t))\log\varphi(t)}{p^2G(t)R(t)\varphi(t)} \\ &\quad - \frac{(p-1)(p-2)r(t)\Phi(h'(t))}{p^2G(t)R(t)\varphi(t)} - \frac{(p-1)^2 r(t)\Phi(h'(t))}{p^2G(t)R(t)\varphi(t)\log\varphi(t)}. \end{aligned}$$

To express $A^{p-2}(t)$ we use the power expansion

$$(1+x)^s = \sum_{j=0}^{\infty}\binom{s}{j}x^j, \quad |x|<1,\ s\in\mathbb{R} \tag{2.21}$$

with

$$x = \frac{1}{pG(t)\varphi(t)} + \frac{1}{pG(t)\varphi(t)\log\varphi(t)}.$$

Note that the applicability of this power expansion is guaranteed by conditions (1.7). Hence

$$\begin{aligned} A^{p-2}(t) &= \sum_{j=0}^{\infty}\binom{p-2}{j}\left[\frac{1}{pG(t)\varphi(t)} + \frac{1}{pG(t)\varphi(t)\log\varphi(t)}\right]^j \\ &= 1 + \frac{p-2}{pG(t)\varphi(t)} + \frac{p-2}{pG(t)\varphi(t)\log\varphi(t)} \\ &\quad + \frac{(p-2)(p-3)}{2p^2G^2(t)\varphi^2(t)} + \frac{(p-2)(p-3)}{p^2G^2(t)\varphi^2(t)\log\varphi(t)} + \frac{(p-2)(p-3)}{2p^2G^2(t)\varphi^2(t)\log^2\varphi(t)} \\ &\quad + \frac{(p-2)(p-3)(p-4)}{6p^3G^3(t)\varphi^3(t)} + o\big(\varphi^{-3}(t)\big) \quad \text{as } t\to\infty. \end{aligned}$$

By a direct computation we obtain

$$\begin{aligned} A^{p-2}(t)B(t) &= (r(t)\big(\Phi(h'(t))\big)'\varphi(t)\log\varphi(t) \\ &\quad + \frac{(p-2)(1-p)\log\varphi(t)}{2p^2G^2(t)\varphi(t)}(r(t)\big(\Phi(h'(t))\big)' - \frac{(p-1)^2\log\varphi(t)}{p^2G(t)R(t)\varphi(t)}r(t)\Phi(h'(t)) \\ &\quad + \frac{(p-2)(1-p)}{p^2G^2(t)\varphi(t)}(r(t)\big(\Phi(h'(t))\big)' - \frac{(p-1)(p-2)}{p^2G(t)R(t)\varphi(t)}r(t)\Phi(h'(t)) \\ &\quad + \frac{(p-2)(1-p)}{2p^2G^2(t)\varphi(t)\log\varphi(t)}(r(t)\big(\Phi(h'(t))\big)' - \frac{(p-1)^2}{p^2G(t)R(t)\varphi(t)\log\varphi(t)}r(t)\Phi(h'(t)) \\ &\quad + \frac{(p-2)(1-p)(p-3)\log\varphi(t)}{3p^3G^3(t)\varphi^2(t)}(r(t)\big(\Phi(h'(t))\big)'(1+o(1)) \\ &\quad - \frac{(p-1)^2(p-2)\log\varphi(t)}{p^3G(t)R(t)\varphi^2(t)}r(t)\Phi(h'(t))(1+o(1)) \quad \text{as } t\to\infty. \end{aligned}$$

Now, using the identities

$$\frac{(r\Phi(h'))'}{G} = \frac{G'}{hG} - \frac{h'}{h^2} \quad \text{and} \quad \frac{r\Phi(h')}{R} = \frac{h'}{h^2},$$

which follow from the definitions of R, G in (1.8), we get

$$\begin{aligned}
\tilde{x}(t)\big(r(t)\Phi(\tilde{x}'(t))\big)' &= h(t)A^{p-2}(t)B(t) \\
&= h(t)(r(t)\big(\Phi(h'(t))\big)'\varphi(t)\log\varphi(t) \\
&\quad + \frac{\log\varphi(t)}{G(t)\varphi(t)}\left[\frac{(p-2)(1-p)}{2p^2}\frac{G'(t)}{G(t)} - \frac{p-1}{2p}\frac{h'(t)}{h(t)}\right] \\
&\quad + \frac{1}{G(t)\varphi(t)}\left[\frac{(p-2)(1-p)}{p^2}\frac{G'(t)}{G(t)}\right] \\
&\quad + \frac{1}{G(t)\varphi(t)\log\varphi(t)}\left[\frac{(p-2)(1-p)}{2p^2}\frac{G'(t)}{G(t)} - \frac{p-1}{2p}\frac{h'(t)}{h(t)}\right] \\
&\quad + \frac{\log\varphi(t)}{G^2(t)\varphi^2(t)}\left[\frac{(p-2)(1-p)(p-3)}{3p^3}\frac{G'(t)}{G(t)} - \frac{2(p-1)(p-2)}{3p^2}\frac{h'(t)}{h(t)}\right] \\
&\quad + \frac{G'(t)}{G^2(t)}o\left(\frac{\log\varphi(t)}{\varphi^2(t)}\right) + \frac{h'(t)}{h(t)G^2(t)}o\left(\frac{\log\varphi(t)}{\varphi^2(t)}\right) \quad \text{as } t\to\infty.
\end{aligned}$$

Finally, we have

$$\tilde{x}(t)\hat{L}[\tilde{x}](t) = \tilde{x}(t)\big(r(t)\Phi(\tilde{x}'(t))\big)' + c(t)h^p(t)\varphi(t)\log\varphi(t) + \frac{\log\varphi(t)}{2qR(t)\varphi(t)}.$$

Using the facts that h is a solution of (1.1), $h'/h = G/R$ and $q = p/(p-1)$, the last two formulas lead to (2.19). □

Lemma 2.4. *Let h be a positive solution of* (1.1) *such that $h'(t) \neq 0$ for large t and that* (1.7) *holds. Further let $\tilde{x}(t) := h(t)(\int^t R^{-1}(s)\,ds)^{\frac{1}{p}}\log^{\frac{2}{p}}(\int^t R^{-1}(s)\,ds)$. Then*

$$\begin{aligned}
\tilde{x}\hat{L}[\tilde{x}] &= \frac{(p-2)(1-p)G'(t)\log^2(\int^t R^{-1}(s)\,ds)}{2p^2G^2(t)(\int^t R^{-1}(s)\,ds)}(1+o(1)) \\
&\quad + \frac{2(p-2)(1-p)\log^2(\int^t R^{-1}(s)\,ds)}{3p^2G(t)R(t)(\int^t R^{-1}(s)\,ds)^2}(1+o(1))
\end{aligned} \tag{2.22}$$

as $t\to\infty$.

Proof. We use notation (2.18) and, suppressing the argument t, we proceed similarly as in the proof of Lemma 2.3. By a direct differentiation of $\tilde{x}$ and since $hG = h'R$, we obtain

$$\begin{aligned}
\tilde{x}' &= h'\varphi^{\frac{1}{p}}\log^{\frac{2}{p}}\varphi + \frac{1}{p}hR^{-1}\varphi^{\frac{1}{p}-1}\log^{\frac{2}{p}}\varphi + \frac{2}{p}hR^{-1}\varphi^{\frac{1}{p}-1}\log^{\frac{2}{p}-1}\varphi \\
&= h'\varphi^{\frac{1}{p}}\log^{\frac{2}{p}}\varphi\left[1 + \frac{1}{pG\varphi} + \frac{2}{pG\varphi\log\varphi}\right].
\end{aligned} \tag{2.23}$$

Let us denote

$$\bar{A} := 1 + \frac{1}{pG\varphi} + \frac{2}{pG\varphi\log\varphi}.$$

Then

$$r\Phi(\tilde{x}') = r\Phi(h')\varphi^{\frac{p-1}{p}}\log^{\frac{2p-2}{p}}\varphi\bar{A}^{p-1}$$

and its differentiation gives

$$\begin{aligned}(r\Phi(\tilde{x}'))' &= (r\Phi(h'))'\varphi^{\frac{p-1}{p}}\log^{\frac{2p-2}{p}}\varphi \bar{A}^{p-1} + \frac{p-1}{p} r\Phi(h')R^{-1}\varphi^{\frac{-1}{p}}\log^{\frac{2p-2}{p}}\varphi\bar{A}^{p-1} \\ &\quad + \frac{2p-2}{p} r\Phi(h')R^{-1}\varphi^{\frac{-1}{p}}\log^{1-\frac{2}{p}}\varphi\bar{A}^{p-1} + (p-1)r\Phi(h')\varphi^{\frac{p-1}{p}}\log^{\frac{2p-2}{p}}\varphi\bar{A}^{p-2}\bar{A}' \\ &= \varphi^{-\frac{1}{p}}\log^{1-\frac{2}{p}}\varphi\bar{A}^{p-2}\Big\{(r\Phi(h'))'\varphi\log\varphi\bar{A} + \frac{p-1}{p} r\Phi(h')R^{-1}\log\varphi\bar{A} \\ &\qquad + \frac{2(p-1)}{p} r\Phi(h')R^{-1}\bar{A} + (p-1)r\Phi(h')\varphi\log\varphi\bar{A}'\Big\},\end{aligned}$$

where

$$\bar{A}' = -\frac{G'}{pG^2\varphi} - \frac{1}{pGR\varphi^2} - \frac{2G'}{pG^2\varphi\log\varphi} - \frac{2}{pGR\varphi^2\log\varphi} - \frac{2}{pGR\varphi^2\log^2\varphi}.$$

Denote the inside of the last curly brackets as $\bar{B}$. With the use of formulas for $\bar{A}$ and $\bar{A}'$ followed by the fact that $G' = h(r\Phi(h'))' + h'r\Phi(h')$ and $hG = h'R$ we get

$$\begin{aligned}\bar{B} &= (r\Phi(h'))'\varphi\log\varphi + \frac{(r\Phi(h'))'\log\varphi}{pG} + \frac{2(r\Phi(h'))'}{pG} + \frac{(p-1)r\Phi(h')\log\varphi}{pR} \\ &\quad - \frac{(p-1)^2 r\Phi(h')\log\varphi}{p^2GR\varphi} + \frac{2(p-1)(2-p)r\Phi(h')}{p^2GR\varphi} + \frac{2(p-1)r\Phi(h')}{pR} \\ &\quad + \frac{2(p-1)(2-p)r\Phi(h')}{p^2GR\varphi\log\varphi} - \frac{(p-1)r\Phi(h')G'\log\varphi}{pG^2} - \frac{2(p-1)r\Phi(h')G'}{pG^2} \\ &= (r(\Phi(h'))'\varphi\log\varphi + \frac{(2-p)(r(\Phi(h'))'\log\varphi}{pG} + \frac{2(2-p)(r(\Phi(h'))'}{pG} \\ &\quad - \frac{(p-1)^2 r\Phi(h')\log\varphi}{p^2GR\varphi} + \frac{2(p-1)(2-p)r\Phi(h')}{p^2GR\varphi} + \frac{2(p-1)(2-p)r\Phi(h')}{p^2GR\varphi\log\varphi}.\end{aligned}$$

Next, since conditions (1.7) hold, we can use the power expansion (2.21) with

$$x = \frac{1}{pG\varphi} + \frac{2}{pG\varphi\log\varphi}$$

and we obtain

$$\begin{aligned}\bar{A}^{p-2} &= 1 + \frac{p-2}{pG\varphi} + \frac{2(p-2)}{pG\varphi\log\varphi} + \frac{(p-2)(p-3)}{2p^2G^2\varphi^2} + \frac{2(p-2)(p-3)}{p^2G^2\varphi^2\log\varphi} \\ &\quad + \frac{2(p-2)(p-3)}{p^2G^2\varphi^2\log^2\varphi} + \frac{(p-2)(p-3)(p-4)}{6p^3G^3\varphi^3} + o(\varphi^{-3}) \quad \text{as } t\to\infty.\end{aligned}$$

Expanding $\bar{A}^{p-2}\bar{B}$ and joining the terms together with respect to φ yields

$$\begin{aligned}\bar{A}^{p-2}\bar{B} &= (r\Phi(h'))'\varphi\log\varphi \\ &\quad + \frac{\log\varphi}{G\varphi}\left(-\frac{(p-1)^2}{p^2}\frac{r\Phi(h')}{R} + \frac{(p-2)(1-p)}{2p^2}\frac{(r\Phi(h'))'}{G}\right) \\ &\quad + \frac{1}{G\varphi}\left(\frac{2(p-1)(2-p)}{p^2}\frac{r\Phi(h')}{R} + \frac{2(p-2)(1-p)}{p^2}\frac{(r\Phi(h'))'}{G}\right) \\ &\quad + \frac{1}{G\varphi\log\varphi}\left(\frac{2(p-1)(2-p)}{p^2}\frac{r\Phi(h')}{R} + \frac{2(p-2)(1-p)}{p^2}\frac{(r\Phi(h'))'}{G}\right)\end{aligned}$$

$$+\frac{\log\varphi}{G^2\varphi^2}\left(\frac{(p-1)^2(2-p)}{p^3}\frac{r\Phi(h')}{R}+\frac{(p-2)(1-p)(p-3)}{3p^2}\frac{(r\Phi(h'))'}{G}\right)$$
$$+\frac{r\Phi(h')}{R}o\left(\frac{\log\varphi}{G^2\varphi^2}\right)+\frac{(r\Phi(h'))'}{G}o\left(\frac{\log\varphi}{G^2\varphi^2}\right)\quad \text{as } t\to\infty.$$

Using the identities

$$\frac{r\Phi(h')}{R}=\frac{h'}{h^2},\qquad \frac{(r\Phi(h'))'}{G}=\frac{G'}{hG}-\frac{h'}{h^2},$$

the above product $\bar{A}^{p-2}\bar{B}$ simplifies to

$$\begin{aligned}\bar{A}^{p-2}\bar{B}&=(r\Phi(h'))'\varphi\log+\frac{\log\varphi}{G\varphi}\left(-\frac{(p-1)}{2p}\frac{h'}{h^2}+\frac{(p-2)(1-p)}{2p^2}\frac{G'}{hG}\right)\\&\quad+\frac{1}{G\varphi}\left(\frac{2(p-2)(1-p)}{p^2}\frac{G'}{hG}\right)+\frac{1}{G\varphi\log\varphi}\left(\frac{2(p-2)(1-p)}{p^2}\frac{G'}{hG}\right)\\&\quad+\frac{\log\varphi}{G^2\varphi^2}\left(\frac{2(p-1)(2-p)}{3p^2}\frac{h'}{h^2}+\frac{(p-2)(1-p)(p-3)}{3p^2}\frac{G'}{hG}\right)\\&\quad+\frac{h'}{h^2}o\left(\frac{\log\varphi}{G^2\varphi^2}\right)+\frac{G'}{hG}o\left(\frac{\log\varphi}{G^2\varphi^2}\right)\quad \text{as } t\to\infty.\end{aligned}$$

Altogether we have

$$\begin{aligned}\tilde{x}\hat{L}[\tilde{x}]&=\tilde{x}(r\Phi(\tilde{x}'))'+c\tilde{x}^p+\frac{\tilde{x}^p}{2qh^pR\varphi^2}\\&=h\log\varphi\bar{A}^{p-2}\bar{B}+ch^p\varphi\log^2\varphi+\frac{\log^2\varphi}{2qR\varphi}.\end{aligned}$$

Since h solves the equation $(r\Phi(h'))'+c\Phi(h)=0$, $\frac{1}{q}=\frac{p-1}{p}$ and $\frac{1}{R}=\frac{h'}{Gh}$, we finally obtain

$$\begin{aligned}\tilde{x}\hat{L}[\tilde{x}]&=\frac{(p-2)(1-p)}{2p^2}\frac{G'\log^2\varphi}{G^2\varphi}+\frac{2(p-2)(1-p)}{p^2}\frac{G'\log\varphi}{G^2\varphi}\\&\quad+\frac{2(p-2)(1-p)}{p^2}\frac{G'}{G^2\varphi}(1+o(1))+\frac{2(p-1)(2-p)}{3p^2}\frac{\log^2\varphi}{GR\varphi^2}(1+o(1)).\end{aligned}$$

as $t\to\infty$. This means that $\tilde{x}\hat{L}[\tilde{x}]$ can be written in the form (2.22). □

3 Oscillation and nonoscillation criteria for (1.10)

The following theorem is an improved version of [10, Theorem 5]. In contrast to that result, we do not need the condition

$$\lim_{t\to\infty}\log^2\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)R(t)G'(t)=0$$

considered in [10] and we have generalized the statement to $\alpha\neq\frac{1}{2}$.

Theorem 3.1. *Suppose that h is a positive solution of (1.1) such that $h'(t)\neq 0$ for large t, (1.7) holds and the integral $\int^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t$ converges. If*

$$\limsup_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s<\frac{1}{q}(-\alpha+\sqrt{2\alpha}),\tag{3.1}$$
$$\liminf_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s>\frac{1}{q}(-\alpha-\sqrt{2\alpha})\tag{3.2}$$

for some $\alpha>0$, then (1.10) is nonoscillatory.

Proof. The idea of the proof is to apply Theorem A to equation (1.10), i.e., $L := \tilde{L}$. We take $\tilde{x}(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}}$. By a direct differentiation and using the fact that $h'R = hG$, we get

$$\tilde{x}'(t) = h'(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)^{\frac{1}{p}}\left[1 + \tfrac{1}{pG(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right].$$

Now we express the functions $\tilde{R}$ and $\tilde{G}$ defined in (2.1) for this concrete $\tilde{x}$ and use (1.8) and (1.7) to obtain

$$\begin{aligned} \tilde{R}(t) &= r(t)\tilde{x}^2(t)|\tilde{x}'(t)|^{p-2} \\ &= r(t)h^2(t)|h'(t)|^{p-2}\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)\left[1 + \tfrac{1}{pG(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right]^{p-2} \\ &= R(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)\big(1 + o(1)\big) \quad \text{as } t \to \infty \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \tilde{G}(t) &= r(t)\tilde{x}(t)\Phi(\tilde{x}'(t)) \\ &= r(t)h(t)\Phi(h'(t))\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)\left[1 + \tfrac{1}{pG(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right]^{p-1} \\ &= G(t)\big(\int^t R^{-1}(s)\,\mathrm{d}s\big)\big(1 + o(1)\big) \quad \text{as } t \to \infty. \end{aligned} \tag{3.4}$$

It follows from (3.3) that

$$\int_T^t \tilde{R}^{-1}(s)\,\mathrm{d}s = (1 + o(1))\log\left(\int^t R^{-1}\right) - K, \quad K \in \mathbb{R},$$

hence conditions (1.7) and (3.4) imply that (2.2) is fulfilled.

Since

$$\tilde{x}(t)\tilde{L}[\tilde{x}](t) = \tilde{x}(t)\hat{L}[\tilde{x}](t) + g(t)|\tilde{x}(t)|^p = \tilde{x}(t)\hat{L}[\tilde{x}](t) + g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s,$$

Lemma 2.2 and the condition for the convergence of $\int^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t$ guarantee that the integral $\int^\infty \tilde{x}(t)\tilde{L}[\tilde{x}](t)\,\mathrm{d}t$ is convergent and we have

$$\begin{aligned} &\int_T^t \tilde{R}^{-1}(s)\ \mathrm{d}s \int_t^\infty \tilde{x}(s)\tilde{L}[\tilde{x}](s)\ \mathrm{d}s \\ &\quad \sim \log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \left(\tilde{x}(s)\hat{L}[\tilde{x}](s) + g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\ \mathrm{d}s \end{aligned} \tag{3.5}$$

as $t \to \infty$. Now we show that

$$\lim_{t\to\infty} \log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \tilde{x}\hat{L}[\tilde{x}](s)\,\mathrm{d}s = 0. \tag{3.6}$$

By (2.17), it is sufficient to show that

$$\lim_{t\to\infty} \log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \frac{1}{G(s)R(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\,\mathrm{d}s = 0 \tag{3.7}$$

and

$$\lim_{t\to\infty} \log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \frac{G'(s)}{G^2(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s = 0. \tag{3.8}$$

Since $\lim_{t\to\infty}\int_t^\infty \frac{1}{GR(\int^t R^{-1})^2}\,\mathrm{d}s = 0$, using L'Hospital's rule and (2.13) we have

$$\begin{aligned}
&\lim_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \frac{1}{G(s)R(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\,\mathrm{d}s \\
&\quad=\lim_{t\to\infty}\frac{-G^{-1}(t)R^{-1}(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^{-2}}{-\log^{-2}(\int^t R^{-1}(s)\,\mathrm{d}s)(\int^t R^{-1}(s)\,\mathrm{d}s)^{-1}R^{-1}(t)} \\
&\quad=\lim_{t\to\infty}\frac{\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}=0,
\end{aligned}$$

hence (3.7) holds. To show (3.8), we use integration by parts

$$\int_t^\infty \frac{G'(s)}{G^2(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s = \frac{1}{G(t)\int^t R^{-1}(t)\,\mathrm{d}t} - \int_t^\infty \frac{1}{G(s)R(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\,\mathrm{d}s,$$

which, together with (2.13) and (3.7), yields to (3.8). Hence (3.6) is proved. Consequently, by (3.5), we obtain

$$\begin{aligned}
&\int_T^t \tilde{R}^{-1}(s)\ \mathrm{d}s\int_t^\infty \tilde{x}(s)\tilde{L}[\tilde{x}](s)\ \mathrm{d}s \\
&\qquad\sim\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}\tag{3.9}$$

as $t\to\infty$. This means that conditions (2.3), (2.4) follow from (3.1), (3.2). All the assumptions of Theorem A are fulfilled, hence (1.10) is nonoscillatory. □

The next statement is an oscillatory counterpart of Theorem 3.1.

Theorem 3.2. *Suppose that h is a positive solution of (1.1) such that $h'(t)\neq 0$ for large t, (1.7) holds, the integral $\int^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t$ converges and let there exist constants γ_1 and γ_2 such that*

$$g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s \geq \frac{\gamma_1|G'(t)|}{G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \frac{\gamma_2}{|G(t)|R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\tag{3.10}$$

for large t, where

$$\gamma_1 > \frac{(p-1)(p-2)}{2p^2}\operatorname{sgn} G',\qquad \gamma_2 > \frac{2(p-1)(p-2)}{3p^2}\operatorname{sgn} G.\tag{3.11}$$

If

$$\liminf_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s > \frac{1}{2q}\tag{3.12}$$

then (1.10) is oscillatory.

Proof. We apply Theorem B with $L := \tilde{L}$. Taking $\tilde{x}(t) := h(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^{\frac{1}{p}}$ we obtain (3.3) and (3.4). Consequently, conditions (1.7) imply that both conditions in (2.6) are satisfied. Similarly to the proof of Theorem 3.1, we conclude that the second condition in (2.5) holds due to Lemma 2.2, since

$$\tilde{x}(t)\tilde{L}[\tilde{x}](t) = \tilde{x}(t)\hat{L}[\tilde{x}](t) + g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s$$

and condition (2.7) follows from (3.9) and (3.12). Concerning the first condition in (2.5), we have from Lemma 2.2 that

$$\begin{aligned}\tilde{x}(t)\tilde{L}[\tilde{x}](t) = &\frac{(p-2)(1-p)G'(t)}{2p^2G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s}(1+o(1))\\ &+\frac{2(1-p)(p-2)}{3p^2G(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2}(1+o(1))\\ &+g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\end{aligned}$$

as $t\to\infty$. Hence, the first condition in (2.5) is ensured by (3.10). Equation (1.10) is oscillatory by Theorem B. □

In the next theorem we handle equation (1.10) in the case, when the perturbation $g(t)$ is of the form

$$g(t) := \frac{\lambda}{h^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}, \quad \lambda\in\mathbb{R}. \tag{3.13}$$

In this special case equation (1.10) becomes conditionally oscillatory.

Theorem 3.3. *Suppose that h is a positive solution of (1.1) such that $h'(t)\neq 0$ for large t and (1.7) holds and consider the equation*

$$(r(t)\Phi(x'))' + \left[c(t) + \frac{1}{h^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\left(\frac{1}{2q}+\frac{\lambda}{\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}\right)\right]\Phi(x)=0. \tag{3.14}$$

If $\lambda\le\frac{1}{2q}$, then (3.14) is nonoscillatory. If $\lambda>\frac{1}{2q}$ and there exists a constant γ such that

$$\frac{1}{R(t)\log^2(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)} \ge \frac{\gamma|G'(t)|}{G^2(t)}, \quad \gamma > \frac{p-2}{p}\operatorname{sgn}G'(t) \tag{3.15}$$

holds for large t, then (3.14) is oscillatory.

Proof. If $\lambda\neq\frac{1}{2q}$, then the statement follows from Theorem 3.1 and Theorem 3.2. Indeed, if $g(t)$ is given by (3.13), then

$$\begin{aligned}\int_T^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t &= \int_T^\infty \frac{\lambda}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}\,\mathrm{d}t\\ &= \frac{\lambda}{\log(\int^T R^{-1}(s)\,\mathrm{d}s)} - \lim_{t\to\infty}\frac{\lambda}{\log(\int^t R^{-1}(s)\,\mathrm{d}s)},\end{aligned}$$

so the integral $\int^\infty g(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t$ is convergent. Consequently, concerning conditions (3.1), (3.2) and (3.12), we have

$$\lim_{t\to\infty}\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty g(s)h^p(s)\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s=\lambda.$$

Hence, if $-\frac{3}{2q}<\lambda<\frac{1}{2q}$, then (3.14) is nonoscillatory by Theorem 3.1, where we take $\alpha=\frac{1}{2}$ in (3.1) and (3.2). If $\lambda\le-\frac{3}{2q}$, the nonoscillation of (3.14) follows form the well-known Sturm comparison theorem. If $\lambda>\frac{1}{2q}$, we use Theorem 3.2. It remains to show that condition (3.15) is sufficient for (3.10). Since $\lambda > \frac{1}{2q}$, there exists $\varepsilon > 0$ such that $\lambda = \frac{1}{2q} + \varepsilon$ and condition (3.10) with $g(t)$ defined in (3.13) can be written as

$$\frac{\frac{1}{2q} + \varepsilon}{R(t)\log^2\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)} \geq \frac{\gamma_1 |G'(t)|}{G^2(t)} + \frac{\gamma_2}{|G(t)|R(t)\int^t R^{-1}(s)\,\mathrm{d}s},$$

where γ_1, γ_2 satisfy (3.11). If we set $\gamma_1 := \frac{\gamma}{2q}$, then

$$\gamma_1 > \frac{(p-2)\operatorname{sgn} G'(t)}{2pq} = \frac{(p-1)(p-2)}{2p^2}\operatorname{sgn} G'$$

and condition (3.15) implies that

$$\frac{1}{2qR(t)\log^2\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)} \geq \frac{\gamma_1 |G'(t)|}{G^2(t)} \quad \text{for large } t.$$

Hence, it remains to show that

$$\frac{\varepsilon}{\log^2\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)} \geq \frac{\gamma_2}{|G(t)|\int^t R^{-1}(s)\,\mathrm{d}s},$$

i.e.,

$$\varepsilon \geq \frac{\gamma_2 \log^2\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)}{|G(t)|\int^t R^{-1}(s)\,\mathrm{d}s}$$

for large t. This inequality is satisfied for any $\gamma_2 \in \mathbb{R}$ since the limit of the function on the right-hand side of this inequality is zero as $t \to \infty$. Oscillation of (3.14) follows from Theorem 3.2.

