MASARYKOVA UNIVERZITA
PŘÍRODOVĚDECKÁ FAKULTA

Habilitation Thesis

BRNO 2018 ZDENĚK SVOBODA

MASARYKOVA UNIVERZITA
PŘÍRODOVĚDECKÁ FAKULTA

ASYMPTOTIC PROPERTIES OF FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH DELAY

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Brno 2018

Bibliografický záznam

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Název práce:	ASYMPTOTIC PROPERTIES FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH DELAY
Akademický rok:	2017/2018
Počet stran:	$6+233$
Klíčová slova:	zpožděné funkcionálně diferenciálná rovnice; kladná řešení; zpožděné maticové funkce; toplogická metoda; princip retraktu; exponenciální stabilita; asymptotický rozklad

Bibliographic Entry

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Title of Thesis: ASYMPTOTIC PROPERTIES FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH DELAY

Academic Year: 2017/2018

Number of Pages: $6+233$

Keywords: delayed functional differential equation; positive solutions; delayed matrix functions; topological mathod; retract principle; exponential stability; asymptotic expansion;

Abstrakt

V této habilitační práci jsou shrnuty výsledky mých 13 vybraných vědeckých prací, které jsou věnovány problematice funkcionálních diferenciálních rovnic a jejich asymptotickým vlastnostem. Výber těchtopublikací může být rozdělen do 3 částí. První se zabývá lineárními systémy s konstantními koeficienty a konstantním zpožděním. Druhá část je věnována toplogické metodě a jejímu užití při studiu asymptotických vlastností zpožděných funkcionálně diferenciálních rovnic. Třetí oblast výzkumu je věnována exponenciální stabiltě zpožděných funkcionálně diferenciálních rovnic.

Abstract

In this habilitation thesis are summarized the results of my 13 selected scientific papers which are devoted to the problems of functional differential equations and their asymptotic properties. The selection of this papers can be divided into three parts. The first deals with linear systems with constant coefficients and constant delays. The second part is devoted to the topological method and its use in the study of asymptotic properties of delayed functional differential equations. The third area of research is devoted to the exponential stability of delayed functional differential equations.

Contents

Chapter 1

Introduction

This thesis is based on a selection of the author's papers since 1998 dealing with the asymptotic properties of delayed differential equations. The selection can be divided into three parts. The first deals with linear systems with constant coefficients and constant delays. In this section, definitions of delayed matrix functions are introduced as a new formalization of the well-known step-by-step method. It is a concretization of the previously used term of the fundamental matrix, see [34]. One of the many motivations is the fact that these systems are canonical equations based on which the entire class of delayed equations can be transformed. These results are shown by F. Neuman [49][50], and others [35], [63].

Papers [38], [39] bring an integral representation of the solutions to a first-order system defining a delayed exponential of matrix and bringing the basic results. This formalization was widely applied, e.g., in boundary-value problems, control problems, and stability problems, modification to discrete equations was performed, generalizations to the case of several delays were developed, etc. (see papers [6], [7], [36], [47], [48]). In the papers [46], [51] the systems with more constant delays and commutative matrices coefficients are investigated and the generalization of delayed exponential of matrix is given. For a second-order system, analogously, two functions are defined - delayed matrix sine and cosine [16]. In [a12] the relationships are given between these functions, which can be understood as a generalization of the well-known Euler's formula, because, for the zero delay, the relationship for delayed matrix functions is reduced to this identity.

First, it is possible to use this identity as a motivation for studying more general systems. The representation of the solutions to the second-order system of n equations is obtained from the representation of the first n components of the solution to the related initial problem of a first-order linear system of $2n$ equations, see [a12].

Second it is possible to obtain the asymptotic properties of the delayed matrix functions sine and cosine from the asymptotic properties of the delayed exponential of matrix. The results given in the papers [59], [a6] describes the asymptotic properties of a delayed matrix exponential for $k \to \infty$ proving that the sequence of values of the delayed matrix exponential at the nodes is approximately represented by a geometric progression. A constant matrix is found such that its matrix exponential is the "quotient" factor that depends on the principal branch of Lambert function, see [a6]. In the paper [a13] it was proved that the spectral norm of the delayed matrix sin and cosine are unbounded for $t \to \infty$ by asymptotic properties of delayed matrix exponential.

And finally, a possible application is shown of delayed matrix functions to formalizing the solution of partial differential equations with constant delay, see [15].

The second part is devoted to the topological method and its use in the study of asymptotic properties of delayed functional differential equations. We describe a modification of the Wazewski's topological method for ordinary differential equations [33] with new use of the topological method applied to delayed functional-differential equations with bounded delay. Introduced by Rybakowski [53], this modification uses a topological method for a system of curves. This idea is further adapted to functional differential systems with unbounded delay and finite memory [a3]. This type of functional-differential equations was given in [43] by using of the p function. Later, the topological principle was used to study the asymptotic properties of neutral differential equations. This was made possible by introducing a system of subsidiary inequalities in the definition of a polyfacial set.

These modifications of the topological method were also used to study the asymptotic properties of delayed systems. First, it was used for the asymptotic integration of a functional-differential equation in which the solution is represented by an asymptotic series. Using a concrete example of one equation, the existence is shown of different asymptotic properties to the solution to this equation as depending on the magnitude of the delay [a11]. Another application of this method made it possible to obtain criteria for the existence of positive solutions to functional differential equations with unbounded delay. The criteria given in [a3], [a4], [a5], [58] generalize the criteria for delayed differential equations with bonded delay.

The topological principle was used to study the asymptotic properties of neutral differential equations as a tool for the verification of a new existence criterion for the positive solutions of one neutral equation, see [a9]. In order to achieve a continuous dependence of solutions, which are moreover continuously differentiable, the definition of a system of initial functions fulfilling the sewing condition was introduced.

The third area of research is devoted to the exponential stability of delayed functional differential equations. The problem has been studied by a number of authors and new results are presented in this section, that are generalizations of previous criteria.

Some existing models with delayed equations.

Delay differential equation arise in many applications in different fields being described in books such as [41] or [62] with many examples. In [41] part of models is introduced by the equation of a showering person

$$\dot{T}_m(t) = -\kappa(T_m(t-h) - T_d), \tag{1.1}$$

$T_m(t)$ denotes the water temperature at the mixer output, h is the positive constant time that the water takes to go from the mixer output to the top of the person's head. T_d is the desired water temperature on the head of the showering person and the coefficient κ depends the person's temperament. The paper also studies the modification of the well-known equation with delayed terms because the new equations are more accurate models of the studied processes. Hutchinson [62] proposed a logistic delay population model of the form

$$\dot{x}(t) = \gamma x(t)\left(1 - \frac{x(t-\tau)}{K}\right) \tag{1.2}$$

where constant γ is the coefficient of linear growth, the constant K is the average population size related to the ability of the environment to sustain the population, x(t) is the population size at time t. The delay $\tau > 0$ means that the food resources at time t are determined by the population size at time $T - h$. For more details, see [25].

Putting $x(t) = K(1 + y(t/h)$, we obtain a new equation for $y(t)$;

$$\dot{y}(t) = -\gamma h y(t-1)(1 + y(t)),$$

which is encountered in the number theory [65]. Gurney et al. [27] proposed a DDE to describe the Nicolson blowflies model.

$$\dot{N}(t) = aN(t-\tau)\mathrm{e}^{bN(t-\tau)} - dN(t) \tag{1.3}$$

where N(t) denotes the size of the population at time t, a is the maximum per capita rate of producing eggs per day, d is the death rate in the adult population, and τ is the time taken from the birth of a member until it becomes mature.

One of the classical equations of non-linear dynamics was formulated by a Dutch physicist Van der Pol. Originally, it was a model for an electrical circuit with a triode valve, and was later extensively studied as a prototype of a rich class of dynamical behavior. This model is described by the equation

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0$$

which may be studied as a system of two first-order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu(1 - x^2)y - x. \end{aligned}$$

Van der Pol found stable oscillations, which he called relaxation-oscillations and which are now known as limit cycles. [2] also studies a Van der Pol's oscillator with delay (feed back). This is the equation

$$\ddot{x}(t) + x(t) - \varepsilon(1 - x^2(t))\dot{x}(t) + kx(t-\tau) = 0,$$

which is also studied by a parameter-expanding method in [40].

Chapter 2

Fundamental matrix for linear systems with constant coefficients and constant delay

This chapter is devoted to linear systems with constant coefficients and delay. Our results are stated in theorems of this chapter are proved in papers [a6], [a12], [a13].

The application of the well-known "step by step" method to solving ordinary differential equations has recently been in the case of linear first-order systems with single constant delay and with constant matrix, formalized using special types of delayed matrices (delayed matrix exponential, delayed matrix sine and delayed matrix cosine). These matrix functions are defined on intervals $(k-1)\tau \le t < k\tau$, $k = 0, 1, \dots$ (where $\tau > 0$ is a delay) as matrix polynomials, and are continuous at the nodes $t = k\tau$, see [39], [16]. The papers [59], [a6] studies the asymptotic properties of a delayed matrix exponential for $k \to \infty$ proving that the sequence of values of the delayed matrix exponential at the nodes is approximately represented by a geometric progression. A constant matrix is found such that its matrix exponential is the "quotient" factor that depends on the principal branch of Lambert function. The formulas derived can be applied to the study of the asymptotic properties of the solutions to linear differential systems with constant matrices and with a single delay.

The well-known "step by step" method is one of the basic concepts for the investigation of linear differential equations and systems with delay. The application of this method to linear first-order systems with single constant delay and with constant matrix of linear terms has recently been formalized using the concept of a delayed matrix exponential e_τ^{Bt} in [38, 39]. For linear second-order "oscillating" systems with constant matrix and with a single constant delay, analogous results have recently been derived using the so-called delayed matrix cosine $\mathrm{Cos}_\tau Bt$ and delayed matrix sine $\mathrm{Sin}_\tau Bt$ in [36]. The above special delayed matrix functions are defined on every interval $(k-1)\tau \le t < k\tau$, $k = 0, 1, \dots$ (where $\tau > 0$ is a delay) as matrix polynomials, and are continuous at nodes $t = k\tau$. Such "step by step" definitions complicate their asymptotic analysis.

2.1 Linear first-order systems

Let B be an $s\times s$ constant matrix, Θ the $s\times s$ null matrix, I the $s\times s$ unit matrix, and let $\tau>0$ be a constant. The delayed matrix exponential e_τ^{Bt} of the matrix B is an $s\times s$ matrix function mapping $\mathbb{R}$ to $\mathbb{R}^{s\times s}$, continuous on $\mathbb{R}\setminus\{-\tau\}$, and defined as follows:

$$e_\tau^{Bt} := \sum_{j=0}^{k} B^j \frac{(t-(j-1)\tau)^j}{j!} \tag{2.1}$$

where $k=\lceil t/\tau\rceil$ is the ceiling function, i.e., the smallest integer greater than or equal to t/τ.

The main property of the delayed matrix exponential e_τ^{Bt} is the following:

$$\frac{\mathrm{d}\, e_\tau^{Bt}}{\mathrm{d}t} = Be_\tau^{B(t-\tau)},\ t\in\mathbb{R}\setminus\{0\}$$

and the matrix $Y(t)=e_\tau^{Bt}$ solves the initial problem for a matrix differential system with a single delay

$$\begin{aligned} \dot{Y}(t) &= BY(t-\tau),\\ Y(t) &= I,\ t\in[-\tau,0]. \end{aligned} \tag{2.2}$$

If $\varphi\colon[-\tau,0]\to\mathbb{R}^n$ is a continuously differentiable vector-function, then the solution of the initial-value problem

$$\dot{y}(t) = By(t-\tau),\ \ t\in[-\tau,\infty), \tag{2.3}$$

$$y(t) = \varphi(t),\ \ t\in[-\tau,0] \tag{2.4}$$

can be represented in the form

$$y(t) = e_\tau^{Bt}\varphi(-\tau) + \int\limits_{-\tau}^{0} e_\tau^{B(t-\tau-s)}\dot{\varphi}(s)]ds. \tag{2.5}$$

This definition illustrates general definition of a fundamental matrix to linear functional differential systems of delayed type given in [34]. For system (2.3), this definition reduces to (details are omitted)

$$X(t) = \begin{cases} B\displaystyle\int_{-\tau}^{t} X(u-\tau)du + I, \text{ for almost all } t\geq -\tau, \\ \Theta, -2\tau\leq t< -\tau \end{cases}. \tag{2.6}$$

Let A be a regular $s\times s$ constant matrix satisfying $AB=BA$ and let $f(t)$ be a continuous function. Then, the solution of the initial-value problem

$$\begin{aligned} \dot{y}(t) &= Ay(t)+By(t-\tau)+f(t),\ t\in[-\tau,\infty),\\ y(t) &= \varphi(t),\ t\in[-\tau,0] \end{aligned}$$

is given by the formula

$$y(t)=e^{A(t+\tau)}e_{\tau}^{B_1 t}\varphi(-\tau)+\int_{-\tau}^{0} e^{A(t-\tau-s)}e_{\tau}^{B_1(t-\tau-s)}e^{A\tau}[\dot{\varphi}(s)-A\varphi(s)]ds$$
$$+\int_{0}^{t} e^{A(t-\tau-s)}e_{\tau}^{B_1(t-\tau-s)}e^{A\tau}f(s)ds \quad (2.7)$$

where $B_1=e^{-A\tau}B$. These results, together with the results for a non homogenous system, are proved in [38, 39].

2.2 Linear second-order systems

The above-mentioned usefullness of the delayed matrix exponential served as a stimulation to look for another delayed matrix functions capable of simply expressing solutions to some linear differential systems with constant coefficients. In [36], delayed matrix functions are defined called the delayed matrix sine $\mathrm{Sin}_{\tau}At$ and delayed matrix cosine $\mathrm{Cos}_{\tau}At$ for $t\in\mathbb{R}$ as

$$\mathrm{Sin}_{\tau}At=\sum_{s=0}^{\lfloor t/\tau\rfloor+1}(-1)^s A^{2s+1}\frac{(t-(s-1)\tau)^{2s+1}}{(2s+1)!} \quad (2.8)$$

and

$$\mathrm{Cos}_{\tau}At=\sum_{s=0}^{\lfloor t/\tau\rfloor+1}(-1)^s A^{2s}\frac{(t-(s-1)\tau)^{2s}}{(2s)!}, \quad (2.9)$$

where $\lfloor\cdot\rfloor$ is the floor function. Both the delayed matrix sine and cosine are the fundamental matrices of a homogeneous second-order linear system with a single delay

$$\ddot{x}(t)=-A^2x(t-\tau). \quad (2.10)$$

In [36] the Cauchy initial value problem is solved for equation (2.10) and the initial condition

$$x(t)=\varphi(t),\ \text{for } -\tau\le t\le 0 \quad (2.11)$$

where $\varphi\in C^2([-\tau,0],\mathbb{R}^n)$. Assuming that the matrix A is regular, a representation of the solution to Cauchy initial problem (2.10), (2.11) is given in the integral form

$$x(t)=\mathrm{Cos}_{\tau}At\ \varphi(-\tau)+A^{-1}\mathrm{Sin}_{\tau}At\ \dot{\varphi}(-\tau)$$
$$+A^{-1}\int_{-\tau}^{0}\mathrm{Sin}_{\tau}A(t-\tau-\xi)\ \ddot{\varphi}(\xi)d\xi. \quad (2.12)$$

The motivation for the study of properties of solutions to second-order linear differential systems is the applicability of this fact to the study of solutions to linear partial differential second-order equations.

In [a12] the relations are studied between the first-order and second-order systems. The solutions to second-order linear differential systems can be regarded as the first n components of the solutions to first-order linear differential systems of $2n$ equations. In [a12] the following useful identities

$$\mathrm{Cos}_{2\tau}A(t-\tau) = \mathrm{Re}\,\mathrm{e}_{\tau}^{(iA)t} \quad \text{and} \quad \mathrm{Sin}_{2\tau}A(t-2\tau) = \mathrm{Im}\,\mathrm{e}_{\tau}^{(iA)t} \tag{2.13}$$

are proved. Equivalently,

$$\mathrm{e}_{\tau}^{(iA)t} = \mathrm{Cos}_{2\tau}A(t-\tau) + i\mathrm{Sin}_{2\tau}A(t-2\tau),$$

which can be understood as a generalization of the well-known Euler formula, since we obtain this formula if we put $A = 1$, $\tau = 0$ in the above identity. For the delayed matrix functions, we have

$$\dot{y}(t) = \mathscr{A}y(t-\tau/2),$$

where

$$\mathscr{A} := \begin{pmatrix} \Theta & A \\ -A & \Theta \end{pmatrix}, \quad y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

is equivalent with (2.10) through the substitution $x(t) = y_1(t)$. In much the same way as above, we can derive (for details we refer to [a12])

$$\mathscr{X}(t) = \mathrm{e}_{\tau/2}^{\mathscr{A}t} = \begin{pmatrix} \mathrm{Cos}_{\tau}A(t-\tau/2) & \mathrm{Sin}_{\tau}A(t-\tau) \\ -\mathrm{Sin}_{\tau}A(t-\tau) & \mathrm{Cos}_{\tau}A(t-\tau/2) \end{pmatrix}.$$

These facts may serve as motivation for the study of a more general Cauchy initial problem

$$\ddot{x}(t) + P\dot{x}(t-\tau) + Qx(t-2\tau) = \theta, \tag{2.14}$$

$$x(t) = \xi(t), \ t \in [-\tau, \tau] \tag{2.15}$$

where P, Q are $n \times n$ constant matrices provided that there exists an $n \times n$ matrix Λ satisfying the equation

$$\Lambda^2 + P\Lambda\exp(-\tau\Lambda) + Q\exp(-2\tau\Lambda) = \Theta. \tag{2.16}$$

We assume that a solution of (2.14) can be found in the form

$$x(t) = \exp(\Lambda t) \tag{2.17}$$

where Λ is a suitable $n \times n$ constant matrix. By substituting (2.17) into (2.14), we get

$$\Lambda^2\exp(2\Lambda t) + P\Lambda\exp(\Lambda(t-\tau)) + Q\exp(\Lambda(t-2\tau)) = \Theta$$

and further simplification gives equation (2.16). Let $Y = \Lambda\exp(\Lambda\tau)$ be a new unknown matrix. Then, equation (2.16) can be written as

$$Y^2 + PY + Q = \Theta. \tag{2.18}$$

As this is a quadratic equation with respect the matrix Y, its solution has three forms:

The first one is

$$\ddot{x}(t) - 2A\dot{x}(t-\tau) + (A^2+B^2)x(t-2\tau) = \theta, \ t \geq \tau, \tag{2.19}$$

$$x^{(i)}(t) = \xi^{(i)}(t), \ i = 0,1, \ t \in [-\tau, \tau] \tag{2.20}$$

where the $n \times n$ matrices A, B commute, i.e., $AB = BA$, the matrix B is regular, and the function $\xi\colon [-\tau, \tau] \to \mathbb{R}^n$ is assumed to be twice continuously differentiable.

Theorem 1. *Let $AB = BA$ and let the matrix B be invertible. Then, the solution of the initial problem* (2.19), (2.20) *can be expressed as*

$$\begin{aligned} x(t) = \Big(&\operatorname{Re} \mathrm{e}_\tau^{(A+iB)t} - \operatorname{Im} \mathrm{e}_\tau^{(A+iB)t} B^{-1} A\Big)\, \xi(-\tau) \\ &+ \Big(\operatorname{Im} \mathrm{e}_\tau^{(A+iB)t}\Big) B^{-1} \dot{\xi}(0) + \int_{-\tau}^{0} \bigg(\Big(\operatorname{Re} \mathrm{e}_\tau^{(A+iB)(t-\tau-s)} \Big)\, \dot{\xi}(s) \\ &+ \Big(\operatorname{Im} \mathrm{e}_\tau^{(A+iB)(t-\tau-s)}\Big) B^{-1} (\ddot{\xi}(s+\tau) - A\dot{\xi}(s)) \bigg)\, \mathrm{d}s \end{aligned} \tag{2.21}$$

where $t \geq \tau$.

The second one is the problem (2.22), (2.20) where

$$\ddot{x}(t) - (A+B)\dot{x}(t-\tau) + ABx(t-2\tau) = \theta,\ t \geq \tau, \tag{2.22}$$

with matrices A and B commuting but the regularity of B not assumed.

Theorem 2. *Let $AB = BA$. Then, the solution to the Cauchy initial problem* (2.22), (2.20) *has the form*

$$\begin{aligned} x(t) = \mathrm{e}_\tau^{At} \xi(-\tau) &+ \mathrm{e}_\tau^{(A,B)t} (\dot{\xi}(0) - A\xi(-\tau)) \\ &+ \int_{-\tau}^{0} \Big(\mathrm{e}_\tau^{A(t-\tau-s)} \dot{\xi}(s) + \mathrm{e}_\tau^{(A,B)(t-\tau-s)} (\ddot{\xi}(s+\tau) - A\dot{\xi}(s)) \Big)\, \mathrm{d}s \end{aligned} \tag{2.23}$$

where $t \geq \tau$ and the matrix function $\mathrm{e}_\tau^{(A,B)t}$ is defined as

$$\mathrm{e}_\tau^{(A,B)t} = \sum_{s=0}^{\lfloor t/\tau \rfloor} \frac{(t-(s-1)\tau)^s}{s!} \sum_{i=0}^{s} A^{s-i} B^i.$$

The third initial problem has the form of a solution to the initial problem given by the initial condition (2.20) and by the equation:

$$\ddot{x}(t) - 2A\dot{x}(t-\tau) + A^2 x(t-2\tau) = \theta,\ t \geq \tau, \tag{2.24}$$

where $\xi\colon [-\tau, \tau] \to \mathbb{R}^n$

Theorem 3. *A solution to initial problem* (2.24), (2.20) *has the form*

$$\begin{aligned} x(t) = \mathrm{e}_\tau^{At} \xi(-\tau) &+ D_A \mathrm{e}_\tau^{At} (\dot{\xi}(0) - A\xi(-\tau)) \\ &+ \int_{-\tau}^{0} \Big(\mathrm{e}_\tau^{A(t-\tau-s)} \dot{\xi}(s) + D_A \mathrm{e}_\tau^{A(t-\tau-s)} (\ddot{\xi}(s+\tau) - A\dot{\xi}(s)) \Big)\, \mathrm{d}s \end{aligned} \tag{2.25}$$

where $t \geq \tau$ and the function $D_A \mathrm{e}_\tau^{At}$ is defined as

$$D_A \mathrm{e}_\tau^{At} = \sum_{s=0}^{\lfloor t/\tau \rfloor} \frac{(t-(s-1)\tau)^s}{s!} s A^{s-1} = \frac{\partial}{\partial A} \mathrm{e}_\tau^{At}.$$

The proofs of the theorems that describe the formalization of a solution to the initial value problem consisting of second-order systems with constant delay, $n \times n$ constant matrices and an initial condition are based on the study of solutions to the initial value problem of a first-order system with constant $2n \times 2n$ matrices and one constant delay. For more details, see [a12].

2.3 Asymptotic properties of the delayed matrix functions and Lambert function

The delayed matrix functions are defined on the intervals $(k-1)\tau \leq t < k\tau$, $k = 0, 1, \ldots$ as matrix polynomials and are continuous at the nodes $t = k\tau$. The asymptotic properties of a delayed matrix exponential are studied for $k \to \infty$ and the sequence of values of the delayed matrix exponential at the nodes is approximately represented by a geometric progression. There is a constant matrix C such that the exponential $e^{C\tau}$ is a "quotient", i.e.

$$\lim_{k\to\infty} e_\tau^{Bk\tau} \left(e_\tau^{B(k+1)\tau} \right)^{-1} = e^{-C\tau}, \tag{2.26}$$

where $(\cdot)^{-1}$ denotes the inverse matrix whose existence is assumed.

In the scalar case, the constant C can be expressed by the principal branch of the Lambert function, named after Johann Heinrich Lambert. He sent his paper [44] to Leonhard Euler, who in [20] introduced the Lambert function as the inverse function to the function

$$f(w) = we^w.$$

Thus, the Lambert function, usually denoted by $W = W(z)$, is defined implicitly by the equation

$$z = W(z)e^{W(z)}. \tag{2.27}$$

Such a function is multi-valued (except for the point $z = 0$). For real arguments $z = x$, $W(x)$ satisfying

$$x > -1/e \qquad W(x) > -1,$$

the equation (2.27) defines a single-valued function $W = W_0(x)$ called the principal branch of the Lambert function $W(z)$, which may be extended to an analytic function in the complex plane except for the real numbers $x < -1/\mathrm{e}$ since the point $-1/\mathrm{e}$ is a branch point of Lambert function. Of all Lambert function branches, the principal branch assumes the greatest real part values. We refer to [11] for a survey of the basic properties of Lambert function.

The Maclaurin expansion of $W_0(x)$ about the point $x = 0$ can be found easily and is given by the series

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n, \tag{2.28}$$

having the radius of convergence $r = 1/e$.

In [59] the following Theorem is proved.

Theorem 4. *Let λ_j, $j = 1,\dots,n$ be the eigenvalues of a matrix A and let its Jordan canonical form be*

$$\mathrm{diag}(\lambda_1,\dots,\lambda_n) = D^{-1}BD, \tag{2.29}$$

where D is a regular matrix. If $|\lambda_j|\tau\mathrm{e} < 1$, $j = 1,\dots,n$, then the sequence

$$\mathrm{e}_\tau^{B(k+1)\tau}(\mathrm{e}_\tau^{Bk\tau})^{-1}, \text{ for } k \to \infty$$

is convergent and (2.26) *holds where*

$$\mathrm{e}^{C\tau} = D\exp\left(\mathrm{diag}(W_0(\lambda_1\tau),\dots,W_0(\lambda_n,\tau)\right)D^{-1}. \tag{2.30}$$

This theorem was proved using Abel's extension (see [1]) of the well-known binomial theorem by comparing the Maclaurin expansions for both terms.

In [a6] the theorem is generalized, since the diagonal shape of the Jordan canonical form is not required. Moreover, the asymptotic equation for the sequence $\{e_\tau^{Bk\tau}\}$ is described in the following theorem.

Theorem 5. *Let $\tau > 0$ and let an $n\times n$ constant matrix $B \not\equiv \Theta$ be given. If the eigenvalues λ_i, $i = 1,\dots,n$ of the matrix B satisfy the inequality $|\lambda_i|\tau\mathrm{e} < 1$, then*

$$\lim_{k\to\infty} e_\tau^{Bk\tau}\exp(-kW_0(B\tau)) = B\tau\left(W_0(B\tau)(I+W_0(B\tau))\right)^{-1}. \tag{2.31}$$

For the asymptotic properties of the exponential $\exp(\lambda x)$, the real part of the complex number λ is fundamental. The set of the complex numbers $z = x+iy$ for which the real part of the Lambert function equals zero is defined in the parametric form

$$x = -v\sin v, \qquad y = \quad v\cos v.$$

This parametric specification follows from fact that $\Re W(x+iy) = u = 0$. Analyzing the part of this curve corresponding to the principal branch $W_0(x+iy)$, we conclude that it is a simple closed curve for the admissible range $v \in [-\pi/2,\pi/2]$. This curve is depicted in Figure 2.1. The real part of the principal branch of the Lambert function is negative for

$$|z| < -\arctan\left(\frac{\mathrm{Re}\,z}{|\mathrm{Im}\,z|}\right). \tag{2.32}$$

This domain is bounded by the above curve (see Figure 2.1). Note that a Lambert function W cannot be expressed in terms of elementary functions. For more details, see [11].
Let $F(k) = \{f_{ij}(k)\}_{i,j=1}^n$ and $G = \{g_{ij(k)}\}_{i,j=1}^n$ be matrices defined for all sufficiently large k. We say that

$$F(k) \asymp G(k),\ k\to\infty \tag{2.33}$$

if

$$f_{ij}(k) = g_{ij}(k)(1+o(1)),\ k\to\infty, \tag{2.34}$$

where $o(1)$ is the Landau order symbol "small" o.

Remark 1. *Let all assumptions of Theorem 5 be valid. From formula* (2.31)*, we get the asymptotic relation*

$$e_\tau^{Bk\tau} \asymp B\tau\exp(kW_0(B\tau))(W_0(B\tau)(I+W_0(B\tau)))^{-1},\ k\to\infty. \tag{2.35}$$

This formula can be useful, e.g., in the investigation of the asymptotic behaviour of the solutions to the problem at nodes $t = k\tau$.

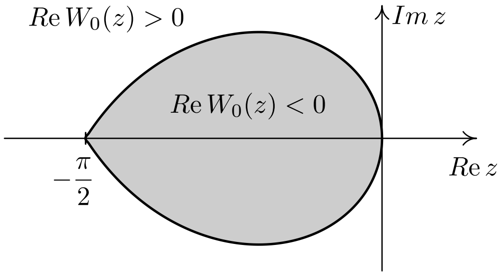

Figure 2.1: The curve $\mathrm{Re}\,W_0(z) = 0$

Some consequences

Recall that the spectral radius $\rho(\cdot)$ is the maximal absolute value of the spectrum of a given matrix and the spectral norm $\|\mathscr{A}\|_\rho = \left(\rho(\mathscr{A}\mathscr{A}^T)\right)^{1/2})$ is defined for a matrix $\mathscr{A}$. The following theorem describes the behaviour of the sequence of values of delayed exponential $e_\tau^{Bk\tau}$ for (discrete) $k \to \infty$ and of delayed exponential e_τ^{Bt} for (continuous) $t \to \infty$, see [a6].

Theorem 6. *Let $\tau > 0$ and let an $n \times n$ constant matrix $B \not\equiv \Theta$ be given. Assume that the eigenvalues λ_i, $i = 1, \ldots, n$ of the matrix B satisfy the inequality $\tau|\lambda_i| < 1/e$, $i = 1, \ldots, n$. The following three statements hold:*

(i) *If all the eigenvalues λ_i, $i = 1, \ldots, n$ satisfy*

$$\tau|\lambda_i| < -\arctan\left(\frac{\mathrm{Re}\,\lambda_i}{|\mathrm{Im}\,\lambda_i|}\right), \tag{2.36}$$

then

$$\lim_{k\to\infty} \rho\left(e_\tau^{Bk\tau}\right) = 0. \tag{2.37}$$

(ii) *If there exists, an index $i_0 \in \{1, \ldots, n\}$ such that*

$$\tau|\lambda_{i_0}| > -\arctan\left(\frac{\mathrm{Re}\,\lambda_{i_0}}{|\mathrm{Im}\,\lambda_{i_0}|}\right), \tag{2.38}$$

then

$$\limsup_{k\to\infty} \left\|e_\tau^{Bk\tau}\right\|_\rho = \infty. \tag{2.39}$$

(iii) *If all the eigenvalues λ_i, $i = 1, \ldots, n$ are real and satisfy*

$$\tau|\lambda_i| > -\arctan\left(\frac{\mathrm{Re}\,\lambda_i}{|\mathrm{Im}\,\lambda_i|}\right), \tag{2.40}$$

then

$$\lim_{t\to\infty} \left\|e_\tau^{Bt}\right\|_\rho = \infty. \tag{2.41}$$

Figure 2.2 details the eigenvalue domain for each case considered.

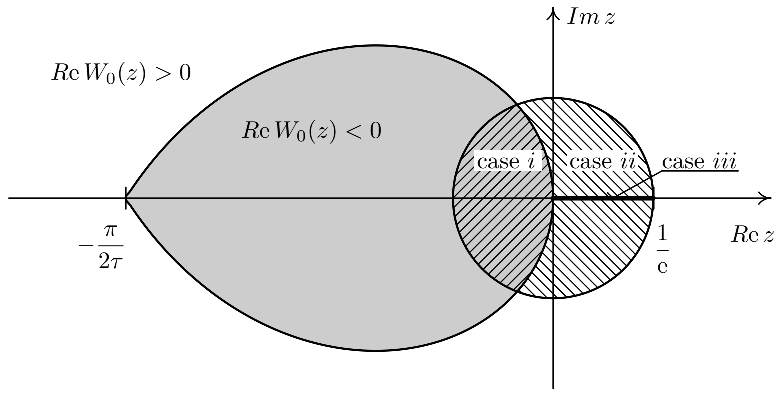

Figure 2.2: Detailed eigenvalue domains

Equation of a showering person System (2.3) often describe mathematical models of real-world phenomena. The solution of the initial problem (2.3), (2.4) is given by formula (2.5). We investigate the long-time behaviour of the solutions generated by constant initial functions, i.e., assume $\varphi(t) \equiv C_\varphi$ for every fixed $t \in [-\tau, 0]$ and $C_\varphi \in \mathbb{R}^n$. Then,

$$\dot{\varphi}(t) \equiv \theta,\ t \in [-\tau, 0],$$

where θ is the null vector. Formula (2.5) becomes

$$y(t) = e_\tau^{Bt}\varphi(-\tau) = e_\tau^{Bt} C_\varphi. \tag{2.42}$$

If all assumptions of Theorem 5 hold, by formula (2.35), we get the asymptotic expression for (2.42) at nodes $t = k\tau$ as $k \to \infty$

$$y(k\tau) = e_\tau^{Bk\tau} C_\varphi \asymp B\tau \exp(kW_0(B\tau))(W_0(B\tau)(I + W_0(B\tau)))^{-1} C_\varphi. \tag{2.43}$$

The above sentence will be used for a model that generalizes the description of the water temperature controlled by the person in the shower, i.e., the generalization of the equation (1.1). Setting $y(t) = T(t) - T_d$ in (1.1), we get

$$\dot{y}(t) = -\gamma y(t - \tau),\ \ t \in [0, \infty). \tag{2.44}$$

Assuming that the water temperature before regulation is constant, i.e., the initial condition is given by the equation

$$y(t) = y_0,\ \ t \in [-\tau, 0], \tag{2.45}$$

the solution of (2.44), (2.45) is

$$y(t) = e_\tau^{-\gamma t} y_0,\ \ t \in [-\tau, \infty)$$

and, if $\gamma\tau\mathrm{e} < 1$, then, by (2.33)–(2.35) and (2.43),

$$y(k\tau) = e_\tau^{-\gamma k\tau} y_0 = -\gamma\tau \exp(kW_0(-\gamma\tau)) \frac{y_0(1 + o(1))}{W_0(-\gamma\tau)(1 + W_0(-\gamma\tau))}$$

as $k \to \infty$. By (2.27), the last formula can be simplified to

$$y(k\tau) = \frac{y_0(1+o(1))}{1+W_0(-\gamma\tau)} \mathrm{e}^{(1+k)W_0(-\gamma\tau)}, \quad k \to \infty.$$

Since, by (2.28),

$$W_0(-\gamma\tau) = -\gamma\tau - (\gamma\tau)^2 - \frac{3}{2}(\gamma\tau)^3 + \cdots,$$

we have $y(k\tau) > 0$ and $\lim_{k\to\infty} y(k\tau) = 0$. This means that the regulated temperature $T(k\tau)$ will tend to the desired value T_d as $k \to \infty$.

The above example can be generalized, e.g., for the two showering persons. Suppose that hot and cold water is supplied in two separate pipes to a bathroom with two showers. Inside the bathroom, each pipe branches into two pipes leading to the shower mixers. A person taking a shower regulates the water temperature flowing from the mixer to the sprinkler. Due to the changes in the water pressure caused by water being regulated by two persons simultaneously, there is a mutual dependence between the temperatures T_1 and T_2 of the water flowing from mixer one to sprinkler one and from mixer two to sprinkler two, respectively. Then, a simple model modeling the behaviour of two showering persons is

$$\dot{T}_1(t) = -\gamma_{11}[T_1(t-\tau) - T_{d1}] + \gamma_{12}[T_2(t-\tau) - T_{d2}], \tag{2.46}$$

$$\dot{T}_2(t) = \gamma_{21}[T_1(t-\tau) - T_{d1}] - \gamma_{22}[T_2(t-\tau) - T_{d2}] \tag{2.47}$$

where $\gamma_{ij} > 0$, $i, j = 1,2$ and T_{di}, $i = 1,2$ are the desired water temperatures agreeable for two showering person, respectively. Substituting $y_i(t) = T_i(t) - T_{di}$ in (2.46), (2.47), we get

$$\dot{y}_1(t) = -\gamma_{11}y_1(t-\tau) + \gamma_{12}y_2(t-\tau), \tag{2.48}$$

$$\dot{y}_2(t) = \gamma_{21}y_1(t-\tau) - \gamma_{22}y_2(t-\tau). \tag{2.49}$$

Assuming that the water temperature before regulation is constant, i.e., the initial condition is given by the equation

$$y_1(t) = y_2(t) = y_0, \ t \in [-\tau, 0], \tag{2.50}$$

the solution of (2.48)–(2.50) is

$$y(t) = (y_1(t), y_2(t))^T = e_\tau^{-\Gamma t} y^0, \ t \in [-\tau, \infty) \tag{2.51}$$

where $y^0 = (y_0, y_0)^T$ and

$$\Gamma = \begin{pmatrix} -\gamma_{11} & \gamma_{12} \\ \gamma_{21} & -\gamma_{22} \end{pmatrix}.$$

Let the eigenvalues

$$\lambda_i = \frac{1}{2}\left[-(\gamma_{11}+\gamma_{22}) + (-1)^i\sqrt{(\gamma_{11}-\gamma_{22})^2 + 4\gamma_{12}\gamma_{21}}\right], \ i = 1,2$$

of the matrix Γ satisfy $|\lambda_i|\tau\mathrm{e} < 1$, $i = 1,2$. Then, by formula (2.43), at nodes $t = k\tau$, the solution (2.51) has the asymptotic behavior

$$y(k\tau) \asymp \Gamma\tau \exp(kW_0(\Gamma\tau))(W_0(\Gamma\tau)(I + W_0(\Gamma\tau)))^{-1} y^0$$

as $k \to \infty$.

Delayed matrix sine and cosine

To describe the asymptotic properties of other delayed matrix functions, we use the equations (2.13). Our recent result has been proved in [a13].

Theorem 7. *Let λ_j, $j = 1, \ldots, n$ be the eigenvalues of a matrix A and let its Jordan canonical form be given by* (2.29). *If $|\lambda_j| < 1/(\mathrm{e}\tau)$, $j = 1, \ldots, n$ and there exists at least one $j = j^* \in \{1, \ldots, n\}$ such that $\lambda_{j^*} \neq 0$, then*

$$\limsup_{t \to \infty} \| \operatorname{Cos}_\tau At \|_\rho = \infty$$

and

$$\limsup_{t \to \infty} \| \operatorname{Sin}_\tau At \|_\rho = \infty.$$

Another direction of research is to show that the condition about eigenvalues of matrix $|\lambda_j| < 1/(\mathrm{e}\tau)$, $j = 1, \ldots, n$ is not necessary and can be weakened.

Chapter 3

Topological method for functional differential equations

The oscillation of solutions and existence of positive solutions are the essential problems encountered when studying the asymptotic properties of differential equations. Many criteria for the existence of positive solutions may be derived applying the retract or the Lyapunoff method to a system of differential equations with unbounded delay but with finite memory in the sense given in [43].

The results for the existence of a solution in a predetermined set are given in [a3]. To arrive at these results, two principles were used. First the retract principle which is often used in the theory of ordinary differential equations (see e.g. [33]) and goes back to Ważewski [64]. For RFDE's with bounded retardation, this principle was modified, e.g., by Rybakowski [53]. Here, Rybakowski's modified result was used, which concerns the existence of at least one curve in a given family of curves, with the graph lying in an open set. Second, an inverse principle was used, which originates in the theory of Lyapunov stability, and for retarded functional differential equations, it was developed by Razumikhin (e.g. [52]).

In both principles the concept system of curves is used.

3.1 The system of curves

If a set $A \subset \mathbb{R} \times \mathbb{R}^n$ is given, then $\operatorname{int} A$, $\overline{A}$ and ∂A denote, as usual, the interior, the closure, and the boundary of A, respectively.

Definition 1. *Let Λ be a topological space, let a subset $\tilde{\Omega} \subset \mathbb{R} \times \Lambda$ be open in $\mathbb{R} \times \Lambda$, and let x be a mapping associating with every $(\delta, \lambda) \in \tilde{\Omega}$ a function $x(\delta, \lambda) : D_{\delta,\lambda} \to \mathbb{R}^n$ where $D_{\delta,\lambda}$ is an interval in $\mathbb{R}$. Assume 1 through 3:*

1) $\delta \in D_{\delta,\lambda}$.

2) *If $t \in \operatorname{int} D_{\delta,\lambda}$, then there is an open neighborhood $\mathscr{O}(\delta, \lambda)$ of (δ, λ) in $\tilde{\Omega}$ such that $t \in D_{\delta',\lambda'}$ holds for all $(\delta', \lambda') \in \mathscr{O}(\delta, \lambda)$.*

3) If (δ',λ'), $(\delta,\lambda)\in\tilde{\Omega}$, *and* $t'\in D_{\delta',\lambda'}$, $t\in D_{\delta,\lambda}$, *then*

$$\lim_{(\delta',\lambda',t')\to(\delta,\lambda,t)} x(\delta',\lambda')(t') = x(\delta,\lambda)(t).$$

If all these conditions are satisfied, then $(\Lambda,\tilde{\Omega},x)$ *is called a* system of curves *in* $\mathbb{R}^n$.

Studying the proof of Theorem 2.1, in [53, p.119], we formulated in [a3] two results **Lemma 1 (Retract Principle)** and **Lemma 2 (Lyapunov Principle)** suitable for our applications to retarded functional-differential equations with unbounded delay.

3.2 The retract method and the Lyapunov method for p-RFDE's

Let us recall the notion of a p function.

Definition 2. *([43, p. 8]) The function* $p\in C[\mathbb{R}\times[-1,0],\mathbb{R}]$ *is called a* p-function *if it has the following properties:*

(i) $p(t,0)=t$*;*

(ii) $p(t,-1)$ *is a nondecreasing function of* t*;*

(iii) there exists a $\sigma\geq-\infty$ *such that* $p(t,\vartheta)$ *is an increasing function for* ϑ *for each* $t\in(\sigma,\infty)$.

Definition 3. *([43, p. 8]) Let* $t_0\in\mathbb{R}$, $A>0$ *and* $y\in C([p(t_0,-1),t_0+A),\mathbb{R}^n)$. *For any* $t\in[t_0,t_0+A)$, *we define function* y_t *by* $y_t(\vartheta)=y(p(t,\vartheta))$, $-1\leq\vartheta\leq0$ *and we write* $y_t\in\mathscr{C}\equiv C[[-1,0],\mathbb{R}^n]$.

We investigate the system

$$\dot{y}(t)=f(t,y_t), \tag{3.1}$$

with a functional $f\in C[[t_0,t_0+A)\times\mathscr{C},\mathbb{R}^n]$, called a system of *$p$-type retarded functional differential equations* (p-RFDE's). The function $y\in C([p(t_0,-1),t_0+A),\mathbb{R}^n)\cap C^1([t_0,t_0+A),\mathbb{R}^n)$ satisfying (3.1) on $[t_0,t_0+A)$ is called a *solution of this system of* p-RFDE's *on* $[[p(t_0,-1),t_0+A)$.

Remark 2. *System (3.1) with* y_t *defined in accordance with Definition 3 is called a system with unbounded delay and with finite memory. Note that the frequently used symbol "* y_t *" (e.g., in accordance with [34, p. 38],* $y_t(s)=y(t+s)$, *where* $-\tau\leq s\leq0$, $\tau>0$, $\tau=\mathrm{const}$*) for an equation with bounded delay is a partial case of the above definition of* y_t. *Indeed, in this case, we can put* $p(t,\vartheta)\equiv t+\tau\vartheta$.

Let Ω be an open subset of $\mathbb{R}\times\mathscr{C}$ and the function $f\in C(\Omega,\mathbb{R}^n)$. If $(t_0,\phi)\in\Omega$, *there exists a solution* $y=y(t_0,\phi)$ of the system of p-RFDE's (3.1) through (t_0,ϕ) (see [43, p. 25]). Moreover, this *solution is unique* if $f(t,\phi)$ is locally Lipschitzian with respect to ϕ ([43, p. 30]) and is *continuable* in the usual sense of extended existence if f is quasibounded ([43, p. 41]). Suppose that the solution $y=y(t_0,\phi)$ of p-RFDE's (3.1) through $(t_0, \phi) \in \Omega$, defined on $[t_0, A]$, is unique. Then, the property of the *continuous dependence* holds, too (see [43, p. 33]), i.e., for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $(s, \psi) \in \Omega$, $|s - t_0| < \delta$ and $\|\psi - \phi\| < \delta$ implies

$$\|y_t(s, \psi) - y_t(t_0, \phi)\| < \varepsilon, \text{ for all } t \in [\zeta, A]$$

where $y(s, \psi)$ is the solution of the system of p- RFDE's (3.1) through (s, ψ), $\zeta = \max\{s, t_0\}$, and $\|\cdot\|$ is the supremum norm in $\mathbb{R}^n$. Note that the system of solutions to (3.1) under the above assumptions is a system of curves in the sense of definition 1 and this fact can be adapted easily for the case of Ω having the form $\Omega = [p^*, \infty) \times \mathscr{C}$ where $p^* \in \mathbb{R}$ and the cross-section $\{(\tilde{t}, \varphi) \in \Omega\}$ being an open set for every $\tilde{t} \in [p^*, \infty)$.

Let l_i, m_j, $i = 1, \dots p$, $j = 1, \dots s$, $p + s > 0$ be real-valued C^1-functions defined on $\mathbb{R} \times \mathbb{R}^n$. The set

$$\tilde{\omega} = \{(t, y) \in [p^*, \infty) \times \mathbb{R}^n, \, l_i(t, y) < 0, \, m_j(t, y) < 0, \text{ for all } i, j \}$$

will be called a *polyfacial set*.

Definition 4. *A polyfacial set $\tilde{\omega}$ is called* regular with respect to equation *(3.1) if* α), β), γ) *below hold:*

α) *If $(t, \phi_t) \in \mathbb{R} \times \mathscr{C}$ and if $(p(t, \vartheta), \phi_t(\vartheta)) \in \tilde{\omega}$ for all $\vartheta \in [-1, 0)$, then $(t, \phi_t) \in \tilde{\Omega}$.*

β) *For all $i = 1, \dots, p$, all $(t, y) \in \partial\tilde{\omega}$ for which $l_i(t, y) = 0$ and for all $\phi_t \in \mathscr{C}$ for which $\phi_t(0) = y$ and $(p(t, \vartheta), \phi_t(\vartheta)) \in \tilde{\omega}$ for all $\vartheta \in [-1, 0)$, it follows that*

$$Dl_i(t, y) \equiv \sum_{r=1}^{n} \frac{\partial l_i}{\partial y_r}(t, y) \cdot f_r(t, \phi_t) + \frac{\partial l_i}{\partial t}(t, y) > 0.$$

γ) *For all $j = 1, \dots, s$, all $(t, y) \in \partial\tilde{\omega}$ for which $m_j(t, y) = 0$ and for all $\phi_t \in \mathscr{C}$ for which $\phi_t(0) = y$ and $(p(t, \vartheta), \phi_t(\vartheta)) \in \tilde{\omega}$ for all $\vartheta \in [-1, 0)$, it follows that*

$$Dm_j(t, y) \equiv \sum_{r=1}^{n} \frac{\partial m_j}{\partial y_r}(t, y) \cdot f_r(t, \phi_t) + \frac{\partial m_j}{\partial t}(t, y) < 0.$$

Lemma 1 (Retract Method)**.** *Let $p > 0$. Let $\tilde{\omega}$ be a nonempty polyfacial set, regular with respect to equation* (3.1)*, let the function $f \in C(\tilde{\Omega}, \mathbb{R}^n)$ be locally Lipschitzian with respect to the second argument, and*

$$W = \{(t, y) \in \partial\tilde{\omega} : \, m_j(t, y) < 0, \, j = 1, \dots, s\}. \tag{3.2}$$

Let Z be a subset of $\tilde{\omega} \cup W$ and let the mapping $q : B = \overline{Z} \cap (Z \cup W) \to \mathscr{C}$ be continuous and such that if $z = (\delta, y) \in B$, then $(\delta, q(z)) \in \tilde{\Omega}$, and :

1) *If $z \in Z \cap \tilde{\omega}$, then $(p(\delta, \vartheta), q(z)(p(\delta, \vartheta))) \in \tilde{\omega}$ for $\vartheta \in [-1, 0]$,*

2) *If $z \in W \cap B$, then $(\delta, q(z)(\delta)) = z$ and $(p(\delta, \vartheta), q(z)(p(\delta, \vartheta))) \in \tilde{\omega}$ for $\vartheta \in [-1, 0)$.*

Let, moreover, $Z \cap W$ be a retract of W, but not a retract of Z. Then, there exists a $z_0 = (\delta_0, y_0) \in Z \cap \tilde{\omega}$ such that $(t, y(\delta_0, q(z_0))(t)) \in \tilde{\omega}$ for every $t \in D_{\delta_0, q(z_0)}$.

Lemma 2 (Lyapunov Method)**.** *Let $p = 0$. Let $\tilde{\omega}$ be a nonempty polyfacial set, regular with respect to equation* (3.1) *and let the function $f \in C(\tilde{\Omega}, \mathbb{R}^n)$ be locally Lipschitzian with respect to the second argument. Let a mapping $q : B \to \mathscr{C}$, $B = \overline{\tilde{\omega}} \cap \{(t^*, y), t^* \in \mathbb{R}, t^* = \text{const}, y \in \mathbb{R}^n\}$ be continuous and such that if $z = (t^*, y) \in B$, then $(t^*, q(z)) \in \tilde{\Omega}$, and:*

1) *If $z \in \tilde{\omega}$, then $(p(t^*, \vartheta), q(z)(p(t^*, \vartheta))) \in \tilde{\omega}$ for $\vartheta \in [-1, 0]$.*

2) *If $z \in \partial\tilde{\omega}$, then $(t^*, q(z)(t^*)) = z$ and $(p(t^*, \vartheta), q(z)(p(t^*, \vartheta))) \in \tilde{\omega}$ for $\vartheta \in [-1, 0)$.*

Then, for every $z_0 = (t^*, y_0) \in B \cap \tilde{\omega}$ and every $t \in D_{t^*, q(z_0)}$:

$$(t, y(t^*, q(z_0))(t)) \in \tilde{\omega}. \tag{3.3}$$

3.3 Retract principle for neutral functional differential equations

This part discusses a problem of extending the retract principle to neutral differential equations. A common basis with the previous results is the reuse of the retract principle for a system of curves. The problem given by the fact that the value of the derivative of a solution depends on the values of the derivative of this solution in the past is solved by modifying the notion of a regular polyfacial set to the notion of a regular polyfacial set with respect to an equation and subsidiary inequalities. Particular problems are solved in [a7] and [a9].

Neutral functional differential equations We consider a neutral functional differential system of the form

$$\dot{y}(t) = f(t, y_t, \dot{y}_t) \tag{3.4}$$

where the symbol $\dot{y}$ (sometimes we use y') stands for the derivative (considered, if necessary, as one-sided). First, we give the necessary auxiliary background for this equation.

Let $\mathscr{C}$ be the set of all continuous functions $\varphi\colon [-h, 0] \to \mathbb{R}^n$ and $\mathscr{C}^1$ be the set of all continuously differentiable functions $\varphi\colon [-h, 0] \to \mathbb{R}^n$.

We assume $t \geq t_0$, $y_t(\theta) = y(t + \theta)$, $\theta \in [-h, 0]$ where $h > 0$ is a constant and $f\colon E_h \to \mathbb{R}^n$ with $E_h := [t_0, \infty) \times \mathscr{C} \times \mathscr{C}$. We pose an initial problem for (3.4):

$$y_{t_0} = \varphi,\ \dot{y}_{t_0} = \dot{\varphi} \tag{3.5}$$

where $\varphi \in \mathscr{C}^1$. The norm of $\varphi \in \mathscr{C}$ is defined as $\|\varphi\|_h := \max_{\theta \in [-h, 0]} \|\varphi(\theta)\|$ and, if $\varphi \in \mathscr{C}^1$, then

$$\|\varphi\|_h := \max_{\theta \in [-h, 0]} \|\varphi(\theta)\| + \max_{\theta \in [-h, 0]} \|\dot{\varphi}(\theta)\|.$$

A function $y\colon [t_0 - h, t_\varphi) \to \mathbb{R}^n$, $t_\varphi \in (t_0, \infty]$, is a solution of (3.4), (3.5) if $y_{t_0} = \varphi$, $\dot{y}_{t_0} = \dot{\varphi}$ and (3.4) is satisfied for any $t \in [t_0, t_\varphi)$. The following result is taken from a book [41, p. 107] by Kolmanovskii and Myshkis.

