MASARYKOVA UNIVERZITA
PŘÍRODOVĚDECKÁ FAKULTA
ÚSTAV MATEMATIKY A STATISTIKY

Habilitační práce

Brno 2015

Sulkhan Mukhigulashvili

MASARYKOVA UNIVERZITA
PŘÍRODOVĚDECKÁ FAKULTA
ÚSTAV MATEMATIKY A STATISTIKY

O některých dvoubodových okrajových úlohách pro funkcionální diferenciální rovnice

Habilitační práce

Sulkhan Mukhigulashvili

Brno 2015

Bibliografický záznam

Autor:	Mukhigulashvili Sulkhan , *Dr. CSc* Vysoké učení technické v Brně, fakulta podnikatelská, Ústav informatiky.
Název práce:	O některých dvoubodových okrajových ulohách pro funkcionální diferenciální rovnice
Obor:	Matematika- matematická analzýa
Akademický rok:	2015/2016
Počet stran:	
Klíčová slova:	Dvoubodová úloha, singulární funkcionální diferenciální rovnice, řešitelnost a jednoznačnost řešení, dvoubodová úloha v rezonančním případě, Fredholmovost úlohy.

Bibliographic Entry

Author	Mukhigulashvili Sulkhan, *Dr. CSc*, Brno University Of Technology, The Faculty of Business and Management, Department Of Information Technology.
Titleof Thesis:	On Some Two-Point Boundary Value Problems For Functional-Differential Equations
Field:	Mathematics- Mathematical analysis
AcademicYear:	2015/2016
Number of Pages:	
Keywords:	Two-poin boundary value problems, singular functional-differential equations, solvability, unique solvability, Fredholm‘s property, BVP at resonance.

Abstrakt

V tomto přehledu jsou obsaženy některé výsledky z jedné monografie a čtyř časopiseckých prací, v nichž jsou vyšetřovány dvoubodové okrajové úlohy pro funkcionální diferenciální rovnice jak v regulárním, tak i sigulárním případě (např. Dirichletova, smíšená, fokální, periodická). Jsou zde uvedeny efektivní podmínky zaručující řešitelnost, jednoznačnou řešitelnost, a také Fredholmovost studovaných úloh. Uvedené výsledky byly v době jejich publikace originální, nové a rozšiřovaly znalosti v daném oboru.

Abstract

The present survey considers four papers and a monograph where various kinds of boundary value problems (Dirichlet, periodic, mixed, focal) for linear and nonlinear functional differential equations are studied in both regular and singular cases. The works mentioned contain results on the solvability, unique solvability and Fredholm property of the problems under consideration. The results obtained, at the time of publication, had been new and contributed essentially to the knowledge of the problems mentioned.

CONTENT

ÚVOD

Předložená práce obsahuje přehled některých mých výsledků týkajících se otázek řešitelnosti, jednoznačné řešitelnosti a korektnosti některých okrajových úloh pro funkcionálně-diferenciální rovnice. Nejprve stručně popíšeme pět vybraných publikací.

V článku ***The Dirichlet BVP for second order nonlinear ordinary differential equation at resonance. Italian J. Of Pure and Appl. Math., 2011, No. 28, 177–204*** je, narozdíl od předešlých prací uvažována nelineární diferenciální rovnice tvaru $u'' = p(t)u + f(t,u) + h(t)$, v níž je funkce f sublineární v druhé proměnné, v případě když homogenní Dirichletova úloha pro rovnici $u'' = p(t)u$ má netriviální řešení (tzv. rezonanční případ). V tomto článku se nepředpokládá, že koeficient p je konstantní funkce, což je omezení pro výsledky tohoto typu obvyklé v existující literatuře. V práci ***A periodic boundary value problem for functional-differential equations of higher order. Georgian Math. J., Vol. 16, 2009, No. 4, 651–665*** **(spoluautor R. Hakl)**, v níž jsou nalezena efektivní kritéria jednoznačné řešitelnosti periodické úlohy pro funkcionálně-diferenciální rovnice vyšších řádu s nemonotonními operátory na pravé straně rovnice, které zlepšují dřívější výsledky autorů Lasota–Opial a Kiguradze–Kusano, a které jsou nezlepšitelné pro rovnice řádu $n \leq 7$. Metoda, která je zde použita k důkazu hlavních tvrzení, je vyvinuta v několika předešlých publikacích. V monografii ***Two-point boundary value problems for second order functional differential equations. Mem. Differential Equations Math. Phys., 20, 2000, 1–112*** je studována Dirichletova a smíšená okrajová úloha pro lineární singulární funkcionálně-diferenciální rovnice druhého řádu. V první kapitole jsou uvedeny postačující podmínky zaručující jednoznačnou řešitelnost daných úloh a je ukázáno, že některé z efektivních podmínek jsou v jistém smyslu nezlepšitelné. V druhé kapitole jsou pak dokázány věty o korektnosti daných úloh. V článku ***Two-point boundary value problems for strongly singular higher-order linear differential equations with deviating arguments. E. J. Qualitative Theory of Diff. Equ., 2012, No. 38, 1–34 (spoluaturka N. Partsvania)*** jsou dokázána tvrzení typu Agawala–Kiguradzeho pro dvoubodové a fokální úlohy pro silně singulární diferenciální rovnice vyšších řádů s odkloněnými argumenty. Tato tvrzení obsahují postačující podmínky zaručující, že studované úlohy mají tzv. Fredholmovu vlastnost. Dále jsou v tomto článku nalezeny efektivní nezlepšitelná kritéria jednoznačné řešitelnosti těchto lineárních úloh. Je známo, že máme-li prostudovanou otázku jednoznačné řešitelnosti dvoubodových okrajových úloh pro lineární diferenciální rovnice, je možné odvodit kritéria řešitelnosti nelineárních úloh, v nichž jsou nelineární rovnice v jistém smyslu „blízké“ odpovídajícím rovnicím lineárním. Výsledky tohoto typu pro nelineární funkcionálně-diferenciální rovnice jsou prezentovány v práci ***The Dirichlet boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Czechoslovak Mathematical Journal, Vol. 63, 2013, No. 1, 235–263***. Pomocí výsledků známých v lineárním případě jsou zde odvozeny efektivní postačující podmínky, zaručující jednoznačnou řešitelnost Dirichletovy úlohy pro silně singulární nelineární funkcionálně-diferenciální rovnice vyšších řádů.

Pro lepší přehlednost a čitelnost textu je na začátku každého oddílu uvedeno označení, které je v něm pak jednotně používáno.

INTRODUCTION

In the present survey I review my several studies exploring solvability, unique solvability and correctness of some boundary value problems for functional-differential equations. First I will briefly characterize five selected studies.

In the ***article The Dirichlet BVP The second Order Nonlinear Ordinary Differential Equation At Resonance. Italian J. Of Pure and Appl. Math., 2011, No. 28, 177-204***, in difference with the previous paper, the nonlinear equation $u''(t) = p(t)u(t) + f(t, u(t)) + h(t)$ under Dirichlet boundary value problem conditions is studied in the case when f is sublinear function in the second argument and the homogeneous linear equation $u''(t) = p(t)u(t)$ under homogeneous Dirichlet boundary value conditions has a nontrivial solution, i.e. in the resonance case. It is noteworthy that unlike this article, similar problems are studied in literature only in the concrete case when $p \equiv Const$. In the paper ***A Periodic Boundary Value Problem For Functional-Differential Equations Of Higher Order (with R. Hakl). Georgian Math. J. Vol.16, (2009), No.4, 651-665***, the efficient sufficient conditions guaranteeing the unique solvability of the periodic problem are established in the case of nonmonotone linear operators, which improve the results of Lasota - Opial and Kiguradze-Kusano and are optimal for $n \leq 7$. The method used for the investigation of the considered problem is based on the method developed in my previous papers. In the monograph ***Two-point boundary value problems for second order functional differential equations. Mem. Differential Equations Math. Phys. 20 (2000), 1-112***, the Dirichlet and mixed problems for second order linear singular functional-differential equations are studied. In the first chapter the sufficient conditions of unique solvability of the named problem are established and some of them are sharp in some sense. The correctness of the above mentioned problem is studied in the second chapter of the monograph. In the paper ***Two-point boundary value problems for strongly singular higher-order linear differential equations with deviating arguments (with N. Partsvania). E. J. Qualitative Theory of Diff. Equ., 2012, No.38, 1-34***, for strongly singular higher-order differential equations with deviating arguments, under two point conjugated and right-focal boundary conditions, Agarval-Kiguradze type theorems are established, which guarantee the presence of Fredholm's property for the above mentioned problems. Also we provide easily verifiable best possible conditions that guarantee the existence of a unique solution of the studied problems. As is known, if we have studied the unique solvability of the linear functional-differential equations under some two-point boundary value problem, it simplifies study of the question of solvability of the same two-point boundary value problem for nonlinear functional-differential equations if the nonlinear equation is in a some sense "close" to this linear equation. Results of this type for the nonlinear functional-differential equations are presented in the study ***The Dirichlet Boundary Value Problems For Strongly Singular Higher-Order Nonlinear Functional-Differential Equations. Czechoslovak Mathematical Journal, vol. 63 (2013), No. 1, pp. 235-263***, where by using the results proved for the linear equations, the efficient sufficient conditions guaranteeing the unique solvability for Dirichlet problem are established for the strongly singular higher-order nonlinear functional differential equations.

The notation used in the survey is introduced separately for every single section at its beginning.

Kapitola 1

Two-point boundary value problems for regular functional-differential equations

1.1 The Dirichlet BVP For The Second Order Nonlinear Ordinary Differential Equation At Resonance

In this chapter first we consider the paper [4], in which on the set $I = [a,\ b]$ where the second order ordinary differential equation

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \tag{1.1.1}$$

are studied under the boundary conditions

$$u(a) = 0, \quad u(b) = 0, \tag{1.1.2}$$

where $h, p \in L([a, b])$ and $f \in K(I \times R; R)$.

By a solution of the problem (1.1.1), (1.1.2) we understand a function $u \in \widetilde{C}'([a, b])$, which satisfies the equation (1.1.1) almost everywhere on I and satisfies the conditions (1.1.2).

Along with (1.1.1), (1.1.2) we consider the homogeneous problem

$$w''(t) = p(t)w(t) \quad \text{for} \quad t \in I, \tag{1.1.3}$$

$$w(a) = 0, \quad w(b) = 0. \tag{1.1.4}$$

The case when the problem (1.1.3), (1.1.4) has the nontrivial solution is still little investigated and in the majority of articles, the authors study the case with p constant in the equation (1.1.1), i.e., when the problem (1.1.1), (1.1.2) and the equation (1.1.3) are of type

$$u''(t) = -\lambda^2 u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in [0, \pi], \tag{1.1.5}$$

$$u(0) = 0, \quad u(\pi) = 0, \tag{1.1.6}$$

and

$$w''(t) = -\lambda^2 w(t) \quad \text{for} \quad t \in [0, \pi] \tag{1.1.7}$$

respectively, with $\lambda = 1$.

In this work, the solvability of the problem (1.1.1), (1.1.2) is studied in the case when the function $p \in L([a, b])$ is not necessarily constant, under the assumption that the homogeneous problem (1.1.3), (1.1.4) has the nontrivial solution with an arbitrary number of zeroes. For the equation (1.1.7), this is the case when λ is not necessarily the first eigenvalue of the problem (1.1.7), (1.1.4), with $a = 0,\ b = \pi$.

Throughout the paper the following notation are used:

$K(I \times R; R)$ is the set of functions $f : I \times R \to R$ satisfying the Carathéodory conditions. Also having the function $w : I \to R$, we put:

$$N_w \overset{def}{=} \{t \in]a, b[: w(t) = 0\},$$
$$\Omega_w^+ \overset{def}{=} \{t \in I : w(t) > 0\}, \quad \Omega_w^- \overset{def}{=} \{t \in I : w(t) < 0\}.$$

Also, to formulate the main results of this paper we need the following definitions:

Definition 1.1.1. Let A be a finite (eventually empty) subset of I. We say that $f \in E(A)$, if $f \in K(I \times R; R)$ and, for any measurable set $G \subseteq I$ and an arbitrary constant $r > 0$, we can choose $\varepsilon > 0$ such that if

$$\int_G |f(s, x)| ds \neq 0 \ \text{ for } \ x \geq r \ (x \leq -r)$$

then

$$\int_{G \setminus U_\varepsilon} |f(s, x)| ds - \int_{U_\varepsilon} |f(s, x)| ds \geq 0 \quad \text{for} \quad x \geq r \ (x \leq -r),$$

where $U_\varepsilon = I \cap \Big(\cup_{k=1}^n]t_k - \varepsilon/2n,\ t_k + \varepsilon/2n[\Big)$ if $A = \{t_1, t_2, ..., t_n\}$, and $U_\varepsilon = \emptyset$ if $A = \emptyset$.

Remark 1.1.1. It is clear that if $f(t,x) \overset{def}{\equiv} f_0(t)g_0(x)$, where $f_0 \in L([a,b])$ and $g_0 \in C(R)$, then $f \in E(A)$ for every finite set $A \subset I$.

Now we can consider the main result of our paper. The first theorem deals with a case when $N_w = \emptyset$, which for problem (1.1.7),(1.1.6) corresponds to the case $\lambda = 1$.

Theorem 1.1.1. *Let w be a nonzero solution of the problem* (1.1.3), (1.1.4),

$$N_w = \emptyset, \tag{1.1.8}$$

there exist a constant $r > 0$, nonnegative functions $f^-, f^+ \in L([a,b])$ and $g, h_0 \in L(I;]0,+\infty[\,)$ such that

$$f(t,x)\text{sgn}x \le g(t)|x| + h_0(t) \quad \textit{for} \quad |x| \ge r \tag{1.1.9}$$

and

$$\begin{aligned} f(t,x) \le -f^-(t) \quad \textit{for} \quad x \le -r, \\ f^+(t) \le f(t,x) \quad \textit{for} \quad x \ge r \end{aligned} \tag{1.1.10}$$

on I. Let, moreover, there exist $\varepsilon > 0$ such that

$$-\int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r||_L \le -\int_a^b h(s)|w(s)|ds \le$$

$$\le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \tag{1.1.11$_1$}$$

where

$$\gamma_r(t) = \sup\{|f(t,x)| : |x| \le r\}. \tag{1.1.12}$$

Then the problem (1.1.1), (1.1.2) *has at least one solution.*

Example 1.1.1. It follows from Theorem 1.1.1 that the equation

$$u''(t) = -\lambda^2 u(t) + \sigma|u(t)|^\alpha \text{sgn}u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi \tag{1.1.13}$$

where $\sigma = 1$, $\lambda = 1$, and $\alpha \in]0,1]$, with the conditions (1.1.6) has at least one solution for every $h \in L([0,\pi])$.

And finally let us consider two theorems for the case when N_w is not necessarily empty set, where in the second theorem we assume, that for the function f the representation $f(t,x) = f_0(t)g_0(x)$ is valid.

Theorem 1.1.2. *Let $i \in \{0,1\}$, w be a nonzero solution of the problem* (1.1.3), (1.1.4)*, $f \in E(N_w)$, there exist a constant $r > 0$ such that the function $(-1)^i f$ is non-decreasing in the second argument for $|x| \geq r$,*

$$(-1)^i f(t,x)\text{sgn}x \geq 0 \quad \textit{for} \quad t \in I, \ |x| \geq r, \tag{1.1.14}$$

$$\int_{\Omega_w^+} |f(s,r)|ds + \int_{\Omega_w^-} |f(s,-r)|ds \neq 0, \tag{1.1.15}$$

and

$$\lim_{|x| \to +\infty} \frac{1}{|x|} \int_a^b |f(s,x)|ds = 0. \tag{1.1.16}$$

Then there exists $\delta > 0$ such that the problem (1.1.1), (1.1.2) *has at least one solution for every h satisfying the condition*

$$\left| \int_a^b h(s)w(s)ds \right| < \delta. \tag{1.1.17}$$

Example 1.1.2. From Theorem 1.1.2 it follows that the problem (1.1.13), (1.1.6) with $\sigma \in \{-1,1\}$, $\lambda \in N$, and $\alpha \in]0,1[$ has at least one solution if $h \in L([0,\pi])$ is such that $\int_0^\pi h(s)\sin\lambda s ds = 0$.

Theorem 1.1.3. *Let $i \in \{0,1\}$, w be a nonzero solution of the problem* (1.1.3),(1.1.4)*, $f(t,x) \overset{def}{\equiv} f_0(t)g_0(x)$ with nonnegative $f_0 \in L([a,b])$, $g_0 \in C(R)$, there exist a constant $r > 0$ such that $(-1)^i g_0$ is non-decreasing for $|x| \geq r$ and*

$$(-1)^i g_0(x)\text{sgn}x \geq 0 \qquad \textit{for} \quad |x| \geq r. \tag{1.1.18}$$

Let, moreover,

$$|g_0(r)| \int_{\Omega_w^+} f_0(s)ds + |g_0(-r)| \int_{\Omega_w^-} f_0(s)ds \neq 0 \tag{1.1.19}$$

and

$$\lim_{|x| \to +\infty} |g_0(x)| = +\infty, \qquad \lim_{|x| \to +\infty} \frac{g_0(x)}{x} = 0. \tag{1.1.20}$$

Then, for every $h \in L(I;R)$, the problem (1.1.1), (1.1.2) *has at least one solution.*

Example 1.1.3. From Theorem 1.1.3 it follows that the equation

$$u''(t) = p_0(t)u(t) + p_1(t)|u(t)|^\alpha \text{sgn}u(t) + h(t) \quad \text{for} \quad t \in I, \tag{1.1.21}$$

where $\alpha \in]0,1[$ and $p_0, p_1, h \in L([a,b])$, with the conditions (1.1.2) has at least one solution provided that $p_1(t) > 0$ for $t \in I$.

1.2 A Periodic Boundary Value Problem For Functional Differential Equations Of Higher Order

One of the most significant problem among two point boundary value problems is the periodic problem. In the paper [5] the problem of existence and uniqueness of solution is studied for the higher-order linear functional-differential equation

$$u^{(n)}(t) = \sum_{i=0}^{n-1} \ell(u^{(i)})(t) + q(t) \tag{1.2.1}$$

under the periodic boundary conditions

$$u^{(j)}(0) = u^{(j)}(\omega) + c_j \qquad (j = 0, \dots, n-1), \tag{1.2.2}$$

where $n \geq 2$, $\ell : C([0,\ \omega]) \to L([0,\ \omega])$ are linear bounded operators, $q \in L([0,\ \omega])$, and $c_j \in R$ $(i, j = 0, \dots, n-1)$.

By a solution to the problem (1.2.1), (1.2.2) we understand a function $u \in \widetilde{C}^{n-1}([0,\ \omega])$, which satisfies the equality (1.2.1) almost everywhere in $[0, \omega]$ and the boundary condition (1.2.2).

The problem on the existence of a periodic solution to ordinary and functional differential equations was studied very intensively in the past. The first important step was made for linear ordinary differential equations of the type

$$u^{(n)}(t) = p(t)u(t) + q(t) \tag{1.2.3}$$

by Lasota and Opial. They showed that the problem (1.2.3), (1.2.2) is uniquely solvable for $n \geq 4$ if the function $p \in L([0,\ \omega])$ has a constant sign, $p \not\equiv 0$, and the inequality

$$\int_o^{\omega} |p(s)|ds < \left(\frac{2}{\omega}\right)^{n-1} \frac{2 \cdot 4 \cdots (n-2)}{1 \cdot 3 \cdots (n-3)} \tag{1.2.4}$$

is fulfilled. This result is far from being optimal

Below we consider conditions guaranteeing the unique solvability of the problem (1.2.1), (1.2.2), even in case when the operators ℓ_i are not monotone, which improve the results of Lasota – Opial and Kiguradze – Kusano and are optimal for $n \leq 7$. The method used for the investigation of the considered problem is based on the method developed in our previous papers for functional differential equations.

Definition 1.2.1. We will say that a linear operator $\ell : C([0,\ \omega]) \to L([0,\ \omega])$ belongs to the set P_ω if it is *non–negative*, i.e., for any non–negative $x \in C([0,\ \omega])$ the inequality $\ell(x)(t) \geq 0$ for $t \in [0,\ \omega]$ is fulfilled.