In the remaining part of the proof we deal with the critical case $\lambda = \frac{1}{2q}$. We use Theorem A with $\tilde{x}(t) := h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \log^{\frac{1}{p}}\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)$ and

$$L(t) := \hat{L}(t) + \frac{1}{2qh^p(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2 \log^2\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)}\Phi(x(t)). \tag{3.16}$$

In this case we have from (2.20)

$$\begin{aligned}\tilde{x}'(t) &= h'(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \log^{\frac{1}{p}}\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\\ &\quad\times\left[1 + \frac{1}{pG(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \frac{1}{pG(t)\int^t R^{-1}(s)\,\mathrm{d}s\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)}\right],\end{aligned}$$

hence, according to (2.1) (suppressing the arguments) we have

$$\begin{aligned}\tilde{R} &= rh^2|h'|^{p-2}\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)\left[1 + \frac{1}{pG\left(\int^t R^{-1}\right)} + \frac{2}{pG\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)}\right]^{p-2}\\ &= R\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)\left(1 + o(1)\right) \quad \text{as } t \to \infty\end{aligned}$$

and

$$\begin{aligned}\tilde{G} &= rh\Phi(h')\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)\left[1 + \frac{1}{pG\left(\int^t R^{-1}\right)} + \frac{2}{pG\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)}\right]^{p-1}\\ &= rh\Phi(h')\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)\left(1 + o(1)\right) \quad \text{as } t \to \infty. \end{aligned} \tag{3.17}$$

These computations and conditions (1.7) imply that

$$\int_T^t \tilde{R}^{-1}(s)\,\mathrm{d}s \sim \log\left(\log\int^t R^{-1}(s)\,\mathrm{d}s\right) \quad \text{as } t\to\infty \tag{3.18}$$

and thus condition (2.2) is satisfied.

Next, since $\frac{p}{p-1}=q$, from (3.16) and Lemma 2.3 we obtain

$$\begin{aligned}
\tilde{x}(t)L[\tilde{x}](t) &= \tilde{x}(t)\hat{L}[\tilde{x}](t) + \frac{1}{2qR(t)(\int^t R^{-1}(s)\,\mathrm{d}s)\log(\int^t R^{-1}(s)\,\mathrm{d}s)} \\
&= \frac{G'(t)\log(\int^t R^{-1}(s)\,\mathrm{d}s)}{G^2(t)\int^t (R^{-1}(s)\,\mathrm{d}s)}\left[\frac{(p-2)(1-p)}{2p^2}+o(1)\right] \\
&\quad + \frac{\log(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2}\left[\frac{2(p-2)(1-p)}{3p^2}+o(1)\right]
\end{aligned} \tag{3.19}$$

as $t\to\infty$. By Lemma 2.1 we have that $\int^\infty \tilde{x}(t)L[\tilde{x}](t)\,\mathrm{d}t$ is convergent. Using L'Hospital's rule we obtain

$$\begin{aligned}
&\lim_{t\to\infty}\log\left(\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\right)\int_t^\infty \frac{\log(\int^s (R^{-1}(\tau)\,\mathrm{d}\tau))}{G(s)R(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\,\mathrm{d}s \\
&= \lim_{t\to\infty}\frac{-\log(\int^t R^{-1}(s)\,\mathrm{d}s)G^{-1}(t)R^{-1}(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^{-2}}{-\log^{-2}(\log(\int^t R^{-1}(s)\,\mathrm{d}s))\log^{-1}(\int^t R^{-1}(s)\,\mathrm{d}s)(\int^t R^{-1}(s)\,\mathrm{d}s)^{-1}R^{-1}(t)} \\
&= \lim_{t\to\infty}\frac{\log^2(\log(\int^t R^{-1}(s)\,\mathrm{d}s))\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s} = 0.
\end{aligned} \tag{3.20}$$

Integration by parts gives

$$\int_t^\infty \frac{G'(s)\log(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)}{G^2(s)\int^s (R^{-1}(\tau)\,\mathrm{d}\tau)}\,\mathrm{d}s = \frac{\log(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \int_t^\infty \frac{1-\log(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)}{G(s)R(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)^2}\,\mathrm{d}s,$$

hence, by Lemma 2.1 and (3.20), we obtain

$$\lim_{t\to\infty}\log\left(\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\right)\int_t^\infty \frac{G'(s)\log(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)}{G^2(s)\int^s (R^{-1}(\tau)\,\mathrm{d}\tau)}\,\mathrm{d}s = 0. \tag{3.21}$$

Consequently, (3.20) and (3.21) imply

$$\lim_{t\to\infty}\int_T^t \tilde{R}^{-1}(s)\,\mathrm{d}s\int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\,\mathrm{d}s = \lim_{t\to\infty}\log\left(\log\int^t R^{-1}(s)\,\mathrm{d}s\right)\int_t^\infty \tilde{x}(s)L[\tilde{x}](s)\,\mathrm{d}s = 0.$$

Hence, conditions (2.3) and (2.4) are satisfied with $\alpha=\frac{1}{2}$ and this means that (3.14) with $\lambda=\frac{1}{2q}$ is nonoscillatory by Theorem A. □

The next result is a nonoscillatory criterion for (1.10) based on Theorem C.

Theorem 3.4. *Let h be a positive solution of* (1.1) *such that $h'(t)\neq 0$ for large t and* (1.7) *holds. If*

$$\limsup_{t\to\infty}\frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)\log^2\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\log(\int^t R^{-1}(s)\,\mathrm{d}s)} < \frac{1}{q}(-\alpha+\sqrt{2\alpha}), \tag{3.22}$$

$$\liminf_{t\to\infty}\frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)\log^2\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\log(\int^t R^{-1}(s)\,\mathrm{d}s)} > \frac{1}{q}(-\alpha-\sqrt{2\alpha}) \tag{3.23}$$

for some $\alpha>0$, $T\in\mathbb{R}$ sufficiently large, then equation (1.10) *is nonoscillatory.*

Proof. Take $\tilde{x}(t) = h(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^{\frac{1}{p}} \log^{\frac{2}{p}}(\int^t R^{-1}(s)\,\mathrm{d}s)$ and $L := \tilde{L}$ in Theorem C.

Using (2.1) and (2.23) we express (suppressing the arguments)

$$\begin{aligned}\tilde{R} &= rh^2|h'|^{p-2}\big(\textstyle\int^t R^{-1}\big)\log^2\big(\int^t R^{-1}\big)\left[1+\frac{1}{pG\left(\int^t R^{-1}\right)}+\frac{2}{pG\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)}\right]^{p-2}\\ &= R\big(\textstyle\int^t R^{-1}\big)\log^2\big(\int^t R^{-1}\big)\big(1+o(1)\big) \quad \text{as } t\to\infty\end{aligned}$$

and

$$\begin{aligned}\tilde{G} &= rh\Phi(h')\big(\textstyle\int^t R^{-1}\big)\log^2\big(\int^t R^{-1}\big)\left[1+\frac{1}{pG\left(\int^t R^{-1}\right)}+\frac{2}{pG\left(\int^t R^{-1}\right)\log\left(\int^t R^{-1}\right)}\right]^{p-1}\\ &= G\big(\textstyle\int^t R^{-1}\big)\log^2\big(\int^t R^{-1}\big)\big(1+o(1)\big) \quad \text{as } t\to\infty.\end{aligned}$$

From these formulas we have that the integral $\int^\infty \tilde{R}^{-1}(t)\,\mathrm{d}t$ is convergent since

$$\begin{aligned}\int_T^\infty \frac{1}{\tilde{R}(s)}\,\mathrm{d}s &= \int_T^\infty \frac{R^{-1}(s)}{\int^s R^{-1}(\tau)\,\mathrm{d}\tau \log^2(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)}(1+o(1))\,\mathrm{d}s\\ &= \frac{1}{\log\int^T R^{-1}(t)\,\mathrm{d}t}(1+o(1)) < \infty.\end{aligned}$$

Next, let us observe that

$$|\tilde{G}(t)|\int_t^\infty \frac{1}{\tilde{R}(s)}\,\mathrm{d}s = |G(t)|\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\log\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)(1+o(1)) \to \infty$$

as $t\to\infty$, hence conditions in (2.8) hold.

Further we are interested in the expression

$$\begin{aligned}&\int_t^\infty \tilde{R}^{-1}(s)\ \mathrm{d}s \int_T^t \tilde{x}(s)L[\tilde{x}](s)\ \mathrm{d}s\\ &\quad\sim \frac{\int_T^t \left(\tilde{x}(s)\hat{L}[\tilde{x}](s) + g(s)h^p(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)\log^2\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\right)\,\mathrm{d}s}{\log(\int^t R^{-1}(s)\,\mathrm{d}s)}\end{aligned}$$

as $t\to\infty$. Since, by Lemma 2.1 and (2.22), the integral $\int_T^\infty \tilde{x}\hat{L}[\tilde{x}]\,\mathrm{d}s$ is convergent, property (3.22) is sufficient for (2.9) and (3.23) is sufficient for (2.10). □

To formulate the oscillatory version of Theorem 3.3 we first prove the following oscillation criterion.

Theorem 3.5. *Suppose that there exist constants γ_1, γ_2 satisfying (3.11) such that*

$$\tilde{g}(t)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s \geq \frac{\gamma_1|G'(t)|}{G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \frac{\gamma_2}{|G(t)|R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2} \tag{3.24}$$

for large t. If

$$\int^\infty \tilde{g}(t)h^p(t)(\textstyle\int^t R^{-1}(s)\,\mathrm{d}s)\log(\int^t R^{-1}(s)\,\mathrm{d}s)\,\mathrm{d}t = \infty, \tag{3.25}$$

then equation

$$\begin{aligned}\bar{L}[x] := (r(t)\Phi(x'))' + \Bigg(c(t) &+ \frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\\ &+\frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)} + \tilde{g}(t)\Bigg)\Phi(x) = 0.\end{aligned} \tag{3.26}$$

is oscillatory.

Proof. Let us take $\tilde{x}(t) := h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \log^{\frac{1}{p}}\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)$ and verify conditions (2.11) and (2.12) in Theorem D. Let us remark that $\tilde{x}$ is chosen to be the same as in the second part of the proof of Theorem 3.3. Condition (2.12) is a direct consequence of (3.17) and (3.18) together with (1.7).

Next, let us consider the operator $L[x]$ given by (3.16). According to the proof of Theorem 3.3, $\int^\infty \tilde{x}(t)L[\tilde{x}](t)\,\mathrm{d}t$ converges. Since $\bar{L}[x] = L[x] + \tilde{g}(t)\Phi(x)$, we have

$$\begin{aligned}\int^\infty \tilde{x}(t)\bar{L}[\tilde{x}](t)\,\mathrm{d}t &= \int^\infty \tilde{x}(t)L[\tilde{x}](t)\,\mathrm{d}t + \int^\infty \tilde{g}(t)|\tilde{x}(t)|^p\,\mathrm{d}t \\ &= \int^\infty \tilde{x}(t)L[\tilde{x}](t)\,\mathrm{d}t + \int^\infty \tilde{g}(t)h^p(t)(\textstyle\int^t R^{-1}(s)\,\mathrm{d}s)\log(\int^t R^{-1}(s)\,\mathrm{d}s)\,\mathrm{d}t = \infty\end{aligned}$$

thanks to (3.25).

Finally, using (3.19), we see that

$$\begin{aligned}\tilde{x}(t)\bar{L}[\tilde{x}](t) = {}& \frac{G'(t)\log(\int^t R^{-1}(s)\,\mathrm{d}s)}{G^2(t)\int^t (R^{-1}(s)\,\mathrm{d}s)}\left[\frac{(p-2)(1-p)}{2p^2} + o(1)\right] \\ &+ \frac{\log(\int^t R^{-1}(s)\,\mathrm{d}s)}{G(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2}\left[\frac{2(p-2)(1-p)}{3p^2} + o(1)\right] \\ &+ \tilde{g}(t)h^p(t)(\textstyle\int^t R^{-1}(s)\,\mathrm{d}s)\log(\int^t R^{-1}(s)\,\mathrm{d}s)\end{aligned}$$

which is, provided (3.11) and (3.24), nonnegative for large t. Hence both the parts of (2.11) hold and the statement follows from Theorem D. □

The oscillatory counterpart of Theorem 3.3 reads as follows. Here, equation (1.10) is seen as a perturbation of (3.14) with $\lambda = \frac{1}{2q}$, i.e., (1.10) is considered as an equation of the form (3.26).

Theorem 3.6. *Suppose that there exist constants γ_1, γ_2 satisfying (3.11) such that*

$$\begin{aligned}&\left(g(t) - \frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)}\right)h^p(t)\int^t R^{-1}(s)\,\mathrm{d}s \\ &\quad\geq \frac{\gamma_1|G'(t)|}{G^2(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \frac{\gamma_2}{|G(t)|R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2}\end{aligned} \tag{3.27}$$

for large t. If

$$\liminf_{t\to\infty} \frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)\log^2\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\log(\int^t R^{-1}(s)\,\mathrm{d}s)} > \frac{1}{2q}, \tag{3.28}$$

then equation (1.10) is oscillatory.

Proof. Equation (1.10) can be rewritten in the form of (3.26) with

$$\tilde{g}(t) = g(t) - \frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2\log^2(\int^t R^{-1}(s)\,\mathrm{d}s)},$$

on which we apply Theorem 3.5. Condition (3.24) follows from (3.27). Next we show that (3.25) holds. From (3.28) we have that there exists $\varepsilon > 0$ and $\tilde{T} > T$ such that

$$\frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}(\tau)\,\mathrm{d}\tau)\log^2\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)\,\mathrm{d}s}{\log(\int^t R^{-1}(s)\,\mathrm{d}s)} > \frac{1}{2q} + \varepsilon, \quad t > \tilde{T}. \tag{3.29}$$

Let $b > \tilde{T}$, then (suppressing some unnecessary arguments)

$$
\begin{aligned}
I &:= \int_T^b \left(g(t) - \frac{1}{2qh^p(t)R(\int^t R^{-1})^2 \log^2(\int^t R^{-1})} \right) h^p(t)(\int^t R^{-1}) \log(\int^t R^{-1}) \, \mathrm{d}t \\
&= \int_T^b g(t)h^p(t)(\int^t R^{-1}) \log(\int^t R^{-1}) \, \mathrm{d}t - \int_T^b \tfrac{1}{2qR(\int^t R^{-1}) \log(\int^t R^{-1})} \, \mathrm{d}t \\
&= \int_T^b \frac{1}{\log(\int^t R^{-1})} g(t)h^p(t)(\int^t R^{-1}) \log^2(\int^t R^{-1}) \, \mathrm{d}t - \int_T^b \tfrac{1}{2qR(\int^t R^{-1}) \log(\int^t R^{-1})} \, \mathrm{d}t.
\end{aligned}
$$

With the use of integration by parts and the notation

$$
K_1 := \int_T^{\tilde{T}} \frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}) \log^2(\int^s R^{-1}) \, \mathrm{d}s}{R(\int^t R^{-1}) \log^2(\int^t R^{-1})} \, \mathrm{d}t
$$

we have

$$
\begin{aligned}
I = &\left[\frac{1}{\log(\int^t R^{-1})} \int_T^t g(s)h^p(s)(\int^s R^{-1}) \log^2(\int^s R^{-1}) \, \mathrm{d}s \right]_T^b + K_1 \\
&+ \int_{\tilde{T}}^b \frac{\int_T^t g(s)h^p(s)(\int^s R^{-1}) \log^2(\int^s R^{-1}) \, \mathrm{d}s}{R(\int^t R^{-1}) \log^2(\int^t R^{-1})} \, \mathrm{d}t - \frac{1}{2q} \left[\log(\log(\int^t R^{-1})) \right]_T^b.
\end{aligned}
$$

With respect to (3.29), we can estimate:

$$
\begin{aligned}
I \geq &\frac{1}{\log(\int^b R^{-1})} \int_T^b g(t)h^p(t)(\int^t R^{-1}) \log^2(\int^t R^{-1}) \, \mathrm{d}t + K_1 \\
&+ \int_{\tilde{T}}^b \frac{\frac{1}{2q} + \varepsilon}{R(\int^t R^{-1}) \log^2(\int^t R^{-1})} \, \mathrm{d}t - \frac{1}{2q} \left[\log(\log(\int^t R^{-1})) \right]_T^b \geq \frac{1}{2q} + \varepsilon + K_1 \\
&+ \left(\frac{1}{2q} + \varepsilon \right) \left[\log(\log(\int^t R^{-1})) \right]_{\tilde{T}}^b - \frac{1}{2q} \log(\log(\int^b R^{-1})) + \tfrac{1}{2q} \log(\log(\int^T R^{-1})) \\
= &\frac{1}{2q} + \varepsilon + K_1 + \varepsilon \log(\log(\int^b R^{-1})) + K_2,
\end{aligned}
$$

where $K_2 = -(\frac{1}{2q} + \varepsilon) \log(\log(\int^{\tilde{T}} R^{-1})) + \frac{1}{2q} \log(\log(\int^T R^{-1}))$ is constant and therefore integral I tends to infinity as $b \to \infty$. □

4 Remarks

Remark 4.1. Let us consider the nonoscillatory Euler type equation with the oscillation constant (1.4). It is known that the function $t^{\frac{p-1}{p}}$ is a solution of this equation. To show how our results apply to perturbations of (1.4), consider the interval $[1, \infty)$ and take the solution

$$
h(t) := \left(\frac{p}{p-1} \right)^{\frac{p-2}{p}} t^{\frac{p-1}{p}}.
$$

Then $R(t) = t$, $G = \frac{p-1}{p}$ and $\int_1^t R^{-1}(s) \, \mathrm{d}s = \log t$, hence conditions (1.7) hold. Consequently,

$$
\frac{1}{2qh^p(t)R(t)\left(\int_1^t R^{-1}(s) \, \mathrm{d}s\right)^2} = \frac{\mu_p}{t^p \log^2 t}
$$

and this means that equation (1.9) becomes the Riemann–Weber type equation (1.5) and results of the previous section reduce to those for the perturbed Riemann–Weber type equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t} + g(t)\right)\Phi(x) = 0. \tag{4.1}$$

In particular, Theorem 3.1 with $\alpha = \frac{1}{2}$ reduces to [1, Corollary 2], see also [2, Theorem 3.3] in the case $n = 1$. Theorem 3.2 is a generalized version of [9, Corollary 1] and also of [2, Theorem 3.3] with $n = 1$. Note that, since $G' = 0$, condition (3.10) simplifies to

$$g(t)t^p \log^3 t \geq \gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2}, \quad \gamma := \gamma_2 \left(\frac{p-1}{p}\right)^{p-3},$$

which is condition (3.15) from [2]. Concerning Theorem 3.3, observe that condition (3.15) is satisfied and equation (3.14) with $\lambda = \frac{1}{2q}$ is equation (1.6) with $n = 2$. Hence, Theorem 3.3 generalizes results of [11] for $n = 2$. Finally, Theorem 3.4 applied to (4.1) is [13, Theorem 3.1] in the case $n = 1$.

Remark 4.2. Based on the results of this paper and their comparison with those for the perturbed Euler type equation discussed in Remark 4.1, we suppose that we can study perturbations of equation (3.14) with $\lambda = \frac{1}{2q}$ and find a perturbation such that the obtained perturbed equation is conditionally oscillatory. More generally, we conjecture that the equation with arbitrary number of iterated logarithmic terms

$$(\Phi(x'))' + \left(c(t) + \sum_{j=0}^{n} \frac{1}{2qh^p(t)R(t)(\int^t R^{-1}(s)\,\mathrm{d}s)^2 \operatorname{Log}_j^2(\int^t R^{-1}(s)\,\mathrm{d}s)}\right)\Phi(x) = 0 \tag{4.2}$$

is conditionally oscillatory (here $\operatorname{Log}_0 t := 1$). This would generalize the result of [11] concerning equation (1.6) and give us the possibility to generalize the oscillation and nonoscillation criteria of this paper to the case when we study perturbations of (4.2), similarly as in [2, 13], where perturbations of (1.6) are studied.

Remark 4.3. Let us comment the particular choice of the functions $\tilde{x}$ in the proofs of our results. Consider the operators $\hat{L}$ and $\tilde{L}$ defined in (1.9) and (1.10), respectively. If $\tilde{x}$ is a solution of (1.9), then $\tilde{x}(t)\tilde{L}[\tilde{x}](t) = g(t)\tilde{x}^p(t)$ and hence, when applying one of the Theorems A, B, C to equation (1.10), the expression $\tilde{x}(s)\tilde{L}[\tilde{x}](s)$ appearing in conditions (2.3), (2.4), (2.7), (2.9), (2.10) is replaced by $g(s)\tilde{x}^p(s)$. It has been shown in [3, 10] that equation (1.9) has a pair of linearly independent solutions that are asymptotically close (as $t \to \infty$) to the functions

$$x_1(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}},$$

$$x_2(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} \log^{\frac{2}{p}}\left(\int^t R^{-1}(s)\,\mathrm{d}s\right).$$

We have taken $\tilde{x} := x_1$ in the proofs of Theorem 3.1 and Theorem 3.2 and $\tilde{x} := x_2$ in the proof of Theorem 3.4. Computations in proofs of these theorems together with Lemma 2.2 and Lemma 2.4 show that in both the cases $\tilde{x} := x_1$ and $\tilde{x} := x_2$, the expression $\tilde{x}\hat{L}[\tilde{x}]$ is small enough such that it does not have an influence on the limits superior and inferior in conditions

(2.3), (2.4), (2.7), (2.9), (2.10), hence the expression $g(s)x_1^p(s)$ appears in (3.1), (3.2) and (3.12), and $g(s)x_2^p(s)$ appears in (3.22) and (3.23).

In the proofs of Theorem 3.3 and Theorem 3.5 we have taken $\tilde{x} := h(\int^t R^{-1})^{\frac{1}{p}} \log^{\frac{1}{p}}(\int^t R^{-1})$, since we conjecture that this function is asymptotically close to one of the solutions of (3.14) with $\lambda = \frac{1}{2q}$. This conjecture is supported by Lemma 2.3 (observe that the left-hand side of identity (2.19) is equal to $\tilde{x}(t)\tilde{L}[\tilde{x}](t))$ and by the asymptotic formulas for equation (1.6) derived in [11].

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Article

Integral Comparison Criteria for Half-Linear Differential Equations Seen as a Perturbation

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Abstract: In this paper, we present further developed results on Hille–Wintner-type integral comparison theorems for second-order half-linear differential equations. Compared equations are seen as perturbations of a given non-oscillatory equation, which allows studying the equations on the borderline of oscillation and non-oscillation. We bring a new comparison theorem and apply it to the so-called generalized Riemann–Weber equation (also referred to as a Euler-type equation).

Keywords: half-linear differential equation; oscillation criteria; modified Riccati technique; Euler-type equation; second-order differential equation

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1. Introduction

In this paper, we continue our research on Hille–Wintner-type comparison criteria for half-linear, second-order differential equations and provide an answer to one of the open problems stated in [1]. We study the equation of the form

$$l[x] := (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\, sgn\, x, \quad p > 1, \tag{1}$$

where r, c are continuous functions and $r(t) > 0$. Equation (1) can be seen as a generalization of the second-order linear Sturm–Liouville linear equation, to which it reduces for $p = 2$, and it is well-known that many techniques for linear equations work effectively for half-linear equations too. Recall that one of the differences between half-linear and linear equations is well-visible in the notation—the attribute "half-linear" refers to the fact that the solution space of (1) has only one of the two linearity properties, where it is homogenous but not additive. On the other hand, classification of solutions and equations in terms of oscillation remains the same—a solution is called oscillatory if it has got infinitely many zeros tending to infinity, and non-oscillatory otherwise; and since oscillatory and non-oscillatory solutions cannot coexist, equations are classified as oscillatory or non-oscillatory according to their solutions. To refer to the most current results of the oscillation theory of (1), let us mention, for example, papers [2–6].

Because we are interested in the qualitative behavior of solutions of (1), we study it on a neighborhood of infinity, that is, on intervals of the form $t \geq t_0$, where t_0 is a real constant. By saying that a condition holds for large t, we mean that there exists such an interval-neighborhood of infinity, where the condition holds.

In our research, we focus on comparison theorems which compare two equations and their oscillatory properties. Let us consider another half-linear equation

$$L[x] := (r(t)(\Phi(x'))' + C(t)\Phi(x) = 0. \tag{2}$$

Comparing the coefficient functions $c(t)$ and $C(t)$ (and even $r(t)$ and its counterpart $R(t)$ in general) pointwise, leads to the Sturm comparison theorems, whereas comparing integrals with coefficient functions aims at Hille–Wintner-type criteria. In formulation of classical Hille–Wintner criteria for half-linear equations, one distinguishes two cases, depending on the behavior of the integral $\int^{\infty} r^{1-q}(t)\,dt$. In case of its divergence and under the assumption that $\int^{\infty} C(t)\,dt < \infty$, the criterion says that if

$$0 \le \int_t^{\infty} c(s)\,ds \le \int_t^{\infty} C(s)\,ds \quad \text{for large } t$$

and (2) is non-oscillatory, then (1) is non-oscillatory too, see [7] or [8] Section 2.3.1. If the integral $\int^{\infty} r^{1-q}(t)\,dt$ converges, denote $\rho(t) := \int_t^{\infty} r^{1-q}(s)\,ds$ and suppose that $c(t) \ge 0$, $C(t) \ge 0$ for large t. If

$$\int_t^{\infty} c(s)\rho^p(s)\,ds \le \int_t^{\infty} C(s)\rho^p(s)\,ds < \infty \quad \text{for large } t,$$

then non-oscillation of (2) implies non-oscillation of (1) (see [9] or [8] Section 2.3.1).