Theorem 8. *Let $f\colon E_h \to \mathbb{R}^n$ be a continuous functional satisfying, in some neighborhood of any point of E_h, the condition*

$$\|f(t,\psi_1,\chi_1) - f(t,\psi_2,\chi_2)\| \le L\|\psi_1 - \psi_2\|_h + \ell\|\chi_1 - \chi_2\|_h \tag{3.6}$$

with constants $L \in [0,\infty)$, $\ell \in [0,1)$. In addition, assume that $\varphi \in \mathscr{C}^1$ and that the sewing condition

$$\dot{\varphi}(0) = f(t_0,\varphi,\dot{\varphi}) \tag{3.7}$$

is fulfilled. Then, there exists a $t_\varphi \in (t_0,\infty]$ such that:

a) *There exists a solution y of* (3.4), (3.5) *on $[t_0 - h, t_\varphi)$.*

b) *On any interval $[t_0 - h, t_1] \subset [t_0 - h, t_\varphi)$, $t_1 > t_0$, this solution is unique.*

c) *If $t_\varphi < \infty$, then $\dot{y}(t)$ has not a finite limit as $t \to t_\varphi^-$.*

d) *The solution y and its derivative $\dot{y}$ depend continuously on f, φ.*

For a particular case of system (3.4) given by

$$f(t,y_t,\dot{y}_t) := \qquad f(t,y(t-h_1(t)),\dots,y(t-h_o(t)),\dot{y}(t-g_1(t)),\dots,\dot{y}(t-g_\ell(t)))$$

where the indices o and ℓ are non-negative, i.e.,

$$\dot{y}(t) = f(t,y(t-h_1(t)),\dots,y(t-h_o(t)),\dot{y}(t-g_1(t)),\dots,\dot{y}(t-g_\ell(t))), \tag{3.8}$$

a more general result can be proved easily by the method of steps (compare [41, pp. 111, 96] and [32]).

Theorem 9. *Let $f\colon [t_0,\infty) \times \mathbb{R}^{o+\ell} \to \mathbb{R}^n$, $h_i\colon [t_0,\infty) \to (0,h]$, $i = 1,\dots,o$ and $g_j\colon [t_0,\infty) \to (0,h]$, $j = 1,\dots,\ell$ be continuous functions. In addition, assume that $\varphi \in \mathscr{C}^1$ and that the sewing condition* (3.7)*, in the case considered, having the form*

$$\dot{\varphi}(0) = f(t_0,\varphi(-h_1(t_0)),\dots,\varphi(-h_o(t_0)),\dot{\varphi}(-g_1(t_0)),\dots,\dot{\varphi}(-g_\ell(t_0))) \tag{3.9}$$

is fulfilled. Then:

a) *There exists a solution y of* (3.4), (3.5) *on $[t_0 - h, \infty)$.*

b) *On any interval $[t_0 - h, t_1] \subset [t_0 - h, \infty)$, $t_1 > t_0$, this solution is unique.*

c) *The solution y and its derivative $\dot{y}$ depend continuously on f, φ.*

Polyfacial set Let $\Lambda = \mathscr{C}^1$, $\tilde{\Omega} \subset \{(t,\lambda) \in [t_0,\infty) \times \mathscr{C}^1$ such that $\dot{\lambda}(0) = f(t_0,\lambda,\dot{\lambda})\}$ and function f satisfy all the assumptions of Theorem 8. In this case, through each $(t_0,\lambda) \in \tilde{\Omega}$, there passes a unique solution $y(t_0,\lambda)$ of (3.4) defined on the maximal interval $[t_0 - h, a_\lambda)$. Let $D_{t_0,\lambda} = [t_0 - h, a_\lambda)$ where $a_\lambda > t_0$. Then, $(\Lambda, \tilde{\Omega}, y)$ is a system of curves in $\mathbb{R}^n$. In [a7] we define the polyfacial set as:

Definition 5. *Let p and s be nonnegative integers, $p+s > 0$, $t_* > t_0$, and let*

$$l_i\colon [t_0 - r, t_*) \to \mathbb{R} \times \mathbb{R}^n,\ i = 1,\dots,p,$$
$$m_j\colon [t_0 - r, t_*) \to \mathbb{R} \times \mathbb{R}^n,\ j = 1,\dots,s$$

be continuously differentiable functions. The set

$$\omega := \{(t,y) \in [t_0 - r, t_*) \times \mathbb{R}^n,\, l_i(t,y) < 0,\ m_j(t,y) < 0, \textit{ for all } i,j\ \}$$

is called a polyfacial set provided that the cross-section

$$\omega \cap \{(t,y)\colon t = t^*, y \in \mathbb{R}^n\}$$

is an open and simply connected set for every fixed $t^* \in [t_0 - r, t_*)$.

In order to prove the existence of a solution of (3.4) lying in a polyfacial set, ω should meet some additional requirements. Because of the neutrality of the equations, we need to be able to foresee the properties of the derivatives of solutions as described by the auxiliary inequalities.

Definition 6. *Let q be a nonnegative integer, $t_* > t_0$, and let*

$$c_k\colon [t_0 - r, t_*) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ k = 1,\dots,q,$$

be continuous functions. A polyfacial set ω is called regular with respect to equation (3.4) *and auxiliary inequalities*

$$c_k(t,y,x) \le 0,\ k = 1,\dots,q \tag{3.10}$$

if α) – δ) below hold:

α) If $(t,\phi) \in \mathbb{R} \times \mathscr{C}^1$ and $(t+\theta, \phi(\theta)) \in \omega$ for $\theta \in [-r,0)$, then $(t,\phi,\dot{\phi}) \in E_r$.

β) If $(t,\phi) \in \mathbb{R} \times \mathscr{C}^1$, $(t+\theta, \phi(\theta)) \in \omega$ for $\theta \in [-r,0)$ and, moreover,

$$c_k(t+\theta, \phi(\theta), \dot{\phi}(\theta)) \le 0,\ \theta \in [-r,0),\ k = 1,\dots,q, \tag{3.11}$$

then also

$$c_k(t+\theta, \phi(\theta), f(t,\phi,\dot{\phi})) \le 0,\ k = 1,\dots,q. \tag{3.12}$$

γ) For all $i = 1,\dots,p$, all $(t,y) \in \partial\omega$ for which $l_i(t,y) = 0$ and for all $\phi \in \mathscr{C}^1$ for which $\phi(0) = y$, $(t+\theta, \phi(\theta)) \in \omega$, $\theta \in [-r,0)$ and

$$c_k(t+\theta, \phi(\theta), \dot{\phi}(\theta)) \le 0,\ \theta \in [-r,0),\ k = 1,\dots,q, \tag{3.13}$$

it follows that:

$$Dl_i(t,y) \equiv \frac{\partial l_i}{\partial t}(t,y) + \sum_{r=1}^{n} \frac{\partial l_i}{\partial y_r}(t,y) \cdot f_r(t,\phi,\dot{\phi}) > 0.$$

δ) *For all $j = 1,\dots,s$, all $(t,y) \in \partial\omega$ for which $m_j(t,y) = 0$ and for all $\phi \in \mathscr{C}^1$ for which $\phi(0) = y$, $(t+\theta, \phi(\theta)) \in \omega$, $\theta \in [-r,0)$ and*

$$c_k(t+\theta, \phi(\theta), \dot{\phi}(\theta)) \le 0,\ \theta \in [-r,0),\ k = 1,\dots,q$$

for all $\theta \in [-1,0)$, it follows that:

$$Dm_j(t,y) \equiv \frac{\partial m_j}{\partial t}(t,y) + \sum_{r=1}^{n} \frac{\partial m_j}{\partial y_r}(t,y) \cdot f_r(t,\phi,\dot{\phi}) < 0.$$

If ω is a polyfacial set, then define the set W used in Lemma 1 (Retract Principle) (see [a3]) as

$$W := \{(t,y) \in \partial\omega :\ m_j(t,y) < 0,\ j = 1,\dots,s\}. \tag{3.14}$$

Moreover, we need to specify the properties of the mapping q in Lemma 1 (Retract Principle) (see [a3]). The following definition describes the admissible behavior of functions with respect to ω. A fixed set of functions generated by this mapping and satisfying the properties listed in the following definition is called a set of initial functions.

Definition 7 (Set of initial functions)**.** *Let Z be a subset of $\omega \cup W$ and let the mapping*

$$q\colon B \to \mathscr{C}^1,\ B := \overline{Z} \cap (Z \cup W)$$

be continuous. We assume that, if $z = (\delta, y) \in B$, then $(\delta, q(z)) \in \tilde{\Omega}$. If moreover,:

1) *For $z \in Z \cap \omega$, we have $(\delta+\theta, q(z)(\theta)) \in \omega$ for $\theta \in [-r,0]$.*

2) *For $z \in W \cap B$, we have $(\delta, q(z)(\delta)) = z$, and*

   either

   2*a*) $(\delta+\theta, q(z)(\theta)) \in \omega$ *for* $\theta \in [-r,0)$

   or

   2*b*) $(\delta+\theta, q(z)(\theta)) \in \overline{\omega}$ *for $\theta \in [-r,0)$ and, for all $\sigma > 0$, there is a $t = t(\sigma,z)$, $\delta < t \le \delta+\sigma$ such that t is within the domain of definition of solution $y(\delta, q(z))$ of* (3.4) *and $(t, y(\delta, q(z))(t)) \notin \overline{\omega}$,*

then such a set of functions is called a set of initial functions for (3.4) *with respect to ω and Z.*

Finally, we will formulate the below theorem as an application of Lemma 1 (Retract Principle) (see [a3]) to a system of neutral equations (3.4). Therefore, its proof is omitted.

Theorem 10. *Let ω be a nonempty polyfacial set, regular with respect to* (3.4) *and inequalities* (3.10)*. Assume that $\phi \in \mathscr{C}^1$ and that the sewing condition* (3.7) *is fulfilled. Let a fixed $t_* \in (t_0, \infty]$ exist such that:*

a) There exists a solution y of (3.4)*,* (3.5) *on $[t_0 - r, t_*)$.*

b) On any interval $[t_0 - r, t_1] \subset [t_0 - h, t_*)$, $t_1 > t_0$, this solution is unique.

c) If $t_* < \infty$, then $\dot{y}(t)$ has not a finite limit as $t \to t_*^-$.

d) The solution y and its derivative $\dot{y}$ depend continuously on f, ϕ.

Assume that q defines a set of initial functions for (3.4) *with respect to ω and Z and that the derivative of every solution $y(\delta, q(z))(t)$ of* (3.4) *defined by any $z = (\delta, x) \in B$ has a finite left limit at every point t provided that*

$$(t, y(\delta, q(z))(t)) \in \overline{\omega}.$$

Let, moreover, $Z \cap W$ be a retract of W, but not a retract of Z. Then, there exists at least one point $z_0 = (\delta_0, x_0) \in Z \cap \omega$ such that a solution $y(\delta_0, q(z_0))(t)$ exists on $[t_0 - r, t_*)$ and

$$(t, y(\delta_0, q(z_0))(t)) \in \omega$$

holds for all $t \in [t_0 - r, t_*)$.

Chapter 4

Applications to nonlinear systems

We start this chapter with a theorem describing the sufficient and necessary conditions for the existence of at least one solution to a given RDFE in a predetermined domain. $\mathbb{R}^n_{\geq 0}$ $\left(\mathbb{R}^n_{>0}\right)$ will denote the set of all component-wise nonnegative (positive) vectors v in $\mathbb{R}^n$, i.e., $v = (v_1, \ldots, v_n) \in \mathbb{R}^n_{\geq 0} \left(\mathbb{R}^n_{>0}\right)$ if and only if $v_i \geq 0$ $(v_i > 0)$ for $i = 1, \ldots, n$. For $u, v \in \mathbb{R}^n$, we write $u \leq v$ if $v - u \in \mathbb{R}^n_{\geq 0}$; $u \ll v$ if $v - u \in \mathbb{R}^n_{>0}$ and $u < v$ if $u \leq v$ and $u \neq v$.

Let p^*, t^* be constants satisfying $p^* = p(t^*, -1)$ for a given p -function. Define vector valued functions $\rho, \delta \in C([p^*, \infty), \mathbb{R}^n)$, satisfying $\rho \ll \delta$ on $[p^*, \infty)$, and continuously differentiable on $[t^*, \infty)$. Put $\Omega := [t^*, \infty) \times \mathscr{C}$ and

$$\omega := \{(t, y) : t \geq p^*, \rho(t) \ll y \ll \delta(t)\}.$$

Definition 8. *A* system of initial functions $\mathscr{S}_{\mathscr{E},\omega}$ *with respect to nonempty sets* $\mathscr{E}$ *and* ω *where* $\mathscr{E} \subset \overline{\omega}$ *is defined as a continuous mapping* $\nu : \mathscr{E} \to \mathscr{C}$ *such that* $a)$ *and* $b)$ *in the following text hold:*

a) *For each* $z = (t, y) \in \mathscr{E} \cap \operatorname{int} \omega$ *and* $\vartheta \in [-1, 0] : (t + \vartheta, \nu(z)(p(t, \vartheta))) \in \omega$.

b) *For each* $z = (t, y) \in \mathscr{E} \cap \partial\omega$ *and* $\vartheta \in [-1, 0) : (t + \vartheta, \nu(z)(p(t, \vartheta))) \in \omega$ *and, moreover,* $(t, \nu(z)(p(t, 0))) = z$.

We denote by $\mathscr{S}^1_{\mathscr{E},\omega}$ *a system of initial functions* $\mathscr{S}_{\mathscr{E},\omega}$ *if all functions* $\nu(z)$, $z = (t, y) \in \mathscr{E}$ *are continuously differentiable on* $[-1, 0)$.

The necessary and sufficient condition for the existence of a positive solution is given by the next theorem.

Theorem 11. *Let* $f \in C(\Omega, \mathbb{R}^n)$ *be locally Lipschitzian with respect to the second argument, quasibounded, and, moreover:*

(i) *For any* $i = 1, \ldots, p$ $(0 \leq p \leq n)$, $t \geq t^*$ *and* $\pi \in C([p(t, -1), t], \mathbb{R}^n)$ *such that* $(\theta, \pi(\theta)) \in \omega$ *for all* $\theta \in [p(t, -1), t)$, $(t, \pi(t)) \in \partial\omega$, *it follows* $(t, \pi_t) \in \Omega$,

$$\dot{\delta}_i(t) < f_i(t, \pi_t) \text{ when } \pi_i(t) = \delta_i(t) \tag{4.1}$$

and

$$\dot{\rho}_i(t) > f_i(t, \pi_t) \text{ when } \pi_i(t) = \rho_i(t). \tag{4.2}$$

(*ii*) *For any* $i = p+1,\dots,n$, $t \ge t^*$ *and* $\pi \in C([p(t,-1),t],\mathbb{R}^n)$ *such that* $(\theta,\pi(\theta)) \in \omega$ *for all* $\theta \in [p(t,-1),t)$, $(t,\pi(t)) \in \partial\omega$, *it follows* $(t,\pi_t) \in \Omega$,

$$\dot{\delta}_i(t) > f_i(t,\pi_t) \text{ when } \pi_i(t) = \delta_i(t) \tag{4.3}$$

and

$$\dot{\rho}_i(t) < f_i(t,\pi_t) \text{ when } \pi_i(t) = \rho_i(t). \tag{4.4}$$

Then, there exists an uncountable set $\mathscr{Y}$ *of solutions to the system* (3.1) *on the interval* $[p^*,\infty)$ *such that, for each* $y \in \mathscr{Y}$,

$$\rho(t) \ll y(t) \ll \delta(t),\ t \in [p^*,\infty). \tag{4.5}$$

The proof is given in [a3].

The above results on the existence of solutions of functional differential equations in a prescribed area were used in two main directions. The first is to prove the existence of a solution when modifying the first Ljapunov method. Here we assume a solution of the perturbed equation in the form of a power series.These coefficients are solutions to a system of linear equations. As the sequence thus obtained is not generally convergent, an asymptotic expansion of this solution is constructed. Then, using the retract method, we prove the existence of a sequence of solutions that are an asymptotic decomposition.

4.1 Asymptotic expansion of solution

The first method of Lyapunov is a well known technique used for studying the asymptotic behavior of ordinary differential equations in the form of a linear system with perturbation. This method uses the solution in the form of a convergent power series, for details see [10]. The results for equations in the implicit form [12] or for integro-differential equations [61] were derived by modifying the first method of Lyapunov. The existence of solutions with a certain asymptotic form were proved in the references cited using Ważewski's topological method. For analogous representations of solutions to a retarded differential equation, see [57], [a10]. The perturbation has a polynomial form in both cases. In this paper, we study an equation in the form

$$\dot{y}(t) = -a(t)y(t) + \sum_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t) \prod_{j=1}^{n} \big(y(\xi_j(t))\big)^{i_j} \tag{4.6}$$

where $\mathbf{i} = (i_1,\dots,i_n)$ is a multiindex, $i_j \ge 0$ are integers and $|\mathbf{i}| = \sum_{j=1}^{n} i_j$. The continuous functions $\xi_j(t)$ satisfy $\xi_j(t) \ge t_0$ for all $t \in [t_0,\infty)$ and the function $\xi(t)$, which is defined as $\xi(t) = \min_{1\le i\le n} \xi_i(t)$, is nondecreasing for $t \ge t_0$. Therefore, all asymptotic relations such as the Landau symbols o, O and the asymptotic equivalence $\sim$ will be considered for $t \to \infty$. This fact will not be pointed out in the sequel.

The function $a(t)$ satisfies the following conditions:

C1 $a(t)$ is continuous and positive on the interval $[t_0,\infty)$ and $1/a(t) = O(1)$,

C2 $(t-\xi(t))\widetilde{a}(t) = o(A(t))$ where the functions $A(t)$, $\widetilde{a}(t)$ are defined as

$$A(t) = \int_{t_0}^{t} a(u)du,\ \widetilde{a}(t) = \max_{u\le t}(a(u)).$$

Further conditions for the continuous functions $c_{\mathbf{i}}(t):\ [t_0,\infty)\to\mathbb{R}$ will be given later. In order to apply the first method of Lyapunov to the equation (4.6) we assume the solution in the form of a formal series

$$y(t,C) = \sum_{n=1}^{\infty} f_n(t)\varphi^n(t,C) \tag{4.7}$$

where $\varphi(t,C)$ is the solution of the homogeneous equation $\dot{y}(t) = -a(t)y(t)$ given by the formula

$$\varphi(t,C) = C\exp\Big(-A(t)\Big),$$

with $f_1(t)\equiv 1$, and the functions $f_k(t)$ for $k = 2,..,n$ being particular solutions to a certain system of auxiliary differential equations. Using Ważewski's topological method in the form used in [a3] for differential equations with unbounded delay and finite memory, we prove the existence of a solution

$$y_n(t,C) \sim f_n(t,C) = \sum_{k=1}^{n} f_k(t)\varphi^k(t,C).$$

To facilitate the specification of the coefficients of the power series which is the product of the power series raised to a power, we use the following notation: $\mathfrak{s} = (\mathfrak{s}_1,\ldots,\mathfrak{s}_n)$ is an ordered n-tuple of sequences $\mathfrak{s}_j = \left\{\mathfrak{s}_j^k\right\}_{k=1}^{\infty}$ of nonnegative integers with a finite sum $|\mathfrak{s}_j| = \sum\limits_{k=1}^{\infty}\mathfrak{s}_j^k$, denoting further

$$\mathfrak{s}! = \prod_{j=1}^{n}\prod_{k=1}^{\infty}\mathfrak{s}_j!^k,\quad \mathbf{i}(\mathfrak{s})! = \prod_{j=1}^{n}|\mathfrak{s}_i|!$$
$$V(\mathfrak{s}) = \sum_{j=1}^{n}\sum_{k=1}^{\infty}k\mathfrak{s}_j^k,\qquad \mathbf{i}(\mathfrak{s}) = (|\mathfrak{s}_1|,\ldots,|\mathfrak{s}_n|).$$

For any ordered n-tuple of sequences (of numbers or functions) $\mathscr{C} = (\mathbf{c}_1,\ldots,\mathbf{c}_n)$ where $\mathbf{c}_j = \{c_j^k\}_{k=1}^{\infty}$, we denote $\mathscr{C}^{\mathfrak{s}} = \prod\limits_{j=1}^{n}\prod\limits_{k=1}^{\infty}\left(c_j^k\right)^{\mathfrak{s}_j^k}$ where $\left(c_j^k\right)^0 = 1$ for every c_j^k. Then, it is possible to write

$$\prod_{j=1}^{n}\left(\sum_{k=1}^{\infty}c_j^k x^k\right)^{i_j} = \sum_{k=|\mathbf{i}|}^{\infty}x^k\sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}\frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathscr{C}^{\mathfrak{s}}\quad \text{where the symbol}\ \sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}$$

denotes the sum over all $\mathfrak{s}$ such that $V(\mathfrak{s}) = k,\ \mathbf{i}(\mathfrak{s}) = \mathbf{i}$ and, for the empty set of $\mathfrak{s}$, this symbol equals 0.

Substituting $y(t)$ into equation (4.6) and matching the coefficients at identical powers $\varphi^k(t,C)$, an auxiliary system is obtained of linear differential equations

$$\dot{f}_k(t) = (k-1)a(t)f_k(t) + \sum_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t) \sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i} \\ V(\mathfrak{s})=k}} \frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!} \mathscr{F}^{\mathfrak{s}} \tag{4.8}$$

where $\mathscr{F}(t)$ is the n-tuple of sequences $\{f_k(\xi_i(t))\exp(k(A(t)-A(\xi_i(t))))\}_{k=1}^{\infty}$, i.e.,

$$\mathscr{F}(t) = \big(\ldots\{f_k(\xi_i(t))\exp(k(A(t)-A(\xi_i(t))))\}_{k=1}^{\infty},\ldots\big).$$

$V(\mathfrak{s}) = k \geq 2$ and $|\mathbf{i}(\mathfrak{s})| \geq 2$ imply $\mathfrak{s}_k^l = 0$ for $l \geq k$. From this follows that the auxiliary system (4.8) is recurrent.

Theorem 12. *Let the functions $c_{\mathbf{i}}(t)$ for all positive τ and $\mathbf{i}$ satisfy*

$$\lim_{t\to\infty} c_{\mathbf{i}}(t)\exp(-\tau A(t)) = 0.$$

Then, there exists a sequence $\{f_k(t)\}_{k=1}^{\infty}$ of solutions of the auxiliary system (4.8)

$$f_k(t) = \int_t^{\infty} -a(s)\exp\left\{-\int_t^s (k-1)a(u)du\right\} \sum_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t) \sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i} \\ V(\mathfrak{s})=k}} \frac{|\mathbf{i}(\mathfrak{s})|!}{\mathbf{i}(\mathfrak{s})!} \mathscr{F}^{\mathfrak{s}} ds \tag{4.9}$$

such that $\lim_{t\to\infty} f_k(t)\exp(-\tau A(t)) = 0$ for all positive τ.

Let $\|.\|$ denote the maximum norm on $C^0[r^*, t_0]$.

The next 2 Theorems are proved in [a11]. The first is a consequence of Theorem 11.

Theorem 13. *Let the assumptions of Theorem 12 hold and let*

$$\lim_{t\to\infty} (f_{k+1}(t))^{-1}\exp(-\tau A(t)) = 0$$

where $\tau < 1$ is a constant. We denote $r^* = \min_{t \geq t_0}(\xi(t))$. Then, for every $C \neq 0$ and $\psi \in C^0[r^*, t_0], \|\psi\| \leq 1, \psi(t_0) = 0$, there exists a solution $y_C(t)$ of equation (4.6) such that

$$|y_C(t) - y_k(t)| \leq \sigma |f_{k+1}(t)\varphi^{k+1}(t,C)| \tag{4.10}$$

for $t \in [t_C, \infty)$ where the functions $f_k(t)$ are solutions (4.9) of system (4.8), $\sigma > 1$ is a constant. t_C is a function of the parameter C and of σ, k.

Theorem 14. *Let the assumptions of Theorem 12 be satisfied and let there exist a sequence $\{K_k\}_{k=1}^{\infty}$, $K_0 = 1$ such that the assumptions of Theorem 13 are satisfied for every K_k, i.e., $\lim_{t\to\infty}(f_{K_k}(t)^{-1})\exp(-\tau A(t)) = 0$. Then, there exists an asymptotic expansion of the solution $y_C(t)$ in the form*

$$y_C(t) \approx \sum_{k=1}^{\infty} F_k(t), \text{ where } F_k(t) = \sum_{l=K_{k-1}}^{K_k - 1} f_l(t)\varphi^l(t,C)$$

and $f_l(t)$ are solutions of (4.9).

These theorems are applicable to (1.2), (1.3) above and are used in the bellow illustrative example

4.2 Example

Consider the equation

$$\dot{y}(t) = -y\cos(ty(\xi(t))) = -y(t) + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{t^{2k}y(t)(y(\xi(t)))^{2k}}{(2k)!}$$

on the interval $[1,\infty)$ together with two various delays by choosing a pair of functions $\xi_1(t) = t - r$, $\xi_2(t) = t - \ln t$

- the first delay $r_1(t) = t - \xi_1(t) = r = r > 0$ is constant
- the second delay $r_2(t) = t - \xi_2(t) = t - (t - \ln t) = \ln t$ is unbounded.

In this case, with $a(t) = 1$, we put $y_0 = 1 \Rightarrow A(t) = t - 1$, $\mathbf{i} = (i_1, i_2)$,

$$c_{(1,2k)} = (-1)^{k+1}\frac{t^{2k}}{(2k)!} \quad \text{(for other multiindices } c_{\mathbf{i}} = 0\text{)}.$$

If we denote $\mathscr{F} = \left(\{f_i(t)\}_{i=1}^{\infty}, \left\{f_i(\xi(t))e^{i(t-\xi(t))}\right\}_{i=1}^{\infty}\right)$, the system of auxiliary differential equations has the form

$$\dot{f}_k(t) = (k-1)f_k(t) + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^{2i}}{(2i)!}\sum_{\substack{\mathbf{i}(\mathfrak{s})=(1,2i)\\ V(\mathfrak{s})=k}}\frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathscr{F}^{\mathfrak{s}}.$$

By induction, we may prove that, for any delay, $f_{2k} = 0$ holds. First, $f_2(t) = 0$ is due to $\dot{f}_2(t) = f_2(t)$.

From $\mathbf{i}(s_1, s_2) = (1, 2l)$, it follows $|s_1| = 1$ and $|s_2| = 2l$. By the induction assumption $f_{2j} = 0$, from $(\{f_i(t)\}_{i=1}^{\infty})^{s_1} \neq 0$, it follows that $V(s_1)$ is odd. From the requirement $\mathscr{F}^{(s_1,s_2)} \neq 0$ and the fact that $2k = V(s_1,s_2) = V(s_1) + V(s_2)$, we deduce that $V(s_2)$ is odd, too. Because

$$V(s_2) = \sum_{k=1}^{\infty}ks_2^k = \sum_{k=1}^{\infty}\underbrace{k+\cdots+k}_{s_2^k}$$

can be interpreted as the sum of $|V(s_2)| = 2l$ numbers. We see that at least one number is even (the sum of an even number of odd numbers is even) and every product on the right-hand side of the auxiliary equation contains zero multiplicands and, for the function f_{2k}, we have

$$\dot{f}_{2k}(t) = f_{2k}(t) \quad \Rightarrow \quad f_{2k} = 0.$$

The asymptotic form of the solutions f_{2k+1} depends on the delay $r_i(t) = t - \xi_i(t)$ but the property

$$f_{2k-1}(t) \sim f_{2k-1}(\xi(t)) \quad \text{holds for both} \quad r_i(t).$$

First, for $r_1(t)$, the solutions have the asymptotic form

$$f_{2k+1} = t^{2k}(c_{2k+1} + O(1/t)),$$

where $c_1 = 1$ and c_{2k+1} are given by the recurrent formula

$$c_{2k+1} = \frac{1}{2k}\sum_{i=1}^{\infty}\frac{(-1)^i}{(2i)!}\sum_{\substack{\mathbf{i}(\mathfrak{s})=(1,2i)\\ V(\mathfrak{s})=2k+1}}\mathscr{C}^{\mathfrak{s}_1}\mathscr{C}_r^{\mathfrak{s}_2},$$

where $\mathscr{C} = \{c_i\}_{i=1}^{\infty}$, $\mathscr{C}_r = \{c_i\exp(ir)\}_{i=1}^{\infty}$.

Second, we have the equation $\exp(k(A(t)-A(\xi(t)))) = \exp(k\ln t) = t^k$. It can be proved by induction that the solutions f_{2k+1} have the asymptotic form

$$f_{2k+1} = -t^{2k+p(k-1)}\left(d(k-1)/2k+O(1/t)\right).$$

The constants $d(k)$ and $p(k)$ satisfy the recurrence formulas

$$d(k) = -d(k-1)/2k, \qquad p(k) = p(k-1)+2k,$$

otherwise

$$d(k) = \frac{(-1)^{k-1}}{2^k(k-1)!} \quad \text{and} \quad p(k) = (k+2)(k-1).$$

By Theorem 14, we obtain the existence of a pair of asymptotic expansions $y_1(t)$, $y_2(t)$ of the solutions for two different delays $r_1(t)$, $r_2(t)$:

$$y_1(t) \approx \sum_{k=1}^{\infty} t^{2(k-1)}c_{2k-1}e^{(2k-1)t}C^{2k-1}$$
$$y_2(t) \approx \sum_{k=1}^{\infty}\frac{(-1)^{k-1}t^{(k+2)(k-1)}}{2^k(k-1)!}e^{(2k-1)t}C^{2k-1}.$$

Fore more details see [a11].

Remark 3. *This example shows the fundamental dependence of the asymptotic properties of the expansion on the magnitude of the delay. For a small delay ($r_1(t)\to 0$), the expansion $y_1(t)$ converges to the expansion of the solution of an ordinary equation*

$$\dot{y}(t) = -y\cos(ty(t)).$$

For a sufficiently large delay $r_2(t) = \ln(t)$, the expansion $y_2(t)$ is the same as for the equation

$$\dot{y}(t) = -y(t)+t^2y(t)y^2(t-\ln t)/2,$$

i.e., the expansions for the perturbation with an infinite sum and for the perturbation with only the first summand are identical.

4.3 Positive solutions of nonlinear system

The next theorem easily follows from the more general Theorem 11 by putting $\rho(t) = 0$ and applying the next definition.

Definition 9. *A functional* $g \in C(\Omega, \mathbb{R}^n)$ *is called* i-strongly decreasing *(or* i-strongly increasing*),* $i \in \{1,2,\dots,n\}$*, if, for each* $(t,\varphi) \in \Omega$ *and* $(t,\psi) \in \Omega$ *such that*

$$\varphi(p(t,\vartheta)) \ll \psi(p(t,\vartheta)), \text{ where } \vartheta \in [-1,0) \text{ and } \varphi_i(p(t,0) = \psi_i(p(t,0)),$$

the inequality

$$g_i(t,\varphi) > g_i(t,\psi) \quad (\text{or } g_i(t,\varphi) < g_i(t,\psi))$$

holds.

Let $k = (k_1,\dots,k_n) \gg 0$ be a constant vector. Let $\lambda(t) = (\lambda_1(t),\dots,\lambda_n(t))$ denote a vector, defined and locally integrable on $[p^*,\infty)$. Define an auxiliary operator

$$T(k,\lambda)(t) := k\mathrm{e}^{\int_{p^*}^t \lambda(s)\mathrm{d}s} = \left(k_1\mathrm{e}^{\int_{p^*}^t \lambda_1(s)\mathrm{d}s},\dots,k_n\mathrm{e}^{\int_{p^*}^t \lambda_n(s)\mathrm{d}s}\right). \tag{4.11}$$

Theorem 15. *Let* $f \in C(\Omega,\mathbb{R}^n)$ *be locally Lipschitzian with respect to the second argument, quasibounded and, moreover:*

(i) *f is i-strongly decreasing if $i = 1,\dots,p$ and i-strongly increasing if $i = p+1,\dots,n$.*

(ii) *$f_i(t,0) \le 0$ for $i = 1,\dots,p$ and $f_i(t,0) \ge 0$ for $i = p+1,\dots,n$ if $(t,0) \in \Omega$.*

Then, for the existence of a positive solution $y = y(t)$ on $[p^*,\infty)$ of the system of p-RFDE's (3.1) *(where $p^* = p(t^*,-1)$), the existence of a positive constant vector k and a locally integrable vector $\lambda : [p^*,\infty) \to \mathbb{R}^n$ continuous on $[p^*,t^*) \cup [t^*,\infty)$ satisfying the system of integral inequalities*

$$\mu_i\lambda_i(t) \ge \frac{\mu_i}{k_i}\mathrm{e}^{-\int_{p^*}^t \lambda_i(s)\mathrm{d}s} \cdot f_i(t, T(k,\lambda)_t), \quad i = 1,\dots,n \tag{4.12}$$

for $t \ge t^*$ with $\mu_i = -1$ for $i = 1,\dots,p$ and $\mu_i = 1$ for $i = p+1,\dots,n$ is necessary and sufficient.

For more details, see [a3]. Here and in [a4] applications to some examples may be found.

The next theorem only gives a sufficient condition for the existence of a positive solution. Let a constant vector $k \gg 0$ and a vector $\lambda(t)$ defined and locally integrable on $[p^*,\infty)$ are given. Then, an operator T is well defined by (4.11). Define for every $i \in \{1,2,\dots,n\}$ two types of subsets of the set $\mathscr{C}$:

$$\mathscr{T}^i := \Big\{\phi \in \mathscr{C} : 0 \ll \phi(\vartheta) \ll T(k,\lambda)_t(\vartheta),\ \vartheta \in [-1,0] \\ \text{except for } \phi_i(0) = k_i\mathrm{e}^{\int_{p^*}^t \lambda_i(s)\mathrm{d}s}\Big\}$$

and

$$\mathscr{T}_i := \{\phi \in \mathscr{C} : 0 \ll \phi(\vartheta) \ll T(k,\lambda)_t(\vartheta),\ \vartheta \in [-1,0] \text{ except for } \phi_i(0) = 0\}.$$

Theorem 16. *Let $f \in C(\Omega, \mathbb{R}^n)$ be locally Lipschitzian with respect to the second argument and quasibounded. Let a constant vector $k \gg 0$ and a vector $\lambda(t)$ defined and locally integrable on $[\,p^*, \infty)$ are given. If, moreover, inequalities*

$$\mu_i \lambda_i(t) > \frac{\mu_i}{k_i} \mathrm{e}^{-\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} \cdot f_i(t, \phi) \tag{4.13}$$

hold for every $i \in \{1, 2, \ldots, n\}$, $(t, \phi) \in [t^*, \infty) \times \mathscr{T}^{\,i}$ and inequalities

$$\mu_i f_i(t, \phi) > 0 \tag{4.14}$$

hold for every $i \in \{1, 2, \ldots, n\}$, $(t, \phi) \in [t^*, \infty) \times \mathscr{T}_i$, where $\mu_i = -1$ for $i = 1, \ldots, p$ and $\mu_i = 1$ for $i = p+1, \ldots, n$, then there exists a positive solution $y = y(t)$ on $[\,p^*, \infty)$ of the system of p-RFDE's (3.1).

These results together with Theorem 15 make it possible to formulate numerous consequences particularly for linear applications. For more details, see [a5].

Chapter 5

Positive solutions of a linear system

Many resulte may be derived for a linear system. Consider a system

$$\dot{y}(t) = L(t, y_t) + h(t), \tag{5.1}$$

where $h \in C([t^*, \infty), \mathbb{R}^n)$, $L \in C(\Omega \times \mathscr{C}, \mathbb{R}^n)$ is a linear functional and y_t is defined in accordance with Definition 3. Then, the bellow Theorems 15 and 16 give corresponding linenar analogies.

Theorem 17. *Let $L \in C(\Omega \times \mathscr{C}, \mathbb{R}^n)$ and, moreover,:*

(i) *For $i = 1, \ldots, p$, L is i-strongly decreasing and $L_i(t,0) + h_i(t) \leq 0$ if $(t,0) \in \Omega$ and*

(ii) *for $i = p+1, \ldots, n$, L is i-strongly increasing and $L_i(t,0) + h_i(t) \geq 0$ for if $(t,0) \in \Omega$.*

Then, the existence of a positive solution $y(t)$ on $[p^*, \infty)$ of the system of p-RFDE's (5.1) (where $p^* = p(t^*, -1)$) is equivalent with the existence of a positive constant vector k and a locally integrable vector $\lambda : [p^*, \infty) \to \mathbb{R}^n$ continuous on $[p^*, t^*) \cup [t^*, \infty)$ satisfying the system of integral inequalities

$$\mu_i \lambda_i(t) \geq \frac{\mu_i}{k_i} \cdot \mathrm{e}^{-\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} \cdot \left(L_i\left(t, T(k, \lambda)_t\right) + h_i(t)\right), \;\; i = 1, \ldots, n \tag{5.2}$$

for $t \geq t^*$ with $\mu_i = -1$ for $i = 1, \ldots, p$ and $\mu_i = 1$ for $i = p+1, \ldots, n$.

Theorem 18. *Let $L \in C(\Omega, \mathbb{R}^n)$ be linear. Let a constant vector $k \gg 0$ and a vector $\lambda(t)$ defined and locally integrable on $[p^*, \infty)$ are given. If, moreover the inequalities*

$$\mu_i \lambda_i(t) > \frac{\mu_i}{k_i} \cdot \mathrm{e}^{-\int_{p^*}^{t} \lambda_i(s)\mathrm{d}s} \cdot \left(L_i\left(t, \phi\right) + h_i(t)\right) \tag{5.3}$$

hold for every $i \in \{1, 2, \ldots, n\}$, $(t, \phi) \in [t^*, \infty) \times \mathscr{T}^i$ and inequalities

$$\mu_i \left(L_i(t, \phi) + h_i(t)\right) > 0 \tag{5.4}$$

hold for every $i \in \{1, 2, \ldots, n\}$, $(t, \phi) \in [t^*, \infty) \times \mathscr{T}_i$, where $\mu_i = -1$ for $i = 1, \ldots, p$ and $\mu_i = 1$ for $i = p+1, \ldots, n$. Then, there exists a positive solution $y = y(t)$ on $[p^*, \infty)$ of the system p-RFDE's (5.1).

The proofs of both theorems may be found in [58]. [a3] consideres the linear system

$$\dot{y} = A(t)y(t) + B(t)y(\tau(t)) \tag{5.5}$$

where $\tau : [t^*, \infty) \to [p^*, \infty)$ is a continuous nondecreasing function and $\tau(t) < t$. In this case, $p(t, \vartheta) = t + \vartheta \cdot (t - \tau(t))$ and $p^* = \tau(t^*)$. With respect to the $n \times n$ matrices $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$, we assume their continuity on $[t^*, \infty)$ and, moreover, the validity of the inequalities:

$$a_{ij}(t) \le 0, b_{ij}(t) \le 0 \quad \text{if } i = 1, \dots, p, \ j = 1, \dots, n, \ t \in [t^*, \infty), \tag{5.6}$$

$$a_{ij}(t) \ge 0, b_{ij}(t) \ge 0 \quad \text{if } i = p+1, \dots, n, \ j = 1, \dots, n, \ t \in [t^*, \infty), \tag{5.7}$$

$$\sum_{j=1}^{n} b_{ij}(t) \ne 0 \quad \text{for every } i = 1, \dots, n \text{ and } t \in [t^*, \infty). \tag{5.8}$$

Here the next Theorem is proved.

Theorem 19. *For the existence of a solution $y = y(t)$ of system (5.5), positive on $[p^*, \infty)$, the necessary and sufficient condition is that there exists a continuous vector $\lambda \in C([p^*, \infty), \mathbb{R}^n)$ such that $\lambda(t) \gg 0$ for $t \ge t^*$, satisfying for $i = 1, \dots, n$ the system of integral inequalities*

$$\lambda_i(t) \ge \mu_i \left(a_{ii}(t) + b_{ii}(t) \mathrm{e}^{-\mu_i \int_{\tau(t)}^{t} \lambda_i(s)\, \mathrm{d}s} \right) + \frac{\mu_i}{k_i} \cdot \sum_{j=1, j \ne i}^{n} k_j \mathrm{e}^{\int_{p^*}^{t} (\mu_j \lambda_j(s) - \mu_i \lambda_i(s))\, \mathrm{d}s} \left(a_{ij}(t) + b_{ij}(t) \mathrm{e}^{-\mu_j \int_{\tau(t)}^{t} \lambda_j(s)\, \mathrm{d}s} \right), \tag{5.9}$$

on $[t^*, \infty)$ with a positive constant vector k and with $\mu_i = -1$ for $i = 1, \dots, p$; $\mu_i = 1$ for $i = p+1, \dots, n$.

Remark 4. *Earlier, sufficient conditions for the existence of bounded solutions of systems and equations of the type (5.5) were given in [9, 8].*

[a3] establishes sufficient conditions for the existence of positive solutions to the following linear system

$$\dot{y}(t) = -A(t)y(p(t, -1)) \tag{5.10}$$

where $A = \{a_{ij}\}$ is an $n \times n$ matrix with entries continuous on $[t^*, \infty)$ satisfying $a_{ij}(t) \ge 0$, $i, j = 1, 2, \dots, n$ and $\sum\limits_{j=1}^{n} a_{ij}(t) > 0$ for every $i = 1, 2, \dots, n$,. The next theorem is proved in [a3].

Theorem 20. *For the existence of a positive solution $y = y(t)$ on $[p^*, \infty)$ (with $p^* = p(t^*, -1)$) of linear system (5.10) a sufficient condition is the existence of a positive constant vector k and a locally integrable function $\lambda^* : [p^*, \infty) \to \mathbb{R}$ continuous on $[p^*, t^*) \cup [t^*, \infty)$ and satisfying the integral inequality*

$$\lambda^*(t) \mathrm{e}^{-\int_{p(t,-1)}^{t} \lambda^*(q) \mathrm{d}q} \ge \max_{i=1,2,\dots,n} \left\{ \frac{1}{k_i} \sum_{j=1}^{n} k_j a_{ij}(t) \right\} \tag{5.11}$$

for $t \ge t^*$.

Inequality (5.11) offers numerous possibilities of finding particular sufficient conditions. We will consider two of them.

Theorem 21. *Let a continuous nondecreasing function $\lambda^* : [p^*,\infty) \to \mathbb{R}$ satisfy the inequality*

$$\lambda^*(t)\mathrm{e}^{-\lambda^*(t)\cdot[t-p(t,-1)]} \geq \max_{i=1,2,\dots,n} \left\{ \frac{1}{k_i} \sum_{j=1}^{n} k_j a_{ij}(t) \right\} \tag{5.12}$$

for $t \geq t^*$, where $k = (k_1, k_2, \dots, k_n)$ is a suitable positive constant vector. Then, linear system (5.10) *has a positive solution $y = y(t)$ on $[p^*,\infty)$ (with $p^* = p(t^*,-1)$).*

Theorem 22. *Let A be a constant matrix that cannot be decomposed. Then, for the existence of a positive solution $y = y(t)$ on $[p^*,\infty)$ (with $p^* = p(t^*,-1)$) of linear system* (5.10)*, it is sufficient if a locally integrable function $\lambda^* : [p^*,\infty) \to \mathbb{R}$, continuous on $[p^*,t^*) \cup [t^*,\infty)$, satisfies the inequality*

$$\lambda^*(t)\mathrm{e}^{-\int_{p(t,-1)}^{t} \lambda^*(q)\mathrm{d}q} \geq \rho(A) \tag{5.13}$$

for $t \geq t^*$, where $\rho(A)$ is the spectral radius of A.

5.1 A scalar equation with discrete delays

Let us study the conditions for the existence of a positive solution to a scalar equation with discrete delays

$$\dot{y}(t) = -\sum_{q=1}^{m} c_q(t) y(p(t,\vartheta_q)) \tag{5.14}$$

with $-1 = \vartheta_1 < \vartheta_2 < \cdots < \vartheta_m = 0$,functions c_q, $q = 1,2,\dots,m$, continuous on $[t^*,\infty)$ that are nonnegative if $q = 1,2,\dots,m-1$ and satisfy inequality $\sum_{q=1}^{m-1} c_q(t) > 0$ for $t \in [t^*,\infty)$.

Theorem 23. *For the existence of a positive solution $y = y(t)$ on $[p^*,\infty)$ (where $p^* = p(t^*,-1)$) of the equation* (5.14) *the existence is necessary and sufficient of a locally integrable function $\lambda^* : [p^*,\infty) \to \mathbb{R}$ continuous on $[p^*,t^*) \cup [t^*,\infty)$ and satisfying the integral inequality*

$$\lambda^*(t) \geq \sum_{q=1}^{m} c_q(t) \mathrm{e}^{\int_{p(t,\vartheta_q)}^{t} \lambda^*(s)\mathrm{d}s} \tag{5.15}$$

for $t \geq t^*$.

Example 1. *Consider equation (5.14) with $m = 3$, $c_3(t) \equiv 0$. Let $c_1(t)$, $c_2(t)$ be positive continuous functions, $\vartheta_1 = -1$, $\vartheta_2 = -1/2$, $\vartheta_3 = 0$ and let the p-function be defined as:*

$$p(t,\theta) = \begin{cases} t + 2\tau\theta & \textit{for } \theta \in (-1/2, 0], \\ 2(t-\tau)(\theta+1) + \sqrt{t}(\theta+1/2)(-2) & \textit{for } \theta \in [-1,-1/2]. \end{cases}$$

Then, equation (5.14) takes the form:

$$\dot{y}(t) = -c_1(t)y(\sqrt{t}) - c_2(t)y(t-\tau), \tag{5.16}$$

where c_1, c_2 are positive continuous functions and the inequality (5.15) has the form:

$$\lambda(t) \geq c_1(t)\exp\left(\int_{\sqrt{t}}^{t}\lambda(s)ds\right) + c_2(t)\exp\left(\int_{t-\tau}^{t}\lambda(s)ds\right).$$

We put $\lambda(t) = 1/t$. Then, we obtain

$$\frac{1}{t} \geq c_1(t)\frac{t}{t-\tau} + c_2(t)\frac{t}{\sqrt{t}}.$$

This inequality (on the interval $[p^*,\infty)$) is a sufficient condition for the existence of a positive solution of equation (5.16) on interval $[(p^*)^2,\infty)$. Also, the equalities

$$c_1(t) = o\left(\frac{1}{t}\right) \text{ and } c_2(t) = o\left(\frac{1}{t\sqrt{t}}\right) \text{ for } t \to \infty$$

are sufficient conditions for the existence of an eventually positive solution of equation (5.16).

Theorem 23 can serve as a source of various sufficient conditions including the well-known sufficient conditions given, e.g., in [19, 32]. It is possible to show several concrete consequences of Theorem 23 concerning the equation

$$\dot{y}(t) = -c(t)y(p(t,-1)) \tag{5.17}$$

with a positive continuous function c. Obviously, equation (5.17) is a particular case of (5.14) if $m = 1$.

Theorem 24. *Let c be a positive continuous function on $[p^*,\infty)$ and let the inequality*

$$\mathrm{e}\cdot\int_{p(t,-1)}^{t} c(s)\mathrm{d}s \leq 1 \tag{5.18}$$

hold on $[t^*,\infty)$ (with $p^* = p(t^*,-1)$). Then, (5.17) *has a positive solution $y = y(t)$ on $[p^*,\infty)$.*

The following corollary follows directly from (5.18).

Corollary 1. *Let all conditions of Theorem 24 be valid and let there exist a nondecreasing function $b(t)$, $t \in [p^*,\infty)$ such that $c(t) \leq b(t)$ holds on $[p^*,\infty)$ and*

$$b(t) \leq \frac{1}{\mathrm{e}\cdot[t-p(t,-1)]} \tag{5.19}$$

holds on $[t^*,\infty)$. Then, (5.17) has a positive solution $y = y(t)$ on $[p^*,\infty)$.

Theorem 25. *Let $c(t)$ be a positive continuous function on $[t^*,\infty)$ and let there exist a positive constant K such that*

$$c(t) \leq K\mathrm{e}^{-K(t-p(t,-1))} \tag{5.20}$$

on $[t^*,\infty)$. Then, (5.17) *has a positive solution $y = y(t)$ on $[p^*,\infty)$ (with $p^* = p(t^*,-1)$).*

Remark 5. *The results presented are sharp. This may be demonstrated, e.g., by the last result. If $p(t,-1) := t-\tau$ with a positive constant τ, $c(t) \equiv c = \mathrm{const}$ and if $K := 1/\tau$, then (5.20) yields a classical result ([32, Theorem 2.2.3]) ensuring the existence of a positive solution:*

$$c\tau\mathrm{e} \leq 1.$$

5.2 Positive solutions to a scalar equation in the critical case

In [a8], the oscillation is discussed of solutions to the equation

$$\dot{y}(t) = -a(t)y(t-\tau(t)), \tag{5.21}$$

where $t \in I := [t_0, \infty)$, $t_0 \in \mathbb{R}$, $a\colon I \to \mathbb{R}^+ := (0, \infty)$ is a continuous function and $\tau\colon I \to \mathbb{R}^+$ is a continuous function such that $t - \tau(t) > t_0 - \tau(t_0)$ if $t > t_0$.

This study has been motivated by what can be found in the [13], [14], [17], [18], [54]. Using an example, it may be shown the that simple generalization of the results given in [13], [14] does not describe the situation completely. In [a8] two criteria are derived.

The first one is based on the criterion for the case of a constant delay derived in [14].

Theorem 26. I) *Let us assume that $a(t) \le a_k(t)$ with*

$$\begin{aligned} a_k(t) := {} & \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} \\ & + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\ldots\ln_k t)^2} \end{aligned} \tag{5.22}$$

if $t \to \infty$ and for an integer $k \ge 0$. Then, there exists a positive solution $x = y(t)$ of (5.21) *with $\tau(t) \equiv \tau = \text{const}$. Moreover,*

$$y(t) < \nu_k(t) := e^{-t/\tau}\sqrt{t\ln t\ln_2 t\ldots\ln_k t}$$

as $t \to \infty$.

II) *Let us assume that*

$$a(t) > a_{k-2}(t) + \frac{\theta\tau}{8e(t\ln t\ln_2 t\ldots\ln_{k-1} t)^2} \tag{5.23}$$

if $t \to \infty$, for an integer $k \ge 2$ and a constant $\theta > 1$. Then, all the solutions of (5.21) *with $\tau(t) \equiv \tau = \text{const}$ oscillate.*

Now we give two possible criteria. For the first, one we define a new auxiliary function $a_{k\tau}(t)$ similarly to (5.22) replayicing constant τ by function $\tau(t)$.

Theorem 27. *Let us assume that*

$$a(t) \le a_{k\tau}(t) \quad \textit{and} \quad \int_{t-\tau(t)}^{t} ds/\tau(s) \le 1$$

if $t \to \infty$ for an integer $k \ge 0$. Let moreover $\tau(t)\ln t\ln_2 t\ldots\ln_k t = o(t)$ as $t \to \infty$. Then, there exists a positive solution $x = y(t)$ of (5.21) *satisfying*

$$y(t) < \sqrt{t\ln t\ln_2 t\ldots\ln_k t}\cdot\exp\int_{t_0-\tau(t_0)}^{t}\left(\frac{-1}{\tau(s)}\right)\mathrm{d}s \tag{5.24}$$

as $t \to \infty$.

Theorem 28. *Let us assume that*

$$a(t) \leq \frac{1}{\tau(t)} \cdot \exp\left(-\int_{t-\tau(t)}^{t} \frac{\mathrm{d}s}{\tau(s)}\right) \tag{5.25}$$

as $t \to \infty$. Then, there exists a positive solution $x = y(t)$ of (5.21). *Moreover,*

$$y(t) < \exp\left(-\int_{t_0-\tau(t_0)}^{t} \frac{\mathrm{d}s}{\tau(s)}\right).$$

Analysis of both criteria To compare Theorem 27 with Theorem 28, we will investigate equation (5.21), where

$$\tau(t) := c + d/t \tag{5.26}$$

and c, d are positive constants, i.e., we consider an equation

$$\dot{y}(t) = -a(t)y(t - c - d/t). \tag{5.27}$$

Application of the first criterion The delay (5.26) is decreasing, tends to c as $t \to \infty$ and satisfies the inequality $\int_{t-\tau(t)}^{t} \frac{\mathrm{d}s}{\tau(s)} < 1$. If

$$a(t) \leq a_{k\tau}(t) \tag{5.28}$$

for an integer $k \geq 0$ as $t \to \infty$ then, by Theorem 27, equation (5.27) has a positive solution. We will develop the first several terms of the asymptotic decomposition of $a_{k\tau}(t)$ with $\tau(t)$ given by (5.26) if $t \to \infty$ and rewrite condition (5.28) to get a sufficient condition for the existence of a positive solution of (5.27) in the form

$$a(t) \leq a_{k\tau}(t) = \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{t} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{c}{8}\right) \cdot \frac{1}{t^2} + o\left(\frac{1}{t^2}\right). \tag{5.29}$$

Application of the second criterion We compute

$$\int_{t-\tau(t)}^{t} \frac{\mathrm{d}s}{\tau(s)} = 1 + \frac{d}{ct} - \frac{d}{c^2} \ln \frac{t}{t-c}\,.$$

Now we are able to asymptotically decompose the right-hand side of inequality (5.25) as $t \to \infty$. We get

$$\frac{1}{\tau(t)} \exp\left(\int_{t-\tau(t)}^{t} \frac{-\mathrm{d}s}{\tau(s)}\right) = \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{t} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{d}{2c}\right) \cdot \frac{1}{t^2} + o\left(\frac{1}{t^2}\right).$$

Finally, by the second criterion, a sufficient condition for the existence of a positive solution of (5.27) is

$$a(t) \leq \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{t} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{d}{2c}\right) \cdot \frac{1}{t^2} + o\left(\frac{1}{t^2}\right). \tag{5.30}$$

Final comparison

Comparing the right-hand sides of expressions (5.29) and (5.30), we see that the first two terms of both decompositions coincide. The quality of every criterion is expressed by the coefficients of the term $1/t^2$, by the coefficient C_2^{I} in the case of expression (5.29) and the coefficient C_2^{II}, i.e.,

$$C_2^{\mathrm{I}} = \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{c}{8} \right) \qquad C_2^{\mathrm{II}} = \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{d}{2c} \right)$$

We conclude $C_2^{\mathrm{I}} < C_2^{\mathrm{II}}$ if $c^2 < 4d$ and $C_2^{\mathrm{I}} > C_2^{\mathrm{II}}$ if $c^2 > 4d$. Thus, we can state.