The following notations is used throughout this part of our survey:
N is a set of all natural numbers.
If $\ell : C([0,\ \omega]) \to L([0,\ \omega])$ is a linear bounded operator, then

$$\|\ell\| = \sup_{\|x\|_C \leq 1} \|\ell(x)\|_L .$$

$$A_0 = 1, \quad A_1 = \frac{1}{15}, \quad A_j = A_1 \sum_{m_1=1}^{2} \sum_{m_2=1}^{m_1+1} \dots \sum_{m_{j-1}=1}^{m_{j-2}+1} \frac{1}{\eta(m_1)\dots\eta(m_{j-1})} ,$$

$$B_1 = \frac{1}{8}, \quad B_j = A_1 \sum_{m_1=1}^{2} \sum_{m_2=1}^{m_1+1} \dots \sum_{m_{j-1}=1}^{m_{j-2}+1} \frac{1}{\eta(m_1)\dots\eta(m_{j-1})} \prod_{i=1}^{m_{j-1}+1} \left(1 + \frac{1}{2i}\right),$$

for $j \geq 2$, where

$$\eta(t) = (2t+1)(2t+3).$$

Let

$$d_0 = 1, \qquad d_1 = 4, \qquad d_2 = 32, \qquad d_3 = 192, \tag{1.2.5}$$

and for $p \in N$ put

$$\begin{gathered} d_{2p+2} = \frac{1}{\max\left\{(h_p(t)h_p(1-t))^{1/2} : 0 \leq t \leq 1\right\}} , \\ d_{2p+3} = \frac{1}{\max\left\{(f_p(s,t)f_p(1-s,1-t))^{1/2} : 0 \leq s \leq 1,\ 0 \leq t \leq 1\right\}} , \end{gathered} \tag{1.2.6}$$

where the functions $f_p : [0,1] \times [0,1] \to R_+$, $h_p : [0,1] \to R_+$ are defined as follows:

$$f_p(s,t) = \sum_{j=0}^{p-1} \alpha_{pj} t^{2(j+1)} + \alpha_{pp} t^{2p+3} s, \qquad h_p(t) = \sum_{j=0}^{p} \beta_{pj} t^{2(j+1)}, \tag{1.2.7}$$

and

$$\begin{gathered} \alpha_{pj} = \frac{A_j}{3 \cdot 4^{j+1} d_{2(p-j)+1}} , \qquad \beta_{pj} = \frac{A_j}{3 \cdot 4^{j+1} d_{2(p-j)}} \qquad (j = 0, \dots, p-1), \\ \alpha_{pp} = \frac{A_p}{3 \cdot 4^{p+1}} , \qquad \beta_{pp} = \frac{B_p}{3 \cdot 4^{p+1}} . \end{gathered} \tag{1.2.8}$$

Now we can formulate our main theorem on unique solvability of problem (1.2.1), (1.2.2).

Theorem 1.2.1. *Let $j \in \{0, 1\}$, the operator ℓ_0 admit the representation $\ell_0 = \ell_{0,1} - \ell_{0,2}$, where $\ell_{0,1}, \ell_{0,2} \in P_\omega$, and let ℓ_i $(i = 1, \dots, n-1)$ be bounded linear operators. Let, moreover, the conditions*

$$||\ell_{0,1}|| + ||\ell_{0,2}|| \neq 0 \tag{1.2.9}$$

$$\frac{\omega^{n-1}}{d_{n-1}}||\ell_{0,1+j}|| + \Omega < 1, \tag{1.2.10}$$

$$\frac{||\ell_{0,1+j}||}{1 - \Omega - \frac{\omega^{n-1}}{d_{n-1}}||\ell_{0,1+j}||} \leq ||\ell_{0,2-j}||, \tag{1.2.11}$$

$$||\ell_{0,2-j}|| \leq \frac{2d_{n-1}}{\omega^{n-1}}\left(1 - \Omega + \sqrt{(1-\Omega)\left(1 - \Omega - \frac{\omega^{n-1}}{d_{n-1}}||\ell_{0,1+j}||\right)}\right) \tag{1.2.12}$$

hold with

$$\Omega = \sum_{i=1}^{n-1} \frac{\omega^{n-1-i}}{d_{n-1-i}}||\ell_i|| \tag{1.2.13}$$

and d_i $(i = 0, \dots, n-1)$ be defined by (1.2.5)–(1.2.8)*. Then the problem* (1.2.1), (1.2.2) *has a unique solution.*

In the case when the operator ℓ_0 is monotone from our theorem it follows:

Corollary 1.2.1. *Let $\sigma \in \{-1, 1\}$ and $\sigma\ell_0 \in P_\omega$. Let, moreover, the conditions*

$$\|\ell_0\| \neq 0, \tag{1.2.14}$$

$$\sum_{i=1}^{n-1} \frac{\omega^{n-1-i}}{d_{n-1-i}}\|\ell_i\| < 1, \tag{1.2.15}$$

and

$$\|\ell_0\| \leq \frac{4d_{n-1}}{\omega^{n-1}}\left(1 - \sum_{i=1}^{n-1} \frac{\omega^{n-1-i}}{d_{n-1-i}}\|\ell_i\|\right) \tag{1.2.16}$$

hold. Then the problem (1.2.1), (1.2.2) *has a unique solution.*

To illustrate our theorem, we consider also one corollary for the equation

$$u^{(n)}(t) = \ell_0(u)(t) + q(t). \tag{1.2.17}$$

Corollary 1.2.2. *Let* $\sigma \in \{-1, 1\}$, $\sigma\ell_0 \in C([0,\ \omega])$. *Let, moreover, the conditions*

$$\|\ell_0\| \neq 0, \tag{1.2.18}$$

and

$$\|\ell_0\| \leq \frac{4d_{n-1}}{\omega^{n-1}} \tag{1.2.19}$$

hold. Then the problem (1.2.17), (1.2.2) *has a unique solution.*

Kapitola 2

Two-Point Boundary Value Problems For Singular Functional-Differential Equations

2.1 Two-point boundary value problems for second order functional-differential equations

First we consider some results from the monograph [1], where the second order linear singular functional-differential equation

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + g(u)(t) + p_2(t) \tag{2.1.1}$$

is studied under the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2 \tag{2.1.2}$$

or

$$u(a) = c_1, \quad u(b) = c_2, \tag{2.1.3}$$

and separately for the case of homogeneous conditions

$$u(a) = 0, \quad u(b) = 0, \tag{2.1.4}$$

$$u(a) = 0, \quad u(b) = 0, \tag{2.1.5}$$

where $c_1, c_2 \in R\, p_j \in L_{loc}(]a,\ b[)$ $(j = 0, 1, 2)$ and $g : C(]a,\ b[) \to L_{loc}(]a,\ b[)$ is a continuous linear operator. In this short survey we consider only four theorems and its corollaries about unique solvability of problems (2.1.1), (2.1.2), and (2.1.1), (2.1.4), from twelve theorems and its corollaries proved in this monograph. We do not consider problems (2.1.1), (2.1.3), and (2.1.1), (2.1.5), and theorems on the correctness of the above mentioned problems.

Throughout the work the following notations are used:

$[x]_+ = \frac{1}{2}\big(|x| + x\big)$, $[x]_- = \frac{1}{2}\big(|x| - x\big)$.

$C(]a,\ b[)$ is the space of continuous and bounded functions $u :]a,\ b[\to R$ with the norm

$$||u||_C = \sup\{|u(t)| : a < t < b\};$$

$\widetilde{C}(]a,\ b[)$ is the set of functions $u :]a,\ b[\to R$ absolutely continuous on each subsegment of $]a,\ b[$.

$\widetilde{C}'(]a,\ b[)$ is the set of functions $u :]a,\ b[\to R$ absolutely continuous on each subsegment of $]a,\ b[$, along with their first order derivative.

$L([a,\ b])$ is the space of summable functions $u : [a,\ b] \to R$ with the norm

$$||u||_L = \int\limits_a^b |u(t)|dt.$$

$L_{+\infty}([a,\ b])$ is the space of essentially bounded functions $u : [a,\ b] \to R$ with the norm

$$||u||_{+\infty} = essup\{|u(t)| : t \in [a,\ b]\}.$$

$L_{loc}(]a,\ b[)$ is the set of measurable functions $u : [a,\ b] \to R$, summable on each subsegment of $]a, b[$.

Let $x, y :]a,\ b[\to]0,\ +\infty[$ be continuous functions.

$C_x(]a,\ b[)$ is the space of continuous $u \in C(]a,\ b[)$ such that

$$||u||_{C,x} = \sup\left\{\frac{|u(t)|}{x(t)} : a < t < b\right\} < +\infty;$$

$L_y(]a,\ b[)$ is the space of functions $u \in L_{loc}(]a,\ b[)$ such that

$$||u||_{L,y} = \int\limits_a^b y(t)|u(t)|dt < +\infty;$$

$\mathcal{L}(C_x,\ L_x)$ is the set of linear operators $h : C_x(]a,\ b[\to L_y(]a,\ b[)$ such that

$$\sup\{|h(x)(\cdot)| : ||u||_{C,x} \le 1\} \in L_y(]a,\ b[);$$

$\sigma : L_{loc}(]a,\ b[) \to \widetilde{C}(]a,\ b[)$ is the operator defined by

$$\sigma(p)(t) = \exp\Big(\int\limits_{\frac{a+b}{2}}^{t} p(s)ds \Big) \quad \text{for} \quad a \le t \le b.$$

If $\sigma(p) \in L([a,\ b])$, then we define the operators σ_1 and σ_2 by

$$\sigma_1(p)(t) = \frac{1}{\sigma(p)(t)} \int\limits_a^t \sigma(p)(s)ds \int\limits_t^b \sigma(p)(s)ds$$

$$\sigma_2(p)(t) = \frac{1}{\sigma(p)(t)} \int\limits_a^t \sigma(p)(s)ds \quad \text{for} \quad a \le t \le b$$

Let, $f, g \in C(]a,\ b[)$ and $c \in [a,\ b]$, than we write

$$f(t) = O(g(t)) \qquad (f(t) = O^*(g(t))) \quad \text{as} \quad t \to c,$$

if

$$\limsup_{t\to c} \frac{|f(t)|}{|g(t)|} < +\infty \quad \Big(0 < \liminf_{t\to c} \frac{|f(t)|}{|g(t)|} \ \text{ and } \ \limsup_{t\to c} \frac{|f(t)|}{|g(t)|} < +\infty\Big).$$

Now note that the problems (2.1.1), (2.1.2), and (2.1.1), (2.1.4) are studied under the assumptions

$$\begin{gathered} p_j \in L_{loc}(]a,\ b[)\ (j = 0, 1, 2), \\ \sigma(p_1) \in L([a,\ b]), \quad p_0 \in L_{\sigma_1(p_1)}([a,\ b]), \end{gathered} \tag{2.1.6}$$

by the method of Nagumo's upper and lower functions, and we find the conditions under which Fredholm's alternative is valid, introduce the sets of nonoscillation $\mathbb{V}_{i,0}$, and describe sets of two-dimensional vector functions $(p_0, p_1) :]a,\ b[\to R^2$, and linear operators h, for which our problem is uniquely solvable.

Definition 2.1.1. We will say that $w \in C(]a,\ b[)$ is the lower (upper) function of the problem (2.1.1), (2.1.2) if:

(a) w' is of the form $w'(t) = w_0(t) + w_1(t)$, where $w_0 :]a,\ b[\to R$ is absolutely continuous on each segment from $]a,\ b[$, the function $w_1 :]a,\ b[\to R$ is nondecreasing (nonincreasing) and its derivative is almost everiwhere equal to zero;

(b) almost everywhere on $]a,\ b[$ the inequality

$$w''(t) \geq p_0(t)w(t) + p_1(t)w'(t) + g(w)(t) + p_2(t)$$
$$(w''(t) \leq p_0(t)w(t) + p_1(t)w'(t) + g(w)(t) + p_2(t))$$

is satisfaed;

(c) there exists the limit $w'(b-)$ and

$$w(a) \leq c_1,\ \ w(b-0) \leq c_2 \qquad (w(a) \geq c_1,\ \ w(b-0) \geq c_2).$$

Definition 2.1.2. We will say that two-dimensional vector function $(p_0, p_1) :]a,\ b[\to R^2$ belongs to the set $\mathbb{V}_{1,0}(]a,\ b[)$ if the conditions (2.1.6) are fulfilled, the solution of the problem

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t), \tag{2.1.7}$$
$$u(a) = 0,\ \ \lim_{t\to a} \frac{u'(t)}{\sigma(p_1)(t)} = 1,$$

has no zeros in the interval $]a,\ b[$ and $u(b-) \geq 0$.

Definition 2.1.3. Let $h : C(]a,\ b[) \to L_{loc}(]a,\ b[)$ be a continuous linear operator. We will say that a two-dimensional vector function $(p_0,\ p_1) :]a,\ b[\to R^2$ belong to the set $\mathbb{V}_{1,0}(]a,\ b[,\ h)$ if the conditions (2.1.6) are satisfied and the problem

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) - h(u)(t)$$
$$u(a) = 0, \qquad u(b-) = 0$$

has a positive upper function w on the segment $[a,\ b]$.

Definition 2.1.4. Let $h : C(]a,\ b[) \to L_{loc}(]a,\ b[)$ be a continuous linear operator. We will say that a two-dimensional vector function $(p_0,\ p_1) :]a,\ b[\to R^2$ belong to the set $\mathbb{V}_{1,\beta}(]a,\ b[,\ h)$ if

$$(p_0,\ p_1) \in \mathbb{V}_{1,0}(]a,\ b[,\ h)$$

and there exists a measurable function $q_\beta :]a,\ b[\to [0,\ +\infty[$ such that

$$\int_a^b |G(t,\ s)|q_\beta(s)ds = O^*(x^\beta(t))$$

as $t \to a,\ b \to b$, where G is Green's function of the problem (2.1.7), (2.1.4), and

$$x(t) = \int_a^t \sigma(p_1)(s)ds \int_t^b \sigma(p_1)(s)ds \tag{2.1.8}$$

for $a \leq t \leq b$.

Now we can consider some basic results of our monograph.

Theorem 2.1.1. *Let*

$$p_2 \in L_{\sigma_1(p_1)}([a,\ b]) \tag{2.1.9}$$

and the constants $\alpha,\ \beta \in [0,\ 1]$ *connected by the inequality*

$$\alpha + \beta \le 1 \tag{2.1.10}$$

be such that

$$(p_0,\ p_1) \in \mathbb{V}_{1,\beta}(]a,\ b[,\ h), \tag{2.1.11}$$

where

$$h \in \mathcal{L}(C_{x^\beta},\ L_{\frac{x^\alpha}{\sigma(p_1)}}) \cap \mathcal{L}(C,\ L_{\sigma_1(p_1)}) \tag{2.1.12}$$

is a nonnegative operator and the function x *is defined by* (2.1.8).

Let moreover a continuous linear operator $g : C(]a,\ b[) \to L_{\sigma_1(p_1)}([a,\ b])$ *be such that for any function* $u \in C(]a,\ b[)$ *almost everywhere in the interval* $]a,\ b[$ *the inequality*

$$|g(u)(t)| \le h(|u|)(t) \tag{2.1.13}$$

is satisfied. Then the problem (2.1.1), (2.1.2) *has one and only one solution.*

From this theorem follows a few efficient sufficient conditions of unique solvability. Let us consider one of these corollaries:

Corollary 2.1.1. *Let the function* x *is defined by* (2.1.8), *the constants* $\alpha,\ \beta \in [0,\ 1]$ *by connected by* (2.1.10), *the function* $p_j :]a,\ b[\to R$ $(j = 0, 1, 2)$ *satisfy conditions* (2.1.6), (2.1.9),

$$[p_0]_- \in L_{\frac{x^\alpha}{\sigma(p_1)}}([a,\ b]), \tag{2.1.14}$$

and for any function $u \in C(]a,\ b[)$ *almost everywhere in the interval* $]a,\ b[$ *the inequality* (2.1.13) *be satisfied, where a nonnegative operator* h *satisfies* (2.1.12).

Let moreover,

$$\Big[\Big(\int\limits_t^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha}\int\limits_a^t \frac{([p_0(s)]_- x^{\beta}(s)+h(x^{\beta})(s))}{\sigma(p_1)(s)}\Big(\int\limits_a^s \sigma(p_1)(\eta)d\eta\Big)^{\alpha}ds+$$
$$\Big(\int\limits_a^t \sigma(p_1)(\eta)d\eta\Big)^{\alpha}\int\limits_t^b \frac{([p_0(s)]_- x^{\beta}(s)+h(x^{\beta})(s))}{\sigma(p_1)(s)}\Big(\int\limits_s^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha}ds\Big] <$$
$$< \frac{4}{\int\limits_a^b \sigma(p_1)(\eta)d\eta}\Big(\frac{\int\limits_a^b \sigma(p_1)(\eta)d\eta}{2}\Big)^{2(\alpha+\beta)}. \tag{2.1.15}$$

Then the problem (2.1.1), (2.1.2) *has one and only one solution.*

The next theorem shows us that in the case of boundary conditions (2.1.4) the singularity of functions p_0, p_1 and operator h, can be stronger as in the case of the boundary conditions (2.1.2).

Theorem 2.1.2. *Let the constants* $\alpha,\ \beta \in [0,\ 1]$ *connected by the inequality* (2.1.10) *be such that*

$$p_2 \in L_{\frac{x^{1-\beta}}{\sigma(p_1)}}([a,\ b]) \tag{2.1.16}$$

and the functions $p_0, p_1 :]a,\ b[\to R$ *satisfy the inclusion* (2.1.11), *where*

$$h \in \mathcal{L}(C_{x^{\beta}},\ L_{\frac{x^{\alpha}}{\sigma(p_1)}}) \tag{2.1.17}$$

is a nonnegative operator and the function x *is defined by* (2.1.8).

Let moreover a continuous linear operator $g : C_{x^{\beta}}(]a,\ b[) \to L_{\frac{x^{\alpha}}{\sigma(p_1)}}([a,\ b])$ *be such that for any function* $u \in C_{x^{\beta}}]a,\ b[)$ *almost everywhere in the interval* $]a,\ b[$ *the inequality* (2.1.13) *is satisfied. Then the problem* (2.1.1), (2.1.4) *has one and only one solution in the space* $C_{x^{\beta}}(]a,\ b[)$.

Corollary 2.1.2. *Let the function* x *is defined by* (2.1.8), *the constants* $\alpha,\ \beta \in [0,\ 1]$ *by connected by* (2.1.10), *the function* $p_j :]a,\ b[\to R$ $(j = 0, 1, 2)$ *satisfy conditions* (2.1.6), (2.1.16),(2.1.14) *and for any function* $u \in C_{x^{\beta}}(]a,\ b[)$ *almost everywhere in the interval* $]a,\ b[$ *the inequality* (2.1.13) *be satisfied, where the nonnegative operator* h *satisfies the inclusion* (2.1.17). *Let moreover* (2.1.15) *be satisfied. Then the problem* (2.1.1), (2.1.4) *has one and only one solution in the space* $C_{x^{\beta}}(]a,\ b[)$.

For clearness we will give two corollaries for the equation

$$u''(t) = g_0(t)u(\tau(t)) + p_2(t), \tag{2.1.18}$$

the first in the case of boundary conditions (2.1.2), and the second in the case of (2.1.4).

Corollary 2.1.3. *Let the constants* $\alpha,\ \beta \in [0,\ 1]$ *by connected by* (2.1.10), $\tau : [a,\ b] \to [a,\ b]$ *be a measurable function and*

$$g_0, p_2 \in L_x([a,\ b]), \tag{2.1.19}$$

where

$$x(t) = (t-a)(b-t) \quad \text{for} \quad a \le t \le b. \tag{2.1.20}$$

Let, moreover,

$$\begin{gathered}\int\limits_a^b |g_0(s)|[(\tau(s)-a)(b-\tau(s))]^\beta[(s-a)(b-s)]^\alpha ds < \\ < 4^{1-\alpha-\beta}(b-a)^{2(\alpha+\beta)-1}.\end{gathered} \tag{2.1.21}$$

Then the problem (2.1.18), (2.1.2) *has one and only one solution.*

Corollary 2.1.4. *Let the constants* $\alpha,\ \beta \in [0,\ 1]$ *by connected by* (2.1.10), $\tau : [a,\ b] \to [a,\ b]$ *be a measurable function and*

$$p_2 \in L_{x^{1-\beta}}([a,\ b]), \tag{2.1.22}$$

where the function x *be defined by* (2.1.20). *Let, moreover condition* (2.1.21) *be satisfied. Then the problem* (2.1.18), (2.1.4) *has one and only one solution in the space* $C_{x^\beta}(]a,\ b[)$.