Inspired by these results, in paper [1] we adopted the view of the perturbation principle, which allows to refine the results on the threshold between oscillation and non-oscillation, and proved the generalized version of the Hille–Wintner criterion in the following setting. Together with (1) and (2), let us have the equation of the same form

$$\tilde{L}[x] := (r(t)(\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0, \tag{3}$$

which is supposed to be non-oscillatory, and let h be its positive principal solution. Equations (1) and (2) can be seen as perturbations of (3). The main result of [1] showed for the case $\int^{\infty} r^{1-q}(t)\,dt = \infty$ that under certain assumptions (see Theorem 1 below), the inequality

$$0 \le \int_t^{\infty} (c(s) - \tilde{c}(s))h^p(s)\,ds \le \int_t^{\infty} (C(s) - \tilde{c}(s))h^p(s)\,ds < \infty$$

together with non-oscillation of (2) ensure non-oscillation of (1).

As an immediate consequence, we obtained a comparison theorem, where in the place of the equation which is being perturbed, we have the half-linear Euler equation

$$(\Phi(y'))' + \frac{\gamma_p}{t^p}\Phi(y) = 0. \tag{4}$$

Here, $\gamma_p = \left(\frac{p-1}{p}\right)^p$ is the so-called oscillation constant of (4), since it is the greatest possible constant for which the Euler equation is non-oscillatory, for larger constants at that place the equation oscillates. Its principal solution is known exactly and is equal to $t^{\frac{p-1}{p}}$. Another well-known equation that lies on the boundary between oscillation and non-oscillation is the so-called (generalized) Riemann–Weber equation (also referred to as the Euler–Weber equation or just the Euler-type equation). However, the principal solution of this equation cannot be expressed explicitly, and only its asymptotic form is known; hence, the criterion from [1] cannot be applied to it. This was the reason for mentioning the open problem in [1], whether the principal solution in the criterion can be replaced by a function, which is, in some sense, only close to it. As the technique concerning the so-called modified Riccati equation has been developed in more depth over the last few years (see, for example, [10]), we can now show that the answer is positive.

The paper is organized as follows. In the next section, we recall the Riccati technique, including the usage of the modified Riccati equation, the concept of the principal solution, technical lemmas, and remind the original theorem from [1] in its full version. In the section with the main results, we state and prove the main theorem and show some of its consequences for Riemann–Weber-type equations. The last part brings several concluding remarks.

2. Preliminaries

Supposing that Equation (1) is non-oscillatory, it is a well-known fact that if x is its solution, then the function $w = r\Phi\left(\frac{x'}{x}\right)$ solves the relevant Riccati equation

$$w'(t) + c(t) + (p-1)r^{1-q}(t)|w(t)|^q = 0, \quad q = \frac{p}{p-1}, \tag{5}$$

on some interval of the form $[T, \infty)$, and conversely, the solvability of (5) on an interval $[T, \infty)$ guarantees non-oscillation of (1). Here, we refer to the basic literature, for example, [8] (Section 1.1.4), for introduction to the theory (see also [11]).

It can be shown (as introduced by [12]) that among all non-oscillatory solutions of (5), there exists the *minimal* one $\tilde{w}$, for which any other solution of (5) satisfies the inequality $w(t) > \tilde{w}(t)$ for large t. Then, the solution of (5) given by

$$\tilde{x} = K \exp\left\{\int^t r^{1-q}(s)\Phi^{-1}(\tilde{w}(s))\, ds\right\},$$

is called "principal" and it is related to the minimal solution of (5) by the formula $\tilde{w} = r\Phi(\tilde{x}'/\tilde{x})$. Note that Φ^{-1} is the inverse operator to Φ and q is the so-called conjugate number to p, and $\frac{1}{p} + \frac{1}{q} = 1$ holds.

The concept of the minimal solution of the Riccati equation is also known from the theory of linear differential equations, where the so-called integral characterization holds. Its possible extension to half-linear equations was studied, for example, in [13,14]. In [14], it was shown that the condition $\int^\infty \frac{dt}{r(t)x^2(t)|x'(t)|^{p-2}} = \infty$ is under certain assumptions necessary or sufficient for x to be the principal solution, but a complete "both-way" integral characterization has not been proven.

Now, let us turn our attention to the modified Riccati technique. Let $h(t)$ be a differentiable function such that $h(t) \geq 0$ and $h'(t) \neq 0$ for large t, and let us use the notation

$$R(t) := r(t)h^2(t)|h'(t)|^{p-2}, \quad G(t) := r(t)h(t)\Phi(h'(t)).$$

It was shown, for example, in [10] (Lemma 4) that a neigborhood of infinity solvability of (5) (and hence, also non-oscillation of (1)) is equivalent to solvability of the so-called modified Riccati equation

$$v'(t) + h(t)l[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(v, G) = 0, \tag{6}$$

where

$$H(v, G) := |v + G|^q - q\Phi^{-1}(G)v - |G|^q.$$

The solution v of the modified Riccati Equation (6) and the solution w of the Riccati Equation (5) satisfy the relation $v = h^p w - G$.

The behavior of $H(v, G)$ was deeply described, for example, in [10], and we present here only those parts of its Lemma 5 and 6, which are relevant for us.

Lemma 1. *The function $H(v, G)$ has the following properties:*

(i) $H(v, G) \geq 0$ *with the equality if and only if* $v = 0$.

(ii) *For every* $L > 0$, *there exist constants* $K_1 = K_1(L) > 0, K_2 = K_2(L) > 0$ *such that*

$$K_1|G(t)|^{q-2}v^2 \leq H(v, G) \leq K_2|G(t)|^{q-2}v^2$$

for any t *and* v *satisfying* $\left|\frac{v}{G}\right| \leq L$.

The nonnegativity of solutions of the modified Riccati Equation (6) was studied in several papers. The following lemma summarizes results which are already adjusted to our needs and based on Lemma 4 and a part of the proof of Theorem 4 in [15] (for more resources see references therein).

Lemma 2. *Let h be a positive, continuously differentiable function, such that $h'(t) \neq 0$ and $h(t)l[h](t) \geq 0$ for large t. Let $\int^\infty R^{-1}(t)\,dt = \infty$ and*

$$(\liminf_{t\to\infty} |G(t)| > 0 \quad \text{and} \quad \limsup_{t\to\infty} |G(t)| < \infty) \quad \text{or} \quad \lim_{t\to\infty} |G(t)| = \infty.$$

Then, all proper solutions of (6) are nonnegative.

Finally, let us present the main theorem of [1]. Note that h is here the principal solution, and assumption (7) is the condition appearing in its possible integral characterization.

Theorem 1. *Let $\int^\infty r^{1-q}(t)\,dt = \infty$. Suppose that Equation (3) is non-oscillatory and possesses a positive principal solution h, such that there exists a finite limit*

$$\lim_{t\to\infty} G(t) =: L > 0$$

and

$$\int^\infty R^{-1}(t)\,dt = \infty. \tag{7}$$

Further, suppose that $0 \leq \int_t^\infty C(s)\,ds < \infty$ and

$$0 \leq \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)\,ds \leq \int_t^\infty (C(s) - \tilde{c}(s))h^p(s)\,ds < \infty,$$

all for large t. If Equation (2) is non-oscillatory, then (1) is also non-oscillatory.

3. Main Results

In this section, we present the main theorem and its corollaries.

Theorem 2. *Suppose that there exists a positive continuously differentiable function $h(t)$ such that $h'(t) \neq 0$ for large t and the following conditions hold:*

$$\int_t^\infty R^{-1}(s)\,ds = \infty, \tag{8}$$

$$h(t)L[h](t) \geq 0, \tag{9}$$

$$\int_t^\infty h(s)\tilde{L}[h](s)\,ds < \infty, \tag{10}$$

$$(\liminf_{t\to\infty} |G(t)| > 0 \quad \text{and} \quad \limsup_{t\to\infty} |G(t)| < \infty) \quad \text{or} \quad \lim_{t\to\infty} |G(t)| = \infty, \tag{11}$$

all for large t.

Let the inequality

$$-\int_t^\infty h(s)\tilde{L}[h](s)\,ds \leq \int_t^\infty (c(s) - \tilde{c}(s))h^p(s)\,ds \leq \int_t^\infty (C(s) - \tilde{c}(s))h^p(s)\,ds < \infty \tag{12}$$

be satisfied. Then, if Equation (2) is non-oscillatory, Equation (1) is non-oscillatory too.

Proof. Suppose that Equation (2) is non-oscillatory. Let x be its solution. Then, the function $w = r\Phi\left(\frac{x'}{x}\right)$ solves on an interval $[T, \infty)$ the relevant Riccati equation

$$w'(t) + C(t) + (p-1)r^{1-q}(t)|w(t)|^q = 0$$

and the function $v = h^p w - G$ solves the modified Riccati equation

$$v'(t) + h(t)L[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(v,G) = 0. \tag{13}$$

Because $H(v,G) \geq 0$ (see Lemma 1) and $h(t)L[h](t) \geq 0$ for large t by (9), we observe that $v'(t) \leq 0$ and the function $v(t)$ is non-increasing for large t. According to Lemma 2, the function $v(t)$ is non-negative, and there exists a non-negative finite limit $\lim_{t\to\infty} v(t)$.

If $\lim_{t\to\infty} |G(t)| = \infty$, then we immediately see that $|\frac{v}{G}| \to 0$ for $t \to \infty$.

Now we show the same for the remaining case if $\liminf_{t\to\infty} |G(t)| > 0$ and $\limsup_{t\to\infty} |G(t)| < \infty$. Integrating (13) over the interval $[T,t]$ yields

$$v(T) - v(t) = \int_T^t h(s)L[h](s) + (p-1)r^{1-q}(s)h^{-q}(s)H(v(s),G(s))\,ds,$$

and hence,

$$v(T) \geq \int_T^t h(s)L[h](s) + (p-1)r^{1-q}(s)h^{-q}(s)H(v(s),G(s))\,ds. \tag{14}$$

Now we have (suppressing the argument)

$$hL[h] = h\{(r\Phi(h'))' + \tilde{c}\Phi(h) + (C-\tilde{c})\Phi(h)\} = h\tilde{L}[h] + (C-\tilde{c})h^p$$

and thanks to (10) and (12), we observe that $\int^\infty h(t)L[h](t)\,dt < \infty$. Let $t \to \infty$ in (14) imply the convergence of the integral

$$\int^\infty (p-1)r^{1-q}(t)h^{-q}(t)H(v(t),G(t))\,dt < \infty.$$

With respect to our assumption—the first part of (11)—there exists a constant $L > 0$ and $T_1 \geq T$ such that $|\frac{v(t)}{G(t)}| < L$ for $t \geq T_1$. By Lemma 1, there exists $K > 0$ such that

$$K|G(t)|^{q-2}v^2(t) \leq H(v(t),G(t)) \quad \text{for} \quad t \geq T_1,$$

which means

$$K\frac{v^2(t)}{R(t)} \leq r^{1-q}(t)h^{-q}(t)H(v(t),G(t)) \quad \text{for} \quad t \geq T_1.$$

Integrate the inequality over the interval $[T_2,\infty)$, where $T_2 \geq T_1$:

$$K_2\int_{T_2}^\infty \frac{v^2(t)}{R(t)}\,dt \leq \int_{T_2}^\infty r^{1-q}(t)h^{-q}(t)H(v(t),G(t))\,dt < \infty.$$

By (8), we see that $v \to 0$ for $t \to \infty$, and hence, v satisfies the integral equation

$$v(t) = \int_t^\infty h(s)L[h](s) + (p-1)r^{1-q}(s)h^{-q}(s)H(v(s),G(s))\,ds. \tag{15}$$

Now let us define the following integral operator

$$F(u) = \int_t^\infty h(s)l[h](s) + (p-1)r^{1-q}(s)h^{-q}(s)H(u(s),G(s))\,ds$$

on the set

$$U = \{u(t), 0 \leq u(t) \leq v(t), t \in [T,\infty)\}.$$

Our aim is to show that F on U fulfills such conditions that it has got a fixed point. Up to this point, first observe that

$$H'_u(u,G) = q\Phi^{-1}(|u+G|) - q\Phi^{-1}(G),$$

and since Φ^{-1} is increasing and $u \geq 0$ on U, it means that $H'_u \geq 0$, that is, H is increasing in the first variable. Let us take functions $u_1(t), u_2(t)$ such that $0 \leq u_1(t) \leq u_2(t) \leq v(t)$, then the inequality $F(u_1) \leq F(u_2)$ holds too. To verify that the operator F maps the set U to itself, we consider the inequality

$$0 \leq F(0) \leq F(u) \leq F(v) \leq v. \tag{16}$$

The middle two inequalities hold on U according to the previous paragraph. Since

$$F(0) = \int_t^\infty h(s)l[h](s)\,ds = \int_t^\infty h(t)\tilde{L}[h](s) + (c(s) - \tilde{c}(s))h^p(s)\,ds,$$

the first inequality in (16) holds by the first inequality in (12). The last inequality

$$F(v) = \int_t^\infty h(s)l[h](s) + (p-1)r^{1-q}(s)h^{-q}(s)H(v(s), G(s))\,ds \leq v$$

follows from the fact that

$$hl[h] = h\tilde{L}[h] + (c - \tilde{c})h^p, \quad hL[h] = h\tilde{L}[h] + (C - \tilde{c})h^p$$

together with (12) and (15).

Furthermore, $F(U)$ is obviously bounded on closed subintervals of $[T, \infty)$. To show that F is uniformly continuous, let $u \in U$ be arbitrary, $\varepsilon > 0$ and take $t_1, t_2 \in [T, \infty]$ such that (without loss of generality) $t_1 < t_2$. Denote $f(t) = h(t)l[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(u(t), G(t))$. We have

$$\begin{aligned}
&|F(u)(t_2) - F(u)(t_1)| \\
&= \left|\int_{t_2}^\infty f(t)\,dt - \int_{t_1}^\infty f(t)\,dt\right| = \left|\int_{t_1}^{t_2} f(t)\,dt\right| \\
&\leq \left|\int_{t_1}^{t_2} h(t)l[h](t)\,dt\right| + \left|\int_{t_1}^{t_2} (p-1)r^{1-q}(t)h^{-q}(t)H(u(t), G(t))\,dt\right|.
\end{aligned}$$

Since both the integrals converge, there exists δ such that each of the integrals in absolute value is less than $\frac{\varepsilon}{2}$ for $|t_2 - t_1| < \delta$ and

$$|F(u)(t_2) - F(u)(t_1)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, F is uniformly continuous. Using the Schauder–Tychonov theorem, there exists a fixed point of F on U such that $F(u) = u$ and u solves the integral equation

$$u(t) = \int_t^\infty h(s)l[h](s) + (p-1)r^{1-q}(s)h^{-q}(s)H(u(s), G(s))\,ds$$

and also the modified Riccati Equation (6), and $w = h^{-p}(u + G)$ is a solution of the Riccati equation joined with (1). Hence, Equation (1) is non-oscillatory. □

As an immediate consequence of the previous theorem, we have the following statement.

Corollary 1. *Let the assumptions of Theorem 2 be satisfied. Then, the oscillation of Equation (1) implies that of (2).*

Now, for the sake of clarity, recall that by log we mean the natural logarithm, $\log_k$ stands for an iterative logarithm, and Log_j is a product of these functions according to the following definition:

$$\log_1 t = \log t, \quad \log_k t = \log_{k-1}(\log t),\ k \geq 2, \quad Log_j t = \Pi_{k=1}^{j} \log_k t.$$

Let us consider the generalized Riemann–Weber half-linear equation with critical coefficients

$$L_{RW}[x] := (\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t}\right)\Phi(x) = 0, \tag{17}$$

where $n \in \mathbb{N}$ and $\mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$. The consequence of Theorem 2 for the case where the non-oscillatory Equation (3), which is being perturbed, is set to be the Equation (17), reads as follows.

Corollary 2. *Suppose that the condition*

$$L\left[t^{\frac{p-1}{p}}\, Log_n^{\frac{1}{p}} t\right] \geq 0 \tag{18}$$

holds for large t*. If the inequality*

$$-\int_t^\infty f\,ds \leq \int_t^\infty (c-\tilde{c})s^{p-1}\, Log_n(s)\, ds \leq \int_t^\infty (C-\tilde{c})s^{p-1}\, Log_n(s)\, ds < \infty,$$

where $f(s)$ *is defined by (19) (see below) and* $\tilde{c}(t) = \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p\, Log_j^2 t}\right)$*, is satisfied, and if Equation (2) is non-oscillatory, then Equation (1) is non-oscillatory too.*

Proof. First, note that Equation (17) is non-oscillatory, and it has, in a certain sense, the largest possible coefficient function $\tilde{c}$, for which the non-oscillation is preserved. Indeed, equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n-1} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + \frac{\mu}{t^p \mathrm{Log}_n^2 t}\right)\Phi(x) = 0$$

is conditionally oscillatory, $\mu = \mu_p$ is its oscillation constant, and it is oscillatory for $\mu > \mu_p$ and non-oscillatory for $\mu \leq \mu_p$. The asymptotic formulas for the two linearly independent non-oscillatory solutions of (17) were derived in [16]. These solutions are asymptotically equivalent to the functions

$$h_1(t) = t^{\frac{p-1}{p}}\, Log_n^{\frac{1}{p}} t, \quad h_2(t) = t^{\frac{p-1}{p}}\, Log_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t$$

and h_1 is asymptotically close to the principal solution.

Let us take $h(t) = h_1(t)$ in Theorem 2 and check the conditions.

Došlý in [17] showed that for $h(t) = t^{\frac{p-1}{p}}\, Log_n^{\frac{1}{p}} t$ and the operator defined in (17), we have

$$h'(t) = \frac{p-1}{p} t^{-\frac{1}{p}}\, Log_n^{\frac{1}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\, Log_i t}\right)$$

and

$$f(t) := h(t)L_{RW}[h](t) = \frac{Log_n t}{t \log^3 t}\left[\frac{2\gamma_p p(2-p)}{3(p-1)^2} + o(1)\right] \quad \text{as } t \to \infty. \tag{19}$$

Thus, (10) holds (the calculation can be found in [17] above the relation (3.9)). The condition in (9) is reduced to (18).

Next, as $t \to \infty$, we have

$$R(t) = h^2(t)|h'(t)|^{p-2} = \left(\frac{p-1}{p}\right)^{2-p} t\, Log_n(t)(1 + o(1))$$

and

$$\int^t R^{-1}(s)\,ds = \left(\frac{p-1}{p}\right)^{2-p} \log_{n+1}(t)(1+o(1))$$

which is divergent for $t \to \infty$, so (8) holds. Further,

$$G(t) = h(t)\Phi(h') = \left(\frac{p-1}{p}\right) Log_{\,n}(t)(1+o(1))$$

and it tends to infinity for $t \to \infty$, and hence, (11) is also satisfied. □

In the next corollary, we apply the results to the generalized Riemann–Weber equation with $n+1$ terms in the sum as the testing Equation (2) in order to obtain a Hille–Wintner-type comparison criterion for the perturbed Riemann–Weber-type equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + g(t)\right)\Phi(x) = 0. \tag{20}$$

Corollary 3. *Let the inequality*

$$-\int_t^\infty f(s)\,ds \le \int_t^\infty g(s)s^{p-1}\, Log_{\,n}(s)\,ds \le \frac{\mu_p}{\log_{n+1}(t)},$$

where $f(t)$ is given by (19)*, hold. Then, Equation* (20) *is non-oscillatory.*

Proof. We take (20) in place of (1),

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n+1} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t}\right)\Phi(x) = 0$$

in place of (2), (17) in place of (3), and $h = h_1 = t^{\frac{p-1}{p}}\, Log_{\,n}^{\frac{1}{p}} t$. Observe that

$$(c - \tilde{c}) = g(t), \quad (C - \tilde{c}) = \frac{\mu_p}{t^p\, Log_{\,n+1}^{\,2}(t)}$$

and

$$(C - \tilde{c})h^p = \frac{\mu_p}{t^p\, Log_{\,n+1}^{\,2}(t)} t^{p-1}\, Log_{\,n}(t) = \frac{\mu_p}{t\, Log_{\,n}(t) \log_{n+1}^2(t)}.$$

The integral

$$\int_t^\infty (C - \tilde{c})h^p = \int_t^\infty \frac{\mu_p}{s\, Log_{\,n}(s) \log_{n+1}^2(s)}\,ds = \frac{\mu_p}{\log_{n+1}(t)},$$

as can be shown by the substitution $\log_{n+1}(s) = u$ and with the use of the fact that $(\log_{n+1}(t))' = \frac{1}{t\, Log_{\,n}(t)}$.

Finally, let us verify the condition (9). We have

$$\begin{aligned} hL[h] &= hL_{RW}[h] + (C - \tilde{c})h^p \\ &= \frac{Log_{\,n} t}{t \log^3 t}\left[\frac{2\gamma_p p(2-p)}{3(p-1)^2} + o(1)\right] + \frac{\mu_p}{t\, Log_{\,n}(t) \log_{n+1}^2(t)} \end{aligned}$$

as $t \to \infty$. Show that $\frac{Log_n t}{t \log^3 t} = o(\frac{1}{t\, Log_n(t) \log^2_{n+1}(t)})$ as $t \to \infty$:

$$\lim_{t\to\infty} \frac{\frac{Log_n t}{t\log^3 t}}{\frac{1}{t\, Log_n(t)\log^2_{n+1}(t)}} = \lim_{t\to\infty} \frac{Log^2_{n+1}(t)}{\log^3 t} = \lim_{t\to\infty} \frac{\log^2_{n+1}(t)}{\log t} \leq \lim_{t\to\infty} \left(\frac{\log_2(t)}{\log^{\varepsilon} t}\right)^{2n},$$

where $\varepsilon = \frac{1}{2n}$. This limit tends to 0 as $t \to \infty$ since $\lim_{t\to\infty} \frac{\log_2(t)}{\log^{\varepsilon} t} = 0$ for $\varepsilon > 0$ (as can be shown by the L'Hospital's rule). Hence, $hL[h] \geq 0$ for large t. $\square$

4. Concluding Remarks

(a) Let us mention that Corollary 3, as the specific application of Theorem 2 to concrete Equations (20) and (17), and the generalized Riemann–Weber equation with $n+1$ terms, brings a result which is in compliance with the Hille–Nehari-type criterion, that was proved in [17] (more on Hille–Nehari-type criteria for (20) can be found also in [18]). Its non-oscillatory part says the following. Suppose that the integral $\int^{\infty} \tilde{c}(t)t^{p-1}\, Log_n t\, dt$ is convergent. If

$$\limsup_{t\to\infty} \log_{n+1} t \int_t^{\infty} \tilde{c}(s)s^{p-1}\, Log_n s\, ds < \mu_p,$$
$$\liminf_{t\to\infty} \log_{n+1} t \int_t^{\infty} \tilde{c}(s)s^{p-1}\, Log_n s\, ds > -3\mu_p,$$

then (20) is non-oscillatory.

(b) Let us observe that Theorem 2 can be applied also to the situation where the Euler Equation (4) is in the position of (3). We can use the exact principal solution $h = t^{\frac{p-1}{p}}$ for which

$$R(t) = \left(\frac{p-1}{p}\right)^{2-p} t(1+o(1)), \quad \int^t R^{-1}(s)\, ds = (1+o(1))\left(\frac{p-1}{p}\right)^{2-p} \log(t),$$

$$G(t) = \left(\frac{p-1}{p}\right)(1+o(1)) \quad \text{and} \quad h(t)L_E[h](t) = 0$$

as $t \to \infty$. Such a corollary was already presented in [1].

(c) Note that the perturbation $(c - \tilde{c})$ does not have to be less than $(C - \tilde{c})$ pointwise (then the Sturm comparison theorem would be sufficient) and c can oscillate around $\tilde{c}$ as long as the integral inequality (12) holds. For results for Riemann–Weber-type half-linear equations with sums of periodic functions instead of constants, see [19].

(d) Finally, comment on the differences between Theorems 1 and 2.

Firstly, in Theorem 2, we do not suppose $\int^{\infty} r^{1-q} = \infty$ anywhere. Next, the main difference is the fact that h is once a principal solution of (3) and once a function which is only close to that principal solution. The condition (8) is in both the theorems, and it is connected with the closeness of functions h to the principal solution. The condition (9), that is, $hL[h] \geq 0$ for large t, does not have its counterpart in Theorem 1, and here we have another difference between the theorems. The reason for this condition is in usage of Lemma 2. The assumption (10) is a variant on the condition $0 \leq \int^{\infty} C(t)\, dt < \infty$ from Theorem 1. The condition (11) is in fact an extension of the assumption of the existence of a finite limit $\lim_{t\to\infty} |G(t)|$. We might ask whether (11) could be replaced just by $\liminf_{t\to\infty} |G(t)| > 0$, but certainly, it can be replaced by a weaker condition of the existence of the limit such that $\lim_{t\to\infty} |G(t)| > 0$. Note that the first part of (11) holds for the case where Equation (3) is the Euler Equation (4), whereas the second part holds for the case of Riemann–Weber type Equation (17). The last difference is in (12) in the very first inequality. In Theorem 1, the integral $\int_t^{\infty} h(s)\tilde{L}[h](s)\, ds$ is equal to 0 trivially, because h is an exact solution of (3).

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Article

Use of the Modified Riccati Technique for Neutral Half-Linear Differential Equations

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Abstract: We study the second-order neutral half-linear differential equation and formulate new oscillation criteria for this equation, which are obtained through the use of the modified Riccati technique. In the first statement, the oscillation of the equation is ensured by the divergence of a certain integral. The second one provides the condition of the oscillation in the case where the relevant integral converges, and it can be seen as a Hille–Nehari-type criterion. The use of the results is shown in several examples, in which the Euler-type equation and its perturbations are considered.

Keywords: half-linear neutral differential equation; oscillation criteria; modified Riccati technique

MSC: 34K11; 34C10

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1. Introduction

In this paper, we study the oscillatory properties of the second-order half-linear neutral differential equation

$$\big(r(t)\Phi(z'(t))\big)' + c(t)\Phi(x(\tau(t))) = 0, \quad z(t) = x(t) + b(t)x(\sigma(t)), \tag{1}$$

where $t \geq t_0$ and $\Phi(x) = |x|^{p-2}x$, $p \in \mathbb{R}$, and $p > 1$. We suppose that the coefficients of the equation satisfy the usual conditions: $r \in C([t_0,\infty),\mathbb{R}^+)$, $b \in C^1([t_0,\infty),\mathbb{R}_0^+)$, $c \in C([t_0,\infty),\mathbb{R}_0^+)$, c is not identically equal to zero in any neighborhood of infinity, and

$$b(t) \leq 1. \tag{2}$$

Concerning the deviating arguments, we assume that $\tau, \sigma \in C^1([t_0,\infty),\mathbb{R})$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \sigma(t) = \infty$ and

$$\tau' \geq 0, \quad \tau(t) \leq t, \quad \sigma(t) \leq t. \tag{3}$$

We also suppose that

$$\int^\infty r^{1-q}(t)\,dt = \infty, \tag{4}$$

where q denotes the conjugate number of p, i.e., $q = \frac{p}{p-1}$, and the symbol $\int^\infty$ means that it does not matter what the lower limit of the integral is if it is large enough, and that the limit process is applied on the upper limit of the integral as it tends to infinity. The above setting and conditions (2)–(4) are intended to hold throughout this whole paper and in all of its statements.