Theorem 29. *The first criterion is more general in the case of $c^2 > 4d$; the second criterion is more general if $c^2 < 4d$.*

Using this example we may show that Theorem 26 cannot be generalized by replacing the constant delay τ by a nonconstant function $\tau(t)$. For more details, see [a8]. This result has been used in some papers.

5.3 A scalar equation with distributed delay

In [a5] we consider the existence of a positive solution of a scalar equation having the distributed delay

$$\dot{y}(t) = -\int_{-1}^{\vartheta_*} c(t,\vartheta)y(p(t,\vartheta))\mathrm{d}\vartheta \tag{5.31}$$

with $\vartheta_* \in (-1,0]$, and continuous $c : [t^*,\infty) \times [-1,\vartheta_*] \to (0,\infty)$. The main results of [a5] are the following.

Theorem 30. *For the existence of a positive solution $y = y(t)$ on $[p^*,\infty)$ (where $p^* = p(t^*,-1)$) of the equation (5.31), the existence is necessary and sufficient of a locally integrable function $\lambda^* : [p^*,\infty) \to \mathbb{R}$ continuous on $[p^*,t^*) \cup [t^*,\infty)$ and satisfying the integral inequality*

$$\lambda^*(t) \geq \int_{-1}^{\vartheta_*} c(t,\vartheta)\mathrm{e}^{\int_{p(t,\vartheta)}^{t} \lambda^*(q)\mathrm{d}q}\mathrm{d}\vartheta \tag{5.32}$$

for $t \geq t^*$.

The following results are consequences of Theorem 30.

Theorem 31. *Let there exist a positive constant K such that inequality*

$$\int_{-1}^{\vartheta_*} c(t,\vartheta)\mathrm{d}\vartheta \leq K\mathrm{e}^{-K\cdot[t-p(t,-1)]} \tag{5.33}$$

holds on $[t^*,\infty)$. Then, equation (5.31) with a positive continuous function c on $[t^*,\infty) \times [-1,\vartheta_*]$ has a positive solution $y = y(t)$ on $[p^*,\infty)$ (where $p^* = p(t^*,-1)$).

Theorem 32. *Let the difference $t-p(t,-1)$ be a nonincreasing function on $[t^*,\infty)$. Then, equation* (5.31) *with a positive continuous function c on $[t^*,\infty)\times[-1,\vartheta_*]$ has a positive solution $y=y(t)$ on $[p^*,\infty)$ (where $p^*=p(t^*,-1)$) if the inequality*

$$\int_{-1}^{\vartheta_*} c(t,\vartheta)\mathrm{d}\vartheta \le \frac{1}{\mathrm{e}\cdot[t-p(t,-1)]} \tag{5.34}$$

holds on $[t^*,\infty)$.

A straightforward consequence of inequality (5.34) is the following corollary.

Corollary 2. *Let all conditions of Theorem 32 be valid and let there exist a function $b:[t^*,\infty)\times[-1,\vartheta_*]\to\mathbb{R}$, nondecreasing in ϑ on $[-1,\vartheta_*]$ for each $t\in[t^*,\infty)$, such that $c(t,\vartheta)\le b(t,\vartheta)$ on $[t^*,\infty)\times[-1,\vartheta_*]$. If, moreover,*

$$b(t,\vartheta_*) \le \frac{1}{\mathrm{e}\cdot[t-p(t,-1)](1+\vartheta_*)} \tag{5.35}$$

holds on $[t^*,\infty)$, then (5.31) has a positive solution $y=y(t)$ on $[p^*,\infty)$.

5.4 Existence of decreasing positive solutions to linear differential equations of neutral type

Using the retract method, a new criterion is derived in [a7] for the existence of positive decreasing solutions to linear differential equations of neutral type: linear neutral differential equation

$$\dot{y}(t) = -c(t)y(t-\tau(t)) + d(t)\dot{y}(t-\delta(t)) \tag{5.36}$$

where $c,d\colon [t_0,\infty)\to[0,\infty)$, $t_0\in\mathbb{R}$ and $\tau,\delta\colon [t_0,\infty)\to(0,r]$, $r\in\mathbb{R}$, $r>0$ are continuous functions and $c(t)+d(t)>0$, $t\in[t_0,\infty)$.

Theorem 33. *For the existence of a positive decreasing solution of* (5.36) *on $[t_0-r,\infty)$, the necessary and sufficient condition is that there exists a continuous function $\lambda\colon [t_0-r,\infty)\to(0,\infty)$ such that inequality*

$$\lambda(t) \ge c(t)\exp\left(\int\limits_{t-\tau(t)}^{t} \lambda(s)ds\right) + d(t)\lambda(t-\delta(t))\exp\left(\int\limits_{t-\delta(t)}^{t} \lambda(s)ds\right),$$

holds for $t\ge t_0$.

Let the functions $c(t)$, $d(t)$ and delays $\tau(t)$, $\delta(t)$ in equation (5.36) be constant, i.e., $c(t)\equiv c=\text{const}$, $d(t)\equiv d=\text{const}$, $\tau(t)\equiv\tau=\text{const}$, $\delta(t)\equiv\delta=\text{const}$, then equation (5.36) becomes

$$\dot{y}(t) = -cy(t-\tau) + d\dot{y}(t-\delta). \tag{5.37}$$

Corollary 3. *For the existence of a positive decreasing solution of* (5.37) *on $[t_0-r,\infty)$, the existence is sufficient of a positive constant λ such that inequality*

$$\lambda \ge ce^{\lambda\tau} + \lambda de^{\lambda\delta} \tag{5.38}$$

holds.

For the choice of $\lambda = 1/\tau$ or $\lambda = 1/\delta$ in (5.38), we get

Corollary 4. *For the existence of a positive decreasing solution of* (5.37) *on* $[t_0 - r, \infty)$ *it is sufficient that either inequality*

$$1 > ce\tau + de^{\delta/\tau} \tag{5.39}$$

or inequality

$$1 > c\delta e^{\tau/\delta} + de \tag{5.40}$$

holds.

Chapter 6

Exponential stability of linear delay differential systems

6.1 Formulation of the problem

In [a1] the uniform exponential stability is studied of linear systems with time varying coefficients

$$\dot{x}_i(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j(h_{ij}^k(t)),\ i = 1,\dots,m \tag{6.1}$$

where $t \geq 0$, m and r_{ij}, $i, j = 1,\dots,m$ are natural numbers, coefficients $a_{ij}^k\colon [0,\infty) \to \mathbb{R}$ and delays $h_{ij}^k\colon [0,\infty) \to \mathbb{R}$ are measurable functions. For the scalar case ($m = 1$), the system (6.1) reduces to a linear differential equation with several delays

$$\dot{x}(t) = -\sum_{k=1}^{r} a_k(t)x(h_k(t)). \tag{6.2}$$

Equation (6.2) is studied in detail, e.g., in [3], [29], [30], [28], [42], and a review on stability results can be found in [4]. For system (6.1), there are not so many results.

The following short overview of the existing results uses the notion of an M-matrix. A square matrix is called a non-singular M-matrix if all its off-diagonal elements are non-positive and its principal minors are positive. (In [5], equivalent definitions can be found.)

An asymptotic stability conditions for the autonomous case of system (6.1) (when $a_{ij}^k(t) \equiv a_{ij}^k$, $h_{ij}^k(t) \equiv t - \tau_{ij}^k$ and a_{ij}^k, τ_{ij}^k are constant) is considered in [31]. In particular, for the system

$$\dot{x}_i(t) = -\sum_{j=1}^{m} a_{ij}x_j(t-\tau_{ij}),\ i = 1,\dots,m, \tag{6.3}$$

where $\tau_{ij} \geq 0$, the following holds (below, a_+ denotes the positive part of a, i.e., $a_+ = \max\{a, 0\}$).

Theorem 1 (Corollary 4.3, [31])**.** *Let*

$$0 < a_{ii}\tau_{ii} < 1 + 1/e\,,\ i = 1,\dots,m$$

and let the $m \times m$ matrix H with components

$$h_{ij} = \begin{cases} \left(\dfrac{1-(a_{ii}\tau_{ii}-1/e)_+}{1+(a_{ii}\tau_{ii}-1/e)_+} \right) a_{ii}, & i=j, \\ -|a_{ij}|, & i \neq j, \end{cases}$$

$i, j = 1, \dots, m$ be a non-singular M-matrix. Then, system (6.3) *is asymptotically stable for any selection of delays τ_{ij}, $i \neq j$, $i, j = 1, \dots, m$.*

In [55], the system (6.3) is also considered and the following derived.

Theorem 2 (Theorem 1.3, [55])**.** *Let*

$$0 \le a_{ii}\tau_{ii} < 3/2\,, \quad i = 1, \dots, m$$

and let the matrix G with components

$$g_{ij} = \begin{cases} -\left(\dfrac{1+a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})/9}{1-a_{ii}\tau_{ii}(3+2a_{ii}\tau_{ii})/9} \right) |a_{ij}|, & i \neq j, \\ a_{ii}, & i = j, \end{cases}$$

be a nonsingular M-matrix. Then, system (6.3) *is asymptotically stable for any selection of delays τ_{ij}, $i \neq j$, $i, j = 1, \dots, m$.*

In [56], the authors consider the non-autonomous system

$$\dot{x}_i(t) = -\sum_{j=1}^{m} a_{ij}(t) x_j(h_{ij}(t)), \; i = 1, \dots, m, \tag{6.4}$$

where $t \in [t_0, \infty)$, $t_0 \in \mathbb{R}$, $a_{ij}(t)$, $h_{ij}(t)$ are continuous functions, $h_{ij}(t) \le t$, and $h_{ij}(t)$ are monotone increasing functions such that $\lim_{t\to\infty} h_{ij}(t) = \infty$, $i, j = 1, \dots, m$.

Theorem 3 (Theorem 2.2, [56])**.** *Assume that, for $t \ge t_0$, there exist non-negative numbers b_{ij}, $i, j = 1, \dots, m$, $i \neq j$ such that $|a_{ij}(t)| \le b_{ij} a_{ii}(t)$, $i, j = 1, \dots, m$, $i \neq j$, $a_{ii}(t) \ge 0$ and*

$$\int^{\infty} a_{ii}(s)ds = \infty, \quad d_i = \limsup_{t\to\infty} \int_{h_{ii}(t)}^{t} a_{ii}(s)ds < 3/2, \quad i = 1, \dots m.$$

Let $\tilde{B} = (\tilde{b}_{ij})_{i,j=1}^{m}$ be an $m \times m$ matrix with entries $\tilde{b}_{ii} = 1$, $i = 1, \dots, m$ and, for $i \neq j$, $i, j = 1, \dots, m$,

$$\tilde{b}_{ij} = \begin{cases} -\left(\dfrac{2+d_i^2}{2-d_i^2} \right) b_{ij}, & \text{if } d_i < 1, \\ -\left(\dfrac{1+2d_i}{3-2d_i} \right) b_{ij}, & \text{if } d_i \ge 1. \end{cases}$$

If $\tilde{B}$ is a nonsingular M-matrix, then system (6.4) *is asymptotically stable.*

Very interesting global asymptotic stability results have been obtained for nonlinear systems of delay differential equations in the recent papers [45, 21, 22].

Paper [a1] considers general system (6.1) deriving the following result.

Theorem 34 ([a1). *, Theorem 4] Let there be constants a_0 and τ such that, for $t \geq t_0$,*

$$a_i^*(t) := \sum_{k=1}^{r_{ii}} a_{ii}^k(t) \geq a_0 > 0,\ 0 \leq t - h_{ij}^k(t) \leq \tau,\ i = 1, \dots, m \tag{6.5}$$

and

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t \geq t_0} \frac{1}{a_i^*(t)} \left[\sum_{k=1}^{r_{ii}} |a_{ii}^k(t)| \int_{\max\{0, h_{ii}^k(t)\}}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds \right.$$
$$\left. + \sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{k=1}^{r_{ij}} |a_{ij}^k(t)| \right] < 1. \tag{6.6}$$

Then, system (6.1) *is uniformly exponentially stable.*

6.2 Preliminaries

The linear system (6.1) for $t \geq t_0$ (assuming $t_0 \geq 0$) is considered with the initial condition

$$x(t) = \varphi(t),\ t \leq t_0, \tag{6.7}$$

under the following assumptions:

(a1) Functions $a_{ij}^k\colon [0, \infty) \to \mathbb{R},\ i, j = 1, \dots, m,\ k = 1, \dots, r_{ij}$ are Lebesgue measurable and essentially bounded functions.

(a2) Functions $h_{ij}^k\colon [0, \infty) \to \mathbb{R},\ i, j = 1, \dots, m,\ k = 1, \dots, r_{ij}$ are Lebesgue measurable functions, $h_{ij}^k(t) \leq t$, and
$$\limsup_{t \to \infty} (t - h_{ij}^k(t)) < \infty.$$

(a3) $\varphi\colon (-\infty, t_0] \to \mathbb{R}^m$ is a Borel measurable bounded vector-function.

Definition 10. *A locally absolutely continuous vector-function $x\colon \mathbb{R} \to \mathbb{R}^m$ is called* a solution to the problem (6.1)*,* (6.7) *for $t \geq t_0$ if its entries x_i, $i = 1, \dots, m$ satisfy equation* (6.1) *for almost all $t \in [t_0, \infty)$ and equality* (6.7) *holds for $t \leq t_0$.*

Definition 11. *Equation* (6.1) *is* uniformly exponentially stable*, if there exist constants $M > 0$ and $\mu > 0$ such that the solution $x\colon \mathbb{R} \to \mathbb{R}^m$ of problem* (6.1)*,* (6.7) *satisfies the inequality*
$$|x(t)| \leq M e^{-\mu(t - t_0)} \sup_{t \leq t_0} |\varphi(t)|,\ t \geq t_0$$
where M and μ do not depend on t_0.

6.3 Statement of results

Let A_i, $i = 1, \dots, m$ be functions defined as

$$A_i(t) := \frac{1}{a_i(t)} \left[\sum_{k=1}^{r_{ii}} a_{ii}^k(t) \int_{\max\{t_0, h_{ii}^k(t)\}}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds + \sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{k=1}^{r_{ij}} |a_{ij}^k(t)| \right]$$

where

$$a_i(t) := \sum_{k=1}^{r_{ii}} a_{ii}^k(t). \tag{6.8}$$

Theorem 35. *Let*

$$a_i(t) \geq a_0 > 0, \ \ i = 1, \dots, m, \ \ t \geq t_0, \tag{6.9}$$

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t \geq t_0} \frac{1}{a_i(t)} \sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{k=1}^{r_{ij}} |a_{ij}^k(t)| < 1 \tag{6.10}$$

and

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t \geq t_0} A_i(t) < 1 + \frac{1}{\mathrm{e}} \, . \tag{6.11}$$

Then, the system (6.1) *is uniformly exponentially stable.*

PROOF Can be found in [a2]. It also includes the corollaries (**Corollary** 1-10) of the Theorem 35 that generalize the corollaries of the Theorem 34 in [A1]. Similarly, corollaries of the Theorem 35 generalize corollaries of Theorem 34 in [a1].

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New exponential stability conditions for linear delayed systems of differential equations

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Appeared 11 August 2016
Communicated by Tibor Krisztin

Abstract. New explicit results on exponential stability, improving recently published results by the authors, are derived for linear delayed systems

$$\dot{x}_i(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j(h_{ij}^k(t)), \qquad i = 1,\dots,m$$

where $t \geq 0$, m and r_{ij}, $i, j = 1,\dots,m$ are natural numbers, $a_{ij}^k\colon [0,\infty) \to \mathbb{R}$ are measurable coefficients, and $h_{ij}^k\colon [0,\infty) \to \mathbb{R}$ are measurable delays. The progress was achieved by using a new technique making it possible to replace the constant 1 by the constant $1 + 1/e$ on the right-hand sides of crucial inequalities ensuring exponential stability.

Keywords: exponential stability, linear delayed differential system, estimate of fundamental function, Bohl–Perron theorem.

2010 Mathematics Subject Classification: 34K20.

1 Introduction

The objective of the present investigation is to derive easily verifiable explicit exponential stability conditions for the following non-autonomous linear delay differential system

$$\dot{x}_i(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j(h_{ij}^k(t)), \qquad i = 1,\dots,m \tag{1.1}$$

where $t \geq 0$, m is a natural number, r_{ij}, $i, j = 1,\dots,m$ are natural numbers, the coefficients $a_{ij}^k\colon [0,\infty) \to \mathbb{R}$ and delays $h_{ij}^k\colon [0,\infty) \to \mathbb{R}$ are measurable functions.

The equation

$$\dot{x}(t) = -\sum_{k=1}^{r} a_k(t)x(h_k(t)), \tag{1.2}$$

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which is a special scalar case of (1.1), has been studied, e.g., in [6, 12, 14, 15, 20, 25]. A review on stability results to equation (1.2) can be found in [7]. Below, we cite some selected results from the above papers or give extracts of them.

From [20, Theorem 1.2], we get the following corollary.

Theorem 1.1. *Let there be constants a_0, A_k and τ_k, $k = 1, 2, \ldots, r$ such that*

$$0 \le a_k(t) \le A_k, \qquad \sum_{k=1}^{r} a_k(t) \ge a_0 > 0, \qquad 0 \le t - h_k(t) \le \tau_k, \qquad t \ge 0.$$

If, moreover,

$$\sum_{k=1}^{r} A_k \tau_k \le 1, \tag{1.3}$$

then the equation (1.2) is uniformly asymptotically stable and the constant 1 on the right-hand side of (1.3) is the best one possible.

A corollary deduced from [20, Theorem 1.1] follows.

Theorem 1.2. *Let there be constants A_k and τ_k, $k = 1, 2, \ldots, r$ such that*

$$a_k(t) \equiv A_k > 0, \qquad 0 \le t - h_k(t) \le \tau_k, \qquad t \ge 0.$$

If, moreover,

$$\sum_{k=1}^{r} A_k \tau_k < \frac{3}{2}, \tag{1.4}$$

then the equation (1.2) is uniformly asymptotically stable and the constant $3/2$ on the right-hand side of (1.4) is the best one possible.

From [25, Corollary 2.4] we get the following theorem.

Theorem 1.3. *Let $a_k(t)$ and $h_k(t)$, $k = 1, \ldots, r$, $t \ge 0$ be continuous functions and*

$$a_k(t) \ge 0, \qquad \int_0^{\infty} \sum_{k=1}^{r} a_k(t)dt = \infty, \qquad 0 < h_1(t) \le h_2(t) \le \cdots \le h_r(t) \le t.$$

If, moreover,

$$\limsup_{t\to\infty} \sum_{k=1}^{r} \int_{h_1(t)}^{t} a_k(s)ds < \frac{3}{2},$$

then the equation (1.2) is asymptotically stable.

The following result reproduces [15, Proposition 4.4].

Theorem 1.4. *Let $a_k(t) \equiv a_k > 0$, $k = 1, 2, \ldots, r$ and let a constant $\alpha \in [0, 1]$ exist such that*

$$\frac{\alpha}{\mathrm{e} \sum\limits_{i=1}^{r} a_i} \le \max_k (t - h_k(t)), \qquad t \ge t_0$$

and

$$\sum_{i=1}^{r} a_i \limsup_{t\to\infty} (t - h_i(t)) < 1 + \frac{\alpha}{\mathrm{e}}.$$

Then, the equation (1.2) is uniformly asymptotically stable.

Now we give a corollary of [7, Lemma 3.1].

Theorem 1.5. *Let $a_k(t)$ be Lebesgue measurable essentially bounded functions and let there be constants a_0 and τ_k, $k = 1, 2, \ldots, r$ such that*

$$a_k(t) \geq 0, \qquad \int_{t_0}^{\infty} \sum_{k=1}^{r} a_k(s)ds = \infty, \qquad 0 \leq t - h_k(t) \leq \tau_k, \qquad t \geq t_0.$$

If, moreover,

$$\limsup_{t\to\infty} \sum_{k=1}^{r} \frac{a_k(t)}{\sum_{i=1}^{r} a_i(t)} \int_{h_k(t)}^{t} \sum_{i=1}^{r} a_i(s)ds < 1 + \frac{1}{e}, \tag{1.5}$$

then the equation (1.2) is uniformly exponentially stable.

Except for the paper [15], the above mentioned papers consider stability problems for scalar equations only. In [15], linear systems with constant matrices are treated. Unfortunately, there are no results on the stability of general systems of the form (1.1), which can be reduced to Theorems 1.1–1.5 in the scalar case. To illustrate this claim, consider several known results.

In [24], the authors consider the non-autonomous system

$$\dot{x}_i(t) = -\sum_{j=1}^{m} a_{ij}(t)x_j(h_{ij}(t)), \qquad i = 1, \ldots, m \tag{1.6}$$

where $t \in [t_0, \infty)$, $t_0 \in \mathbb{R}$, $a_{ij}(t)$, $h_{ij}(t)$ are continuous functions, $h_{ij}(t) \leq t$, $h_{ij}(t)$ are monotone increasing and such that $\lim_{t\to\infty} h_{ij}(t) = \infty$, $i, j = 1, \ldots, m$.

Theorem 1.6 ([24, Theorem 2.2])**.** *Assume that, for $t \geq t_0$, there exist non-negative numbers b_{ij}, $i, j = 1, \ldots, m$, $i \neq j$ such that $|a_{ij}(t)| \leq b_{ij}a_{ii}(t)$, $i, j = 1, \ldots, m$, $i \neq j$, $a_{ii}(t) \geq 0$ and*

$$\int^{\infty} a_{ii}(s)ds = \infty, \qquad d_i = \limsup_{t\to\infty} \int_{h_{ii}(t)}^{t} a_{ii}(s)ds < 3/2, \qquad i = 1, \ldots m.$$

Let $\tilde{B} = (\tilde{b}_{ij})_{i,j=1}^{m}$ be an $m \times m$ matrix with entries $\tilde{b}_{ii} = 1$, $i = 1, \ldots, m$ and, for $i \neq j$, $i, j = 1, \ldots, m$,

$$\tilde{b}_{ij} = \begin{cases} -\left(\dfrac{2 + d_i^2}{2 - d_i^2}\right) b_{ij}, & \text{if } d_i < 1, \\[2ex] -\left(\dfrac{1 + 2d_i}{3 - 2d_i}\right) b_{ij}, & \text{if } d_i \geq 1. \end{cases}$$

If $\tilde{B}$ is a nonsingular M-matrix, then system (1.6) is asymptotically stable.

This theorem can be viewed as a certain generalization of Theorems 1.2 and 1.3 to systems but only for the case of "one delay" ($r_{ij} = 1$, $i, j = 1, \ldots, m$).

Paper [13] gives a generalization of Theorem 1.4 to linear systems with constant coefficients and delays.

In our recent paper [8], we considered general system (1.1) deriving the following result.

Theorem 1.7 ([8, Theorem 4])**.** *Let there be constants a_0 and τ such that, for $t \geq t_0$,*

$$a_i^*(t) := \sum_{k=1}^{r_{ii}} a_{ii}^k(t) \geq a_0 > 0, \qquad 0 \leq t - h_{ij}^k(t) \leq \tau, \qquad i = 1, \ldots, m \tag{1.7}$$

and

$$\max_{i=1,\dots,m}\ \operatorname*{ess\,sup}_{t\ge t_0}\frac{1}{a_i^*(t)}\left[\sum_{k=1}^{r_{ii}}|a_{ii}^k(t)|\int_{\max\{0,h_{ii}^k(t)\}}^{t}\sum_{j=1}^{m}\sum_{l=1}^{r_{ij}}|a_{ij}^l(s)|ds+\sum_{\substack{j=1\\ j\ne i}}^{m}\sum_{k=1}^{r_{ij}}|a_{ij}^k(t)|\right]<1. \tag{1.8}$$

Then, the system (1.1) *is uniformly exponentially stable.*

Requiring that all assumptions of Theorem 1.5 and Theorem 1.7 are valid simultaneously, condition (1.8) in Theorem 1.7 turns, in the case of equation (1.2) where $a_k(t)\ge 0$, into

$$\operatorname*{ess\,sup}_{t\ge t_0}\frac{1}{\sum_{k=1}^{r}a_k(t)}\sum_{k=1}^{r}a_k(t)\int_{\max\{0,h_k(t)\}}^{t}\sum_{l=1}^{r}a_l(s)ds<1$$

and, for t_0 sufficiently large, coincides with the left-hand side of inequality (1.5).

Nevertheless, Theorem 1.7 is not an extension of Theorem 1.5 to system (1.1) since the right-hand side in the inequality (1.8) is equal to 1 instead of $1+1/e$ on the right-hand side of inequality (1.5) in Theorem 1.5.

The aim of the paper is to improve all the results of [8] and replace the constant 1 by the constant $1+1/e$ not only on the right-hand side of inequality (1.8), but in all explicit stability conditions derived in [8]. The only limitation in this paper in comparison with paper [8] is the condition

$$a_{ii}^k(t)\ge 0,\qquad i=1,\dots,m,\qquad k=1,\dots,r_{ii}. \tag{1.9}$$

Since this condition does not necessarily hold for equations considered in [8], all results of this paper and in [8] are independent.

Our approach is based on estimates of the fundamental solution for scalar delay differential equations and on the Bohl–Perron type result. Some ideas and schemes of [8] are utilized as well.

2 Preliminaries

Let $t_0\ge 0$. We consider an initial problem

$$x(t)=\varphi(t),\qquad t\le t_0 \tag{2.1}$$

for (1.1) where $\varphi=(\varphi_1,\dots,\varphi_m)^T\colon(-\infty,t_0]\to\mathbb{R}^m$ is a vector-function. Throughout the rest of the paper, we assume (a1)–(a3) where

(a1) $a_{ij}^k\colon[0,\infty)\to\mathbb{R}$, $i,j=1,\dots,m$, $k=1,\dots,r_{ij}$ are Lebesgue measurable and essentially bounded functions, $a_{ii}^k(t)\ge 0$;

(a2) $h_{ij}^k\colon[0,\infty)\to\mathbb{R}$, $i,j=1,\dots,m$, $k=1,\dots,r_{ij}$ are Lebesgue measurable functions, $h_{ij}^k(t)\le t$, and $t-h_{ij}^k(t)\le K$, $t\ge 0$ where K is a positive constant;

(a3) $\varphi\colon(-\infty,t_0]\to\mathbb{R}^m$ is a Borel measurable bounded vector-function.

For a vector $x=(x_1,\dots,x_m)^T\in\mathbb{R}^m$, we define $|x|:=\max_{i=1,\dots,m}|x_i|$.

Remark 2.1. The function φ in (2.1) is defined on $(-\infty, t_0]$. By (a2), there exists a positive constant K such that $t - h_{ij}^k(t) \le K$, $i,j = 1,\dots,m$, $k = 1,\dots,r_{ij}$. Thus, the domain of the definition of the initial function φ in (2.1) in the following consideration can be, in principle, restricted to the finite interval $[t_0 - K, t_0]$. In the following computations, it is often necessary to estimate differences $t - \max\{t_0, h_{ii}^k(t)\}$ (or similar) from above. Obviously,

$$t - \max\{t_0, h_{ii}^k(t)\} \le K.$$

Definition 2.2. A locally absolutely continuous vector-function $x\colon \mathbb{R} \to \mathbb{R}^m$ is called *a solution of the problem* (1.1), (2.1) for $t \ge t_0$, if its components $x_i(t)$, $i = 1,\dots,m$ satisfy (1.1) for almost all $t \in [t_0, \infty)$ and (2.1) holds for $t \le t_0$.

Definition 2.3. Equation (1.1) is called uniformly exponentially stable if there exist constants $M > 0$ and $\mu > 0$ such that the solution $x\colon \mathbb{R} \to \mathbb{R}^m$ of (1.1), (2.1) satisfies

$$|x(t)| \le M\, e^{-\mu(t-t_0)} \sup_{t \le t_0} |\varphi(t)|, \qquad t \ge t_0$$

where M and μ do not depend on t_0.

A non-homogeneous system

$$\dot{x}_i(t) = -\sum_{j=1}^{m} \sum_{k=1}^{r_{ij}} a_{ij}^k(t) x_j(h_{ij}^k(t)) + f_i(t), \qquad i = 1,\dots,m \tag{2.2}$$

where $f_i\colon [0,\infty) \to \mathbb{R}$ is a Lebesgue measurable locally essentially bounded function together with the initial problem

$$x(t) = \theta, \qquad t \le t_0, \tag{2.3}$$

where $\theta = (0,\dots,0)^T \in \mathbb{R}^m$, will be used together with homogeneous system (1.1).

In what follows, $\mathbf{L}_\infty^m[t_0,\infty)$ denotes the space of all essentially bounded real vector-functions $y\colon [t_0,\infty) \to \mathbb{R}^m$ with the essential supremum norm

$$\|y\|_{\mathbf{L}_\infty^m} = \operatorname*{ess\,sup}_{t \ge t_0} |y(t)|.$$

As $\mathbf{C}^m[t_0,\infty)$ we denote the space of all continuous m-dimensional bounded real vector-functions on $[t_0,\infty)$ equipped with the supremum norm.

The proof of our main result uses the Bohl–Perron type result ([1–5, 11, 16]).

Theorem 2.4. *If the solution of initial problem* (2.2), (2.3) *belongs to* $\mathbf{C}^m[t_0,\infty)$ *for any* $f \in \mathbf{L}_\infty^m[t_0,\infty)$, $f = (f_1,\dots,f_m)^T$, *then equation* (1.1) *is uniformly exponentially stable.*

Note that, without loss of generality, we can assume $f(t) \equiv \theta$ on the interval $[t_0, t_1]$ for some $t_1 > t_0$ in Lemma 2.4.

Consider the scalar homogeneous initial problem

$$\dot{x}(t) = -\sum_{k=1}^{r} a_k(t) x(h_k(t)), \qquad t \ge s \ge t_0, \tag{2.4}$$

$$x(t) = 0, \qquad t < s, \qquad x(s) = 1, \tag{2.5}$$

where $a_k\colon [0,\infty) \to \mathbb{R}$, $k = 1,\dots,r$ are Lebesgue measurable and essentially bounded functions, $h_k\colon [0,\infty) \to \mathbb{R}$, $k = 1,\dots,r$ are Lebesgue measurable functions, $h_k(t) \le t$.

Definition 2.5. A solution $x = X(t,s)$ of (2.4), (2.5) is called the fundamental function of (1.1).

The associated non-homogeneous equation to (2.4) is

$$\dot{x}(t) = -\sum_{k=1}^{r} a_k(t)x(h_k(t)) + f(t), \qquad t \geq t_0. \tag{2.6}$$

We will need the following representation formula (see, e.g. [1–5]) for solution of (2.6) (with a locally Lebesgue integrable right-hand side f) satisfying the initial problem

$$x(t) = 0, \qquad t \leq t_0. \tag{2.7}$$

Theorem 2.6. *The solution of initial problem* (2.6), (2.7) *is given by the formula*

$$x(t) = \int_{t_0}^{t} X(t,s)f(s)ds. \tag{2.8}$$

The following lemma is taken from [12].

Theorem 2.7. *Let* $a_k(t) \geq 0$ *and*

$$\int_{\min_k\{h_k(t)\}}^{t} \sum_{k=1}^{r} a_k(s)ds \leq \frac{1}{\mathrm{e}}$$

where $t \geq t_0$*,* $k = 1,\dots,r$*. Then, the fundamental function* $X(t,s)$ *of* (2.4) *satisfies* $X(t,s) > 0$ *for* $t \geq s \geq t_0$*.*

We will finish this section by an auxiliary result from [6]. In its formulation, $X(t,s)$ is the fundamental function of (2.4).

Theorem 2.8. *Let* $a_k(t) \geq 0$*,* $X(t,s) > 0$*,* $t \geq s \geq t_0$*,* $t - h_k(t) \leq K$*,* $t \geq t_0$*,* $k = 1,\dots,r$*. Then,*

$$0 \leq \int_{t_0}^{t} X(t,s)\left(\sum_{k=1}^{r} a_k(s)\right)\xi(s)ds \leq 1, \qquad t \geq t_0,$$

where ξ *is the characteristic function of the interval* $[t_0 + K, \infty)$*.*

3 Main result

The main result (Theorem 3.1 below) gives sufficient conditions for the uniform exponential stability to system (1.1). We underline that this theorem is a significant improvement to Theorem 1.7 because almost the same expression is estimated by the constant $1 + 1/\mathrm{e}$ on the right-hand side of inequality (3.4) rather than by the constant 1 on the right-hand side of inequality (1.8).

Let A_i, $i = 1,\dots,m$ be functions defined as

$$A_i(t) := \frac{1}{a_i(t)}\left[\sum_{k=1}^{r_{ii}} a_{ii}^k(t)\int_{\max\{t_0,h_{ii}^k(t)\}}^{t} \sum_{j=1}^{m}\sum_{l=1}^{r_{ij}} |a_{ij}^l(s)|ds + \sum_{\substack{j=1\\ j\neq i}}^{m}\sum_{k=1}^{r_{ij}} |a_{ij}^k(t)|\right]$$

where

$$a_i(t) := \sum_{k=1}^{r_{ii}} a_{ii}^k(t). \tag{3.1}$$

Theorem 3.1 (Main result)**.** *Let*

$$a_i(t) \geq a_0 > 0, \qquad i = 1, \dots, m, \qquad t \geq t_0, \tag{3.2}$$

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t \geq t_0} \frac{1}{a_i(t)} \sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{k=1}^{r_{ij}} |a_{ij}^k(t)| < 1 \tag{3.3}$$

and

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t \geq t_0} A_i(t) < 1 + \frac{1}{\mathrm{e}}\,. \tag{3.4}$$

Then, the system (1.1) *is uniformly exponentially stable.*

Proof. Define auxiliary functions $H_i^k \colon [t_0, \infty) \to \mathbb{R}$, $i = 1, \dots, m$, $k = 1, \dots, r_{ii}$ as follows:

i) If

$$\int_{h_{ii}^k(t)}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds \leq \frac{1}{\mathrm{e}}\,, \tag{3.5}$$

then

$$H_i^k(t) := h_{ii}^k(t).$$

ii) If

$$\int_{h_{ii}^k(t)}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds > \frac{1}{\mathrm{e}}\,, \tag{3.6}$$

then $H_i^k(t)$ is a unique solution of an implicit equation

$$\int_{H_i^k(t)}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds = \frac{1}{\mathrm{e}}\,.$$

Consider the problem (2.2), (2.3) assuming that

$$f_i(t) \equiv 0 \quad \text{ if } t \in [t_0, t_0 + K],\ i = 1, \dots, m. \tag{3.7}$$

Condition (3.7) implies that for the solution of the problem (2.2), (2.3) we have $x_i(t) = 0$, $i = 1, \dots, m$ if $t \in [t_0, t_0 + K]$.

System (2.2) can be transformed to

$$\begin{aligned} \dot{x}_i(t) = &- \sum_{k=1}^{r_{ii}} a_{ii}^k(t) x_i(H_i^k(t)) + \sum_{k=1}^{r_{ii}} a_{ii}^k(t) \int_{h_{ii}^k(t)}^{H_i^k(t)} \dot{x}_i(s) ds \\ &- \sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{k=1}^{r_{ij}} a_{ij}^k(t) x_j(h_{ij}^k(t)) + f_i(t), \qquad t \geq t_0,\ i = 1, \dots, m. \end{aligned} \tag{3.8}$$

It is easy to see that (due to (2.3)) system (3.8) is equivalent with

$$\begin{aligned} \dot{x}_i(t) = &- \sum_{k=1}^{r_{ii}} a_{ii}^k(t) x_i(H_i^k(t)) + \sum_{k=1}^{r_{ii}} a_{ii}^k(t) \int_{\max\{t_0, h_{ii}^k(t)\}}^{H_i^k(t)} \dot{x}_i(s) ds \\ &- \sum_{\substack{j=1 \\ j \neq i}}^{m} \sum_{k=1}^{r_{ij}} a_{ij}^k(t) x_j(h_{ij}^k(t)) + f_i(t), \qquad t \geq t_0,\ i = 1, \dots, m. \end{aligned} \tag{3.9}$$

Moreover, utilizing (2.2), (3.9), it can be transformed to

$$\begin{aligned}\dot{x}_i(t) = & -\sum_{k=1}^{r_{ii}} a_{ii}^k(t)x_i(H_i^k(t)) \\ & -\sum_{k=1}^{r_{ii}} a_{ii}^k(t)\int_{\max\{t_0,h_{ii}^k(t)\}}^{H_i^k(t)} \sum_{j=1}^{m}\sum_{l=1}^{r_{ij}} a_{ij}^l(s)x_j(h_{ij}^l(s))ds \\ & -\sum_{\substack{j=1\\ j\neq i}}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(t)x_j(h_{ij}^k(t)) + p_i(t), \qquad t\geq t_0,\ i=1,\dots,m \end{aligned} \tag{3.10}$$

where

$$p_i(t) = f_i(t) + \sum_{k=1}^{r_{ii}} a_{ii}^k(t)\int_{\max\{t_0,h_{ii}^k(t)\}}^{H_i^k(t)} f_i(s)ds.$$

By assumption (a2), the definition of H_i^k (note that $h_{ii}^k(t)\leq H_i^k(t)\leq t$), and (3.7) we get

$$p_i(t)\equiv 0 \quad \text{if } t\leq t_0+K.$$

Let $X_i(t,s)$, $i=1,\dots,m$ be the fundamental function (see Definition 2.5) of the scalar initial-value problem

$$\begin{aligned}&\dot{x}_i(t) = -\sum_{k=1}^{r_{ii}} a_{ii}^k(t)x_i(H_i^k(t)), \qquad t\geq t_0,\\ &x_i(t)=0, \qquad t\leq t_0.\end{aligned}$$

By virtue of (a1), the definition of $H_i^k(t)$, $i=1,\dots,m$ and Lemma 2.7, we have $X_i(t,s)>0$, $t\geq s\geq t_0$, $i=1,\dots,m$. Using formula (2.8) in Lemma 2.6, from (3.10), we get

$$\begin{aligned}x_i(t) = -\int_{t_0}^{t} X_i(t,s)\Bigg[&\sum_{k=1}^{r_{ii}} a_{ii}^k(s)\int_{\max\{t_0,h_{ii}^k(s)\}}^{H_i^k(s)} \sum_{j=1}^{m}\sum_{l=1}^{r_{ij}} a_{ij}^l(\tau)x_j(h_{ij}^l(\tau))d\tau \\ &+\sum_{\substack{j=1\\ j\neq i}}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k(s)x_j(h_{ij}^k(s))\Bigg]ds + g_i(t), \qquad t\geq t_0,\ i=1,\dots,m\end{aligned} \tag{3.11}$$

where

$$g_i(t) = \int_{t_0}^{t} X_i(t,s)p_i(s)ds$$

and

$$p_i(t) = g_i(t)\equiv 0 \quad \text{if } t\leq t_0+K.$$

Next, we explain why g_i, $i=1,\dots,m$ are essentially bounded functions. By (a1), properties of f_i and H_i^k, $i = 1, \dots, m$, definition (1.7), Remark 2.1, and Lemma 2.8, we deduce

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geq t_0} |g_i(t)| \\
&\quad = \operatorname*{ess\,sup}_{t\geq t_0} \left| \int_{t_0}^{t} X_i(t,s) p_i(s) ds \right| \\
&\quad = \operatorname*{ess\,sup}_{t\geq t_0+K} \left| \int_{t_0}^{t} X_i(t,s) p_i(s) ds \right| \\
&\quad \leq \operatorname*{ess\,sup}_{t\geq t_0+K} \int_{t_0}^{t} X_i(t,s) a_i(s) \frac{|p_i(s)|}{a_i(s)} ds \leq \operatorname*{ess\,sup}_{t\geq t_0+K} \frac{|p_i(t)|}{a_i(t)} \\
&\quad \leq \frac{1}{a_0} \operatorname*{ess\,sup}_{t\geq t_0+K} |p_i(t)| \\
&\quad \leq \frac{1}{a_0} \left(\operatorname*{ess\,sup}_{t\geq t_0+K} |f_i(t)| + \operatorname*{ess\,sup}_{t\geq t_0+K} \sum_{k=1}^{r_{ii}} a_{ii}^k(t) \operatorname*{ess\,sup}_{t\geq t_0+K} |f_i(t)| \cdot \operatorname*{ess\,sup}_{t\geq t_0+K} (H_i^k(t) - \max\{t_0, h_{ii}^k(t)\}) \right) \\
&\quad < \infty.
\end{aligned}$$

System (3.11) can be written in an operator form

$$x_i(t) = (G_i x)(t) + g_i(t), \qquad t \geq t_0,\ i = 1, \dots, m$$

where

$$\begin{aligned}
(G_i x)(t) = -\int_{t_0}^{t} X_i(t,s) &\left[\sum_{k=1}^{r_{ii}} a_{ii}^k(s) \int_{\max\{t_0, h_{ii}^k(s)\}}^{H_i^k(s)} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} a_{ij}^l(\tau) x_j(h_{ij}^l(\tau)) d\tau \right. \\
&\left. + \sum_{\substack{j=1 \\ j\neq i}}^{m} \sum_{k=1}^{r_{ij}} a_{ij}^k(s) x_j(h_{ij}^k(s)) \right] ds, \qquad t \geq t_0,\ i = 1, \dots, m
\end{aligned}$$

or as

$$x = Gx + g \tag{3.12}$$

where

$$G\colon \mathbf{L}_\infty^m \to \mathbf{L}_\infty^m, \qquad (Gx)(t) = ((G_1 x)(t), \dots, (G_m x)(t))^T$$

and $g(t) = (g_1(t), \dots, g_m(t))^T$. Estimate the norm $\|G\|_{\mathbf{L}_\infty^m}$ of the operator G. Since $x_i(t) \equiv 0$, if $t \in [t_0, t_0 + K]$, $i = 1, \dots, m$, then

$$|(G_i x)(t)| \leq \int_{t_0+H}^{t} X_i(t,s) a_i(s) \mathcal{A}_i(s) ds \cdot \|x\|_{\mathbf{L}_\infty}, \qquad i = 1, \dots, m$$

where

$$\mathcal{A}_i(t) := \frac{1}{a_i(t)} \left[\sum_{k=1}^{r_{ii}} a_{ii}^k(t) \int_{\max\{t_0, h_{ii}^k(t)\}}^{H_i^k(t)} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds + \sum_{\substack{j=1 \\ j\neq i}}^{m} \sum_{k=1}^{r_{ij}} |a_{ij}^k(t)| \right].$$

Hence, by Lemma 2.8,

$$\|G\|_{L_\infty^m} \leq \max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\geq t_0} \mathcal{A}_i(t) \tag{3.13}$$

If (3.5) holds, then $H_i^k(t) = h_{ii}^k(t)$, $i = 1, \dots, m$, $k = 1, \dots, r_{ii}$ and, consequently,

$$\mathcal{A}_i(t) \le \frac{1}{a_i(t)} \left[\sum_{\substack{j=1\\ j\neq i}}^{m} \sum_{k=1}^{r_{ij}} |a_{ij}^k(t)| \right].$$

By (3.3) we get

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\ge t_0} \mathcal{A}_i(t) \le \max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\ge t_0} \frac{1}{a_i(t)} \left[\sum_{\substack{j=1\\ j\neq i}}^{m} \sum_{k=1}^{r_{ij}} |a_{ij}^k(t)| \right] < 1. \tag{3.14}$$

If (3.6) is valid, then

$$\int_{H_i^k(t)}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds = \frac{1}{\mathrm{e}}.$$

Hence

$$\begin{aligned}
&\frac{1}{a_i(t)} \sum_{k=1}^{r_{ii}} a_{ii}^k(t) \int_{\max\{t_0, h_{ii}^k(t)\}}^{H_i^k(t)} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds \\
&\quad = \frac{1}{a_i(t)} \sum_{k=1}^{r_{ii}} a_{ii}^k(t) \left[\int_{\max\{t_0, h_{ii}^k(t)\}}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds - \int_{H_i^k(t)}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds \right] \\
&\quad = \frac{1}{a_i(t)} \sum_{k=1}^{r_{ii}} a_{ii}^k(t) \left[\int_{\max\{t_0, h_{ii}^k(t)\}}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds - \frac{1}{\mathrm{e}} \right] \\
&\quad = \frac{1}{a_i(t)} \sum_{k=1}^{r_{ii}} a_{ii}^k(t) \int_{\max\{t_0, h_{ii}^k(t)\}}^{t} \sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} |a_{ij}^l(s)| ds - \frac{1}{\mathrm{e}}.
\end{aligned} \tag{3.15}$$

In this case, using (3.15) and (3.4), we get

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\ge t_0} \mathcal{A}_i(t) \le \max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\ge t_0} \left(A_i(t) - \frac{1}{\mathrm{e}} \right) < 1. \tag{3.16}$$

Finally, from (3.13), (3.14) and (3.16), we deduce $\|G\|_{\mathbf{L}_\infty^m} < 1$. Therefore, the operator equation (3.12) has a unique solution $x \in \mathbf{L}_\infty^m$ This solution solves the system (2.2) and belongs to the space $\mathbf{C}^m[t_0, \infty)$. By Lemma 2.4, system (1.1) is uniformly exponentially stable. □

4 Corollaries to the main result

The purpose of this part is to consider some special cases of the system (1.1) and from Theorem 3.1, deduce simple corollaries on uniform exponential stability. In the proofs, we verify the assumptions of Theorem 3.1 for the case considered. It is often obvious and we omit the unnecessary details.

Corollary 4.1. *Assume that*

$$a_{ii}(t) \ge a_0 > 0, \qquad i = 1, \dots, m, \qquad t \ge t_0, \tag{4.1}$$

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\geq t_0} \frac{1}{a_{ii}(t)} \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ij}(t)| < 1 \tag{4.2}$$

and

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\geq t_0} \left[\int_{\max\{t_0,h_{ii}(t)\}}^{t} \sum_{j=1}^{m} |a_{ij}(s)| ds + \frac{1}{a_{ii}(t)} \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ij}(t)| \right] < 1 + \frac{1}{\mathrm{e}}. \tag{4.3}$$

Then, the system

$$\dot{x}_i(t) = -\sum_{j=1}^{m} a_{ij}(t) x_i(h_{ij}(t))), \qquad i = 1, \dots, m \tag{4.4}$$

is uniformly exponentially stable.

Proof. Let $r_{ij} = 1$, $a_{ij}^k(t) = a_{ij}(t)$, $h_{ij}^k(t) = h_{ij}(t)$, $a_i(t) = a_{ii}(t)$, $i, j = 1, \dots, m$. Then, the system (1.1) reduces to (4.4) and we can apply Theorem 3.1 since assumptions (3.2), (3.3) and (3.4) are, in the particular case, reduced to assumptions (4.1), (4.2) and (4.3). □

Corollary 4.2. *Assume that, for $t \geq t_0$, we have $a_{ii}^k(t) \geq 0$,*

$$\sum_{k=1}^{r_{ii}} a_{ii}^k(t) \geq \alpha_i > 0, \qquad |a_{ij}^k(t)| \leq a_{ij}^k, \qquad t - h_{ij}^k(t) \leq \tau_{ij}^k$$

where $i, j = 1, \dots, m$, $k = 1, \dots, r_{ij}$, α_i, a_{ij}^k, τ_{ij}^k are constants,

$$\max_{i=1,\dots,m} \frac{1}{\alpha_i} \sum_{\substack{j=1\\ j\neq i}}^{m} \sum_{k=1}^{r_{ij}} a_{ij}^k < 1, \tag{4.5}$$

and

$$\max_{i=1,\dots,m} \frac{1}{\alpha_i} \left[\left(\sum_{k=1}^{r_{ii}} a_{ii}^k \tau_{ii}^k \right) \left(\sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} a_{ij}^l \right) + \sum_{\substack{j=1\\ j\neq i}}^{m} \sum_{k=1}^{r_{ij}} a_{ij}^k \right] < 1 + \frac{1}{\mathrm{e}}. \tag{4.6}$$

Then, the system (1.1) is uniformly exponentially stable.

Proof. We have for $t \geq t_0$

$$A_i(t) \leq \frac{1}{\alpha_i} \left[\sum_{k=1}^{r_{ii}} a_{ii}^k \left(\sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} a_{ij}^l \right) \tau_{ii}^k + \sum_{\substack{j=1\\ j\neq i}}^{m} \sum_{k=1}^{r_{ij}} a_{ij}^k \right] = \frac{1}{\alpha_i} \left[\left(\sum_{k=1}^{r_{ii}} a_{ii}^k \tau_{ii}^k \right) \left(\sum_{j=1}^{m} \sum_{l=1}^{r_{ij}} a_{ij}^l \right) + \sum_{\substack{j=1\\ j\neq i}}^{m} \sum_{k=1}^{r_{ij}} a_{ij}^k \right]$$

and (4.6) implies (3.4). □

Corollary 4.3. *Assume that $a_{ii}(t) \geq \alpha_i > 0$, $|a_{ij}(t)| \leq a_{ij}$, $t - h_{ij}(t) \leq \tau_{ij}$ for $i, j = 1, \dots, m$ and $t \geq t_0$ where α_i, a_{ij}, and τ_{ij} are constants and*

$$\max_{i=1,\dots,m} \frac{1}{\alpha_i} \sum_{\substack{j=1\\ j\neq i}}^{m} a_{ij} < 1, \qquad \max_{i=1,\dots,m} \left[\tau_{ii} \sum_{j=1}^{m} a_{ij} + \frac{1}{\alpha_i} \sum_{\substack{j=1\\ j\neq i}}^{m} a_{ij} \right] < 1 + \frac{1}{\mathrm{e}}. \tag{4.7}$$

Then, the system (4.4) is uniformly exponentially stable.

Proof. This result follows from Corollary 4.1. □

Now we give stability conditions for the following linear autonomous system with constant delays

$$\dot{x}_i(t) = -\sum_{j=1}^{m}\sum_{k=1}^{r_{ij}} a_{ij}^k x_j(t-\tau_{ij}^k), \qquad i=1,\dots,m. \tag{4.8}$$

Corollary 4.4. *Assume that $a_{ii}^k \geq 0$, conditions (4.5) and (4.6) hold where*

$$\alpha_i := \sum_{k=1}^{r_{ii}} a_{ii}^k > 0, \qquad i=1,\dots,m.$$

Then, the autonomous system (4.8) is uniformly exponentially stable.

Proof. This follows directly from Corollary 4.2. □

Consider the linear autonomous system with constant delays

$$\dot{x}_i(t) = -\sum_{j=1}^{m} a_{ij} x_j(t-\tau_{ij}), \qquad i=1,\dots,m. \tag{4.9}$$

Corollary 4.5. *Assume that $a_{ii} > 0$ and inequalities (4.7) hold where $\alpha_i = a_{ii}$, $i=1,\dots,m$. Then, the autonomous system (4.9) is uniformly exponentially stable.*

Proof. This follows directly from Corollary 4.3. □

Corollary 4.6. *Assume that $m=1$, $a_k(t) \geq 0$, $k=1,\dots,r$ and, for $t \geq t_0$, at least one of the following conditions hold (a_0, a_i and τ_i, $i=1,\dots,r$ are constants):*

1) $\sum_{k=1}^{r} a_k(t) \geq a_0 > 0$,

$$\operatorname*{ess\,sup}_{t\geq t_0} \frac{1}{\sum_{k=1}^{r} a_k(t)} \left[\sum_{k=1}^{r} a_k(t) \int_{\max\{t_0,h_k(t)\}}^{t} \sum_{l=1}^{r} a_l(s)ds \right] < 1 + \frac{1}{\mathrm{e}}\,. \tag{4.10}$$

2) $a_i(t) \equiv a_i$, $\sum_{i=1}^{r} a_i > 0$, $t - h_i(t) \leq \tau_i$, $i=1,\dots,r$, *and*

$$\sum_{i=1}^{r} a_i\tau_i < 1 + \frac{1}{\mathrm{e}}\,. \tag{4.11}$$

Then, the scalar equation (1.2) is uniformly exponentially stable.