In the monograph a different method of study of our boundary value problems is also developed, the method of minimums and maximums. The next two nonimprovable theorems are proved by this method

Theorem 2.1.3. *Let* $\gamma \in [0,\ 1]$

$$p_2 \in L_x([a,\ b]) \tag{2.1.23}$$

and

$$g \in \mathcal{L}(C,\ L_{x^\gamma}) \tag{2.1.24}$$

be a nonnegative operator, where the function x *is defined by* (2.1.20).

Let, moreover, there exist constants $\alpha, \beta \in [0,\ 1/2]$ *such that*

$$0 \le \beta < 1 - \gamma, \tag{2.1.25}$$

$$\alpha + \beta \le 1/2 \tag{2.1.26}$$

and

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s) ds < 2^\beta \frac{16}{b-a} \Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \tag{2.1.27}$$

Then the problem (2.1.18), (2.1.2) *has one and only one solution.*

Theorem 2.1.4. *Let* $\gamma \in [0,\ 1],\ \delta \in]0, 1-\gamma[$,

$$p_2 \in L_{x^\gamma}([a,\ b]) \tag{2.1.28}$$

and

$$g \in \mathcal{L}(C_{x^\delta},\ L_{x^\gamma}) \tag{2.1.29}$$

be a nonnegative operator, where the function x is defined by (2.1.20).

Let, moreover, there exist constants $\alpha \in [0,\ 1/2], \beta \in]0,\ 1/2]$ *such that the conditions* (2.1.26) *and*

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s) ds \le 2^\beta \frac{16}{b-a} \Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)} \tag{2.1.30}$$

are satisfied. Then the problem (2.1.18), (2.1.4) *has one and only one solution in the space* $C_{x^\delta}(]a,\ b[)$.

The conditions (2.1.27) and (2.1.30) are unimprovable in the sense that it cannot be replaced by the conditions

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s) ds < 2^\beta \frac{16}{b-a} \Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)} + \varepsilon$$

$$\Big(\int\limits_a^b x^\alpha(s) g(x^\beta)(s) ds \le 2^\beta \frac{16}{b-a} \Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)} + \varepsilon\Big).$$

2.2 Two-point boundary value problems for strongly singular higher-order linear differential equations with deviating arguments

In this section we consider the main results from the paper [2], where the differential equation with deviating arguments

$$u^{(n)}(t)=\sum_{j=1}^{m}p_j(t)u^{(j-1)}(\tau_j(t))+q(t)\quad \text{for}\quad a<t<b, \tag{2.2.1}$$

is studied with the two-point boundary value conditions

$$u^{(i-1)}(a)=0\ (i=1,\cdots,m),\quad u^{(j-1)}(b)=0\ (j=1,\cdots,n-m), \tag{2.2.2}$$

$$u^{(i-1)}(a)=0\ (i=1,\cdots,m),\quad u^{(j-1)}(b)=0\ (j=m+1,\cdots,n). \tag{2.2.3}$$

Here $n\geq 2$, m is the integer part of $n/2$, $-\infty<a<b<+\infty$, $p_j,q\in L_{loc}(]a,b[)$ $(j=1,\cdots,m)$, and $\tau_j:]a,b[\to]a,b[$ are measurable functions. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we denote the right (the left) limit of the function $u^{(j-1)}$ at the point $a\,(b)$. Problems (1.2.9), (1.2.10), and (1.2.9), (2.2.3) are said to be strong singular if some or all the coefficients of (1.2.9) are non-integrable on $[a,b]$, having singularities at the end-points of this segment and the conditions

$$\int_a^b (s-a)^{n-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+ds<+\infty,$$

$$\int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)|ds<+\infty\quad (j=2,\cdots,m), \tag{2.2.4}$$

$$\int_a^b (s-a)^{n-m-1/2}(b-s)^{m-1/2}|q(s)|ds<+\infty,$$

in the case of conditions (2.2.2), and

$$\int_a^b (s-a)^{n-1}[(-1)^{n-m}p_1(s)]_+ ds < +\infty,$$

$$\int_a^b (s-a)^{n-j}|p_j(s)|ds < +\infty \quad (j=2,\cdots,m), \tag{2.2.5}$$

$$\int_a^b (s-a)^{n-m-1/2}|q(s)|ds < +\infty,$$

in the case of conditions (2.2.3), are not fulfilled.

Here we consider only the case of problem (2.2.1), (2.2.2), by using the following notations

$R^+ = [0, +\infty[$;

$[x]$ is an integer part of x;

$L_{\alpha,\beta}(]a,b[)$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y :]a,\, b[\to R$, with the norm

$$||y||_{L_{\alpha,\beta}} = \int_a^b (s-a)^\alpha(b-s)^\beta|y(s)|ds;$$

$L([a,b]) = L_{0,0}(]a,b[),\ L^2([a,b]) = L^2_{0,0}(]a,b[)$;

$M(]a,b[)$ is the set of measurable functions $\tau :]a,b[\to]a,b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ is the Banach space of functions $y \in L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$, satisfying

$$\mu_1 \equiv \max\Big\{\Big[\int_a^t (s-a)^\alpha\Big(\int_s^t y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : a \le t \le \frac{a+b}{2}\Big\} +$$

$$+ \max\Big\{\Big[\int_t^b (b-s)^\beta\Big(\int_t^s y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : \frac{a+b}{2} \le t \le b\Big\} < +\infty,$$

The norm in this space is defined by the equality $||\cdot||_{\widetilde{L}^2_{\alpha,\beta}} = \mu_1$.

$\widetilde{C}^k(]a,\, b[)$ is a set of functions $u : [0,\omega] \to R$, which are absolutely continuous together with their derivatives up to the k-th order.

$\widetilde{C}^{n-1,\,m}(]a,\,b[)$ is the space of functions $y \in \widetilde{C}^{n-1}_{loc}(]a,\,b[)$, satisfying

$$\int\limits_a^b |y^{(m)}(s)|^2 ds < +\infty. \tag{2.2.6}$$

When $n = 2m$, we assume that

$$p_j \in L_{loc}(]a,\,b[) \ \ (j = 1, \cdots, m), \tag{2.2.7}$$

and if $n = 2m+1$, we assume that along with (2.2.7), the condition

$$\limsup_{t\to b} \Big|(b-t)^{2m-1}\int\limits_{t_1}^t p_1(s)ds\Big| < +\infty \ \ (t_1 = \frac{a+b}{2}) \tag{2.2.8}$$

is fulfilled.

Along with (1.2.9), we consider the homogeneous equation

$$v^{(n)}(t) = \sum_{j=1}^m p_j(t) v^{(j-1)}(\tau_j(t)) \quad \text{for} \quad a < t < b. \tag{1.1$_0$}$$

In the case where conditions (2.2.4) and (2.2.5) are violated, the question on the presence of the Fredholm's property for problem (2.2.1), (2.2.2) in some subspace of the space $\widetilde{C}^{n-1}_{loc}(]a,\,b[)$ remains so far open. This question is answered in Theorem 2.2.1 formulated below which contains optimal in a certain sense conditions guaranteeing the Fredholm's property for problem (2.2.1), (2.2.2) in the space $\widetilde{C}^{n-1,\,m}(]a,\,b[)$.

A solution of problem (2.2.1), (2.2.2) is sought in the space $\widetilde{C}^{n-1,\,m}(]a,\,b[)$.

In order to formulate the above-mentioned theorem we need following definitions:

Let $h_j :]a,\,b[\times]a,\,b[\to R_+$ and $f_j : R\times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1,\dots,m)$ be the functions and the operators, respectively, defined by the equalities

$$\begin{aligned} h_1(t,s) &= \Big|\int\limits_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+ d\xi\Big|, \\ h_j(t,s) &= \Big|\int\limits_s^t (\xi-a)^{n-2m} p_j(\xi) d\xi\Big| \quad (j = 2, \cdots, m), \end{aligned} \tag{2.2.9}$$

and,

$$f_j(c,\tau_j)(t,s)=\Big|\int\limits_s^t (\xi-a)^{n-2m}|p_j(\xi)|\Big|\int\limits_\xi^{\tau_j(\xi)}(\xi_1-c)^{2(m-j)}d\xi_1\Big|^{1/2}d\xi\Big| \quad (j=1,\cdots,m). \tag{2.2.10}$$

Let, moreover,

$$m!! = \begin{cases} 1 & \text{for} \quad m \le 0 \\ 1\cdot 3\cdot 5\cdots m & \text{for} \quad m \ge 1 \end{cases},$$

if $m = 2k+1$.

Definition 2.2.1. We will say that problem (2.2.1), (2.2.2) has the Fredholm's property in the space $\widetilde{C}^{n-1,m}(]a,\, b[)$, if the unique solvability of the corresponding homogeneous problem (1.1_0), (1.2.10) in that space implies the unique solvability of problem (2.2.1), (2.2.2) for every $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a, b[)$.

Theorem 2.2.1. *Let there exist $a_0 \in]a,\, b[,\ \ b_0 \in]a_0,\, b[$, numbers $l_{kj} > 0,\ \gamma_{kj} > 0$, and functions $\tau_j \in M(]a,\, b[)$ $(k=0,1,\ j=1,\dots,m)$ such that*

$$\begin{gathered} (t-a)^{2m-j}h_j(t,s) \le l_{0j} \quad \textit{for} \quad a < t \le s \le a_0, \\ \limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_{0j}}f_j(a,\tau_j)(t,s) < +\infty, \end{gathered} \tag{2.2.11}$$

$$\begin{gathered} (b-t)^{2m-j}h_j(t,s) \le l_{1j} \quad \textit{for} \quad b_0 \le s \le t < b, \\ \limsup_{t\to b}(b-t)^{m-\frac{1}{2}-\gamma_{1j}}f_j(b,\tau_j)(t,s) < +\infty, \end{gathered} \tag{2.2.12}$$

and

$$\sum_{j=1}^m \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\ l_{kj} < 1 \quad (k=0,1). \tag{2.2.13}$$

Let, moreover, (1.1_0), (2.2.2) have only the trivial solution in the space $\widetilde{C}^{n-1,m}(]a,\, b[)$. Then problem (2.2.1), (2.2.2) has the unique solution u for every $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\ b[)$, and there exists a constant r, independent of q, such that

$$||u^{(m)}||_{L^2} \le r||q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}. \tag{2.2.14}$$

Remark 2.2.1. There exists an example which demonstrates that strict inequality (2.2.13) is sharp because it cannot be replaced by nonstrict one.

The next theorem (the theorem of unique solvability) is proved on the basis of Theorem 2.2.1 which gives us the sharp sufficient conditions under which our problem has the Fredholm's property.

Theorem 2.2.2. *Let there exist numbers* $t^* \in]a, b[$, $l_{kj} > 0$, $\bar{l}_{kj} \geq 0$, *and* $\gamma_{kj} > 0$ $(k = 0, 1;\ j = 1, \ldots, m)$ *such that along with*

$$\begin{aligned} B_0 \equiv \sum_{j=1}^{m} \Big(&\frac{(2m-j)2^{2m-j+1}\, l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \\ + &\frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\, \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \Big) < \frac{1}{2}, \end{aligned} \tag{2.2.15}$$

$$\begin{aligned} B_1 \equiv \sum_{j=1}^{m} \Big(&\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \\ + &\frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < \frac{1}{2}, \end{aligned} \tag{2.2.16}$$

the conditions

$$\begin{gathered} (t-a)^{2m-j} h_j(t,s) \leq l_{0j}, \\ (t-a)^{m-\gamma_{0j}-1/2} f_j(a,\tau_j)(t,s) \leq \bar{l}_{0j} \ \textit{for} \quad a<t\leq s\leq t^*, \end{gathered} \tag{2.2.17}$$

$$\begin{gathered} (b-t)^{2m-j} h_j(t,s) \leq l_{1j}, \\ (b-t)^{m-\gamma_{1j}-1/2} f_j(b,\tau_j)(t,s) \leq \bar{l}_{1j} \quad \textit{for} \quad t^*\leq s\leq t<b \end{gathered} \tag{2.2.18}$$

hold. Then for every $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a, b[)$ *problem* (2.2.1), (2.2.2) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,\, b[)$.

To illustrate this theorem, we consider the problem (2.1.18), (2.1.4). From Theorem 2.2.2, with $n = 2$, $m = 1$, $t^* = (a+b)/2$, $\gamma_{01} = \gamma_{11} = 1/2$, $l_{01} = l_{11} = \kappa_0$, $\bar{l}_{01} = \bar{l}_{11} = \sqrt{2}\kappa_1/\sqrt{b-a}$, we get

Corollary 2.2.1. *Let function* $\tau \in M(]a, b[)$ *be such that*

$$\begin{gathered} 0 \leq \tau(t) - t \leq \frac{2^6}{(b-a)^6}(t-a)^7 \quad \textit{for} \quad a < t \leq \frac{a+b}{2}, \\ -\frac{2^6}{(b-a)^6}(b-t)^7 \leq t - \tau(t) \leq 0 \quad \textit{for} \quad \frac{a+b}{2} \leq t < b. \end{gathered} \tag{2.2.19}$$

Moreover, let function $p :]a, b[\to R$ *and constants* κ_0, κ_1 *be such that*

$$-\frac{2^{-2}(b-a)^2\kappa_0}{[(b-t)(t-a)]^2} \leq g_0(t) \leq \frac{2^{-7}(b-a)^6\kappa_1}{[(b-t)(t-a)]^4} \quad \textit{for} \quad a < t \leq b \tag{2.2.20}$$

and

$$4\kappa_0 + \kappa_1 < \frac{1}{2}. \tag{2.2.21}$$

Then for every $p_2 \in \widetilde{L}^2_{0,\,0}(]a, b[)$ *problem* (2.1.18), (2.1.4) *is uniquely solvable in the space* $\widetilde{C}^{1,\,1}(]a,\, b[)$.

2.3 The Dirichlet Boundary Value Problems For Strongly Singular Higher-Order Non-linear Functional-Differential Equations

Now we'll consider the paper [3] that is based on the results received for the linear equations. Namely, let us consider the article *The Dirichlet Boundary Value Problems For Strongly Singular Higher-Order Nonlinear Functional-Differential Equations* dealing with the issue of solvability of nonlinear functional-differential equation

$$u^{(n)}(t) = F(u)(t) \tag{2.3.1}$$

under the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \ (i = 1, \cdots, m), \quad u^{(i-1)}(b) = 0 \ (i = 1, \cdots, n-m). \tag{2.3.2}$$

Here $n \geq 2$, m is the integer part of $n/2$, $-\infty < a < b < +\infty$, and the operator F acting from the set of $(m-1)$-th time continuously differentiable on $]a,\, b[$ functions, to the set $L_{loc}(]a,\, b[)$. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we denote the right (the left) limit of the function $u^{(j-1)}$ at the point $a\,(b)$.

The singular ordinary differential and functional-differential equations, have been studied with sufficient completeness under different boundary conditions, but the equation (2.3.1), even under the boundary condition (2.3.2), is not studied in the case when the operator F has the form

$$F(x)(t) = \sum_{j=1}^{m} p_j(t) x^{(j-1)}(\tau_j(t)) + f(x)(t), \tag{2.3.3}$$

where the singularity of the functions $p_j : L_{loc}(]a,\, b[)$ be such that the inequalities

$$\begin{gathered} \int_a^b (s-a)^{n-1}(b-s)^{2m-1}[(-1)^{n-m} p_1(s)]_+ ds < +\infty, \\ \int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)| ds < +\infty \quad (j = 2, \cdots, m), \end{gathered} \tag{2.3.4}$$

are not fulfilled (in this case we sad that the linear part of the operator F is a strongly singular), the operator f continuously acting from $C_1^{m-1}(]a,\, b[)$ to $L_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}(]a,\, b[)$, and the inclusion

$$\sup\{f(x)(t) : ||x||_{C_1^{m-1}} \leq \rho\} \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\, b[). \tag{2.3.5}$$

holds.

For the description of the result on the solvability of problem (2.3.1), (2.3.2) we need the following notations and definitions:

$L_n(]a,b[)$ is the Banach space of $y \in L_{loc}(]a,b[)$ functions, with the norm

$$||y||_{L_n} = \sup\Big\{[(s-a)(b-t)]^{m-1/2}\int\limits_s^t (\xi-a)^{n-2m}|y(\xi)|d\xi : \quad a < s \le t < b\Big\} < +\infty.$$

$C^{n-1}_{loc}(]a,\, b[)$, $(\widetilde{C}^{n-1}_{loc}(]a,\, b[))$ is the space of the functions $y :]a,\ b[\to R$, which are continuous (absolutely continuous) together with $y',\ y'',\ \cdots, y^{(n-1)}$ on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.

$C^{m-1}_1(]a,\, b[)$ is the Banach space of the functions $y \in C^{m-1}_{loc}(]a,\, b[)$, such that

$$\begin{aligned} &\limsup_{t\to a} \frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \ (i = 1, \cdots, m), \\ &\limsup_{t\to b} \frac{|x^{(i-1)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \ (i = 1, \cdots, n-m), \end{aligned} \tag{2.3.6}$$

with the norm:

$$||x||_{C^{m-1}_1} = \sum_{i=1}^m \sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \ : \ \ a < t < b\Big\},$$

where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

$\widetilde{C}^{m-1}_1(]a,\, b[)$ is the Banach space of the functions $y \in \widetilde{C}^{m-1}_{loc}(]a,\, b[)$, such that conditions $\int\limits_a^b (y^m(s))^2 ds < +\infty$, and (2.3.6) hold, with the norm:

$$||x||_{\widetilde{C}^{m-1}_1} = \sum_{i=1}^m \sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \ : \ \ a < t < b\Big\} + \Big(\int\limits_a^b |x^{(m)}(s)|^2 ds\Big)^{1/2}.$$

$D_n(]a,\, b[\times R^+)$ is the set of such functions $\delta :]a,\, b[\times R^+ \to L_n(]a,\, b[)$ that $\delta(t,\, \cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,\, b[$, and $\delta(\cdot,\, \rho) \in L_n(]a,\, b[)$ for any $\rho \in R^+$.

$D_{2n-2m-2,\, 2m-2}(]a,\, b[\times R^+)$ is the set of such functions $\delta :]a,\, b[\times R^+ \to \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\, b[)$ that $\delta(t,\, \cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,\, b[$, and $\delta(\cdot,\, \rho) \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\, b[)$ for any $\rho \in R^+$.

In this paper, we prove a priori boundedness principle for the problem (2.3.1), (2.3.2) in the case where the operator has form (2.3.3). For formulate this a priori boundedness principle we have to define the set

$$A_\gamma = \{x \in \widetilde{C}_1^{m-1}(]a,\, b[) : ||x||_{\widetilde{C}_1^{m-1}} \leq \gamma\} \tag{2.3.7}$$

for any $\gamma > 0$, and the operator $P : C_1^{m-1}(]a,\, b[) \times C_1^{m-1}(]a,\, b[) \to L_{loc}(]a,\ b[)$ by the equality

$$P(x,y)(t) = \sum_{j=1}^{m} p_j(x)(t) y^{(j-1)}(\tau_j(t)) \qquad \text{for} \qquad a < t < b, \tag{2.3.8}$$

where $p_j : C_1^{m-1}(]a,\, b[) \to L_{loc}(]a,\ b[)$, $\tau_j \in M(]a,\, b[)$, and introduce the definition:

Definition 2.3.1. Let γ_0 and γ be the positive numbers. We said that the continuous operator $P : C_1^{m-1}(]a,\, b[) \times C_1^{m-1}(]a,\, b[) \to L_n(]a,\ b[)$ to be γ_0, γ consistent with boundary condition (2.3.2) if:

i. for any $x \in A_{\gamma_0}$ and almost all $t \in]a,\ b[$ the inequality

$$\sum_{j=1}^{m} |p_j(x)(t) x^{(j-1)}(\tau_j(t))| \leq \delta(t,\, ||x||_{\widetilde{C}_1^{m-1}}) ||x||_{\widetilde{C}_1^{m-1}} \tag{2.3.9}$$

holds, where $\delta \in D_n(]a,\, b[\times R^+)$.

ii. for any $x \in A_{\gamma_0}$ and $q \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\ b[)$ the equation

$$y^{(n)}(t) = \sum_{j=1}^{m} p_j(x)(t) y^{(j-1)}(\tau_j(t)) + q(t) \tag{2.3.10}$$

under boundary conditions (2.3.2), has the unique solution y in the space $\widetilde{C}^{n-1,\,m}(]a,\, b[)$ and

$$||y||_{\widetilde{C}_1^{m-1}} \leq \gamma ||q||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}. \tag{2.3.11}$$

In the sequel it will always be assumed that the operator F_p defined by equality

$$F_p(x)(t) = |F(x)(t) - \sum_{j=1}^{m} p_j(x)(t) x^{(j-1)}(\tau_j(t))(t)|,$$

continuously acting from $C_1^{m-1}(]a,\, b[)$ to $L_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}(]a,\ b[)$, and

$$\widetilde{F}_p(t,\rho) \equiv \sup\{F_p(x)(t) : ||x||_{C_1^{m-1}} \leq \rho\} \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\ b[) \tag{2.3.12}$$

for each $\rho \in [0,\, +\infty[$.