A differential equation is called neutral if it contains the highest-order derivative of an unknown function both with and without delay. This means that the rate of growth depends on the current state and the state in the past, as well as on the rate of change in the past, which enables a suitable description for many real processes. For example, the process of growth of a human population ([1]) or a population of *Daphnia magna* ([2]) can be modeled by neutral differential equations. Neutral Equation (1) is called half-linear, as its solution space is homogenous but not additive (it only has half of the linearity properties), and it can also be classified as Emden–Fowler equation. Neutral half-linear/Emden–Fowler equations arise in a variety of real-world problems, such as in the study of p-Laplace equations, non-Newtonian fluid theory, the turbulent flow of a polytrophic gas in a porous medium, and so forth (see, for example, [3–6]).

In recent years, the qualitative theory of Equation (1) has attracted considerable attention, and it has been studied under condition (4), for example, in [7–11] (see also the references therein). For the case where the integral in (4) converges, let us refer to [12,13]. If $b(t) \equiv 0$, then the studied equation becomes a delayed half-linear equation, and its oscillation results are provided, for example, in [14–20].

By a solution of (1), we mean a differentiable function $x(t)$ that is eventually not identically equal to zero, such that $r(t)\Phi(z'(t))$ is differentiable and (1) holds for $t \geq t_0$. Equation (1) is said to be oscillatory if it does not have a solution that is eventually positive or negative.

In this paper, we formulate new oscillation criteria for Equation (1). One of them can be classified as a Hille–Nehari type statement. Our results are based on the modification of the Riccati technique. Instead of the usual Riccati inequality, we use the so-called modified Riccati inequality. The modified Riccati technique has been used in the theory of ordinary half-linear differential equations of the form

$$\big(r(t)\Phi(x'(t))\big)' + c(t)\Phi(x(t)) = 0, \tag{5}$$

and it has been revealed that it is a useful tool that can be regarded as a replacement of the missing half-linear version of the transformation formula known from the classical oscillation theory of linear equations. For the related results concerning this method, we refer to [21–23] and the references given therein. We point out that, within the same approach, Hille–Nehari-type criteria for (5) were last studied in [24]. In Ref. [18], the modified Riccati technique was extended and applied to half-linear differential equations with delay:

$$\big(r(t)\Phi(x'(t))\big)' + c(t)\Phi(x(\tau(t))) = 0. \tag{6}$$

Here, we show that the method can also be extended for neutral half-linear equations and used to derive some oscillation criteria for (1).

This paper is organized as follows. In the next section, we introduce the modified Riccati technique and formulate some preliminary results. In Section 3, we present our main results, the oscillation criteria for (1), and in the last section, we apply the results to a perturbed equation of the Euler type.

2. Preliminaries

We start with the properties of the eventually positive solutions of (1) that are ensured with condition (4). By the function Φ^{-1}, we mean the inverse function to Φ, i.e., $\Phi^{-1}(x) = |x|^{q-2}x$.

Lemma 1. *Suppose that $x(t)$ is a solution of (1) that is positive on $[t_0, \infty)$. Then, there exists $T > t_0$ such that*

$$z(t) > 0, \quad z'(t) > 0, \quad \big(r(t)\Phi(z'(t))\big)' \leq 0, \tag{7}$$

$$\frac{z'(\tau(t))}{z'(t)} \geq \Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right) \tag{8}$$

and

$$\big(r(t)\Phi(z'(t))\big)' \leq -c(t)\Phi[z(\tau(t))(1 - b(\tau(t))] \tag{9}$$

for $t \geq T$.

Proof. Condition (7) is a well-known statement and its proof can be found, e.g., in [7] (Lemma 3). Because $r(t)\Phi(z'(t))$ is non-increasing, we have

$$r(t)\Phi(z'(t)) \leq r(\tau(t))\Phi(z'(\tau(t))),$$

which can be rearranged into (8). Now, we observe that $x(\sigma(\tau(t))) \leq z(\sigma(\tau(t))) \leq z(\tau(t))$ and, in view of (1),

$$\begin{aligned} 0 &= \big(r(t)\Phi(z'(t))\big)' + c(t)\Phi[z(\tau(t)) - b(\tau(t))x(\sigma(\tau(t)))] \\ &\geq \big(r(t)\Phi(z'(t))\big)' + c(t)\Phi[z(\tau(t))(1 - b(\tau(t)))], \end{aligned}$$

which implies (9). □

Grace et al. showed in [10] that, under some additional assumptions, condition (8) can be strengthened. Similarly to in [25], they considered the sequence

$$g_0(\varrho) := 1, \quad g_{n+1}(\varrho) := e^{\varrho g_n(\varrho)}, \quad n = 0, 1, 2, \ldots, \tag{10}$$

where ϱ is a positive constant. For $\varrho \in (0, \frac{1}{e}]$, the sequence is increasing and bounded above, and $\lim_{t\to\infty} g_n(\varrho) = g(\varrho) \in [1, e]$, where $g(\varrho)$ is a real root of the equation

$$g(\varrho) = e^{\varrho g(\varrho)}. \tag{11}$$

With the use of this sequence and the notation

$$\begin{aligned} \mathcal{Q}(t) &:= \Phi(1 - b(\tau(t)))c(t), \quad \mathcal{R}(t) := \int_{t_1}^{t} r^{1-q}(s)\,ds, \\ \tilde{\mathcal{R}}(t) &:= \mathcal{R}(t) + \frac{1}{p-1}\int_{t_1}^{t} \mathcal{R}(s)\Phi(\mathcal{R}(\tau(s)))\mathcal{Q}(s)\,ds \end{aligned}$$

for $t \geq t_1$, where t_1 is large enough, Grace et al. proved the following lemma.

Lemma 2 ([10], Lemma 4)**.** *Assume that τ is strictly increasing, Equation (1) has a positive solution $x(t)$ on $[t_0, \infty)$, and the condition*

$$\int_{\tau(t)}^{t} \mathcal{Q}(s)\Phi(\tilde{\mathcal{R}}(s))\,ds \geq \varrho \tag{12}$$

holds for some $\varrho > 0$ and a t that is large enough. Then,

$$\frac{z'(\tau(t))}{z'(t)} \geq \Phi^{-1}\left(\frac{g_n(\varrho)r(t)}{r(\tau(t))}\right)$$

for every n and t that are large enough, where $g_n(\varrho)$ is defined by (10).

Now, let us turn our attention to the Riccati technique. By our assumptions, conditions (2)–(4) hold, and we suppose that Equation (1) has an eventually positive solution $x(t)$. Take

$$w(t) = r(t)\Phi\left(\frac{z'(t)}{z(\tau(t))}\right). \tag{13}$$

By a direct differentiation, we have

$$w'(t) = \frac{(r(t)\Phi(z'(t)))'}{\Phi(z(\tau(t))} - (p-1)r(t)\tau'(t)\frac{\Phi(z'(t))z'(\tau(t))}{|z(\tau(t))|^p}$$

which, with the use of (9), gives

$$w'(t) \le -c(t)\Phi(1-b(\tau(t))) - (p-1)r^{1-q}(t)\tau'(t)\frac{z'(\tau(t))}{z'(t)}|w(t)|^q.$$

Assuming that there exists a positive function $f(t)$ (one possible choice is $\Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right)$ by (8)) such that

$$\frac{z'(\tau(t))}{z'(t)} \ge f(t) > 0, \tag{14}$$

we obtain the Riccati-type inequality of the form

$$w'(t) \le -c(t)\Phi(1-b(\tau(t))) - (p-1)r^{1-q}(t)\tau'(t)f(t)|w(t)|^q. \tag{15}$$

Next, we introduce the modified Riccati technique. Let $h(t)$ be a positive differentiable function, and put

$$G(t) = r(t)h(\tau(t))\Phi\left(\frac{h'(\tau(t))}{f(t)}\right). \tag{16}$$

Using the modified Riccati transformation

$$v(t) = h^p(\tau(t))w(t) - G(t), \tag{17}$$

we obtain the so-called modified Riccati inequality (18) that is derived in the next lemma.

Lemma 3. *Suppose that Equation (1) has an eventually positive solution $x(t)$ and w is defined by (13). Let f be a positive function satisfying (14), let h be a positive differentiable function, and let G be defined by (16). Then, the function $v(t)$, given by (17), satisfies the inequality*

$$v'(t) + C(t) + (p-1)r^{1-q}(t)\tau'(t)h^{-q}(\tau(t))f(t)H(v(t),G(t)) \le 0, \tag{18}$$

where

$$C(t) = h(\tau(t))\left[\left(r(t)\Phi\left(\frac{h'(\tau(t))}{f(t)}\right)\right)' + c(t)\Phi(h(\tau(t)))\Phi(1-b(\tau(t)))\right] \tag{19}$$

and

$$H(v,G) = |v+G|^q - q\Phi^{-1}(G)v - |G|^q. \tag{20}$$

Proof. By a direct differentiation, we obtain

$$\begin{aligned} v'(t) &= ph^{p-1}(\tau(t))h'(\tau(t))\tau'(t)w(t) + h^p(\tau(t))w'(t) - G'(t) \\ &= ph'(\tau(t))\tau'(t)h^{-1}(\tau(t))(v(t)+G(t)) + h^p(\tau(t))w'(t) - G'(t), \end{aligned}$$

and with the use of (15), we have (suppressing the argument t)

$$\begin{aligned} v' &\le ph'(\tau)\tau'h^{-1}(\tau)(v+G) - G' + h^p(\tau)\left[-c\Phi(1-b(\tau)) - (p-1)r^{1-q}\tau' f|w|^q\right] \\ &= ph'(\tau)\tau'h^{-1}(\tau)(v+G) - G' - h^p(\tau)c\Phi(1-b(\tau)) - (p-1)r^{1-q}\tau'h^{-q}(\tau)f|v+G|^q \\ &= -\tilde{C} - (p-1)r^{1-q}\tau'h^{-q}(\tau)f\left[|v+G|^q - q\Phi^{-1}(G)v - |G|^q\right], \end{aligned}$$

where

$$\begin{aligned}\tilde{C} &= -ph'(\tau)\tau' h^{-1}(\tau)(v+G) + G' + h^p(\tau)c\Phi(1-b(\tau)) \\ &\quad + (p-1)r^{1-q}\tau' h^{-q}(\tau) f\Big[q\Phi^{-1}(G)v + |G|^q\Big] \\ &= h^p(\tau)c\Phi(1-b(\tau)) + G' - ph'(\tau)\tau' h^{-1}(\tau)v - ph'(\tau)\tau' h^{-1}(\tau) r h(\tau)\Phi\left(\frac{h'(\tau)}{f}\right) \\ &\quad + pr^{1-q}\tau' h^{-q}(\tau) f r^{q-1} h^{q-1}(\tau)\frac{h'(\tau)}{f} v + (p-1)r^{1-q}\tau' h^{-q}(\tau) f r^q h^q(\tau)\frac{|h'(\tau)|^p}{f^p} \\ &= h^p(\tau)c\Phi(1-b(\tau)) + G' - r|h'(\tau)|^p\tau' f^{1-p}.\end{aligned}$$

Since

$$G'(t) = \left(r(t)\Phi\left(\frac{h'(\tau(t))}{f(t)}\right)\right)' h(\tau(t)) + r(t)|h'(\tau(t))|^p\tau'(t) f^{1-p}(t),$$

we have

$$\begin{aligned}\tilde{C}(t) &= h^p(\tau(t))c(t)\Phi(1-b(\tau(t))) + \left(r(t)\Phi\left(\frac{h'(\tau(t))}{f(t)}\right)\right)' h(\tau(t)) \\ &= h(\tau(t))\left[\left(r(t)\Phi\left(\frac{h'(\tau(t))}{f(t)}\right)\right)' + c(t)\Phi(h(\tau(t)))\Phi(1-b(\tau(t)))\right].\end{aligned}$$

Hence, $\tilde{C}(t) = C(t)$ and the lemma is proved. □

Similarly to in [18], we have the following two statements. In the first one, we formulate estimates for the function $H(v,G)$ from (20). Note that by applying these estimates in (18), we obtain an inequality that is, in fact, the Riccati inequality associated with a certain ordinary linear equation. The second statement gives sufficient conditions for the eventual non-negativity of the solutions to (18). By studying the proof of the original statement in [18], one can easily see that it also holds for the neutral version of the modified Riccati inequality (18).

Lemma 4 ([21], Lemma 5 and Lemma 6)**.** *The function $H(v,G)$ defined by (20) is non-negative and $H(v,G) = 0$ if and only if $v = 0$. Furthermore, if $\liminf_{t\to\infty} |G(t)| > 0$ and $v(t) \to 0$ for $t \to \infty$, then*

$$H(v(t),G(t)) = \frac{q(q-1)}{2}|G(t)|^{q-2}v^2(t)(1+o(1)), \quad \text{as } t\to\infty.$$

Finally, for every $T > 0$, there exists a constant $K > 0$ such that

$$H(v(t),G(t)) \geq K|G(t)|^{q-2}v^2(t)$$

for any t and v satisfying $|v(t)/G(t)| \leq T$.

Lemma 5 ([18], Lemma 2.5)**.** *Let h be a positive continuously differentiable function such that $h' \neq 0$ for large t and $C(t) \geq 0$ for large t. Moreover, let either*

$$\limsup_{t\to\infty} |G(t)| < \infty \tag{21}$$

and

$$\int^\infty \frac{f(t)\tau'(t)}{r^{q-1}(t)h^q(\tau(t))}\,dt = \infty \tag{22}$$

or

$$\lim_{t\to\infty} |G(t)| = \infty \tag{23}$$

and

$$\int^{\infty} \frac{\Phi(f(t))\tau'(t)}{r(t)h^2(\tau(t))|h'(\tau(t))|^{p-2}}\,dt = \infty. \tag{24}$$

Then, all possible proper solutions (i.e., solutions that exist in a neighborhood of infinity) of (18) are eventually nonnegative.

3. Main Results

Theorem 1. *Let f be a positive function, let G and H be defined by (16) and (20), respectively, and let the conditions of Lemma 5 be satisfied. If*

$$\int^{\infty} C(t)\,dt = \infty, \tag{25}$$

then Equation (1) is either oscillatory or, in every neighborhood of ∞, there exists t^* such that $\frac{z'(\tau(t^*))}{z'(t^*)f(t^*)} < 1$ for all solutions of (1).

Proof. Suppose, by a contradiction, that there exists $T \ge t_0$ such that (1) has a solution $x(t)$ that is positive for $t \in [T, \infty)$, and condition (14) holds on this interval. Then, $v(t)$ defined by (17) satisfies (18), and hence,

$$v'(t) \le -C(t) - (p-1)r^{1-q}(t)\tau'(t)h^{-q}(\tau(t))f(t)H(v(t), G(t)).$$

Integrating the inequality from $t_1 \ge T$ to t, we get

$$v(t) \le v(t_1) - \int_{t_1}^{t} C(s)\,ds - (p-1)\int_{t_1}^{t} r^{1-q}(s)\tau'(s)h^{-q}(\tau(s))f(s)H(v(s), G(s))\,ds.$$

Since the last subtracted term is nonnegative, we have

$$v(t) \le v(t_1) - \int_{t_1}^{t} C(s)\,ds$$

and letting $t \to \infty$, we are led to a contradiction with non-negativity of $v(t)$ by Lemma 5. □

Denote

$$R(t) = r(t)\frac{1}{\tau'(t)}h^2(\tau(t))|h'(\tau(t))|^{p-2}f^{1-p}(t). \tag{26}$$

Under the assumptions of the paper, according to (8), we can take $f(t) = \Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right)$ and the functions $G, C,$ and R to get the following form:

$$\begin{aligned}
G_1(t) &= r(\tau(t))h(\tau(t))\Phi(h'(\tau(t))),\\
C_1(t) &= h(\tau(t))\left[(r(\tau(t))\Phi(h'(\tau(t))))' + c(t)\Phi(h(\tau(t)))\Phi(1-b(\tau(t)))\right],\\
R_1(t) &= r(\tau(t))\frac{1}{\tau'(t)}h^2(\tau(t))|h'(\tau(t))|^{p-2}.
\end{aligned}$$

In this special case of the function f, we can formulate a version of Theorem 1 as follows.

Corollary 1. *Let h be a positive continuously differentiable function such that $h' \ne 0$ for large t and $C_1(t) \ge 0$ for large t. Moreover, let either*

$$\limsup_{t\to\infty} |G_1(t)| < \infty \quad \text{and} \quad \int^{\infty} \frac{\tau'(t)}{r^{q-1}(\tau(t))h^q(\tau(t))}\,dt = \infty \tag{27}$$

or

$$\lim_{t\to\infty} |G_1(t)| = \infty \quad \text{and} \quad \int^{\infty} R_1^{-1}(t)\,dt = \infty.$$

If

$$\int^{\infty} C_1(t)\, dt = \infty, \tag{28}$$

then Equation (1) *is oscillatory.*

The second and last theorem is of the Hille–Nehari type and concerns the case where the integral in (25) is convergent. We present a version with the general function f, and G, C, and R are given by (16), (19), and (26); however, one can also formulate the special case of the theorem with the function $f(t) = \Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right)$ and G_1, C_1, and R_1, similarly to in Corollary 1. Recall that the same types of results for half-linear Equation (5) were proved in [26], and for delayed half-linear Equation (6), comparison theorems providing qualitatively similar results were presented in [18].

Theorem 2. *Let h be a positive continuously differentiable function such that $h' \neq 0$ for large t and let f be a positive function such that $C(t) \geq 0$ for large t, $\int^{\infty} R^{-1}(t)\, dt = \infty$, $\liminf_{t\to\infty} G(t) > 0$, and*

$$\limsup_{t\to\infty} G(t) < \infty \quad \text{or} \quad \lim_{t\to\infty} G(t) = \infty. \tag{29}$$

Suppose that $\int^{\infty} C(t)\, dt < \infty$. If

$$\liminf_{t\to\infty} \int^{t} R^{-1}(s)\, ds \int_t^{\infty} C(s)\, ds > \frac{1}{2q}, \tag{30}$$

then Equation (1) *is either oscillatory or in every neighborhood of ∞, there exists t^* such that $\frac{z'(\tau(t^*))}{z'(t^*)f(t^*)} < 1$ for all solutions of* (1)*.*

Proof. Suppose, by a contradiction, that there exists $T \geq t_0$ such that (1) has a solution $x(t)$ that is positive for $t \in [T, \infty)$, and condition (14) holds on this interval. All conditions of Lemma 5 are satisfied. Indeed, conditions (21) and (23) are given in (29), condition (24) is, in fact, $\int^{\infty} R^{-1}(t)\, dt = \infty$, and (22) can be written in the form $\int^{\infty} R^{-1}(t)|G(t)|^{2-q}\, dt = \infty$; this follows from (24) and the fact that $0 < \liminf_{t\to\infty} |G(t)| \leq \limsup_{t\to\infty} |G(t)| < \infty$. With respect to Lemma 5, the function $v(t)$ defined by (17) is eventually non-negative. We show that $\lim_{t\to\infty} v(t) = 0$. It follows from (18) that $v'(t) \leq 0$; hence, the limit exists and is non-negative and finite. Integrating (18) from T_1 to t ($T_1 \geq T$) yields

$$v(T_1) - v(t) \geq \int_{T_1}^{t} C(s)\, ds + (p-1) \int_{T_1}^{t} r^{1-q}(s)\tau'(s)h^{-q}(\tau(s))f(s)H(v(s), G(s))\, ds.$$

Since $v(t) \geq 0$, we have

$$v(T_1) \geq \int_{T_1}^{t} C(s)\, ds + (p-1) \int_{T_1}^{t} r^{1-q}(s)\tau'(s)h^{-q}(\tau(s))f(s)H(v(s), G(s))\, ds.$$

Both the integrals in the inequality are non-negative, and letting $t \to \infty$, we see that the integral

$$\int_{T_1}^{t} r^{1-q}(s)\tau'(s)h^{-q}(\tau(s))f(s)H(v(s), G(s))\, ds$$

is convergent. With respect to conditions $\liminf_{t\to\infty} G(t) > 0$ and $\lim_{t\to\infty} v(t) < \infty$, there exists a positive constant M and $T_2 \geq T_1$ such that $|\frac{v(t)}{G(t)}| < M$ for $t \geq T_2$. According to Lemma 4, there exists $K > 0$ such that

$$K|G(t)|^{q-2}v^2(t) \leq H(v(t), G(t)) \quad \text{for } t \geq T_2.$$

Since $R(t)|G(t)|^{q-2} = \frac{r^{q-1}(t)h^q(\tau(t))}{\tau'(t)f(t)}$, we have

$$K\frac{v^2(t)}{R(t)} \le r^{1-q}(t)\tau'(t)h^{-q}(\tau(t))f(t)H(v(t),G(t)) \quad \text{for } t \ge T_2.$$

Integrating the last inequality from T_3 to t ($T_3 \ge T_2$) and letting $t \to \infty$, we obtain

$$K\int_{T_3}^{\infty} \frac{v^2(t)}{R(t)}\,dt \le \int_{T_3}^{\infty} r^{1-q}(t)\tau'(t)h^{-q}(\tau(t))f(t)H(v(t),G(t))\,dt < \infty.$$

As $\int^{\infty} R^{-1}(t)\,dt = \infty$, the last inequality implies that $\lim_{t\to\infty} v(t) = 0$.
Now, we integrate (18) from t to ∞ to obtain

$$v(t) \ge \int_t^{\infty} C(s)\,ds + (p-1)\int_t^{\infty} r^{1-q}(s)\tau'(s)h^{-q}(\tau(s))f(s)H(v(s),G(s))\,ds.$$

Let $\varepsilon > 0$. According to Lemma 4, there exists a T_4 large enough such that

$$\begin{aligned} H(v(t),G(t)) \ge& \left(\frac{q(q-1)}{2} - \varepsilon\right)v^2(t)|G(t)|^{q-2} \\ =& \left(\frac{q(q-1)}{2} - \varepsilon\right)r^{q-2}(t)h^{q-2}(\tau(t))|h'(\tau(t))|^{2-p}f^{p-2}(t)v^2(t) \end{aligned}$$

for $t \ge T_4$. Hence,

$$v(t) \ge \int_t^{\infty} C(s)\,ds + (p-1)\left(\frac{q(q-1)}{2} - \varepsilon\right)\int_t^{\infty} \frac{v^2(s)}{R(s)}\,ds$$

for $t \ge T_4$. Denoting $\tilde{\varepsilon} = (p-1)\varepsilon$, the inequality becomes

$$v(t) \ge \int_t^{\infty} C(s)\,ds + \left(\frac{q}{2} - \tilde{\varepsilon}\right)\int_t^{\infty} R^{-1}(s)v^2(s)\,ds$$

and multiplication by $\mathcal{R}(t) = \int^t R^{-1}(s)\,ds$ gives

$$\mathcal{R}(t)v(t) \ge \mathcal{R}(t)\int_t^{\infty} C(s)\,ds + \left(\frac{q}{2} - \tilde{\varepsilon}\right)\mathcal{R}(t)\int_t^{\infty} \frac{R^{-1}(s)}{\mathcal{R}^2(s)}(\mathcal{R}(s)v(s))^2\,ds. \tag{31}$$

With respect to (30), there exists $\delta > 0$ such that $\liminf_{t\to\infty} \mathcal{R}(t)\int_t^{\infty} C(s)\,ds \ge \frac{1}{2q} + \delta$. Furthermore, we observe that $\mathcal{R}(t)\int_t^{\infty} \frac{R^{-1}(s)}{\mathcal{R}^2(s)}\,ds = 1$. There are two possible options: either $\liminf_{t\to\infty} \mathcal{R}(t)v(t) < \infty$ or $\liminf_{t\to\infty} \mathcal{R}(t)v(t) = \infty$. Let us discuss both cases.

(a) Suppose that $\liminf_{t\to\infty} \mathcal{R}(t)v(t) := L < \infty$. From (31), one can see that $L > 0$. For every $\bar{\varepsilon} > 0$, there exists a T_5 large enough so that $(\mathcal{R}(t)v(t))^2 > (1-\bar{\varepsilon})^2L^2$ for $t \ge T_5$. Estimations of the terms in (31) give

$$(1+\bar{\varepsilon})L \ge \frac{1}{2q} + \delta + \left(\frac{q}{2} - \tilde{\varepsilon}\right)(1-\bar{\varepsilon})^2L^2.$$

Letting $\tilde{\varepsilon}, \bar{\varepsilon} \to 0$ gives a contradiction, since then,

$$L \ge \frac{1}{2q} + \delta + \frac{q}{2}L^2 \Leftrightarrow \frac{q}{2}\left(L - \frac{1}{q}\right)^2 + \delta \le 0.$$

(b) In the case where $\liminf_{t\to\infty} \mathcal{R}(t)v(t) = \infty$, we denote $m(t) = \inf_{t\le s}\{\mathcal{R}(s)v(s)\}$. Then, from (31), it follows that

$$\mathcal{R}(t)v(t) \geq \frac{1}{2q} + \delta + \left(\frac{q}{2} - \tilde{\varepsilon}\right)m^2(t).$$

Since the function m is nondecreasing, we have for $s \geq t$:

$$\mathcal{R}(s)v(s) \geq \frac{1}{2q} + \delta + \left(\frac{q}{2} - \tilde{\varepsilon}\right)m^2(s) \geq \frac{1}{2q} + \delta + \left(\frac{q}{2} - \tilde{\varepsilon}\right)m^2(t),$$

and hence,

$$m(t) \geq \frac{1}{2q} + \delta + \left(\frac{q}{2} - \tilde{\varepsilon}\right)m^2(t) > \left(\frac{q}{2} - \tilde{\varepsilon}\right)m^2(t),$$

which gives a contradiction with the assumption $\liminf_{t\to\infty} \mathcal{R}(t)v(t) = \infty$.

The proof is finished. □

4. Examples

Consider the Euler-type equation

$$(\Phi(x(t) + b_0 x(\sigma(t)))')' + \frac{\gamma}{t^p}\Phi(x(\lambda t)) = 0, \tag{32}$$

where $\lambda \in (0,1)$, $\sigma(t) \leq t$, $\lim_{t\to\infty} \sigma(t) = \infty$, and $b_0 \in [0,1)$. Equation (32) is of the form (1), where $r(t) = 1$, $c(t) = \frac{\gamma}{t^p}$, $\tau(t) = \lambda t$, and $b(t) = b_0$.