Proof. Let condition 1) be true. Then, inequality (3.4) turns into inequality (4.10) for $m=1$. Let condition 2) be true. Since $a_i(t) \equiv a_i$, inequality (4.10) is transformed to

$$\operatorname*{ess\,sup}_{t\geq t_0} \sum_{k=1}^{r} a_k(t - \max\{t_0, h_k(t)\}) < 1 + \frac{1}{\mathrm{e}}\,.$$

Since

$$\operatorname*{ess\,sup}_{t\geq t_0} \sum_{k=1}^{r} a_k(t - \max\{t_0, h_k(t)\}) \leq \operatorname*{ess\,sup}_{t\geq t_0} \sum_{k=1}^{r} a_k\tau_k$$

inequality (4.11) implies (4.10). □

Now we consider two particular cases of system (1.1),

$$\dot{X}(t) = -B(t)X(h(t)) \tag{4.12}$$

and

$$\dot{X}(t) = -A(t)X(t) - B(t)X(h(t)) \tag{4.13}$$

where $A(t) = (a_{ij}(t))_{i,j=1}^m$, $B(t) = (b_{ij}(t))_{i,j=1}^m$ are $m \times m$ matrices with Lebesgue measurable and locally essentially bounded entries

$$a_{ij} \colon [0, \infty) \to \mathbb{R}, \qquad b_{ij} \colon [0, \infty) \to \mathbb{R}, \qquad i, j = 1, \dots, m$$

and $X(t) = (x_1(t), \dots, x_m(t))^T$. Assume that, for the delay $h \colon [0, \infty) \to \mathbb{R}$, the relevant adaptation of condition (a2) holds, i.e., h is Lebesgue measurable, $h(t) \le t$ and $t - h(t) \le K$, $t \in [0, \infty)$ and $\limsup_{t\to\infty}(t - h(t)) < \infty$.

The following two Corollaries 4.7 and 4.8 deal with the exponential stability of systems (4.12), (4.13).

Corollary 4.7. *Assume that, for $t \ge t_0$, at least one of the conditions hold (b_0, τ, α_i and b_{ij}^*, $i, j = 1, \dots, r$ are constants):*

a) $b_{ii}(t) \ge b_0 > 0$, $i = 1, \dots, m$,

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t \ge t_0} \frac{1}{b_{ii}(t)} \sum_{\substack{j=1 \\ j \ne i}}^m |b_{ij}(t)| < 1,$$

and

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t \ge t_0} \left[\int_{\max\{t_0, h(t)\}}^t \sum_{j=1}^m |b_{ij}(s)| ds + \frac{1}{b_{ii}(t)} \sum_{\substack{j=1 \\ j \ne i}}^m |b_{ij}(t)| \right] < 1 + \frac{1}{\mathrm{e}} .$$

b) $b_{ii}(t) \ge \alpha_i > 0$, $|b_{ij}(t)| \le b_{ij}^*$, $t - h(t) \le \tau$, $i, j = 1, \dots, m$,

$$\max_{i=1,\dots,m} \frac{1}{\alpha_i} \sum_{\substack{j=1 \\ j \ne i}}^m b_{ij}^* < 1, \qquad \max_{i=1,\dots,m} \left[\tau \sum_{j=1}^m b_{ij}^* + \frac{1}{\alpha_i} \sum_{\substack{j=1 \\ j \ne i}}^m b_{ij}^* \right] < 1 + \frac{1}{\mathrm{e}} .$$

Then, the system (4.12) *is uniformly exponentially stable.*

Proof. System (4.12) can be written in the form

$$\dot{x}_i(t) = -\sum_{j=1}^m b_{ij}(t) x_j(h(t)), \qquad i = 1, \dots, m.$$

Now, the corollary directly follows from Corollaries 4.1 and 4.3. □

Corollary 4.8. *Assume that, for $t \ge t_0$,*

$$a_{ii}(t) \ge 0, \qquad b_{ii}(t) \ge 0, \qquad a_{ii}(t) + b_{ii}(t) \ge a_0 > 0, \qquad i = 1, \dots, m,$$

where a_0 is a constant,

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\geq t_0} \frac{1}{a_{ii}(t)+b_{ii}(t)} \sum_{\substack{j=1\\ j\neq i}}^{m} (|a_{ij}(t)|+|b_{ij}(t)|) < 1,$$

and

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\geq t_0} \frac{1}{a_{ii}(t)+b_{ii}(t)} \left[b_{ii}(t) \int_{\max\{t_0,h(t)\}}^{t} \sum_{j=1}^{m} (|a_{ij}(s)|+|b_{ij}(s)|)ds + \sum_{\substack{j=1\\ j\neq i}}^{m} (|a_{ij}(t)|+|b_{ij}(t)|) \right]$$
$$< 1+\frac{1}{\mathrm{e}} \,. \tag{4.14}$$

Then, the system (4.13) is uniformly exponentially stable.

Proof. We can write system (4.13) as

$$\dot{x}(t) = -\sum_{j=1}^{m} a_{ij}(t)x_j(t)) - \sum_{j=1}^{m} b_{ij}(t)x_j(h(t)), \qquad i=1,\dots,m$$

and use Theorem 3.1 for the choice $r_{ii}=2$, $a^1_{ij}(t)=a_{ij}(t)$, $a^2_{ij}(t)=b_{ij}(t)$, $h^1_{ij}(t)=t$, $h^2_{ij}(t)=h(t)$, $i,j=1,\dots,m$. Hence, $a_i(t)=a_{ii}(t)+b_{ii}(t)$, $i=1,\dots,m$ and inequality (4.14) coincides with (3.4). □

Consider particular cases of systems (4.12), (4.13)

$$\dot{X}(t) = -BX(t-\tau) \tag{4.15}$$

and

$$\dot{X}(t) = -AX(t) - BX(t-\tau) \tag{4.16}$$

where $A=(a_{ij})^m_{i,j=1}$ and $B=(b_{ij})^m_{i,j=1}$ are constant matrices, $\tau>0$, and $a_{ii}\geq 0$, $b_{ii}\geq 0$, $i=1,\dots,m$.

Corollary 4.9. *Assume that $b_{ii}>0$, $i=1,2,\dots,m$, and*

$$\max_{i=1,\dots,m} \frac{1}{b_{ii}} \sum_{\substack{j=1\\ j\neq i}}^{m} |b_{ij}| < 1, \qquad \max_{i=1,\dots,m} \left[\tau \sum_{j=1}^{m} |b_{ij}| + \frac{1}{b_{ii}} \sum_{\substack{j=1\\ j\neq i}}^{m} |b_{ij}| \right] < 1+\frac{1}{\mathrm{e}} \,.$$

Then, the system (4.15) is uniformly exponentially stable.

Proof. This follows from Corollary 4.7 (b) where $\alpha_i = b_{ii}$. □

Corollary 4.10. *Assume that $a_{ii}\geq 0, b_{ii}\geq 0, a_{ii}+b_{ii}>0$,*

$$\frac{1}{a_{ii}+b_{ii}} \sum_{\substack{j=1\\ j\neq i}}^{m} (|a_{ij}|+|b_{ij}|) < 1,$$

and

$$\frac{1}{a_{ii}+b_{ii}} \left[\tau b_{ii} \sum_{j=1}^{m} (|a_{ij}|+|b_{ij}|) + \sum_{\substack{j=1\\ j\neq i}}^{m} (|a_{ij}|+|b_{ij}|) \right] < 1+\frac{1}{\mathrm{e}} \tag{4.17}$$

for $i=1,\dots,m$. Then, the system (4.16) is uniformly exponentially stable.

Proof. Estimating the left-hand side of inequality (4.14) in the case of system (4.16) and using (4.17), we obtain

$$\max_{i=1,\dots,m} \operatorname*{ess\,sup}_{t\geq t_0} \frac{1}{a_{ii}(t)+b_{ii}(t)} \left[b_{ii}(t) \int_{\max\{t_0,h(t)\}}^{t} \sum_{j=1}^{m} (|a_{ij}(s)|+|b_{ij}(s)|)ds + \sum_{\substack{j=1\\ j\neq i}}^{m} (|a_{ij}(t)|+|b_{ij}(t)|) \right]$$

$$\leq \max_{i=1,\dots,m} \frac{1}{a_{ii}+b_{ii}} \left[\tau b_{ii} \sum_{j=1}^{m} (|a_{ij}|+|b_{ij}|) + \sum_{\substack{j=1\\ j\neq i}}^{m} (|a_{ij}|+|b_{ij}|) \right] < 1+\frac{1}{\mathrm{e}}\,.$$

Therefore, inequality (4.14) holds and Corollary 4.10 is a consequence of Corollary 4.8. □

5 Concluding remarks

First we will compare the stability results obtained in the paper with some known result. Let system (1.1) be of the form

$$\begin{aligned} \dot{x}_1(t) &= -a_{11}(t)x_1(h_{11}(t)) - a_{12}(t)x_2(h_{12}(t)),\\ \dot{x}_2(t) &= -a_{21}(t)x_1(h_{21}(t)) - a_{22}(t)x_2(h_{22}(t)). \end{aligned} \tag{5.1}$$

Here, $m = 2$ and $r_{ij} = 1$, $i, j = 1, 2$. Assume that there are constants α_i, A_{ij}, τ_{ij}, $i, j = 1, 2$ such that $0 < \alpha_i \leq a_{ii}(t)$, $|a_{ij}(t)| \leq A_{ij}$ and $t - h_{ij}(t) \leq \tau_{ij} \leq K$ and, for a constant $q \in (0, 1)$, $|a_{12}(t)| \leq q a_{11}$ and $|a_{21}(t)| \leq q a_{22}$, $t \in [t_0, \infty)$. Then, (3.2) and (3.3) hold. Inequality (3.4) holds if

$$\begin{aligned} (A_{11}+A_{12})\tau_{11} + \frac{A_{12}}{\alpha_1} &< 1+\frac{1}{\mathrm{e}},\\ (A_{22}+A_{21})\tau_{22} + \frac{A_{21}}{\alpha_2} &< 1+\frac{1}{\mathrm{e}}. \end{aligned} \tag{5.2}$$

By Theorem 3.1, system (5.1) is uniformly exponential stable. The above assumptions are valid, e.g., for the choice

$$a_{ii}(t) \equiv A_{ii} = \alpha_i = 0.1, \qquad a_{ij}(t) \equiv A_{ij} = 0.099, \quad i \neq j, \qquad \tau_{ij} = 1.89 \tag{5.3}$$

in (5.1) if $i, j = 1, 2$.

Apply Theorem 1.6 if $t - h_{ij}(t) \equiv \tau_{ij} \leq K$, $a_{ii}(t) \equiv A_{ii} = \alpha_i > 0$, $a_{ij}(t) \equiv A_{ij}$ if $i \neq j$, $i, j = 1, 2$ in (5.1). Let $0 < a_{12} = b_{12}a_{11}$ and $0 < a_{21} = b_{21}a_{22}$, $t \in [t_0, \infty)$. We get $d_i = A_{ii}\tau_{ii}$, $i = 1, 2$. If $d_i < 1$, then

$$\begin{aligned} \tilde{b}_{12} &= -\left(\frac{2+A_{11}^2\tau_{11}^2}{2-A_{11}^2\tau_{11}^2}\right)\frac{A_{12}}{A_{11}},\\ \tilde{b}_{21} &= -\left(\frac{2+A_{22}^2\tau_{22}^2}{2-A_{22}^2\tau_{22}^2}\right)\frac{A_{21}}{A_{22}}. \end{aligned}$$

Theorem 1.6 implies (recall that a square matrix is a nonsingular M-matrix if its inverse is a positive matrix)) the following result. If

$$A_{ii}\tau_{ii} < 1, \qquad \tilde{b}_{12}\tilde{b}_{21} < 1,$$

then system (5.1) is asymptotically stable.

Let (5.3) is set in (5.1). Then,

$$A_{ii}\tau_{ii} = 0.189 < 1, \qquad \tilde{b}_{12}\tilde{b}_{21} \doteq 1.053 \not< 1$$

and Theorem 1.6 is not applicable.

It is not difficult to derive examples when conditions (5.2) hold, but stability conditions of another known results are not valid.

The stability conditions derived in the paper are written in the form of inequalities with the right-hand sides which are equal the constant $1 + 1/e$. As we mentioned in the introduction, the purpose of this paper was to improve all the results of [8] with the extra condition (1.9). The first open problem is to remove this condition in all statements of this paper.

Nevertheless, there is another challenge for a possible continuation of investigations. Analysing some stability results (e.g. [18, Theorem 5.9]) where in the inequalities considered, the constant $3/2$ plays a significant role as a non-improvable bound, an open problem arises, if we can expect that our results can be improved by replacing the constant $1 + 1/e$ by the constant $3/2$ in the inequalities used. An alternative problem is to prove or disprove that, for the general case of variable coefficients and delays, the constant $1 + 1/e$ is the best one possible.

For further results on the stability of linear delay differential systems, we refer, e.g., to the review paper [23] and to [19, 21]. Recent results on global asymptotic stability for delay differential systems can be found in [9, 10, 17, 22].

Another research challenge is the following. In this paper and in all known papers on the stability of linear delay differential systems, the conditions sufficient for stability involve only diagonal delays. It will be interesting to obtain stability conditions such that all delays are utilized in the relevant inequalities.

As noted in [8], only few necessary stability conditions are known for systems. One of the interesting problems is the following. To prove or disprove the following conjecture: if system (1.1) is asymptotically stable, then the sum of the diagonal elements is nonnegative, i.e.,

$$\sum_{i=1}^{m}\sum_{k=1}^{r_{ii}} a_{ii}^k(t) \geq 0, \qquad t \geq t_0.$$

Finally, we recall a problem tacitly mentioned in the introduction – for system (1.1), derive stability results that could be reduced to Theorems 1.1–1.5 in the scalar case.

Acknowledgements

The second and the third author were supported by the Grant 201/11/0768 of the Czech Grant Agency (Prague). The fourth author was supported by the grant FEKT-S-14-2200 of Faculty of Electrical Engineering and Communication, Brno University of Technology.

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Journal of Computational and Applied Mathematics 147 (2002) 315–331

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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An existence criterion of positive solutions of p-type retarded functional differential equations

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Received 20 February 2001; received in revised form 4 February 2002

Abstract

The conditions of existence of a positive solution (i.e., a solution with positive coordinates on a considered interval) of systems of retarded functional equations in the case of unbounded delay with finite memory are discussed. A general criterion for nonlinear case is given as well as its application to a linear system. Illustrative special cases are considered too. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 34K20; 34K25

Keywords: Positive solution; Delayed equation; Retract principle; p-Function

1. Introduction

The main aim of this paper is to give conditions for the existence of solutions with positive coordinates for systems of retarded functional differential equations (RFDEs) with unbounded delay and with finite memory. Before the formulation of the results of this paper let us give a short survey of the known results. In [11], a criterion concerning the existence of positive solution for the equation

$$\dot{x}(t) + p(t)x(t-\tau(t)) = 0 \tag{1}$$

is given, where $p,\tau \in C([t_0,\infty),\mathbb{R}_+)$, $\tau(t) \leqslant t$, $\lim_{t\to\infty}(t-\tau(t)) = \infty$ and $\mathbb{R}_+ = [0,\infty)$. A function x is called a solution of Eq. (1) corresponding to an initial point $t_1 \geqslant t_0$ (or with respect to t_1) if x is defined and is continuous on $[T_1,\infty)$, $T_1 = \inf_{t\geqslant t_1}\{t-\tau(t)\}$, differentiable on $[t_1,\infty)$, and satisfies (1) for $t \geqslant t_1$.

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PII: S0377-0427(02)00439-9

Theorem A (Erbe et al. [11, p. 29]). *Eq.* (1) *has a positive solution with respect to* t_1 *if and only if there exists a continuous function* $\lambda(t)$ *on* $[T_1,\infty)$ *such that* $\lambda(t)>0$ *for* $t\geqslant t_1$ *and*

$$\lambda(t)\geqslant p(t)\mathrm{e}^{\int_{t-\tau(t)}^{t}\lambda(s)\,\mathrm{d}s},\quad t\geqslant t_1.$$

This result was generalized for nonlinear systems of RFDEs in [6]. Some results in this direction are formulated in [13] and in [1] as well. Positive solutions of Eq. (1) in the critical case were studied e.g., in [5,7–11]. Unfortunately, results of [6] hold for systems with bounded retardation only. In the present paper we investigate the problem of existence of *positive solutions* (i.e., problem of existence of solutions having all its coordinates positive on considered intervals) for nonlinear systems of RFDEs with unbounded delay but with finite memory in the sense given in [16]. Let us recall this notion.

Definition 1 (Lakshmikamthan et al. [16, p. 8]). The function $p\in C[\mathbb{R}\times[-1,0],\mathbb{R}]$ is called a *p-function* if it has the following properties:

(i) $p(t,0)=t$;
(ii) $p(t,-1)$ is a nondecreasing function of t;
(iii) there exists a $\sigma\geqslant-\infty$ such that $p(t,\vartheta)$ is an increasing function for ϑ for each $t\in(\sigma,\infty)$.

Remark 1. Let us note that conditions (i) and (iii) imply property (iv) (introduced as an additional property in [16, p. 8]): $t=p(t,0)>p(t,-1)$ for $t\in(\sigma,\infty)$. In the following, we will suppose that t is sufficiently large, i.e., that (iii) holds on the considered intervals.

In the theory of RFDEs the symbol y_t, which expresses "*taking into account*", the history of the process $y(t)$ considered, is used. With the aid of p-functions the symbol y_t is defined as follows:

Definition 2 (Lakshmikamthan et al. [16, p. 8]). Let $t_0\in\mathbb{R}$, $A>0$ and $y\in C([p(t_0,-1),t_0+A),\mathbb{R}^n)$. For any $t\in[t_0,t_0+A)$, we define y_t by $y_t(\vartheta)=y(p(t,\vartheta))$, $-1\leqslant\vartheta\leqslant0$ and write $y_t\in\mathscr{C}\equiv C[[-1,0],\mathbb{R}^n]$.

In this paper we investigate the system

$$\dot y(t)=f(t,y_t),\tag{2}$$

where $f\in C[[t_0t_0+A)\times\mathscr{C},\mathbb{R}^n]$. This system is called the system of *p-type retarded functional differential equations* (p-RFDEs). The function $y\in C([p(t_0,-1),t_0+A),\mathbb{R}^n)\cap C^1([t_0,t_0+A),\mathbb{R}^n)$ satisfying (2) on $[t_0,t_0+A)$ is called a *solution of this system* p-RFDEs *on* $[[p(t_0,-1),t_0+A)$.

Remark 2. System (2) with y_t defined in accordance with Definition 2 is called a system *with unbounded delay with finite memory*. Note that the frequently used symbol "y_t" (e.g., in accordance with [14, p. 38], $y_t(s)=y(t+s)$, where $-\tau\leqslant s\leqslant0$, $\tau>0$, $\tau=\text{const}$) for equation with bounded delay is a partial case of the above definition of y_t. Indeed, in this case we can put $p(t,\vartheta)\equiv t+\tau\vartheta$.

Suppose that Ω is an open subset of $\mathbb{R} \times \mathscr{C}$ and the function $f \in C(\Omega, \mathbb{R}^n)$. If $(t_0, \phi) \in \Omega$, then *there exists a solution* $y = y(t_0, \phi)$ of the system p-RFDEs (2) through (t_0, ϕ) (see [16, p. 25]). Moreover, this *solution is unique* if $f(t, \phi)$ is locally Lipschitzian with respect to ϕ [16, p. 30] and is *continuable* in the usual sense of extended existence if f is quasibounded (see [16, p. 41]). Suppose that the solution $y = y(t_0, \phi)$ of p-RFDEs (2) through $(t_0, \phi) \in \Omega$, defined on $[t_0, A]$, is unique. Then the property of the *continuous dependence* holds too (see [16, p. 33]), i.e., for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $(s, \psi) \in \Omega$, $|s - t_0| < \delta$ and $\|\psi - \phi\| < \delta$ imply

$$\|y_t(s, \psi) - y_t(t_0, \phi)\| < \varepsilon, \quad \text{for all } t \in [\zeta, A],$$

where $y(s, \psi)$ is the solution of the system p-RFDEs (2) through (s, ψ), $\zeta = \max\{s, t_0\}$ and $\|\cdot\|$ is the supremum norm in $\mathbb{R}^n$. Note that these results can be adapted easily for the case (which will be used in the sequel) when Ω has the form $\Omega = [p^*, \infty) \times \mathscr{C}$ where $p^* \in \mathbb{R}$ and the cross-section $\{(\tilde{t}, \varphi) \in \Omega\}$ is an open set for every $\tilde{t} \in [p^*, \infty)$.

The paper is organized as follows. In Section 2, a general nonlinear case is considered and the main result of the paper is presented together with its nonlinear applications. Applications to a linear system and scalar linear equations are given in Section 3. Proofs of the results (and corresponding auxiliary material) are collected in Section 4. The method used in the proof of the main result also permits to conclude that positive solutions of nonlinear equations exist on half-infinity interval. This is an additional advantage of the results presented.

2. Nonlinear case

With $\mathbb{R}^n_{\geqslant 0}$ ($\mathbb{R}^n_{>0}$) we denote the set of all component-wise nonnegative (positive) vectors v in $\mathbb{R}^n$, i.e., $v = (v_1, \ldots, v_n) \in \mathbb{R}^n_{\geqslant 0}(\mathbb{R}^n_{>0})$ if and only if $v_i \geqslant 0$ ($v_i > 0$) for $i = 1, \ldots, n$. For $u, v \in \mathbb{R}^n$ we write $u \leqslant v$ if $v - u \in \mathbb{R}^n_{\geqslant 0}$; $u \ll v$ if $v - u \in \mathbb{R}^n_{>0}$ and $u < v$ if $u \leqslant v$ and $u \neq v$.

2.1. General nonlinear case

Let p^*, t^* be constants satisfying $p^* = p(t^*, -1)$ for a given p-function. Let us introduce vectors $\rho, \delta \in C([p^*, \infty), \mathbb{R}^n) \cap C^1([t^*, \infty), \mathbb{R}^n)$ satisfying $\rho \ll \delta$ on $[p^*, \infty)$. Let us suppose $\Omega \subseteq (t_0, \infty) \times \mathscr{C}$ with $t_0 \leqslant t^*$ and let us put

$$\omega := \{(t, y) \colon t \geqslant p^*,\ \rho(t) \ll y \ll \delta(t)\}.$$

Theorem 1. *Suppose $f \in C(\Omega, \mathbb{R}^n)$ is locally Lipschitzian with respect to the second argument, quasibounded and, moreover*:

(i) *For any $i = 1, \ldots, p$ (with $p \in \{0, 1, \ldots, n\}$), $t \geqslant t^*$ and $\pi \in C([p(t, -1), t], \mathbb{R}^n)$ such that $(\theta, \pi(\theta)) \in \omega$ for all $\theta \in [p(t, -1), t)$, $(t, \pi(t)) \in \partial\omega$ it follows $(t, \pi_t) \in \Omega$,*

$$\delta_i'(t) < f_i(t, \pi_t) \quad \textit{when } \pi_i(t) = \delta_i(t) \tag{3}$$

and

$$\rho_i'(t) > f_i(t, \pi_t) \quad \textit{when } \pi_i(t) = \rho_i(t). \tag{4}$$

(ii) *For any* $i = p + 1, \ldots, n$, $t \geqslant t^*$ *and* $\pi \in C([p(t,-1), t], \mathbb{R}^n)$ *such that* $(\theta, \pi(\theta)) \in \omega$ *for all* $\theta \in [p(t,-1), t)$, $(t, \pi(t)) \in \partial\omega$ *it follows* $(t, \pi_t) \in \Omega$,

$$\delta_i'(t) > f_i(t, \pi_t) \quad \text{when } \pi_i(t) = \delta_i(t) \tag{5}$$

and

$$\rho_i'(t) < f_i(t, \pi_t) \quad \text{when } \pi_i(t) = \rho_i(t). \tag{6}$$

Then there exists an uncountable set $\mathscr{Y}$ *of solutions of system* (2) *on the interval* $[p^*, \infty)$ *such that for each* $y \in \mathscr{Y}$

$$\rho(t) \ll y(t) \ll \delta(t), \quad t \in [p^*, \infty). \tag{7}$$

Remark 3. The number p in the formulation of Theorem 1 and in the following can also be equal to 0 or n. In such cases the corresponding conditions (either (i) or (ii)) are omitted. Note that in the case $\rho(t) \geqslant 0$ we deal, as follows from (7), with positive solutions.

Definition 3. We say that the functional $g \in C(\Omega, \mathbb{R})$ is *strongly decreasing* (*strongly increasing*) with respect to the second argument on Ω if for each $(t, \varphi) \in \Omega$ and $(t, \psi) \in \Omega$ such that

$$\varphi(p(t, \vartheta)) \ll \psi(p(t, \vartheta)), \quad \vartheta \in [-1, 0),$$

the inequality

$$g(t, \varphi) > g(t, \psi) \quad (\text{or } g(t, \varphi) < g(t, \psi))$$

holds.

Let $k \gg 0$ μ be constant vectors, $\mu_i = -1$ for $i = 1, \ldots, p$ and $\mu_i = 1$ for $i = p + 1, \ldots, n$. Let $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ denote a vector, having continuous entries on $[p^*, \infty)$. Define

$$T(k, \lambda)(t) \equiv k\, \mathrm{e}^{\mu \int_{p^*}^{t} \lambda(s)\,\mathrm{d}s} = (k_1\, \mathrm{e}^{\mu_1 \int_{p^*}^{t} \lambda_1(s)\,\mathrm{d}s}, \ldots, k_n\, \mathrm{e}^{\mu_n \int_{p^*}^{t} \lambda_n(s)\,\mathrm{d}s}).$$

Theorem 2 (Main result). *Suppose* $\Omega = [t^*, \infty) \times \mathscr{C}$, $f \in C(\Omega, \mathbb{R}^n)$ *is locally Lipschitzian with respect to the second argument*, *quasibounded and*, *moreover*:

(i) $f(t, 0) \equiv 0$ *if* $t \geqslant t^*$.
(ii) *The functional* f_i *is strongly decreasing if* $i = 1, \ldots, p$ *and strongly increasing if* $i = p+1, \ldots, n$ *with respect to the second argument on* Ω.

Then for the existence of a positive solution $y = y(t)$ *on* $[p^*, \infty)$ *of the system p-RFDEs* (2) *a necessary and sufficient condition is that there exists a vector* $\lambda \in C([p^*, \infty), \mathbb{R}^n)$, *such that* $\lambda \gg 0$ *on* $[t^*, \infty)$, *satisfying the system of integral inequalities*

$$\lambda_i(t) \geqslant \frac{\mu_i}{k_i}\, \mathrm{e}^{-\mu_i \int_{p^*}^{t} \lambda_i(s)\,\mathrm{d}s} f_i(t, T(k, \lambda)_t), \quad i = 1, \ldots, n \tag{8}$$

for $t \geqslant t^*$, *with a positive constant vector* k *and with* $\mu_i = -1$ *for* $i = 1, \ldots, p$; $\mu_i = 1$ *for* $i = p+1, \ldots, n$.

2.2. Nonlinear applications

Consider a nonlinear integro-differential equation

$$\dot{y}(t) = a\int_0^t L(s)y(t-s)\,\mathrm{d}s + by^2(t), \quad t \geqslant \tilde{\varepsilon} > 0 \tag{9}$$

with continuous function $L:[0,\infty) \to \mathbb{R}^+ = (0,\infty)$, $a \in \{-1,1\}$, $b \in \mathbb{R}$ and $\operatorname{sign} b = \operatorname{sign} a$. Note that similar classes of equations are used for describing the dynamics of a single species of population (see e.g. [12]). The following theorem (the proof of which is a consequence of the main result) holds:

Theorem 3. *For the existence of a positive solution $y = y(t)$ on $[0,\infty)$ of Eq.* (9) satisfying *Eq.* (9) *on the prescribed interval $[\tilde{\varepsilon},\infty)$, the existence of the function $\lambda \in C([0,\infty),\mathbb{R})$, positive on $[\tilde{\varepsilon},\infty)$ and satisfying here the integral inequality*

$$\lambda(t) \geqslant \int_0^t L(s)\mathrm{e}^{-a\int_{t-s}^t \lambda(u)\,\mathrm{d}u}\,\mathrm{d}s + abk\,\mathrm{e}^{a\int_0^t \lambda(s)\,\mathrm{d}s} \tag{10}$$

with a positive constant k, is a necessary and sufficient condition.

Remark 4. As an addition to this theorem note that every positive solution $y = y(t)$ of Eq. (9) with $a = 1$ and with any $b \in \mathbb{R}$, which is defined on interval $[0,A]$, remains positive on its maximal interval of existence $[0,B] \subseteq [0,\infty)$ with $B > A$ (see the proof of Theorem 3).

Theorem 4. *Consider the equation of type* (9) *with $a = -1$, i.e., the equation*

$$\dot{y}(t) = -\int_0^t L(s)y(t-s)\,\mathrm{d}s + by^2(t), \quad t \geqslant \tilde{\varepsilon} > 0, \tag{11}$$

where $b < 0$ and suppose $L(t) \leqslant l\,\mathrm{e}^{-\nu t}$, $t \in [0,\infty)$ with positive constants l,ν and with $\nu > 2\sqrt{l}$. Then there exists a positive solution $y = y(t)$ of Eq. (11) *on* $[0,\infty)$, *satisfying Eq.* (11) *on* $[\tilde{\varepsilon},\infty)$.

3. Linear case

The main result can be applied easily to various classes of linear delayed systems and can serve as a source for various new criteria.

3.1. Linear delayed system

Let us consider the linear system

$$\dot{y} = A(t)y(t) + B(t)y(\tau(t)), \tag{12}$$

where $\tau:[t^*,\infty) \to [p^*,\infty)$ is a continuous nondecreasing function and $\tau(t) < t$. In this case, $p(t,\vartheta) = t + \vartheta\cdot(t-\tau(t))$ and $p^* = \tau(t^*)$. With respect to $n\times n$ matrices $A(t) = (a_{ij}(t))$, $B(t) = (b_{ij}(t))$

we suppose their continuity on $[t^*,\infty)$ and, moreover, the validity of inequalities:

$$a_{ij}(t) \leqslant 0,\ b_{ij}(t) \leqslant 0 \qquad \text{if } i=1,\dots,p, \quad j=1,\dots,n, \quad t\in[t^*,\infty), \tag{13}$$

$$a_{ij}(t) \geqslant 0,\ b_{ij}(t) \geqslant 0 \qquad \text{if } i=p+1,\dots,n, \quad j=1,\dots,n, \quad t\in[t^*,\infty), \tag{14}$$

$$\sum_{j=1}^{n} b_{ij}(t) \neq 0 \quad \text{for every } i=1,\dots,n \text{ and } t\in[t^*,\infty). \tag{15}$$

Theorem 5. *For the existence of a solution $y=y(t)$ of system* (12), *positive on $[p^*,\infty)$, a necessary and sufficient condition is that there exists a continuous vector $\lambda\in C([p^*,\infty),\mathbb{R}^n)$ such that $\lambda(t)\gg 0$ for $t\geqslant t^*$, satisfying the system of integral inequalities*

$$\begin{aligned}\lambda_i(t) \geqslant{} & \mu_i\left(a_{ii}(t)+b_{ii}(t)\mathrm{e}^{-\mu_i\int_{\tau(t)}^{t}\lambda_i(s)\,\mathrm{d}s}\right)\\ & +\frac{\mu_i}{k_i}\sum_{j=1,j\neq i}^{n} k_j\,\mathrm{e}^{\int_{p^*}^{t}(\mu_j\lambda_j(s)-\mu_i\lambda_i(s))\,\mathrm{d}s}\left(a_{ij}(t)+b_{ij}(t)\mathrm{e}^{-\mu_j\int_{\tau(t)}^{t}\lambda_j(s)\,\mathrm{d}s}\right), \quad i=1,\dots,n\end{aligned} \tag{16}$$

on $[t^*,\infty)$ with a positive constant vector k and with $\mu_i=-1$ for $i=1,\dots,p$; $\mu_i=1$ for $i=p+1,\dots,n$.

Remark 5. Let us remark that sufficient conditions for the existence of bounded solutions of systems and equations of type (12) were given in [3,4].

3.2. Scalar linear applications

Let us consider the scalar linear equation with delay

$$\dot{y}(t) = -\int_{\tau(t)}^{t} K(t,s)y(s)\,\mathrm{d}s, \tag{17}$$

where $K:[t^*,\infty)\times[p^*,\infty)\to\mathbb{R}^+$ is a continuous function, and $\tau:[t^*,\infty)\to[p^*,\infty)$ is a nondecreasing function with $\tau(t)<t$.

Theorem 6. *Eq.* (17) *has a positive solution $y=y(t)$ on $[p^*,\infty)$ if and only if there exists a function $\lambda\in C([p^*,\infty),\mathbb{R})$, such that $\lambda(t)>0$ for $t\geqslant t^*$ and*

$$\lambda(t) \geqslant \int_{\tau(t)}^{t} K(t,s)\mathrm{e}^{\int_s^t \lambda(u)\,\mathrm{d}u}\,\mathrm{d}s \tag{18}$$

on the interval $[t^*,\infty)$.

Inequality (18) can be used for finding sufficient conditions for the existence of a positive solution of Eq. (17). Let us give two of them.

In the case when $\tau(t)\equiv p^*<t^*$ and $K(t,s)\equiv c(t)$ for every $t\in[t^*,\infty)$, Eq. (17) takes the form

$$\dot{y}(t) = -c(t)\int_{p^*}^{t} y(s)\,\mathrm{d}s. \tag{19}$$

Theorem 7. *For the existence of a solution of Eq.* (19), *positive on* $[p^*, \infty)$, *the inequality*

$$c(t) \leqslant \frac{\delta^2}{e^{\delta(t-p^*)} - 1}, \quad t \in [t^*, \infty) \tag{20}$$

with a positive constant δ *is a sufficient condition.*

In the case when $\tau(t) \equiv t - l$, $l \in \mathbb{R}^+$ and $K(t,s) \equiv c(t)$ for every $t \in [t^*, \infty)$, Eq. (17) takes the form

$$\dot{y}(t) = -c(t) \int_{t-l}^{t} y(s)\,ds. \tag{21}$$

Theorem 8. *For the existence of a solution of Eq.* (21), *positive on* $[t^* - l, \infty)$, *the inequality*

$$c(t) \leqslant M, \quad t \in [t^*, \infty) \tag{22}$$

is sufficient for $M = \alpha(2-\alpha)/l^2 = \text{const}$ *with a constant* α *being the positive root of the equation* $2 - \alpha = 2e^{-\alpha}$. (*The approximate values are* $\alpha \doteq 1.5936$ *and* $M \doteq 0.6476/l^2$.)

4. Auxiliary material and proofs

4.1. Retract principle and Lyapunov-type principle

The proof of Theorem 1 is made with the aid of the retract principle. This principle, well known and often used in the theory of ordinary differential equations (see e.g. [15]), goes back to Ważewski [19]. For RFDEs with bounded retardation, this principle was modified e.g. in Rybakowski [18]. Here, we use Rybakowski's modified result (Lemma 1 below) which concerns the existence of at least one curve in a given family of curves, with graph lying in an open set. Then this lemma is applied to systems of p-RFDEs (Lemma 3 below). Except this, the inverse principle is used (Lemmas 2 and 4). This principle has the origin in the theory of Lyapunov stability and for retarded functional differential equations, it was developed by Razumikhin (e.g., [17]).

If a set $A \subset \mathbb{R} \times \mathbb{R}^n$ is given, then $\operatorname{int} A$, $\bar{A}$ and ∂A denote, as usual, the interior, the closure, and the boundary of A, respectively.

Definition 4. Let Λ be a topological space, let a subset $\tilde{\Omega} \subset \mathbb{R} \times \Lambda$ be open in $\mathbb{R} \times \Lambda$, and let x be a mapping, associating with every $(\delta, \lambda) \in \tilde{\Omega}$ a function $x(\delta, \lambda) : D_{\delta,\lambda} \to \mathbb{R}^n$ where $D_{\delta,\lambda}$ is an interval in $\mathbb{R}$. Assume (1)–(3)

(1) $\delta \in D_{\delta,\lambda}$.
(2) If $t \in \operatorname{int} D_{\delta,\lambda}$, then there is open neighbourhood $\mathcal{O}(\delta, \lambda)$ of (δ, λ) in $\tilde{\Omega}$ such that $t \in D_{\delta',\lambda'}$ holds for all $(\delta', \lambda') \in \mathcal{O}(\delta, \lambda)$.
(3) If (δ', λ'), $(\delta, \lambda) \in \tilde{\Omega}$, and $t' \in D_{\delta',\lambda'}$, $t \in D_{\delta,\lambda}$, then

$$\lim_{(\delta',\lambda',t') \to (\delta,\lambda,t)} x(\delta', \lambda')(t') = x(\delta, \lambda)(t).$$

If all these conditions are satisfied, then $(\Lambda, \tilde{\Omega}, x)$ is called a *system of curves* in $\mathbb{R}^n$.

Studying the proof of Theorem 2.1 in [18, p. 119], we get the following formulation of it, suitable for our applications.

Lemma 1 (Retract principle). *Let $(\Lambda, \tilde{\Omega}, x)$ be a system of curves in $\mathbb{R}^n$. Let $\tilde{\omega}$, W, Z be sets. Assume that conditions* (1)–(4) *below hold*:

(1) (*a*) $\tilde{\omega} \subset [p^*, \infty) \times \mathbb{R}^n$ *where* $p^* \in \mathbb{R}$ *and the cross-section* $\{(\tilde{t}, y) \in \tilde{\omega}\}$ *is an open set for every* $\tilde{t} \in [p^*, \infty)$, $W \subset \partial\tilde{\omega}$,
(*b*) $z \subset \tilde{\omega} \cup W$, $Z \cap W$ *is a retract of* W, *but not a retract of* Z.
(2) *There is a continuous map* $q : B \to \Lambda$, *where* $B = \bar{Z} \cap (Z \cup W)$, *such that for any* $z = (\delta, y) \in B$: $(\delta, q(z)) \in \tilde{\Omega}$, *and if also* $z \in W$, *then* $x(\delta, q(z))(\delta) = y$.
(3) *Let* A *be the set of all* $z = (\delta, y) \in Z \cap \tilde{\omega}$ *such that for fixed* $(\delta, y) \in A$ *there is a* $t > \delta$, $t \in D_{\delta, q(z)}$ *and* $(t, x(\delta, q(z))(t)) \notin \tilde{\omega}$.
Assume that for every $z = (\delta, y) \in A$ *there is a* $t(z), t(z) > \delta$, *such that*:

(a) $t(z) \in D_{\delta, q(z)}$ *and for all* t, $\delta \leqslant t < t(z)$: $(t, x(\delta, q(z))(t)) \in \tilde{\omega}$,
(b) $(t(z), x(\delta, q(z))(t(z))) \in W$,
(c) *for any* $\sigma > 0$, *there is a* $t = t(\sigma, z)$, $t(z) < t \leqslant t(z) + \sigma$, *such that* $t \in D_{\delta, q(z)}$ *and* $(t, x(\delta, q(z))(t)) \notin \bar{\tilde{\omega}}$.
(4) *For any* $z = (\delta, y) \in W \cap B$, *and all* $\sigma > 0$, *there is a* $t = t(\sigma, z)$, $\delta < t \leqslant \delta + \sigma$ *such that* $t \in D_{\delta, q(z)}$ *and* $(t, x(\delta, q(z))(t)) \notin \tilde{\omega}$.

Then there is a $z_0 = (\delta_0, y_0) \in Z \cap \tilde{\omega}$ *such that for every* $t \in D_{\delta_0, q(z_0)}$:
$$(t, x(\delta_0, q(z_0))(t)) \in \tilde{\omega}.$$

Proof of the following lemma is obvious and is therefore omitted.

Lemma 2 (Lyapunov principle). *Let $(\Lambda, \tilde{\Omega}, x)$ be a system of curves in $\mathbb{R}^n$ and $\tilde{\omega}$ be a set. Assume that conditions* (1)–(4) *below hold*:

(1) $\tilde{\omega} \subset [p^*, \infty) \times \mathbb{R}^n$ *where* $p^* \in \mathbb{R}$ *and the cross-section* $\{(\tilde{t}, y) \in \tilde{\omega}\}$ *is an open set for every* $\tilde{t} \in [p^*, \infty)$.
(2) *There is a continuous map* $q : B \to \Lambda$, *where* $B = \bar{\tilde{\omega}} \cap \{(t^*, y), t^* \in \mathbb{R},\ t^* = \text{const},\ t^* > p^*,\ y \in \mathbb{R}^n\}$, *such that for any* $z = (t^*, y) \in B$: $(t^*, q(z)) \in \tilde{\Omega}$, *and if also* $z \in \partial\tilde{\omega}$, *then* $x(t^*, q(z))(t^*) = y$.
(3) *For every* $z = (t^*, y) \in B \cap \tilde{\omega}$ *with property* $(t, x(t^*, q(z))(t)) \in \tilde{\omega}$ *for all* t *within an interval* $t^* < t < t(z)$ *and* $(t(z), x(t^*, q(z))(t(z))) \in \partial\tilde{\omega}$, $t(z) \in D_{t^*, q(z)}$ *there is a* σ *such that* $t(z) + \sigma \in D_{t^*, q(z)}$ *and* $(t, x(t^*, q(z))(t)) \in \tilde{\omega}$ *for all* t, $t(z) < t < t(z) + \sigma$.
(4) *For any* $z = (t^*, y) \in B \cap \partial\tilde{\omega}$, *and all* $\sigma > 0$, *there is a* $t = t(\sigma, z)$, $\delta < t \leqslant \delta + \sigma$ *such that* $t \in D_{t^*, q(z)}$ *and* $(t, x(t^*, q(z))(t)) \in \tilde{\omega}$.

Then for every $z_0 = (t^*, y_0) \in B \cap \tilde{\omega}$ *and every* $t \in D_{t^*, q(z_0)}$:
$$(t, x(t^*, q(z_0))(t)) \in \bar{\tilde{\omega}}. \tag{23}$$

4.2. Regular polyfacial set, retract, and Lyapunov methods for p-RFDEs

Let $\Lambda = \mathscr{C}$. Let $\tilde{\Omega}$ be open in $\mathbb{R} \times \mathscr{C}$, $f \in C(\tilde{\Omega}, \mathbb{R}^n)$ and through each $(\delta, \lambda) \in \tilde{\Omega}$ there exists a unique solution $y(\delta, \lambda)$ of (2) defined on maximal interval $[\delta, a)$, $\delta < a \leqslant \infty$. Let $D_{\delta,\lambda} = [\delta, a)$. Then $(\Lambda, \tilde{\Omega}, y)$ is a system of curves in $\mathbb{R}^n$ in the sense of Definition 4.

Let l_i, m_j, $i = 1, \ldots, p$, $j = 1, \ldots, s$, $p + s > 0$ be real-valued C^1-functions defined on $\mathbb{R} \times \mathbb{R}^n$. The set

$$\tilde{\omega} = \{(t, y) \in [p^*, \infty) \times \mathbb{R}^n,\ l_i(t, y) < 0,\ m_j(t, y) < 0, \text{ for all } i, j\}$$

will be called a *polyfacial set*.

Definition 5. A polyfacial set $\tilde{\omega}$ is called *regular with respect to Eq.* (2) if $(\alpha), (\beta), (\gamma)$ below hold:

(α) If $(t, \phi_t) \in \mathbb{R} \times \mathscr{C}$ and if $(p(t, \vartheta), \phi_t(\vartheta)) \in \tilde{\omega}$ for all $\vartheta \in [-1, 0)$, then $(t, \phi_t) \in \tilde{\Omega}$.

(β) For all $i = 1, \ldots, p$, all $(t, y) \in \partial\tilde{\omega}$ for which $l_i(t, y) = 0$ and for all $\phi_t \in \mathscr{C}$ for which $\phi_t(0) = y$ and $(p(t, \vartheta), \phi_t(\vartheta)) \in \tilde{\omega}$ for all $\vartheta \in [-1, 0)$, it follows that

$$Dl_i(t, y) \equiv \sum_{r=1}^{n} \frac{\partial l_i}{\partial y_r}(t, y) f_r(t, \phi_t) + \frac{\partial l_i}{\partial t}(t, y) > 0.$$

(γ) For all $j = 1, \ldots, s$, all $(t, y) \in \partial\tilde{\omega}$ which $m_j(t, y) = 0$ and for all $\phi_t \in \mathscr{C}$ which $\phi_t(0) = y$ and $(p, (t, \vartheta), \phi_t(\vartheta)) \in \tilde{\omega}$ for all $\vartheta \in [-1, 0)$, it follows that

$$Dm_j(t, y) \equiv \sum_{r=1}^{n} \frac{\partial m_j}{\partial y_r}(t, y) f_r(t, \phi_t) + \frac{\partial m_j}{\partial t}(t, y) < 0.$$

The following lemma concerning the existence of a solution of Eq. (2) with graph remaining in the set $\tilde{\omega}$ on its maximal existence interval, will play a crucial role in the proof of Theorem 1.

Lemma 3 (Retract method). *Let $p > 0$. Let $\tilde{\omega}$ be a nonempty polyfacial set, regular with respect to Eq.* (2), *let the function $f \in C(\tilde{\Omega}, \mathbb{R}^n)$ be locally Lipschitzian with respect to the second argument, and*

$$W = \{(t, y) \in \partial\tilde{\omega}\colon m_j(t, y) < 0,\ j = 1, \ldots, s\}. \tag{24}$$

Let Z be a subset of $\tilde{\omega} \cup W$ and let mapping $q : B = \bar{Z} \cap (Z \cup W) \to \mathscr{C}$ be continuous and such that if $z = (\delta, y) \in B$, then $(\delta, q(z)) \in \tilde{\Omega}$, and:

(1) *if $z \in Z \cap \tilde{\omega}$, then $(p(\delta, \vartheta), q(z)(p(\delta, \vartheta))) \in \tilde{\omega}$ for $\vartheta \in [-1, 0]$,*

(2) *if $z \in W \cap B$, then $(\delta, q(z)(\delta)) = z$ and $(p(\delta, \vartheta), q(z)(p(\delta, \vartheta))) \in \tilde{\omega}$ for $\vartheta \in [-1, 0)$.*

Let, moreover, $Z \cap W$ be a retract of W, but not a retract of Z. Then there exists a $z_0 = (\delta_0, y_0) \in Z \cap \tilde{\omega}$ such that $(t, y(\delta_0, q(z_0))(t)) \in \tilde{\omega}$ for every $t \in D_{\delta_0, q(z_0)}$.

Proof. We prove the lemma using Lemma 1. Conditions (1) and (2) of Lemma 1 are obviously satisfied. Let us verify conditions (3) and (4).

Verification of condition (3): Let $z=(\delta, y)\in A$, and let $t(z)$ be the smallest of all $t\geqslant\delta$ such that $t\in D_{\delta,q(z)}$ and $(t, y(\delta, q(z))(t))\notin\tilde{\omega}$. Since $(\delta, y(\delta, q(z))(\delta))=(\delta, q(z)(p(t,0)))\in\tilde{\omega}$, it follows that $\delta<t(z)<\infty$. Obviously, $(t(z), y(\delta, q(z))(t(z)))\in\partial\tilde{\omega}$ and moreover for $\delta\leqslant t<t(z)$ it holds: $(t, y(\delta, q(z))(t))\in\tilde{\omega}$, hence (3a) is satisfied.

Let $\phi_t\equiv y_{t(z)}(\delta, q(z))$. Obviously $\phi_t\in\mathscr{C}$. Then $(t(z)), \phi_t)\in\tilde{\Omega}$, and $(t(z), \phi(t(z)))=(t(z), y(\delta, q(z))(t(z)))\in\partial\tilde{\omega}$, and

$$(p(t(z),\vartheta), \phi(p(t(z),\vartheta)))\in\tilde{\omega}, \quad \text{for } \vartheta\in[-1,0).$$

To prove condition (3b) suppose, on the contrary, that $(t(z), \phi(p(t(z),0)))\notin W$. Since $(t(z), \phi(p(t(z),0)))\in\partial\tilde{\omega}$ it follows $m_{j_0}(t(z), \phi(p(t(z),0)))=0$ for some $j_0\in\{1,\dots,s\}$. Hence, inequality (γ) in Definition 5 is satisfied. Since $y(\delta, q(z))(t)$ is differentiable in t for $t>\delta$, this inequality becomes

$$Dm_{j_0}(t, y(\delta, p(z))(t))|_{t=t(z)}<0,$$

i.e., for some $\sigma>0$ and all $0<h<\sigma$:

$$\begin{aligned} &m_{j_0}(t(z)-h, y(\delta, q(z))(t(z)-h)) \\ &\quad > m_{j_0}(t(z), y(\delta, q(z))(t(z)))=m_{j_0}(t(z), \phi(p(t(z),0))=m_{j_0}(t(z), \phi_t(0))=0. \end{aligned}$$

Hence, $(t(z)-h, y(\delta, q(z))(t(z)-h))\notin\bar{\tilde{\omega}}$. This is a contradiction to (3a). Then $(t(z), \phi_t(0))\in W$ and, therefore, (3b) is satisfied.

It follows that $l_{i_0}(t(z), \phi(t(z),0))=0$ for some $i_0\in\{1,\dots,p\}$. Applying (β) of Definition 5, we get

$$Dl_{i_0}(t, y(\delta, p(z))(t))|_{t=t(z)}>0,$$

hence, for some $\sigma>0$ and all $0<h<\sigma$:

$$l_{i_0}(t(z)+h, y(\delta, q(z))(t(z)+h))>l_{i_0}(t(z), y(\delta, q(z))(t(z)))=l_{i_0}(t(z), \phi(p(t(z),0))=0.$$

Hence $(t(z)+h, y(\delta, q(z))(t(z)+h))\notin\bar{\tilde{\omega}}$ and (3c) is satisfied.

Verification of condition (4): If $z=(\delta, y)\in W\cap B$, then there is a $i_0\in\{1,\dots,p\}$ such that $l_{i_0}(\delta, y)=0$. Let $\phi=q(z)$, then $(\delta+\vartheta, \phi(p(\delta,\vartheta)))\in\tilde{\omega}$, for all $\vartheta\in[-1,0)$. Hence, the derivative from the right

$$Dl_{i_0}(t, y(\delta, p(z))(t))|_{t=\delta+0}>0.$$

This implies the existence of some $\sigma>0$ such that for all $0<h<\sigma$:

$$l_{i_0}(\delta+h, y(\delta, q(z))(\delta+h))>l_{i_0}(\delta, y(\delta, q(z))(\delta))=l_{i_0}(\delta, \phi(p(\delta,0)))=0,$$

i.e., $(\delta+h, y(\delta, q(z))(\delta+h))\notin\bar{\tilde{\omega}}$ for $0<h,\sigma$. So, condition (4) of Lemma 1 holds and the Lemma 1 is valid in the described situation. From its conclusion, the conclusion of Lemma 3 follows. □

Lemma 4 (Lyapunov method). *Let* $p=0$. *Let* $\tilde{\omega}$ *be a nonempty polyfacial set, regular with respect to Eq.* (2) *and let the function* $f\in C(\tilde{\Omega}, \mathbb{R}^n)$ *be locally Lipschitzian with respect to the second argument. Let mapping* $q: B\to\mathscr{C}$, $B=\bar{\tilde{\omega}}\cap\{(t^*, y),\ t^*\in\mathbb{R},\ t^*=\text{const},\ y\in\mathbb{R}^n\}$ *be continuous and such that if* $z=(t^*, y)\in B$, *then* $(t^*, q(z))\in\tilde{\Omega}$, *and*:

(1) *If* $z \in \tilde{\omega}$, *then* $(p(t^*, \vartheta), q(z)(p(t^*, \vartheta))) \in \tilde{\omega}$ *for* $\vartheta \in [-1, 0]$.
(2) *If* $z \in \partial\tilde{\omega}$. *then* $(t^*, q(z)(t^*)) = z$ *and* $(p(t^*, \vartheta), q(z)(p(t^*, \vartheta))) \in \tilde{\omega}$ *for* $\vartheta \in [-1, 0)$.