Then the following theorem is valid

Theorem 2.3.1. *Let the operator P be γ_0, γ consistent with boundary condition* (2.3.2), *and there exists a positive number $\rho_0 \le \gamma_0$, such that*

$$||\widetilde{F}_p(\,\cdot,\,\min\{2\rho_0,\gamma_0\})||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \le \frac{\gamma_0}{\gamma}. \tag{2.3.13}$$

Let moreover, for any $\lambda \in]0,\,1[$, an arbitrary solution $x \in A_{\gamma_0}$ of the equation

$$x^{(n)}(t) = (1-\lambda)P(x,\,x)(t) + \lambda F(x)(t) \tag{2.3.14}$$

under the conditions (2.3.2), *admits the estimate*

$$||x||_{\widetilde{C}_1^{m-1}} \le \rho_0. \tag{2.3.15}$$

Then problem (2.3.1), (2.3.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a,\,b[)$.*

On the bases of this theorem we can prove some efficient theorems. Let us consider one of them. In order to consider them we define the operators: $h_j : C_1^{m-1}(]a,\,b[)\times]a,\,b[\times]a,\,b[\to L_{loc}(]a,\,b[\times]a,\,b[)$, $\quad f_j : C_1^{m-1}(]a,\,b[) \times [a,\,b] \times M(]a,b[) \to C_{loc}(]a,\,b[\times]a,\,b[)$ $(j = 1,\ldots,m)$ by the equalities

$$\begin{gathered}
h_1(x,t,s) = \Big|\int\limits_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(x)(\xi)]_+ d\xi\Big|,\\
h_j(x,t,s) = \Big|\int\limits_s^t (\xi-a)^{n-2m}p_j(x)(\xi)d\xi\Big| \quad (j=2,\cdots,m),
\end{gathered} \tag{2.3.16}$$

and

$$\begin{gathered}
f_j(x,c,\tau_j)(t,s) =\\
= \Big|\int\limits_s^t (\xi-a)^{n-2m}|p_j(x)(\xi)|\Big|\int\limits_\xi^{\tau_j(\xi)}(\xi_1-c)^{2(m-j)}d\xi_1\Big|^{1/2}d\xi\Big|.
\end{gathered} \tag{2.3.17}$$

Theorem 2.3.2. *Let the continuous operator $P : C_1^{m-1}(]a,\,b[)\times C_1^{m-1}(]a,\,b[) \to L_n(]a,\,b[)$ admits to the condition* (2.3.9) *where $\delta \in D_n(]a,\,b[\times R^+)$, $\tau_j \in M(]a,\,b[)$ and the numbers $\gamma_0,\, t^* \in]a,\,b[,\; l_{kj} > 0,\; \bar{l}_{kj} > 0,\; \gamma_{kj} > 0\,(k = 1,2;\; j = 1,\cdots,m)$, be such that the inequalities*

$$\begin{gathered}
(t-a)^{2m-j}h_j(x,t,s) \le l_{0j},\\
\limsup_{t\to a}(t-a)^{m-\frac{1}{2}-\gamma_{0j}}f_j(x,a,\tau_j)(t,s) \le \bar{l}_{0j}
\end{gathered} \tag{2.3.18}$$

for $a < t \le s \le t^*$, $||x||_{\widetilde{C}_1^{m-1}} \le \gamma_0$,

$$\begin{gathered}(b-t)^{2m-j} h_j(x,t,s) \le l_{1j},\\ \limsup_{t\to b}(b-t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(x,b,\tau_j)(t,s) \le \overline{l}_{1j}\end{gathered} \tag{2.3.19}$$

for $t^* \le s \le t < b$, $||x||_{\widetilde{C}_1^{m-1}} \le \gamma_0$, *and conditions* (2.2.15), (2.2.16) *hold. Let moreover the operator* F *and function* $\eta \in D_{2n-2m-2,\,2m-2}(]a,\,b[\times R^+)$ *be such that condition*

$$|F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)| \le \eta(t,\,||x||_{\widetilde{C}_1^{m-1}}), \tag{2.3.20}$$

and inequality

$$||\eta(\cdot,\,\gamma_0)||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} < \frac{\gamma_0}{r_n}, \tag{2.3.21}$$

be fulfilled, where

$$r_n = \Big(1 + \sum_{j=1}^{m} \frac{2^{m-j+1/2}}{(m-j)!(2m-2j+1)^{1/2}(b-a)^{m-j+1/2}}\Big)\times$$

$$\times \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0,\,B_1\})(2m-1)!!}.$$

Then problem (2.3.1), (2.3.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,\,b[)$.

To illustrate this theorem, we consider its corollary for the equation

$$u''(t) = -\frac{\lambda |u(t)|^k}{[(t-a)(b-t)]^{2+k/2}} u(\tau(t)) + q(x)(t), \tag{2.3.22}$$

where $\lambda,\,k \in R^+$, the function $\tau \in M(]a,\,b[)$, the operator $q : C_1^{m-1}(]a,\,b[) \to \widetilde{L}^2_{0,0}(]a,\;b[)$ is continuous and

$$\eta(t,\,\rho) \equiv \sup\{|q(x)(t)| : ||x||_{\widetilde{C}_1^{m-1}} \le \rho\} \in \widetilde{L}^2_{0,0}(]a,\;b[).$$

Than from Theorem 2.3.2 it follows

Corollary 2.3.1. *Let the function* $\tau \in M(]a,\,b[)$, *the continuous operator* $q : C_1^{m-1}(]a,\,b[) \to \widetilde{L}^2_{0,0}(]a,\;b[)$, *and the numbers* $\gamma_0 > 0$, $\lambda \ge 0$, $k > 0$, *by such that*

$$|\tau(t) - t| \le \begin{cases}(t-a)^{3/2} & \textit{for} \;\; a < t \le (a+b)/2\\ (b-t)^{3/2} & \textit{for} \;\; (a+b)/2 \le t < b\end{cases}, \tag{2.3.23}$$

$$||\eta(t,\, \gamma_0)||_{\widetilde{L}^2_{0,\,0}} \leq$$

$$\leq \Big(1+\sqrt{\frac{2}{b-a}}\Big)^{-1}\frac{(b-a)^2-16\lambda\gamma_0^k(1+[2(b-a)]^{1/4})}{2(1+b-a)(b-a)^2}, \qquad (2.3.24)$$

and

$$\lambda < \frac{(b-a)^2}{32\gamma_0^k(1+[2(b-a)]^{1/4})}. \qquad (2.3.25)$$

Then the problem (2.3.22), (2.3.2) *is solvable.*

Literatura

[1] Mukhigulashvili S., *Two-point boundary value problems for second order functional differential equations,* Mem. Differential Equations Math. Phys., 20 (2000), 1-112.

[2] Mukhigulashvili S., Partsvania N., *Two-point boundary value problems for strongly singular higher-order linear differential equations with deviating arguments,* E. J. Qualitative Theory of Diff. Equ., 2012, No.38, 1-34.

[3] Mukhigulashvili S., *The Dirichlet Boundary Value Problems For Strongly Singular Higher-Order Nonlinear Functional-Differential Equations,* Czechoslovak Mathematical Journal, vol. 63 (2013), No. 1, pp. 235-263.

[4] Mukhigulashvili S., *The Dirichlet BVP The second Order Nonlinear Ordinary Differential Equation At Resonance,* Italian J. Of Pure and Appl. Math., 2011, No.28, 177-204,

[5] Mukhigulashvili S., Hakl R., *A Periodic Boundary Value Problem For Functional-Differential Equations Of Higher Order,* Georgian Math. J. Vol.16, (2009), No.4, 651-665.

The reader can see the literature used in this works in the articles attached to the survey.

THE DIRICHLET BVP FOR THE SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION AT RESONANCE

Sulkhan Mukhigulashvili

Permanent addresses:
1. Institute of Mathematics
Academy of Sciences of the Czech Republic
Žižkova 22, 616 62 Brno
Czech Republic

2. I. Chavchavadze State University
Faculty of physics and mathematics
I. Chavchavadze Str. No. 32, 0179 Tbilisi
Georgia
e-mail: mukhig@ipm.cz

Abstract. Efficient sufficient conditions are established for the solvability of the Dirichlet problem

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad a \le t \le b,$$

$$u(a) = 0, \quad u(b) = 0,$$

where $h, p \in L([a, b]; R)$ and $f \in K([a, b] \times R; R)$, in the case where the linear problem

$$u''(t) = p(t)u(t), \quad u(a) = 0, \quad u(b) = 0$$

has nontrivial solutions.

Key words and phrases: nonlinear ordinary differential equation, Dirichlet problem at resonance.

2000 Mathematics Subject Classification: 34B15, 34C15, 34C25.

1. Introduction

Consider on the set $I = [a, b]$ the second order nonlinear ordinary differential equation

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in I \tag{1.1}$$

with the boundary conditions

$$u(a) = 0, \quad u(b) = 0, \tag{1.2}$$

where $h, p \in L(I; R)$ and $f \in K(I \times R; R)$.

By a solution of the problem (1.1), (1.2) we understand a function $u \in \widetilde{C}'(I, R)$, which satisfies the equation (1.1) almost everywhere on I and satisfies the conditions (1.2).

Along with (1.1), (1.2) we consider the homogeneous problem

$$w''(t) = p(t)w(t) \quad \text{for} \quad t \in I, \tag{1.3}$$

$$w(a) = 0, \quad w(b) = 0. \tag{1.4}$$

At present, the foundations of the general theory of two-point boundary value problems are already laid and problems of this type are studied by many authors and investigated in detail (see, for instance, [1], [4], [5], [8], [12],[13], [14]-[16], [17] and references therein). On the other hand, in all of these works, only the case when the homogeneous problem (1.3), (1.4) has only a trivial solution is studied. The case when the problem (1.3), (1.4) has also the nontrivial solution is still little investigated and in the majority of articles, the authors study the case with p constant in the equation (1.1), i.e., when the problem (1.1), (1.2) and the equation (1.3) are of type

$$u''(t) = -\lambda^2 u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in [0, \pi], \tag{1.5}$$

$$u(0) = 0, \quad u(\pi) = 0, \tag{1.6}$$

and

$$w''(t) = -\lambda^2 w(t) \quad \text{for} \quad t \in [0, \pi] \tag{1.7}$$

respectively, with $\lambda = 1$. (see, for instance, [2], [3], [4], [6]-[11], [14]-[16], and references therein).

In the present paper, we study solvability of the problem (1.1), (1.2) in the case when the function $p \in L(I; R)$ is not necessarily constant, under the assumption that the homogeneous problem (1.3), (1.4) has the nontrivial solution with an arbitrary number of zeroes. For the equation (1.7), this is the case when λ is not necessarily the first eigenvalue of the problem (1.7), (1.4), with $a = 0$, $b = \pi$.

The obtained results are new and generalize some well-known results (see,[2], [3], [4], [6], [10]).

The following notation is used throughout the paper: N is the set of all natural numbers. R is the set of all real numbers, $R_+ = [0, +\infty[$. $C(I; R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$. $\widetilde{C}'(I; R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivatives. $L(I; R)$ is the Banach space of Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)| ds$.

$K(I \times R; R)$ is the set of functions $f : I \times R \to R$ satisfying the Carathéodory conditions, i.e., $f(\cdot, x) : I \to R$ is a measurable function for all $x \in R$, $f(t, \cdot) : R \to R$ is a continuous function for almost all $t \in I$, and for every $r > 0$ there exists $q_r \in L(I; R_+)$ such that $|f(t,x)| \le q_r(t)$ for almost all $t \in I$, $|x| \le r$.

Having $w : I \to R$, we put: $N_w \stackrel{def}{=} \{t \in]a,b[: w(t) = 0\}$,

$$\Omega_w^+ \stackrel{def}{=} \{t \in I : w(t) > 0\},$$
$$\Omega_w^- \stackrel{def}{=} \{t \in I : w(t) < 0\},$$

and $[w(t)]_+ = (|w(t)| + w(t))/2$, $[w(t)]_- = (|w(t)| - w(t))/2$ for $t \in I$.

Definition 1.1. Let A be a finite (eventually empty) subset of I. We say that $f \in E(A)$, if $f \in K(I \times R; R)$ and, for any measurable set $G \subseteq I$ and an arbitrary constant $r > 0$, we can choose $\varepsilon > 0$ such that if

$$\int_G |f(s,x)| ds \neq 0 \quad \text{for} \quad x \ge r \ (x \le -r)$$

then

$$\int_{G \setminus U_\varepsilon} |f(s,x)| ds - \int_{U_\varepsilon} |f(s,x)| ds \ge 0 \quad \text{for} \quad x \ge r \ (x \le -r),$$

where $U_\varepsilon = I \cap \left(\bigcup_{k=1}^{n}]t_k - \varepsilon/2n, \ t_k + \varepsilon/2n[\right)$ if $A = \{t_1, t_2, ..., t_n\}$, and $U_\varepsilon = \emptyset$ if $A = \emptyset$.

Remark 1.1. If $f \in K(I \times R; R)$ then $f \in E(\emptyset)$.

Remark 1.2. It is clear that if $f_1 \in L(I; R)$ and $f(t,x) \stackrel{def}{\equiv} f_1(t)$ then $f \in E(A)$ for every finite set $A \subset I$.

Remark 1.3. It is clear that if $f(t,x) \stackrel{def}{\equiv} f_0(t) g_0(x)$, where $f_0 \in L(I; R)$ and $g_0 \in C(I; R)$, then $f \in E(A)$ for every finite set $A \subset I$.

The example below shows that there exists a function $f \in K(I \times R; R)$ such that $f \notin E(\{t_1, ..., t_k\})$ for some points $t_1, ..., t_k \in I$.

Example 1.1. Let $f(t,x) = |t|^{-1/2} g(t,x)$ for $t \in [-1, 0[\cup]0, 1]$, $x \in R$, and $f(0, .) \equiv 0$, where $g(-t, x) = g(t,x)$ for $t \in]-1, 1]$, $x \in R$, and

$$g(t,x) = \begin{cases} x & \text{for} \quad x \le 1/t, \ t > 0 \\ 1/t & \text{for} \quad x > 1/t, \ t > 0 \end{cases}.$$

Then $f \in K([0,1] \times R; R)$ and it is clear that $f \notin E(\{0\})$ because, for every $\varepsilon > 0$, if $x \ge 1/\varepsilon$ then $\int_\varepsilon^1 f(s,x) ds - \int_0^\varepsilon f(s,x) ds = 4(\varepsilon^{-1/2} - x^{1/2}) - 2 < 0$.

2. Main results

Theorem 2.1. *Let w be a nonzero solution of the problem* (1.3), (1.4),

$$N_w = \emptyset, \tag{2.1}$$

there exist a constant $r > 0$, functions $f^-, f^+ \in L(I; R_+)$ and $g, h_0 \in L(I;]0, +\infty[\,)$ such that

$$f(t,x)\mathrm{sgn}x \le g(t)|x| + h_0(t) \quad \textit{for} \quad |x| \ge r \tag{2.2}$$

and

$$\begin{gathered} f(t,x) \le -f^-(t) \quad \textit{for} \quad x \le -r, \\ f^+(t) \le f(t,x) \quad \textit{for} \quad x \ge r \end{gathered} \tag{2.3}$$

on I. Let, moreover, there exist $\varepsilon > 0$ such that

$$-\int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r||_L \le -\int_a^b h(s)|w(s)|ds \le$$

$$\le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \tag{2.4$_1$}$$

where

$$\gamma_r(t) = \sup\{|f(t,x)| : |x| \le r\}. \tag{2.5}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Example 2.2. It follows from Theorem 2.1 that the equation

$$u''(t) = -\lambda^2 u(t) + \sigma|u(t)|^\alpha \mathrm{sgn}u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi \tag{2.6}$$

where $\sigma = 1$, $\lambda = 1$, and $\alpha \in]0, 1]$, with the conditions (1.6) has at least one solution for every $h \in L([0, \pi], R)$.

Theorem 2.2. *Let w be a nonzero solution of the problem* (1.3), (1.4), *condition* (2.1) *hold, there exist a constant $r > 0$, functions $f^-, f^+ \in L(I; R_+)$ and $q \in K(I \times R; R_+)$ such that q is non-decreasing in the second argument,*

$$|f(t,x)| \le q(t,x) \quad \textit{for} \quad |x| \ge r, \tag{2.7}$$

$$\begin{gathered} f^-(t) \le f(t,x) \quad \textit{for} \quad x \le -r, \\ f(t,x) \le -f^+(t) \quad \textit{for} \quad x \ge r \end{gathered} \tag{2.8}$$

on I, and

$$\lim_{|x| \to +\infty} \frac{1}{x} \int_a^b q(s,x)ds = 0. \tag{2.9}$$

Let, moreover, there exist $\varepsilon > 0$ *such that*

$$-\int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r||_L \le \int_a^b h(s)|w(s)|ds \le$$

$$\le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \tag{2.4$_2$}$$

where γ_r *is defined by* (2.5). *Then the problem* (1.1), (1.2) *has at least one solution.*

Example 2.3. From Theorem 2.2 it follows that the problem (2.6), (1.6) with $\sigma = -1$, $\lambda = 1$, and $\alpha \in]0, 1[$ has at least one solution for every $h \in L([0, \pi]; R)$.

Remark 2.4. If $f \not\equiv 0$ the condition (2.4$_i$) of Theorem 2.i ($i = 1, 2$) can be replaced by

$$\begin{aligned} -\int_a^b f^-(s)|w(s)|ds < (-1)^i \int_a^b h(s)|w(s)|ds < \\ < \int_a^b f^+(s)|w(s)|ds, \end{aligned} \tag{2.10$_i$}$$

because, from (2.10$_i$) there follows the existence of a constant $\varepsilon > 0$ such that the condition (2.4$_i$) is satisfied.

Theorem 2.3. *Let* $i \in \{0, 1\}$, w *be a nonzero solution of the problem* (1.3), (1.4), $f \in E(N_w)$, *there exist a constant* $r > 0$ *such that the function* $(-1)^i f$ *is non-decreasing in the second argument for* $|x| \ge r$,

$$(-1)^i f(t, x)\mathrm{sgn} x \ge 0 \quad for \quad t \in I, \ |x| \ge r, \tag{2.11}$$

$$\int_{\Omega_w^+} |f(s, r)|ds + \int_{\Omega_w^-} |f(s, -r)|ds \ne 0, \tag{2.12}$$

and

$$\lim_{|x| \to +\infty} \frac{1}{|x|} \int_a^b |f(s, x)|ds = 0. \tag{2.13}$$

Then there exists $\delta > 0$ *such that the problem* (1.1), (1.2) *has at least one solution for every* h *satisfying the condition*

$$\left| \int_a^b h(s)w(s)ds \right| < \delta. \tag{2.14}$$

Corollary 2.1. *Let the assumptions of Theorem* 2.3 *be satisfied and*

$$\int_a^b h(s)w(s)ds = 0. \tag{2.15}$$

Then the problem (1.1), (1.2) *has at least one solution.*

Example 2.4. From Theorem 2.3 it follows that the problem (2.6), (1.6) with $\sigma \in \{-1,1\}$, $\lambda \in N$, and $\alpha \in]0,1[$ has at least one solution if $h \in L([0,\pi],R)$ is such that $\int_0^\pi h(s)\sin\lambda s ds = 0$.