Example 1. *Take $h(t) = t^{\frac{p-1}{p}}$; then, by a direct computation, we have*

$$G_1(t) = \left(\frac{p-1}{p}\right)^{p-1}, \quad R_1(t) = \left(\frac{p-1}{p}\right)^{p-2} t \tag{33}$$

and

$$\int^{\infty} \frac{\tau'(t)}{r^{q-1}(\tau(t))h^q(\tau(t))}\,dt = \int^{\infty} \frac{1}{t}\,dt \to \infty.$$

Hence, condition (27) is satisfied. Furthermore, we have

$$\begin{aligned} C_1(t) &= (\lambda t)^{\frac{p-1}{p}}\left[\left(\left(\frac{p-1}{p}\right)^{p-1}(\lambda t)^{\frac{1-p}{p}}\right)' + c(t)\Phi(1-b_0)\left((\lambda t)^{\frac{p-1}{p}}\right)^{p-1}\right] \\ &= (\lambda t)^{\frac{p-1}{p}}\left[-\left(\frac{p-1}{p}\right)^{p}\lambda^{\frac{1}{p}-1}t^{\frac{1}{p}-2} + c(t)\Phi(1-b_0)(\lambda t)^{p-2+\frac{1}{p}}\right] \\ &= t^{-1}\left[-\left(\frac{p-1}{p}\right)^{p} + c(t)\Phi(1-b_0)\lambda^{p-1}t^p\right] \\ &= t^{-1}\left[-\left(\frac{p-1}{p}\right)^{p} + \gamma\Phi(1-b_0)\lambda^{p-1}\right]. \end{aligned} \tag{34}$$

The positivity of the expression $-\left(\frac{p-1}{p}\right)^p + \gamma\Phi(1-b_0)\lambda^{p-1}$ implies (28). So, by Corollary 1, Equation (32) is oscillatory if

$$\gamma > \left(\frac{p-1}{p}\right)^p \frac{1}{\lambda^{p-1}\Phi(1-b_0)}. \tag{35}$$

This corresponds to the result known in the case where $b_0 = 0$ (equations with delay, see [18]) and also with the case where $b_0 = 0$ and $\lambda = 1$ (ordinary equations).

Example 2. *Condition (35) can even be strengthened for a class of Equation (32), which satisfies (12) with $\varrho \in (0, \frac{1}{e})$. By a direct computation (or see [10]), one can show that for (32),*

$$\varrho = (1-b_0)^{p-1}\gamma\lambda^{p-1}\left(1 + \frac{1}{p-1}\lambda^{p-1}(1-b_0)^{p-1}\gamma\right)^{p-1}\log\left(\frac{1}{p-1}\right).$$

Note that here and in what follows, the symbol log *stands for the natural logarithm. If $\varrho \in (0, \frac{1}{e})$, according to Lemma 2, we can use in place of the positive function $f(t)$ from (14) the function $\Phi^{-1}\left(\frac{g(\varrho)r(t)}{r(\tau(t))}\right)$, where $g(\varrho)$ is defined by (11). Then, functions R, G, and C become*

$$\begin{aligned} G_2(t) &= r(\tau(t))h(\tau(t))\Phi(h'(\tau(t)))(g(\varrho))^{-1}, \\ C_2(t) &= h(\tau(t))\left[\left(r(\tau(t))\frac{\Phi(h'(\tau(t)))}{g(\varrho)}\right)' + c(t)\Phi(h(\tau(t)))\Phi(1-b_0)\right], \\ R_2(t) &= r(\tau(t))\frac{1}{\tau'(t)}h^2(\tau(t))|h'(\tau(t))|^{p-2}(g(\varrho))^{-1}. \end{aligned}$$

In our setting for Equation (32) and again with $h(t) = t^{\frac{p-1}{p}}$, these read as

$$G_2(t) = \left(\frac{p-1}{p}\right)^{p-1}(g(\varrho))^{-1}, \quad R_1(t) = \left(\frac{p-1}{p}\right)^{p-2}(g(\varrho))^{-1}t$$

and

$$C_2(t) = t^{-1}\left[-\left(\frac{p-1}{p}\right)^{p}(g(\varrho))^{-1} + \gamma\Phi(1-b_0)\lambda^{p-1}\right].$$

Since (21) and (22) hold and (25) is implied by the positivity of the expression

$$\left[-\left(\frac{p-1}{p}\right)^{p}(g(\varrho))^{-1} + \gamma\Phi(1-b_0)\lambda^{p-1}\right],$$

Equation (32) is oscillatory, according to Theorem 1, provided that

$$\gamma > \left(\frac{p-1}{p}\right)^{p}\frac{1}{(g(\varrho))\lambda^{p-1}\Phi(1-b_0)}.$$

This corresponds to the condition derived in [10].

In the theory of ordinary equations, i.e., in the case where $\lambda = 1$ and $b_0 = 0$ in (32), it is known that Equation (32) oscillates if and only if (35) holds. This means that the constant $\gamma = \left(\frac{p-1}{p}\right)^p$ is the critical constant between oscillation and non-oscillation of (32), and it is natural to study perturbations of the Euler-type equation with this critical constant and to find critical constants in the added terms. This corresponds to the concept of conditional oscillation (for details, see [23,27] and the references given therein). In the case of the delayed and neutral equations, there is not such a boundary between oscillation and non-oscillation of (32). However, based on the results known from the ordinary case, let us study the neutral version of the Euler–Weber-type equation

$$(\Phi(x(t) + b_0x(\sigma(t)))')' + \left(\frac{\left(\frac{p-1}{p}\right)^p}{\lambda^{p-1}\Phi(1-b_0)t^p} + \frac{\mu}{t^p\log^2 t}\right)\Phi(x(\lambda(t))) = 0. \qquad (36)$$

Example 3. *For (36), we use Theorem 2 with* $f(t) = \Phi^{-1}\left(\frac{r(t)}{r(\tau(t))}\right)$*. Similarly to in Example 1, we take* $h(t) = t^{\frac{p-1}{p}}$*, we obtain (33), and, with the use of the relevant coefficient function* $c(t)$ *in (34), we have*

$$C_1(t) = \frac{\mu}{t\log^2 t}\Phi(1-b_0)\lambda^{p-1}$$

and

$$\int_t^\infty C_1(s)\,ds = \int_t^\infty \frac{\mu\Phi(1-b_0)\lambda^{p-1}}{s\log^2 s}\,ds = \frac{\mu\Phi(1-b_0)\lambda^{p-1}}{\log t}.$$

Since $R_1^{-1} = \left(\frac{p}{p-1}\right)^{p-2}\frac{1}{t}$*, condition (30) becomes*

$$\left(\frac{p}{p-1}\right)^{p-2}\mu\Phi(1-b_0)\lambda^{p-1} > \frac{1}{2q}$$

which, by Theorem 2, implies that Equation (36) is oscillatory if

$$\mu > \frac{1}{2\Phi(1-b_0)\lambda^{p-1}}\left(\frac{p-1}{p}\right)^{p-1}. \tag{37}$$

Note that in the ordinary case where $\lambda = 1$ *and* $b_0 = 0$*, the constant from (37) is critical, which means that (36) is oscillatory if and only if* $\mu > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$*.*

Example 4. *Let us consider the perturbation of the Euler–Weber equation. Take* $h(t) = t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t$ *and consider Equation (1) with* $r(t) = 1$*,* $\tau(t) = \lambda t$*, and* $b(t) = 0$*. Observe that*

$$h'(t) = \frac{p-1}{p}t^{-\frac{1}{p}}\log^{\frac{1}{p}} t\left[1 + \frac{1}{(p-1)\log t}\right].$$

By a direct computation, we see that

$$G_1(t) = \left(\frac{p-1}{p}\right)^{p-1}\log(\lambda t)(1+o(1)),$$
$$R_1(t) = \left(\frac{p-1}{p}\right)^{p-2}t\log(\lambda t)(1+o(1))$$

as $t \to \infty$*. With the use of the power expansion formula*

$$(1+x)^s = 1 + sx + \frac{s(s-1)}{2}x^2 + \frac{s(s-1)(s-2)}{6}x^3 + o(x^3) \quad \text{as} \quad x \to 0,$$

one can show that for

$$c(t) = \left(\frac{p}{p-1}\right)^p\frac{1}{\lambda^{p-1}t^p} + \frac{1}{2}\left(\frac{p}{p-1}\right)^{p-1}\frac{1}{\lambda^{p-1}t^p\log^2(\lambda t)} + \frac{\mu}{t^p\log^2(\lambda t)\log^2(\log(\lambda t))},$$

we have

$$C_1(t) = \frac{\mu\lambda^{p-1}}{t\log(\lambda t)\log^2(\log(\lambda t))}(1+o(1)) \quad \text{as } t \to \infty.$$

Because

$$\int R_1^{-1}(t)\,dt \sim \left(\frac{p-1}{p}\right)^{2-p}\log(\log(\lambda t))$$

and

$$\int C_1(t)\,dt \sim -\frac{\mu\lambda^{p-1}}{\log(\log(\lambda t))},$$

condition (30) *becomes*

$$\left(\frac{p-1}{p}\right)^{2-p}\mu\lambda^{p-1} > \frac{1}{2q}.$$

Since $\frac{1}{q} = \frac{p-1}{p}$*, the considered equation is oscillatory if*

$$\mu > \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}\frac{1}{\lambda^{p-1}}.$$

5. Conclusions

The aim of this paper was to study how the modified Riccati technique can be applied to Equation (1), which has not been tried for neutral equations before, and what results this approach can provide. According to our results, the modified Riccati method is applicable to Equation (1), and it can be used to find new criteria, for whose proofs it is enough to manipulate the modified Riccati inequality. We have presented two new oscillation criteria and illustrated their uses in examples dealing with a half-linear Euler-type equation and its perturbations.

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Research Article

Differential Transform Algorithm for Functional Differential Equations with Time-Dependent Delays

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An algorithm using the differential transformation which is convenient for finding numerical solutions to initial value problems for functional differential equations is proposed in this paper. We focus on retarded equations with delays which in general are functions of the independent variable. The delayed differential equation is turned into an ordinary differential equation using the method of steps. The ordinary differential equation is transformed into a recurrence relation in one variable using the differential transformation. Approximate solution has the form of a Taylor polynomial whose coefficients are determined by solving the recurrence relation. Practical implementation of the presented algorithm is demonstrated in an example of the initial value problem for a differential equation with nonlinear nonconstant delay. A two-dimensional neutral system of higher complexity with constant, nonconstant, and proportional delays has been chosen to show numerical performance of the algorithm. Results are compared against Matlab function DDENSD.

1. Introduction

Functional differential equations (FDEs) are used to model processes and phenomena which depend on past values of the modelled entities. Indicatively, we mention models describing machine tool vibrations [1], predator-prey type models [2], and models used in economics [3]. Further models and details can be found for instance in [4, 5] or [6].

Differential transformation (DT), a semianalytical approach based on Taylor's theorem, has been proved to be efficient in solving a variety of initial value problems (IVPs), ranging from ordinary to functional, partial, and fractional differential equations [7–11]. However, there is no publication about systematic application of DT to IVP for differential equations with nonconstant delays which are functions of the independent variable.

In this paper, we present an extension of DT to a class of IVPs for delayed differential equations with analytic right-hand side. Albeit the analyticity assumption seems to be quite restrictive, it is reasonable to develop theory for such class of equations [12, 13].

The paper is organised as follows. In Section 2, we define the subject of our study and briefly describe the methods we combine to solve the studied problem, including recalling necessary results of previous studies. Section 3 contains the main results of the paper, including algorithm description, new theorems, examples, and comparison of numerical results. In Section 4, we briefly summarise what has been done in the paper.

2. Methods

2.1. Problem Statement. The problem studied in this paper is to find a solution on a given finite interval $[t_0, T] \subset [0, \infty)$ to an IVP for the following system of p functional differential equations of n-th order with multiple delays $\alpha_1(t), \ldots, \alpha_r(t)$:

$$\mathbf{u}^{(n)}(t) = \mathbf{f}\Big(t, \mathbf{u}(t), \mathbf{u}'(t), \ldots, \mathbf{u}^{(n-1)}(t), \mathbf{u}_1(\alpha_1(t)), \ldots, \mathbf{u}_r(\alpha_r(t))\Big), \tag{1}$$

where $\mathbf{u}^{(n)}(t) = (u_1^{(n)}(t), \ldots, u_p^{(n)}(t))^T$, $\mathbf{u}^{(k)}(t) = (u_1^{(k)}(t), \ldots, u_p^{(k)}(t))$, $k = 0, 1, \ldots, n-1$, and $\mathbf{f} = (f_1, \ldots, f_p)^T$ are p-dimensional vector functions, $\mathbf{u}_i(\alpha_i(t)) = (\mathbf{u}(\alpha_i(t)), \mathbf{u}'(\alpha_i(t)), \ldots, \mathbf{u}^{(m_i)}(\alpha_i(t)))$ are $(m_i \cdot p)$-dimensional vector functions, $m_i \le n$, $i = 1, 2, \ldots, r$, $r \in \mathbb{N}$, and f_j: $[t_0, T)$ are real functions for $j = 1, 2, \ldots, p$, where $\omega = \sum_{i=1}^{r} m_i$.

We assume that each $\alpha_i(t) = t - \tau_i(t)$, where $\tau_i(t) \ge \tau_{i0} > 0$ for $t \in [t_0, T]$, $i = 1, 2, \ldots, r$, is in general a real function, that is, a time-dependent or time-varying delay. Constant and proportional delays are considered as special cases. In case that some α_i is a proportional delay, we do not require the condition $\tau_i(t) \ge \tau_{i0} > 0$ to be valid at 0 if $t_0 = 0$.

Let $t^* = \min_{1 \le i \le r}\left\{\inf_{t \in [t_0, T]}(\alpha_i(t))\right\}$ and $m = \max\{m_1, m_2, \ldots, m_r\}$; hence, $t^* \le t_0$ and $m \le n$. If $m < n$, we have a retarded system (1); otherwise, if $m = n$, we call the system neutral. Furthermore, if $t^* < 0$, initial vector function $\Phi(t) = (\phi_1(t), \ldots, \phi_p(t))^T$ must be prescribed on the interval $[t^*, t_0]$.

DT algorithm for the case $t^* = t_0 = 0$ with all delays being proportional is described in [14]. DT algorithm for the case $t^* < t_0$ when all delays are constant is introduced in [15]. In this paper, we develop the algorithm for the case $t^* < t_0$ when at least one delay is nonconstant.

To have a complete IVP, we consider system (1) together with initial conditions:

$$\begin{aligned} &\mathbf{u}(t_0) = \mathbf{v}_0, \\ &\mathbf{u}'(t_0) = \mathbf{v}_1, \ldots, \mathbf{u}^{(n-1)}(t_0) = \mathbf{v}_{n-1}, \end{aligned} \tag{2}$$

and, since $t^* < t_0$, also subject to initial vector function $\Phi(t)$ on interval $[t^*, t_0]$ such that

$$\Phi(t_0) = \mathbf{u}(t_0), \ldots, \Phi^{(n-1)}(t_0) = \mathbf{u}^{(n-1)}(t_0). \tag{3}$$

We consider the IVPs (1)–(3) under the following hypotheses:

(H1) We assume that all the functions $\phi_j(t)$, $j = 1, \ldots, p$, are analytic in $[t^*, t_0]$, the functions $\alpha_i(t)$, $i = 1, \ldots, r$, are analytic in $[t_0, T]$ and the functions f_j, $j = 1, \ldots, p$, are analytic in an open set containing $[t_0, T] \times [\mathbf{u}(t_0), \mathbf{u}(T)] \times \ldots \times [\mathbf{u}_r(\alpha_r(t_0)), \mathbf{u}_r(\alpha_r(T))]$.

(H2) If $\alpha_i(t) = q_i t$ and $m_i = n$ in f_j for some $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, p\}$, that is, jth equation is neutral with respect to the proportional delay α_i, we assume that $u_l^{(n)}(\alpha_i(t)) \equiv 0$ for $l \in \{1, \ldots, p\}$, $l \ne j$. This hypothesis is included since if it is not fulfilled, the existence of unique solution of IVP could be violated.

We note that these assumptions imply that the IVP (1)–(3) has a unique solution in the interval $[t_0, T]$.

2.2. Method of Steps. The basic idea of our approach is to combine DT and the general method of steps. The method of steps enables us to replace the terms including delays with initial vector function $\Phi(t)$ and its derivatives. Then, the original IVP for the delayed or neutral system of differential equations is turned into IVP for a system of ordinary differential equations.

For the sake of clarity, we include a simple explanatory example. Suppose that we have a system with three delays, one of each type considered: $\alpha_1(t) = t - \tau_1(t)$, $\alpha_2(t) = t - \tau_2$, and $\alpha_3(t) = q_3 t$. We have to distinguish two cases:

(a) If $t_0 = 0$, applying the method of steps turns system (1) into

$$\begin{aligned} \mathbf{u}^{(n)}(t) = \mathbf{f}\Big(&t, \mathbf{u}(t), \ldots, \mathbf{u}^{(n-1)}(t), \\ &\mathbf{\Phi}_1(t - \tau_1(t)), \mathbf{\Phi}_2(t - \tau_2), \mathbf{u}_3(q_3 t)\Big), \end{aligned} \tag{4}$$

while

(b) If $t_0 > 0$, system (1) is simplified to

$$\begin{aligned} \mathbf{u}^{(n)}(t) = \mathbf{f}\Big(&t, \mathbf{u}(t), \ldots, \mathbf{u}^{(n-1)}(t), \\ &\mathbf{\Phi}_1(t - \tau_1(t)), \mathbf{\Phi}_2(t - \tau_2), \mathbf{\Phi}_3(q_3 t)\Big), \end{aligned} \tag{5}$$

where

$$\begin{aligned} \mathbf{\Phi}_1(t - \tau_1(t)) &= \Big(\Phi(t - \tau_1(t)), \Phi'(t - \tau_1(t)), \ldots, \\ &\qquad \Phi^{(m_1)}(t - \tau_1(t))\Big), \\ \mathbf{\Phi}_2(t - \tau_2) &= \Big(\Phi(t - \tau_2), \Phi'(t - \tau_2), \ldots, \Phi^{(m_2)}(t - \tau_2)\Big), \\ \mathbf{u}_3(q_3 t) &= \Big(\mathbf{u}(q_3 t), \mathbf{u}'(q_3 t), \ldots, \mathbf{u}^{(m_3)}(q_3 t)\Big), \\ \mathbf{\Phi}_3(q_3 t) &= \Big(\Phi(q_3 t), \Phi'(q_3 t), \ldots, \Phi^{(m_3)}(q_3 t)\Big), \end{aligned} \tag{6}$$

and $m_l \le n$ for $l = 1, 2, 3, 4$. More details on the general method of steps can be found, for instance, in monographs [4] or [6].

Continuation of the method of steps algorithm for equations with constant delays $\tau_1, \ldots, \tau_r$ is described in [15]. Briefly summarised, the interval $[t_0, T]$ is divided into subintervals $I_l = [t_{l-1}, t_l]$, $l = 1, \ldots, K$, where $t_K = T$ and t_l, $l = 1, \ldots, K-1$, are the principal discontinuity points which is the set of points $t_{\rho,\sigma}$, such that $t_{0,1} = t_0$ and for $\rho, \sigma \ge 1$, $t_{\rho,\sigma}$ are the minimal roots with odd multiplicity of r equations:

$$t_{\rho,(\sigma-1)r+\mu} - \tau_\mu = t_{\rho-1,\sigma}, \quad \mu = 1, \ldots, r. \tag{7}$$

If nonconstant nonproportional delays α_i appear in system (1), the principal set of discontinuity points is defined as follows:

Definition 1. The principal discontinuity points for the solutions of system (1) are given by the set of points $t_{\rho,\sigma}$, such that $t_{0,1} = t_0$ and for $\rho, \sigma \geq 1$, $t_{\rho,\sigma}$ are the minimal roots with odd multiplicity of r equations:

$$\alpha_\mu\left(t_{\rho,(\sigma-1)r+\mu}\right) = t_{\rho-1,\sigma}, \quad \mu = 1, \ldots, r. \tag{8}$$

Similar to the case of constant delays, we break the interval $[t_0, T]$ into subintervals $I_l = [t_{l-1}, t_l]$, $l = 1, \ldots, K$. We start with the mesh grid $\{t_0, \ldots, t_K\}$ formed by the principal discontinuity points calculated using Definition 1. To improve convergence or performance of the algorithm, there is a possibility to refine the mesh grid by inserting other points into it. For more details on the principal discontinuity points and mesh grid, we refer to the monograph [16].

2.3. Differential Transformation

Definition 2. Differential transformation of a real function $u(t)$ at a point $t_0 \in \mathbb{R}$ is $\mathscr{D}\{u(t)\}[t_0] = \{U(k)[t_0]\}_{k=0}^{\infty}$, where $U(k)[t_0]$, $k-$th component of the differential transformation of the function $u(t)$ at t_0, $k \in \mathbb{N}_0$, is defined as

$$U(k)[t_0] = \frac{1}{k!}\left[\frac{\mathrm{d}^k u(t)}{\mathrm{d}t^k}\right]_{t=t_0}, \tag{9}$$

provided that the original function $u(t)$ is analytic in a neighbourhood of t_0.

Definition 3. Inverse differential transformation of $\{U(k)[t_0]\}_{k=0}^{\infty}$ is defined as

$$u(t) = \mathscr{D}^{-1}\{\{U(k)[t_0]\}_{k=0}^{\infty}\}[t_0] = \sum_{k=0}^{\infty} U(k)[t_0](t-t_0)^k. \tag{10}$$

In applications, the function $u(t)$ is expressed by a finite sum

$$u(t) = \sum_{k=0}^{N} U(k)[t_0](t-t_0)^k. \tag{11}$$

As we can observe in (10), DT is based on Taylor series; hence, any theorem about convergence of Taylor series may be used. However, we would like to point out the paper [17] where the finest general explicit *a priori* error estimates are given.

The following formulas are listed, e.g., in [18] and will be used in Section 3.3.

Lemma 1. *Assume that $F(k)[t_0]$ and $U(k)[t_0]$ are differential transformations of functions $f(t)$ and $u(t)$, respectively*:

$$\begin{gathered}
If\ f(t) = \frac{\mathrm{d}^n u(t)}{\mathrm{d}t^n},\ then\ F(k)[t_0] = \frac{(k+n)!}{k!}U(k+n)[t_0]. \\
If\ f(t) = t^n,\ then\ F(k)[0] = \delta(k-n), \\
where\ \delta(k-n) = \delta_{kn}\ is\ the\ Kronecker\ delta. \\
If\ f(t) = \mathrm{e}^{\lambda t},\ then\ F(k)[0] = \frac{\lambda^k}{k!}.
\end{gathered} \tag{12}$$

Remark 1. Similar formulas can be obtained using numerical approach called Functional Analytical Technique based on Operator Theory [19, 20].

The main disadvantage of many papers about DT is that there are almost no examples of equations with non-polynomial nonlinear terms containing unknown function $u(t)$ like, for instance, $f(u) = \sqrt[5]{1+u^3}$ or $f(u) = e^{\sqrt{\sin u}}$. However, DT of components containing nonlinear terms can be obtained in a consistent way using the algorithm described in [21].

Theorem 1. *Let g and f be real functions analytic near t_0 and $g(t_0)$, respectively, and let h be the composition $h(t) = (f \circ g)(t) = f(g(t))$. Denote $\mathscr{D}\{g(t)\}[t_0] = \{G(k)\}_{k=0}^{\infty}$, $\mathscr{D}\{f(t)\}[g(t_0)] = \{F(k)\}_{k=0}^{\infty}$, and $\mathscr{D}\{(f \circ g)(t)\}[t_0] = \{H(k)\}_{k=0}^{\infty}$ as the differential transformations of functions g, f, and h at t_0, $g(t_0)$, and t_0, respectively. Then, the numbers $H(k)$ in the sequence $\{H(k)\}_{k=0}^{\infty}$ satisfy the relations $H(0) = F(0)$ and*

$$H(k) = \sum_{l=1}^{k} F(l) \cdot \widehat{B}_{k,l}(G(1), \ldots, G(k-l+1)), \quad \text{for } k \geq 1, \tag{13}$$

where $\widehat{B}_{k,l}(\widehat{x}_1, \ldots, \widehat{x}_{k-l+1})$ are the partial ordinary Bell polynomials.

The following Lemma proved in [21] is useful when calculating partial ordinary Bell polynomials.

Lemma 2. *The partial ordinary Bell polynomials $\widehat{B}_{k,l}(\widehat{x}_1, \ldots, \widehat{x}_{k-l+1})$, $l = 1, 2, \ldots, k \geq l$, satisfy the recurrence relation*

$$\widehat{B}_{k,l}(\widehat{x}_1, \ldots, \widehat{x}_{k-l+1}) = \sum_{i=1}^{k-l+1} \frac{i \cdot l}{k}\widehat{x}_i \widehat{B}_{k-i,l-1}(\widehat{x}_1, \ldots, \widehat{x}_{k-i-l+2}), \tag{14}$$

where $\widehat{B}_{0,0} = 1$ and $\widehat{B}_{k,0} = 0$ for $k \geq 1$.

3. Results and Discussion

3.1. Algorithm Description. Recall system (1)

$$\mathbf{u}^{(n)}(t) = \mathbf{f}\left(t, \mathbf{u}(t), \mathbf{u}'(t), \ldots, \mathbf{u}^{(n-1)}(t), \mathbf{u}_1(\alpha_1(t)), \ldots, \mathbf{u}_r(\alpha_r(t))\right), \tag{15}$$

with initial conditions

$$\mathbf{u}(t_0) = \mathbf{v}_0, \mathbf{u}'(t_0) = \mathbf{v}_1, \ldots, \mathbf{u}^{(n-1)}(t_0) = \mathbf{v}_{n-1}, \tag{16}$$

and initial vector function $\Phi(t)$ on interval $[t^*, t_0]$ satisfying

$$\Phi(t_0) = \mathbf{u}(t_0), \ldots, \Phi^{(n-1)}(t_0) = \mathbf{u}^{(n-1)}(t_0). \tag{17}$$

Further recall that in Section 2.2, we broke the interval $[t_0, T]$ into subintervals $I_l = [t_{l-1}, t_l]$, $l = 1, \ldots, K$. Define $I_0 = [t^*, t_0]$.