Then for every $z_0 = (t^*, y_0) \in B \cap \tilde{\omega}$ *and every* $t \in D_{t^*, q(z_0)}$:

$$(t, y(t^*, q(z_0))(t)) \in \tilde{\omega}. \tag{25}$$

Proof. We prove the lemma using Lemma 2. Conditions (1) and (2) of Lemma 2 are obviously satisfied. Let us verify condition (3).

Suppose, on the contrary, that for a $z = (t^*, y) \in B$ with property $(t, y(t^*, q(z))(t)) \in \tilde{\omega}$ for all, $t, t^* < t < t(z)$ and $(t(z), y(t^*, q(z))(t(z)) \in \partial\tilde{\omega}$, $t(z) \in D_{t^*, q(z_0)}$ there is a $\sigma^* > 0$ such that $t(z) + \sigma^* \in D_{t^*, q(z_0)}$ and $(t, y(t^*, q(z))(t)) \notin \tilde{\omega}$ for all t, $t(z) < t < t(z) + \sigma^*$.

Let us put $\phi_t \equiv y_{t(z)}(t^*, q(z))$. Then $\phi_t \in \mathscr{C}$, $(t(z)), \phi_t) \in \tilde{\Omega}$, $(t(z), \phi(t(z))) = (t(z), y(t^*, q(z))(t(z))) \in \partial\tilde{\omega}$, and

$$(p(t(z), \vartheta), \phi(p(t(z), \vartheta))) \in \tilde{\omega} \quad \text{for } \vartheta \in [-1, 0).$$

Since $(p(t(z), 0), \phi(p(t(z), 0))) = (t(z), \phi(t(z))) \in \partial\tilde{\omega}$, it follows $m_{j_0}(t(z), \phi(p(t(z), 0))) = 0$ for some $j_0 \in \{1, \ldots, s\}$. Hence inequality (γ) in Definition 5 is satisfied and, similarly as in the proof of Lemma 3, the inequality

$$Dm_{j_0}(t, y(t^*, p(z))(t))|_{t=t(z)} < 0$$

leads to a contradiction. Thus, condition (3) of Lemma 2 holds.

Let us verify condition (4). If $z = (t^*, y) \in B \cap \partial\tilde{\omega}$, then there is a $j_0 \in \{1, \ldots, s\}$ such that $m_{j_0}(t^*, y) = 0$. Let $\phi = q(z)$. Then $(t^* + \vartheta, \phi(p(t^*, \vartheta))) \in \tilde{\omega}$, for all $\vartheta \in [-1, 0)$. Hence, the derivative from the right

$$Dm_{j_0}(t, y(t^*, p(z))(t))|_{t=t^*+0} < 0.$$

This inequality implies the validity of condition (4) of Lemma 2. From its conclusion (see formula (23)) we have

$$(t, y(t^*, q(z_0))(t)) \in \bar{\tilde{\omega}}$$

for every $z_0 = (t^*, y_0) \in B \cap \tilde{\omega}$ and every $t \in D_{t^*, q(z_0)}$. The stronger inequality (25) can be proved by the method used above, since if $(t^0, y(t^*, q(z_0))(t^0)) \in \partial\tilde{\omega}$ with $t^0 \in D_{t^*, q(z_0)}$, then $m_{j_0}(t^0, y(t^*, q(z_0))(t^0)) = 0$ for some $j_0 \in \{1, \ldots, s\}$. This fact again leads to a contradiction. The lemma is proved. □

Proof of Theorem 1. Suppose $p > 0$. Let us define the auxiliary functions

$$l_i(t, y) \equiv l_i(t, y_i) \equiv (y_i - \rho_i(t))(y_i - \delta_i(t)), \quad i = 1, \ldots, p,$$

$$m_j(t, y) \equiv m_j(t, y_{p+j}) \equiv (y_{p+j} - \rho_{p+j}(t))(y_{p+j} - \delta_{p+j}(t)), \quad j = 1, \ldots, r$$

with $p + r = n$. Then

$$\omega = \{(t, y): t \geqslant p^*, \ l_i(t, y) < 0, \ m_j(t, y) < 0, \ i = 1, \ldots, p, \ j = 1, \ldots, r\}.$$

At first we show that the set ω is a regular polyfacial set with respect to system (2). Condition (α) of Definition 5 holds obviously (we suppose that $\tilde{\omega} \equiv \omega$ and $\tilde{\Omega} \equiv \Omega$ is put here). Let us compute

$$Dl_i(t,y) = (y_i - \rho_i(t))(f_i(t,\pi_t) - \delta_i'(t)) + (y_i - \delta_i(t))(f_i(t,\pi_t) - \rho_i'(t)),$$

where $i = 1,\dots,p$ and

$$Dm_j(t,y) = (y_{p+j} - \rho_{p+j}(t))(f_{p+j}(t,\pi_t) - \delta'_{p+j}(t)) \\ + (y_{p+j} - \delta_{p+j}(t))(f_{p+j}(t,\pi_t) - \rho'_{p+j}(t)),$$

where $j = 1,\dots,r$. In view of (3) and (4) we get for $(t,y) \in \partial\omega$ and $i = 1,\dots,p$

$$Dl_i(t,y)|_{y_i=\delta_i(t)} = (\delta_i(t) - \rho_i(t))(f_i(t,\pi_t) - \delta_i'(t))|_{y_i=\delta_i(t)} > 0,$$

$$Dl_i(t,y)|_{y_i=\rho_i(t)} = (\rho_i(t) - \delta_i(t))(f_i(t,\pi_t) - \rho_i'(t))|_{y_i=\delta_i(t)} > 0$$

and in view of (5) and (6) we get for $(t,y) \in \partial\omega$ and $j = 1,\dots,r$

$$Dm_j(t,y)|_{y_{p+j}=\delta_{p+j}(t)} = (\delta_{p+j}(t) - \rho_{p+j}(t))(f_{p+j}(t,\pi_t) - \delta'_{p+j}(t))|_{y_{p+j}=\delta_{p+j}(t)} < 0,$$

$$Dm_j(t,y)|_{y_{p+j}=\rho_{p+j}(t)} = (\rho_{p+j}(t) - \delta_{p+j}(t))(f_{p+j}(t,\pi_t) - \rho'_{p+j}(t))|_{y_{p+j}=\rho_{p+j}(t)} < 0.$$

So, conditions (β) and (γ) of Definition 5 are valid and ω is a regular polyfacial set with respect to system (2).

Let us show now that Lemma 1 (where $\tilde{\omega} \equiv \omega$ and $\tilde{\Omega} \equiv \Omega$ is put) holds. Define the set

$$Z \equiv \{t^*, y_1,\dots,y_p, y^0_{p+1},\dots,y^0_n) : l_i(t^*, y_i) \leqslant 0,\ i = 1,\dots,p, \\ m_j(t^*, y^0_{p+j}) < 0,\ y^0_{p+j} = \text{const},\ j = 1,\dots,r\}$$

and a mapping of the set

$$W = \{(t,y) \in \partial\omega : m_j(t,y) < 0,\ j = 1,\dots,r\}$$

(see formula (24)) into $Z \cap W$:

$$W \ni (t, y_1,\dots,y_p, y_{p+1},\dots,y_n) \mapsto (t^*, \tilde{y}_1,\dots,\tilde{y}_p, y^0_{p+1},\dots,y^0_n) \in Z \cap W$$

with

$$\tilde{y}_i = \delta_i(t^*) + (y_i - \delta_i(t))\,\frac{\rho_i(t^*) - \delta_i(t^*)}{\rho_i(t) - \delta_i(t)}, \quad i = 1,\dots,p.$$

This mapping is continuous (points of the set $Z \cap W$ are mapped into itself) and, consequently, $Z \cap W$ is a retract of W. The set $Z \cap W$ is not a retract of Z because in the case $p > 1$ the boundary of p-dimensional ball is not its retract (see e.g. [2]) and in the case $p = 1$, the set $Z \cap W$ consists of two disjoint nonempty subsets and, consequently, is not a retract of Z.

It is easy to define the mapping $q : B = \bar{Z} \cap (Z \cup W) \to \mathscr{C}$, for $z = (t^*, y_1,\dots,y_n) \in B$ as

$$q_i(z)(\vartheta) = \delta_i(p(t^*,\vartheta)) + h(\vartheta)\,\frac{\delta_i(t^*) - y_i}{\delta_i(t^*) - \rho_i(t^*)}\,(\rho_i(p(t^*,\vartheta)) - \delta_i(p(t^*,\vartheta))), \tag{26}$$

where h is any function such that $h \in C([-1,0],(0,1])$, $h(t) = 1 \Leftrightarrow t = 0$. This mapping is continuous and for all $\vartheta \in [-1,0)$ the inequality $\rho(p(t^*,\vartheta)) \ll q(z)(\vartheta) \ll \delta(p(t^*,\vartheta))$ holds. Moreover $(t^*, q(z)(0)) = z$.

All the assumptions of Lemma 3 are fulfilled. Then there exists a point $z_0 = (t^*, y_0) \in Z \cap \omega$ such that the graph of the corresponding solution $y(t^*, q(z_0))(t)$ of system (2) belongs to the set ω for each $t \in D_{t^*,q(z_0)}$. Since in ω existence and unicity of every initial problem is guaranteed, we conclude $D_{t^*,q(z_0)} = [t^*, \infty)$, i.e. inequalities (7) hold on $[t^*, \infty)$. Taking into account the properties of initial functions and quasiboundedness of f, we conclude that inequalities (7) hold even on the larger interval $[p^*, \infty)$.

Let $p = 0$. In this case the proof can be simplified without using the topological principle. Putting $\tilde{\omega} \equiv \omega$, $\tilde{\Omega} \equiv \Omega$ and defining the mapping $q : B \to \mathscr{C}$ with $B = \bar{\tilde{\omega}} \cap \{(t^*, y),\ t^* \in \mathbb{R},\ t^* = \text{const},\ y \in \mathbb{R}^n\}$ by formula (26) we see that all assumptions of Lemma 4 are valid. From its conclusion (and from the last steps of the previous part of the proof) we conclude that inequalities (7) hold. The theorem is proved. □

Proof of Theorem 2. *Necessity*: Let $y(t) \in C([p^*, \infty), \mathbb{R}^n)$ be a positive solution of system (2) on $[t^*, \infty)$. It can be shown easily that for every $i = 1, \ldots, n$ there exist continuous function $\lambda_i \in C([p^*, \infty), \mathbb{R})$ such that

$$y_i(t) = k_i\, e^{\mu_i \int_{p^*}^{t} \lambda_i(s)\, ds}, \quad t \in [p^*, \infty) \tag{27}$$

i.e.,

$$y(t) = k\, e^{\mu \int_{p^*}^{t} \lambda(s)\, ds}, \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n),\ \ t \in [p^*, \infty)$$

with $k_i = y_i(p^*) > 0$. System (2) turns, by means of (27), into the following system of integro-functional equations

$$\lambda_i(t) = \frac{\mu_i}{k_i}\, e^{-\mu_i \int_{p^*}^{t} \lambda_i(s)\, ds} f_i(t, T(k, \lambda)_t), \quad t \geqslant t^*,\ \ i = 1, \ldots, n. \tag{28}$$

The necessity of condition (8) is now in view of (27), (28), (i) and (ii) obvious since inequalities (8) hold and

$$\lambda_i(t) \equiv \frac{y_i'(t)}{\mu_i y_i(t)} = \frac{f_i(t, T(k, \lambda)_t)}{\mu_i y_i(t)} > 0, \quad t \geqslant t^*,\ \ i = 1, \ldots, n.$$

Sufficiency: This part of the proof follows immediately from Theorem 1 for $\rho(t) \equiv 0$ and $\delta(t) \equiv k \exp(\mu \int_{p^*}^{t} \lambda(s)\, ds)$. Indeed, in this case inequality (3) holds since, for $i = 1, \ldots, p$ and $\pi_i(t) = \delta_i(t)$ (in view of (8) and condition (ii) of Theorem 2), we get for $t \geqslant t^*$:

$$\begin{aligned}
\delta_i'(t) - f_i(t, \pi_t) &= k_i \mu_i \lambda_i(t) e^{\mu_i \int_{p^*}^{t} \lambda_i(s)\, ds} - f_i(t, \pi_t) \\
&= -k_i \lambda_i(t) e^{-\int_{p^*}^{t} \lambda_i(s)\, ds} - f_i(t, \pi_t) \leqslant [\text{in view of (8)}] \leqslant f_i(t, T(k, \lambda)_t) - f_i(t, \pi_t) \\
&< [\text{in view of (ii) since } T(k, \lambda)_t(\vartheta) > \pi_t(\vartheta) \text{ for } \vartheta \in [-1, 0) \text{ and } T_i(k, \lambda)(0) = \pi_t(0)] \\
&< f_i(t, \pi_t) - f_i(t, \pi_t) = 0.
\end{aligned}$$

Inequality (4) holds too since, for $i = 1, \ldots, p$; $\pi_i(t) = \rho_i(t) = 0$ in view of conditions (i) and (ii) of Theorem 2, we get for $t \geqslant t^*$

$$\rho_i'(t) - f_i(t, \pi_t) = -f_i(t, \pi_t) > 0.$$

Inequalities (5) and (6) can be verified in a similar manner. Theorem 2 is proved. □

Proof of Theorem 3. We show that the proof is a consequence of Theorem 2 if $n = 1$ and

$$f(t, y_t) \equiv a \int_0^t L(s) y(t-s)\,\mathrm{d}s + b y^2(t)$$

is put in its formulation. In our case $t^* = \tilde{\varepsilon}$, $p^* = 0$ and we can put $\Omega = (0, \infty) \times \mathscr{C}$. Inequality (10) follows from Inequality (8). The functional $f(t, y_t)$ (which is obviously quasibounded) is for $a = 1$ (and $b > 0$) strongly increasing with respect to the second argument on Ω and for $a = -1$ (and $b < 0$) strongly decreasing with respect to the second argument on Ω in the sense of Definition 3. In the first case we put $\mu = 1$, in the second one $\mu = -1$. Except this, if $a = 1$ and $b \in \mathbb{R}$ is arbitrary, every solution $y(t)$ which is positive for $0 \leqslant t \leqslant A$ remains positive for every $t > A$ on its maximal interval of existence $[0, B)$. Obviously, supposition $y(t_1) = 0$ for a $t_1 \in (A, B)$ and $y(t) > 0$ on $[0, t_1)$ leads to a contradiction, since

$$\dot{y}(t_1) = \int_0^{t_1} L(s) y(t_1 - s)\,\mathrm{d}s > 0$$

and, consequently, for $t < t_1$ (if t is sufficiently close to t_1) we get $y(t) < 0$. This contradicts the supposition of positively of $y(t)$ on $[0, t_1)$. $\square$

Proof of Theorem 4. This proof uses Theorem 3. Inequality (10) will hold if there exists a positive $\lambda(t)$ satisfying the inequalities

$$\lambda(t) \geqslant l \int_0^t \mathrm{e}^{-vs + \int_{t-s}^t \lambda(u)\,\mathrm{d}u}\,\mathrm{d}s - bk\,\mathrm{e}^{-\int_0^t \lambda(s)\,\mathrm{d}s} \geqslant \int_0^t L(s) \mathrm{e}^{\int_{t-s}^t \lambda(u)\,\mathrm{d}u}\,\mathrm{d}s - bk\,\mathrm{e}^{-\int_0^t \lambda(s)\,\mathrm{d}s}$$

on $[\tilde{\varepsilon}, \infty)$ with a positive constant k. Supposing $\lambda(t) \equiv \lambda = \text{const}$, $\lambda \neq v$ we get

$$\lambda \geqslant \frac{l}{v - \lambda}(1 - \mathrm{e}^{(\lambda - v)t}) - bk\,\mathrm{e}^{-\lambda t}, \quad t \in [\tilde{\varepsilon}, \infty). \tag{29}$$

Suppose $v - \lambda > 0$. Then Inequality (29) holds if

$$\lambda \geqslant \frac{l}{v - \lambda} - bk$$

or

$$f(\lambda) \equiv \lambda - \frac{l}{v - \lambda} \geqslant -bk.$$

Since the right side of this inequality can be made sufficiently small (due to the positive number k which can be chosen sufficiently small), it is enough to take $\lambda = \lambda^*$ such that $f(\lambda^*) > 0$. Since the equation

$$f'(\lambda) \equiv 1 - \frac{l}{(v - \lambda)^2} = 0$$

has the roots $\lambda_{1,2} = v \pm \sqrt{l}$, we can put $\lambda^* = v - \sqrt{l}$. Then $\lambda^* > 0$, $v - \lambda^* > 0$ and $f(\lambda^*) = v - 2\sqrt{l} > 0$. The theorem is proved. $\square$

Proof of Theorem 5. Theorem 5 follows from Theorem 2 if $\Omega = [t^*, \infty) \times \mathscr{C}$,

$$f(t, y_t) = (f_1(t, y_t), \ldots, f_n(t, y_t))$$

and

$$f(t, y_t) \equiv \sum_{j=1}^{n} [a_{ij}(t)y_j(t) + b_{ij}(t)y_j(\tau(t))], \quad i = 1, \dots, n$$

is put in its formulation. Note that the system of integral inequalities (8) turns into the system of inequalities (16). Conditions (13) and (15) ensure that the functional $f_i(t, y_t)$ is strongly decreasing on Ω for $i = 1, \dots, p$ and conditions (14) and (15) ensure that the functional $f_i(t, y_t)$ is strongly increasing on Ω for $i = p+1, \dots, n$ in the sense of Definition 3. Condition (15) gives a guarantee that in every row of matrix B there is at least one nonzero element for every $t \in [t^*, \infty)$. This property is sufficient for the validity of Definition 3. The theorem is proved. □

Proof of Theorem 6. We will use Theorem 2 again. The functional

$$f(t, y_t) = -\int_{\tau(t)}^{t} K(t,s)y(s)\,\mathrm{d}s$$

is strongly decreasing with respect to the second argument on $\Omega = [t^*, \infty) \times \mathscr{C}$ in the sense of Definition 3. Integral inequality (8) with $n = i = 1$, $\mu = -1$ takes the form

$$\lambda(t) \geqslant \mathrm{e}^{\int_{p^*}^{t} \lambda(u)\,\mathrm{d}u} \int_{\tau(t)}^{t} K(t,s)\mathrm{e}^{-\int_{p^*}^{t} \lambda(u)\,\mathrm{d}u}\,\mathrm{d}s, \quad t \in [t^*, \infty).$$

From this inequality, inequality (18) follows. The theorem is proved. □

Proof of Theorem 7. In the case considered, inequality (18) takes the form

$$\lambda(t) \geqslant c(t) \int_{p^*}^{t} \mathrm{e}^{\int_{s}^{t} \lambda(u)\,\mathrm{d}u}\,\mathrm{d}s, \quad t \in [t^*, \infty). \tag{30}$$

We will look for a constant solution of this inequality, i.e., we put $\lambda(t) \equiv \lambda = \text{const}$. Then

$$\begin{aligned} \lambda &\geqslant c(t) \int_{p^*}^{t} \mathrm{e}^{\lambda(t-s)}\,\mathrm{d}s = c(t)\mathrm{e}^{\lambda t} \left(\frac{\mathrm{e}^{-\lambda s}}{-\lambda} \right) \Bigg|_{p^*}^{t} \\ &= c(t)\mathrm{e}^{\lambda t} \left(-\frac{1}{\lambda} \right) (\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda p^*}) = \frac{c(t)}{\lambda} (\mathrm{e}^{\lambda(t-p^*)} - 1), \quad t \in [t^*, \infty) \end{aligned}$$

or

$$c(t) \leqslant \frac{\lambda^2}{\mathrm{e}^{\lambda(t-p^*)} - 1}, \quad t \in [t^*, \infty).$$

It is now clear that the value $\lambda = \delta > 0$ satisfies Inequality (30) and Inequality (20) is a consequence of Theorem 6. The theorem is proved. □

Proof of Theorem 8. In the case considered, $p^* = t^* - l$ and Inequality (18) takes the form

$$\lambda(t) \geqslant c(t) \int_{t-l}^{t} \mathrm{e}^{\int_{s}^{t} \lambda(u)\,\mathrm{d}u}\,\mathrm{d}s, \quad t \in [t^*, \infty).$$

Supposing $\lambda(t) \equiv \lambda = \text{const}$, we get

$$\lambda \geqslant c(t)\int_{t-l}^{t} e^{\lambda(t-s)}\,ds = c(t)e^{\lambda t}\left(\frac{e^{-\lambda s}}{-\lambda}\right)\Bigg|_{t-l}^{t}$$
$$= c(t)e^{\lambda t}\left(-\frac{1}{\lambda}\right)(e^{-\lambda t} - e^{-\lambda(t-l)}) = \frac{c(t)}{\lambda}(e^{\lambda l} - 1), \quad t \in [t^*, \infty)$$

or

$$c(t) \leqslant \frac{\lambda^2}{e^{\lambda l} - 1} = \frac{1}{l^2}\,\frac{(\lambda l)^2}{e^{\lambda l} - 1} \equiv \frac{1}{l^2}\,g(\lambda l), \quad t \in [t^*, \infty).$$

Let us look for the maximum of the function

$$g(x) = \frac{x^2}{e^x - 1}$$

in $(0, \infty)$. Since $g(0^+) = g(+\infty) = 0$ and $g(x) > 0$ for $x \in (0, \infty)$ this maximum exists. Since

$$g'(x) = \frac{x}{(e^x - 1)^2}\,[e^x(2 - x) - 2],$$

the maximum is reached in the point $x = \alpha$ satisfying the equation

$$e^{\alpha} = \frac{2}{2 - \alpha}$$

and $g(\alpha) = \alpha(2 - \alpha)$. So inequality (22) is a consequence of inequality (18). Easy numerical computation shows that $\alpha \doteq 1.5936$, $g(\alpha) \doteq 0.6476$. Theorem 8 is now a consequence of Theorem 6. □

Acknowledgement

This research was supported by the Grant 201/99/0295 of the Grant Agency of Czech Republic and by the Project ME 423 of Ministry of Education of Czech Republic.

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Existence of Strictly Decreasing Positive Solutions of Linear Differential Equations of Neutral Type

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Abstract

The paper is concerned with a linear neutral differential equation

$$\dot{y}(t) = -c(t)y(t-\tau(t)) + d(t)\dot{y}(t-\delta(t))$$

where $c\colon [t_0,\infty) \to (0,\infty)$, $d\colon [t_0,\infty) \to [0,\infty)$, $t_0 \in \mathbb{R}$ and $\tau,\delta\colon [t_0,\infty) \to (0,r]$, $r \in \mathbb{R}$, $r > 0$ are continuous functions. A new criterion is given for the existence of positive strictly decreasing solutions. The proof is based on the Rybakowski variant of a topological Ważewski principle suitable for differential equations of the delayed type. Unlike in the previous investigations known, this time the progress is achieved by using a special system of initial functions satisfying a so-called sewing condition. The result obtained is extended to more general equations. Comparisons with known results are given as well.

Keywords: Neutral equation, delay, positive solution, sewing condition
AMS 2010 classification: Primary 34K40; 34K25; 34K12.

1 Introduction

The aim of the paper is to give a criterion for the existence of positive strictly decreasing solutions to the linear neutral differential equation

$$\dot{y}(t) = -c(t)y(t-\tau(t)) + d(t)\dot{y}(t-\delta(t)) \tag{1}$$

where $c\colon [t_0,\infty) \to (0,\infty)$, $d\colon [t_0,\infty) \to [0,\infty)$, $t_0 \in \mathbb{R}$, and $\tau,\delta\colon [t_0,\infty) \to (0,r]$, $r \in \mathbb{R}$, $r > 0$ are continuous functions.

The existence of positive solutions of functional differential equations of delayed type is a classical problem which is satisfactorily solved for various classes of equations in numerous papers and books. We should note, however, that the positivity of solutions to neutral differential equations is investigated to a degree less than that of the positivity of solutions of non-neutral equations with delay.

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Some results on the existence of positive solutions for delayed differential equations and their systems are summarized, e.g., in [1, 2, 3, 23, 24, 25].

Let us cite one of the nice classical implicit results on the existence of a positive solution of a linear equation with delay ([39], see also [23, Theorem 2.1.4] and [2, Theorem 2.2.13]), which serves as a source for various explicit sufficient positivity criteria. Consider the equation

$$\dot{y}(t) + p(t)y(t - \tau(t)) = 0 \tag{2}$$

where $p, \tau \colon [t_0, \infty) \to \mathbb{R}_+$, $\mathbb{R}_+ := [0, \infty)$ are continuous functions, $\tau(t) \le t$ and $\lim\limits_{t\to\infty} (t - \tau(t)) = \infty$. Set $T_0 = \inf\limits_{t \ge t_0} \{t - \tau(t)\}$. A function y is called a solution of (2) with respect to initial point t_0 if y is defined and continuous on $[T_0, \infty)$, differentiable on $[t_0, \infty)$, and satisfies (2) for $t \ge t_0$.

Theorem 1. *Equation* (2) *has a positive solution with respect to* t_0 *if and only if there exists a continuous function* $\lambda(t)$ *on* $[T_0, \infty)$ *such that* $\lambda(t) > 0$ *for* $t \ge t_0$ *and*

$$\lambda(t) \ge p(t) \exp\left(\int\limits_{t-\tau(t)}^{t} \lambda(s)ds \right), \quad t \ge t_0. \tag{3}$$

The above criterion was generalized for systems of linear and nonlinear differential equations with bounded delay in [9] and for nonlinear systems of differential equations with unbounded delay and with finite memory in [16]. Positive solutions of (2) in the so-called critical case were studied, e.g., in [5, 11, 12, 17, 19, 22, 35] and an overview of some sufficient conditions to equation (2) in the critical case is given in a recent paper [4]. Asymptotic formulas describing two classes of asymptotically different positive solutions are analyzed, e.g., in [13, 14] and [15]. The problem of positive solutions is also investigated in further numerous papers such as [6, 7, 8, 10, 20, 21, 29, 37] and the references therein.

To describe the main result of the paper we should note that, to the best of our knowledge, there is no extension of the implicit-type (with respect to λ) result given by Theorem 1, where the key role is played by inequality (3), to neutral equations of the type (1) if the solutions are understood as continuously differentiable functions (see Definition 1) below. In this direction, we will show that in the case of equation (1), inequality (3) can be replaced by

$$\lambda(t) \ge c(t) \exp\left(\int\limits_{t-\tau(t)}^{t} \lambda(s)ds \right) + d(t)\lambda(t - \delta(t)) \exp\left(\int\limits_{t-\delta(t)}^{t} \lambda(s)ds \right), \tag{4}$$

$t \ge t_0$, where $\lambda \colon [t_0 - r, \infty) \to (0, \infty)$. Strictly speaking, Theorem 1 for $p > 0$ deals with strictly decreasing positive solutions. Our method gives the same statement in this sense. Namely, inequality (4) is necessary and sufficient for the existence of a positive and strictly decreasing solution of equation (1).

The topological (retract) method of T. Ważewski [38], which was successfully modified to retarded differential equations by K.P. Rybakowski (see, e.g., [33, 34]) serves as a theoretical tool to prove the main result. For a nice overview of topological principle, we also refer to [36]. In [18] the retract principle was modified for neutral functional differential equations. This modification should make it possible to use this in the present paper. Even if [18] contains an illustrative example, showing how this modification works, there is one serious problem restricting the classes of equations suitable for considering by it. Below, we explain the heart of the matter.

We consider a neutral functional differential system of the form

$$\dot{y}(t) = f(t, y_t, \dot{y}_t) \tag{5}$$

where the symbol $\dot{y}$ stands for the derivative (considered, if necessary, as one-sided). Sometimes we use the symbol y' as well (if there is no doubt whether the derivative is one-sided or not).

Let $\mathcal{C}$ be the set of all continuous functions $\phi\colon [-r, 0] \to \mathbb{R}^n$ and $\mathcal{C}^1$ be the set of all continuously differentiable functions $\phi\colon [-r, 0] \to \mathbb{R}^n$. Assume $t \geq t_0$, $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$ and $f\colon E_r \to \mathbb{R}^n$ with $E_r := [t_0, \infty) \times \mathcal{C} \times \mathcal{C}$.

We pose an initial problem for (5):

$$y_{t_0} = \phi, \ \ \dot{y}_{t_0} = \dot{\phi} \tag{6}$$

where $\phi \in \mathcal{C}^1$. The norm of $\phi \in \mathcal{C}$ is defined as $\|\phi\|_r := \max\limits_{\theta\in[-r,0]} \|\phi(\theta)\|$ and, if $\phi \in \mathcal{C}^1$, then

$$\|\phi\|_r := \max_{\theta\in[-r,0]} \|\phi(\theta)\| + \max_{\theta\in[-r,0]} \|\phi'(\theta)\|$$

where $\|\cdot\|$ is the Euclidean norm.

In the literature there are various definitions of a solution to neutral differential equations. In the paper, as a solution of (5), (6), we assume a continuously differentiable function within the meaning of the following definition.

Definition 1. *A continuously differentiable function* $y\colon [t_0 - r, t_\phi) \to \mathbb{R}^n$, $t_\phi \in (t_0, \infty]$, *is a solution of* (5), (6) *if* $y_{t_0} = \phi$, $\dot{y}_{t_0} = \dot{\phi}$ *and* (5) *is satisfied for any* $t \in [t_0, t_\phi)$.

V. Kolmanovskii and A. Myshkis [28] considered the initial-value problem for neutral differential equations (5), (6). Although this problem should be expected have a continuously differentiable solution on an interval $[t_0, t_\phi)$, in general, this is not true. Even if the functional f and the initial function ϕ are arbitrarily smooth, and the initial problem can be solved by the method of steps, the continuous solution may, generally speaking, have jumps of the derivative for arbitrarily large t. Such jumps will be absent if the initial function ϕ satisfies the *sewing condition*

$$\dot{\phi}(0) = f(t_0, \phi, \dot{\phi}). \tag{7}$$

Theorem 2. *[28, p.107] Let* $f\colon E_r \to \mathbb{R}^n$ *be a continuous functional satisfying, in some neighborhood of any point of* E_r, *the Lipschitz condition*

$$\|f(t, \psi_1, \chi_1) - f(t, \psi_2, \chi_2)\| \leq L_1\|\psi_1 - \psi_2\|_r + L_2\|\chi_1 - \chi_2\|_r$$

with constants $L_i \in [0,\infty)$, $i = 1, 2$. Assume also $\phi \in \mathcal{C}^1$ and the sewing condition (7) *being fulfilled. Then, there exists a $t_\phi \in (t_0, \infty]$ such that:*

a) *There exists a solution y of* (5), (6) *on $[t_0 - r, t_\phi)$.*
b) *On any interval $[t_0 - r, t_1] \subset [t_0 - r, t_\phi)$, $t_1 > t_0$, this solution is unique.*
c) *If $t_\phi < \infty$, then $\dot{x}(t)$ has not a finite limit as $t \to t_\phi^-$.*
d) *The solution y and $\dot{y}$ depend continuously on ϕ.*

For a particular case of system (5) given by

$$\begin{aligned} \dot{y}(t) &= f(t, y_t, \dot{y}_t) \\ &:= f(t, y(t-h_1(t)), \dots, y(t-h_o(t)), \dot{y}(t-g_1(t)), \dots, \dot{y}(t-g_\ell(t))), \end{aligned}$$

where indices $o \geq 0$ and $\ell \geq 1$, a more general result can be proved easily by the method of steps (compare [28, pages 111, 96, and 15]).

Theorem 3. *Let*

$$f\colon [t_0,\infty) \times \mathbb{R}^{o+\ell} \to \mathbb{R}^n,$$

$$h_i\colon [t_0,\infty) \to (0, r], \;\; i = 1, \dots, o \quad \textit{and} \quad g_j\colon [t_0,\infty) \to (0, r], \;\; j = 1, \dots, \ell$$

be continuous functions. Assume also $\phi \in \mathcal{C}^1$ and the sewing condition (7)*, in the case considered having the form*

$$\dot{\phi}(0) = f(t_0, \phi(-h_1(t_0)), \dots, \phi(-h_o(t_0)), \dot{\phi}(-g_1(t_0)), \dots, \dot{\phi}(-g_\ell(t_0))) \qquad (8)$$

being fulfilled. Then:

a) *There exists a solution y of* (5), (6) *on $[t_0 - r, \infty)$.*
b) *On any interval $[t_0 - r, t_1] \subset [t_0 - r, \infty)$, $t_1 > t_0$, this solution is unique.*
c) *The solution y and $\dot{y}$ depend continuously on ϕ.*

To succeed in applying Theorem 2 (or Theorem 3) to prove the existence and uniqueness of a continuously differentiable (by Definition 1) solution, the sewing condition (7) (or (8)) must be fulfilled. If not, then, generally speaking, a solution has no continuous derivative and certainly, it has no two-sided derivative for $t = t_0$. To define an initial function that satisfies the sewing condition is usually not an easy task. The above weighty circumstance when applying the retract principle to neutral functional differential equations, follows from the necessity to satisfy the sewing condition. When the retract principle is used, it is necessary to construct not only one initial function but a set of functions, called the set of initial functions, satisfying several assumptions. One of the assumption is that *every* function of this set must satisfy a *sewing condition.* So, from above it follows that, technically, is not easy to construct such a set. In the present paper, we perform, for the case of linear neutral differential equation (1), the relevant construction of a set of initial functions when dealing with a criterion for a solution to be positive. This is an important progress as, eventually, we are able to prove that such a positive solution is continuously differentiable (in the meaning of Definition 1).

The rest of the paper is structured as follows. In Part 2 we give a generalization of the retract principle to neutral functional differential equations, previously developed in [18]. The main result (a criterion for the existence of a positive strictly decreasing and continuously differentiable solution of neutral differential equation (1)) is given in Part 3 where a special construction of a system of initial functions satisfying the sewing condition is also developed. For a more general equation than (1), a criterion for the existence of a positive strictly decreasing and continuously differentiable solution is formulated in Part 4. Some open questions, corollaries and remarks as well as comparisons with some of the previous results are listed in Part 5.

2 Retract Method

This part provides necessary background. It is mainly taken from papers [18] and [34]. Note that the underlying ideas are based, in addition to the paper of the founder T. Ważewski [38], on the so-called Razumikhin condition in the theory of stability, e.g., [30, 31, 32], and on Razumikhin's type of extension of Ważewski's principle by K.P. Rybakowski [33, 34]). Mentioned are the necessary changes of the original versions, making it possible to prove a criterion for the existence of positive solutions to equation (1).

If a set $A \subset \mathbb{R} \times \mathbb{R}^n$ is given, then $\operatorname{int} A$, $\overline{A}$ and ∂A denote, as usual, the interior, the closure, and the boundary of A, respectively.

Definition 2. *(compare [18, 34]) Let Λ be a topological space, let a subset $\tilde{\Omega} \subset \mathbb{R} \times \Lambda$ be open in $\mathbb{R} \times \Lambda$, and let x be a mapping associating with every $(\delta, \lambda) \in \tilde{\Omega}$ a function $x(\delta, \lambda)\colon D_{\delta,\lambda} \to \mathbb{R}^n$ where $D_{\delta,\lambda}$ is an interval in $\mathbb{R}$. Assume* (1)–(3)*:*

(1) $\delta \in D_{\delta,\lambda}$.

(2) *If $t \in \operatorname{int} D_{\delta,\lambda}$, then there is an open neighbourhood $\mathcal{O}(\delta, \lambda)$ of (δ, λ) in $\tilde{\Omega}$ such that $t \in D_{\delta',\lambda'}$ holds for all $(\delta', \lambda') \in \mathcal{O}(\delta, \lambda)$.*

(3) *If (δ', λ'), $(\delta, \lambda) \in \tilde{\Omega}$, and $t' \in D_{\delta',\lambda'}$, $t \in D_{\delta,\lambda}$, then*

$$\lim_{(\delta',\lambda',t') \to (\delta,\lambda,t)} x(\delta', \lambda')(t') = x(\delta, \lambda)(t).$$

Then, $(\Lambda, \tilde{\Omega}, x)$ is called a system of curves in $\mathbb{R}^n$.

Definition 3. *If $A \subset A^*$ are any two sets of a topological space and $\pi\colon A^* \to A$ is a continuous mapping from A^* onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a* retraction *of A^* onto A. If there exists a retraction of A^* onto A, A is called a* retract *of A^*.*

Lemma 1. *(compare [18, 34]) Let $(\Lambda, \tilde{\Omega}, x)$ be a system of curves in $\mathbb{R}^n$. Let $\tilde{\omega}$, W, Z be sets. Assume the below conditions* (1)–(4)*:*

(1) a) $\tilde{\omega} \subset [t_0 - r, t_*) \times \mathbb{R}^n$, $t_* > t_0$, *the cross-section* $\{(\tilde{t}, y) \in \tilde{\omega}\}$ *is an open simply connected set for every* $\tilde{t} \in [t_0 - r, t_*)$, *and* $W \subset \partial\tilde{\omega}$,

b) $Z \subset \tilde{\omega} \cup W$, $Z \cap W$ *is a retract of* W, *but not a retract of* Z.

(2) *There is a continuous map* $q\colon B \to \Lambda$ *where* $B = \overline{Z} \cap (Z \cup W)$ *such that, for any* $z = (\delta, y) \in B$, $(\delta, q(z)) \in \tilde{\Omega}$, *and, if also* $z \in W$, *then* $x(\delta, q(z))(\delta) = y$.

(3) *Let* A *be the set of all* $z = (\delta, y) \in Z \cap \tilde{\omega}$ *such that, for fixed* $(\delta, y) \in A$, *there is a* $t > \delta$, $t \in D_{\delta, q(z)}$ *and* $(t, x(\delta, q(z))(t)) \notin \tilde{\omega}$.

Assume that, for every $z = (\delta, y) \in A$, *there is a* $t(z)$, $t(z) > \delta$, *such that:*

a) $t(z) \in D_{\delta, q(z)}$ *and, for all* t, $\delta \le t < t(z)$, $(t, x(\delta, q(z))(t)) \in \tilde{\omega}$,

b) $(t(z), x(\delta, q(z))(t(z))) \in W$,

c) *For any* $\sigma > 0$, *there is a* t, $t(z) < t \le t(z) + \sigma$ *such that* $t \in D_{\delta, q(z)}$ *and* $(t, x(\delta, q(z))(t)) \notin \overline{\tilde{\omega}}$.

(4) *For any* $z = (\delta, y) \in W \cap B$ *and all* $\sigma > 0$, *there is a* t, $\delta < t \le \delta + \sigma$ *such that* $t \in D_{\delta, q(z)}$ *and* $(t, x(\delta, q(z))(t)) \notin \overline{\tilde{\omega}}$.

Then, there is a $z_0 = (\delta_0, y_0) \in Z \cap \tilde{\omega}$ *such that, for every* $t \in D_{\delta_0, q(z_0)}$,

$$(t, x(\delta_0, q(z_0))(t)) \in \tilde{\omega}. \tag{9}$$

Remark 1. *Let*

$$\Lambda = \mathcal{C}^1,\ \ \tilde{\Omega} \subset \{(t, \lambda) \in [t_0, \infty) \times \mathcal{C}^1 \text{ such that } \dot{\lambda}(0) = f(t_0, \lambda, \dot{\lambda})\}$$

and function f *satisfies all the assumptions of Theorem 2. In this case, through each* $(t_0, \lambda) \in \tilde{\Omega}$, *there exists a unique solution* $y(t_0, \lambda)$ *of* (5) *defined on its maximal interval* $[t_0 - r, a_\lambda)$. *Let* $D_{t_0, \lambda} = [t_0 - r, a_\lambda)$ *where* $a_\lambda > t_0$. *Then,* $(\Lambda, \tilde{\Omega}, y)$ *is a system of curves in* $\mathbb{R}^n$ *within the meaning of Definition 2. A similar remark holds when all the assumptions of Theorem 3 are satisfied.*

Usually, when applying Lemma 1 to prove the existence of a solution of a given system with the graph staying in a prescribed domain $\tilde{\omega}$, the form of $\tilde{\omega}$ should be specified. As a standard shape of such a domain, used in numerous investigations, serves the so-called polyfacial set defined below.

Definition 4. *Let* p *and* s *be nonnegative integers,* $p + s > 0$, $t_* > t_0$, *and let*

$$\begin{aligned} l_i &\colon [t_0 - r, t_*) \to \mathbb{R} \times \mathbb{R}^n,\ \ i = 1, \dots, p, \\ m_j &\colon [t_0 - r, t_*) \to \mathbb{R} \times \mathbb{R}^n,\ \ j = 1, \dots, s \end{aligned}$$

be continuously differentiable functions. The set

$$\omega := \{(t, y) \in [t_0 - r, t_*) \times \mathbb{R}^n,\ l_i(t, y) < 0,\ \ m_j(t, y) < 0,\ \text{for all } i, j\ \}$$

is called a polyfacial set provided that the cross-section

$$\omega \cap \{(t, y) \colon t = t^*, y \in \mathbb{R}^n\}$$

is an open and simply connected set for every fixed $t^* \in [t_0 - r, t_*)$.

When $p = 0$ in Definition 4, the functions l_i, $i = 1, \ldots, p$ are not defined. Similarly, if $s = 0$, the functions m_j, $j = 1, \ldots, s$ are omitted. In order to prove the existence of a solution of (5) satisfying the property (9), a polyfacial set ω should meet some additional requirements. We can characterize such requirements as properties guaranteeing the properties of solutions of system (5), formulated for the system of curves $(\Lambda, \tilde{\Omega}, x)$ in Lemma 1. Because of the neutrality of the equations, we need to be able to foresee the properties of the derivatives of solutions as described by auxiliary inequalities.

Definition 5. *(compare [18]) Let q be a nonnegative integer, $t_* > t_0$, and let*

$$c_k \colon [t_0 - r, t_*) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \;\; k = 1, \ldots, q,$$

be continuous functions. A polyfacial set ω is called regular with respect to Eq. (5) *and auxiliary inequalities*

$$c_k(t, y, x) \leq 0, \;\; k = 1, \ldots, q \tag{10}$$

if α) – δ) below hold:

α) If $(t, \phi) \in \mathbb{R} \times \mathcal{C}^1$ and $(t + \theta, \phi(\theta)) \in \omega$ for $\theta \in [-r, 0)$, then $(t, \phi, \dot{\phi}) \in E_r$.

β) If $(t, \phi) \in \mathbb{R} \times \mathcal{C}^1$, $(t + \theta, \phi(\theta)) \in \omega$ for $\theta \in [-r, 0)$ and, moreover,

$$c_k(t + \theta, \phi(\theta), \dot{\phi}(\theta)) \leq 0, \;\; \theta \in [-r, 0), \quad k = 1, \ldots, q, \tag{11}$$

then also

$$c_k(t + \theta, \phi(\theta), f(t, \phi, \dot{\phi})) \leq 0, \;\; k = 1, \ldots, q. \tag{12}$$

γ) For all $i = 1, \ldots, p$, all $(t, y) \in \partial\omega$ for which $l_i(t, y) = 0$ and for all $\phi \in \mathcal{C}^1$ for which $\phi(0) = y$, $(t + \theta, \phi(\theta)) \in \omega$, $\theta \in [-r, 0)$ and

$$c_k(t + \theta, \phi(\theta), \dot{\phi}(\theta)) \leq 0, \;\; \theta \in [-r, 0), \quad k = 1, \ldots, q, \tag{13}$$

it follows that:

$$Dl_i(t, y) \equiv \frac{\partial l_i}{\partial t}(t, y) + \sum_{r=1}^{n} \frac{\partial l_i}{\partial y_r}(t, y) \cdot f_r(t, \phi, \dot{\phi}) > 0.$$

δ) For all $j = 1, \ldots, s$, all $(t, y) \in \partial\omega$ for which $m_j(t, y) = 0$ and for all $\phi \in \mathcal{C}^1$ for which $\phi(0) = y$, $(t + \theta, \phi(\theta)) \in \omega$, $\theta \in [-r, 0)$ and

$$c_k(t + \theta, \phi(\theta), \dot{\phi}(\theta)) \leq 0, \;\; \theta \in [-r, 0), \quad k = 1, \ldots, q$$

for all $\theta \in [-1, 0)$, it follows that:

$$Dm_j(t, y) \equiv \frac{\partial m_j}{\partial t}(t, y) + \sum_{r=1}^{n} \frac{\partial m_j}{\partial y_r}(t, y) \cdot f_r(t, \phi, \dot{\phi}) < 0.$$

If ω is a polyfacial set, then define the set W used in Lemma 1 as

$$W := \{(t, y) \in \partial\omega : \ m_j(t, y) < 0, \, j = 1, \ldots, s\}. \tag{14}$$

Moreover, we need to specify the properties of the mapping q in Lemma 1. The following definition describes the admissible behavior of functions with respect to ω. A fixed set of functions generated by this mapping and satisfying properties gathered in the following definition is called a set of initial functions.

Definition 6 (Set of initial functions). *Let Z be a subset of $\omega \cup W$ and let the mapping*

$$q \colon B \to \mathcal{C}^1, \ \ B := \overline{Z} \cap (Z \cup W)$$

be continuous. We assume that, if $z = (\delta, y) \in B$, then $(\delta, q(z)) \in \tilde{\Omega}$. If moreover

1) *For $z \in Z \cap \omega$, we have $(\delta + \theta, q(z)(\theta)) \in \omega$ for $\theta \in [-r, 0]$.*

2) *For $z \in W \cap B$, we have $(\delta, q(z)(\delta)) = z$ and*

   either

   2a) *$(\delta + \theta, q(z)(\theta)) \in \omega$ for $\theta \in [-r, 0)$*

   or

   2b) *$(\delta + \theta, q(z)(\theta)) \in \overline{\omega}$ for $\theta \in [-r, 0)$ and, for all $\sigma > 0$, there is a $t = t(\sigma, z)$, $\delta < t \leq \delta + \sigma$ such that t is within the domain of definition of solution $x(\delta, q(z))$ of (5) and $(t, x(\delta, q(z))(t)) \notin \overline{\omega}$,*

then such a set of functions is called a set of initial functions for (5) with respect to ω and Z.

Finally, we will formulate the below theorem as an application of Lemma 1 for a system of neutral equations (5). Therefore, its proof is omitted.

Theorem 4. *Let ω be a nonempty polyfacial set, regular with respect to (5) and inequalities (10). Assume $\phi \in \mathcal{C}^1$ and the sewing condition (7) being fulfilled. Let a fixed $t_* \in (t_0, \infty]$ exist such that:*

a) There exists a solution y of (5), (6) on $[t_0 - r, t_*)$.

b) On any interval $[t_0 - r, t_1] \subset [t_0 - h, t_*)$, $t_1 > t_0$, this solution is unique.

c) If $t_* < \infty$, then $\dot{y}(t)$ has not a finite limit as $t \to t_*^-$.

d) The solution y and $\dot{y}$ depend continuously on ϕ.

Assume that q defines a set of initial functions for (5) with respect to ω and Z and that the derivative of every solution $x(\delta, q(z))(t)$ of (5) defined by any $z = (\delta, x) \in B$ has a finite left limit at every point t provided that

$$(t, x(\delta, q(z))(t)) \in \overline{\omega}.$$

Let, moreover, $Z \cap W$ be a retract of W, but not a retract of Z. Then, there exists at least one point $z_0 = (\delta_0, x_0) \in Z \cap \omega$ such that a solution $x(\delta_0, q(z_0))(t)$ exists on $[t_0 - r, t_*)$ and

$$(t, x(\delta_0, q(z_0))(t)) \in \omega$$

holds for all $t \in [t_0 - r, t_*)$.

3 Main Result

In this section we give a criterion (sufficient and necessary conditions) for the existence of a positive and strictly decreasing solution of the equation (1).

Equation (1) is a particular case of equation (5) if the functional f in the right-hand side of (5) is specified as

$$f(t, \phi, \dot{\phi}) := -c(t)\phi(-\tau(t)) + d(t)\dot{\phi}(-\delta(t)).$$

Such a functional f is used in the remaining part of the paper.

Theorem 5. *For the existence of a positive strictly decreasing solution of* (1) *on $[t_0 - r, \infty)$, a necessary and sufficient condition is that there exists a continuous function $\lambda \colon [t_0 - r, \infty) \to (0, \infty)$ such that inequality* (4) *holds for $t \geq t_0$.*

Proof. Necessity. Let a continuously differentiable positive strictly decreasing solution $y = y(t)$ of (1) be given on $[t_0 - r, \infty)$. From (1) we conclude $\dot{y}(t) < 0$ for every $t \in [t_0, \infty)$. We show that $y(t)$ can be expressed in the form

$$y(t) = \exp\left(-\int_{t_0}^{t} \lambda(s)\mathrm{d}s\right), \quad t \geq t_0 - r \tag{15}$$

where λ satisfies all conditions formulated in the theorem. Taking the derivative of y, we get

$$\dot{y}(t) = -\lambda(t)\exp\left(-\int_{t_0}^{t} \lambda(s)\mathrm{d}s\right), \quad t \geq t_0 - r \tag{16}$$

and, therefore,

$$\lambda(t) := -\frac{\dot{y}(t)}{y(t)}, \quad t \geq t_0 - r. \tag{17}$$

It can be seen from (15)-(17) that $\lambda(t) > 0$ if $t \geq t_0 - r$. Substitute (15) into (1), assuming $t \geq t_0$, and divide the equation obtained by $\exp\left(-\int_{t_0}^{t} \lambda(s)\mathrm{d}s\right)$. We get

$$\lambda(t) = c(t)\exp\left(\int_{t-\tau(t)}^{t} \lambda(s)ds\right) + d(t)\lambda(t - \delta(t))\exp\left(\int_{t-\delta(t)}^{t} \lambda(s)ds\right)$$

where $t \geq t_0$. This means that inequality (4) holds.

SUFFICIENCY. In this part we make use of Theorem 4. The proof is divided into five steps.

STEP 1. DEFINITION OF THE POLYFACIAL SET ω. We set $n = p = 1$, $s = 0$, $t_* = \infty$ and

$$l(t,y) = l_1(t,y) = y\left(y - \nu \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right)\right)$$

where $y \in \mathbb{R}$, $\nu > 1$ is a constant and λ satisfies inequality (4). Then, the set

$$\omega := \{(t,y) \in [t_0 - r, \infty) \times \mathbb{R},\ l(t,y) < 0\} \tag{18}$$

is a polyfacial set within the meaning of Definition 4 since, for every fixed $t^* \in [t_0 - r, \infty)$, the set

$$\omega \cap \{(t,y)\colon t = t^*, y \in \mathbb{R}\} = \left\{(t,y)\colon t = t^*, 0 < y < \nu \exp\left(-\int\limits_{t_0}^{t^*} \lambda(s)ds\right)\right\}$$

is open and simply connected.

STEP 2. REGULARITY OF ω. Set $q = 1$. Define a function

$$c\colon [t_0 - r, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

as

$$c(t,y,x) = x\left(x + \nu\lambda(t) \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right)\right), \tag{19}$$

and identify $c = c_1$.

We show that the set ω defined by (18) is regular with respect to equation (1) and auxiliary inequality $c(t,y,x) \le 0$ by Definition 5. Therefore, we will verify all its assumptions $\alpha) - \delta)$ (denoted below as $\alpha^*) - \delta^*)$).

$\alpha^*)$ If $(t,\phi) \in \mathbb{R} \times \mathcal{C}^1$ and $(t+\theta, \phi(\theta)) \in \omega$ for $\theta \in [-r, 0)$, then the functional f is defined at $(t, \phi, \dot{\phi})$. Thus, point $\alpha)$ of Definition 5 holds.

$\beta^*)$ Let $(t,\phi) \in \mathbb{R} \times \mathcal{C}^1$, $(t+\theta, \phi(\theta)) \in \omega$ for $\theta \in [-r, 0)$ and

$$c(t+\theta, \phi(\theta), \dot{\phi}(\theta)) \le 0,\ \ \theta \in [-r, 0). \tag{20}$$

From (19) and (20) we get

$$-\nu\lambda(t+\theta) \exp\left(-\int\limits_{t_0}^{t+\theta} \lambda(s)ds\right) \le \dot{\phi}(\theta) \le 0,\ \ \theta \in [-r, 0). \tag{21}$$

In addition, we have

$$f(t,\phi,\dot{\phi}) = -c(t)\phi(-\tau(t)) + d(t)\dot{\phi}(-\delta(t)) < 0 \tag{22}$$

since $c(t) > 0$ and $\phi(-\tau(t)) > 0$. Now using the definition of ω (18) and inequalities (21), (4), we get

$$\begin{aligned}
f(t,\phi,\dot{\phi}) =& -c(t)\phi(-\tau(t)) + d(t)\dot{\phi}(-\delta(t)) \\
\geq& -\nu c(t)\exp\left(-\int\limits_{t_0}^{t-\tau(t)}\lambda(s)ds\right) \\
&-\nu d(t)\lambda(t-\delta(t))\exp\left(-\int\limits_{t_0}^{t-\delta(t)}\lambda(s)ds\right) \\
=&\nu\exp\left(-\int\limits_{t_0}^{t}\lambda(s)ds\right)\left(-c(t)\exp\left(\int\limits_{t-\tau(t)}^{t}\lambda(s)ds\right)\right. \\
&\left.-d(t)\lambda(t-\delta(t))\exp\left(\int\limits_{t-\delta(t)}^{t}\lambda(s)ds\right)\right) \\
\geq& -\nu\lambda(t)\exp\left(-\int\limits_{t_0}^{t}\lambda(s)ds\right).
\end{aligned} \tag{23}$$

Combining (22) and (23), we obtain

$$-\nu\lambda(t)\exp\left(-\int\limits_{t_0}^{t}\lambda(s)ds\right) \leq f(t,\phi,\dot{\phi}) < 0. \tag{24}$$

A consequence of (24) is the inequality

$$\begin{aligned}
&c(t+\theta,\phi(\theta),f(t,\phi,\dot{\phi})) \\
&\qquad = f(t,\phi,\dot{\phi})\left(f(t,\phi,\dot{\phi})+\nu\lambda(t)\exp\left(-\int\limits_{t_0}^{t}\lambda(s)ds\right)\right) \leq 0.
\end{aligned}$$

Thus, point β) of Definition 5 holds.