Theorem 2.4. *Let $i \in \{0,1\}$, w be a nonzero solution of the problem* (1.3),(1.4), $f(t,x) \overset{def}{\equiv} f_0(t)g_0(x)$ *with $f_0 \in L(I;R_+)$, $g_0 \in C(R;R)$, there exist a constant $r > 0$ such that $(-1)^i g_0$ is non-decreasing for $|x| \geq r$ and*

$$(-1)^i g_0(x)\text{sgn}x \geq 0 \qquad \textit{for} \quad |x| \geq r. \tag{2.16}$$

Let, moreover,

$$|g_0(r)| \int_{\Omega_w^+} f_0(s)ds + |g_0(-r)| \int_{\Omega_w^-} f_0(s)ds \neq 0 \tag{2.17}$$

and

$$\lim_{|x|\to+\infty} |g_0(x)| = +\infty, \qquad \lim_{|x|\to+\infty} \frac{g_0(x)}{x} = 0. \tag{2.18}$$

Then, for every $h \in L(I;R)$, the problem (1.1), (1.2) *has at least one solution.*

Example 2.5. From Theorem 2.4 it follows that the equation

$$u''(t) = p_0(t)u(t) + p_1(t)|u(t)|^\alpha \text{sgn}u(t) + h(t) \quad \text{for} \quad t \in I, \tag{2.19}$$

where $\alpha \in]0,1[$ and $p_0, p_1, h \in L(I;R)$, with the conditions (1.2) has at least one solution provided that $p_1(t) > 0$ for $t \in I$.

Theorem 2.5. *Let $i \in \{0,1\}$ and w be a nonzero solution of the problem* (1.3), (1.4). *Let, moreover, there exist constants $r > 0$, $\varepsilon > 0$, and functions $\alpha, f^+, f^- \in L(I;R_+)$ such that the conditions*

$$\begin{aligned} (-1)^i f(t,x) &\leq -f^-(t) \quad \textit{for} \quad x \leq -r, \\ f^+(t) &\leq (-1)^i f(t,x) \quad \textit{for} \quad x \geq r, \end{aligned} \tag{2.20$_i$}$$

$$\sup\{|f(t,x)| : x \in R\} \leq \alpha(t) \tag{2.21}$$

hold on I, and let

$$
\begin{aligned}
&-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds + \varepsilon||\alpha||_L \leq \\
(2.22_i) \qquad &\leq (-1)^{i+1}\int_a^b h(s)w(s)ds \leq \\
&\leq \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds - \varepsilon||\alpha||_L.
\end{aligned}
$$

Then the problem (1.1), (1.2) *has at least one solution.*

Remark 2.5. If $f \not\equiv 0$ then the condition (2.22_i) $(i = 1, 2)$ of Theorem 2.5 can be replaced by

$$
\begin{aligned}
&-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds < \\
(2.23_i) \qquad &< (-1)^{i+1}\int_a^b h(s)w(s)ds < \\
&< \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds.
\end{aligned}
$$

because from (2.23_i) there follows the existence of a constant $\varepsilon > 0$ such that the condition (2.22_i) is satisfied.

Remark 2.6. If $\widetilde{f}(t) = \min\{f^+(t), f^-(t)\}$ then the condition (2.22_i) of Theorem 2.5 can be replaced by

$$\left|\int_a^b h(s)w(s)ds\right| \leq \int_a^b \widetilde{f}(s)|w(s)|ds - \varepsilon||\alpha||_L.$$

Example 2.6. From Theorem 2.5 it follows that the equation

$$u''(t) = -\lambda^2 u(t) + \frac{|u(t)|^\alpha}{1 + |u(t)|^\alpha}\mathrm{sgn} u(t) + h(t) \quad \text{for} \quad 0 \leq t \leq \pi, \tag{2.24}$$

where $\lambda \in N$ and $\alpha \in]0, +\infty[$, with the conditions (1.6) has at least one solution if $h \in L([0, \pi], R)$ is such that $|h(t)| < 1$ for $0 \leq t \leq \pi$.

3. Problem (1.5), (1.6).

Throughout this section we will assume that $a = 0$, $b = \pi$, and $I = [0, \pi]$. Since the functions $\beta \sin \lambda t$ $(\beta \in R)$ are nontrivial solutions of the problem (1.7), (1.4), from Theorems 2.1–2.5 it immediately follows:

Corollary 3.2. *Let $\lambda = 1$ and all the assumptions of Theorem* 2.1 (*resp. Theorem* 2.2) *except* (2.1) *be fulfilled with $w(t) = \sin t$. Then the problem* (1.5), (1.6) *has at least one solution.*

Now, note that

$$N_{\sin\lambda t}=\begin{cases}\emptyset & \text{for} \quad \lambda=1\\ \{\pi n/\lambda : n=1,...,\lambda-1\} & \text{for} \quad \lambda\geq 2\end{cases}.$$

Corollary 3.3. *Let $i\in\{0,1\}$, $\lambda\in N$, $f\in E(N_{\sin\lambda t})$, there exist a constant $r>0$ such that the function $(-1)^i f$ is non-decreasing in the second argument for $|x|\geq r$, and let the conditions* (2.11)–(2.13) *be fulfilled with $w(t)=\sin\lambda t$. Then there exists $\delta>0$ such that the problem* (1.5), (1.6) *has at least one solution for every $h\in L(I;R)$ satisfying the condition $|\int_0^\pi h(s)\sin\lambda s ds|<\delta$.*

Corollary 3.4. *Let $i\in\{0,1\}$, $\lambda\in N$, and let all the assumptions of Theorem* 2.4 *be fulfilled with $w(t)=\sin\lambda t$. Then, for any $h\in L(I;R)$, the problem* (1.5), (1.6) *has at least one solution.*

Corollary 3.5. *Let $i\in\{0,1\}$, $\lambda\in N$ and let there exist a constant $r>0$ such that* (2.20_i)–(2.22_i) *be fulfilled with $w(t)=\sin\lambda t$. Then the problem* (1.5), (1.6) *has at least one solution.*

Remark 3.7. If $f\not\equiv 0$ then in Corollary 3.2 (resp. Corollary 3.5), the condition (2.4_i) (resp. (2.22_i)) can be replaced by the condition (2.10_i) (resp. (2.23_i)) with $w(t)=\sin t$ (resp. $w(t)=\sin\lambda t$).

4. Auxiliary propositions

Let $u_n\in\widetilde{C}'(I;R)$, $\|u_n\|_C\neq 0$ $(n\in N)$, w be an arbitrary solution of the problem (1.3), (1.4), and $r>0$. Then, for every $n\in N$, we define:

$$A_{n,1}\overset{def}{=}\{t\in I:|u_n(t)|\leq r\},\qquad A_{n,2}\overset{def}{=}\{t\in I:|u_n(t)|>r\},$$

$$B_{n,i}\overset{def}{=}\{t\in A_{n,2}:\operatorname{sgn}u_n(t)=(-1)^{i-1}\operatorname{sgn}w(t)\}\quad(i=1,2),$$

$$C_{n,1}\overset{def}{=}\{t\in A_{n,2}:|w(t)|\geq 1/n\},\qquad C_{n,2}\overset{def}{=}\{t\in A_{n,2}:|w(t)|<1/n\},$$

$$D_n\overset{def}{=}\{t\in I:|w(t)|>r||u_n||_C^{-1}+1/2n\},$$

$$A^{\pm}_{n,2}\overset{def}{=}\{t\in A_{n,2}:\pm u_n(t)>r\},\quad B^{\pm}_{n,i}\overset{def}{=}A^{\pm}_{n,2}\cap B_{n,i},$$

$$C^{\pm}_{n,i}\overset{def}{=}A^{\pm}_{n,2}\cap C_{n,i}\quad(i=1,2),\ D^{\pm}_n\overset{def}{=}\{t\in I:\pm w(t)>r||u_n||_C^{-1}+1/2n\},$$

From these definitions it is clear that, for any $n\in N$, we have

$$A_{n,1}\cap A_{n,2}=\emptyset, A^+_{n,2}\cap A^-_{n,2}=\emptyset,\quad B_{n,1}\cap B_{n,2}=\emptyset,\ C_{n,1}\cap C_{n,2}=\emptyset,$$

$$D^+_n\cap D^-_n=\emptyset,\ B^+_{n,2}\cap B^-_{n,2}=\emptyset,\ C^+_{n,i}\cap C^-_{n,i}=\emptyset\ (i=1,2), \tag{4.1}$$

and

$$A_{n,1}\cup A_{n,2}=I,\ A^+_{n,2}\cup A^-_{n,2}=A_{n,2},\ B_{n,1}\cup B_{n,2}=A_{n,2}\setminus N_w,$$

$$\begin{gathered}C_{n,1}\cup C_{n,2}=A_{n,2},\ B^+_{n,2}\cup B^-_{n,2}=B_{n,2},\ C^{\pm}_{n,1}\cup C^{\pm}_{n,2}=A^{\pm}_{n,2},\\ C^+_{n,i}\cup C^-_{n,i}=C_{n,i}\ (i=1,2),\ D^+_n\cup D^-_n=D_n.\end{gathered} \tag{4.2}$$

Lemma 4.1. *Let* $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $r > 0$, w *be an arbitrary nonzero solution of the problem* (1.3), (1.4), *and*

$$||u_n||_C \geq 2rn \quad \text{for} \quad n \in N, \tag{4.3}$$

$$||v_n - w||_C \leq 1/2n \quad \text{for} \quad n \in N, \tag{4.4}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$. *Then, for any* $n_0 \in N$, *we have*

$$D_{n_0}^+ \subset A_{n,2}^+, \quad D_{n_0}^- \subset A_{n,2}^- \quad \text{for} \quad n \geq n_0, \tag{4.5}$$

$$C_{n_0,1}^+ \subset D_n^+ \quad C_{n_0,1}^- \subset D_n^- \quad \text{for} \quad n \geq n_0. \tag{4.6}$$

Moreover

$$\lim_{n\to+\infty} \text{mes} A_{n,1} = 0, \qquad \lim_{n\to+\infty} \text{mes} A_{n,2} = \text{mes} I, \tag{4.7}$$

$$C_{n,1} \subset B_{n,1}, \qquad B_{n,2} \subset C_{n,2}, \tag{4.8}$$

$$B_{n,2}^+ \subset C_{n,2}^+, \qquad B_{n,2}^- \subset C_{n,2}^-, \tag{4.9}$$

$$C_{n,1}^+ \subset B_{n,1}^+, \qquad C_{n,1}^- \subset B_{n,1}^-, \tag{4.10}$$

$$\begin{gathered} \lim_{n\to+\infty} \text{mes} C_{n,1} = \lim_{n\to+\infty} \text{mes} B_{n,1} = \text{mes} I, \\ \lim_{n\to+\infty} \text{mes} C_{n,2} = \lim_{n\to+\infty} \text{mes} B_{n,2} = 0, \end{gathered} \tag{4.11}$$

$$r < |u_n(t)| \leq ||u_n||_C/2n \qquad \text{for} \quad t \in B_{n,2}, \tag{4.12}$$

$$|u_n(t)| \geq ||u_n||_C/2n > r \qquad \text{for} \quad t \in C_{n,1}, \tag{4.13}$$

$$C_{n,2}^{\pm} = \{t \in A_{n,2} : 0 \leq \pm w(t) < 1/n\}, \tag{4.14$_1$}$$

$$C_{n,1}^{\pm} \subset \Omega_w^{\pm}, \quad \lim_{n\to+\infty} \text{mes} C_{n,1}^{\pm} = \text{mes} \Omega_w^{\pm}. \tag{4.15}$$

Proof. From the unique solvability of the Cauchy problem for the equation (1.3) it follows that the set N_w is finite. Consequently, we can assume that $N_w = \{t_1, ..., t_k\}$. Let also $t_0 = a$, $t_{k+1} = b$ and $T_n \overset{def}{=} I \cap \left(\bigcup_{i=0}^{k+1} [t_i - 1/n, \ t_i + 1/n] \right)$. We first show that, for every $n_0 \in N$, there exists $n_1 > n_0$ such that

$$A_{n,1} \subseteq T_{n_0} \quad \text{for} \quad n \geq n_1. \tag{4.16}$$

Suppose on the contrary that, for some $n_0 \in N$, there exists the sequence $t'_{n_j} \in A_{n_j,1}$ $(j \in N)$ with $n_j < n_{j+1}$, such that $t'_{n_j} \notin T_{n_0}$ for $j \in N$. Without loss of generality we can assume that $\lim\limits_{j\to+\infty} t'_{n_j} = t'_0$. Then from the conditions (4.3), (4.4), the definition of the set $A_{n,1}$ and the equality $w(t'_0) = (w(t'_0) - w(t'_{n_j})) + (w(t'_{n_j}) - v_{n_j}(t'_{n_j})) + v_{n_j}(t'_{n_j})$, we get $|w(t'_0)| = 0$, i.e., $t'_0 \in \{t_0, t_1, ..., t_{k+1}\}$. But this contradicts the condition $t'_{n_j} \notin T_{n_0}$ and thus (4.16) is true. Since $\lim\limits_{n\to+\infty} \text{mes} T_n = 0$, it follows from (4.2) and (4.16) that (4.7) is valid.

Let $t_0 \in D^+_{n_0}$. Then from (4.4) it follows that

$$\frac{u_n(t_0)}{||u_n||_C} \geq w(t_0) - |v_n(t_0) - w(t_0)| > \frac{r}{||u_{n_0}||_C} + \frac{1}{2n_0} - \frac{1}{2n} \geq \frac{r}{||u_{n_0}||_C}$$

for $n \geq n_0$, and thus $t_0 \in A^+_{n,2}$ for $n \geq n_0$, i.e., $D^+_{n_0} \subset A^+_{n,2}$ for $n \geq n_0$. The second relation of (4.5) can be proved analogously. Now suppose that $t_0 \in C_{n,1}$ and $t_0 \notin B_{n,1}$. Then, in view of (4.1) and (4.2), it is clear that $t_0 \in B_{n,2}$, and thus

$$|v_n(t_0) - w(t_0)| = |v_n(t_0)| + |w(t_0)| > 1/n \,, \tag{4.17}$$

which contradicts (4.4). Consequently, $C_{n,1} \subset B_{n,1}$ for $n \in N$. This, together with the relations $C_{n,2} = A_{n,2} \setminus C_{n,1}$, $B_{n,2} \subseteq A_{n,2} \setminus B_{n,1}$, implies $B_{n,2} \subset C_{n,2}$, i.e., (4.8) holds. The conditions (4.9) and (4.10) follow immediately from (4.8). In view of the fact that $\lim\limits_{n\to+\infty} \text{mes} C_{n,i} = (2-i)\text{mes} I$, from (4.8) we get (4.11). Now, let $t_0 \in B_{n,2}$ and suppose that $|v_n(t_0)| > 1/2n$. Then from (4.4) we obtain the contradiction $1/2n \geq |v_n(t_0) - w(t_0)| = |v_n(t_0)| + |w(t_0)| > 1/2n$. Thus $\dfrac{|u_n(t_0)|}{||u_n||_C} = |v_n(t_0)| \leq \dfrac{1}{2n}$ and using the definitions of the sets $B_{n,2}$ and $A_{n,2}$ we obtain (4.12). Also, from the inequality $\dfrac{|u_n(t)|}{||u_n||_C} = |v_n(t)| \geq |w(t)| - |v_n(t) - w(t)|$ by (4.3), (4.4) and the definition of the sets $C_{n,1}$ and $A_{n,2}$ we obtain (4.13).

Let there exist $t_0 \in C^+_{n,2}$ such that $t_0 \notin \{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\}$. Then from the definition of the sets $C_{n,2}$ and the inclusion $C^+_{n,2} \subset C_{n,2}$ we get $-1/n < w(t) < 0$ and $t_0 \in A^+_{n,2}$. In this case the inequality (4.17) is fulfilled, which contradicts (4.4). Therefore $C^+_{n,2} \subset \{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\}$. Let now $t_0 \in \{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\}$ and $t_0 \notin C^+_{n,2}$. Then from the definition of the set $C_{n,2}$ and (4.2) it is clear that $t_0 \in C^-_{n,2}$, i.e., $t_0 \in A^-_{n,2}$, and that the inequality (4.17) holds, which contradicts (4.4). Therefore $\{t \in A_{n,2} : 0 \leq w(t) \leq 1/n\} \subset C^+_{n,2}$. From the last two inclusions it follows that (4.14_1) holds for $C^+_{n,2}$. From (4.2) and (4.14_1) for $C^+_{n,1}$ it is clear that (4.14_1) is true for $C^-_{n,1}$ too. Analogously one can prove that

$$C^{\pm}_{n,1} = \{t \in A_{n,2} : \pm w(t) \geq 1/n\} \quad \text{for} \quad n \in N. \tag{4.14$_2$}$$

From (4.14_2), the definition of the sets $D^{\pm}_n$ and (4.3) we obtain (4.6). From the definition of the set $\Omega^{\pm}_w$ and (4.14_2) we have $C^{\pm}_{n,1} \subset \Omega^{\pm}_w$. Hence

$$\text{mes} C^{\pm}_{n,1} \leq \text{mes} \Omega^{\pm}_w.$$

On the other hand $C^{\pm}_{n,1} = \{t \in I : \pm w(t) \ge 1/n\} \setminus (I \setminus A_{n,2})$ and thus

$$\mathrm{mes} C^{\pm}_{n,1} \ge \mathrm{mes}\Omega^{\pm}_{w} - \mathrm{mes}(I \setminus A_{n,2}).$$

In view of (4.7) from last two inequalities we conclude that (4.15) holds. ∎

Lemma 4.2. *Let $i \in \{1, 2\}$, $r > 0$, $k \in N$, w_0 be a nonzero solution of the problem (1.3), (1.4), $N_{w_0} = \{t_1, ..., t_k\}$, the function $f_1 \in E(N_{w_0})$ be non-decreasing in the second argument for $|x| \ge r$, and*

$$f_1(t, x)\mathrm{sgn} x \ge 0 \qquad \text{for} \quad t \in I,\ |x| \ge r. \tag{4.18}$$

Then:

a) If $G \subset I$ and

$$\int_G |f_1(s, (-1)^i r)w_0(s)|ds \ne 0, \tag{4.19}$$

then there exist $\delta_0 > 0$ and $\varepsilon_1 > 0$ such that

$$\mathbb{I}(G, U_\varepsilon, x) \overset{def}{\equiv} \int\limits_{G\setminus U_\varepsilon} |f_1(s, x)w_0(s)|ds - \int\limits_{U_\varepsilon} |f_1(s, x)w_0(s)|ds \ge \delta_0 \tag{4.20}$$

for $(-1)^i x \ge r$ and $0 < \varepsilon \le \varepsilon_1$, where $U_\varepsilon = I \cap \Big(\cup_{j=1}^{k} [t_j - \varepsilon/2k,\ t_j + \varepsilon/2k]\Big)$.

b) If $u_n \in \widetilde{C}'(I; R)$ $(n \in N)$, $r > 0$, w is an arbitrary nonzero solution of the problem (1.3), (1.4), and the condition (4.3) holds, then there exist $\varepsilon_2 \in]0, \varepsilon_1]$ and $n_0 \in N$ such that

$$\mathbb{I}(D^{+}_{n}, U^{+}_{\varepsilon}, x) \ge -\frac{\delta_0}{2} \quad \text{for} \quad x \ge r, \tag{4.21$_1$}$$

$$\mathbb{I}(D^{-}_{n}, U^{-}_{\varepsilon}, x) \ge -\frac{\delta_0}{2} \quad \text{for} \quad x \le -r \tag{4.21$_2$}$$

for $n \ge n_0$ and $0 < \varepsilon \le \varepsilon_2$, where $U^{\pm}_{\varepsilon} = \{t \in U_\varepsilon : \pm w(t) \ge 0\}$.