Then, we are looking for a solution $\mathbf{u}(t)$ of the IVP (1)–(3) in the form

$$\mathbf{u}(t) = \begin{cases} \mathbf{u}_{I_1}(t), & t \in I_1, \\ \mathbf{u}_{I_2}(t), & t \in I_2, \\ \vdots & \\ \mathbf{u}_{I_K}(t), & t \in I_K, \end{cases} \tag{18}$$

where solution $\mathbf{u}_{I_j}$ in the jth interval I_j is obtained in the following way. We solve the following equation:

$$\begin{aligned} \mathbf{u}_{I_j}^{(n)}(t) = f\Big(&t, \mathbf{u}_{I_j}(t), \mathbf{u}'_{I_j}(t), \ldots, \mathbf{u}_{I_j}^{(n-1)}(t), \\ &\mathbf{u}_{j,1}(\alpha_1(t)), \ldots, \mathbf{u}_{j,r}(\alpha_r(t))\Big), \end{aligned} \tag{19}$$

where

$$\mathbf{u}_{j,i}(\alpha_i(t)) = \left(\mathbf{u}_{I_l}(\alpha_i(t)), \mathbf{u}'_{I_l}(\alpha_i(t)), \ldots, \mathbf{u}_{I_l}^{(m_i)}(\alpha_i(t))\right), \tag{20}$$

if $\alpha_i(t) \in I_l$, for $t \in I_j$, $l \in \{1, \ldots, j\}$, $j \in \{1, \ldots. K\}$.
In case that $\alpha_i(t) \in I_0 = [t^*, t_0]$ for $t \in I_j$, then again

$$\mathbf{u}_{j,i}(\alpha_i(t)) = \left(\phi(\alpha_i(t)), \phi'(\alpha_i(t)), \ldots, \phi^{(m_i)}(\alpha_i(t))\right). \tag{21}$$

Application of DT at t_{j-1} to equation (19) yields a system of recurrence algebraic equations:

$$\mathbf{U}_{I_j}(k+n)\left[t_{j-1}\right] = \mathbf{F}\Big(k, \mathbf{U}_{I_j}(k), \mathbf{U}_{I_j}(k+1), \ldots, \mathbf{U}_{I_j}(k+n-1)\Big), \tag{22}$$

where the function $\mathbf{F}$ is the DT of the righthand side of equation (19) and involves application of Theorem 1.

Next, we transform the initial conditions (2). Following Definition 2, we derive

$$\mathbf{U}_{I_j}(k)\left[t_{j-1}\right] = \frac{1}{k!}\mathbf{u}_{I_j}^{(k)}\left(t_{j-1}\right), \quad \text{for } k = 0, 1, \ldots, n-1,\ j \in \{1, \ldots, K\}. \tag{23}$$

Using (22) with (23) and then inverse transformation rule, we obtain approximate solution to (19) in the form of Taylor series:

$$\mathbf{u}_{I_j}(t) = \sum_{k=0}^{\infty} \mathbf{U}_{I_j}(k)\left[t_{j-1}\right]\left(t - t_{j-1}\right)^k, \quad t \in I_j, \tag{24}$$

for all $j \in \{1, \ldots, K\}$.

To transform (20) correctly, we need the following theorem.

Theorem 2. *Let $\alpha_i(t) \in I_l$ for $t \in I_j$, where $l \in \{1, \ldots, j-1\}$. Let $p \in \mathbb{N}$. Denote $\mathscr{D}\{\alpha_i(t)\}[t_{j-1}] = \{A_i(k)[t_{j-1}]\}_{k=0}^{\infty}$. Then,*

$$\begin{aligned} &\mathscr{D}\left\{u_{I_l}^{(p)}(\alpha_i(t))\right\}\left[t_{j-1}\right] \\ &= \left\{\sum_{y=0}^{k} \frac{(y+p)!}{y!} U_{I_l}(y+p)\left[\alpha_i\left(t_{j-1}\right)\right]\right. \\ &\quad \left. \cdot \widehat{B}_{k,y}\left(A_i(1)\left[t_{j-1}\right], \ldots, A_i(k-y+1)\left[t_{j-1}\right]\right)\right\}_{k=0}^{\infty}, \end{aligned} \tag{25}$$

where $\widehat{B}_{0,0} = 1$, $\widehat{B}_{k,0} = 0$ for $k \geq 1$, and

$$\begin{aligned} U_{I_l}(y)\left[\alpha_i\left(t_{j-1}\right)\right] = \sum_{x=0}^{\infty} \binom{x+y}{x} \left(\alpha_i\left(t_{j-1}\right) - t_{l-1}\right)^x \\ \cdot U_{I_l}(x+y)\left[t_{l-1}\right], \end{aligned} \tag{26}$$

for $y \geq 0$.

Proof. To prove (25) with $p = 0$, we use Theorem 1 with $f(t) = u_{I_l}(t)$, $g(t) = \alpha_i(t)$, and $h(t) = (f \circ g)(t)$. We immediately get

$$\begin{aligned} H(k)\left[t_{j-1}\right] = \sum_{y=1}^{k} U_{I_l}(y)\left[\alpha_i\left(t_{j-1}\right)\right] \\ \cdot \widehat{B}_{k,y}\left(A_i(1)\left[t_{j-1}\right], \ldots, A_i(k-y+1)\left[t_{j-1}\right]\right), \end{aligned} \tag{27}$$

for $k \geq 1$. For $k = 0$, Theorem 1 yields $H(0)\ [t_{j-1}] = U_{I_l}(0)[\alpha_i(t_{j-1})] = U_{I_l}(0)[\alpha_i(t_{j-1})] \cdot \widehat{B}_{0,0}(A_i(1)[t_{j-1}])$. Now, (25) for $p > 0$ is a consequence of Lemma 1 and it remains to prove (26). We recall that

$$\mathbf{u}_{I_l}(t) = \sum_{k=0}^{\infty} \mathbf{U}_{I_l}(k)\left[t_{l-1}\right]\left(t - t_{l-1}\right)^k, \quad t \in I_l. \tag{28}$$

As the assumption was that $\alpha_i(t_{j-1}) \in I_l$, we may apply Definition 2 to (28) and obtain

$$\begin{aligned} U_{I_l}(y)\left[\alpha_i\left(t_{j-1}\right)\right] = \frac{1}{y!}\left[\frac{\mathrm{d}^y u_{I_l}(t)}{\mathrm{d}t^y}\right]_{t=t_0} \frac{1}{y!} \sum_{z=y}^{\infty} \frac{z!}{(z-y)!} U_{I_l}(z)\left[t_{l-1}\right] \\ \cdot \left(t_0 - t_{l-1}\right)^{z-y}. \end{aligned} \tag{29}$$

Substituting $t_0 = \alpha_i(t_{j-1})$ and $z = x + y$ gives (26). □

3.2. New DT Formulas. In the applications, we also use the following DT formulas.

Theorem 3. *Assume that $F(k)[t_0]$ is the differential transformation of the function $f(t)$ and $r \in \mathbb{R}$:*

(a) *If $f(t) = t^r$, then $F(k)[t_0] = \binom{r}{k} t_0^{r-k}$ for all t such that $|t - t_0| < |t_0|$,*

where $\binom{r}{k} = r(r-1)\ldots(r-k+1)/k! = (r)_k/k!$ and $(r)_k$ is the Pochhammer symbol.

(b) *If $f(t) = \ln(t)$, then $F(k)[t_0] = (-1)^{k-1}/(k \cdot t_0^k)$ for $k \geq 1$.*

Proof

(a) Recall the Newton's generalisation of the binomial formula: if x and y are real numbers with $|x| > |y|$, and r is any complex number, one has

$$(x + y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k, \tag{30}$$

where $\binom{r}{k} = r(r-1)\ldots(r-k+1)/k!$. Let us rewrite t^r as $t^r = (t - t_0 + t_0)^r = (t_0 + (t - t_0))^r$. Applying (30) yields

$$t^r = \sum_{k=0}^{\infty} \binom{r}{k} t_0^{r-k} (t - t_0)^k. \tag{31}$$

(b) We start by proving the formula

$$(\ln(t))^{(k)} = \frac{(-1)^{k-1}(k-1)!}{t^k}, \tag{32}$$

by induction. For $k = 1$, we have $(\ln(t))' = 1/t$; hence, (32) is valid. Suppose that (32) holds for k. Then,

$$\begin{aligned} (\ln(t))^{(k+1)} &= \left((\ln(t))^{(k)}\right)' = \left(\frac{(-1)^{k-1}(k-1)!}{t^k}\right)' \\ &= (-1)^{k-1}(k-1)!\left(t^{-k}\right)' \\ &= (-1)^{k-1}(k-1)!(-k)t^{-k-1} \\ &= \frac{(-1)^k k!}{t^{k+1}}. \end{aligned} \tag{33}$$

Thus, formula (32) is valid for all $k \in \mathbb{N}$. Now by Definition 2,

$$\begin{aligned} F(k)[t_0] &= \frac{1}{k!}\left[\frac{\mathrm{d}^k \ln(t)}{\mathrm{d}t^k}\right]_{t=t_0} = \frac{1}{k!}\left[\frac{(-1)^{k-1}(k-1)!}{t^k}\right]_{t=t_0} \\ &= \frac{(-1)^{k-1}}{k \cdot t_0^k}. \end{aligned} \tag{34}$$

□

3.3. Applications. In this section, we introduce two test problems and show how the practical implementation of the presented algorithm looks like in concrete examples. Comparison of numerical results is given in Section 3.4.

As the first test problem, we choose an IVP for a scalar equation with one nonconstant delay where the exact solution is known to be the exponential function e^t. The purpose of including this example is to compare results obtained by DT against values of the exact solution and also against results obtained by Matlab function DDENSD which has been designed to approximate solutions to IVP for neutral delayed differential equations.

Example 1. Consider the delayed equation:

$$u'(t) = u(t) - t + u(\ln(t)), \tag{35}$$

with the initial condition

$$u(1) = e, \tag{36}$$

and with the initial function

$$\phi(t) = e^t, \quad t \in [0, 1]. \tag{37}$$

First we find the differential transform of the initial condition (36) which is $U(0)[1] = e$. Further denote $\mathscr{D}\{e^t\}[0] = \{E(k)[0]\}_{k=0}^{\infty}$ as the transformation of the exponential function with the center at 0 and $\mathscr{D}\{\ln(t)\}[1] = \{F(k)[1]\}_{k=0}^{\infty}$ as the transformation of the logarithmic function at 1, respectively. Then, Lemma 1 and Theorem 3 yield

$$\begin{aligned} E(k)[0] &= \frac{1}{k!}, \\ F(k)[1] &= \frac{(-1)^{k-1}}{k}, \quad \text{for } k \geq 1. \end{aligned} \tag{38}$$

For $t \in [1, e]$, equation (35) is transformed into

$$(k+1)U(k+1)[1] = U(k)[1] - \delta(k) - \delta(k-1) + H(k)[1], \tag{39}$$

where

$$\begin{aligned} H(k)[1] &= \sum_{l=1}^{k} E(l)[0]\widehat{B}_{k,l}(F(1)[1], \ldots, F(k-l+1)[1]), \\ &\qquad \text{for } k \geq 1, \\ H(0)[1] &= E(0)[0] = 1. \end{aligned} \tag{40}$$

We have

$$\begin{aligned}
U(1)[1] &= e - 1 - 0 + 1 = e,\\
H(1)[1] &= E(1)[0] \cdot F(1)[1] = 1 \cdot 1 = 1,\\
U(2)[1] &= \frac{1}{2}\left(U(1)[1] - 0 - 1 + H(1)[1]\right)\\
&= \frac{1}{2}(e - 1 + 1) = e \cdot \frac{1}{2},\\
H(2)[1] &= E(1)[0] \cdot F(2)[1] + E(2)[0] \cdot (F(1)[1])^2\\
&= 1 \cdot \left(-\frac{1}{2}\right) + \frac{1}{2} \cdot 1 = 0,\\
U(3)[1] &= \frac{1}{3}\left(U(2)[1] + H(2)[1]\right) = \frac{1}{3} \cdot e \cdot \frac{1}{2} = e \cdot \frac{1}{3!},\\
&\vdots
\end{aligned} \tag{41}$$

Using the inverse transformation, we see that for $t \in [1, e]$,

$$u(t) = e\left(1 + (t-1) + \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3!} \dots\right) = e \cdot e^{t-1} = e^t, \tag{42}$$

which corresponds to the exact solution to the IVPs (35)–(37).

In the second step of the method of steps, i.e., in the interval $t \in [e, e^e]$, we know that $u(t) = e^t$ for $t \in [1, e]$ and equation (35) is transformed into

$$(k+1)U(k+1)[e] = U(k)[e] - e\delta(k) - \delta(k-1) + H(k)[e], \tag{43}$$

where

$$\begin{aligned}
H(0)[e] &= U(0)[1],\\
H(k)[e] &= \sum_{l=1}^{k} U(l)[1]\widehat{B}_{k,l}\left(F(1)[e], \dots, F(k-l+1)[e]\right),\\
&\qquad \text{for } k \geq 1.
\end{aligned} \tag{44}$$

Here, $F(k)[e]$, according to Theorem 3, are coefficients of Taylor series of logarithmic function with the center at e:

$$\begin{aligned}
F(0)[e] &= 1,\\
F(k)[e] &= \frac{(-1)^{k-1}}{k \cdot e^k}, \quad \text{for } k \geq 1.
\end{aligned} \tag{45}$$

Taking the values calculated in the first step and substituting them into the recurrence formulas (43) and (44), we obtain

$$\begin{aligned}
U(0)[e] &= u(e) = e^e,\\
U(1)[e] &= U(0)[e] - e\delta(0) - \delta(-1) + H(0)[e]\\
&= e^e - e - 0 + e = e^e,\\
H(1)[e] &= U(1)[1] \cdot F(1)[e] = e \cdot \frac{1}{e} = 1,\\
U(2)[e] &= \frac{1}{2}\left(U(1)[e] - 0 - 1 + H(1)[e]\right)\\
&= \frac{1}{2}\left[e^e - 1 + 1\right] = \frac{1}{2}e^e,\\
H(2)[e] &= U(1)[1] \cdot \widehat{B}_{2,1}\left(F(1)[e], F(2)[e]\right)\\
&\quad + U(2)[1] \cdot \widehat{B}_{2,2}\left(F(1)[e]\right)\\
&= e \cdot F(2)[e] + \frac{e}{2} \cdot (F(1)[e])^2\\
&= e \cdot \frac{1}{2}\left(\frac{-1}{e^2}\right) + \frac{e}{2} \cdot \left(\frac{1}{e}\right)^2 = 0,\\
U(3)[e] &= \frac{1}{3}\left(U(2)[e] + H(2)[e]\right) = \frac{1}{3!}e^e,\\
H(3)[e] &= U(1)[1] \cdot \widehat{B}_{3,1}\left(F(1)[e], F(2)[e], F(3)[e]\right)\\
&\quad + U(2)[1] \cdot \widehat{B}_{3,2}\left(F(1)[e], F(2)[e]\right) + U(3)[1]\\
&\quad \cdot \widehat{B}_{3,3}\left(F(1)[e]\right)\\
&= e \cdot F(3)[e] + \frac{e}{2} \cdot 2 \cdot F(1)[e] \cdot F(2)[e] + \frac{e}{3!}(F(1)[e])^3\\
&= e \cdot \frac{2}{e^3}\frac{1}{3!} + e \cdot \frac{1}{e} \cdot \frac{(-1)}{e^2} \cdot \frac{1}{2} + \frac{e}{6} \cdot \frac{1}{e^3} = \frac{1}{e^2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{6}\right) = 0,\\
U(4)[e] &= \frac{1}{4}\left(U(3)[e] + H(3)[e]\right) = \frac{1}{4!}e^e.
\end{aligned} \tag{46}$$

Hence, for $t \in [e, e^e]$, we have

$$\begin{aligned}
u(t) &= e^e + e^e(t-e) + \frac{1}{2}e^e(t-e)^2 + \frac{1}{3!}e^e(t-e)^3\\
&\quad + \frac{1}{4!}e^e(t-e)^4 + \dots\\
&= e^e\left(1 + (t-e) + \frac{(t-e)^2}{2} + \frac{(t-e)^3}{3!} + \frac{(t-e)^4}{4!} + \dots\right)\\
&= e^e \cdot e^{t-e} = e^t,
\end{aligned} \tag{47}$$

which again coincides with the exact solution to problems (35)–(37).

In the second application, we have chosen an IVP for a nonlinear system of neutral delayed differential equations taken from the fully open access paper [18]. There are several reasons to test the proposed algorithm on the particular problem. The first is that the problem involves a nonlinear system of neutral equations of high complexity whose exact solution is unknown. Secondly, the proposed algorithm is a complete differential transform version of the algorithm presented in [18] where modified Adomian formula has been used. Furthermore, the calculations done in [18] are shown only for the first step of the method of steps up to the first principal discontinuity point, whereas we continue calculations beyond that point in this paper. Last but not least, we want to verify performance and reproduce values obtained by DT and published in [18]. Rebenda et al. [18] has been submitted 4 years ago for the first time, and since that time, the Maple source code has been lost.

Example 2. Consider a nonlinear system of neutral delayed differential equations:

$$u_1''' = u_1'''(t-2)u_1\left(\frac{t}{3}\right) + \sqrt[3]{(u_1(t))^2} + u_2'\left(t - \frac{1}{2}e^{-t}\right),$$
$$u_2''' = \frac{1}{2}u_2'''\left(\frac{t}{2}\right) + u_2'(t-1)u_1\left(\frac{t}{3}\right), \tag{48}$$

with initial functions

$$\phi_1(t) = e^t,$$
$$\phi_2(t) = t^2, \tag{49}$$

for $t \in [-2, 0]$, and initial conditions

$$u_1(0) = 1,$$
$$u_1'(0) = 1,$$
$$u_1''(0) = 1,$$
$$u_2(0) = 0,$$
$$u_2'(0) = 0,$$
$$u_2''(0) = 2. \tag{50}$$

For $t \in [0, t_1]$, where $t_1 \approx 0,351734$ is the minimal root of $t - (1/2)e^{-t} = 0$, using the method of steps, we obtain

$$u_1''' = e^{(t-2)}u_1\left(\frac{t}{3}\right) + \sqrt[3]{(u_1(t))^2} + 2t - e^{-t},$$
$$u_2''' = \frac{1}{2}u_2'''\left(\frac{t}{2}\right) + 2(t-1)u_1\left(\frac{t}{3}\right). \tag{51}$$

We need to find the differential transform of the considered problem. We notice that system (2) contains nonlinear term $h(t) = \sqrt[3]{(u_1(t))^2}$. To get DT of this term, $\mathscr{D}\{h(t)\}[0] = \{H_1(k)[0]\}_{k=0}^{\infty}$, and we apply Theorem 1. First, applying DT to system (2) at $t_0 = 0$, we get the recurrent system:

$$(k+1)(k+2)(k+3)U_1(k+3)[0]$$
$$= e^{-2}\sum_{l=0}^{k}\frac{1}{l!}\left(\frac{1}{3}\right)^{k-l}U_1(k-l)[0] + H_1(k)[0] + 2\delta(k) - \frac{(-1)^k}{k!}, \tag{52}$$

$$(k+1)(k+2)(k+3)\left(1 - \frac{1}{2^{k+1}}\right)U_2(k+3)[0]$$
$$= 2\left(\frac{1}{3}\right)^{k-1}U_1(k-1)[0] - \frac{2}{3^k}U_1(k)[0]. \tag{53}$$

Denote $g(t) = t^{2/3}$; then, $h(t) = (g \circ u_1)(t)$, and following Theorem 1, we obtain

$$H_1(0)[0] = G_1(0)[1],$$
$$H_1(k)[0] = \sum_{l=1}^{k} G_1(l)[1]\widehat{B}_{k,l}(U_1(1)[0], \dots, U_1(k-l+1)[0]), \tag{54}$$

for $k \geq 1$, where $\mathscr{D}\{g(t)\}[1] = \{G_1(k)[1]\}_{k=0}^{\infty}$ and, Theorem 3 being applied, $G_1(k)[1] = \binom{2/3}{k}$ for $k \geq 0$. Furthermore, the transformed initial conditions are

$$U_1(0)[0] = 1,$$
$$U_1(1)[0] = 1,$$
$$U_1(2)[0] = \frac{1}{2},$$
$$U_2(0)[0] = 0,$$
$$U_2(1)[0] = 0,$$
$$U_2(2)[0] = 1. \tag{55}$$

Using them, we compute the first three coefficients of the nonlinear term $h(t)$:

$$H_1(0)[0] = G_1(0)[1] = 1,$$
$$H_1(1)[0] = G_1(1)[1] \cdot B_{1,1}(U_1(1)[0]) = \frac{2}{3} \cdot 1 = \frac{2}{3},$$
$$H_1(2)[0] = G_1(1)[1] \cdot B_{2,1}(U_1(1)[0], U_1(2)[0])$$
$$+ G_1(2)[1] \cdot B_{2,2}(U_1(1)[0]) = \frac{2}{3}\cdot\frac{1}{2} - \frac{1}{9}\cdot 1 = \frac{2}{9}. \tag{56}$$

Solving recurrent systems (52) and (53), we get

$$k = 0:\ U_1(3)[0] = \frac{1}{6}\left(e^{-2}U_1(0)[0] + H_1(0)[0] + 1\right)$$
$$= \frac{2 + e^{-2}}{6},$$
$$U_2(3)[0] = \frac{1}{3}(-2U_1(0)[0]) = -\frac{2}{3}.$$
$$k = 1:\ U_1(4)[0] = \frac{e^{-2}}{24}\left(\frac{1}{3}U_1(1)[0] + U_1(0)[0]\right)$$
$$+ \frac{1}{24}(H_1(1)[0] + 1) = \frac{4e^{-2} + 5}{72},$$
$$U_2(4)[0] = \frac{1}{18}\left(2U_1(0)[0] - \frac{2}{3}U_1(1)[0]\right) = \frac{2}{27}.$$
$$k = 2:\ U_1(5)[0] = \frac{e^{-2}}{60}\left(\frac{1}{9}U_1(2)[0] + \frac{1}{3}U_1(1)[0]\right.$$
$$\left. + \frac{1}{2}U_1(0)[0]\right) + \frac{1}{60}\left(H_1(2)[0] - \frac{1}{2}\right)$$
$$= \frac{16e^{-2} - 5}{1080},$$
$$U_2(5)[0] = \frac{2}{105}\left(\frac{2}{3}U_1(1)[0] - \frac{2}{9}U_1(2)[0]\right) = \frac{2}{189}. \tag{57}$$

Using the inverse DT (Definition 3), we get approximate solution for the IVPs (48)–(50) on the interval $[0, t_1]$:

$$u_{1,I_1}(t) = 1 + t + \frac{1}{2}t^2 + \frac{2+e^{-2}}{6}t^3 + \frac{4e^{-2}+5}{72}t^4 + \frac{16e^{-2}-5}{1080}t^5 + \dots, \tag{58}$$

$$u_{2,I_1}(t) = t^2 - \frac{2}{3}t^3 + \frac{2}{27}t^4 + \frac{2}{189}t^5 + \dots,$$

which is exactly the same approximate solution which has been obtained in [18].

The second step brings us to solving the given IVP on the interval $[t_1, t_2]$, where t_2 is the minimal root of $t - (1/2)e^{-t} = t_1$, $t_2 \approx 0,620556$. Now taking into account that both proportional delays $q_1 t = (1/3)t$ and $q_2 t = (1/2)t$ and also the time-dependent delay $t - \tau_1(t) = t - (1/2)e^{-t}$ map the interval $[t_1, t_2]$ into the interval $[0, t_1]$, system (2) becomes

$$\begin{aligned} u_1''' &= e^{(t-2)}u_{1,I_1}\left(\frac{t}{3}\right) + \sqrt[3]{(u_1(t))^2} + u'_{2,I_1}\left(t - \frac{1}{2}e^{-t}\right), \\ u_2''' &= \frac{1}{2}u'''_{2,I_1}\left(\frac{t}{2}\right) + 2(t-1)u_{1,I_1}\left(\frac{t}{3}\right). \end{aligned} \tag{59}$$

Denote $\mathcal{D}\{h(t)\}[t_1] = \{H_2(k)[t_1]\}_{k=0}^{\infty}$ and $\mathcal{D}\{f(t)\}[t_1] = \{F_2(k)[t_1]\}_{k=0}^{\infty}$, where $f(t) = u_2'(t - (1/2)e^{-t})$. By application of Theorem 2 to corresponding terms, system (2) transformed at $t_0 = t_1$ reads as

$$\begin{aligned} &(k+1)(k+2)(k+3)U_1(k+3)[t_1] \\ &= e^{-2}\sum_{l=0}^{k}\frac{e^{t_1}}{l!}\left(\frac{1}{3}\right)^{k-l}\sum_{x=0}^{\infty}\binom{x+k-l}{x}U_1(x+k-l)[0]\left(\frac{t_1}{3}\right)^x \\ &\quad + H_2(k)[t_1] + F_2(k)[t_1], \end{aligned} \tag{60}$$

$$\begin{aligned} &(k+1)(k+2)(k+3)U_2(k+3)[t_1] \\ &= \left(\frac{1}{2}\right)^{k+1}(k+3)(k+2)(k+1)\sum_{x=0}^{\infty}\binom{x+k+3}{x} \\ &\quad \cdot U_2(x+k+3)[0]\left(\frac{t_1}{2}\right)^x + 2\sum_{l=0}^{k}\left[(t_1-1)\delta(l) + \delta(l-1)\right] \\ &\quad \cdot \left(\frac{1}{3}\right)^{k-l}\sum_{x=0}^{\infty}\binom{x+k-l}{x}U_1(x+k-l)[0]\left(\frac{t_1}{3}\right)^x. \end{aligned} \tag{61}$$

Now denote $\mathcal{D}\{g(t)\}[u_1(t_1)] = \{G_2(k)[u_1(t_1)]\}_{k=0}^{\infty}$; then, according to Theorem 3, $G_2(k)[u_1(t_1)] = \binom{2/3}{k}(u_1(t_1))^{2/3-k}$ for $k \geq 0$ and Theorem 1 implies

$$\begin{aligned} H_2(0)[t_1] &= G_2(0)[u_1(t_1)] = \sqrt[3]{u_1(t_1)^2}, \\ H_2(k)[t_1] &= \sum_{l=1}^{k}G_2(l)[u_1(t_1)] \cdot \widehat{B}_{k,l}(U_1(1)[t_1], \dots, \\ &\quad U_1(k-l+1)[t_1]), \quad \text{for } k \geq 1. \end{aligned} \tag{62}$$

Further denote $e(t) = t - (1/2)e^{-t}$ and $\mathcal{D}\{e(t)\}[t_1] = \{E_2(k)[t_1]\}_{k=0}^{\infty}$. Then, $f(t) = (u_2' \circ e)(t)$ and, since $e(t_1) = 0$, Theorem 1 in combination with Lemma 1 yields

$$\begin{aligned} E_2(k)[t_1] &= t_1 \cdot \delta(k) + \delta(k-1) - \frac{1}{2}\cdot\frac{e^{-t_1}(-1)^k}{k!}, \quad k \geq 0, \\ F_2(0)[t_1] &= U_2(1)[0] = 0, \\ F_2(k)[t_1] &= \sum_{l=1}^{k}(l+1)U_2(l+1)[0] \cdot \widehat{B}_{k,l}(E_2(1)[t_1], \dots, \\ &\quad E_2(k-l+1)[t_1]), \quad \text{for } k \geq 1. \end{aligned} \tag{63}$$

To get the initial data $U_1(k)[t_1]$ and $U_2(k)[t_1]$ for $k = 0, 1, 2$, we have to transform

$$u_i(t) = U_i(0)[0] + U_i(1)[0]t + U_i(2)[0]t^2 + U_i(3)[0]t^3 + \dots, \tag{64}$$

at t_1, $i = 1, 2$. For $k = 0, 1, 2$, we have

$$U_i(k)[t_1] = U_i(0)[0] + U_i(1)[0]\binom{1}{k}t_1^{1-k} + U_i(2)[0]\binom{2}{k}t_1^{2-k} + U_i(3)[0]\binom{3}{k}t_1^{3-k} + \dots, \tag{65}$$

i.e.,

$$\begin{aligned} U_i(0)[t_1] &= \sum_{k=0}^{\infty}U_i(k)[0]t_1^k, \\ U_i(1)[t_1] &= \sum_{k=0}^{\infty}(k+1)U_i(k+1)[0]t_1^k, \\ U_i(2)[t_1] &= \sum_{k=0}^{\infty}\binom{k+2}{2}U_i(k+2)[0]t_1^k. \end{aligned} \tag{66}$$

The initial values at t_1 will be approximated by taking finite sums in computer evaluations of the infinite sums above. Observe that $u_1(t_1) = U_1(0)[t_1]$.