γ^*) Let $\phi \in C^1([-r,0],\mathbb{R})$ be such that $(t+\theta,\phi(\theta)) \in \omega$ for $\theta \in [-r,0)$ and $(t,\phi(0)) \in \partial\omega$. Then, either

$$\phi(0) = 0 \tag{25}$$

or

$$\phi(0) = \nu\exp\left(-\int\limits_{t_0}^{t}\lambda(s)ds\right). \tag{26}$$

Moreover, we assume that (13) holds, i.e.,

$$\begin{aligned}
&c(t+\theta,\phi(\theta),\dot{\phi}(\theta)) \\
&= \dot{\phi}(\theta)\left(\dot{\phi}(\theta)+\nu\lambda(t+\theta)\exp\left(-\int_{t_0}^{t+\theta}\lambda(s)ds\right)\right) \leq 0,\ \ \theta \in [-r,0).
\end{aligned} \tag{27}$$

Let (25) be true. We will use the properties $\phi(-\tau(t)) > 0$ (it follows from definition (18) of the set ω) and $\dot{\phi}(-\delta(t)) \le 0$ (it is a consequence of (27)) to get

$$\begin{aligned} Dl(t,y) = Dl(t,0) &= \frac{\partial l}{\partial t}(t,0) + \frac{\partial l}{\partial y}(t,0) \cdot f(t,\phi,\dot{\phi}) \\ &= -\nu \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right)(-c(t)\phi(-\tau(t)) + d(t)\dot{\phi}(-\delta(t))) > 0. \end{aligned}$$

Let (26) be true. We will use the properties

$$\phi(-\tau(t)) < \nu \exp\left(-\int_{t_0}^{t-\tau(t)} \lambda(s)ds\right)$$

(it follows from definition (18) of the set ω) and

$$\dot{\phi}(-\delta(t)) \ge -\nu\lambda(t-\delta(t)) \exp\left(-\int_{t_0}^{t-\delta(t)} \lambda(s)ds\right)$$

(it is a consequence of (27)).

Then,

$$\begin{aligned} Dl(t,y) &= Dl\left(t, \nu \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right)\right) \\ &= \frac{\partial l}{\partial t}\left(t, \nu \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right)\right) + \frac{\partial l}{\partial y}\left(t, \nu \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right)\right) f(t,\phi,\dot{\phi}) \\ &= \nu \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right) \\ &\quad \cdot \left(\nu\lambda(t) \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right) - c(t)\phi(-\tau(t)) + d(t)\dot{\phi}(-\delta(t))\right) \\ &> \nu \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right)\left(\nu\lambda(t) \exp\left(-\int\limits_{t_0}^{t} \lambda(s)ds\right)\right. \\ &\quad \left. -\nu c(t) \exp\left(-\int\limits_{t_0}^{t-\tau(t)} \lambda(s)ds\right) - \nu d(t)\lambda(t-\delta(t)) \exp\left(-\int\limits_{t_0}^{t-\delta(t)} \lambda(s)ds\right)\right) \\ &\ge \nu^2 \exp\left(-2\int\limits_{t_0}^{t} \lambda(s)ds\right)\left(\lambda(t) - c(t) \exp\left(\int\limits_{t-\tau(t)}^{t} \lambda(s)ds\right)\right. \\ &\quad \left. - d(t)\lambda(t-\delta) \exp\left(\int\limits_{t-\delta(t)}^{t} \lambda(s)ds\right)\right) \ge \text{ [by (4)] } \ge 0. \end{aligned}$$

Thus, point γ) of Definition 5 holds.

δ^*) There is no function of the type $m(t,y)$ in the definition (18) of polyfacial set ω.

We conclude that the set ω defined by (18) is regular by Definition 5 with respect to equation (1) and auxiliary inequality $c(t,y,x) \le 0$.

Step 3. Using Theorem 4 - sets W and Z. To apply Theorem 4, we define the set W in accordance with (14) as

$$W := \{(t,y) \in \partial\omega : \ m_j(t,y) < 0,\ j = 1,\dots,s\} = \{(t,y) \in \partial\omega\}$$

since no function of the type m_j, $j = 1,\dots,s$ is used. Moreover, define

$$Z := \{(t,y) \in \omega \cup W : t = t_0\} = \{(t_0,y) : y \in [0,1]\}.$$

Obviously, $Z \cap W$ is a retract of W, but not a retract of Z.

Step 4. Using Theorem 4 - initial functions for (1). Now we will construct a set of initial functions for (1) with respect to ω and Z such that every initial function ϕ satisfies the sewing condition (7), i.e.

$$S(t_0,\phi) = 0 \tag{28}$$

where

$$S(t_0,\phi) := f(t_0,\phi,\dot\phi) - \dot\phi(0) = -c(t_0)\phi(-\tau(t_0)) + d(t_0)\dot\phi(-\delta(t_0)) - \dot\phi(0).$$

Define for any $z = (t_0,y) \in Z$ (recall that $y \in [0,1]$) two initial functions $\varphi_y^{\max}, \varphi_y^{\min} \in C^1[-r,0]$:

$$\varphi_y^{\max}(s) := \nu \exp\left(-\int\limits_{t_0}^{t_0+s} \lambda(u)\mathrm{d}u\right) - \nu + y,$$

$$\varphi_y^{\min}(s) := \frac{1}{2}ks^2 + y$$

where, for a constant $\varepsilon \in (0,1)$,

$$k := \frac{\varepsilon}{r} \cdot \min_{-r \le \theta \le 0} \lambda(t_0+\theta) \exp\left(-\int\limits_{t_0}^{t_0+\theta} \lambda(u)\mathrm{d}u\right) > 0.$$

Obviously, $\varphi_y^{\max}(0) = y$, $\quad \varphi_y^{\min}(0) = y$. For $s \in [-r,0)$, we prove

$$0 < \varphi_y^{\min}(s) < \varphi_y^{\max}(s) < \nu \exp\left(-\int\limits_{t_0}^{t_0+s} \lambda(u)\mathrm{d}u\right). \tag{29}$$

The left-hand inequality in (29) holds since $y \in [0,1]$ and $k > 0$. The right-hand inequality in (29) holds since $-\nu + y < 0$. To prove the middle inequality in (29), we define a function

$$\Psi(s) := \varphi_y^{\min}(s) - \varphi_y^{\max}(s), \quad s \in [-r, 0].$$

Then, for $s \in [-r, 0)$.

$$\begin{aligned} \Psi'(s) = \frac{\varepsilon}{r} s \min_{-r \le \theta \le 0} \lambda(t_0 + \theta) \exp\left(- \int_{t_0}^{t_0+\theta} \lambda(u)\mathrm{d}u \right) \\ + \nu\lambda(t_0 + s) \exp\left(- \int_{t_0}^{t_0+s} \lambda(u)\mathrm{d}u \right) \\ \ge -\varepsilon \min_{-r \le \theta \le 0} \lambda(t_0 + \theta) \exp\left(- \int_{t_0}^{t_0+\theta} \lambda(u)\mathrm{d}u \right) \\ + \nu\lambda(t_0 + s) \exp\left(- \int_{t_0}^{t_0+s} \lambda(u)\mathrm{d}u \right) > 0. \end{aligned}$$

Therefore,

$$\varphi_y^{\min}(s) - \varphi_y^{\max}(s) = \Psi(s) < \Psi(0) = 0, \quad s \in [-r, 0)$$

and the middle inequality in (29) is proved.

Moreover, the following chain of inequalities obviously hold

$$\begin{aligned} 0 \ge \dot{\varphi}_y^{\min}(s) = ks \ge -kr = -\varepsilon \min_{-r \le \theta \le 0} \lambda(t_0 + \theta) \exp\left(- \int_{t_0}^{t_0+\theta} \lambda(u)\mathrm{d}u \right) \\ > -\nu\lambda(t_0 + s) \exp\left(- \int_{t_0}^{t_0+s} \lambda(u)\mathrm{d}u \right) = \dot{\varphi}_y^{\max}(s), \quad s \in [-r, 0]. \end{aligned} \tag{30}$$

We show that the values $S(t_0, \varphi_y^{\max})$, $S(t_0, \varphi_y^{\min})$ take opposite signs. Using (4), we get

$$\begin{aligned} S(t_0, \varphi_y^{\max}) &= -c(t_0) \left(\nu \exp\left(- \int_{t_0}^{t_0-\tau(t_0)} \lambda(s)\mathrm{d}s \right) - \nu + y \right) \\ &\quad - \nu d(t_0)\lambda(t_0 - \delta(t_0)) \exp\left(- \int_{t_0}^{t_0-\delta(t_0)} \lambda(s)\mathrm{d}s \right) + \nu\lambda(t_0) \\ &= -\nu c(t_0) \exp\left(\int_{t_0-\tau(t_0)}^{t_0} \lambda(s)\mathrm{d}s \right) \\ &\quad - \nu d(t_0)\lambda(t_0 - \delta(t_0)) \exp\left(\int_{t_0-\delta(t_0)}^{t_0} \lambda(s)\mathrm{d}s \right) \end{aligned}$$

$$+\nu\lambda(t_0)+c(t_0)(\nu-y)>0, \tag{31}$$

and

$$S(t_0,\varphi_y^{\min})=-c(t_0)\left(\frac{k}{2}(-\tau(t_0))^2+y\right)-d(t_0)k\delta(t_0)<0. \tag{32}$$

Define a one-parameter family of functions φ_y^α depending on a parameter $\alpha\in[0,1]$ as

$$\varphi_y^\alpha(s):=\alpha\varphi_y^{\max}(s)+(1-\alpha)\varphi_y^{\min}(s),\ \ s\in[-r,0]. \tag{33}$$

Then, by (31) and (32),

$$S(t_0,\varphi_y^0)S(t_0,\varphi_y^1)=S(t_0,\varphi_y^{\min})S(t_0,\varphi_y^{\max})<0.$$

The operator

$$\begin{aligned}S(t_0,\varphi_y^\alpha)=&-c(t_0)\left[\alpha\varphi_y^{\max}(-\tau(t_0))+(1-\alpha)\varphi_y^{\min}(-\tau(t_0))\right]\\&+d(t_0)\left[\alpha\dot\varphi_y^{\max}(-\delta(t_0))+(1-\alpha)\dot\varphi_y^{\min}(-\delta(t_0))\right]-\dot\varphi_y^\alpha(0),\end{aligned} \tag{34}$$

where

$$\dot\varphi_y^\alpha(0)=\alpha\dot\varphi_y^{\max}(0)+(1-\alpha)\dot\varphi_y^{\min}(0)=-\alpha\nu\lambda(t_0),$$

is strongly monotone with respect to α, since, due to (4),

$$\begin{aligned}\frac{\partial}{\partial\alpha}S(t_0,\varphi_y^\alpha)=&-c(t_0)\left[\varphi_y^{\max}(-\tau(t_0))-\varphi_y^{\min}(-\tau(t_0))\right]\\&+d(t_0)\left[\dot\varphi_y^{\max}(-\delta(t_0))-\dot\varphi_y^{\min}(-\delta(t_0))\right]+\nu\lambda(t_0)\\=&-c(t_0)\left[\nu\exp\left(-\int\limits_{t_0}^{t_0-\tau(t_0)}\lambda(u)\mathrm{d}u\right)-\nu+y-\frac{k}{2}(-\tau(t_0))^2-y\right]\\&+d(t_0)\left[-\nu\lambda(t_0-\delta(t_0))\exp\left(-\int\limits_{t_0}^{t_0-\delta(t_0)}\lambda(u)\mathrm{d}u\right)-k(-\delta(t_0))\right]+\nu\lambda(t_0)\\=&\ \nu\left[\lambda(t_0)-c(t_0)\exp\left(-\int\limits_{t_0}^{t_0-\tau(t_0)}\lambda(u)\mathrm{d}u\right)\right.\\&\left.-d(t_0)\lambda(t_0-\delta(t_0))\exp\left(-\int\limits_{t_0}^{t_0-\delta(t_0)}\lambda(u)\mathrm{d}u\right)\right]\\&+\nu c(t_0)+\frac{k}{2}c(t_0)\tau^2(t_0))+kd(t_0)\delta(t_0)>0.\end{aligned}$$

Then, there exists a unique value $\alpha=\alpha_y\in[0,1]$ such that $S(t_0,\varphi_y^{\alpha_y})=0$, i.e., the sewing condition (28) is true. This value, as can be seen in (34), is defined by the formula

$$\alpha_y=\frac{c(t_0)\varphi_y^{\min}(-\tau(t_0))-d(t_0)\dot\varphi_y^{\min}(-\delta(t_0))}{c(t_0)\Psi(-\tau(t_0))-d(t_0)\dot\Psi(-\delta(t_0))+\nu\lambda(t_0)}$$

and depends continuously on y since $\varphi_y^{\min}$, $\varphi_y^{\max}$ and Ψ depend continuously on y. Therefore, the function

$$\varphi_y^{\alpha_y}(s) = \alpha_y \varphi_y^{\max}(s) + (1-\alpha_y)\varphi_y^{\min}(s)$$
$$= \alpha_y \left[\nu \exp\left(-\int\limits_{t_0}^{t_0+s} \lambda(u)\mathrm{d}u\right) - \nu + y\right] + (1-\alpha_y)\left[\frac{1}{2}ks^2 + y\right], \ \ s \in [-r, 0]$$

is continuous with respect to y as well.

Applying (30), we see that, for any function $\varphi_y^{\alpha_y}(s)$, $s \in [-r, 0]$, defined by (33), we have:

$$\begin{aligned}\dot{\varphi}_y^{\alpha_y}(s) =& \alpha_y \dot{\varphi}_y^{\max}(s) + (1-\alpha_y)\dot{\varphi}_y^{\min}(s)\\ &\le \alpha_y \dot{\varphi}_y^{\min}(s) + (1-\alpha_y)\dot{\varphi}_y^{\min}(s) = \dot{\varphi}_y^{\min}(s), \ \ s \in [-r, 0],\\ \dot{\varphi}_y^{\alpha_y}(s) =& \alpha_y \dot{\varphi}_y^{\max}(s) + (1-\alpha_y)\dot{\varphi}_y^{\min}(s)\\ &\ge \alpha_y \dot{\varphi}_y^{\max}(s) + (1-\alpha_y)\dot{\varphi}_y^{\max}(s) = \dot{\varphi}_y^{\max}(s), \ \ s \in [-r, 0].\end{aligned}$$

Step 5. Using Theorem 4 - initial functions for (1) and mapping q. By Definition 6, we will construct a continuous mapping $q\colon B \to \mathcal{C}^1$ where the set B is defined in Lemma 1, point (2) and, in our case, becomes

$$B = \overline{Z} \cap (Z \cup W) = Z.$$

Then, q maps the set Z into the space of initial functions satisfying the sewing condition. Define such a mapping $q\colon B \to C^1[-r, 0]$ for every $z = (t_0, y) \in B$ by the formula

$$q(z) = q((t_0, y)) = \varphi_y^{\alpha_y}. \tag{35}$$

This mapping is continuous and

$$\begin{aligned}&(t_0 + \theta, q(z)(\theta))\\ &\qquad = (t_0 + \theta, \alpha_y \varphi_y^{\max}(\theta) + (1-\alpha_y)\varphi_y^{\min}(\theta)) \in \omega \quad \text{for} \quad \theta \in [-r, 0),\end{aligned}$$

$$(t_0, q(z)(0)) = (t_0, \alpha_y \varphi_y^{\max}(0) + (1-\alpha_y)\varphi_y^{\min}(0)) = z.$$

The mapping q satisfies conditions 1) and $2a$) of Definition 6. All assumptions of Theorem 4 are now fulfilled. Therefore, there exists at least one point $z_0 = (t_0, y_0) \in Z \cap \omega$ such that a solution $x(t_0, q(z_0))(t)$ of (1) exists on $[t_0 - r, \infty)$ and

$$(t, x(t_0, q(z_0))(t)) \in \omega \tag{36}$$

holds for all $t \in [t_0 - r, \infty)$. Because of the shape of ω, such a solution is positive and, by (4), it is strictly decreasing. $\square$

Remark 2. *Let all assumptions of Theorem 5 be true. From its proof (see* (36) *and the definition* (18) *of the set* ω*) we deduce that, if* (4) *holds for* $t \ge t_0$*, then there exist a positive strictly decreasing solution* $y = y(t)$ *of* (1) *on* $[t_0 - r, \infty)$ *satisfying the inequalities*

$$0 < y(t) < \exp\left(-\int_{t_0}^{t} \lambda(s)ds\right), \ \ t \in [t_0 - r, \infty). \tag{37}$$

Moreover, from formulas (11) *and* (12) *of Definition 5, such a solution satisfies the inequalities*

$$c(t, y(t), \dot{y}(t)) \le 0, \ \ t \in [t_0 - r, \infty),$$

i.e.,

$$-\lambda(t)\exp\left(-\int_{t_0}^{t} \lambda(s)ds\right) \le \dot{y}(t) \le 0, \ \ t \in [t_0 - r, \infty). \tag{38}$$

Due to the linearity of (1)*, the coefficient* ν *is omitted in* (37) *and* (38)*.*

4 Generalization

Consider an equation

$$\dot{y}(t) = -\sum_{i=1}^{m} c_i(t)y(t - \tau_i(t)) + \sum_{j=1}^{r} d_j(t)\dot{y}(t - \delta_j(t)) \tag{39}$$

where $c_i, d_j\colon [t_0, \infty) \to [0, \infty)$, and $\tau_i, \delta_j\colon [t_0, \infty) \to (0, r]$ are continuous functions. Moreover, assume $\sum_{i=1}^{m} c_i(t) > 0$, $t \in [t_0, \infty)$. Obviously, equation (39) is more general than equation (1). Now we will formulate a generalization of Theorem 5. We omit its proof since it is similar to that of Theorem 5. Note that the system of initial functions can be used in the proof without any changes.

Theorem 6. *For the existence of a positive strictly decreasing solution of* (39) *on* $[t_0 - r, \infty)$*, a necessary and sufficient condition is that there exists a continuous function* $\lambda\colon [t_0 - r, \infty) \to (0, \infty)$ *such that the inequality*

$$\lambda(t) \ge \sum_{i=1}^{m} c_i(t)\exp\left(\int\limits_{t-\tau_i(t)}^{t} \lambda(s)ds\right) + \sum_{j=1}^{r} d_j(t)\lambda(t - \delta_j(t))\exp\left(\int\limits_{t-\delta_j(t)}^{t} \lambda(s)ds\right)$$

holds for $t \ge t_0$*. Moreover, if this inequality holds, then there exists a positive strictly decreasing solution* $y = y(t)$ *of* (39) *on* $[t_0 - r, \infty)$ *satisfying inequalities* (37) *and* (38)*.*

5 Concluding discussions

From the proof of Theorem 5, we conclude that a positive solution (if inequality (4) holds) is generated by a function from a one-parameter family of functions $\varphi_y^{\alpha_y}$, defined by formula (35) where the parameter $y \in [0,1]$. More specifically, as it follows from points (2) and (3) of Lemma 1, we can restrict the values of the parameter y only to values $y \in (0,1)$. In this connection, the following open problem arises.

Open Problem 1. *How to compute a value (values) of parameter* $y = y^* \in (0,1)$ *such that the initial function* $\varphi_{y^*}^{\alpha_{y^*}}$ *determines a positive solution of equation* (1) *(or* (39)*) indicated in Theorem 5?*

A solution to this open problem can have certain importance, e.g., in numerical computations.

Because of the linearity of considered equations and the existence of a positive solution, we conclude that there exists a one-parameter family of linearly dependent positive solutions of equation (1) on interval $[t_0 - r, \infty)$.

It is easy to explain, that there exists a one-parameter family of linearly independent positive solutions of equation (1) on $[t_0 - r, \infty)$. Looking again at the proof of Theorem 5, we emphasize that the definition of the function $\varphi_y^{\min}$ depends (through the constant k) on a parameter $\varepsilon \in (0,1)$. Therefore, each function in the system of initial functions $\varphi_y^{\alpha_y}$ where $y \in (0,1)$, relevant to a choice of ε, is linearly independent on an interval $[t_0 - r, t_0]$ of every function in the system of initial functions $\varphi_y^{\alpha_y}$ constructed for a different choice of ε. Consequently, positive solutions defined by different initial functions, being linearly independent on interval $[t_0 - r, t_0]$ are linearly independent positive solutions of equation (1) on $[t_0 - r, \infty)$. One cannot, however, conclude that such a type of linear independence on the interval $[t_0 - r, \infty)$ implies the existence of a one-parameter family of linearly independent positive solutions of equation (1) on every interval $[t_1 - r, \infty)$ where $t_1 \geq t_0$. This assertion can be wrong due to, e.g., the effect of solution pasting (we refer to [26, Part 3.5]). A similar discussion applies to the function $\varphi_y^{\max}$ and the parameter ν. Nevertheless, we formulate the following open problem connected with this topic.

Open Problem 2. *Indicate sufficient conditions for the existence of at least a one-parameter family of linearly independent positive solutions of equation* (1) *(or* (39)*) on every interval* $[t_1 - r, \infty)$ *where* $t_1 \geq t_0$.

Obviously, Theorem 5 is a generalization of Theorem 1 to neutral differential equations. Now, we will restrict our discussion only to equation (1) and its special cases although it is easy to formulate corresponding remarks to more general equation (39) and its special cases.

Let the functions $c(t)$, $d(t)$ and delays $\tau(t)$, $\delta(t)$ in equation (1) be constant, i.e., $c(t) \equiv c = \text{const}$, $d(t) \equiv d = \text{const}$, $\tau(t) \equiv \tau = \text{const}$, $\delta(t) \equiv \delta = \text{const}$ and equation (1) becomes

$$\dot{y}(t) = -cy(t-\tau) + d\dot{y}(t-\delta). \tag{40}$$

Then, Theorem 5 is formulated as

Theorem 7. *For the existence of a positive strictly decreasing solution of* (40) *on* $[t_0 - r, \infty)$*, a necessary and sufficient condition is that there exists a continuous function* $\lambda\colon [t_0 - r, \infty) \to (0, \infty)$ *such that inequality*

$$\lambda(t) \geq c \exp\left(\int\limits_{t-\tau}^{t} \lambda(s)ds\right) + d\lambda(t-\delta)\exp\left(\int\limits_{t-\delta}^{t} \lambda(s)ds\right) \tag{41}$$

holds for $t \geq t_0$*.*

From Theorem 7 and formula (41) where $\lambda(t) \equiv \lambda = \text{const}$, we immediately get the following corollaries. These criteria are well-known, we refer, e.g., to [24, Theorem 5.2.10, Corollary 5.2.11], [25, Theorem 6.7.1]. Similar criteria can be found, e.g., in [1, Corollary 6.5], [2, Theorem 3.5.3] and [23, Theorem 3.2.3].

Corollary 1. *For the existence of a positive strictly decreasing solution of* (40) *on* $[t_0 - r, \infty)$ *it is sufficient the existence of a positive constant* λ *such that inequality*

$$\lambda \geq ce^{\lambda\tau} + \lambda de^{\lambda\delta} \tag{42}$$

holds.

For the choice $\lambda = 1/\tau$ or $\lambda = 1/\delta$ in (42), we get

Corollary 2. *For the existence of a positive strictly decreasing solution of* (40) *on* $[t_0 - r, \infty)$ *it is sufficient that either inequality*

$$1 > ce\tau + de^{\delta/\tau} \tag{43}$$

or inequality

$$1 > c\delta e^{\tau/\delta} + de \tag{44}$$

hold.

Corollaries 1, 2 can be improved in view of Remark 2 (formulas (37), (38)) in the sense that if inequalities (42), (43), (44) are valid, then on $[t_0, \infty)$ there exist a positive solution vanishing for $t \to \infty$ and having negative and vanishing for $t \to \infty$ continuous derivative.

Remark 3. *In the paper we regard solutions of equation* (1) *as continuously differentiable functions satisfying the given equation everywhere. As noted, e.g., in [28, p. 107] it leads to some complications, since the sewing condition must be valid for continuously differentiable initial functions. In the proof of Theorem 5, a modification of the retract principle suitable for neutral differential equations was used. This principle, to be successfully applied, needs not only one initial function, but a whole family of initial functions satisfying the sewing condition. Therefore, the crucial moment of the proof was a special construction of such a family of initial functions.*

To compare our results with, e.g., those given in [1, Theorem 6.1] we emphasize that the definition of a solution substantially differs (a solution is defined as an absolutely continuous function satisfying the equation almost everywhere).

In [1, 3, 23, 24, 25] part of the results is devoted to the existence of positive solutions of neutral equations having, e.g., the form

$$(y(t) + P(t)y(t-\tau))' + Q(t)y(t-\sigma) = 0, \ \ t \geq t_0$$

under various conditions for P and Q. The substantial difference is that the delays in the equation, unlike those in our investigation, are constant. Thus, the results derived in the cited sources are, in principle, not applicable to equation (1).

Acknowledgements

The authors greatly appreciate the work of the anonymous referee, whose comments and suggestions have helped to improve the paper in many aspects. The first author was supported by the Czech Science Foundation under the project 16-08549S. The research of the second author was carried out under the project CEITEC 2020 (LQ1601) with financial support from the Ministry of Education, Youth and Sports of the Czech Republic under the National Sustainability Programme II. The work of the first author was realized in CEITEC - Central European Institute of Technology with research infrastructure supported by the project CZ.1.05/1.1.00/02.0068 financed from European Regional Development Fund.

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Computers and Mathematics with Applications 56 (2008) 556–564

An International Journal
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Explicit criteria for the existence of positive solutions for a scalar differential equation with variable delay in the critical case

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Received 28 February 2007; received in revised form 28 December 2007; accepted 4 January 2008

Abstract

A scalar linear differential equation with time-dependent delay $\dot{x}(t) = -a(t)x(t - \tau(t))$ is considered, where $t \in I := [t_0, \infty)$, $t_0 \in \mathbb{R}$, $a : I \to \mathbb{R}^+ := (0, \infty)$ is a continuous function and $\tau : I \to \mathbb{R}^+$ is a continuous function such that $t - \tau(t) > t_0 - \tau(t_0)$ if $t > t_0$. The goal of our investigation is to give sufficient conditions for the existence of positive solutions as $t \to \infty$ in the critical case in terms of inequalities on a and τ. A generalization of one known final (in a certain sense) result is given for the case of τ being not a constant. Analysing this generalization, we show, e.g., that it differs from the original statement with a constant delay since it does not give the best possible result. This is demonstrated on a suitable example.

Keywords: Positive solution; Delayed equation; Critical case; Infinite delay; p-function

1. Preliminaries

In this paper we consider a scalar linear differential equation with time-dependent delay

$$\dot{x}(t) = -a(t)x(t - \tau(t)), \tag{1}$$

where $t \in I := [t_0, \infty)$, $t_0 \in \mathbb{R}$, $a : I \to \mathbb{R}^+ := (0, \infty)$ is a continuous function and $\tau : I \to \mathbb{R}^+$ is a continuous function such that $t - \tau(t) > t_0 - \tau(t_0)$ if $t > t_0$. The goal of our investigation is to give sufficient conditions for the existence of positive solutions of (1) as $t \to \infty$ in terms of inequalities on a and τ. In the literature, several results have been derived with the aid of a suitable estimation of function a. A final result (in a certain sense) in one of the directions pursued is given in [1] for the case of a constant delay. Namely, it holds.

Theorem 1. **(I)** *Let us assume that $a(t) \le a_k(t)$ with*

$$a_k(t) := \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\ldots\ln_k t)^2} \tag{2}$$

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0898-1221/$ - see front matter
doi:10.1016/j.camwa.2008.01.015

if $t \to \infty$ *and an integer* $k \geq 0$. *Then there exists a positive solution* $x = x(t)$ *of* (1) *with* $\tau(t) \equiv \tau = \text{const}$. *Moreover,*

$$x(t) < v_k(t) := \mathrm{e}^{-t/\tau}\sqrt{t \ln t \ln_2 t \cdots \ln_k t}$$

as $t \to \infty$.

(II) *Let us assume that*

$$a(t) > a_{k-2}(t) + \frac{\theta\tau}{8e(t \ln t \ln_2 t \cdots \ln_{k-1} t)^2} \tag{3}$$

if $t \to \infty$, *an integer* $k \geq 2$ *and a constant* $\theta > 1$. *Then all the solutions of* (1) *with* $\tau(t) \equiv \tau = \text{const}$ *oscillate.*

In this theorem, $\ln_k t := \ln(\ln_{k-1} t)$, $k \geq 1$, $\ln_0 t := t$ and it is assumed that $t > \exp_{k-2} 1$ where $\exp_k t := \exp(\exp_{k-1} t)$, $k \geq 1$, $\exp_0 t := t$, and $\exp_{-1} t := 0$.

Theorem 1 can be applied to what is called the critical case since inequalities (2) and (3) are almost opposite. With respect to the critical case, we refer (in addition to the paper, mentioned above) to the papers [2–5] and the book [6]. We give a generalization of the first part of Theorem 1 for the case of τ being not a constant. As a tool for this generalization, we use the results on the existence of positive solutions for retarded functional differential equations with unbounded delay and finite memory. The necessary relevant information is given in Section 2. The generalization of Theorem 1 is given in Section 3. Analysing this generalization, we conclude that it differs from the original statement with a constant delay since it does not give the best possible result. To show this, in Section 3 we formulate another sufficient condition of positivity and in Section 4 we show that, for a class of delays, it yields a better result. Finally, in Section 5 we explain why a generalization of Theorem 1 (i.e., generalization in both its parts) for the case of τ being not a constant is not possible. Other results concerning the existence of positive solutions, may, for example, be found in [7–20].

2. Positive solutions of equations with p-functions

A continuous function $p: \mathbb{R} \times [-1, 0] \to \mathbb{R}$ is called a p-function if it has the following properties [21, p. 8]: $p(t, 0) = t$, $p(t, -1)$ is a nondecreasing function of t, and there exists a $\sigma \geq -\infty$ such that $p(t, \vartheta)$ is an increasing function for ϑ for each $t \in (\sigma, \infty)$. Throughout the following text, we assume $\sigma = t_0$. We define $p_0 := p(t_0, -1)$.

We consider a differential equation with p-functions

$$\dot{x}(t) = -\sum_{q=1}^{m} c_q(t) x(p(t, \vartheta_q)), \tag{4}$$

where $\vartheta_q = \text{const}$, $q = 1, \ldots, m$, $-1 = \vartheta_1 < \vartheta_2 < \cdots < \vartheta_m = 0$, functions $c_q: [t_0, \infty) \to \mathbb{R}_+ := [0, \infty)$ are continuous and $\sum_{q=1}^{m-1} c_q(t) > 0$ for $t \in [t_0, \infty)$. We will use one result derived in [14] concerning necessary and sufficient conditions of the existence of positive solutions for equations with p-functions:

Theorem 2. *A positive solution* $x = x(t)$ *on* $[p_0, \infty)$ *of* (4) *exists if and only if a locally integrable function* $\lambda: [p_0, \infty) \to \mathbb{R}$ *exists continuous on* $[p_0, t_0) \cup [t_0, \infty)$ *and satisfying the integral inequality*

$$\lambda(t) \geq \sum_{q=1}^{m} c_q(t) \mathrm{e}^{\int_{p(t,\vartheta_q)}^{t} \lambda(s)\mathrm{d}s}$$

for $t \geq t_0$. *Moreover,* $x(t) < \exp\left(-\int_{p_0}^{t} \lambda(s)\mathrm{d}s\right)$.

Eq. (1) is a particular case of (4). This becomes clear if we define

$$p(t, \vartheta) := \begin{cases} t + 2\vartheta\tau(t) \quad \text{if } -1/2 \leq \vartheta \leq 0, \\ t_0 - \tau(t_0) + 2(1 + \vartheta)\,(t - \tau(t) - (t_0 - \tau(t_0))) \\ \quad \text{if } -1 \leq \vartheta \leq -1/2, \end{cases}$$

$m = 3$, $\vartheta_1 = -1$, $\vartheta_2 = -1/2$, $\vartheta_3 = 0$, $c_1(t) = 0$, $c_2(t) = a(t)$ and $c_3(t) = 0$. Then $p_0 = t_0 - \tau(t_0)$ and Theorem 2 reduces to:

Theorem 3. *A positive solution $x = x(t)$ on $[t_0 - \tau(t_0), \infty)$ of the Eq. (1) exists if and only if a locally integrable function $\lambda\colon [t_0 - \tau(t_0), \infty) \to \mathbb{R}$ exists continuous on $[t_0 - \tau(t_0), t_0) \cup [t_0, \infty)$ and satisfying the integral inequality*

$$\lambda(t) \geq a(t) \mathrm{e}^{\int_{t-\tau(t)}^{t} \lambda(s) \mathrm{d}s} \tag{5}$$

for $t \geq t_0$. Moreover, $x(t) < \exp\left(-\int_{t_0-\tau(t_0)}^{t} \lambda(s)\mathrm{d}s\right)$.

This theorem will be used for finding two explicit criteria for the existence of positive solutions of (1).

Remark 1. The above specification of p-function is obviously not unique. One can put e.g. $p(t, \vartheta) := t + \vartheta\tau(t)$, $m = 2$, $\vartheta_1 = -1$, $\vartheta_2 = 0$, $c_1(t) = a(t)$ and $c_2(t) = 0$ and assume that $t - \tau(t)$ is a nondecreasing function of t on I rather than assuming $t - \tau(t) > t_0 - \tau(t_0)$ if $t > t_0$ as above. Then, $p_0 = t_0 - \tau(t_0)$ and Theorem 2 reduces to Theorem 3 again.

3. Criteria of existence of positive solutions

3.1. *First criterion — a generalization of* Theorem 1, *part* I

Now we give a generalization of Theorem 1 with the aid of a suitable auxiliary function more general than the function $a_k(t)$ given by (2). The form of this new function formally copies the old one, but now delay τ will be a function. The proof needs some auxiliary results. Below, symbols O and o mean the Landau order symbols. If real functions f_1, f_2, f_3 are defined as $t \to \infty$, then the relation $f_1(t) = f_2(t) + O(f_3(t))$ means that there exists a positive constant M such that

$$|f_1(t) - f_2(t)| \leq M|f_3(t)|$$

as $t \to \infty$, and the relation $f_1(t) = f_2(t) + o(f_3(t))$ is equivalent with

$$\lim_{t\to\infty} \frac{f_1(t) - f_2(t)}{f_3(t)} = 0$$

if $f_3(t) \neq 0$.

Lemma 1. *Let $\tau(t) = o(t)$ as $t \to \infty$. Then*

$$(t - \tau(t))^\sigma = t^\sigma \left[1 - \frac{\sigma\tau(t)}{t} + \frac{\sigma(\sigma-1)\tau^2(t)}{2t^2} - \frac{\sigma(\sigma-1)(\sigma-2)\tau^3(t)}{6t^3} + O\left(\frac{\tau^4(t)}{t^4}\right)\right] \tag{6}$$

for $t \to \infty$ and any fixed $\sigma \in \mathbb{R}$.

Proof. This can be verified easily using the binomial formula. □

Lemma 2. *Let $\tau(t)\ln t = o(t)$ as $t \to \infty$. Then*

$$[\ln(t - \tau(t))]^{\frac{1}{2}} = (\ln t)^{\frac{1}{2}} \left[1 - \frac{\tau(t)}{2t\ln t} - \frac{\tau^2(t)}{4t^2\ln t}\left(1 + \frac{1}{2\ln t}\right) + O\left(\frac{\tau^3(t)}{t^3\ln t}\right)\right]$$

as $t \to \infty$.

Proof. For $t \to \infty$ we have

$$\begin{aligned}[\ln(t - \tau(t))]^{\frac{1}{2}} &= (\ln t)^{\frac{1}{2}} \left[1 + \frac{1}{\ln t}\ln\left(1 - \frac{\tau(t)}{t}\right)\right]^{\frac{1}{2}} \\ &= (\ln t)^{\frac{1}{2}} \left[1 - \frac{1}{\ln t}\left(\frac{\tau(t)}{t} + \frac{\tau^2(t)}{2t^2} + O\left(\frac{\tau^3(t)}{t^3}\right)\right)\right]^{\frac{1}{2}}.\end{aligned}$$

The proof can be finished by expanding the expression in square brackets using the binomial formula. □

Lemma 3. *Let $\tau(t)\ln t \ln_2 t \dots \ln_k t = o(t)$ as $t \to \infty$. Then*

$$[\ln_k(t-\tau(t))]^{\frac{1}{2}} = (\ln_k t)^{\frac{1}{2}}\left[1 - \frac{\tau(t)}{2t\ln t\ln_2 t\cdots\ln_{k-1} t\ln_k t}\right.$$
$$-\frac{\tau^2(t)}{4t^2\ln t\ln_2 t\cdots\ln_{k-1} t\ln_k t}\left(1+\frac{1}{\ln t}+\cdots+\frac{1}{\ln t\ln_2 t\cdots\ln_{k-1} t}\right.$$
$$\left.\left.+\frac{1}{2\ln t\ln_2 t\cdots\ln_{k-1} t\ln_k t}\right)+O\left(\frac{\tau^3(t)}{t^3\ln t\ln_2 t\cdots\ln_k t}\right)\right] \tag{7}$$

for $t \to \infty$ and any fixed $k \geq 1$.

Proof. For $k = 1$, the proof follows from Lemma 2. Suppose that (7) holds with for $k-1$ (instead of k) and $(k-1) \geq 1$. We use it for the representation of $\ln_{k-1}(t-\tau(t))$ in the relation

$$[\ln_k(t-\tau(t))]^{\frac{1}{2}} = (\ln_k t)^{\frac{1}{2}}\left[1+\frac{1}{\ln_k t}\left(\ln\frac{\ln_{k-1}(t-\tau(t))}{\ln_{k-1} t}\right)\right]^{\frac{1}{2}}.$$

We get

$$[\ln_k(t-\tau(t))]^{\frac{1}{2}} = (\ln_k t)^{\frac{1}{2}}\left[1+\frac{2}{\ln_k t}\ln\left(1-\frac{\tau(t)}{2t\ln t\ln_2 t\cdots\ln_{k-2} t\ln_{k-1} t}\right.\right.$$
$$-\frac{\tau^2(t)}{4t^2\ln t\ln_2 t\cdots\ln_{k-2} t\ln_{k-1} t}\left(1+\frac{1}{\ln t}+\cdots+\frac{1}{\ln t\ln_2 t\cdots\ln_{k-2} t}\right.$$
$$\left.\left.\left.+\frac{1}{2\ln t\ln_2 t\cdots\ln_{k-2} t\ln_{k-1} t}\right)+O\left(\frac{\tau^3(t)}{t^3\ln t\ln_2 t\dots\ln_{k-1} t}\right)\right)\right]^{\frac{1}{2}}.$$

After decomposing logarithm $\ln(1-\cdots)$ into its Taylor's polynomial, we expand the expression in square brackets by the binomial formula. Then, using only the necessary terms, we get the representation (7). □

Let us consider now a linear equation

$$\dot{x}(t) = -A(t)x(t-\tau(t)) \tag{8}$$

with $A\colon I \to \mathbb{R}$.

Lemma 4 (*[1]*). *Let $a(t) \leq A(t)$ on I and (8) have a positive solution $x = \mu(t)$ on $[t_0-\tau(t_0),\infty)$. Then (1) has a positive solution $x = x(t)$ on $[t_0-\tau(t_0),\infty)$ and, moreover, $x(t) < \mu(t)$ holds.*

Now we define a new auxiliary function

$$a_{k\tau}(t) := \frac{1}{e\tau(t)}+\frac{\tau(t)}{8et^2}+\frac{\tau(t)}{8e(t\ln t)^2}+\frac{\tau(t)}{8e(t\ln t\ln_2 t)^2}+\cdots+\frac{\tau(t)}{8e(t\ln t\ln_2 t\dots\ln_k t)^2} \tag{9}$$

for $t \to \infty$ and an integer $k \geq 0$.

Theorem 4. *Let us assume that $a(t) \leq a_{k\tau}(t)$ and $\int_{t-\tau(t)}^{t} \mathrm{d}s/\tau(s) \leq 1$ if $t \to \infty$ and an integer $k \geq 0$. Let moreover $\tau(t)\ln t\ln_2 t\dots\ln_k t = o(t)$ as $t \to \infty$. Then there exists a positive solution $x = x(t)$ of (1) satisfying*

$$x(t) < \sqrt{t\ln t\ln_2 t\dots\ln_k t}\cdot\exp\int_{t_0-\tau(t_0)}^{t}\left(\frac{-1}{\tau(s)}\right)\mathrm{d}s \tag{10}$$

as $t \to \infty$.

Proof. Let us consider an auxiliary equation

$$\dot{x}(t) = -a_{k\tau}(t)x(t-\tau(t)) \tag{11}$$

and let us prove the existence of a positive solution. We verify (5) for $a(t) := a_{k\tau}(t)$ and

$$\lambda(t) := \lambda_k(t) = \frac{1}{\tau(t)} - \frac{1}{2t} - \frac{1}{2t\ln t} - \frac{1}{2t\ln t\ln_2 t} - \cdots - \frac{1}{2t\ln t\ln_2 t\ldots\ln_k t}.$$

With the aid of Lemma 1 (substituting $\sigma = 1/2$ in (6)), Lemmas 2 and 3 we estimate the exponential term on the right-hand side of (5). We obtain

$$\begin{aligned}
\exp\int_{t-\tau(t)}^{t}\lambda_k(s)\mathrm{d}s &= \exp\int_{t-\tau(t)}^{t}\frac{\mathrm{d}s}{\tau(s)}\cdot\sqrt{\frac{t-\tau(t)}{t}\cdot\frac{\ln(t-\tau(t))}{\ln t}\cdots\frac{\ln_k(t-\tau(t))}{\ln_k t}}\\
&\le \mathcal{E} := e\left[1-\frac{\tau(t)}{2t}-\frac{\tau^2(t)}{8t^2}-\frac{\tau^3(t)}{16t^3}+O\left(\frac{\tau^4(t)}{t^4}\right)\right]\\
&\quad\times\left[1-\frac{\tau(t)}{2t\ln t}-\frac{\tau^2(t)}{4t^2\ln t}\left(1+\frac{1}{2\ln t}\right)+O\left(\frac{\tau^3(t)}{t^3\ln t}\right)\right]\\
&\quad\cdots\\
&\quad\times\left[1-\frac{\tau(t)}{2t\ln t\ln_2 t\ldots\ln_{k-1}t\ln_k t}-\frac{\tau^2(t)}{4t^2\ln t\ln_2 t\ldots\ln_{k-1}t\ln_k t}\right.\\
&\quad\times\left(1+\frac{1}{\ln t}+\cdots+\frac{1}{\ln t\ln_2 t\ldots\ln_{k-1}t}+\frac{1}{2\ln t\ln_2 t\ldots\ln_{k-1}t\ln_k t}\right)\\
&\quad\left.+\ O\left(\frac{\tau^3(t)}{t^3\ln t\ln_2 t\ldots\ln_{k-1}t\ln_k t}\right)\right].
\end{aligned}$$

After some simplification, we get

$$\begin{aligned}
\exp\int_{t-\tau(t)}^{t}\lambda_k(s)\mathrm{d}s \le \mathcal{E} = e\Bigg[1&-\frac{\tau(t)}{2t}\left(1+\frac{1}{\ln t}+\cdots+\frac{1}{\ln t\ldots\ln_k t}\right)\\
&-\frac{\tau^2(t)}{8t^2}\left(1+\frac{1}{(\ln t)^2}+\cdots+\frac{1}{(\ln t\ldots\ln_k t)^2}\right)-\frac{\tau^3(t)}{16t^3}+O\left(\frac{\tau^3(t)}{t^3\ln t}\right)\Bigg].
\end{aligned}$$

Now we have, for the right-hand side $\mathcal{R}$ of (5),

$$\begin{aligned}
\mathcal{R} &\le \left[\frac{1}{\tau(t)}+\frac{\tau(t)}{8t^2}+\frac{\tau(t)}{8(t\ln t)^2}+\frac{\tau(t)}{8(t\ln t\ln_2 t)^2}+\cdots+\frac{\tau(t)}{8(t\ln t\ln_2 t\ldots\ln_k t)^2}\right]\mathrm{e}^{-1}\cdot\exp\int_{t-\tau(t)}^{t}\lambda_k(s)\mathrm{d}s\\
&\le \frac{1}{\tau(t)}-\frac{1}{2t}-\frac{1}{2t\ln t}-\frac{1}{2t\ln t\ln_2 t}-\cdots-\frac{1}{2t\ln t\ln_2 t\ldots\ln_k t}-\frac{\tau^2(t)}{8t^3}+O\left(\frac{\tau^2(t)}{t^3\ln t}\right).
\end{aligned}$$

Comparing the left-hand side $\mathcal{L}$ of (5) and the right-hand side $\mathcal{R}$ of (5) we conclude that for $\mathcal{L} \ge \mathcal{R}$

$$0 \ge -\frac{\tau^2(t)}{8t^3}+O\left(\frac{\tau^2(t)}{t^3\ln t}\right)$$

is sufficient. This inequality obviously holds as $t \to \infty$. Therefore, (5) is valid and (11) has a positive solution $x = \mu_k(t)$. Now it remains to apply Lemma 4 with $A(t) := a_{k\tau}(t)$. Consequently, (1) has a positive solution $x = x(t)$ that satisfies the inequality $x(t) < \mu_k(t)$ as $t \to \infty$. For $\mu_k(t)$, we have an estimate

$$\begin{aligned}
\mu_k(t) &< \exp\left(-\int_{t_0-\tau(t_0)}^{t}\lambda_k(s)\mathrm{d}s\right)\\
&= \left(\frac{t\ln t\ldots\ln_k t}{(t_0-\tau(t_0))\ln(t_0-\tau(t_0))\ldots\ln_k(t_0-\tau(t_0))}\right)^{\frac{1}{2}}\exp\left(-\int_{t_0-\tau(t_0)}^{t}\frac{1}{\tau(s)}\mathrm{d}s\right).
\end{aligned}$$

From the linearity of (1), it follows that there exists a positive solution satisfying (10). □

3.2. *Second criterion*

The second sufficient condition for the existence of a positive solution can be derived from inequality (5).

Theorem 5. *Let us assume that*

$$a(t) \le \frac{1}{\tau(t)} \cdot \exp\left(-\int_{t-\tau(t)}^{t} \frac{\mathrm{d}s}{\tau(s)}\right) \tag{12}$$

as $t \to \infty$. Then there exists a positive solution $x = x(t)$ of (1). Moreover,

$$x(t) < \exp\left(-\int_{t_0-\tau(t_0)}^{t} \frac{\mathrm{d}s}{\tau(s)}\right).$$

Since the statement of Theorem 5 is a straightforward consequence of (5) with $\lambda(t) := 1/\tau(t)$, no proof is necessary. We remark only that, for $\tau(t) = \tau = \text{const}$, inequality (12) gives a classical sufficient condition for the existence of positive solutions, namely, the condition $a(t) \le 1/(\tau e)$.

4. Analysis of both criteria

To compare Theorem 4 with Theorem 5, we will investigate equation (1), where

$$\tau(t) := c + d/t \tag{13}$$

and c, d are positive constants, i.e., we consider an equation

$$\dot{x}(t) = -a(t)x(t - c - d/t). \tag{14}$$

4.1. Application of the first criterion

The delay (13) is decreasing, tends to c as $t \to \infty$ and satisfies the inequality

$$\int_{t-\tau(t)}^{t} \frac{\mathrm{d}s}{\tau(s)} < 1.$$

If

$$a(t) \le a_{k\tau}(t) \tag{15}$$

for an integer $k \ge 0$ as $t \to \infty$ then, by Theorem 4, Eq. (14) has a positive solution. We will first develop several terms of the asymptotic decomposition of $a_{k\tau}(t)$ with $\tau(t)$ given by (13) if $t \to \infty$ and rewrite condition (15). We get sufficient condition for the existence of a positive solution of (14) in the form

$$a(t) \le a_{k\tau}(t) = \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{t} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{c}{8}\right) \cdot \frac{1}{t^2} + o\left(\frac{1}{t^2}\right). \tag{16}$$

Remark 2. The right-hand side of (16) was obtained only with the aid of two terms of expression (9) and does not explicitly contain index k. In other words, we used only the necessary (for our following analysis) part of the expression (9). Therefore, our decomposition and, consequently, inequality (16) holds for every $k \ge 0$.

4.2. Application of the second criterion

We compute

$$\int_{t-\tau(t)}^{t} \frac{\mathrm{d}s}{\tau(s)} = \int_{t-c-d/t}^{t} \frac{\mathrm{d}s}{c + d/s} = \left[\frac{s}{c} - \frac{d}{c^2}\ln(cs + d)\right]_{t-c-d/t}^{t}$$
$$= 1 + \frac{d}{ct} - \frac{d}{c^2}\ln\frac{t}{t-c}.$$

Now we are able to asymptotically decompose the right-hand side of inequality (12) as $t \to \infty$. We get

$$
\begin{aligned}
\frac{1}{\tau(t)} \cdot \quad \exp\left(-\int_{t-\tau(t)}^{t} \frac{\mathrm{d}s}{\tau(s)}\right) &= \frac{1}{c+d/t} \cdot \exp\left(-1 - \frac{d}{ct} + \frac{d}{c^2}\ln\frac{t}{t-c}\right) \\
&= \frac{1}{ec} \cdot \frac{1}{1+d/(ct)} \cdot \left(\frac{t-c}{t}\right)^{-d/c^2} \cdot e^{-d/(ct)} \\
&= \text{[to decompose the third term, we use Lemma 1 with} \sigma = -d/c^2 \text{ and } \tau(t) \equiv c \text{ in (6)]} \\
&= \frac{1}{ec} \cdot \left(1 - \frac{d}{ct} + \frac{d^2}{c^2t^2} + o\left(\frac{1}{t^2}\right)\right) \cdot \left(1 + \frac{d}{ct} + \frac{d(d+c^2)}{2c^2t^2} + o\left(\frac{1}{t^2}\right)\right) \\
&\quad \times \left(1 - \frac{d}{ct} + \frac{d^2}{2c^2t^2} + o\left(\frac{1}{t^2}\right)\right) \\
&= \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{t} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{d}{2c}\right) \cdot \frac{1}{t^2} + o\left(\frac{1}{t^2}\right).
\end{aligned}
$$

Finally, by the second criterion, the sufficient condition for the existence of a positive solution of (14) is

$$
a(t) \le \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{t} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{d}{2c}\right) \cdot \frac{1}{t^2} + o\left(\frac{1}{t^2}\right). \tag{17}
$$

4.3. Final comparison

Comparing the right-hand sides of expressions (16) and (17), we see that the first two terms of both decompositions coincide. The quality of every criterion is expressed by the coefficients of the term $1/t^2$, i.e., by the coefficient

$$
C_2^{\mathrm{I}} = \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{c}{8}\right)
$$

in the case of expression (16) and by the coefficient

$$
C_2^{\mathrm{II}} = \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{d}{2c}\right)
$$

in the case of expression (17). We conclude $C_2^{\mathrm{I}} < C_2^{\mathrm{II}}$ if $c^2 < 4d$ and $C_2^{\mathrm{I}} > C_2^{\mathrm{II}}$ if $c^2 > 4d$. Thus, we have

Theorem 6. *The first criterion is more general in the case of $c^2 > 4d$; the second criterion is more general if $c^2 < 4d$.*

5. Theorem 1 cannot be generalized for variable delay

Let us formulate the following natural conjecture which is a generalization of Theorem 1 for variable delay (we omit the inequality for a positive solution):

Conjecture 1. *Let us assume $\int_{t-\tau(t)}^{t} \mathrm{d}s/\tau(s) \le 1$ as $t \to \infty$.*
(a) *If*

$$
a(t) \le a_{k\tau}(t)
$$

with $a_{k\tau}(t)$ defined by formula (9) for $t \to \infty$ and an integer $k \ge 0$, then there exists a positive solution $x = x(t)$ of (1).
(b) *If*

$$
a(t) > a_{k-2,\tau}(t) + \frac{\theta\tau(t)}{8e(t\ln t\ln_2 t \ldots \ln_{k-1} t)^2} \tag{18}
$$

for $t \to \infty$, an integer $k \ge 2$ and a constant $\theta > 1$, then all the solutions of (1) oscillate.