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \ne 0$ such that $w(t) = \beta w_0(t)$ and thus $N_w = N_{w_0}$.

a) For any $\alpha \in R_+$, we put $G_1 = ([a, a + \alpha] \cup [b - \alpha, b]) \cap G$. In view of the condition (4.19), we can choose $\alpha \in]0, (b - a)/2[$ such that if $G_2 = G \setminus G_1$, $t_a = \inf\{G_2\}$ and $t_b = \sup\{G_2\}$, then

$$a < t_a, \quad t_b < b, \tag{4.22}$$

and $\int_{G_1} |f_1(s, (-1)^i r)w_0(s)|ds \ne 0$, $\int_{G_2} |f_1(s, (-1)^i r)|ds \ne 0$. From these inequalities, by virtue of conditions (4.18) and $f_1 \in E(N_{w_0})$, where f_1 is non-decreasing in the second argument, there follows the existence of $\delta_0 > 0$ and $\varepsilon^* > 0$ such that

$$\int_{G_2\setminus U_{\varepsilon^*}} |f_1(s, x)|ds - \int_{U_{\varepsilon^*}} |f_1(s, x)|ds \ge 0 \quad \text{for} \quad (-1)^i x \ge r, \tag{4.23}$$

$$\int_{G_1\setminus U_{\varepsilon^*}} |f_1(s,x)w_0(s)|ds \geq \delta_0 \quad \text{for} \quad (-1)^i x \geq r. \tag{4.24}$$

Now we put $I^* = [t_a^*,\ t_b^*]$, where $t_a^* = \dfrac{a+\min(t_a,t_1)}{2}$ and $t_b^* = \dfrac{\max(t_k,t_b)+b}{2}$. In view of (4.22), we obtain

$$G_2 \subset I^*, \quad N_{w_0} \subset I^*, \quad w_0(t_a^*) \neq 0,\ w_0(t_b^*) \neq 0. \tag{4.25}$$

Then it is clear that there exists $\gamma_1 > 0$ such that, for any $\gamma \in]0,\gamma_1[$, the equation $|w_0(t)| = \gamma$ has only $t_{\gamma,i}, t_{\gamma,i}^* \in I^*$ $(i = 1,...,k)$ solutions such that

$$t_{\gamma,i} < t_i < t_{\gamma,i}^* \quad (i = 1,...,k), \tag{4.26}$$

$$|w_0(t)| \leq \gamma \quad \text{for} \quad t \in H_\gamma, \qquad |w_0(t)| > \gamma \quad \text{for} \quad t \in I^* \setminus H_\gamma, \tag{4.27}$$

where $H_\gamma = \bigcup\limits_{i=1}^{k} [t_{\gamma,i}, t_{\gamma,i}^*]$, and

$$\lim_{\gamma\to+0} t_{\gamma,i} = \lim_{\gamma\to+0} t_{\gamma,i}^* = t_i \quad (i = 1,...,k). \tag{4.28}$$

The relations (4.26) and (4.28) imply that there exist $\gamma \in]0,\gamma_1]$ and $\varepsilon_1 \in]0,\varepsilon^*]$ such that

$$U_{\varepsilon_1} \subset H_\gamma \subset U_{\varepsilon^*}. \tag{4.29}$$

Moreover, in view of the inclusion $G_1 \subset G$, it is clear that

$$G \setminus U_{\varepsilon_1} = \big[(G\setminus G_1)\setminus U_{\varepsilon_1}\big] \cup \big(G_1 \setminus U_{\varepsilon_1}\big),\ \big[(G\setminus G_1)\setminus U_{\varepsilon_1}\big] \cap \big(G_1\setminus U_{\varepsilon_1}\big) = \emptyset,$$

and thus

$$\mathbb{I}(G, U_{\varepsilon_1}, x) = \int_{G_1\setminus U_{\varepsilon_1}} |f_1(s,x)w_0(s)|ds + \mathbb{I}(G_2, U_{\varepsilon_1}, x) \quad \text{for} \quad (-1)^i x \geq r.$$

By virtue of (4.23), (4.25), (4.27), and (4.29), we get

$$\mathbb{I}(G_2, U_{\varepsilon_1}, x) \geq \gamma\Big(\int_{G_2\setminus H_\gamma} |f_1(s,x)|ds - \int_{H_\gamma} |f_1(s,x)|ds\Big) \geq$$

$$\geq \gamma\Big(\int_{G_2\setminus U_{\varepsilon^*}} |f_1(s,x)|ds - \int_{U_{\varepsilon^*}} |f_1(s,x)|ds\Big) \geq 0$$

for $(-1)^i x \geq r$. In view of the last two relations, (4.24), (4.29), and the fact that $U_\varepsilon \subset U_{\varepsilon_1}$ for $\varepsilon \leq \varepsilon_1$, we conclude that the inequality (4.20) holds.

b) First consider the case when

$$\int_{D_n^+} |f_1(s,x)w_0(s)|ds = 0 \ \text{ for } \ x \geq r,\ n \in N. \tag{4.30}$$

From (4.3) and the definitions of the sets $D_n^{\pm}$ and $U_\varepsilon^{\pm}$ we get

$$\lim_{n\to+\infty} \operatorname{mes}(U_\varepsilon^{\pm} \setminus D_n^{\pm}) = 0. \tag{4.31}$$

Then, in view of (4.30) and the fact that for any $\varepsilon > 0$ and $n \in N$

$$U_\varepsilon^{\pm} = (U_\varepsilon^{\pm} \cap D_n^{\pm}) \cup (U_\varepsilon^{\pm} \setminus D_n^{\pm}), \quad (U_\varepsilon^{\pm} \cap D_n^{\pm}) \cap (U_\varepsilon^{\pm} \setminus D_n^{\pm}) = \emptyset, \tag{4.32}$$

we have $\int_{U_\varepsilon^+} |f_1(s,x)w_0(s)|ds = \int_{U_\varepsilon^+ \setminus D_n^+} |f_1(s,x)w_0(s)|ds$ for $x \geq r$, $n \in N$, and $\varepsilon > 0$. Thus by virtue of (4.31), we get $\int_{U_\varepsilon^+} |f_1(s,x)w_0(s)|ds = 0$. From the last equality and (4.30) we conclude that

$$I(D_n^+, U_\varepsilon^+, x) = 0 \quad \text{for} \quad x \geq r,\ n \in N,\ \varepsilon > 0. \tag{4.33}$$

Therefore, in this case the condition (4.21_1) is true.

Now consider the case when for some $r_1 \geq r$ there exists $n_0 \in N$ such that

$$\int_{D_n^+} |f_1(s,x)w_0(s)|ds \neq 0 \ \text{ for } \ x \geq r_1,\ n \geq n_0. \tag{4.34}$$

It is clear that there exist $\eta > 0$ and $\varepsilon_2 \in]0, \varepsilon_1]$ such that

$$\int_{U_\varepsilon^+} |f_1(s,x)w_0(s)|ds \leq \frac{\delta_0}{2} \ \text{ for } \ r \leq x \leq r_1 + \eta,\ \varepsilon \leq \varepsilon_2,$$

and thus

$$I(D_n^+, U_\varepsilon^+, x) \geq -\frac{\delta_0}{2} \quad \text{for} \quad r \leq x \leq r_1 + \eta,\ n \geq n_0,\ \varepsilon \leq \varepsilon_2. \tag{4.35}$$

On the other hand, from (4.34) it is clear that $\int_{D_{n_0}^+} |f_1(s, r_1 + \eta)w_0(s)|ds \neq 0$. Therefore, from the item $a)$ of our lemma with $G = D_n^+$, and the inclusions $D_{n_0}^+ \subset D_n^+$, $U_\varepsilon^+ \subset U_\varepsilon$ for $n \geq n_0$, $\varepsilon > 0$, we get $I(D_n^+, U_\varepsilon^+, x) \geq \delta_0$ for $x \geq r_1 + \eta$, $n \geq n_0$, $0 < \varepsilon \leq \varepsilon_2$. From this inequality and (4.35) we obtain (4.21_1) in second case too.

Analogously one can prove (4.21_2). ■

Lemma 4.3. *Let all the conditions of Lemma* 4.1 *be fulfilled and there exist* $r > 0$ *such that the condition* (4.18) *holds, where* $f_1 \in K(I \times R; R)$. *Then*

$$\liminf_{n\to+\infty} \int_s^t f_1(\xi, u_n(\xi)) \operatorname{sgn} u_n(\xi) d\xi \geq 0 \quad \textit{for} \quad a \leq s < t \leq b. \tag{4.36}$$

Proof. Let

$$\gamma_r^*(t) \overset{def}{=} \sup\{|f_1(t,x)| : |x| \leq r\} \quad \text{for} \quad t \in I. \tag{4.37}$$

Then, according to (4.1), (4.2), and (4.18), we obtain the estimate

$$\int_s^t f_1(\xi, u_n(\xi)) \operatorname{sgn} u_n(\xi) d\xi \geq$$

$$\geq -\int_{[s,t]\cap A_{n,1}} \gamma_r^*(\xi) d\xi + \int_{[s,t]\cap A_{n,2}} |f_1(\xi, u_n(\xi))| d\xi$$

for $a \leq s < t \leq b$, $n \in N$. This estimate and (4.7) imply (4.36). ■

Lemma 4.4. *Let $r > 0$, the functions $f_1 \in K(I \times R; R)$, $h_1 \in L(I; R)$, $f^+, f^- \in L(I; R_+)$ be such that*

$$f_1(t,x) \le -f^-(t) \quad \text{for} \quad x \le -r, \qquad f^+(t) \le f_1(t,x) \quad \text{for} \quad x \ge r \tag{4.38}$$

on I, and there exist a nonzero solution w_0 of the problem (1.3), (1.4) *and $\varepsilon > 0$ such that*

$$N_{w_0} = \emptyset \tag{4.39}$$

and

$$-\int_a^b f^-(s)|w_0(s)|ds + \varepsilon||\gamma_r^*||_L \le -\int_a^b h_1(s)|w_0(s)|ds \le$$

$$\le \int_a^b f^+(s)|w_0(s)|ds - \varepsilon||\gamma_r^*||_L, \tag{4.40}$$

where γ_r^* is defined by (4.37)*. Then, for every nonzero solution w of the problem* (1.3), (1.4), *and functions $u_n \in \widetilde{C}'(I; R)$ $(n \in N)$ such that the conditions* (4.3),

$$|v_n^{(i)}(t) - w^{(i)}(t)| \le 1/2n \quad \text{for} \quad t \in I, \ n \in N, \ (i = 0, 1) \tag{4.41}$$

where $v_n(t) = u_n(t)||u_n||_C^{-1}$ for $t \in I$ and

$$u_n(a) = 0, \quad u_n(b) = 0 \tag{4.42}$$

are fulfilled, there exists $n_1 \in N$ such that

$$\mathbb{M}_n(w) \overset{def}{\equiv} \int_a^b (h_1(s) + f_1(s, u_n(s)))w(s)ds \ge 0 \quad \text{for} \quad n \ge n_1. \tag{4.43}$$

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \ne 0$ such that $w(t) = \beta w_0(t)$. Also, it is not difficult to verify that all the assumptions of Lemma 4.1 are satisfied for the function $w(t) = \beta w_0(t)$. From the unique solvability of the Cauchy problem for the equation (1.3) and the conditions (1.4) we conclude that $w'(a) \ne 0$ and $w'(b) \ne 0$. Therefore, in view of (4.41) and (4.42), there exists $n_2 \in N$ such that

$$u_n(t)\mathrm{sgn}\beta w_0(t) > 0 \qquad \text{for} \quad n \ge n_2, \ a < t < b. \tag{4.44}$$

Moreover, by (4.1) and (4.2) we get the estimate

$$\frac{\mathbb{M}_n(w)}{|\beta|} \ge -\int_{A_{n,1}} \gamma_r^*(s)|w_0(s)|ds + \sigma\int_a^b h_1(s)w_0(s)ds + +\sigma\int_{A_{n,2}} f_1(s, u_n(s))w_0(s)ds, \tag{4.45}$$

where γ_r^* is given by (4.37) and $\sigma = \mathrm{sgn}\beta$. Now note that $f^- \equiv 0$, $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Then by virtue of (4.7), we see that there exist $\varepsilon > 0$ and $n_1 \in N$ ($n_1 \geq n_2$) such that $\int_a^b f^{\pm}(s)|w_0(s)|ds - \frac{\varepsilon}{2}||\gamma_r^*||_L \leq \int_{A_{n,2}} f^{\pm}(s)|w_0(s)|ds$ and $\frac{\varepsilon}{2}||\gamma_r^*||_L \geq \int_{A_{n,1}} \gamma_r^*(s)|w_0(s)|ds$ for $n \geq n_1$. By these inequalities, (4.3), (4.38) and (4.44), from (4.45) we obtain

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\gamma_r^*||_L + \int_a^b h_1(s)|w_0(s)|ds + \int_a^b f^+(s)|w_0(s)|ds$$

if $n \geq n_1$, $\sigma w_0(t) \geq 0$, and

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon||\gamma_r^*||_L - \int_a^b h_1(s)|w_0(s)|ds + \int_a^b f^-(s)|w_0(s)|ds$$

if $n \geq n_1$, $\sigma w_0(t) \leq 0$. From the last two estimates in view of (4.40) it follows that (4.43) is valid. ∎

Lemma 4.5. *Let w_0 be a nonzero solution of the problem* (1.3), (1.4), $r > 0$, *the function $f_1 \in E(N_{w_0})$ be non-decreasing in the second argument for $|x| \geq r$, condition* (4.18) *hold, and*

$$\int_{\Omega_{w_0}^+} |f_1(s,r)|ds + \int_{\Omega_{w_0}^-} |f_1(s,-r)|ds \neq 0. \tag{4.46}$$

Then there exist $\delta > 0$ and $n_1 \in N$ such that if

$$\left|\int_a^b h_1(s)w_0(s)ds\right| < \delta \tag{4.47}$$

then, for every nonzero solution w of the problem (1.3), (1.4) *and the functions $u_n \in \widetilde{C}'(I;R)$ ($n \in N$) fulfilling the conditions* (4.3), (4.41), (4.42), *the inequality* (4.43) *holds.*

Proof. It is not difficult to verify that all the assumption of Lemma 4.1 are satisfied. Then, by the definition of the sets $B_{n,1}$, $B_{n,2}$, the conditions (4.1), (4.2), and (4.18), we obtain the estimate

$$\int_a^b f_1(s,u_n(s))w(s)ds \geq -\int_{A_{n,1}} \gamma_r^*(s)|w(s)|ds + \widehat{\mathbb{M}}_n(w), \tag{4.48}$$

where

$$\widehat{\mathbb{M}}_n(w) \overset{def}{\equiv} -\int_{B_{n,2}} |f_1(s,u_n(s))w(s)|ds + \int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds.$$

On the other hand, from the unique solvability of the Cauchy problem for the equation (1.3) it is clear that

$$w'(a) \neq 0, \quad w'(b) \neq 0, \quad w'(t_i) \neq 0 \qquad \text{for } \ i = 1, ..., k \tag{4.49}$$

if $N_{w_0} = \{t_1, ..., t_k\}$. Now note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Consequently,

$$\Omega_w^{\pm} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta > 0 \quad \text{and} \quad \Omega_w^{\mp} = \Omega_{w_0}^{\pm} \quad \text{if} \quad \beta < 0. \tag{4.50}$$

Then in view of (4.15) and (4.46), there exists $n_2 \geq n_0$ such that

$$\int_{C^+_{n_2,1}} |f_1(s,r)w_0(s)|ds \neq 0 \ \text{ and/or } \int_{C^-_{n_2,1}} |f_1(s,-r)w_0(s)|ds \neq 0. \tag{4.51}$$

From (4.51), in view of (4.6), it follows that

$$\int_{D_n^+} |f_1(s,r)w_0(s)|ds \neq 0 \quad \text{for} \quad n \geq n_2 \tag{4.52$_1$}$$

and/or

$$\int_{D_n^-} |f_1(s,-r)w_0(s)|ds \neq 0 \quad \text{for} \quad n \geq n_2. \tag{4.52$_2$}$$

Consequently, all the assumptions of Lemma 4.2 are satisfied with $G = D_n^+$ and/or $G = D_n^-$. Therefore, there exist $\varepsilon_0 \in]0, \varepsilon_2[$, $n_3 \geq n_2$, and $\delta_0 > 0$ such that

$$\begin{gathered} \mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) \geq \delta_0 \ \text{ for } \ x \geq r, \quad n \geq n_3, \\ \mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) \geq -\delta_0/2 \ \text{ for } \ x \leq -r, \ \ n \geq n_3 \end{gathered} \tag{4.53}$$

if (4.52$_1$) holds, and

$$\begin{gathered} \mathbb{I}(D_n^-, U_{\varepsilon_0}^-, x) \geq \delta_0 \quad \text{for} \quad x \leq -r, \ n \geq n_3, \\ \mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) \geq -\delta_0/2 \quad \text{for} \quad x \geq r, \ \ n \geq n_3 \end{gathered} \tag{4.54}$$

if (4.52$_2$) holds.

On the other hand, the definition of the set U_ε and (4.14$_1$), imply that there exists $n_4 > n_3$, such that

$$C_{n,2}^+ \subset U_{\varepsilon_0}^+, \quad C_{n,2}^- \subset U_{\varepsilon_0}^- \quad \text{for} \quad n \geq n_4. \tag{4.55}$$

By these inclusions, (4.2), and (4.5) we obtain

$$C_{n,1}^+ = A_{n,2}^+ \setminus C_{n,2}^+ \supset D_{n_4}^+ \setminus U_{\varepsilon_0}^+, \ C_{n,1}^- = A_{n,2}^- \setminus C_{n,2}^- \supset D_{n_4}^- \setminus U_{\varepsilon_0}^+ \tag{4.56}$$

for $n \geq n_4$. First suppose that $N_{w_0} \neq \emptyset$ and there exists $n \geq n_4$ such that

$$B_{n,2} \neq \emptyset. \tag{4.57}$$

Then, by taking into account that f_1 is non-decreasing in the second argument for $|x| \geq r$, (4.3), (4.12), (4.18) and the definitions of the sets $B^+_{n,2}, B^-_{n,2}$, we get

$$
\begin{gathered}
|f_1(t, u_n(t))| = f_1(t, u_n(t)) \leq \\
\leq f_1\left(t, \frac{\|u_n\|_C}{2n}\right) = \left|f_1\left(t, \frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for } t \in B^+_{n,2}, \\
|f_1(t, u_n(t))| = -f_1(t, -u_n(t)) \leq \\
\leq -f_1(t, -\frac{\|u_n\|_C}{2n}) = \left|f_1\left(t, -\frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for } t \in B^-_{n,2}.
\end{gathered} \tag{4.58}
$$

Analogously, from (4.3), (4.13), (4.18), and the definitions of the sets $C^+_{n,1}, C^-_{n,1}$, we obtain the estimates

$$
\begin{gathered}
|f_1(t, u_n(t))| \geq \left|f_1\left(t, \frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for} \quad t \in C^+_{n,1}, \\
|f_1(t, u_n(t))| \geq \left|f_1\left(t, -\frac{\|u_n\|_C}{2n}\right)\right| \quad \text{for} \quad t \in C^-_{n,1}.
\end{gathered} \tag{4.59}
$$

Then from (4.1), (4.2), (4.9), (4.58) and respectively from (4.1), (4.2), (4.8), and (4.59) we have

$$
\int_{B_{n,2}} |f_1(s, u_n(s))w(s)| ds \leq
$$

$$
\leq \int_{B^+_{n,2}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w(s)\right| ds + \int_{B^-_{n,2}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w(s)\right| ds \leq \tag{4.60}
$$

$$
\leq \int_{C^+_{n,2}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w(s)\right| ds + \int_{C^-_{n,2}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w(s)\right| ds
$$

and respectively

$$
\int_{B_{n,1}} |f_1(s, u_n(s))w(s)| ds \geq \int_{C_{n,1}} |f_1(s, u_n(s))w(s)| ds \geq
$$

$$
\geq \int_{C^+_{n,1}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w(s)\right| ds + \int_{C^-_{n,1}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w(s)\right| ds. \tag{4.61}
$$

If the condition (4.57) holds, from (4.60) and (4.61) we obtain

$$
\frac{\widehat{\mathbb{M}}_n(w)}{|\beta|} \geq \left(\int_{C^+_{n,1}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds - \int_{C^+_{n,2}} \left|f_1\left(s, \frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds\right)
$$

$$
+\left(\int_{C^-_{n,1}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds - \int_{C^-_{n,2}} \left|f_1\left(s, -\frac{\|u_n\|_C}{2n}\right) w_0(s)\right| ds\right),
$$

Whence, by (4.55) and (4.56) we get

$$\frac{\widehat{\mathbb{M}}_n(w)}{|\beta|} \geq \mathbb{I}\left(D^+_{n_4}, U^+_{\varepsilon_0}, \frac{\|u_n\|_C}{2n}\right) + \mathbb{I}\left(D^-_{n_4}, U^-_{\varepsilon_0}, -\frac{\|u_n\|_C}{2n}\right) \tag{4.62}$$

for $n \geq n_4$. From (4.62) by (4.53) and (4.54) we obtain

$$\widehat{\mathbb{M}}_n(w) \geq \frac{\delta_0|\beta|}{2} \qquad \text{for} \quad n \geq n_4. \tag{4.63}$$

On the other hand, in view of (4.10), (4.18), the definition of the sets $A_{n,2}, B_{n,1}$, and the fact that f_1 is non-decreasing in the second argument, we obtain the estimate

$$\int_{B_{n,1}} |f_1(s, u_n(s))w(s)|ds \geq$$

$$\geq \int_{B^+_{n,1}} |f_1(s,r)w(s)|ds + \int_{B^-_{n,1}} |f_1(s,-r)w(s)|ds \geq \tag{4.64}$$

$$\geq \int_{C^+_{n,1}} |f_1(s,r)w(s)|ds + \int_{C^-_{n,1}} |f_1(s,-r)w(s)|ds.$$

Now suppose that there exists $n \geq n_4$ such that

$$B_{n,2} = \emptyset. \tag{4.65}$$

Then from (4.51) and (4.64), (4.65) there follows the existence of $\delta^* > 0$ such that $\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta^*$. From this inequality and (4.63) it follows that, in both cases when (4.57) or (4.65) are fulfilled, the inequality

$$\widehat{\mathbb{M}}_n(w) \geq |\beta|\delta \quad \text{for} \quad n \geq n_4 \tag{4.66}$$

holds with $\delta = \min\{\delta_0/2, \delta^*\}$. From (4.48) by (4.7) and (4.66), we see that for any $\varepsilon \in]0, \delta[$ there exists $n_1 > n_4$ such that

$$\int_a^b f_1(s, u_n(s))w(s)ds \geq |\beta|(\delta - \varepsilon) \quad \text{for} \quad n \geq n_1,$$

and thus

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq \delta - \varepsilon - \left|\int_a^b h_1(s)w_0(s)ds\right| \quad \text{for} \quad n \geq n_1. \tag{4.67}$$

If $N_{w_0} = \emptyset$ then $|w(t)| > 0$ for $a < t < b$ and in view of (4.3), (4.41), (4.42) and (4.49), the condition (4.65) holds, i.e., the inequality (4.67) also holds.