Now let us compute the first few values of $H_2(k)[t_1]$. Denote $\widehat{B}_{k,l} = \widehat{B}_{k,l}(U_1(1)[t_1], \dots, U_1(k-l+1)[t_1])$. Then, the first values of $\widehat{B}_{k,l}$ are

$$\widehat{B}_{0,0} = 1,$$
$$\widehat{B}_{1,1} = U_1(1)[t_1] \cdot \widehat{B}_{0,0} = U_1(1)[t_1],$$
$$\widehat{B}_{2,1} = \frac{1}{2}U_1(1)[t_1] \cdot \widehat{B}_{1,0} + U_1(2)[t_1] \cdot \widehat{B}_{0,0} = U_1(2)[t_1],$$
$$\widehat{B}_{2,2} = U_1(1)[t_1] \cdot \widehat{B}_{1,1} = (U_1(1)[t_1])^2, \tag{67}$$

and the coefficients H_2 for $k = 0, 1, 2$ are

$$H_2(0)[t_1] = u_1(t_1),$$
$$H_2(1)[t_1] = \frac{2}{3}(u_1(t_1))^{-1/3} \cdot \widehat{B}_{1,1} = \frac{2}{3}(u_1(t_1))^{-1/3} \cdot U_1(1)[t_1],$$
$$H_2(2)[t_1] = \frac{2}{3}(u_1(t_1))^{-1/3} \cdot \widehat{B}_{2,1} - \frac{1}{9}(u_1(t_1))^{-4/3} \cdot \widehat{B}_{2,2} = \frac{2}{3}(u_1(t_1))^{-1/3} \cdot U_1(2)[t_1] - \frac{1}{9}(u_1(t_1))^{-4/3} \cdot (U_1(1)[t_1])^2. \tag{68}$$

Let us turn our attention to the first few values of $F_2(k)[t_1]$. Starting with $E_2(k)[t_1]$,

$$E_2(0)[t_1] = t_1 - \frac{1}{2}e^{-t_1} = 0,$$
$$E_2(1)[t_1] = 1 + \frac{1}{2}e^{-t_1} = 1 + t_1, \tag{69}$$
$$E_2(2)[t_1] = -\frac{1}{2} \cdot \frac{e^{-t_1}}{2} = -\frac{t_1}{2}.$$

Now, $\widehat{B}_{k,l}$ are $\widehat{B}_{k,l}(E_2(1)[t_1], \ldots, E_2(k-l+1)[t_1])$:

$$\widehat{B}_{0,0} = 1,$$
$$\widehat{B}_{1,1} = E_2(1) \cdot \widehat{B}_{0,0} = 1 + t_1,$$
$$\widehat{B}_{2,1} = \frac{1}{2}E_2(1) \cdot \widehat{B}_{1,0} + E_2(2) \cdot \widehat{B}_{0,0} = -\frac{t_1}{2}, \tag{70}$$
$$\widehat{B}_{2,2} = E_2(1) \cdot \widehat{B}_{1,1} = (1 + t_1)^2.$$

Finally, coefficients F_2 for $k = 0, 1, 2$ are

$$F_2(0)[t_1] = U_2(1)[0] = 0,$$
$$F_2(1)[t_1] = 2U_2(2)[0] \cdot \widehat{B}_{1,1} = 2(1 + t_1),$$
$$F_2(2)[t_1] = 2U_2(2)[0] \cdot \widehat{B}_{2,1} + 3U_2(3)[0] \cdot \widehat{B}_{2,2} = -t_1 - 2(1 + t_1)^2. \tag{71}$$

At this moment, we substitute H_2 and F_2 into systems (60) and (61). The next three coefficients at t_1 for U_1 are

$$k = 0\colon U_1(3)[t_1] = \frac{1}{6}\left(e^{-2}e^{t_1} \cdot u_1\left(\frac{t_1}{3}\right) + u_1(t_1)\right),$$
$$k = 1\colon U_1(4)[t_1] = \frac{1}{24}\left(e^{-2}\left(e^{t_1} \cdot \frac{1}{3} \cdot \sum_{x=0}^{\infty}(x+1)U_1(x+1)[0]\left(\frac{t_1}{3}\right)^x + e^{t_1} \cdot u_1\left(\frac{t_1}{3}\right)\right) + \frac{2}{3} \cdot (u_1(t_1))^{-1/3} \cdot U_1(1)[t_1] + 2(1 + t_1)\right),$$
$$k = 2\colon U_1(5)[t_1] = \frac{1}{60}e^{-2}\left(e^{t_1}\left(\frac{1}{3}\right)^2 \cdot \sum_{x=0}^{\infty}\binom{x+2}{x}U_1(x+2)[0] \cdot \left(\frac{t_1}{3}\right)^x + e^{t_1} \cdot \frac{1}{3} \cdot \sum_{x=0}^{\infty}(x+1)U_1(x+1)[0]\left(\frac{t_1}{3}\right)^x + \frac{e^{t_1}}{2} \cdot u_1\left(\frac{t_1}{3}\right)\right) + \frac{1}{60}\left(\frac{2}{3}(u_1(t_1))^{-1/3} \cdot (U_1(2)[t_1]) - \frac{1}{9}(u_1(t_1))^{-4/3} \cdot (U_1(1)[t_1])^2 - t_1 - 2(1 + t_1)^2\right), \tag{72}$$

and for U_2, we obtain

$$k = 0\colon\ U_2(3)[t_1] = \frac{1}{2}\left(\sum_{x=0}^{\infty}\binom{x+3}{x}U_2(x+3)[0]\left(\frac{t_1}{2}\right)^x\right) + \frac{1}{3}(t_1 - 1) \cdot u_1\left(\frac{t_1}{3}\right),$$
$$k = 1\colon\ U_2(4)[t_1] = \frac{1}{4}\left(\sum_{x=0}^{\infty}\binom{x+4}{x}U_2(x+4)[0]\left(\frac{t_1}{2}\right)^x\right) + \frac{1}{12}\left((t_1 - 1) \cdot \frac{1}{3} \cdot \sum_{x=0}^{\infty}(x+1)U_1(x+1)[0]\left(\frac{t_1}{3}\right)^x + u_1\left(\frac{t_1}{3}\right)\right),$$
$$k = 2\colon\ U_1(5)[t_1] = \frac{1}{8}\left(\sum_{x=0}^{\infty}\binom{x+5}{x}U_2(x+5)[0]\left(\frac{t_1}{2}\right)^x\right) + \frac{1}{30}\left((t_1 - 1) \cdot \frac{1}{9} \cdot \sum_{x=0}^{\infty}\binom{x+2}{x}U_1(x+2)[0]\left(\frac{t_1}{3}\right)^x + \frac{1}{3}\sum_{x=0}^{\infty}(x+1)U_1(x+1)[0]\left(\frac{t_1}{3}\right)^x\right). \tag{73}$$

Using the inverse DT, again we get approximate solution for the IVPs (48), (49), and (50) on the interval $[t_1, t_2]$:

Table 1: Example 1, error analysis of u in $[1, e]$.

	Exact solution	DT 5	DT 10	DT 25	Matlab DDENSD
t	e^t	$\lvert u - e^t \rvert$	$\lvert u - e^t \rvert$	$\lvert u - e^t \rvert$	$\lvert u - e^t \rvert$
1	2.7183	0	0	0	0
1.4296	4.1769	0.0000	$6.4890E-12$	$8.8818E-16$	$1.6976E-5$
1.8591	6.4182	0.0017	$1.0381E-8$	$8.8818E-16$	$3.6095E-5$
2.2887	9.8622	0.0211	$1.2409E-6$	$3.5527E-15$	$9.3438E-5$
e	15.1543	0.1273	$3.0578E-5$	$1.2861E-12$	$1.4012E-4$

Table 2: Example 1, error analysis of u in $[e, e^e]$.

	Exact solution	DT 5	DT 10	DT 25	Matlab DDENSD
t	e^t	$\lvert u - e^t \rvert$	$\lvert u - e^t \rvert$	$\lvert u - e^t \rvert$	$\lvert u - e^t \rvert$
e	15.1543	$3.5527E-15$	$3.5527E-15$	$3.5527E-15$	0
5.8273	339.4331	32.3491	1.333	$2.2737E-13$	0.0743
8.9363	$7.6028E+3$	$4.4755E+3$	397.5509	$2.2638E-5$	3.1741
12.0453	$1.7029E+5$	$1.5373E+5$	$5.6777E+4$	0.9751	112.2467
e^e	$3.8143E+6$	$3.7554E+6$	$2.6576E+6$	$2.0358E+3$	$3.1395E+3$

$$\begin{aligned} u_{1,I_2}(t) &= U_1(0)[t_1] + U_1(1)[t_1](t-t_1) + U_1(2)[t_1] \\ &\quad \cdot (t-t_1)^2 + U_1(3)[t_1](t-t_1)^3 + \ldots, \\ u_{2,I_2}(t) &= U_2(0)[t_1] + U_2(1)[t_1](t-t_1) + U_2(2)[t_1] \\ &\quad \cdot (t-t_1)^2 + U_2(3)[t_1](t-t_1)^3 + \ldots. \end{aligned} \tag{74}$$

As the calculations are getting more complicated, all the calculations have been done numerically only.

3.4. Numerical Results and Discussion. Table 1 shows comparison of results for Example 1 obtained by DT algorithm with the orders of Taylor polynomials of the approximate solution $N = 5, 10, 25$ to results of Matlab function DDENSD in the interval $[1, e]$. Since the exact solution is known, absolute errors illustrate precision of each algorithm setting. All numbers are rounded to four decimal places. We see that DDENSD performs satisfactory well and DT for $N = 10, 25$ does even better, whereas DT for $N = 5$ does not show satisfactory precision.

Table 2 brings the same comparison in the second interval $[e, e^e]$. We can observe a fast growth rate of the function values of the exact solution, which leads to the growth of absolute errors and loss of precision in all settings. It indicates that at the end of the considered interval $[e, e^e]$, the rate of precision would be better seen using relative errors.

Implementation of DT in Matlab in case of Example 2 produces numerical results which are listed in Table 3. The results of DT with order of the Taylor polynomial $N = 10$ are compared to values obtained by DT combined with modified Adomian formula in [18] and to values produced by Matlab function DDENSD.

First, we should say that the function DDENSD had difficulty at 0 where the value of the delayed argument $t/2$ was equal to the argument itself. Hence, to make DDENSD work, we replaced $t/2$ by $t/2 - 10^{-16}$ in the second equation of (2). Our hypothesis is that the reason of the DDENSD failure is a combination of two facts: the second equation is neutral with respect to a proportional delay and the interval where the problem is considered contains 0.

Second, we should mention that the numerical results for DDENSD were obtained by looking for approximate solutions on the whole interval $[0, t_2]$. When trying to follow the method of steps, i.e., using DDENSD on $[0, t_1]$ and then on $[t_1, t_2]$, the results on the second interval $[t_1, t_2]$ did not correspond to reality: there was a discontinuity in u_2 at t_1.

Furthermore, we recall that the values taken from [18] have been computed using symbolic software Maple and the source code of the computation has been lost.

Now, we can see a very good concordance of all algorithms in numerical values of the second component u_2, while we observe a growing distance between the values of the first component u_1 computed by presented DT algorithm and values computed by the other two algorithms. As u_1 has exponential characteristics, we interpret the growing distance as growing lack of precision of DT algorithm which is based on approximation by Taylor polynomials. We suppose that dividing the intervals $[0, t_1]$ and $[t_1, t_2]$ into smaller subintervals, i.e., refining the mesh grid, and applying the DT algorithm on those smaller intervals consecutively will improve the performance of the presented algorithm.

Although it seems that the algorithm used in [18] shows better performance than the one presented in this paper, we cannot claim it with certainty as the source code got lost and we are not able to reproduce the data. Moreover, the approach used in [18] involves calculations of symbolic derivatives which makes it difficult to implement in numerical software like Matlab.

Table 3: Example 2, comparison of presented DT, DT in [18] and Matlab function DDENSD for u_1 and u_2 in the first step for $t \in [0, t_1]$, and presented DT and DDENSD in the second step for $t \in [t_1, t_2]$.

Method	DT 10	DT 10	Matlab DDENSD	Matlab DDENSD	DT [18]	DT [18]
t	u_1	u_2	u_1	u_2	u_1	u_2
0.00	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
0.05	1.0513	0.0024	1.0513	0.0024	1.0513	0.0024
0.10	1.1054	0.0093	1.1050	0.0093	1.1051	0.0093
0.15	1.1625	0.0203	1.1614	0.0203	1.1618	0.0203
0.20	1.2230	0.0348	1.2204	0.0348	1.2209	0.0348
0.25	1.2871	0.0524	1.2822	0.0524	1.2832	0.0524
0.30	1.3552	0.0726	1.3469	0.0726	1.3481	0.0726
0.35	1.4277	0.0951	1.4146	0.0951	1.4160	0.0951
0.3520	1.4306	0.0960	1.4174	0.0960	—	—
0.3904	1.4896	0.1146	1.4717	0.1146	—	—
0.4289	1.5514	0.1340	1.5280	0.1340	—	—
0.4673	1.6161	0.1541	1.5864	0.1541	—	—
0.5057	1.6839	0.1747	1.6469	0.1748	—	—
0.5441	1.7548	0.1957	1.7098	0.1957	—	—
0.5826	1.8290	0.2169	1.7750	0.2169	—	—
0.6210	1.9065	0.2380	1.8428	0.2381	—	—

4. Conclusion

In the paper, we presented an algorithm which makes use of the differential transformation to initial value problems for systems of delayed or neutral differential equations with nonconstant delays. Two examples have been chosen to validate and test the algorithm. Numerical comparison of the presented semianalytical approach to Matlab function DDENSD brought interesting and promising results.

Example 1 showed expected and reliable behaviour of the differential transform in the first step of the method of steps and expected deviation in the numerical results from values of the exact solution in the second step. Furthermore, we could observe a good concordance between the presented algorithm and DDENSD.

After facing difficulties with DDENSD in Example 2, we could confirm a very good concordance of both differential transform and DDENSD in values of the component u_2 which has a polynomial character on the considered intervals. On the other hand, we observed a growing discrepancy between the two methods in values of the component u_1 which has an exponential character. Our conclusion is that the disagreement is caused by large lengths of the intervals where the approximate solution is computed using the differential transform and that refining the mesh grid is necessary to obtain better performance.

Further investigation will be focused on experimenting with different densities of mesh grids and studying convergence of the algorithm to find the optimal mesh grid. Numerical experiments will be focused on tuning the performance on problems with high complexity whose exact solutions are known and subsequently on applications to nonartificial real-life problems whose exact solutions are unknown.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Supplementary Materials

We have included the commented Matlab source code for the first step of the method of steps in Example 1 as supplementary material. (*Supplementary Materials*)

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Applications of the Differential Transform to second-order half-linear Euler equations

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Abstract

The purpose of the paper is to show applications of the differential transform to second-order half-linear Euler equations with and without delay. The case of proportional delay is considered. Finding a numerical solution to an initial value problem is reduced to solving recurrence relations. The outputs of the recurrence relations are coefficients of the Taylor series of the solution. Validity of the presented algorithm is demonstrated on concrete examples of initial value problems. Numerical results are compared with solutions produced by Matlab function "ddesd".

AMS (MOS) subject classification: 34K28, 34K07, 34A45, 65L03, 65L05.
Keywords: Half-linear Euler equation; differential transform; method of steps; differential equation with delay

1 Introduction

Half-linear Euler type equations have been studied extensively in terms of the qualitative properties. However, combination of research methodologies is one of the successful generators of new ideas, results, and insights. The aim of this paper is to complement the theoretical results about qualitative behavior of solutions - asymptotic formulas or oscillatory properties - to half-linear Euler equations with and without delay with finding an approximate solution to the initial value problem numerically. Motivated by the current progress in research on the differential transform, the purpose of the paper is to investigate how the differential transform algorithm can be applied to half-linear Euler equations with a proportional delay and without delay.

The half-linear equation without delay can be achieved as a transformation of partial differential equations that contain the so-called p-Laplacian

$$\Delta_p u := \operatorname{div}(||\nabla u||^{p-2}\nabla u),$$

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where for $u(x) = u(x_1, \dots, x_N)$, $N \in \mathbb{N}$, the symbol ∇u stands for the Hamilton nabla operator and div represents the divergence operator. Origins of p-Laplacian are described, for example, in the paper [2]. Accordingly, the history of p-Laplacian is closely linked to applications in the filtration of fluids through porous media and nonlinear non-Newtonian fluid dynamics. Another application can be found, for example, in the paper [1]. The p-Laplacian is used to model a non-homogenous diffusion to determine the height of a growing pile of noncohesive sand, where an ordinary differential equation arises in the limit case of "infinitely fast/slow" diffusion (see also [14]).

The paper is organized as follows. In Section 2, we briefly summarize the results about qualitative properties of solutions to half-linear Euler equations with and without delay. Section 3 is devoted to theory and results on the differential transform. Sections 4 and 5 contain the main results of the paper. Numerical algorithms of the differential transform are adapted and applied to ordinary and delayed half-linear Euler equations. Concrete examples are illustrated by numerical results. Section 6 concludes the paper with a summary and outlines possibilities for further research.

2 Half-linear differential equations

Half-linear differential equation of the second order is an equation of the form

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{1}$$

where $r(t), c(t)$ are continuous functions and $r(t) > 0$. For $p = 2$, the equation (1) reduces to the second-order linear Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0.$$

Also from this point of view, the study of the properties of its generalized form (1) is a natural direction of research.

The name "half-linear" has its origin in the fact that the space of solutions to (1) is homogeneous but not additive. The qualitative theory of half-linear differential equations has been studied extensively during the last decades. For the summary of the results up to 2005, we recommend the book [7]. More recent results can be found, for example, in [15, 32, 6, 26, 12, 16, 27] and references therein.

From the qualitative point of view, half-linear differential equations of the form (1) can be divided into two classes. A half-linear equation is either oscillatory, which means that every its nontrivial solution has infinitely many zeros that form a sequence tending to infinity, or non-oscillatory, which means that every solution has constant sign in a neighbourhood of infinity. We recall that an oscillatory solution and a non-oscillatory solution to (1) cannot exist at the same time, which is a direct consequence of Sturm separation theorem [7, p. 16, Theorem 1.2.3].

The subject of our interest in this paper is the second-order half-linear Euler equation

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}} x^{\alpha}(t) = 0, \tag{2}$$

as well as the second-order half-linear Euler equation with a proportional delay

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}} x^{\alpha}(\lambda t) = 0, \tag{3}$$

where $\alpha > 0$ is a quotient of two odd positive numbers, $\gamma \in (0, \infty)$ and $\lambda \in (0, 1)$.

Euler equation (2) and its generalized forms belong to the most studied half-linear differential equations, see, for example [5, 8, 4, 10, 27].

For $r(t) = 1, c(t) = \frac{\gamma}{t^{\alpha+1}}$ and $p = \alpha + 1$, the equation (1) is reduced to the Euler equation (2). Half-linear Euler equation is conditionally oscillatory with the oscillation constant

$$\gamma_{\alpha} = \left(\frac{\alpha}{\alpha + 1} \right)^{\alpha+1},$$

that is, (2) is oscillatory if $\gamma > \gamma_{\alpha}$ and non-oscillatory if $\gamma < \gamma_{\alpha}$.

If $\gamma = \gamma_{\alpha}$, the equation (2) is also non-oscillatory and the two linearly independent solutions forming the solution space have the form

$$x_1(t) = t^{\frac{\alpha}{\alpha+1}},$$
$$x_2(t) \sim t^{\frac{\alpha}{\alpha+1}} \log^{\frac{2}{\alpha+1}}(t),$$

where $\sim$ means the asymptotic equivalence as $t \to \infty$. It means that one of the solutions is known explicitly, whereas we only have an asymptotic formula for the second one. For details we refer to [7, Section 1.4.2].

Asymptotic formulas for the two linearly independent solutions are known also in the case $\gamma < \gamma_{\alpha}$ (see [13]). For $i = 1, 2$ these are of the form

$$x_i(t) \sim t^{\lambda_i^{\frac{1}{\alpha}}} \quad \text{as} \quad t \to \infty,$$

where λ_i are the zeros of the equation

$$|\lambda|^{1+\frac{1}{\alpha}} - \lambda + \gamma = 0.$$

The Euler equation with a proportional delay (3) can be seen as a special case of the delayed half-linear equation

$$\big(r(t)\Phi(x'(t))\big)' + c(t)\Phi(x(\tau(t))) = 0, \tag{4}$$

where $r(t)$, $c(t)$, $\tau(t)$ are continuous functions on $[t_0, \infty)$, $r(t) > 0$, and τ is a delay function satisfying

$$\tau(t) \le t, \quad \tau'(t) \ge 0, \quad \lim_{t\to\infty} \tau(t) = \infty.$$

In contrast to the non-delayed case (1), delayed half-linear equations may have oscillatory and non-oscillatory solutions simultaneously. A consequence of this fact is that techniques applicable to half-linear equations without delay (1) often cannot be applied to delayed half-linear equations. In particular, only Riccati type inequality is available instead of the Riccati equation. Moreover, to prove that a solution is oscillatory is easier than to prove that it is non-oscillatory.

If we choose $r(t) = 1, c(t) = \frac{\gamma}{t^{\alpha+1}}$, $p - 1 = \alpha$, and $\tau(t) = \lambda t$ in (4), we obtain the delayed Euler equation (3). Criteria providing conditions on γ under which (3) has only oscillatory solutions were studied, for example, in [9] and [11].

3 Differential Transform

The Differential transform (DT) is a semi-analytical method based on Taylor's theorem. Its history dates back to 1970s to the work of G. E. Pukhov [17]. It has been shown that DT is convenient for solving a variety of initial value problems (IVPs), covering the range from ordinary to functional, partial and fractional differential equations [22, 29, 25, 28, 24, 19]. In particular, results on the differential equations with proportional, constant and non-constant delays can be found in [30], [23] and [21].

The differential transform of a real function $u(t)$ at a point $t_0 \in \mathbb{R}$ that is analytic in a neighbourhood of t_0 is

$$\mathcal{D}\{u(t)\}[t_0] = \{U(k)[t_0]\}_{k=0}^{\infty}. \tag{5}$$

Here $U(k)[t_0]$ is the kth component of the differential transform of the function $u(t)$ at t_0, $k \in \mathbb{N}_0$, that is defined by

$$U(k)[t_0] = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}. \tag{6}$$

The inverse differential transform of $\{U(k)[t_0]\}_{k=0}^{\infty}$ is defined by

$$u(t) = \mathcal{D}^{-1}\Big\{\{U(k)[t_0]\}_{k=0}^{\infty}\Big\}[t_0] = \sum_{k=0}^{\infty} U(k)[t_0](t-t_0)^k. \tag{7}$$

In applications, the function $u(t)$ is approximated by the finite sum

$$u(t) = \sum_{k=0}^{N} U(k)[t_0](t-t_0)^k.$$

As we can see from (7), DT is related to the Taylor series. It means that the results about convergence of Taylor series may be used to decide on convergence of the DT algorithms. However, we refer to the particular paper [31] where the optimal general explicit *a-priori* error estimates are given.

The following results will be used in the application sections 4 and 5.

- Assume that $u(t)$ is a real analytic function near t_0. Then (6) implies that

$$U(0)[t_0] = u(t_0), U(1)[t_0] = u'(t_0), U(2)[t_0] = \frac{u''(t_0)}{2}, U(3)[t_0] = \frac{u'''(t_0)}{3!}, \dots. \tag{8}$$

The relationships (8) will be used by transforming initial conditions.

- Assume that $U(k)[t_0]$ is the kth component of the differential transform of the real analytic function $u(t)$ at t_0. Then

$$\mathcal{D}\{u'(t)\}[t_0] = \{(k+1)U(k+1)[t_0]\}_{k=0}^{\infty}. \tag{9}$$

Proof: Using (7), we can write

$$u'(t) = \frac{d}{dt}\sum_{k=0}^{\infty} U(k)[t_0](t-t_0)^k = \sum_{k=1}^{\infty} kU(k)[t_0](t-t_0)^{k-1} = \sum_{k=0}^{\infty}(k+1)U(k+1)[t_0](t-t_0)^k . \square$$

- [21] Assume that $F(k)[t_0]$ is the kth component of the differential transform of the function $f(t)$ at t_0 and $r \in \mathbb{R}$.

$$\text{If} \quad f(t) = t^r, \quad \text{then} \quad F(k)[t_0] = \binom{r}{k} t_0^{r-k} \tag{10}$$

for all t such that $|t-t_0| < |t_0|$, where $\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!} = \frac{(r)_k}{k!}$, and $(r)_k$ represents the Pochhammer symbol.