Comparing the results in Section 4, we can conclude that the Conjecture 1 does not hold. This can be proved by showing that Conjecture 1 is false for at least one variable delay. We will show that it does not hold for an equation of the type (14) with variable delay (13). We set $c = d = 1$, $\tau(t) = 1 + 1/t$, $k = 2$,

$$a(t) := \frac{1}{e}\left(1 - \frac{1}{t} + \frac{4}{3}\cdot\frac{1}{t^2}\right)$$

and consider an equation of the type (14), i.e.,

$$\dot{x}(t) = -\frac{1}{e}\left(1 - \frac{1}{t} + \frac{4}{3}\cdot\frac{1}{t^2}\right) x\left(t - 1 - \frac{1}{t}\right). \tag{19}$$

We will verify inequality (18). Due to Remark 2 and the decomposition (16), we have

$$a_{0\tau}(t) + \frac{\theta\tau(t)}{8e(t\ln t)^2} = \frac{1}{e}\left(1 - \frac{1}{t} + \frac{9}{8}\cdot\frac{1}{t^2}\right) + o\left(\frac{1}{t^2}\right)$$

as $t \to \infty$. Inequality (18) holds since

$$\begin{aligned} a(t) &= \frac{1}{e}\left(1 - \frac{1}{t} + \frac{4}{3}\cdot\frac{1}{t^2}\right) > a_{0\tau}(t) + \frac{\theta\tau(t)}{8e(t\ln t)^2} \\ &= \frac{1}{e}\left(1 - \frac{1}{t} + \frac{9}{8}\cdot\frac{1}{t^2}\right) + o\left(\frac{1}{t^2}\right) \end{aligned}$$

as $t \to \infty$. Then all the solutions of (19) should oscillate by Conjecture 1, part (b). In our case, however,

$$a(t) = \frac{1}{e}\left(1 - \frac{1}{t} + \frac{4}{3}\cdot\frac{1}{t^2}\right) < \frac{1}{e}\left(1 - \frac{1}{t} + \frac{3}{2}\cdot\frac{1}{t^2}\right) + o\left(\frac{1}{t^2}\right)$$

as $t \to \infty$ and inequality (17) holds. Then, by Theorem 5, Eq. (19) has a positive solution as $t \to \infty$. This is a contradiction with Conjecture 1.

Acknowledgment

This research was supported by the Council of Czech Government MSM 0021630503, MSM 0021630519 and MSM 0021630529, and by the Grant 201/08/0469 of Czech Grant Agency.

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DIBLÍK J. , SVOBODA Z., ŠMARDA Z.,, Retract principle for neutral functional differential equations. *Nonlinear Analysis, Theory, Methods and Applications.* 2009. 2009(71). p. e1393 (18 p.). This article may be published with the publisher's consent

DEMONSTRATIO MATHEMATICA
Vol. XXVIII No 1 1995

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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A DELAYED DIFFERENTIAL EQUATION

1. Introduction

In this paper we consider the asymptotic expansion of solutions of delayed differential equations

$$g(t)\dot{y}(t) = -ay(t) + \sum_{i+j=2}^{N} c_{ij}(t)y^i(t)y^j(t-r), \tag{1}$$

where $N \geq 2$ is an integer, $a > 0$, $r > 0$ are constants, $g(t) : R_+^0 \to R_+$, $c_{ij}(t) : R_+^0 \to R$ are continuous functions (further conditions will be given latter). The purpose of this paper is to prove that for each real parameter C and function $\psi \in B_o = \{\psi \in C^0[-r, 0], \|\psi\| \leq 1,\ \psi(0) = 0\}$ which describe the power of the set of solutions, there is a solution $y(t) = y(t, C, \psi)$ of 1 which may be at $t \to \infty$ represented by asymptotic series (symbol $\approx$ denotes the asymptotic expansions)

$$y(t, C, \psi) \approx \sum_{k=1}^{\infty} f_k(t)\varphi^k(t, C) \tag{2}$$

where $\varphi(t, C)$ is the solution of equation $g(t)\dot{y}(t) = -ay(t)$, given by the formula $\varphi(t, C) = C \exp \int_0^t \frac{-a}{g(u)}\, du$, $f_1(t) \equiv 1$ and the functions $f_k(t)$ for $k = 2, \ldots, n$ are particular solutions of some system of auxiliary differential equations. To prove our results we will use Ważewski's topological method in the form, proposed by K. Rybakowski [5], which may be used for differential equations with retarded arguments. The first Lyapunoff's method is often used to construct the solutions of ordinary differential equations in the form of power-like series. Such a way is not possible here. First lefthand ends of existence intervals of partial sums can tend to infinity and, secondly, if it does not happen, the partial sums need not to converge uniformly. The modification of the first Lyapunoff's method were used in [6], [1].

Delayed differential equations appear in many technical problems. The form of equation (1) include some equations which have been recently considered. For example the logistic equation with recruitment delays

$$\dot{x}(t) = x(t-r)(A - Bx(t))$$

which were considered by Gopalsamy [2], with regard to the applications on ecology. After substitution $x(t) = \frac{A}{B} + y(t)$ have the form of equation (1), where $g(t) = 1$, $a = A$, $c_{11} = -B$ and $c_{ij} = 0$ for $i \neq 1$, $j \neq 1$, $N = 2$. Moreover also one branch of the equation partially solved with respect to derivative in Diblik's work [1] have (after solving with respect to derivatives) the form of the equation (1), in which are not terms with retarded arguments.

2. Preliminaries

To describe simply coefficients of power series raised to a power, it is suitable to denote: α, β — are sequences of nonnegative integers with finite sumation.

Let $\alpha = \{\alpha_k\}_{k=1}^{\infty}$, then we denote

$$|\alpha| = \sum_{k=1}^{\infty} \alpha_k, \; V(\alpha) = \sum_{k=1}^{\infty} k\alpha_k, \; \alpha! = \prod_{k=1}^{\infty} \alpha_k!, \; \max(\alpha) = \max\{k \mid \alpha_k \neq 0\}.$$

Let $\mathbf{a} = \{a_k\}_{k=1}^{\infty}$ be any sequence (of numbers or functions). We define

$$\mathbf{a}^{\alpha} = \prod_{k=1}^{\infty} a_k^{\alpha_k}, \qquad \text{where } a_k^0 = 1 \text{ for every } a_k.$$

Then it is possible to prove

$$\Big(\sum_{k=1}^{\infty} a_k \mathbf{x}^k\Big)^n = \sum_{k=n}^{\infty} \mathbf{x}^k \sum_{n}^{k} \frac{n!}{\alpha!}\mathbf{a}^{\alpha},$$

where $\sum_n^k$ denotes the sumation over all sequences such that $|\alpha| = n$, $V(\alpha) = k$. As we work with the product of the power series raised to a power, we denote $\sum_{i,j}^k$ is the sumation over all couples (α, β) such that $V(\alpha) + V(\beta) = k$, $|\alpha| = i$, $|\beta| = j$.

Throughout this paper $g(t)$, $G(t)$ denote functions such that

C1. $g(t) \in C^0[0,\infty)$, $g(t) > 0$ for $t \geq t_0$ and $g(t) = O(1)$ as $t \to \infty$.
C2. $G(t) = o(g(t))$ as $t \to \infty$, where $G(t) = (\int_0^t g^{-1}(u)\,du)^{-1}$
C3. there is a constant $\lambda > 0$ such that

$$\frac{g(t) - g(t-r)}{g(t-r)} = o(G^{\lambda}(t)) \quad \text{as } t \to \infty.$$

This condition enables us to consider relative large class of functions: $g(t)$ may be constant, a periodical function (r is a period) or there is a positive $\lim_{t\to\infty} g(t)$ and if $\lim_{t\to\infty} g(t) = 0$ in addition then the function $g(t)$ must satisfy

$$\int_0^t g(u)\,du = o(g^k(t)) \quad \text{as } t \to \infty,\ k > 0 \text{ is a constant.}$$

LEMMA 1. *Let functions $g(t)$, $G(t)$ satisfy the conditions* **C1, C2, C3.** *Then:*

1. $G(t) \sim G(t-K)$ *as* $t \to \infty$ *where* K *is any constant*
2. $g(t)(g^{-1}(t-ir) - g^{-1}(t-ir+r)) = o(G^\lambda(t))$ *as* $t \to \infty$.

Proof.

$$\lim_{t\to\infty} \frac{G(t)}{G(t-K)} = \lim_{t\to\infty} \frac{\int_0^t g^{-1}(u)\,du - \int_{t-K}^t g^{-1}(u)\,du}{\int_0^t g^{-1}(u)\,du} =$$

$$= 1 - \lim_{t\to\infty} G(t) \int_{t-K}^{t} g^{-1}(u)\,du = 1,$$

therefore the function $G(t)$ is a decraesing function and we obtain

$$\lim_{t\to\infty} G(t) \int_{t-K}^{t} g^{-1}(u)\,du \le \lim_{t\to\infty} \int_{t-K}^{t} G(u)g^{-1}(u)\,du \le K o(1) = 0.$$

Moreover for $t \to \infty$ using **C3** we get

$$g(t) = g(t-ir)\prod_{j=1}^{i}(1 + o(G^\lambda(t-jr))) = g(t-ir)\prod_{j=1}^{i}(1 + o(G^\lambda(t)(1+o(1)))$$

$$= g(t-ir)\sum_{j=1}^{i}\binom{i}{j}1^j(o(G^\lambda(t)(1+o(1))))^{i-j} = g(t-ir)(1+o(G^\lambda(t)).$$

Thus

$$\frac{g(t) - g(t-ir)}{g(t-ir)} = o(G^\lambda(t)).$$

Eventually we get

$$g(t)(g^{-1}(t-ir) - g^{-1}(t-ir+r)) = \frac{g(t)-g(t-ir)}{g(t-ir)} - \frac{g(t)-g(t-ir+r)}{g(t-ir+r)} = o(G^\lambda(t)).$$

LEMMA 2. *Let the coefficients of equation*

$$g(t)\dot{y}(t) = Ky(t) + E(t)f(t) \tag{3}$$

satisfy:

1. $K > 0$ *is a constant,*
2. *the functions* $g(t)$, $G(t) = (\int_0^t g^{-1}(u)\,du)^{-1}$ *fulfill* **C1, C2, C3,**
3. $E(t) \equiv \exp \sum_{i=1}^{n} \int_{t-ir}^{t-ir+r} \frac{K_i}{g(u)}\,du$, *where* $K_i > 0$ *are constants,*
4. *the function* $f(t)$ *has the asymptotic form* $f(t) = G^{\gamma}(t)b(t) + O(G^{\gamma+\varepsilon_1}(t))$, *as* $t \to \infty$ *where* $\varepsilon_1 > 0$, γ *are constants,* $b(t) \in C^1[t_0, \infty)$ *and moreover:* $b(t) = o(G^{\tau}(t))$ *as* $t \to \infty$, *for all* $\tau > 0$, $g(t)\dot{b}(t) = o(G^{\delta}(t))$ *as* $t \to \infty$, *where* $\delta > 0$ *is a constant.*

Then there exists the solution $Y(t)$ *of* (3) *such that, the following asymptotic relations hold*

$$Y(t) = E(t)G^{\gamma}(t)\Big(-\frac{b(t)}{K} + O(G^{\varepsilon}(t))\Big) \quad \dot{Y}(t) = O(g^{-1}(t)G^{\gamma+\varepsilon}(t)),$$

where $0 < \varepsilon < \min(\lambda, \varepsilon_1, \delta, 1)$ *is a constant.*

Proof. After the subtitution $y(t) = x(t)E(t)$ the equation (3) has form:

$$(4)\quad g(t)\dot{x}(t) = \Big(K + \sum_{i=1}^{n} K_i g(t)(g^{-1}(t-ir) - g^{-1}(t-ir+r))\Big)x(t) + f(t).$$

We define the domain $\Omega = \{(x,t) | t > t_0,\ u(x,t) < 0\}$, where $u(x,t) = (ax + G^{\gamma}(t)b(t))^2 - G^{2(\gamma+\varepsilon)}(t)$. The assumptions of Picard-Lindelöf's theorem are locally satisfied in the domain Ω, therefore throught each $(x,t) \in \Omega$ goes a unique solution of (4). Using the assumptions 1, 2, 3, 4 we compute the trajectory derivative $\dot{u}(x,t)$ along the solution $x(t)$ of (3) on the bound $\partial\Omega$:

$$\dot{u}(x,t) = \frac{2}{g(t)}\{KG^{2(\gamma+\varepsilon)}(t) - G^{2(\gamma+\varepsilon+1)}(t) + G^{2(\gamma+\varepsilon+\lambda)}(t)o(1) \pm$$
$$\pm\, G^{2\gamma+\varepsilon}[G^{\lambda}(t)b(t)o(1) + KG^{\varepsilon_1}(t)O(1) - \gamma b(t)G(t) + G^{\delta}(t)o(1)]\}.$$

For sufficiently large t the construction of the number ε implies

$$\operatorname{sign} \dot{u}(x,t) = \operatorname{sign} \frac{2a}{g(t)} G^{2(\gamma+\varepsilon)}(t) = 1.$$

Then according to Ważewski's principle [4, p. 282] there is at least one solution $x(t)$ of (4) such that $x(t) \in \Omega$. The asymptotic form of the solution $x(t)$ and also $y(t) = E(t)x(t)$ is obtained from the construction of the domain Ω.

3. Main results

Let the formal solution of equation (1) be expressed in the form (2), where $\varphi(t, C)$ is the general solution of the equation $g(t)\dot{y}(t) = -ay(t)$, consequently $\varphi(t, C) \equiv C \exp \int_{t_0}^{t} \frac{-a}{g(s)}\,ds$, where C is a constant and $f_1(t) = 1$, $f_k(t)$ for $k \geq 2$ are unknown functions for the time being. After substituting

$y(t,C)$ in the equation (1) and comparing coefficients of the same powers $\varphi^k(t,C)$ we obtain an auxiliary system of linear differential equations:

$$(5_k) \qquad g(t)\dot{f}_k(t) = a(k-1)f_k(t) + \sum_{i+j=2}^{N} c_{ij}(t) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!}\mathbf{f}^{\alpha}(t)\mathbf{h}^{\beta}(t)$$

where

$$\mathbf{f}(t) = \{f_k(t)\}_{k=1}^{\infty}, \qquad \mathbf{h}(t) = \{h_k(t)\}_{k=1}^{\infty} = \left\{ f_k(t-r)\exp \int_{t-r}^{t} \frac{ak}{g(s)}\, ds \right\}_{k=1}^{\infty}.$$

As $V(\alpha) + V(\beta) = k$ and $|\alpha| + |\beta| \geq 2$ yields $\alpha_l = 0$ and $\beta_l = 0$ for $l \geq k$, the auxiliary system (5_k) is recurrent. Therefore we may define recurrently two sequences of functions:

$$\mathbf{p}(t) = \{p_k(t)\}_{k=1}^{\infty}, \quad \mathbf{q}(t) = \{q_k(t)\}_{k=1}^{\infty} = \left\{ p_k(t-r)\exp \int_{t-r}^{t} \frac{ak}{g(s)}\, ds \right\}_{k=1}^{\infty},$$

$$p_1(t) = 1,$$

$$p_k(t) = \frac{1}{a(k-1)} \sum_{i+j=2}^{N} c_{ij}(t) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!}\mathbf{p}^{\alpha}(t)\mathbf{q}^{\beta}(t).$$

If $|\beta| \neq 0$, then the expresion $\exp \int_{t-r}^{t} \frac{ak}{g(s)}\, ds$ is included in $\mathbf{q}^{\beta}(t)$ and also in $p_k(t)$. Now using Lemma 2 we describe the asymptotic behaviour of particular solutions of the system 5_k.

THEOREM 1. *Let the functions $p_k(t)$ have the asymptotic form*

$$p_k(t) = E_k(t)G^{\gamma_k}(t)(b_k(t) + O(G^{\varepsilon_k}(t)))$$

as $t \to \infty$ where $\varepsilon_k > 0$, γ_k are constants, $b_k(t) \in C^1[t_k, \infty)$, $b_k(t) = o(g^{\tau}(t))$ as $t \to \infty$ for any positive τ, $g(t)\dot{b}_k(t) = o(g^{\lambda_k}(t))$, as $t \to \infty$, $\lambda_k > 0$ is a constant.

$$E_k(t) = \exp \sum_{i=1}^{n_k} K_k^i \int_{t-ir}^{t-ir+r} \frac{ds}{g(s)}.$$

Assume further there is a sequence $\{\nu_k\}_{k=1}^{\infty}$ such that

$$\nu_k \in (\gamma_k, \gamma_k + \min(\lambda, \delta_k, 1, \varepsilon_k - \Delta_k^*)),$$

where $\Delta_k^* = \max(\Delta_1, \ldots, \Delta_{k-1})$, $\Delta_1 = 0$, $\Delta_l = \gamma_l + \varepsilon_l - \nu_l$ for $l = 2, \ldots, k-1$.

Then the coeficients $f_k(t)$ of the series (2), *which are the solutions of the auxiliary system* (5_k), *i. e.*

(6_k) $\quad f_k(t) =$

$$= \int_t^{\infty} \frac{-1}{g(s)} \sum_{i+j=2}^{N} c_{ij}(s) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!} \mathbf{f}^{\alpha}(s)\mathbf{h}^{\beta}(s) \exp\left\{ - \int_t^{s} \frac{a(k-1)}{g(u)}\, du \right\} ds$$

can be expressed in the asymptotic form

$$(7_k) \qquad \begin{aligned} f_k(t) &= E_k(t)G^{\gamma_k}(t)\Big(-\frac{b_k(t)}{a(k-1)} + O(g^{\nu_k}(t))\Big) \\ \dot{f}_k(t) &= \frac{1}{g(t)} E_k(t) O(G^{\nu_k}(t)). \end{aligned}$$

Proof. The formulas (6_k) are obtained by integrating the system (5_k). The convergence of (6_k) is evident. It remains to show the asymptotic estimate (7_k). This will be done by induction.

For $k = 2$ the coefficients of the equation (5_2) satisfy the requirements of Lemma 2, thus the solution (6_2) has the form (7_2).

In spite of $\mathbf{f}(t)$ being substituted instead of $\mathbf{p}(t)$ and $\mathbf{h}(t)$ being substituted instead of $\mathbf{q}(t)$, in the recurrent definition of $p_k(t)$, the asymptotic form

$$p_k^*(t) = g^{\gamma_k}(t)(b_{1k}(t) + O(b_{0k}^*(t)g^{\varepsilon_k - \Delta_k^*}(t)))G_k(t)$$

has the same asymptotic properties like $p_k(t)$. Therefore the equation (5_k) satisfies the assumptions of Lemma 2, then (6_k) takes the form (7_k) and the theorem is proved.

Remark. The necessary condition for satisfying assumptions of Theorem 1 is $\lim_{t\to\infty} c_{ij}(t)\exp(-\tau G^{-1}(t)) = 0$. This is satisfied for example if functions $c_{ij}(t)$ have the same asymptotic behaviour $p_k(t)$.

We shall denote

$$y_n(t) = \sum_{k=1}^{n} f_k(t)\varphi^k(t,C) \quad \text{and} \quad \sum_n(t) = \sum_{i+j=2}^{N} c_{ij}(t) \sum_{i,j}^{k} \frac{i!j!}{\alpha!\beta!}\mathbf{f}^{\alpha}(t)\mathbf{h}^{\beta}(t).$$

THEOREM 2. *Let the assumptions of Theorem* 1 *hold and suppose that*

$$\lim_{t\to\infty} f_{n+1}^{-1}(t)\exp(-\tau G^{-1}(t)) = 0,$$

where $\tau < 1$ *is a constant. Then for every* $C \neq 0$ *and* $\psi \in C^0[-r,0]$, $\|\psi\| \le 1$, $\psi(0) = 0$ *there exists a solution* $y_C(t)$ *of equation (1) such that*

$$(8) \qquad |y_C(t) - y_n(t)| \le \delta |f_{n+1}(t)\varphi^{n+1}(t,C)|$$

for $t \in [t_C, \infty)$ *where coefficients* $f_k(t)$ *are the solutions* (6_k) *of the system* (5_k), $\delta > 1$ *is a constant,* t_C *is a function of the parametr* C *and of* δ, n.

Proof. The existence of solution $y_C(t)$ which satisfies the inequality (8) will be proved by Ważewski's principle for retarded functional differential equations $\dot{y} = f(t, y_t)$, where y_t denotes an element of $C^0 = C^0[-r, 0]$ defined as $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$ for any continuous mapping y from an interval $[-r+t, t]$ into R. For this method see [5]. The function $f(t, y_t)$: $R \times C^0[-r, 0] \to R$, defined by a formula

$$f(t,\phi) = \frac{1}{g(t)}\Big(-a\phi(0) + \sum_{i+j=2}^{N} c_{ij}(t)\phi^i(0)\phi^j(-r)\Big)$$

is continuous and Lipschitzian in ψ in each compact set in Ω^ε, where

$$\Omega^\varepsilon = \{(t,\psi) | t > t_0 - r;\ \|\varphi - y_{nt}\| < A(t)\},$$

$$\|\phi\| = \sup_{-r\le\theta\le 0} |\phi(\theta)| \quad \text{and} \quad A(t) = (\varepsilon+1) \max_{t-r\le\theta\le t} (f_{n+1}(\theta)\varphi^{n+1}(\theta, C))$$

for a positive constant ε. Thus for any $(t,\phi) \in \Omega^\varepsilon$ there exists the unique solution of the equation $y = f(t, y_t)$ [3,p. 42].

We shall prove that $\omega = \{(y,t) \mid l(y,t) < 0,\ t > t_C\}$, where $l(y,t) = (y - y_n(t))^2 - (\delta f_{n+1}(t)\varphi^{n+1}(t,C))^2$ is the regular polyfacial set with respect to the equation $\dot{y} = f(t, y_t)$, where $f(t,\phi)$ is defined as above. Then for all $K \in (-1, 1)$

$$\dot{l}(y,t) = \frac{2}{g(t)}\Big((\pm\delta\varphi^{n+1}(t,C)|f_{n+1}(t)|)\Big[-a(y_n(t) \pm \delta\varphi^{n+1}(t,C)|f_{n+1}(t)|) +$$
$$+ \sum_{i+j=2}^{N} c_{ij}(t)(y_n(t) \pm \delta|f_{n+1}(t)|\varphi^{n+1}(t,C))^i \times (y_n(t-r) +$$
$$+ K\delta|f_{n+1}(t-r)|\varphi^{n+1}(t-r,C))^j + a y_n(t) - \sum_{k=1}^{n}\varphi^k(t,C)\sum_k(t)\Big] -$$
$$- (\delta\varphi^{n+1}(t,C))^2[-a(n+1)f_{n+1}^2(t) + g(t)\dot{f}_{n+1}(t)f_{n+1}(t)]\Big).$$

Using binomial theorem for i, j-power in sumation $\sum_{i+j=2}^{N}$ we obtain

$$\dot{l}(y,t) = \frac{2}{g(t)}\Big((\delta\varphi^{n+1}(t,C))^2[-a(n+1)f_{n+1}^2(t) - g(t)\dot{f}_{n+1}(t)f_{n+1}(t)] \pm$$
$$\pm \delta\varphi^{n+1}(t,C)\Big[-\sum_{k=1}^{n}\varphi^k(t,C)\sum_k(t) + \sum_{i+j=2}^{N} c_{ij}(t)(y_n^i(t)y_n^j(t-r) +$$
$$+ y_n^i(t)\varphi^{n+1}(t,C)V_1(t) + y_n^j(t-r)\varphi^{n+1}(t,C)V_2(t) + \varphi^{2n+2}(t,C)V_1(t)V_2(t))\Big]\Big),$$

where

$$V_1(t)=\sum_{l=0}^{j-1}\binom{j}{l}(-1)^{j-l}y_n^l(t-r)\left(K\delta|f_{n+1}(t-r)|\exp\int\limits_{t-r}^{t}\frac{a(n+1)}{g(s)}\,ds\right)^{j-l}\times$$
$$\times(\varphi^{n+1}(t,C))^{j-l-1},$$
$$V_2(t)=\sum_{l=0}^{i-1}\binom{i}{l}(-1)^{i-l}y_n^l(t)(\delta|f_{n+1}(t)|)^{i-l}(\varphi^{n+1}(t,C))^{i-l-1}.$$

Therefore

$$\{(\alpha,\beta)\mid V(\alpha)+V(\beta)\le n+1,\ |\alpha|+|\beta|\ge 2,\ \max(\alpha)\le n,\ \max(\beta)\le n\}=$$
$$\{(\alpha,\beta)|V(\alpha)+V(\beta)\le n+1,|\alpha|+|\beta|\ge 2\},$$

we obtain

$$\sum_{i+j=2}^{N}c_{ij}(t)y_n^i(t)y_n^j(t-r)=\sum_{k=1}^{n=1}\varphi^k(t,C)\sum_k(t)+$$
$$+\sum_{k=n+2}^{nN}\varphi^k(t,C)\sum_{i+j=2}^{N}c_{ij}(t)\sum_{i_n j_n}^{k}\frac{i!j!}{\alpha!\beta!}\mathbf{f}^\alpha(t)\mathbf{h}^\beta(t),$$

where $\sum_{i_nj_n}^k$ denotes the sumation ower all (α,β) such that $V(\alpha)+V(\beta)=k$, $|\alpha|=i$, $|\beta|=j$, $\max(\alpha)\le n$, $\max(\beta)\le n$.

Eventualy we get

$$\dot{l}(y,t)=\frac{2}{g(t)}\varphi^{2n+2}(t,C)\times$$
$$\times\Big[(anf_{n+1}^2(t)-g(t)\dot{f}_{n+1}(t)f_{n+1}(t))\delta^2\pm\delta f_{n+1}(t)\sum_{n+1}(t)\Big]\pm$$
$$\pm\,\delta\varphi^{2n+3}(t,C)\Big[\sum_{k=n+2}^{nN}\varphi^{k-n-2}(t,C)\sum_{i+j=2}^{N}c_{ij}(t)\sum_{i_nj_n}^{k}\frac{i!j!}{\alpha!\beta!}\mathbf{f}^\alpha(t)\mathbf{h}^\beta(t)\times$$
$$\times\sum_{i+j=2}^{N}c_{ij}(t)(+y_n^i(t)V_1(t)+y_n^j(t-r)V_2(t)+\varphi^{n+1}(t,C)V_1(t)V_2(t))\Big].$$

For sufficiently large $t>t_C$ and $\delta>1$ we deduce that

$$\operatorname{sign}\dot{l}(y,t)=\operatorname{sign}(anf_{n+1}^2(t)-g(t)\dot{f}_{n+1}(t)f_{n+1}(t)).$$

As $\lim_{t\to\infty}g(t)\frac{\dot{f}_{n+1}(t)}{f_{n+1}(t)}=\lim_{t\to\infty}(G^{\nu_{n+1}-\lambda_{n+1}}(t)=0$ we obtain

$$\operatorname{sign}\dot{l}(y,t)=\operatorname{sign}anf_{n+1}^2(t)=1.$$

Consequently ω is the polyfacial set regular with respect to the equation (1), $W = \partial\omega$, $Z = \{(y, t_C) \mid l(y, t_C) \leq 0\}$.

We define $p : B = \overline{Z} \cap (Z \cup W) = Z \to C$ as:

$$p(z) = (y - y_n(t_C))(1 - |\psi|)\frac{(f_{n+1}(t)\varphi^{n+1}(t, C))_{t_C}}{r|f_{n+1}(t_C)\varphi^{n+1}(t_C, C)|} + (y_n(t))_{t_C} \text{ for } z = (y, t_C).$$

The mapping $p(z)$ is evidently continuos and for every $z \in B$ $p(z)$ satisfies:

$$(t_C + \theta, p(z)(\theta)) \in \omega \quad \text{for } \theta \in [-r, 0).$$

Moreover it holds: $Z \cap W$ is a retract of W but $Z \cap W$ is not a retract of Z. Then all assumptions of Ważewski principle for retarded functional differential equations are satisfied and thus there exists at least one solution $y_C(t)$ of (1) such that $y_C(t) \in \omega$ for $t > t_C$. The asymptotic form of the solution $y_C(t)$ is obtained from the construction of the domain ω and proof is complete.

Remark. As the relation $h_k(t) = f_k(t - r)\exp\int_{t-r}^{t}\frac{ak}{g(s)}\,ds$ is used in the definition of the sequences $\mathbf{f}(t)$ and $\mathbf{h}(t)$ and the function $h_k(t)$ is used in the definition of $f_{k+1}(t)$ the lefthand end of the existence interval of the function $f_{k+1}(t)$ is greater by r then the lefthand end of the existence interval of $f_k(t)$. If lefthand ends of the existence intervals of the functions $c_{ij}(t)$ are finite then lefthand ends of the existence intervals of the functions $f_k(t)$ must tend to infinity.

COROLLARY. *If all asumptions of Theorem 2 are satisfied for every n, then there exists the asymptotic expansion of the solution $y_C(t)$ in the form*

$$y_C(t) \approx \sum_{n=1}^{\infty} f_n(t)\varphi^n(t, C),$$

where the coefficients $f_n(t)$ are the solutions (5_n).

Proof. As

$$\lim_{t\to\infty}\frac{f_{n+1}(t)\varphi^{n+1}(t, C)}{f_n(t)\varphi^n(t, C)} =$$

$$= \lim_{t\to\infty} G^{\gamma_{n+1}-\gamma_n}(t)\frac{b_{1n+1}(t) + O(g^{\nu_{n+1}-\gamma_{n+1}}(t))}{b_{1n}(t) + O(g^{\nu_n-\gamma_n}(t))}\varphi(t, C) = 0$$

the assertion is proved.

EXAMPLE. We consider the equation:

$$\frac{1}{t}\dot{y}(t) = -2y(t) + y^2(t-1) + t\sin t\, y^2(t)y(t-1).$$

In this case we have $a = 2$, $r = 1$, $g(t) = \frac{1}{t}$, $\lambda = 2$, $\psi(t) = -1$. Then the auxiliary system (4_k) have a form:

$$\frac{1}{t}\dot{f}_n(t) = 2(n-1)f_n(t) + \frac{1}{(n-2)!}\exp(a_n t + b_n)(1 + O(t^{-0.9})).$$

Using Lemma 2 we obtain:

$$f_n(t) = \frac{1}{2(n-1)!}\exp(a_n t + b_n)(1 + O(t^{-0.9})),$$

where $a_n = n^2 + 2n - 2$ and $b_n = -\frac{1}{6}(2n^3 + 3n^2 - 11n + 6)$. Then according the Theorem 2 and corollary we obtain

$$y_C(t) \approx \sum_{n=1}^{\infty} \frac{C}{2(n-1)!}\exp\left(a_n t + b_n - \frac{t^2}{2}\right).$$

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Received August 10, 1992.

137 (2012) MATHEMATICA BOHEMICA No. 2, 239–248

ASYMPTOTIC PROPERTIES OF ONE DIFFERENTIAL EQUATION WITH UNBOUNDED DELAY

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(Received October 15, 2009)

Abstract. We study the asymptotic behavior of the solutions of a differential equation with unbounded delay. The results presented are based on the first Lyapunov method, which is often used to construct solutions of ordinary differential equations in the form of power series. This technique cannot be applied to delayed equations and hence we express the solution as an asymptotic expansion. The existence of a solution is proved by the retract method.

Keywords: asymptotic expansion, retract method

MSC 2010: 34E05

1. Introduction

The first method of Lyapunov is a well known technique used to study the asymptotic behavior of ordinary differential equations in the form of a linear system with perturbation. This method uses the solution in the form of a convergent power series, for details see [1]. The results for equations in the implicit form [2] or for integro-differential equations [8] were derived by modifying the first method of Lyapunov. The existence of solutions with a certain asymptotic form were proved in the results cited using Ważewski's topological method. For analogous representations of solutions for a retarded differential equation, see [6], [7]. The perturbation has a polynomial form in both cases. In this paper, we study an equation in the form

$$\dot{y}(t) = -a(t)y(t) + \sum_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t) \prod_{j=1}^{n} \bigl(y(\xi_j(t))\bigr)^{i_j} \tag{1.1}$$

where $\mathbf{i} = (i_1, \ldots, i_n)$ is a multiindex, $i_j \geqslant 0$ are integers and $|\mathbf{i}| = \sum\limits_{j=1}^{n} i_j$. The continuous functions $\xi_j(t)$ satisfy $t > \xi_j(t) \geqslant r_0$ for all $t \in [t_0, \infty)$ and the function $\xi(t)$, which is defined as $\xi(t) = \min\limits_{1 \leqslant i \leqslant n} \xi_i(t)$, is nondecreasing for $t \geqslant t_0$. Therefore, all asymptotic relations such as the Landau symbols o, O and the asymptotic equivalence $\sim$ will be considered for $t \to \infty$. This fact will not be pointed out in the sequel.

The function $a(t)$ satisfies the following conditions:

(C1) $a(t)$ is continuous and positive on the interval $[t_0, \infty)$ and $1/a(t) = O(1)$,

(C2) $(t - \xi(t))\widetilde{a}(t) = o\,(A(t))$ where the functions $A(t)$, $\widetilde{a}(t)$ are defined as $A(t) = \int_{t_0}^{t} a(u)\,\mathrm{d}u$, $\widetilde{a}(t) = \max\limits_{u \leqslant t}(a(u))$.

Further conditions for continuous functions $c_{\mathbf{i}}(t)\colon [t_0, \infty) \to \mathbb{R}$ will be given later. In order to apply the first method of Lyapunov to the equation (1.1) we assume the solution in the form of a formal series

$$y(t, C) = \sum_{n=1}^{\infty} f_n(t)\varphi^n(t, C) \tag{1.2}$$

where $\varphi(t, C)$ is the solution of the homogeneous equation $\dot{y}(t) = -a(t)y(t)$ given by the formula $\varphi(t, C) = C\exp(-A(t))$, the function $f_1(t) \equiv 1$, and the functions $f_k(t)$ for $k = 2, \ldots, n$ are particular solutions of a certain system of auxiliary differential equations. Using Ważewski's topological method in the form as used in [3] and [4] for differential equations with unbounded delay and finite memory, we prove the existence of a solution $y_n(t, C) \sim Y_n(t, C) = \sum\limits_{k=1}^{n} f_k(t)\varphi^k(t, C)$.

2. Preliminaries

Lemma 2.1. *Let a function $a(t)$ satisfy conditions* (C1), (C2). *Then*

$$A(t) \sim A(\xi^i(t)) \text{ as } t \to \infty \text{ for any integer } i \in \mathbb{N} \tag{2.1}$$

where $\xi^1(t) = \xi(t)$, and for $i > 1$, the functions $\xi^i(t)$ are defined by

$$\xi^{i+1}(t) = \xi(\xi^i(t)).$$

P r o o f. First, we see that, by virtue of condition (C2), the assertion is true for $i = 1$:

$$\int_{\xi(t)}^{t} a(u)\,\mathrm{d}u \leqslant (t - \xi(t))\widetilde{a}(t) = o(A(t)) \text{ and } \lim_{t\to\infty} \frac{A(\xi(t))}{A(t)}$$
$$= 1 - \lim_{t\to\infty} \frac{\int_{\xi(t)}^{t} a(u)\,\mathrm{d}u}{A(t)} = 1.$$

The assumption $\xi(t) \not\to \infty$ for $t \to \infty$ implies that there exists a constant $\xi(\infty)$ and condition (C2) is not satisfied. If $\xi(t) \to \infty$ for $t \to \infty$, then $\xi^i(t) \to \infty$ for $t \to \infty$, too. Now we use the assertion for $i = 1$ substituting $\xi^i(t)$ for t and the proof follows by induction. $\square$

Remark 2.1. Note that condition (C1) implies the divergence of the integral $\int_{t_0}^{\infty} a(u)\,\mathrm{d}u$, which has two consequences.

First, the function $\varphi(t, C)$ satisfies the relation $\varphi^k(t, C) = o\big(\varphi^l(t, C)\big)$ for $k > l$, which guarantees that the sequence $\{\varphi^n(t, C)\}_{n=1}^{\infty}$ is asymptotic.

Second, the divergence implies the relation $1/A(t) = o(1)$ which is suitable for asymptotic estimation.

In order to specify the asymptotic behavior of the solution of the auxiliary equations we consider the equation

$$\dot{y}(t) = na(t)y(t) + f(t) \tag{2.2}$$

where $n > 0$ is a constant and the properties of the function $f(t)$ are described by a function $k(t)$, a constant K, and the relations

(F1) $\lim\limits_{t\to\infty} f(t)\exp\left(\tau k(t)\right) = 0$ for all $\tau < K$,

(F2) $\lim\limits_{t\to\infty} |f(t)|\exp\left(\tau k(t)\right) = \infty$ for all $\tau > K$.

The asymptotic behavior of the solution of equation (2.2) depends on the relation between the functions $k(t)$ and $na(t)$.

Lemma 2.2. *Let either $k(s) - k(t) = o(\int_t^s na(u)\,\mathrm{d}u)$ or $k(s) - k(t) = O(\int_t^s na(u)\,\mathrm{d}u)$ and $K = 0$ where K is the constant used in assumptions* (F1), (F2). *Now if the function $f(t)$ satisfies assumption* (F1), *then there exists at least one solution $Y(t)$ of equation* (2.2) *satisfying also assumption* (F1). *If the function $f(t)$, moreover, satisfies assumption* (F2), *then the solution $Y(t)$ also satisfies assumption* (F2).

Proof. We may rewrite assumptions (F1), (F2) for the function $f(t)$ satisfying them so that, for sufficiently large t and constants $\tau_1, \tau_2 > 0$, the function $f(t)$ satisfies the inequality

$$\exp\left((K - \tau_2)k(t)\right) \leqslant |f(t)| \leqslant \exp\left((K + \tau_1)k(t)\right),$$

and also, for the desired solution $Y(t) = \int_t^{\infty} -f(s)\exp\int_t^s -na(u)\,\mathrm{d}u\,\mathrm{d}s$, we have estimates of the solution of equation (2.2)

$$\begin{aligned}\exp((K+\tau_1)k(t))\int_t^{\infty}\exp\Big\{-(K+\tau_1)(k(t)-k(s))-\int_t^s na(u)\,\mathrm{d}u\Big\}\,\mathrm{d}s \geqslant |Y(t)|\\ \geqslant \exp((K-\tau_2)K(t))\int_t^{\infty}\exp\Big\{-(K-\tau_2)\tau(k(s)-k(t))-\int_t^s na(u)\,\mathrm{d}u\Big\}\,\mathrm{d}s.\end{aligned}$$

Now utilizing the assumptions of this lemma, we see that the asymptotic behavior of exponents involved in both integrands are the same as the asymptotic behavior of the function $\int_t^s na(u)\,\mathrm{d}u$. As the function $(na(t))^{-1}$ is bounded, the integral $\int_t^s na(u)\,\mathrm{d}u$ is divergent for $s\to\infty$ and the integrals on both sides of the inequalities are convergent and there exist constants A_1, A_2 such that

$$A_1\exp\left((K-\tau_2)k(t)\right)\leqslant |Y(t)|\leqslant A_2\exp\left((K+\tau_1)k(t)\right).$$

Assumption (F1) implies the second inequality, which ensures the convergence and thus the existence of the integral defining $Y(t)$ which is the solution of the given equation. $\square$

To make the specification of the coefficients of the power series which is the product of the power series raised to a power easier, we use the following notation: $\mathfrak{s}=(\mathfrak{s}_1,\dots,\mathfrak{s}_n)$ is an ordered n-tuple of sequences $\mathfrak{s}_j=\left\{\mathfrak{s}_j^k\right\}_{k=1}^{\infty}$ of nonnegative integers with a finite sum $|\mathfrak{s}_j|=\sum\limits_{k=1}^{\infty}\mathfrak{s}_j^k$, and we denote $\mathfrak{s}!=\prod\limits_{j=1}^{n}\prod\limits_{k=1}^{\infty}\mathfrak{s}_j^k!$, $\mathbf{i}(\mathfrak{s})!=\prod\limits_{j=1}^{n}|\mathfrak{s}_i|!$, $V(\mathfrak{s})=\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{\infty}k\mathfrak{s}_j^k$, $\mathbf{i}(\mathfrak{s})=(|\mathfrak{s}_1|,\dots,|\mathfrak{s}_n|)$. For any ordered n-tuple of sequences (of numbers or functions) $\mathcal{C}=(\mathbf{c}_1,\dots,\mathbf{c}_n)$ where $\mathbf{c}_j=\{c_j^k\}_{k=1}^{\infty}$, we denote $\mathcal{C}^{\mathfrak{s}}=\prod\limits_{j=1}^{n}\prod\limits_{k=1}^{\infty}\left(c_j^k\right)^{\mathfrak{s}_j^k}$ where $\left(c_j^k\right)^0=1$ for every c_j^k. Then it is possible to write

$$\prod_{j=1}^{n}\left(\sum_{k=1}^{\infty}c_j^k x^k\right)^{i_j}=\sum_{k=|\mathbf{i}|}^{\infty}x^k\sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}\frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathcal{C}^{\mathfrak{s}}$$

where the symbol $\sum\limits_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}$ denotes the sum over all $\mathfrak{s}$ such that $V(\mathfrak{s})=k$, $\mathbf{i}(\mathfrak{s})=\mathbf{i}$ and, for empty set of $\mathfrak{s}$, this symbol equals 0.

3. Main results

We assume that the formal solution of equation (1.1) is expressed in the form (1.2) where $\varphi(t,C)$ is the general solution of the equation $\dot{y}(t)=-a(t)y(t)$. Consequently, $\varphi(t,C)=C\exp(-A(t))$ where $C\neq 0$ is a constant, $f_1(t)=1$ and $f_k(t)$, $k\geqslant 2$ for the time being are unknown functions. Substituting $y(t)$ in equation (1.1) and matching the coefficients at the same powers $\varphi^k(t,C)$, we obtain an auxiliary system of linear differential equations

$$\dot{f}_k(t)=(k-1)a(t)f_k(t)+\sum_{|\mathbf{i}|=2}^{\infty}c_{\mathbf{i}}(t)\sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}\frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathcal{F}^{\mathfrak{s}}\tag{3.1}$$

where $\mathcal{F}(t)$ is the n-tuple of sequences $\{f_k(\xi_i(t))\exp(k(A(t)-A(\xi_i(t))))\}_{k=1}^{\infty}$ i.e. $\mathcal{F}(t)=\big(\ldots\{f_k(\xi_i(t))\exp(k(A(t)-A(\xi_i(t))))\}_{k=1}^{\infty},\ldots\big)$. The facts $V(\mathfrak{s})=k\geqslant 2$ and $|\mathbf{i}(\mathfrak{s})|\geqslant 2$ imply $\mathfrak{s}_i^l=0$ for $l\geqslant k$. Moreover, the auxiliary system (3.1) is recurrent.

Theorem 3.1. *For the functions* $c_{\mathbf{i}}(t)$, *let* $\lim\limits_{t\to\infty} c_{\mathbf{i}}(t)\exp(-\tau A(t))=0$ *for all positive* τ. *Then there exists a sequence* $\{f_k(t)\}_{k=1}^{\infty}$ *of solutions of the auxiliary system* (3.1)

$$f_k(t)=\int_t^{\infty}-a(s)\exp\Big\{-\int_t^s(k-1)a(u)\,\mathrm{d}u\Big\}\sum_{|\mathbf{i}|=2}^{\infty}c_{\mathbf{i}}(t)\sum_{\substack{\mathbf{i}(\mathfrak{s})=\mathbf{i}\\ V(\mathfrak{s})=k}}\frac{|\mathbf{i}(\mathfrak{s})|!}{\mathbf{i}(\mathfrak{s})!}\mathcal{F}^{\mathfrak{s}}\,\mathrm{d}s \tag{3.2}$$

such that $\lim\limits_{t\to\infty} f_k(t)\exp(-\tau A(t))=0$ *for all* τ.

Proof. Formula (3.2) can be obtained by integrating the system (3.1). When applying Lemma 2.2, we put $k(t)=A(t)$. Condition (C2) proves that for the function $y(t)$ satisfying assumption (F1) of Lemma 2.2, the function $y(\xi^j(t))$ satisfies this assumption, too. Therefore, the sum and the product of functions verifying assumption (F1) of Lemma 2.1 satisfy the assumptions of Lemma 2.2. Using Lemma 2.2, we can then easily show the convergence of (3.2) and the desired property. □

Remark 3.1. An assertion analogous to the one of Theorem 3.1 with the property described by assumption (F2) of Lemma 2.2 cannot be proved as the sum of functions verifying the assumption (F2) need not satisfy this assumption.

Let $\|\cdot\|$ denote the maximum norm on $C^0[r^*,t_0]$. Moreover, we denote

$$y_k(t)=\sum_{l=1}^{k}f_l(t)\varphi^l(t,C),\qquad \sum^{k}(t)=\sum_{|\mathbf{i}|=2}^{\infty}c_{\mathbf{i}}(t)\sum_{\substack{\mathbf{i}(\alpha)=\mathbf{i}\\ V(\alpha)=k}}\frac{\mathbf{i}(\alpha)!}{\alpha!}\mathcal{F}^{\alpha}.$$

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *hold and let*

$$\lim_{t\to\infty} f_{k+1}^{-1}(t)\exp(-\tau A(t))=0$$

where $\tau<1$ *is a constant. We denote* $r^*=\min\limits_{t\geqslant t_0}(\xi(t))$. *Then for every* $C\neq 0$ *and* $\psi\in C^0[r^*,t_0]$, $\|\psi\|\leqslant 1$, $\psi(t_0)=0$, *there exists a solution* $y_C(t)$ *of equation* (1.1) *such that*

$$|y_C(t)-y_k(t)|\leqslant\sigma|f_{k+1}(t)\varphi^{k+1}(t,C)| \tag{3.3}$$

for $t\in[t_C,\infty)$ *where the functions* $f_k(t)$ *are solutions* (3.2) *of system* (3.1), $\sigma>1$ *is a constant.* t_C *is a function of the parameter* C *and of* σ,k.

P r o o f. The existence of the solution $y_C(t)$ is proved by Theorem 1 in [3], which is based on the retract method and the second method of Lyapunov. A sufficient condition for the existence of a solution of the equation with unbounded delay and finite memory is described there. The theory of this type of equations (referred to as p-type retarded functional differential equation) is given in [5]. In this case we put $p(t,\vartheta) = t + \vartheta(t-\psi(t))$ and the function on the right hand side of the equation $f(t, y_t)\colon\ \mathbb{R}\times C^0[-1,0] \to \mathbb{R}$ is defined by the formula:

$$f(t,\psi) = -a(t)\psi(p(t,0)) + \sum_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t)\prod_{l=1}^{n}\psi^{i_l}(p(t,\vartheta_{i_l}(t)))$$

where $\vartheta_{i_l}(t) = -(t-\xi_{i_l}(t))/(t-\xi(t))$. The set ω used in Theorem 1 is defined as

$$\omega = \{(y,t)\colon\ y_k(t) - \sigma|f_{k+1}|(t)\varphi^{k+1} < y < y_k(t) + \sigma|f_{k+1}(t)|\varphi^{k+1},\ t > t_C\}.$$

Note that the numbers p, n used in Theorem 1 in [3] equal 1 and, consequently, the indices of functions δ, ϱ are omitted, i.e., $\delta = y_k(t) + \sigma|f_{k+1}|(t)\varphi^{k+1}(t,C)$ and $\varrho = y_k(t) - \sigma|f_{k+1}|(t)\varphi^{k+1}(t,C)$. We verify the inequalities

$$\delta'(t) > f(t,\pi) \text{ and } \varrho'(t) < f(t,\pi)$$

where $\pi \in C([p(t,-1),t],\mathbb{R})$ is such that $(\theta,\pi(\theta)) \in \omega$ for all $\theta \in [p(t,-1),t)$ and $\pi(t) = \delta(t)$ or $\pi(t) = \varrho(t)$, respectively, for a sufficiently large t. As the sequence $\{\varphi^k(t,C)\}_{k=1}^{\infty}$ is asymptotic, we can rearrange the terms in these inequalities with respect to the powers of the functions $\varphi^k(t,C)$. We verify the first inequality.

First, for sufficiently large t, $f_{k+1}\varphi^{k+1}(t,C) \neq 0$ and the derivative $\delta'(t)$ exists:

$$\begin{aligned}\delta'(t) = {} & \sum_{l=1}^{k}\left(f_l'(y) - la(t)f_l(t)\right)\varphi^l(t,C)\\ & + \sigma\,\mathrm{sign}(f_{k+1}(t))\left(f_{k+1}'(t) - (k+1)a(t)f_{k+1}(t)\right)\varphi^{k+1}(t,C).\end{aligned}$$

Second, for $\pi(t) = \delta(t)$ there exist suitable positive constants such that

$$\begin{aligned}f(t,\pi_t) = {} & -a(t)\left(y_k(t) + \sigma|f_{k+1}(t)|\varphi^{k+1}(t,C)\right)\\ & + \sum_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t)\prod_{l=1}^{n}\left(y_k(t) + K_l\sigma|f_{k+1}(t)|\varphi^{k+1}(t,C)\right)^{i_l}.\end{aligned}$$

Since the system (3.1) is recurrent, the coefficients at $\varphi^l(t,C)$ after substituting $y(t,C)$ in the form (1.2) and $y(t) = y_k(t) \pm \sigma|f_{k+1}|(t)\varphi^{k+1}(t,C)$ in the sum

$\sum\limits_{|\mathbf{i}|=2}^{\infty} c_{\mathbf{i}}(t)\big(\mathbf{y}(\xi(t))\big)^{\mathbf{i}}$ coincide for $l = 1, \ldots, k+1$, i.e.

$$\begin{aligned} f(t,\pi_t) = & -a(t)\Bigg(\sum_{l=1}^{k} f_l(t)\varphi^l(t,C) + \sigma|f_{k+1}|(t)\varphi^{k+1}(t,C)\Bigg) \\ & + \sum_{j=1}^{k+1}\overset{j}{\sum}(t)\varphi(t,C)^j + \varphi(t,C)^{k+2}R(t) \end{aligned}$$

where $R(t)$ is a function satisfying $\lim\limits_{t\to\infty} R(t)\exp(-\tau\int_{t_0}^{t}\mathrm{d}u/g(u)) = 0$ for all positive τ.

Now we can evaluate the sign of the difference $\delta'(t) - f(t,\pi_t)$ (with $\pi(t) = \delta(t)$):

$$\begin{aligned} \delta'(t) - f(t,\pi_t) = & \sum_{l=1}^{k}\Big(f_l'(y) - \frac{(l-1)f_l(t)}{g(t)} - \overset{l}{\sum}(t)\Big)\varphi^l(t,C) \\ & + \Big[\sigma\operatorname{sign}(f_{k+1}(t))\Big(f_{k+1}'(t) - \frac{kf_{k+1}(t)}{g(t)}\Big) - \overset{k+1}{\sum}(t)\Big]\varphi^{k+1}(t,C) - \varphi(t,C)^{k+2}R(t). \end{aligned}$$

The functions $f_k(t)$ are solutions of (3.1) for $l = 1, \ldots, k$. Therefore, the minimal power of $\varphi(t,C)$ in the difference $\delta'(t) - f(t,\pi_t)$ is $k+1$. Moreover, the term $\varphi(t,C)^{k+2}R(t)$ and higher powers are very small for sufficiently large t, the sign of this difference is given by the factor at the power $\varphi(t,C)^{k+1}$, i.e.

$$\begin{aligned} \operatorname{sign}(\delta'(t) - f(t,\pi_t)) &= \sigma\operatorname{sign}(f_{k+1}(t))\Big(f_{k+1}'(t) - \frac{kf_{k+1}(t)}{g(t)}\Big) - \overset{k+1}{\sum}(t) \\ &= \sigma\operatorname{sign}(f_{k+1}(t))\overset{k+1}{\sum}(t) - \overset{k+1}{\sum}(t) = \sigma\operatorname{sign}(f_{k+1}(t))\overset{k+1}{\sum}(t). \end{aligned}$$

Due to definition (3.2) of $f_{k+1}(t)$, we obtain $\operatorname{sign}(\delta'(t) - f(t,\pi_t)) = -1$ and the inequality $\delta'(t) > f(t,\pi_t)$ holds, too. A similar consideration for the difference $\varrho'(t) - f(t,\pi_t)$ (with $\pi(t) = \varrho(t)$) gives $\varrho'(t) < f(t,\pi_t)$. Now we may use Theorem 1 in [3] to obtain the existence of a solution satisfying the estimate (3.6). $\square$

Theorem 3.3. *Let the assumptions of Theorem* 3.1 *be satisfied and let there exist a sequence* $\{K_k\}_{k=1}^{\infty}$, $K_0 = 1$ *such that the assumptions of Theorem* 3.2 *are satisfied for every* K_k, *i.e.,* $\lim\limits_{t\to\infty} f_{K_k}^{-1}(t)\exp(-\tau A(t)) = 0$. *Then there exists an asymptotic expansion of the solution* $y_C(t)$ *in the form*

$$y_C(t) \approx \sum_{k=1}^{\infty} F_k(t), \;\; \text{where } F_k(t) = \sum_{l=K_{k-1}}^{K_k - 1} f_l(t)\varphi^l(t,C)$$

and $f_l(t)$ *are solutions of* (3.2).