Consequently, since $\varepsilon > 0$ is arbitrary, the inequality (4.43) from (4.67) and (4.47) follows. ■

Lemma 4.6. *Let w_0 be a nonzero solution of the problem* (1.3), (1.4), $r > 0$, *and the conditions* (4.18), (4.47) *hold with* $f_1(t,x) \stackrel{def}{\equiv} f_0(t)g_1(x)$, *where* $f_0 \in L(I;R_+)$, $\int_a^b |f_0(s)|ds \neq 0$ *and a non-decreasing function* $g_1 \in C(R;R)$ *be such that*

$$\lim_{|x|\to+\infty} |g_1(x)| = +\infty. \tag{4.68}$$

Then, for every nonzero solution w of the problem (1.3), (1.4) *and functions* $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ *fulfilling the conditions* (4.3), (4.41), (4.42), *the inequality* (4.43) *holds.*

Proof. From the assumptions of our lemma it is clear that the relations (4.48)–(4.56), (4.58)-(4.61) and (4.64) with $f_1(t,x) = f_0(t)g_1(x)$ and $w(t) = \beta w_0(t)$ $(\beta \neq 0)$ are fulfilled.

Assuming $\int_{C^+_{n_2,1}} |f_1(s,r)w_0(s)|ds \neq 0$, the condition (4.52_1) is satisfied i.e., (4.53) holds.

Now notice that from (4.15) and the equality $C^+_{n,1} = \Omega^+_w \setminus (\Omega^+_w \setminus C^+_{n,1})$ it follows that there exist $\varepsilon > 0$ and $n_0 \in N$ such that

$$\int_{C^+_{n,1}} |f_0(s)w_0(s)|ds \geq \int_{\Omega^+_w} |f_0(s)w_0(s)|ds - \varepsilon > 0 \tag{4.69}$$

for $n \geq n_0$.

First consider the case when there exists $n \geq n_4$ such that the condition (4.65) holds. Without loss of generality we can assume that $n_4 > n_0$. Then by (4.50), (4.64), (4.65) and (4.69), we obtain

$$\widehat{\mathbb{M}}_n(w) \geq |\beta||g_1(r)| \left(\int_{\Theta_\beta} |f_0(s)w_0(s)|ds - \varepsilon \right) > 0, \tag{4.70}$$

where $\Theta_\beta = \Omega^+_{w_0}$ if $\beta > 0$ and $\Theta_\beta = \Omega^-_{w_0}$ if $\beta < 0$.

Consider now the case when there exists $n \geq n_4$ such that (4.57) holds. From (4.3) and the definition of the set D^+_n it follows that $D^+_n \subset D^+_{n+1}$, and since g_1 is non-decreasing, from (4.53) we obtain $\mathbb{I}(D^+_n, U^+_{\varepsilon_0}, x) \geq |g_1(r)|\mu = \mathbb{I}(D^+_{n_4}, U^+_{\varepsilon_0}, r) \geq \delta_0$ for $x \geq r$, with $\mu = \int_{D^+_{n_4}\setminus U^+_{\varepsilon_0}} |f_0(s)w_0(s)|ds - \int_{U^+_{\varepsilon_0}} |f_0(s)w_0(s)|ds$. By the last inequality, (4.3), (4.53), and (4.62) we get $\mu > 0$ and

$$\widehat{\mathbb{M}}_n(w) \geq |\beta|(|g_1(r)|\mu - \delta_0/2). \tag{4.71}$$

Applying (4.70), (4.71) in (4.48) and taking (4.7) into account, we conclude that there exist $\varepsilon_1 > 0$ and $n_1 \geq n_4$ such that

$$|\beta| \left(|g_1(r)|\mu_1 - \frac{\delta_0}{2} - \varepsilon_1 \right) \leq \int_a^b f_1(s,u_n(s))w(s)ds \quad \text{for} \quad n \geq n_1$$

with $\mu_1 = \min(\mu, \int_{\Omega^+_{w_0}} |f_0(s)w_0(s)|ds - \varepsilon)$. From (4.68) and the last inequality it is clear that, for any function h_1, we can choose $r > 0$ such that the inequality (4.43) will be true. Analogously one can prove (4.43) in the case when $\int_{C^-_{n_2,1}} |f_1(s,r)w_0(s)|ds \neq 0$. ∎

Lemma 4.7. *Let $r > 0$, there exist functions $\alpha, f^-, f^+ \in L(I, R_+)$ such that the condition* (4.38) *is satisfied,*

$$\sup\{|f_1(t,x)| : x \in R\} = \alpha(t) \quad \text{for} \quad t \in I, \tag{4.72}$$

and there exist a nonzero solution w_0 of the problem (1.3), (1.4) *and $\varepsilon > 0$ such that*

$$\begin{aligned} -\int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds + \varepsilon||\alpha||_L \leq \\ \leq -\int_a^b h_1(s)w_0(s)ds \leq \\ \leq \int_a^b (f^-(s)[w_0(s)]_- + f^+(s)[w_0(s)]_+)ds - \varepsilon||\alpha||_L. \end{aligned} \tag{4.73}$$

Then, for every nonzero solution w of the problem (1.3), (1.4) *and functions $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$ fulfilling the conditions* (4.3), (4.41), *and* (4.42), *there exists $n_1 \in N$ such that the inequality* (4.43) *holds.*

Proof. First note that, for any nonzero solution w of the problem (1.3), (1.4), there exists $\beta \neq 0$ such that $w(t) = \beta w_0(t)$. Moreover, it is not difficult to verify that all the assumptions of Lemma4.1 are satisfied for the function $w(t) = \beta w_0(t)$. From (4.1), (4.2), and (4.72) we get

$$\begin{aligned} \mathbb{M}_n(w) \geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{B_{n,1}} f_1(s,u_n)w(s)ds + \\ + \int_a^b h_1(s)w(s)ds. \end{aligned} \tag{4.74}$$

On the other hand, by the definition of the set $B_{n,1}$ we obtain

$$\text{sgn} u_n(t) = \text{sgn} w(t) \quad \text{for} \quad t \in B^+_{n,1} \cup B^-_{n,1}. \tag{4.75}$$

Hence, by (4.1), (4.2), (4.10), (4.38), and (4.75), from (4.74) we obtain the estimate

$$\begin{aligned} \mathbb{M}_n(w) - \int_a^b h_1(s)w(s)ds \geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \\ + \int_{B^+_{n,1}} f^+(s)|w(s)|ds + \int_{B^-_{n,1}} f^-(s)|w(s)|ds \geq \\ \geq -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{C^+_{n,1}} f^+(s)|w(s)|ds + \int_{C^-_{n,1}} f^-(s)|w(s)|ds. \end{aligned} \tag{4.76}$$

Now, note that $f^- \equiv 0$ and $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Therefore by (4.7), (4.11), (4.15), and the inclusions $C^+_{n,1} \subset \Omega^+_w$, $C^-_{n,1} \subset \Omega^-_w$, we see that there exist $\varepsilon > 0$

and $n_1 \in N$ such that

$$
\begin{aligned}
&\frac{1}{3}\varepsilon\|\alpha\|_L \geq \int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w_0(s)|ds \\
&\int_{\Omega_w^\pm} f^\pm(s)|w_0(s)|ds - \frac{1}{3}\varepsilon\|\alpha\|_L \leq \int_{C_{n,1}^\pm} f^\pm(s)|w_0(s)|ds
\end{aligned}
\tag{4.77}
$$

for $n \geq n_1$. By virtue of (4.76) and (4.77), we obtain

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon\|\alpha\|_L + \int_{\Omega_w^+} f^+(s)|w_0(s)|ds +$$

$$+ \int_{\Omega_w^-} f^-(s)|w_0(s)|ds + \sigma \int_a^b h_1(s)w_0(s)ds$$

for $n \geq n_1$, where $\sigma = \operatorname{sgn}\beta$. Now, by taking into account that

$$\int_{\Omega_w^\pm} l(s)|w_0(s)|ds = \int_{\Omega_{w_0}^\pm} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_\pm ds$$

if $\beta > 0$ and

$$\int_{\Omega_w^\pm} l(s)|w_0(s)|ds = \int_{\Omega_{w_0}^\mp} l(s)|w_0(s)|ds = \int_a^b l(s)[w_0(s)]_\mp ds$$

if $\beta < 0$ for an arbitrary $l \in L(I, R)$, from the last inequalities we get

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon\|\alpha\|_L + \int_a^b (f^+(s)[w_0(s)]_+ + f^-(s)[w_0(s)]_-)ds +$$

$$+ \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1$$

if $\sigma = 1$, and

$$\frac{\mathbb{M}_n(w)}{|\beta|} \geq -\varepsilon\|\alpha\|_L + \int_a^b (f^+(s)[w_0(s)]_- + f^-(s)[w_0(s)]_+)ds -$$

$$- \int_a^b h_1(s)w_0(s)ds \quad \text{for} \quad n \geq n_1$$

if $\sigma = -1$. From the last inequalities and (4.73) we immediately obtain (4.43). ∎

Now we consider the definitions of the sets $V_{10}((a,b))$ introduced and described in [12] (see [Definition 1.3, p. 2350])

Definition 4.2. We say that the function $p \in L([a,b])$ belongs to the set $V_{10}((a,b))$ if for any function p^* satisfying the inequality $p^*(t) \geq p(t)$ for $t \in I$ the unique solution of the initial value problem

$$u''(t) = p^*(t)u(t) \quad \text{for} \quad t \in I, \quad u(a) = 0, \quad u'(a) = 1, \tag{4.78}$$

has no zeros in the set $]a, b]$.

Lemma 4.8. *Let $i \in \{1,2\}$, $p \in L(I;R)$, $p_n(t) = p(t) + (-1)^i/n$, and $w_n \in \widetilde{C}'(I;R)$ $(n \in N)$ be a solution of the problem*

$$w_n''(t) = p_n(t)w_n(t) \quad \text{for} \quad t \in I, \quad w_n(a) = 0, \quad w_n(b) = 0. \tag{4.79$_n$}$$

Then:

a) There exists $n_0 \in N$ such that the problem (4.79_n) has only the zero solution for $n \geq n_0$.

b) If $i = 2$ and $N_w = \emptyset$, where w is a solution of the problem (1.3), (1.4), then the inclusion $p_n \in V_{10}((a,b))$ for every $n \in N$ holds.

Proof. *a)* Let $N^*_{w_n}$ be the number of zeros of the function w_n on I. Assume on the contrary that there exists a sequence $\{w_n\}_{n\geq n_0}^{+\infty}$ of nonzero solutions of the problem (4.79_n).

If $i = 1$ then from the facts that $p_n(t) < p_{n+1}(t)$ and $w_n \not\equiv 0$, by Sturm's comparison theorem, we obtain $N^*_{w_n} - N^*_{w_{n+1}} \geq 1$ $(n \in N)$. Now notice that, in view of (4.79_n), the inequality $N^*_{w_n} \geq 2$ holds. Hence there exist $k_0 \geq 2$ and $n_0 \geq 2$ such that $N^*_{w_{n_0}} = k_0$. Therefore, we obtain the contradiction $k_0 = N^*_{w_{n_0}} > N^*_{w_{n_0}} - N^*_{w_{n_0+k_0}} = (N^*_{w_{n_0}} - N^*_{w_{n_0+1}}) + (N^*_{w_{n_0+1}} - N^*_{w_{n_0+2}}) + \ldots + (N^*_{w_{n_0+k_0-1}} - N^*_{w_{n_0+k_0}}) \geq k_0$.

If $i = 2$, from the fact that $p_{n-1}(t) > p_n(t) > p(t)$ and $w_n \not\equiv 0$, by Sturm's comparison theorem, we obtain $N^*_{w_n} - N^*_{w_{n-1}} \geq 1$ and $N^*_w \geq N^*_{w_n} - 1$ $(n \in N)$ if w is a nonzero solution of the equation (1.3). Now notice that, in view of (4.79_n), the inequality $N^*_{w_n} \geq 2$ holds for every $n \in N$. Therefore, if we denote $N^*_w = k_0$, we obtain the contradiction $k_0 = N^*_w \geq N^*_{w_{n+k_0}} - 1 > N^*_{w_{n+k_0}} - N^*_{w_n} \geq k_0$.

The contradiction obtained proves the item *a)* of our lemma.

b) Assume on the contrary that there exists $n \in N$ such that $p_n \notin V_{10}([a,b])$. If $p^*(t) \geq p_n(t)$ and u is a solution of the problem (4.78), then there exists $t_0 \in]a,b]$ such that $u(t_0) = 0$. Since $p(t) < p^*(t)$, by Sturm's comparison theorem, we obtain that w, the solution of the problem (1.3), (1.4), has a zero in the interval $]a, t_0[$, which contradicts our assumption $N_w = \emptyset$. The contradiction obtained proves the item *b)* of our lemma. ■

5. Proof of the main results

Proof of Theorem 2.1. Let $p_n(t) = p(t) + 1/n$ and, for any $n \in N$, consider the problem

$$u_n''(t) = p_n(t)u_n(t) + f(t, u_n(t)) + h(t) \quad \text{for} \quad t \in I, \tag{5.1}$$

$$u_n(a) = 0, \quad u_n(b) = 0. \tag{5.2}$$

In view of the condition (2.1) and Lemma 4.8, the inclusion $p_n \in V_{10}((a,b))$ holds for every $n \in N$. On the other hand, from the conditions (2.2) and (2.3) we find

$$0 \leq f(t,x)\mathrm{sgn}x \leq g(t)|x| + h_0(t) \quad \text{for } t \in I, \ |x| \geq r. \tag{5.3}$$

Then the inclusion $p_n \in V_{10}((a,b))$, as is well-known (see [12, Theorem 2.2, p.2367]), guarantees that the problem (5.1), (5.2) has at least one solution, suppose u_n. In view of the condition (2.2), without loss of generality we can assume that there exists $\varepsilon^* > 0$ such that $h_0(t) \geq \varepsilon^*$ on I. Then $g(t)|x| + h_0(t) \geq \varepsilon^*$ for $x \in R$, $t \in I$. Consequently, it is not difficult to verify that u_n is also a solution of the equation

$$u_n''(t) = (p_n(t) + p_0(t, u_n(t))\mathrm{sgn}u_n(t))u_n(t) + p_1(t, u_n(t)) \tag{5.4}$$

with $p_0(t,x) = \dfrac{f(t,x)g(t)}{g(t)|x| + h_0(t)}$, $p_1(t,x) = h(t) + \dfrac{f(t,x)h_0(t)}{g(t)|x| + h_0(t)}$.

Now assume that

$$\lim_{n\to+\infty} ||u_n||_C = +\infty \tag{5.5}$$

and $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then

$$v_n''(t) = (p_n(t) + p_0(t, u_n(t))\mathrm{sgn}u_n(t))v_n(t) + \frac{1}{||u_n||_C} p_1(t, u_n(t)), \tag{5.6}$$

$$v_n(a) = 0 \qquad v_n(b) = 0, \tag{5.7}$$

and

$$||v_n||_C = 1 \tag{5.8}$$

for any $n \in N$. In view of the condition (5.3), the functions $p_0, p_1 \in K(I \times R; R)$ are bounded respectively by the functions $g(t)$ and $h(t) + h_0(t)$. Therefore, from (5.6), by virtue of (5.5), (5.7) and (5.8), we see that there exists $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently in view of (5.8), by Arzela-Ascoli lemma, without loss of generality we can assume that there exists $w \in \widetilde{C}'(I, R)$ such that $\lim\limits_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t)$ $(i = 0, 1)$ uniformly on I. From the last equality and (5.5) there follows the existence of an increasing sequence $\{\alpha_k\}_{k=1}^{+\infty}$ of a natural numbers, such that $||u_{\alpha_k}||_C \geq 2rk$ and $||v_{\alpha_k}^{(i)} - w^{(i)}||_C \leq 1/2k$ for $k \in N$. Without loss of generality we can suppose that $u_n \equiv u_{\alpha_n}$ and $v_n \equiv v_{\alpha_n}$. In this case we see that u_n and v_n are the solutions of the problems (5.1), (5.2) and (5.6), (5.7) respectively with $p_n(t) = p(t) + 1/\alpha_n$ for $t \in I$, $n \in N$, and that the inequalities

$$||u_n||_C \geq 2rn, \quad ||v_n^{(i)} - w^{(i)}||_C \leq 1/2n \quad \text{for} \quad n \in N \tag{5.9}$$

are fulfilled. Analogously, since the functions $p_0, p_1 \in K(I \times R; R)$ are bounded, in view of (5.5), we can assume without loss of generality that there exists a function $\widetilde{p} \in L(I; R)$ such that

$$\lim_{n\to+\infty} \frac{1}{||u_n||_C^j} \int_a^t p_j(s, u_n(s))\mathrm{sgn}u_n(s)ds = (1-j)\int_a^t \widetilde{p}(s)ds \tag{5.10$_j$}$$

uniformly on I for $j = 0, 1$. By virtue of (5.8)–(5.10$_j$) $(j = 0, 1)$, from (5.6) we obtain

$$w''(t) = (p(t) + \widetilde{p}(t))w(t), \tag{5.11}$$

$$w(a) = 0, \quad w(b) = 0, \tag{5.12}$$

and

$$||w||_C = 1. \tag{5.13}$$

From the conditions (2.3) and (5.9) it is clear that all the assumptions of Lemma 4.3 with $f_1(t, x) = f(t, x)$ are satisfied, and thus we obtain from (5.10$_j$) $(j = 0)$ the relation $\int_s^t \widetilde{p}(\xi)d\xi \geq 0$ for $a \leq s < t \leq b$, i.e.,

$$\widetilde{p}(t) \geq 0 \quad \text{for} \quad t \in I. \tag{5.14}$$

Now assume that $\widetilde{p} \not\equiv 0$ and w_0 is a solution of the problem (1.3), (1.4). Then using Sturm's comparison theorem for the equations (1.3) and (5.11), from (5.14) we see that there exists a point $t_0 \in]a, b[$ such that $w_0(t_0) = 0$, which contradicts (2.1). This contradiction proves that $\widetilde{p} \equiv 0$. Consequently, w is a solution of the problem (1.3), (1.4). Multiplying the equations (5.1) and (1.3) respectively by w and $-u_n$, and therefore integrating their sum from a to b, in view of the conditions (5.2) and (1.4), we obtain

$$-\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = \int_a^b (h(s) + f(s, u_n(s)))w(s)ds \tag{5.15}$$

for $n \geq n_0$. Therefore by virtue of (5.9) we get

$$\int_a^b (h(s) + f(s, u_n(s)))w(s)ds < 0 \quad \text{for} \quad n \geq n_0. \tag{5.16}$$

On the other hand, in view the conditions (2.1)-(2.4$_1$), (5.2), and (5.9) it is clear that all the assumption of Lemma 4.4 with $f_1(t, x) = f(t, x)$, $h_1(t) = h(t)$ are fulfilled. Therefore, the inequality (4.43) is true, which contradicts (5.16). This contradiction proves that (5.5) does not hold and thus there exists $r_1 > 0$ such that $||u_n||_C \leq r_1$ for $n \in N$. Consequently, from (5.1) and (5.2) it is clear that there exists $r_1' > 0$ such that $||u_n'||_C \leq r_1'$ and $|u_n''(t)| \leq \sigma(t)$ for $t \in I$, $n \in N$, where $\sigma(t) = (1+|p(t)|)r_1+|h(t)|+\gamma_{r_1}(t)$. Hence, by Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $u_0 \in \widetilde{C}'(I; R)$ such that $\lim\limits_{n\to+\infty} u_n^{(i)}(t) = u_0^{(i)}(t)$ $(i = 0, 1)$ uniformly on I. Therefore, it follows from (5.1) and (5.2) that u_0 is a solution of the problem (1.1), (1.2). ∎

Proof of Theorem 2.2. Let $p_n(t) = p(t) - 1/n$ and, for any $n \in N$, consider the problems (5.1), (5.2) and (4.79_n). In view of Lemma 4.8, the problem (4.79_n) has only the zero solution for every $n \geq n_0$. Therefore, as is well-known (see [9, Theorem 1.1, p.345]), from the conditions (2.7), (2.9) it follows that the problem (5.1), (5.2) has at least one solution, suppose u_n.