- [23] Assume that $F(k)$, $G(k)$ are the kth components of differential transforms of functions $f(t)$, $g(t)$ at a point t_0. Differential transform of a product $f(t)g(t)$ at t_0 is

$$\mathcal{D}\{f(t)g(t)\}[t_0] = \left\{\sum_{l=0}^{k} F(l)G(k-l)\right\}_{k=0}^{\infty}. \tag{11}$$

- [18] Let g and f be real functions analytic near t_0 and $g(t_0)$, respectively, and let h be the composition $h(t) = (f \circ g)(t) = f(g(t))$. Denote $\mathcal{D}\{g(t)\}[t_0] = \{G(k)\}_{k=0}^{\infty}$, $\mathcal{D}\{f(t)\}[g(t_0)] = \{F(k)\}_{k=0}^{\infty}$ and $\mathcal{D}\{(f \circ g)(t)\}[t_0] = \{H(k)\}_{k=0}^{\infty}$ the differential transforms of functions g, f and h at t_0, $g(t_0)$ and t_0, respectively. Then the numbers $H(k)$ in the sequence $\{H(k)\}_{k=0}^{\infty}$ satisfy the relations $H(0) = F(0)$ and

$$H(k) = \sum_{l=1}^{k} F(l) \cdot \hat{B}_{k,l}\big(G(1), \dots, G(k-l+1)\big) \ \text{ for } k \geq 1, \tag{12}$$

where $\hat{B}_{k,l}(\hat{x}_1, \dots, \hat{x}_{k-l+1})$ are the partial ordinary Bell polynomials.

- [18] The partial ordinary Bell polynomials $\hat{B}_{k,l}(\hat{x}_1, \ldots, \hat{x}_{k-l+1})$, $l = 1, 2, \ldots$, $k \geq l$, satisfy the recurrence relation

$$\hat{B}_{k,l}(\hat{x}_1, \ldots, \hat{x}_{k-l+1}) = \sum_{i=1}^{k-l+1} \frac{i \cdot l}{k} \hat{x}_i \hat{B}_{k-i,l-1}(\hat{x}_1, \ldots, \hat{x}_{k-i-l+2}),$$

where $\hat{B}_{0,0} = 1$ and $\hat{B}_{k,0} = 0$ for $k \geq 1$. The first few polynomials $\hat{B}_{k,l}(\hat{x}_1, \ldots)$ are:

$$\begin{array}{llll} \hat{B}_{0,0} = 1 & & & \\ \hat{B}_{1,0} = 0 & \hat{B}_{1,1} = \hat{x}_1 & & \\ \hat{B}_{2,0} = 0 & \hat{B}_{2,1} = \hat{x}_2 & \hat{B}_{2,2} = (\hat{x}_1)^2 & \\ \hat{B}_{3,0} = 0 & \hat{B}_{3,1} = \hat{x}_3 & \hat{B}_{3,2} = 2\hat{x}_1\hat{x}_2 & \hat{B}_{3,3} = (\hat{x}_1)^3. \end{array} \tag{13}$$

- [20] Let $\mathcal{D}\{f(t)\}[t_0] = \{F(k)[t_0]\}_{k=0}^{\infty}$ be the differential transform of the function $f(t)$ at t_0. Then the components $F(k)[t_1]$ of the differential transform $\mathcal{D}\{f(t)\}[t_1] = \{F(k)[t_1]\}_{k=0}^{\infty}$ of $f(t)$ at $t_1 > t_0$ may be expressed as

$$F(k)[t_1] = \sum_{j=0}^{\infty} \binom{k+j}{j} (t_1 - t_0)^j F(k+j)[t_0], \quad k \geq 0. \tag{14}$$

4 Application of the differential transform to the Euler equation

Consider the initial value problem for the half-linear Euler equation of the form

$$((x')^{\alpha})' + \frac{\gamma}{t^{\alpha+1}} x^{\alpha} = 0, \quad x(t_0) = a, \quad x'(t_0) = b, \tag{15}$$

in the case when α is a quotient of two positive odd numbers.

The main goal of this section is to obtain image of the equation (15) under the differential transform at t_0. We apply the formulas introduced in the Section

3. Number of the applied formula appears in parentheses above the "=" sign.

$$
\begin{aligned}
\mathcal{D}\{x(t)\} &\overset{(5)}{=} \{X(k)\}_{k=0}^{\infty}\,, \quad X(0) \overset{(8)}{=} a, \quad X(1) \overset{(8)}{=} b,\\
\mathcal{D}\{x'(t)\} &\overset{(9)}{=} \{(k+1)X(k+1)\}_{k=0}^{\infty}\,,\\
\mathcal{D}\{(x'(t))^{\alpha}\} &\overset{(12)}{=} \left\{\sum_{l=1}^{k}\binom{\alpha}{l}(X(1))^{\alpha-l}\hat{B}_{k,l}(2X(2),3X(3),\ldots)\right\}_{k=1}^{\infty} =: \{H_1(k)\}_{k=1}^{\infty}\,,\\
H_1(0) &= \binom{\alpha}{0}(X(1))^{\alpha},\\
\mathcal{D}\{((x'(t))^{\alpha})'\} &\overset{(9)}{=} \{(k+1)H_1(k+1)\}_{k=0}^{\infty}\,,\\
\mathcal{D}\{(x(t))^{\alpha}\} &\overset{(12)}{=} \left\{\sum_{l=1}^{k}\binom{\alpha}{l}(X(0))^{\alpha-l}\hat{B}_{k,l}(X(1),X(2),\ldots)\right\}_{k=1}^{\infty} =: \{H_2(k)\}_{k=1}^{\infty}\,,\\
H_2(0) &= \binom{\alpha}{0}(X(0))^{\alpha},\\
\mathcal{D}\left\{\frac{1}{t^{\alpha+1}}\right\} &\overset{(10)}{=} \left\{\binom{-\alpha-1}{k}t_0^{-\alpha-1-k}\right\}_{k=0}^{\infty} =: \{F_1(k)\}_{k=0}^{\infty}\,.
\end{aligned}
$$

Now, we use of the property (11) for transforming the product. The image of the equation (15) transformed at $t = t_0$ reads as

$$(k+1)H_1(k+1) + \gamma\sum_{l=0}^{k}F_1(l)H_2(k-l) = 0, \quad k \in \mathbb{N}_0. \tag{16}$$

We demonstrate the use of this result on a concrete example.

Example 1 As a testing example, we take the Euler equation (2) with $\alpha = 3$:

$$((x')^3)' + \frac{\gamma}{t^4}x^3 = 0.$$

We know that if $\gamma = \gamma_3 = \left(\frac{3}{4}\right)^4$, then $x_0(t) = t^{3/4}$ is a solution to the initial value problem

$$((x')^3)' + \frac{\left(\frac{3}{4}\right)^4}{t^4}x^3 = 0, \quad x(1) = 1, \quad x'(1) = \frac{3}{4}. \tag{17}$$

For brevity, we will use $X(k)$ instead of $X(k)[1]$, $k \in \mathbb{N}_0$. Transformed equation (16) expands for (17) with $\alpha = 3, t_0 = 1, \gamma = \left(\frac{3}{4}\right)^4$ to

$$
\begin{aligned}
0 = (k+1)&\sum_{l=1}^{k+1}\binom{3}{l}(X(1))^{3-l}\hat{B}_{k+1,l}(2X(2),\ldots)\\
+\gamma&\sum_{l=0}^{k-1}\binom{-4}{l}\sum_{j=1}^{k-l}\binom{3}{j}(X(0))^{3-j}\hat{B}_{k-l,j}(X(1),\ldots) + \gamma\binom{-4}{k}\binom{3}{0}(X(0))^3,
\end{aligned}
$$

where the middle term applies only for $k \geq 1$. Taking into account (13), we can write the preceding equality in a more compact form

$$
\begin{aligned}
0 = (k+1) &\sum_{l=1}^{k+1} \binom{3}{l} (X(1))^{3-l} \hat{B}_{k+1,l}(2X(2), \ldots) \\
&+ \gamma \sum_{l=0}^{k} \binom{-4}{l} \sum_{j=0}^{k-l} \binom{3}{j} (X(0))^{3-j} \hat{B}_{k-l,j}(X(1), \ldots), \quad k \in \mathbb{N}_0.
\end{aligned}
$$

We start with $k = 0$:

$$
1 \cdot \binom{3}{1} (X(1))^2 \hat{B}_{1,1}(2X(2), \ldots) + \gamma \binom{-4}{0} \binom{3}{0} (X(0))^3 = 0
$$

$$
\Rightarrow 3(X(1))^2 2X(2) + \gamma = 0 \Rightarrow X(2) = \frac{-\gamma}{6(X(1))^2} = \frac{\left(\frac{3}{4}\right)^4}{6\left(\frac{3}{4}\right)^2} = -\frac{3}{32}.
$$

Then we substitute $k = 1$:

$$
\begin{aligned}
&2\left(\binom{3}{1}(X(1))^2 \hat{B}_{2,1}(2X(2), \ldots) + \binom{3}{2} X(1) \hat{B}_{2,2}(2X(2), \ldots)\right) \\
&+ \gamma \left(\binom{-4}{0}\binom{3}{1}(X(0))^2 \hat{B}_{1,1}(X(1), \ldots) + \binom{-4}{1}\binom{3}{0}(X(0))^3\right) = 0
\end{aligned}
$$

$$
\Rightarrow 2(3(X(1))^2 \cdot 3X(3) + 3X(1)(2X(2))^2) + \gamma(3(X(0))^2 X(1) - 4(X(0))^3) = 0
$$

$$
\Rightarrow X(3) = \cdots = \frac{5}{128}.
$$

We continue with $k = 2$:

$$
\begin{aligned}
&3\left(\binom{3}{1}(X(1))^2 \hat{B}_{3,1}(2X(2), \ldots) + \binom{3}{2} X(1) \hat{B}_{3,2}(2X(2), \ldots) + \binom{3}{3}(X(1))^0 \hat{B}_{3,3}(2X(2), \ldots)\right) \\
&+ \gamma \left(\binom{-4}{0}\left[\binom{3}{1}(X(0))^2 \hat{B}_{2,1}(X(1), \ldots) + \binom{3}{2} X(0) \hat{B}_{2,2}(X(1), \ldots)\right]\right. \\
&\left. + \binom{-4}{1}\binom{3}{1}(X(0))^2 \hat{B}_{1,1}(X(1), \ldots) + \binom{-4}{2}\binom{3}{0}(X(0))^3\right) = 0,
\end{aligned}
$$

that is

$$
\begin{aligned}
&3\left(3(X(1))^2 4X(4) + 3X(1)2 \cdot 2X(2)3X(3) + (2X(2))^3\right) \\
&+ \gamma \left([3(X(0))^2 X(2) + 3X(0)(X(1))^2] - 4 \cdot 3(X(0))^2 X(1) + \frac{(-4)(-5)}{2}(X(0))^3\right) = 0
\end{aligned}
$$

$$
\Rightarrow X(4) = \cdots = -\frac{45}{2048}.
$$

Now recall that the exact solution to the initial value problem (17) is $x_0(t) = t^{3/4}$. Taylor series expansion of $t^{3/4}$ at $t_0 = 1$ is

$$
1 + \frac{3}{4}(t-1) - \frac{3}{32}(t-1)^2 + \frac{5}{128}(t-1)^3 - \frac{45}{2048}(t-1)^4 + O((t-1)^5).
$$

We can observe that using the differential transform algorithm we got the first several coefficients of Taylor expansion of the exact solution. The Taylor expansion is valid for $|t-1| \leq 1$ since we used the formula (10) in the derivation of the transformed equation (16).

Remark 1 *A different approach to find recurrence relations for obtaining the coefficients of a Taylor series of the solution is presented in papers about the Parker-Sochacki method (see for example [3, 31]). The main idea is to transform an ordinary differential equation into a system of first order ordinary differential equations with nothing but polynomials on the righthand side. Such a method can be applied to a wide class of ordinary differential equations. However, describing the process of finding the polynomial form is not simple and the polynomial system might not be unique. A-priori error estimates presented in [31] can be applied especially to this polynomial form of the equation. Within this context, choosing the transformations* $y_1 = x(t), y_2 = x'(t), y_3 = t^{-4}, y_4 = t, y_5 = 1/x'(t)$*, the initial value problem* (17) *can be rewritten in the following polynomial form:*

$$\begin{aligned}
y_1' &= y_2, & y_1(1) &= 1,\\
y_2' &= -\frac{1}{3}\left(\frac{3}{4}\right)^4 y_1^3\ y_3\ y_5^2, & y_2(1) &= \frac{3}{4},\\
y_3' &= -4y_3^2\ y_4^3, & y_3(1) &= 1,\\
y_4' &= 1, & y_4(1) &= 1,\\
y_5' &= \frac{1}{3}\left(\frac{3}{4}\right)^4 y_1^3\ y_3\ y_5^4, & y_5(1) &= \frac{4}{3}.
\end{aligned}$$

5 Application of the differential transform to the Euler equation with a proportional delay

Consider the initial value problem

$$((x'(t))^\alpha)' + \frac{\gamma}{t^{\alpha+1}}(x(\lambda t))^\alpha = 0, \quad x(t_0) = a, \quad x'(t_0) = b, \tag{18}$$

with $\lambda \in (0,1)$ and the initial function

$$\phi(t) = a + b(t - t_0), \quad t \in (0, t_0] = I_0. \tag{19}$$

Let $t_i = \frac{t_0}{\lambda^i}$ and $I_i = [t_{i-1}, t_i]$, $i \in \mathbb{N}_0$. If $t \in I_i$ then λt lies in I_{i-1}. We follow the process of combining the differential transform with the method of steps for delayed differential equations described in the paper [23].

For $t \in I_1 = [t_0, t_1]$ we determine the solution of the initial value problem (18), (19) as $x_1(t)$. The differential transform of $x_1(t)$ at t_0 will be $X_1(k)[t_0]$,

$k \in \mathbb{N}_0$. Since λt for $t \in I_1$ falls into I_0, we substitute the initial function (19) for $x(\lambda t)$ and rewrite equation (18) in the form

$$((x'(t))^\alpha)' + \frac{\gamma}{t^{\alpha+1}}(a + b(\lambda t - t_0))^\alpha = 0, \quad x(t_0) = a, \quad x'(t_0) = b. \tag{20}$$

Because

$$\phi(\lambda t) = a + b(\lambda t - t_0) = a + bt_0(\lambda - 1) + b\lambda(t - t_0),$$

we have

$$\mathcal{D}\{\phi(\lambda t)\}[t_0] = \{a + bt_0(\lambda - 1), b\lambda, 0, 0, \ldots\},$$

and the kth component of the differential transform of $(\phi(\lambda t))^\alpha$ at t_0 is

$$\begin{aligned} \mathcal{D}\{(\phi(\lambda t))^\alpha\}(k)[t_0] &\overset{(12)+(13)}{=} \sum_{l=0}^{k} \binom{\alpha}{l} (a + bt_0(\lambda - 1))^{\alpha - l} \hat{B}_{k,l}(b\lambda, 0, 0, \ldots) \\ &= \binom{\alpha}{k} (a + bt_0(\lambda - 1))^{\alpha - k} (b\lambda)^k, \ k \geq 0. \end{aligned}$$

After we use the formula (11), the equation (20) transformed at t_0 reads as

$$\begin{aligned} 0 = (k+1) &\sum_{l=1}^{k+1} \binom{\alpha}{l} (X_1(1)[t_0])^{\alpha - l} \hat{B}_{k+1,l}(2X_1(2)[t_0], 3X_1(3)[t_0] \ldots) \\ &+ \gamma \sum_{l=0}^{k} \binom{-\alpha - 1}{l} (t_0)^{-\alpha - 1 - l} \binom{\alpha}{k - l} (a + bt_0(\lambda - 1))^{\alpha - k + l} (b\lambda)^{k - l}. \end{aligned} \tag{21}$$

The initial conditions are transformed to

$$X_1(0)[t_0] = a, \quad X_1(1)[t_0] = b.$$

Substitution for $k = 0, 1, \ldots$ into (21) provides recurrence relations from which one can successively calculate $X_1(k)[t_0]$ for $k \geq 2$. The solution on the interval I_1 is then

$$x_1(t) = \sum_{k=0}^{\infty} X_1(k)[t_0](t - t_0)^k.$$

In applications, we use a computing software to calculate the coefficients of the Taylor series. It means that this series as well as any other series will be truncated. The solution x_1 will become an approximate solution.

Notice that with a general initial function ϕ, equation (21) would have the form

$$\begin{aligned} (k+1) &\sum_{l=1}^{k+1} \binom{\alpha}{l} (X_1(1)[t_0])^{\alpha - l} \hat{B}_{k+1,l}(2X_1(2)[t_0], 3X_1(3)[t_0] \ldots) \\ &+ \gamma \sum_{l=0}^{k} \binom{-\alpha - 1}{l} (t_0)^{-\alpha - 1 - l} \mathcal{D}\{(\phi(\lambda t))^\alpha\}(k)[t_0] = 0. \end{aligned}$$

Now we proceed with the second step. Take $t \in I_2 = [t_1, t_2]$, denote $x_2(t)$ the approximate solution on I_2, and let $X_2(k)[t_1]$ for $k \geq 0$ be the differential transform of x_2 at t_1. Since λt for $t \in I_2$ lies in I_1, we substitute for $x(\lambda t)$ the function $x_1(\lambda t)$ and rewrite equation (18) in the form

$$((x'(t))^\alpha)' + \frac{\gamma}{t^{\alpha+1}}(x_1(\lambda t))^\alpha = 0, \quad x(t_1) = x_1(t_1), \quad x'(t_1) = x_1'(t_1). \tag{22}$$

Since $\lambda t = \lambda t_1 + \lambda(t - t_1)$, we have

$$\mathcal{D}\{\lambda t\} = \{\lambda t_1, \lambda, 0, 0, \ldots\}.$$

Recalling that $\lambda t_1 = t_0$, we have the following expression for the kth component of the differential transform of $x_1(\lambda t)$ at t_1:

$$\mathcal{D}\{x_1(\lambda t)\}(k)[t_1] \overset{(12)}{=} \sum_{l=1}^{k} X_1(l)[\lambda t_1]\hat{B}_{k,l}(\lambda, 0, 0, \ldots) = X_1(k)[t_0]\lambda^k =: G(k)[t_1], k \geq 1,$$

$$\mathcal{D}\{x_1(\lambda t)\}(0)[t_1] = X_1(0)[t_0] = a =: G(0)[t_1].$$

Next, for $(x_1(\lambda t))^\alpha$ we get

$$\mathcal{D}\{(x_1(\lambda t))^\alpha\}(k)[t_1] \overset{(12)+(13)}{=} \sum_{l=0}^{k} \binom{\alpha}{l} (G(0)[t_1])^{\alpha-l}\hat{B}_{k,l}(G(1)[t_1], G(2)[t_1], \ldots), k \geq 0.$$

Again, we use the product formula (11) and the equation (22) transformed at t_1 reads as

$$0 = (k+1)\sum_{l=1}^{k+1}\binom{\alpha}{l}(X_2(1)[t_1])^{\alpha-l}\hat{B}_{k+1,l}(2X_2(2)[t_1], 3X_2(3)[t_1]\ldots) \tag{23}$$
$$+\gamma\sum_{l=0}^{k}\binom{-\alpha-1}{l}(t_1)^{-\alpha-1-l}\sum_{j=0}^{k-l}\binom{\alpha}{j}(G(0)[t_1])^{\alpha-j}\hat{B}_{k-l,j}(G(1)[t_1], G(2)[t_1], \ldots).$$

According to (14), the initial conditions are

$$X_2(0)[t_1] = X_1(0)[t_1] = \sum_{k=0}^{\infty} X_1(k)[t_0](t_1 - t_0)^k,$$
$$X_2(1)[t_1] = X_1(1)[t_1] = \sum_{k=0}^{\infty}(k+1)X_1(k+1)[t_0](t_1 - t_0)^k.$$

The approximate solution x_2 for $t \in I_2$ is then

$$x_2(t) = \sum_{k=0}^{\infty} X_2(k)[t_1](t - t_1)^k.$$

Further steps for $t \in I_i$, $i \geq 3$ lead again to the recurrence relation (23). The only differences will appear in indeces of the Taylor coefficients (that is, X_2

becomes X_i) and centres of the Taylor expansion (that is, t_1 becomes t_{i-1}). The process of calculation of the Taylor coefficients in the ith step follows the pattern of the second step.

Example 2 To demonstrate the described algorithm, we choose the following concrete values of the parameters:

$$\alpha = 3, \quad t_0 = 1, \quad \gamma = \left(\frac{3}{4}\right)^4, \quad a = 1, \quad b = \frac{3}{4}, \quad \lambda = 0.8. \tag{24}$$

Notice that the constants α, t_0 and γ have the exactly same values as in Example 1. Comparison with the solution $x_0(t) = t^{3/4}$ to the non-delayed problem (17) studied in Example 1 will allow us to observe the effect of the delay and the chosen initial function. This is important because it is not possible to find exact solution to the initial value problem (18), (19) with the constants (24) in terms of elementary functions.

The equation (18) with the constants (24) becomes

$$((x'(t))^3)' + \left(\frac{3}{4}\right)^4 \frac{1}{t^4}(x(0.8t))^3 = 0, \quad x(1) = 1, \quad x'(1) = \frac{3}{4}, \tag{25}$$

and the initial function is

$$\phi(t) = 1 + \frac{3}{4}(t-1), \quad t \in (0, 1]. \tag{26}$$

Recalling the fact that $t_i = \frac{t_0}{\lambda^i}$, the first step of the algorithm takes place on the interval $[t_0, t_1] = [1, \frac{10}{8}]$ whereas the second step takes place on the interval $[t_1, t_2] = [\frac{10}{8}, \frac{100}{64}]$. The results of the simulation in Matlab, version 2019b, are shown in Table 1 and Figure 2. The first column of Table 1 presents the values of t in the interval $[t_0, t_2]$ where the comparison is done. In the second column we have values of the approximate solution to the IVP (25), (26) found by using the differential transform algorithm. Here x_1 represents the solution on $[t_0, t_1]$ and x_2 the solution on $[t_1, t_2]$. In both cases, the order of the Taylor polynomial that represents the approximate solution is 5. We note that the accuracy influenced by the chosen order is good enough. The difference between the 5th order approximate solution and higher order approximate solutions on the observed interval is less than 10^{-6}. The third column contains values of the approximate solution computed by the built-in Matlab function designed for solving delay differential equations "ddesd". The fourth column shows the difference between these two numerical solutions at given points. In the fifth column we present values of the exact solution to the initial value problem (17), that is, to the non-delayed half-linear Euler equation. At the end of the interval $[t_0, t_2]$, we can observe that the solution to the half-linear Euler equation with delay tends to grow faster than the solution to the half-linear Euler equation without delay. This fact is illustrated also in Figure 2.

Experimenting with other values of γ, we have noticed that neither "ddesd" nor a straightforward implementation of our algorithm can successfully calculate

Table 1: Comparison of DT (order 5) and Matlab

	$x_1(x_2)$	x_{ddesd}	$x_{ddesd} - x_i$	$x_0 = t^{3/4}$
$t_0 = 1.0000$	1.0000	1.0000	0.0	1.0000
1.0370	1.0621	1.0621	0.0147E - 3	1.0277
1.0741	1.1234	1.1235	0.1103E - 3	1.0551
$t_1 \approx 1.1111$	1.1840	1.1843	0.3515E - 3	1.0822
1.1523	1.2591	1.2595	0.0004	1.1122
1.1934	1.3340	1.3339	-0.0002	1.1418
$t_2 \approx 1.2346$	1.4089	1.4075	-0.0013	1.1712

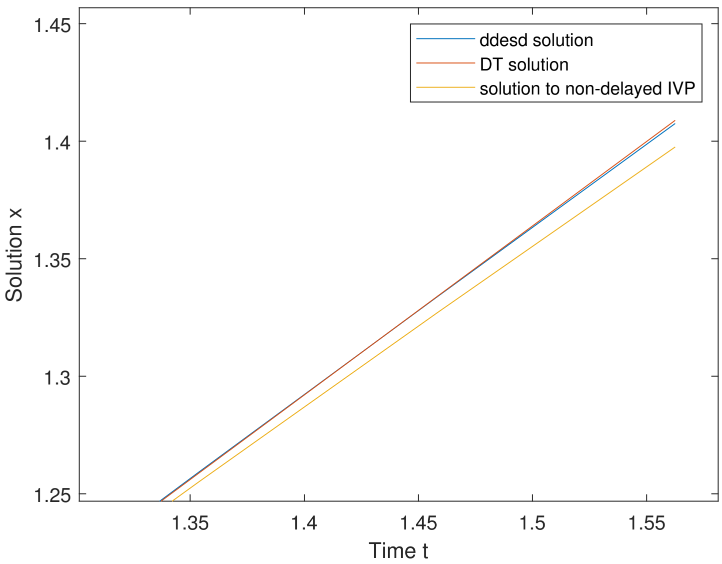
the approximate solution on the interval $[t_0, t_2]$ if γ increases to 1.9. Similar situation happens if we keep γ at the constant value 1 and try to calculate on the interval $[t_0, 3.0]$: Matlab returns an error message. Such behavior suggests that more research in this direction is needed.

6 Conclusion

We presented how the differential transform algorithm can be applied to obtain numerical solutions to second-order half-linear Euler equations without delay and with a proportional delay. The described algorithm includes a modification for different types of initial functions. Applicability of the algorithm is demonstrated on an example of the Euler equation without delay. The coefficients of the Taylor expansion obtained by the differential transform coincide with coefficients of Taylor expansion of the exact solution. Numerical simulation on an example of the delayed equation with concrete values of parameters was performed. The simulation values are in a good agreement with simulations produced by the Matlab routine “ddesd”. All obtained results confirm that the presented algorithm is efficient and convenient for finding approximate solutions to the studied initial value problems.

Our experiment shows that the differential transform method in combination with the method of steps is well applicable to delayed half-linear Euler equations. One of the advantages is that the obtained approximate solution is in the form of a Taylor polynomial. That is different from the outcome of the Matlab function “ddesd”, where the result is a set of approximate function values of the solution. We also found out that both “ddesd” function and our procedure do not work on larger intervals in some cases, particularly with increasing γ. On the other hand, the combination of the differential transform and the method of steps can be theoretically elongated easily. The practical implementation of the algorithm, however, has to deal with the limits of division by numbers close to zero, especially when increasing the order of the Taylor polynomial. The idea

Figure 1: Comparison of approximate solutions to (25), (26) and the exact solution to (17)



of scaling the coefficients to the same dimension by the length of the considered interval might help to overcome this obstacle.

Finally, the knowledge of approximate solutions to initial value problems for half-linear Euler type equations can complement the qualitative theory and motivate further theoretical results. Lat but not least, the lack of success with computations on larger intervals using Matlab gives a strong motivation for continuing research on approximate solutions.

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