P r o o f. Since the assumptions of Theorem 3.2 are fulfilled for every K_k, there exists a solution $y_C(t)$ satisfying the inequality in this theorem. Then the existence of an asymptotic expansion follows from the fact that the sequence $\{F_k\}^\infty$ is asymptotic, i.e., $\lim_{t\to\infty} F_{k+1}(t)/F_k(t) = 0$ and the assertion is proved. $\square$

E x a m p l e 1. We study the asymptotic properties of the solutions of the equation

$$\dot{y}(t) = -y\cos(ty(\xi(t)) = -y(t) + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{t^{2k}y(t)(y(\xi(t)))^{2k}}{(2k)!}$$

on the interval $[1,\infty)$ for two various delays $r_1(t) = r > 0$, i.e., $\xi_1(t) = t - r$, and $r_2(t) = \ln t$, i.e., $\xi_2(t) = t - \ln t$. In this case we have $a(t) = 1$, $A(t) = t - 1$, $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2)$, $c_{(1,2k)} = (-1)^{k+1}t^{2k}/(2k)!$ (for other multiindices $c_{\mathbf{i}} = 0$). If we denote $\mathcal{F} = (\{f_i(t)\}_{i=1}^\infty, \{f_i(\xi(t))\mathrm{e}^{i(t-\xi(t))}\}_{i=1}^\infty)$, the system of auxiliary differential equations of the form

$$\dot{f}_k(t) = (k-1)f_k(t) + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{t^{2i}}{(2i)!}\sum_{\substack{h\mathbf{i}(\mathfrak{s})=(1,2i)\\ V(\mathfrak{s})=k}}\frac{\mathbf{i}(\mathfrak{s})!}{\mathfrak{s}!}\mathcal{F}^{\mathfrak{s}}$$

has a particular solution $f_{2k} = 0$. First, $f_2(t) = 0$ is due to $\dot{f}_2(t) = f_2(t)$. We will prove by induction that the equation for the function f_{2k} has the form $\dot{f}_{2k}(t) = f_{2k}(t)$, therefore, the odd ($|\mathbf{i}(\mathfrak{s})| = 1 + 2l$) sum of odd exponents (due to the induction hypothesis) is not even (2k) and every product on the right-hand side of the auxiliary equation contains zero multiplicands (f_{2i}). The asymptotic form of the solutions f_{2k+1} depends on the delay $r_i(t)$ but the property $f_{2k-1}(t) \sim f_{2k-1}(\xi(t))$ holds for both $r_i(t)$.

First, for $r_1(t)$ the solutions have the asymptotic form $f_{2k+1} = t^{2k}(c_{2k+1} + O(1/t))$, where $c_1 = 1$ and c_{2k+1} are given by the recurrent formula

$$c_{2k+1} = \frac{1}{2k}\sum_{i=1}^{\infty}\frac{(-1)^i}{(2i)!}\sum_{\substack{\mathbf{i}(\mathfrak{s})=(1,2i)\\ V(\mathfrak{s})=2k+1}}\mathcal{C}^{\mathfrak{s}_1}\mathcal{C}_r^{\mathfrak{s}_2}, \quad \text{where } \mathcal{C} = \{c_i\}_{i=1}^\infty,\ \mathcal{C}_r = \{c_i\exp(ir)\}_{i=1}^\infty.$$

Second, we have the relation $\exp(k(A(t) - A(\xi(t)))) = \exp(k\ln t) = t^k$ for the delay $r_2(t)$ and the function f_3 satisfies the equation $\dot{f}_3(t) = 2f_3(t) + \frac{1}{2}t^4$ and we obtain the solution $f_3(t) = t^4(-\frac{1}{4} + O(1/t))$. Applying induction for the solutions f_{2k+1} in the form $f_{2k-1}(t) = t^{p(k)}(d(k) + O(1/t))$, we see that the main power of t in the sum on the right hand side of the equation for f_{2k-1} is at the product $t^2 f_1(t)f_1(\xi(t))tf_{2k-3}(\xi(t))t^{2k-3} = t^{2k+p(k-1)}(d(k-1) + O(1/t))$ and we obtain the equation $\dot{f}_{2k+1}(t) = 2kf_{2k+1}(t) + t^{2k+p(k-1)}(d(k-1) + O(1/t))$. The solution $f_{2k-1}(t)$

has the asymptotic form $f_{2k+1} = -t^{2k+p(k-1)}\left(d(k-1)/2k + O(1/t)\right)$. The constants $d(k)$ and $p(k)$ satisfy the recurrent formulas $d(k) = -d(k-1)/2k$, $p(k) = p(k-1)+2k$, otherwise $d(k) = (-1)^{k-1}2^{-k}/(k-1)!$ and $p(k) = (k+2)(k-1)$. By Theorem 3.3, we obtain the existence of a pair of asymptotic expansions $y_1(t)$, $y_2(t)$ of the solutions for two different delays $r_1(t)$, $r_2(t)$:

$$y_1(t) \approx \sum_{k=1}^{\infty} t^{2(k-1)} c_{2k-1} \mathrm{e}^{(2k-1)t} C^{2k-1},$$
$$y_2(t) \approx \sum_{k=1}^{\infty} \frac{(-1)^{k-1} t^{(k+2)(k-1)}}{2^k (k-1)!} \mathrm{e}^{(2k-1)t} C^{2k-1}.$$

R e m a r k 3.2. This example shows a fundamental dependence of the asymptotic properties of the expansion on the magnitude of the delay. For a small delay ($r_1(t) \to 0$), the expansion $y_1(t)$ converges to the expansion of the solution of an ordinary equation $\dot{y}(t) = -y\cos(ty(t))$. For a sufficiently large delay $r_2(t) = \ln(t)$, the expansion $y_2(t)$ is the same as for the equation $\dot{y}(t) = -y(t) + t^2 y(t) y^2(t - \ln t)/2$, i.e., the expansions for the perturbation with infinite sum and for the perturbation with only the first summand are the same.

A c k n o w l e d g e m e n t. This paper was supported by the Grant 201/08/0469 of the Czech Grant Agency (Prague) and by the Czech Ministry of Education in the frame of project MSM002160503 Research Intention MIKROSYN New Trends in Microelectronic Systems and Nanotechnologies.

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SVOBODA Z., Representation of solutions of linear differential systems of second-order with constant delays. *Neliniini Kolyvannya* **19** No 1 (2016), pp. 129–141, ISSN 1562-3076.

Electronic Journal of Qualitative Theory of Differential Equations
2017, No. 89, 1–15; https://doi.org/10.14232/ejqtde.2017.1.89 www.math.u-szeged.hu/ejqtde/

Asymptotic unboundedness of the norms of delayed matrix sine and cosine

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Received 7 March 2017, appeared 12 December 2017
Communicated by Mihály Pituk

Abstract. In the paper, the asymptotic properties of recently defined special matrix functions called delayed matrix sine and delayed matrix cosine are studied. The asymptotic unboundedness of their norms is proved. To derive this result, a formula is used connecting them with what is called delayed matrix exponential with asymptotic properties determined by the main branch of the Lambert function.

Keywords: delay, delayed matrix functions, Lambert function, unboundedness.

2010 Mathematics Subject Classification: 34K06, 34K07.

1 Introduction

Recently, a new formalization has been developed of the well-known method of steps [12, 13] for solving the initial-value problem for linear differential equations with constant coefficients and a single delay through special matrix functions called delayed matrix functions [6, 15, 20]. Using this method, representations have been found of solutions of homogeneous and non-homogeneous systems, and some stability and control problems were solved in [5, 16]. Also, a generalization has been developed to discrete systems and applied in [4, 21].

Let A be a nonzero $n \times n$ constant matrix, $\tau > 0$ and let $\lfloor \cdot \rfloor$ be the floor function. The delayed matrix exponential, defined in [15], is a matrix polynomial on every interval $[(k-1)\tau, k\tau)$, $k = 0, 1, \dots$, defined by

$$\mathrm{e}_\tau^{At} = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} A^s \, \frac{(t-(s-1)\tau)^s}{s!} \,. \tag{1.1}$$

The delayed matrix exponential equals to zero matrix Θ if $t < -\tau$, the unit matrix I on $[-\tau, 0]$, and is the fundamental matrix of a homogeneous linear system with a single delay

$$\dot{x}(t) = Ax(t-\tau). \tag{1.2}$$

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For the proof, we refer to [15]. In [15], too, a representation is derived of the solution of the Cauchy initial problem (1.2), (1.3), where

$$x(t) = \varphi(t), \qquad -\tau \le t \le 0, \tag{1.3}$$

and $\varphi\colon [-\tau, 0] \to \mathbb{R}^n$ is continuously differentiable.

Fundamental matrix (1.1) serves as a nice illustration of the general definition of a fundamental matrix to linear functional differential systems of delayed type [12,13]. For system (1.2), this definition reduces to (details are omitted)

$$X(t) = \begin{cases} A\displaystyle\int_{-\tau}^{t} X(u-\tau)du + I, \text{ for almost all } t \ge -\tau, \\ \Theta, -2\tau \le t < -\tau \end{cases} \tag{1.4}$$

and its step-by-step application gives

$$X(t) = \mathrm{e}_{\tau}^{At}, \qquad t \ge -2\tau.$$

With its usefulness, the delayed matrix exponential stimulated the search for other delayed matrix functions capable of simply expressing solutions of some linear differential systems with constant coefficients. In [6], solutions of a homogeneous second-order linear system with single delay

$$\ddot{x}(t) = -A^2 x(t-\tau). \tag{1.5}$$

are expressed through delayed matrix functions called the delayed matrix sine $\mathrm{Sin}_{\tau} At$ and delayed matrix cosine $\mathrm{Cos}_{\tau} At$ defined for $t \in \mathbb{R}$ as

$$\mathrm{Sin}_{\tau} At = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} (-1)^s A^{2s+1} \frac{(t-(s-1)\tau)^{2s+1}}{(2s+1)!} \tag{1.6}$$

and

$$\mathrm{Cos}_{\tau} At = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} (-1)^s A^{2s} \frac{(t-(s-1)\tau)^{2s}}{(2s)!}\,. \tag{1.7}$$

Matrices (1.6) and (1.7) are related to the $2n \times 2n$ fundamental matrix $\mathcal{X}(t)$ of $2n$-dimensional system

$$\dot{y}(t) = \mathcal{A} y(t - \tau/2),$$

where

$$\mathcal{A} := \begin{pmatrix} \Theta & A \\ -A & \Theta \end{pmatrix}, \qquad y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

equivalent with (1.5) through the substitution $x(t) = y_1(t)$. In much the same way as above, we can derive (for details we refer to [24])

$$\mathcal{X}(t) = \mathrm{e}_{\tau/2}^{\mathcal{A}t} = \begin{pmatrix} \mathrm{Cos}_{\tau} A(t-\tau/2) & \mathrm{Sin}_{\tau} A(t-\tau) \\ -\mathrm{Sin}_{\tau} A(t-\tau) & \mathrm{Cos}_{\tau} A(t-\tau/2) \end{pmatrix}.$$

The paper aims to prove the asymptotic unboundedness of the norms of delayed matrix sine and delayed matrix cosine. This is done by utilizing relations between these functions and the delayed matrix exponential. The proof is based on the properties of the main branch of the Lambert function.

Therefore, we at first describe the necessary properties of the delayed exponential of a matrix and the Lambert function in Part 2. Then, in Part 3, the main result on the asymptotic properties of delayed matrix sine and delayed matrix cosine is proved.

2 Delayed matrix exponential and Lambert function

To explain clearly the relationship between delayed linear differential equations and Lambert function, we first consider the scalar case. Let $n = 1$, $A = (a)$. Then, the fundamental matrix to the scalar case of the system (1.2), i.e., of

$$\dot{x}(t) = ax(t - \tau) \tag{2.1}$$

is defined by (1.1) as

$$e_\tau^{at} = \sum_{s=0}^{\lfloor t/\tau \rfloor + 1} a^s \frac{(t - (s-1)\tau)^s}{s!}.$$

and its values at nodes $t = k\tau$, $k = 0, 1, \ldots$ are

$$\begin{aligned} e_\tau^{ak\tau} &= \sum_{s=0}^{k+1} a^s \frac{(k\tau - (s-1)\tau)^s}{s!} = \sum_{s=0}^{k} a^s \frac{(k+1-s)^s \tau^s}{s!} \\ &= 1 + a\frac{k\tau}{1!} + a^2 \frac{(k-1)^2\tau^2}{2!} + \cdots + a^{k-1}\frac{2^{k-1}\tau^{k-1}}{k!} + a^k \frac{\tau^k}{k!}. \end{aligned}$$

Assume that there exists a real solution c of a transcendental equation

$$c = ae^{-c\tau}, \tag{2.2}$$

i.e., that there exists a solution $x(t) = e^{ct}$ of (2.1). Moreover, assume that, for a real root c of (2.2), we have

$$e_\tau^{ak\tau} \sim e^{ck\tau} = 1 + c\frac{k\tau}{1!} + c^2\frac{k^2\tau^2}{2!} + \cdots + c^n \frac{k^n \tau^n}{n!} + \cdots$$

when $k \to \infty$. Then,

$$\frac{e_\tau^{a(k+1)\tau}}{e_\tau^{ak\tau}} \sim \frac{e^{c(k+1)\tau}}{e^{ck\tau}} = e^{c\tau}, \qquad k \to \infty. \tag{2.3}$$

Analyzing equation (2.3), provided it is valid, we can expect that, in a general case, the sequence of values of delayed matrix exponential at nodes $t = k\tau$, $k \to \infty$ is approximately represented by a "geometric progression" with the ordinary exponential of a constant matrix serving as a "quotient" factor.

It is reasonable to expect that such a constant matrix can be expressed by the principal branch of the Lambert function since (2.2) can be rewritten as

$$c\tau e^{c\tau} = a\tau \tag{2.4}$$

or as

$$c\tau = W(a\tau) \tag{2.5}$$

where W is the well-known Lambert W-function [3] (its properties given below are taken from this paper), defined as the inverse function to the function

$$z = f(w) = we^w, \tag{2.6}$$

i.e., $w = W(z)$. If $z = x + iy$ and $w = u + iv$, then (2.6) yields

$$x + iy = (u + iv)e^{u+iv} \tag{2.7}$$

and

$$x = e^u(u\cos v - v\sin v), \qquad y = e^u(u\sin v + v\cos v). \tag{2.8}$$

The Lambert W-function is multi-valued (except for the point $z = 0$). For real $z = x > -1/e$ and $w = u > -1$, equation (2.6) defines a single-valued function $w = W_0(x)$. The function $W_0(x)$ can be extended to the whole complex plane as a holomorphic function $W_0(z)$ except for the values $x < -1/e$ and $y = 0$. The extension $w = W_0(z)$ is called the principal branch of the Lambert function.

The range of values of the principal branch $W = W_0(z)$ is bounded by a parametric curve [3, p. 343]

$$\ell = \frac{-v}{\tan v} + iv, \qquad -\pi < v < \pi \tag{2.9}$$

and equals to the domain

$$\mathcal{L} := \left\{ (u, v) \in \mathbb{C} \colon u \geq -1, |v| \leq |v^*| < \pi \quad \text{where} \quad \frac{-v^*}{\tan v^*} = u \right\}.$$

For more details about the Lambert W-function, see [3].

The asymptotic properties of $\exp(W_0(z))$ are, in principle, determined by the real part of $W_0(z)$. Let $z = x + iy$ and

$$W_0(x + iy) = \operatorname{Re} W_0(x + iy) + i \operatorname{Im} W_0(x + iy) = u + iv.$$

The set of complex numbers $z = x + iy$ such that $\operatorname{Re} W_0(z) = u = 0$, i.e., (see (2.7), (2.8)),

$$x + iy = iv \exp(iv)$$

is a closed curve $\tilde{\ell}$:

$$x = -v\sin v, \qquad y = v\cos v \tag{2.10}$$

where, as it is clear from the definition of $\mathcal{L}$, $|v^*| = \pi/2$ for $u = 0$ and $|v| \leq \pi/2$. We have (as a consequence of (2.8))

$$\operatorname{Re} W_0(z) < 0$$

if z lies within the interior of this curve and

$$\operatorname{Re} W_0(z) > 0 \tag{2.11}$$

for numbers z of its exterior. From (2.10) it follows easily that the exterior domain to $\tilde{\ell}$ is specified by the inequality

$$|z| > -\arctan\left(\frac{\operatorname{Re} z}{|\operatorname{Im} z|}\right). \tag{2.12}$$

Lemma 2.1. *For complex numbers $z = x + iy$, $z \neq 0$ with $x \geq 0$,*

$$|\operatorname{Im} W_0(z)| < \frac{\pi}{2}. \tag{2.13}$$

Proof. First, from (2.9) and definition of $\mathcal{L}$, we obtain inequality $|v| = |\mathrm{Im}\, W_0(z)| < \pi$, therefore,

$$v \sin v > 0. \tag{2.14}$$

Secondly, for $w = u + iv = W_0(z)$, the inequality $u < 0$ implies $|v| < \pi/2$ (see the definition of $\mathcal{L}$) and, in this case, (2.13) holds. This guarantees that $\mathrm{sign}(u \cos v) = \mathrm{sign}\, u$. Applying (2.8) and the assumption that x is nonnegative, we obtain

$$\mathrm{e}^u(u \cos v - v \sin v) = x \geq 0 \Rightarrow u \geq 0 \Rightarrow \arg W_0(z) \mathrm{Im}\, W_0(z) \geq 0.$$

This fact also implies

$$|\arg W_0(z) + \mathrm{Im}\, W_0(z)| = |\arg W_0(z)| + |\mathrm{Im}\, W_0(z)|. \tag{2.15}$$

Equation (2.6) yields

$$z = w\mathrm{e}^w = W_0(z)\mathrm{e}^{W_0(z)}.$$

Therefore,

$$\arg z = \arg W_0(z) + \mathrm{Im}\, W_0(z)$$

and, due to relation, (2.15) we also have

$$|\arg z| = |\arg W_0(z)| + |\mathrm{Im}\, W_0(z)|. \tag{2.16}$$

For $z \neq 0$ with non-negative real parts, we have $\mathrm{Re}\, W_0(z) > 0$ by (2.11), from (2.14), we deduce $\arg W_0(z) \neq 0$, $\mathrm{Im}\, W_0(z) \neq 0$, and, utilizing (2.16), we also have

$$\pi/2 \geq |\arg z| = |\arg W_0(z)| + |\mathrm{Im}\, W_0(z)| > |\mathrm{Im}\, W_0(z)|. \qquad \square$$

Reverting to equation (2.3), we can expect that, in some cases, there exists a constant $n \times n$ matrix C such that

$$\lim_{k \to \infty} \mathrm{e}_\tau^{A(k+1)\tau} (\mathrm{e}_\tau^{Ak\tau})^{-1} = \mathrm{e}^{C\tau}, \tag{2.17}$$

provided that the matrices $\mathrm{e}_\tau^{Ak\tau}$ are nonsingular (this property will be assumed throughout the paper). One of such cases is analysed in [23] where the following is proved.

Theorem 2.2. *Let λ_j, $j = 1, \ldots, n$ be the eigenvalues of the matrix A and let its Jordan canonical form be*

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_n) = D^{-1}AD \tag{2.18}$$

where D is a regular matrix. If

$$|\lambda_j| < 1/(\mathrm{e}\tau),$$

$j = 1, \ldots, n$, then the sequence

$$\mathrm{e}_\tau^{A(k+1)\tau} (\mathrm{e}_\tau^{Ak\tau})^{-1}, \;\; k \to \infty$$

converges, (2.17) holds and

$$\mathrm{e}^{C\tau} = D \exp\left(\mathrm{diag}(W_0(\lambda_1\tau), \ldots, W_0(\lambda_n\tau)\right) D^{-1}. \tag{2.19}$$

Note that from (2.19) we immediately get explicit form of C since

$$C\tau = D\left(\mathrm{diag}(W_0(\lambda_1\tau), \ldots, W_0(\lambda_n, \tau)\right) D^{-1}$$

and

$$C = D \,\mathrm{diag}\left(W_0(\lambda_1\tau)/\tau, \ldots, W_0(\lambda_n\tau)/\tau\right) D^{-1}.$$

3 Main result

The asymptotic properties of the delayed matrix sine and cosine can be deduced from the relations with the delayed exponential of a matrix. We give relevant formulas that are similar to the well-known Euler identity. Namely, for an arbitrary $n \times n$ matrix A and $t \in \mathbb{R}$, we have

$$\operatorname{Sin}_\tau A(t-\tau) = \operatorname{Im} \mathrm{e}_{\tau/2}^{iAt} = \frac{1}{2i}\left(\mathrm{e}_{\tau/2}^{iAt} - \mathrm{e}_{\tau/2}^{-iAt}\right) \tag{3.1}$$

and

$$\operatorname{Cos}_\tau A\left(t-\frac{\tau}{2}\right) = \operatorname{Re} \mathrm{e}_{\tau/2}^{iAt} = \frac{1}{2}\left(\mathrm{e}_{\tau/2}^{iAt} + \mathrm{e}_{\tau/2}^{-iAt}\right). \tag{3.2}$$

Formulas (3.1), (3.2) can be proved directly using the definitions of e_τ^{At}, $\operatorname{Sin}_\tau At$ and $\operatorname{Cos}_\tau At$ given by formulas (1.1), (1.6) and (1.7) (for the proof we refer to [24]). Below, we use the spectral norm of a matrix defined as

$$\|A\|_S = \sqrt{\lambda_{\max}(A^* A)} \tag{3.3}$$

where A^* denotes the conjugate transpose of A and $\lambda_{\max}$ is the largest eigenvalue of the matrix $A^* A$. The main result of the paper follows.

Theorem 3.1. *Let λ_j, $j = 1, \ldots, n$ be the eigenvalues of the matrix A and let its Jordan canonical form be given by (2.18). If $|\lambda_j| < 1/(\mathrm{e}\tau)$, $j = 1, \ldots, n$ and there exists at least one $j = j^* \in \{1, \ldots, n\}$ such that $\lambda_{j^*} \neq 0$, then*

$$\limsup_{t\to\infty} \|\operatorname{Cos}_\tau At\|_S = \infty$$

and

$$\limsup_{t\to\infty} \|\operatorname{Sin}_\tau At\|_S = \infty.$$

Proof. We will only prove the assertion for $\operatorname{Cos}_\tau At$ as the proof for $\operatorname{Sin}_\tau At$ is analogous. Using equation (3.2), we derive the assertion of the theorem utilizing the asymptotic properties of the delayed exponential of matrix $\mathrm{e}_{\tau/2}^{iAt}$. From the assumption (2.18), we readily get

$$(iA)^k = D \operatorname{diag}((i\lambda_1)^k, \ldots, (i\lambda_n)^k) D^{-1}, \quad k \geq 0$$

and, using the associativity, we may express $\mathrm{e}_{\tau/2}^{iAk\tau/2}$ (with the aid of definition (1.1)) as

$$\mathrm{e}_{\tau/2}^{Aik\tau/2} = D \operatorname{diag}\left(\mathrm{e}_{\tau/2}^{\lambda_1 ik\tau/2}, \ldots, \mathrm{e}_{\tau/2}^{\lambda_n ik\tau/2}\right) D^{-1}. \tag{3.4}$$

For a natural number ℓ we define

$$F_k^\ell(A) := \mathrm{e}_{\tau/2}^{Ai(k+\ell)\tau/2} (\mathrm{e}_{\tau/2}^{Aik\tau/2})^{-1}.$$

By Theorem 2.2 (formula (2.17)) and by (2.19), we have

$$\lim_{k\to\infty} F_k^1(A) = D \exp\left(\operatorname{diag}(W_0(\lambda_1 i\tau/2), \ldots, W_0(\lambda_n i\tau/2)\right) D^{-1}. \tag{3.5}$$

From

$$F_k^\ell(a) = \prod_{l=1}^{\ell} F_{k-l-1}^1(A),$$

we obtain

$$\lim_{k\to\infty} F_k^{\ell}(A) = \lim_{k\to\infty} \prod_{l=1}^{\ell} F_k^{\ell}(A) = \prod_{l=1}^{\ell} \lim_{k\to\infty} F_k^{\ell}(A)$$
$$= \left(D \exp\left(\mathrm{diag}(W_0(\lambda_1 i\tau/2), \dots, W_0(\lambda_n i\tau/2)\right) D^{-1}\right)^{\ell}.$$

Imagine, for a while, that the matrix A is a 1×1 matrix, i.e., $A = (a)$. Then, from (3.5) (with $\lambda = a$, $D := (1)$), we get

$$F_k^1(a) = (\exp(W_0(ai\tau/2)))\,(1 + v_a(k)) \tag{3.6}$$

where k is an arbitrary natural number and $v = v_a(k)$ is a real discrete function such that

$$\lim_{k\to\infty} v_a(k) = 0. \tag{3.7}$$

Applying formula (3.6) ℓ times, we obtain

$$F_k^{\ell}(A) = (\exp(W_0(ai\tau/2)))^{\ell} \prod_{l=1}^{\ell} (1 + v_a(k-1+l)).$$

Now we can derive a similar formula in the case of an $n\times n$ matrix A. First, utilizing (3.6), we obtain:

$$\begin{aligned}
F_k^1(A) &= D\,\mathrm{diag}\left(\mathrm{e}_{\tau/2}^{\lambda_1 i(k+1)\tau/2}, \dots, \mathrm{e}_{\tau/2}^{\lambda_n i(k+1)\tau/2}\right) D^{-1} \\
&\quad \times D\,\mathrm{diag}\left(\left(\mathrm{e}_{\tau/2}^{\lambda_1 ik\tau/2}\right)^{-1}, \dots, \left(\mathrm{e}_{\tau/2}^{\lambda_n ik\tau/2}\right)^{-1}\right) D^{-1} \\
&= D\,\mathrm{diag}\left(\mathrm{e}_{\tau/2}^{\lambda_1 i(k+1)\tau/2}\left(\mathrm{e}_{\tau/2}^{\lambda_1 ik\tau/2}\right)^{-1}, \dots, \mathrm{e}_{\tau/2}^{\lambda_n i(k+1)\tau/2}\left(\mathrm{e}_{\tau/2}^{\lambda_n ik\tau/2}\right)^{-1}\right) D^{-1} \\
&= D\,\mathrm{diag}\,((\exp(W_0(\lambda_1 i\tau/2)))\,(1 + v_{\lambda_1}(k)), \dots \\
&\quad \dots, (\exp(W_0(\lambda_n i\tau/2)))\,(1 + v_{\lambda_n}(k)))\, D^{-1} \\
&= D\,\mathrm{diag}\,(\exp(W_0(\lambda_1 i\tau/2)), \dots, \exp(W_0(\lambda_n i\tau/2)))\, D^{-1} \\
&\quad \times D\,\mathrm{diag}\,((1 + v_{\lambda_1}(k)), \dots, (1 + v_{\lambda_n}(k)))\, D^{-1} \\
&= D\,\mathrm{diag}\,(\exp(W_0(\lambda_1 i\tau/2)), \dots, \exp(W_0(\lambda_n i\tau/2)))\, D^{-1} M(k)
\end{aligned} \tag{3.8}$$

where the matrix $M(k)$ is defined as

$$M(k) := D\,\mathrm{diag}((1 + v_{\lambda_1}(k)), \dots, (1 + v_{\lambda_n}(k)))D^{-1}.$$

Denote

$$\mathrm{e}^{W_0(iA)\tau/2} := D\,\mathrm{diag}\,(\exp(W_0(\lambda_1 i\tau/2)), \dots, \exp(W_0(\lambda_n i\tau/2)))\, D^{-1}.$$

This matrix commutes with $M(k)$ since

$$\begin{aligned}
\mathrm{e}^{W_0(iA)\tau/2} M(k) &= D\,\mathrm{diag}(\exp(W_0(\lambda_1 i\tau/2)), \dots, \exp(W_0(\lambda_n i\tau/2)))D^{-1} \\
&\quad \times D\,\mathrm{diag}((1 + v_1(k)), \dots, (1 + v_n(k)))D^{-1} \\
&= D\,\mathrm{diag}((1 + v_1(k)), \dots, (1 + v_n(k)))D^{-1} \\
&\quad \times D\,\mathrm{diag}(\exp(W_0(\lambda_1 i\tau/2)), \dots, \exp(W_0(\lambda_n i\tau/2)))D^{-1} \\
&= M(k)\mathrm{e}^{W_0(iA)\tau/2}.
\end{aligned}$$

Utilizing (3.4), (3.6), and (3.8), we derive

$$\begin{aligned}
F_k^\ell(A) &= \mathrm{e}_{\tau/2}^{Ai(k+\ell)\tau/2}(\mathrm{e}_{\tau/2}^{Ai(k+\ell-1)\tau/2})^{-1}\cdots\mathrm{e}_{\tau/2}^{Ai(k+2)\tau/2}(\mathrm{e}_{\tau/2}^{Ai(k+1)\tau/2})^{-1}\mathrm{e}_{\tau/2}^{Ai(k+1)\tau/2}(\mathrm{e}_{\tau/2}^{Aik\tau/2})^{-1}\\
&= D\,\mathrm{diag}\left(\mathrm{e}_{\tau/2}^{\lambda_1 i(k+\ell)\tau/2}\left(\mathrm{e}_{\tau/2}^{\lambda_1 i(k+\ell-1)\tau/2}\right)^{-1},\dots,\mathrm{e}_{\tau/2}^{\lambda_n i(k+\ell)\tau/2}\left(\mathrm{e}_{\tau/2}^{\lambda_n i(k+\ell-1)\tau/2}\right)^{-1}\right)D^{-1}\\
&\quad\times D\,\mathrm{diag}\left(\mathrm{e}_{\tau/2}^{\lambda_1 i(k+\ell-1)\tau/2}\left(\mathrm{e}_{\tau/2}^{\lambda_1 i(k+\ell-2)\tau/2}\right)^{-1},\dots\right.\\
&\quad\left.\dots,\mathrm{e}_{\tau/2}^{\lambda_n i(k+\ell-1)\tau/2}\left(\mathrm{e}_{\tau/2}^{\lambda_n i(k+\ell-2)\tau/2}\right)^{-1}\right)D^{-1}\\
&\quad\dots\\
&\quad\times D\,\mathrm{diag}\left(\mathrm{e}_{\tau/2}^{\lambda_1 i(k+1)\tau/2}\left(\mathrm{e}_{\tau/2}^{\lambda_1 ik\tau/2}\right)^{-1},\dots,\mathrm{e}_{\tau/2}^{\lambda_n i(k+1)\tau/2}\left(\mathrm{e}_{\tau/2}^{\lambda_n ik\tau/2}\right)^{-1}\right)D^{-1}\\
&= \mathrm{e}^{W_0(iA)\tau/2}M(k+\ell-1)\mathrm{e}^{W_0(iA)\tau/2}M(k+\ell-2)\cdots\mathrm{e}^{W_0(iA)\tau/2}M(k)\\
&= \left(\mathrm{e}^{W_0(iA)\tau/2}\right)^\ell\prod_{l=0}^{\ell-1}M(k+l).
\end{aligned}\tag{3.9}$$

It is easy to see that the values of functions $\mathrm{e}_{\tau/2}^{\lambda_l ik\tau/2}$, $\exp(\ell W_0(\lambda_l i\tau/2))$ $(l=1,\dots,n)$ and the values of the same functions with complex conjugate arguments are complex conjugate too. Applying this fact to $\mathrm{Cos}_\tau A\,((k+\ell-1)\tau/2)=\mathrm{Re}\,(\mathrm{e}_{\tau/2}^{iA(k+\ell)\tau/2})$ (see (3.2)), we get (utilizing (3.4), (3.9)):

$$\begin{aligned}
\mathrm{Re}\,\left(\mathrm{e}_{\tau/2}^{iA(k+\ell)\tau/2}\right) &= \frac{1}{2}\left(\mathrm{e}_{\tau/2}^{iA(k+\ell)\tau/2}+\mathrm{e}_{\tau/2}^{-iA(k+\ell)\tau/2}\right)\\
&= \frac{1}{2}\left(D\,\mathrm{diag}\left(\mathrm{e}_{\tau/2}^{\lambda_1 ik\tau/2},\dots,\mathrm{e}_{\tau/2}^{\lambda_n ik\tau/2}\right)D^{-1}\left(\mathrm{e}^{W_0(iA)\tau/2}\right)^\ell\prod_{l=0}^{\ell-1}M(k+l)\right.\\
&\qquad\left.+D\,\mathrm{diag}\left(\mathrm{e}_{\tau/2}^{-\lambda_1 ik\tau/2},\dots,\mathrm{e}_{\tau/2}^{-\lambda_n ik\tau/2}\right)D^{-1}\left(\mathrm{e}^{W_0(-iA)\tau/2}\right)^\ell\prod_{l=0}^{\ell-1}M(k+l)\right)\\
&= \frac{1}{2}D\,\mathrm{diag}\left(\mathrm{e}_{\tau/2}^{\lambda_1 ik\tau/2}\exp(\ell W_0(\lambda_1 i\tau/2))\right.\\
&\quad+\mathrm{e}_{\tau/2}^{-\lambda_1 ik\tau/2}\exp(-\ell W_0(\lambda_1 i\tau/2)),\dots,\mathrm{e}_{\tau/2}^{\lambda_n ik\tau/2}\exp(\ell W_0(\lambda_n i\tau/2))\\
&\quad\left.+\,\mathrm{e}_{\tau/2}^{-\lambda_n ik\tau/2}\exp(-\ell W_0(\lambda_n i\tau/2))\right)D^{-1}\prod_{l=0}^{\ell-1}M(k+l)\\
&= D\,\mathrm{diag}\left(\mathrm{Re}\,\left(\mathrm{e}_{\tau/2}^{\lambda_1 ik\tau/2}\exp(\ell W_0(\lambda_1 i\tau/2))\right),\dots\right.\\
&\quad\left.\dots,\mathrm{Re}\,\left(\mathrm{e}_{\tau/2}^{\lambda_n ik\tau/2}\exp(\ell W_0(\lambda_n i\tau/2))\right)\right)D^{-1}\prod_{l=0}^{\ell-1}M(k+l)\\
&= D\,\mathrm{diag}\left(\mathrm{Re}\,\left(\mathrm{e}_{\tau/2}^{\lambda_1 ik\tau/2}\exp(\ell W_0(\lambda_1 i\tau/2))\right)\prod_{l=0}^{\ell-1}(1+v_{\lambda_1}(k+l)),\dots\right.\\
&\quad\left.\dots,\mathrm{Re}\,\left(\mathrm{e}_{\tau/2}^{\lambda_n ik\tau/2}\exp(\ell W_0(\lambda_n i\tau/2))\right)\prod_{l=0}^{\ell-1}(1+v_{\lambda_n}(k+l))\right)D^{-1}.
\end{aligned}\tag{3.10}$$

Now we use the well-known formula $\mathrm{Re}(z_1z_2)=|z_1||z_2|\cos(\arg z_1+\arg z_2)$ for complex numbers z_1, z_2. Set

$$z_1=z_1(k,\lambda_l):=\mathrm{e}_{\tau/2}^{\lambda_l ik\tau/2},\quad z_2=z_2(\lambda_l):=\exp(\ell W_0(\lambda_l i\tau/2)),$$

where $l \in \{1, \dots, n\}$, and denote

$$\alpha_1(k,\lambda_l) := \arg z_1(k,\lambda_l) = \arg\left(\mathrm{e}_{\tau/2}^{\lambda_l ik\tau/2}\right),$$
$$\alpha_2(\lambda_l) := \arg z_2(\lambda_l) = \arg\left(\exp(\ell W_0(\lambda_l i\tau/2))\right).$$

From the facts that the spectral radius is less or equal any matrix norm, the following inequality for the spectral norm holds

$$\begin{aligned}\|\mathrm{Cos}_\tau A\left((k+\ell-1)\tau/2\right)\|_S &\geq \rho\left(\mathrm{Cos}_\tau A\left((k+\ell-1)\tau/2\right)\right)\\ &= \rho\left(\mathrm{Re}\left(\mathrm{e}_{\tau/2}^{iA(k+\ell)\tau/2}\right)\right) = \rho_{k+\ell}.\end{aligned} \tag{3.11}$$

The similar matrices have same spectra and the spectral radii. The spectrum of diagonal matrix consists to elements of the diagonal and using (3.10), we obtain

$$\begin{aligned}\rho_k &= \max_{j=1,\dots,n}\left\{\left|\mathrm{Re}\left(\mathrm{e}_{\tau/2}^{\lambda_j ik\tau/2}\exp(\ell W_0(\lambda_j i\tau/2))\right)\prod_{l=0}^{\ell-1}(1+v_{\lambda_j}(k+l))\right|\right\}\\ &\geq (1+v^*(k))^\ell \max_{j=1,\dots,n}\left\{\left|\mathrm{Re}\left(\mathrm{e}_{\tau/2}^{\lambda_j ik\tau/2}\exp(\ell W_0(\lambda_j i\tau/2))\right)\right|\right\}\end{aligned} \tag{3.12}$$

where

$$v^*(k) := \min_{j=1,\dots,n;\, l=0,\dots,\ell-1}\left\{v_{\lambda_j}(k+l)\right\}$$

and, by (3.7),

$$\lim_{k\to\infty} v^*(k) = 0. \tag{3.13}$$

Applying (3.11) and (3.12) we obtain the inequality

$$\begin{aligned}\|\mathrm{Cos}_\tau A\left((k+\ell-1)\tau/2\right)\|_S &\geq (1+v^*(k))^\ell \max_{j=1,\dots,n}\left\{\left|\mathrm{Re}\left(\mathrm{e}_{\tau/2}^{\lambda_j ik\tau/2}\exp(\ell W_0(\lambda_j i\tau/2))\right)\right|\right\}\\ &\geq (1+v^*(k))^\ell \max_{j=1,\dots,n}\left\{\left|\mathrm{e}_{\tau/2}^{\lambda_j ik\tau/2}\right|\left|\exp(\ell W_0(\lambda_j i\tau/2))\right|\left|\cos\left(\alpha_1(k,\lambda_l)+\alpha_2(\lambda_l)\right)\right|\right\}.\end{aligned}$$

Assume that $j = j^* \in \{1,\dots,n\}$ is fixed and that the eigenvalue $\lambda_{j^*} \neq 0$ of the matrix A is real. Then, the number $z^* = i\lambda_{j^*}\tau/2$ lies in the exterior domain of $\tilde{\ell}$ since inequality (2.12) holds, i.e.,

$$|z^*| = |i\lambda_{j^*}\tau/2| > -\arctan\left(\frac{\mathrm{Re}\, z^*}{|\mathrm{Im}\, z^*|}\right) = -\arctan 0 = 0 \tag{3.14}$$

and, by (2.11),

$$\mathrm{Re}\, W_0(z^*) = \mathrm{Re}\, W_0(i\lambda_{j^*}\tau/2) > 0. \tag{3.15}$$

Now assume that $j = j^* \in \{1,\dots,n\}$ is fixed and that the eigenvalue $\lambda_{j^*} \neq 0$ of the matrix A is a complex number. Since $\overline{\lambda}_{j^*}$ is an eigenvalue of A as well, we can assume that $\lambda_{j^*} = x - iy$ where $y > 0$. Then, the number $z^* = i\lambda_{j^*}\tau/2$ lies in the exterior domain of $\tilde{\ell}$ since inequality (2.12) holds, i.e.,

$$|z^*| = |i\lambda_{j^*}\tau/2| = \frac{\tau}{2}|ix+y| = \frac{\tau}{2}\sqrt{x^2+y^2} > -\arctan\left(\frac{\mathrm{Re}\, z^*}{|\mathrm{Im}\, z^*|}\right) = -\arctan\left(\frac{y}{|x|}\right)$$

where $\arctan(y/|x|) > 0$. Then, by (2.11),

$$\mathrm{Re}\, W_0(z^*) = \mathrm{Re}\, W_0(i\lambda_{j^*}\tau/2) > 0. \tag{3.16}$$

From (3.15) and (3.16), it follows that there exists an eigenvalue λ_{j^*} of A and a constant $\widetilde{C}$ such that

$$\operatorname{Re} W_0(i\lambda_{j^*}\tau/2) > \widetilde{C} > 0. \tag{3.17}$$

Utilizing (3.1), (3.2) (where $A := (\lambda_{j^*})$ and $t = k\tau/2$) we derive

$$e_{\tau/2}^{\lambda_{j^*} ik\tau/2} = \mathrm{Cos}_\tau \lambda_{j^*}(k-1)\tau/2) + i\,\mathrm{Sin}_\tau \lambda_{j^*}(k/2-1)\tau. \tag{3.18}$$

Let $k = k^*$ be such that

$$\mathrm{Cos}_\tau \lambda_{j^*}(k^*-1)\tau/2) \neq 0. \tag{3.19}$$

It is easy to see that such a k^* always exists and note that it can be assumed greater than an arbitrarily given sufficiently large positive integer. Then (3.18), implies

$$\alpha_1(k^*, \lambda_{j^*}) \neq \pm\frac{\pi}{2}. \tag{3.20}$$

By (2.13), we have $|\alpha_2(\lambda_{j^*})| < \pi/2$. With regard to $\alpha_2(\lambda_{j^*})$, we consider two cases below:

a) Let $\alpha_2(\lambda_{j^*}) \neq 0$. Then, each interval $[\pi/2 + 2s\pi, \pi/2 + 2s\pi + \pi]$, where $s = 0, 1, \dots,$ contains at least two elements of an equidistant sequence

$$\{\alpha_1(k^*, \lambda_{j^*}) + n\alpha_2(\lambda_{j^*})\}_{n=-\infty}^{\infty}$$

and, in each interval, there exists an element of this sequence α^s such that

$$|\alpha^s - \pi/2| > \frac{\pi}{4}, \ \ |\alpha^s - \pi/2 - \pi| > \frac{\pi}{4}$$

and

$$|\cos(\alpha^s)| > \sqrt{2}/2. \tag{3.21}$$

b) Let $\alpha_2(\lambda_{j^*}) = 0$. Then, (3.20) implies

$$|\cos\alpha^s| = |\cos\alpha_1(k^*, \lambda_{j^*})| \neq 0. \tag{3.22}$$

Therefore, in both cases **a)** and **b)**, there exists a sequence of positive integers $\{\ell_l\}_{l=1}^{\infty}$ such that $\lim_{l\to\infty} = \infty$ and (due to (3.17), (3.21) and (3.22)) for all sufficiently large ℓ_l

$$|\exp(\ell_l W_0(i\lambda_{j^*}\tau/2))||\cos(\alpha_1(k^*, \lambda_{j^*}) + \ell_l\alpha_2(\lambda_{j^*}))| > M\exp(\ell_l C\tau/2) \tag{3.23}$$

where

$$M := \begin{cases} \dfrac{\sqrt{2}}{2}, & \text{if } \ \alpha_2(\lambda_{j^*}) \neq 0, \\ |\cos\alpha_1(k^*, \lambda_{j^*})|, & \text{if } \ \alpha_2(\lambda_{j^*}) = 0 \end{cases}$$

and C is a constant satisfying $0 < C < \widetilde{C}$. Moreover, from (3.13), it follows that, for every sufficiently large k, there exists a constant C_0 satisfying $0 < C_0 < C$ such that

$$1 + v^*(k) > \exp(-C_0\tau/2). \tag{3.24}$$

From (3.12), (3.23), (3.24), we can derive

$$
\begin{aligned}
\|\mathrm{Cos}_\tau A\,((k^*+\ell_l-1)\tau/2)\|_S &\geq (1+v^*(k^*))^{\ell_l}\left|\mathrm{e}_{\tau/2}^{\lambda_{j^*}ik^*\tau/2}\right| \\
&\quad\times\left|\exp(\ell_l W_0(\lambda_{j^*}i\tau/2))\right|\left|\cos\left(\alpha_1(k^*,\lambda_{j^*})+\alpha_2(\lambda_{j^*})\right)\right| \\
&\geq \exp(-\ell_l C_0\tau/2)\left|\mathrm{e}_{\tau/2}^{\lambda_{j^*}ik^*\tau/2}\right| M\exp(\ell_l C\tau/2) \\
&= M\left|\mathrm{e}_{\tau/2}^{\lambda_{j^*}ik^*\tau/2}\right|\exp(\ell_l(C-C_0)\tau/2).
\end{aligned}
$$

Finally, we conclude

$$
\begin{aligned}
\limsup_{t\to\infty}\|\,\mathrm{Cos}_\tau\,At\|_S &\geq \lim_{l\to\infty}\|\mathrm{Cos}_\tau A\,((k^*+\ell_l-1)\tau/2)\,\|_S \\
&\geq \lim_{l\to\infty} M\left|\mathrm{e}_{\tau/2}^{\lambda_{j^*}ik^*\tau/2}\right|\exp(\ell_l(C-C_0)\tau/2) \\
&= \infty.
\end{aligned}
$$

An analogous assertion can also be obtained for $\mathrm{Sin}_\tau\,At$. The scheme of the proof in this case remains the same with the following minor modifications. In (3.10) the imaginary parts of the complex expressions considered is used instead of their real parts. The relation (3.10) turns into

$$
\begin{aligned}
\mathrm{Sin}_\tau A\,((k+\ell-2)\tau/2) = D\,\mathrm{diag}\Bigg(&\mathrm{Im}\,\left(\mathrm{e}_{\tau/2}^{\lambda_1 ik\tau/2}\exp(\ell W_0(\lambda_1 i\tau/2))\right)\prod_{l=0}^{\ell-1}(1+v_{\lambda_1}(k+l)),\dots \\
&\dots,\mathrm{Im}\,\left(\mathrm{e}_{\tau/2}^{\lambda_n ik\tau/2}\exp(\ell W_0(\lambda_n i\tau/2))\right)\prod_{l=0}^{\ell-1}(1+v_{\lambda_n}(k+l))\Bigg)D^{-1}
\end{aligned}
$$

and the estimation (3.12) has the form

$$
\begin{aligned}
&\|\mathrm{Sin}_\tau A\,((k+\ell-2)\tau/2)\|_S \\
&\qquad\geq (1+v^*(k))^{\ell}\max_{j=1,\dots,n}\left\{\left|\mathrm{e}_{\tau/2}^{\lambda_j ik\tau/2}\right|\left|\exp(\ell W_0(\lambda_j i\tau/2))\right|\left|\sin\left(\alpha_1(k,\lambda_l)+\alpha_2(\lambda_l)\right)\right|\right\}.
\end{aligned}
$$

In (3.19), Sin_τ instead of Cos_τ is used and the constant M must be redefined as

$$
M := \begin{cases} \dfrac{\sqrt{2}}{2}, & \text{if}\quad \alpha_2(\lambda_{j^*})\neq 0, \\ |\sin\alpha_1(k^*,\lambda_{j^*})|, & \text{if}\quad \alpha_2(\lambda_{j^*})=0. \end{cases}
$$

□

4 Concluding remarks

In this part, we discuss some connections with previous results and facts. The author is grateful to the referee for drawing attention to several topics which are discussed below.

i) Relationship with a linear ordinary non-delayed system. In the paper, properties of delayed matrix exponential and the Lambert W-function are used to prove that spectral norms of delayed matrix sine and delayed matrix cosine are unbounded for $t\to\infty$. This property is proved under the assumption that the spectral radius $\rho(A)$ of the matrix A is less that $1/(\mathrm{e}\tau)$.

Many papers bring results on so-called special solutions of delayed differential systems (we refer, e.g., to [1,2,7–11,14,17–19,22] and to the references therein) approximating, in a certain sense, all solutions of a given system. One of the conditions guaranteeing the existence of special solutions is often (restricted to system (1.2)) the inequality

$$\|A\| < 1/(e\tau)$$

where $\|\cdot\|$ is an arbitrary norm. The totality of all special solutions is only an n-parameter family where n equals the number of equations of the system. Moreover, it is often stated that, in such a case, some properties (such as stability properties) of solutions of the initial system are the same as those for solutions of a corresponding system of ordinary differential equations.

Because of the well-known inequality $\rho(A) \le \|A\|$, it is generally not possible from an assumed inequality $\rho(A) < 1/(e\tau)$ to deduce $\|A\| < 1/(e\tau)$. Nevertheless, for the spectral norm (3.3) used in the paper, we get (under the conditions of Theorem 3.1),

$$\rho(A) = \|A\|_S < 1/(e\tau).$$

It means that, in a way, the properties of solutions of (1.2) are close, in a meaning, to properties of an ordinary differential system and (1.2) is asymptotically ordinary. I.e., every solution of system (1.2) is asymptotically close to a solution of a system of ordinary differential equations.

The construction of such a linear non-delayed system is described, e.g., in [1, Theorem 2.4] (see also the Summary part in [17]). However, to find such a system is, in general, not an easy task. The formula defining the matrix of ordinary differential system ([1, formula (2.8)] or [17, formula (2.10)]) is a series of recurrently defined matrices and to find its sum is not always possible (we refer to [7, Theorem 1.2], [17, part 4]).

In the case of a constant matrix, the fundamental matrix $X_o(t)$ of the corresponding ordinary differential system equals an ordinary matrix exponential $X_o(t) = \exp(\Lambda_0 t)$ where the matrix Λ_0 is a unique solution of the matrix equation

$$\Lambda = A\exp(-\Lambda\tau)$$

such that $\|\Lambda_0\|\,\tau < 1$ (see the proof of statement (i) of the Theorem in [17]). So, an analysis of the asymptotic behavior of the solutions of system (1.2) reduces, in a meaning, to an analysis of the asymptotic behavior of solutions of a system of ordinary differential equations $x' = \Lambda_0 x$, i.e., analysis of the properties of the matrix Λ_0. Tracing the proof of Theorem 3.1, we can assert that the investigation of properties of the matrix Λ_0 is, in our case, performed by using the properties of Lambert W-function defined in Part 2 (see also the motivation example (2.1) and formulas (2.2)–(2.5)).

ii) Existence of a root of characteristic equation with positive real part. Let $n = 1$ and $A = (a)$ in (1.5). Then, the characteristic equation (derived by substituting $x = \exp(\lambda t)$) equals

$$\lambda^2 = -a^2\exp(-\tau\lambda) \tag{4.1}$$

and is equivalent with

$$\frac{\lambda\tau}{2}\exp\left(\frac{\lambda\tau}{2}\right) = \pm\frac{ia\tau}{2}.$$

Utilizing the Lambert W-function, the last equation can be written as (see (2.4), (2.5))

$$\frac{\lambda\tau}{2} = W\left(\pm\frac{ia\tau}{2}\right),$$

therefore, all roots of (4.1) are values of the Lambert function. For

$$z = z_{\pm} = \pm ia\tau/2,$$

inequality (2.12), which determines the domain of the points for which the principal branch of the Lambert function W_0 has positive real parts (inequality (2.11)), holds (see also (3.14), (3.15)). Thus, we conclude that the unboundedness of the delayed matrix sine and cosine is related to the existence of a root of characteristic equation with positive real part.

iii) Asymptotic behavior of the fundamental matrix solution by using the characteristic equation. As noted in the Introduction, the general definition of a fundamental matrix to linear functional differential systems of delayed type in [12,13] yields (in the simple case of the matrix of the system with single delay being a constant matrix) a delayed matrix exponential by formula (1.4). Delayed matrix sine and cosine can be expressed through delayed matrix exponential by formulas (3.1), (3.2). Therefore, both Theorem 2.2 and Theorem 3.1, formulate the asymptotic properties of the relevant fundamental matrix solutions depending on the properties of the eigenvalues of the matrix A and, consequently, through the properties of the roots of the characteristic equation described by the Lambert W-function. It is an open question if the method used in the paper can be extended to matrices A with Jordan canonical forms different from (2.18) in order to get further results on the behavior of the fundamental matrix solution.

Acknowledgements

The author would like to thank the referee for helpful suggestions incorporated in this paper. This research was carried out under the project CEITEC 2020 (LQ1601) with financial support from the Ministry of Education, Youth and Sports of the Czech Republic under the National Sustainability Programme II.

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