Now assume that (5.5) holds and put $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then the conditions (5.7) and (5.8) are fulfilled, and

$$v_n''(t) = p_n(t)v_n(t) + \frac{1}{||u_n||_C}(f(t, u_n(t))) + h(t)). \tag{5.17}$$

In view the conditions (2.7) and (2.9), from (5.17) there follows the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently, in view (5.8) by Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $w \in \widetilde{C}'(I, R)$ such that $\lim\limits_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t) \quad (i = 0, 1)$ uniformly on I. Analogously as in the proof of Theorem 2.1, we can find a sequence $\{\alpha_k\}_{n=1}^{+\infty}$ of natural numbers such that, if we suppose $u_n = u_{\alpha_n}$ then the conditions (5.9) will by true when the functions u_n and v_n are the solutions of the problems (5.1), (5.2) and (5.17), (5.7) respectively with $p_n(t) = p(t) - 1/\alpha_n$ for $t \in I$, $n \in N$. From (5.17), by virtue of (5.7), (5.9) and (2.9), we obtain that w is a solution of the problem (1.3), (1.4). In a similar manner as the condition (5.15) in the proof of Theorem 2.1, we show that

$$\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = \int_a^b (h(s) + f(s, u_n(s)))w(s)ds \tag{5.18}$$

for $n \geq n_0$. Now note that, in view of the conditions (2.1), (2.8), (2.4_2), (5.2), and (5.9), all the assumptions of Lemma 4.4 with $f_1(t, x) = -f(t, x)$, $h_1(t) = -h(t)$ are satisfied. Hence, analogously as in the proof of Theorem 2.1, from (5.18) we show that the problem (1.1), (1.2) has at least one solution. ∎

Proof of Theorem 2.3. Let $p_n(t) = p(t) + (-1)^i/n$ and for any $n \in N$, consider the problems (5.1), (5.2) and (4.79_n). In view of the condition (2.13) and the fact that $(-1)^i f(t, x)$ is non-decreasing in the second argument for $|x| \geq r$, we obtain

$$\lim_{n\to+\infty} \frac{1}{||z_n||_C}\int_a^b |f(s, z_n(s))|ds = 0 \tag{5.19}$$

for an arbitrary sequence $z_n \in C(I; R)$ with $\lim\limits_{n\to+\infty} ||z_n||_C = +\infty$. Moreover, in view of Lemma 4.8, the problem (4.79_n) has only the zero solution for every $n \geq n_0$. Therefore, as it is well-known (see [9, Theorem 1.1, p. 345]), from the inequality (5.19) it follows that the problem (5.1), (5.2) has at least one solution, suppose u_n.

Now assume that (5.5) is fulfilled and put $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then (5.7), (5.8) and (5.17) are also fulfilled. Hence, by the conditions (5.8) and (5.19), from (5.17) we get the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently, in view of (5.8) by the Arzela-Ascoli lemma, without loss of generality we can assume that there exists a function $w \in \widetilde{C}'(I, R)$ such that $\lim\limits_{n\to+\infty} v_n^{(i)}(t) = w^{(i)}(t) \quad (i = 0, 1)$ uniformly on I. Analogously as in the proof of Theorem 2.1, we can find a sequence $\{\alpha_k\}_{n=1}^{+\infty}$ of natural numbers such that, assuming $u_n = u_{\alpha_n}$, the conditions (5.9) is true and the functions u_n and v_n are the solutions of the problems (5.1), (5.2) and (5.17), (5.7) respectively with $p_n(t) = p(t) + (-1)^i/\alpha_n$ for $t \in I,\ n \in N$. From (5.17), by virtue of (5.7), (5.9) and (2.13), we obtain that w is a solution of the problem (1.3), (1.4). In a similar manner as the condition (5.15) in the proof of Theorem 2.1, we show

$$-\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = (-1)^i\int_a^b (h(s) + f(s, u_n(s)))w(s)ds \tag{5.20}$$

for $n \in N \geq n_0$. Now note that, in view the conditions (2.11), (2.12), (2.14), (5.2), and (5.9), all the assumptions of Lemma 4.5 with $f_1(t, x) = (-1)^i f(t, x),\ h_1(t) = (-1)^i h(t)$ are satisfied. Hence, analogously as in the proof of Theorem 2.1, from (5.20) by Lemma 4.5 we obtain that the problem (1.1), (1.2) has at least one solution. ∎

Proof of Corollary 2.1. From the condition (2.15) we immediately obtain (2.14). Therefore all the conditions of Theorem 2.3 are fulfilled. ∎

Proof of Theorem 2.4. The proof is the same as the proof of Theorem 2.3. The only difference is that we use Lemma 4.6 instead of Lemma 4.5. ∎

Proof of Theorem 2.5. From (2.21) it is clear that, for an arbitrary sequence $z_n \in C(I; R)$ such that $\lim\limits_{n\to+\infty} ||z_n||_C = +\infty$, the equality (5.19) is holds. From (5.19) and Lemma 4.7, analogously as in the proof of Theorem 2.3, we show that the problem (1.1), (1.2) has at least one solution. ∎

Acknowledgement. The research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan N0.AV0Z10190503 and the Grant No. 201/06/0254 of the Grant Agency of the Czech Republic.

References

[1] R. P. Agarwal, I. Kiguradze, *Two-point boundary value problems for higher-order linear differential equations with strong singularities.* Boundary Value Problems, 2006, 1-32; Article ID 83910.

[2] S. Ahmad, *A resonance problem in which the nonlinearity may grow linearly.* Proc. Amer. Math. Soc., 92 (1984), 381–384.

[3] M. Arias, *Existence results on the one-dimensional Dirichlet problem suggested by the piecewise linear case.* Proc. Amer. Math. Soc., 97, no. 1 (1986), 121–127.

[4] C. De Coster, P. Habets *Upper and Lower Solutions in the theory of ODE boundary value problems.* Nonlinear Analysis And Boundary Value Problems For Ordinary Differential Equations, Springer, Wien, New York, no. 371 (1996), 1–119.

[5] R. Conti , *Equazioni differenziali ordinarie quasilineari con condizioni lineari.* Ann. Mat. Pura ed Appl., no. 57 (1962), 49–61.

[6] E. Landesman, A. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance,* J. Math. Mech., 19 (1970), 609–623.

[7] P. Drabek, *On the resonance problem with nonlinearity which has arbitrary linear growth.* J. Math. Anal. Appl., 127 (1987), 435–442.

[8] P. Drabek, *Solvability and bifurkations of nonlinear equations.* University of West Bohemia Pilsen (1991), 1–231.

[9] R. Iannacci, M.N. Nkashama *Nonlinear two point boundary value problems at resonance without Landesman-Lazer condition.* Proc. Amer. Math. Soc., 106, no. 4 (1989), 943–952.

[10] R. Iannacci, M.N. Nkashama *Nonlinear boundary value problems at resonance.* Nonlinear Anal., 6 (1987), 919–933.

[11] R. Kannan, J.J. Nieto, M.B. Ray *A Class of Nonlinear Boundary Value Problems Without Landesman-Lazer Condition.* J. Math. Anal. Appl., 105 (1985), 1–11.

[12] I. Kiguradze, B. Shekhter, *Singular boundary value problems for second order ordinary differential equations.* (Russian) Itogi Nauki Tekh., ser. Sovrem. Probl. Mat., Noveish. Dostizheniya, 30 (1987), 105–201; English transl.: J. Sov. Math., 43, no. 2, (1988), 2340–2417.

[13] I. Kiguradze, *Nekotorie Singularnie Kraevie Zadachi dlja Obiknovennih Differencialnih Uravneni.* Tbilisi University (1975), 1–351.

[14] I. Kiguradze, *On a singular two-point boundary value problem.* (Russian) Differentsial' nye Uravneniya, 5 (1969), No. 11, 2002-2016; English transl.: Differ. Equations, 5 (1969), 1493-1504.

[15] I. Kiguradze, *On some singular boundary value problems for nonlinear second order ordinary differential equations.* (Russian) Differentsial' nye Uravneniya 4 (1968), No. 10, 1753-1773; English transl.: Differ. Equations, 4 (1968), 901-910.

[16] I. Kiguradze, *On a singular boundary value problem.* J. Math. Anal. Appl., 30, no. 3, (1970), 475-489.

[17] J. Kurzveil, *Generalized ordinary differential equations.* Czechoslovak Math. J., 8, no. 3, (1958), 360–388.

Accepted: 18.06.2009

Georgian Mathematical Journal
Volume 16 (2009), Number 4, 651–665

A PERIODIC BOUNDARY VALUE PROBLEM FOR FUNCTIONAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER

ROBERT HAKL AND SULKHAN MUKHIGULASHVILI

Abstract. On the interval $[0,\omega]$, consider the periodic boundary value problem

$$u^{(n)}(t)=\sum_{i=0}^{n-1}\ell_i(u^{(i)})(t)+q(t),$$
$$u^{(j)}(0)=u^{(j)}(\omega)+c_j \qquad (j=0,\dots,n-1),$$

where $n\geq 2$, $\ell_i : C\big([0,\omega];R\big)\to L\big([0,\omega];R\big)$ $(i=0,\dots,n-1)$ are linear bounded operators, $q\in L\big([0,\omega];R\big)$, $c_j\in R$ $(j=0,\dots,n-1)$. The effective sufficient conditions guaranteeing the unique solvability of the considered problem are established.

2000 Mathematics Subject Classification: 34K06, 34K10.
Key words and phrases: Functional differential equation, boundary value problem, periodic solution.

Statement of the Problem

Consider the problem on the existence and uniqueness of a solution to the equation

$$u^{(n)}(t)=\sum_{i=0}^{n-1}\ell_i(u^{(i)})(t)+q(t) \qquad \text{for } 0\leq t\leq\omega \tag{0.1}$$

satisfying the periodic boundary conditions

$$u^{(j)}(0)=u^{(j)}(\omega)+c_j \qquad (j=0,\dots,n-1), \tag{0.2}$$

where $n\geq 2$, $\ell_i : C\big([0,\omega];R\big)\to L\big([0,\omega];R\big)$ are linear bounded operators, $q\in L\big([0,\omega];R\big)$, and $c_j\in R$ $(i,j=0,\dots,n-1)$.

By a solution to problem (0.1), (0.2) we understand a function $u\in\widetilde{C}^{n-1}\big([0,\omega];R\big)$, which satisfies equality (0.1) almost everywhere on $[0,\omega]$ and the boundary condition (0.2).

It is well-known that if the linear operators $\ell_i : C\big([0,\omega];R\big)\to L\big([0,\omega];R\big)$ $(i=0,\dots,n-1)$ are strongly bounded, i.e., if there exist summable functions $\eta_i:[0,\omega]\to[0,+\infty[$ such that

$$|\ell_i(x)(t)|\leq\eta_i(t)\|x\|_C \qquad \text{for } 0\leq t\leq\omega,\quad x\in C\big([0,\omega];R\big),$$

then the following theorem on the Fredholm property is valid (see, e.g., [1,10,18])

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

Theorem 0.1. *Problem* (0.1), (0.2) *is uniquely solvable iff the corresponding homogeneous problem*

$$v^{(n)}(t)=\sum_{i=0}^{n-1}\ell_i(v^{(i)})(t), \tag{0.3}$$

$$v^{(j)}(0)=v^{(j)}(\omega) \qquad (j=0,\dots,n-1), \tag{0.4}$$

has only the trivial solution.

The above-mentioned Fredholm property for functional differential equations with general bounded linear operators (i.e., not necessarily strongly bounded) had not been investigated before 2000 despite of the fact that in 1972 H. H. Schaefer [17, Theorem 4] proved that there do exist linear bounded operators $\ell: C\big([0,\omega];R\big)\to L\big([0,\omega];R\big)$ which are not strongly bounded. The first important steps in this direction were made by Bravyi in [2], and later in [5], where, among others, the Fredholm property was proved for the first order boundary value problems for functional differential equations with general bounded linear operators. These results were generalized for the n-th order functional differential systems in [7]. Therefore, Theorem 0.1 is also valid if ℓ_i $(i=0,\dots,n-1)$ are bounded (not necessarily strongly bounded) linear operators.

The problem on the existence of a periodic solution to ordinary and functional differential equations was studied very intensively in the past. The first important step was made for linear ordinary differential equations of the type

$$u^{(n)}(t)=p(t)u(t)+q(t) \tag{0.5}$$

by Lasota and Opial in [11]. They showed that problem (0.5), (0.2) is uniquely solvable for $n\geq 4$ if a function $p\in L\big([0,\omega];R\big)$ has the constant sign, $p\not\equiv 0$, and the inequality

$$\int_0^\omega |p(s)|ds<\left(\frac{2}{\omega}\right)^{n-1}\frac{2\cdot 4\cdots(n-2)}{1\cdot 3\cdots(n-3)} \tag{0.6}$$

is fulfilled. This result is far from being optimal, and in [12], condition (0.6) was improved to

$$\int_0^\omega |p(s)|ds<\frac{2}{\omega}\left(\frac{2\pi}{\omega}\right)^{n-2}. \tag{0.7}$$

The next step was made by Kiguradze and Kusano in [8], where the results of [11,12] were essentially improved. In particular, they proved following propositions.

Proposition 0.1. *Let either* $n=2m$, $(-1)^{m-1}p(t)\geq 0$ *for* $t\in[0,\omega]$, $p(t)\not\equiv 0$ *or* $n=2m-1$, $\sigma p(t)\geq 0$ *for* $t\in[0,\omega]$, $p(t)\not\equiv 0$, *where* $\sigma\in\{-1,1\}$. *Then problem* (0.5), (0.2) *has a unique solution.*

Proposition 0.2. *Let* $n=2m$, $(-1)^m p(t)\geq 0$ *for* $t\in[0,\omega]$, $p(t)\not\equiv 0$ *and inequality* (0.7) *be fulfilled. Then problem* (0.5), (0.2) *has a unique solution.*

Other results on the existence of a periodic solution to differential equations of higher order can be found, e.g., in [3, 9, 13, 15, 16].

However, condition (0.7) in Proposition 0.2 is not yet optimal and, moreover, Proposition 0.1 is not true for functional differential equations, which follows from the fact that the equation with deviating argument

$$u'''(t) = -|\cos t| u(\tau(t))$$

with

$$\tau(t) = \begin{cases} \pi/2 & \text{for } t \in [0, \pi/2[\cup]3\pi/2, 2\pi] \\ 3\pi/2 & \text{for } t \in [\pi/2, 3\pi/2[\end{cases},$$

has a nonzero 2π-periodic solution $\sin t$.

Below we will establish the new conditions guaranteeing the unique solvability of problem (0.1), (0.2), which improve the results of Lasota–Opial and Kiguradze–Kusano and are optimal for $n \leq 7$. The method used for the investigation of the considered problem is based on the method developed in our previous papers (see [3, 4, 13–16]) for functional differential equations.

The following notation is used throughout the paper:

N is a set of all natural numbers.

R is a set of all real numbers, $R_+ = [0, +\infty[$.

$C\big([0, \omega]; R\big)$ is a Banach space of continuous functions $u : [0, \omega] \to R$ with the norm

$$\|u\|_C = \max\big\{|u(t)| : t \in [0, \omega]\big\}.$$

$L\big([0, \omega]; R\big)$ is a Banach space of Lebesgue integrable functions $p : [0, \omega] \to R$ with the norm

$$\|p\|_L = \int_0^\omega |p(s)| ds.$$

$\widetilde{C}^k\big([0, \omega]; R\big)$ is a set of functions $u : [0, \omega] \to R$ which are absolutely continuous together with their derivatives up to k-th order.

If $\ell : C\big([0, \omega]; R\big) \to L\big([0, \omega]; R\big)$ is a linear bounded operator, then

$$\|\ell\| = \sup_{\|x\|_C \leq 1} \|\ell(x)\|_L .$$

$[x]_+ = \frac{1}{2}\big(|x| + x\big)$, $[x]_- = \frac{1}{2}\big(|x| - x\big)$.

$[x]$ is an integer part of x.

All equalities and inequalities between the measurable functions are understood as lying almost everywhere in an appropriate interval.

Definition 0.1. We will say that a linear operator $\ell : C\big([0, \omega]; R\big) \to L\big([0, \omega]; R\big)$ belongs to the set $\mathcal{P}_\omega$ if it is *non–negative*, i.e., for any non–negative $x \in C\big([0, \omega]; R\big)$ the inequality $\ell(x)(t) \geq 0$ for $0 \leq t \leq \omega$ is fulfilled.

In the sequel, the following notation is used:

$$A_0 = 1, \quad A_1 = \frac{1}{15}, \quad A_j = A_1 \sum_{m_1=1}^{2} \sum_{m_2=1}^{m_1+1} \dots \sum_{m_{j-1}=1}^{m_{j-2}+1} \frac{1}{\eta(m_1)\dots\eta(m_{j-1})},$$

$$B_1 = \frac{1}{8}, \quad B_j = A_1 \sum_{m_1=1}^{2} \sum_{m_2=1}^{m_1+1} \dots \sum_{m_{j-1}=1}^{m_{j-2}+1} \frac{1}{\eta(m_1)\dots\eta(m_{j-1})} \prod_{i=1}^{m_{j-1}+1} \left(1 + \frac{1}{2i}\right),$$

for $j \geq 2$, where

$$\eta(t) = (2t+1)(2t+3).$$

Let

$$d_0 = 1, \qquad d_1 = 4, \qquad d_2 = 32, \qquad d_3 = 192, \tag{0.8}$$

and for $p \in N$ put

$$\begin{gathered} d_{2p+2} = \frac{1}{\max\left\{(h_p(t)h_p(1-t))^{1/2} : 0 \leq t \leq 1\right\}}, \\ d_{2p+3} = \frac{1}{\max\left\{(f_p(s,t)f_p(1-s,1-t))^{1/2} : 0 \leq s \leq 1,\ 0 \leq t \leq 1\right\}}, \end{gathered} \tag{0.9}$$

where the functions $f_p : [0,1] \times [0,1] \to R_+$, $h_p : [0,1] \to R_+$ are defined as follows:

$$f_p(s,t) = \sum_{j=0}^{p-1} \alpha_{pj} t^{2(j+1)} + \alpha_{pp} t^{2p+3} s, \qquad h_p(t) = \sum_{j=0}^{p} \beta_{pj} t^{2(j+1)}, \tag{0.10}$$

and

$$\begin{gathered} \alpha_{pj} = \frac{A_j}{3 \cdot 4^{j+1} d_{2(p-j)+1}}, \quad \beta_{pj} = \frac{A_j}{3 \cdot 4^{j+1} d_{2(p-j)}} \quad (j = 0, \dots, p-1), \\ \alpha_{pp} = \frac{A_p}{3 \cdot 4^{p+1}}, \qquad \beta_{pp} = \frac{B_p}{3 \cdot 4^{p+1}}. \end{gathered} \tag{0.11}$$

Now we formulate the result from [6] in the form suitable for us.

Theorem 0.2. *Let $k \in N$, $v \in \widetilde{C}^k\big([0,\omega]; R\big)$, $v^{(i)}(0) = v^{(i)}(\omega)$ $(i = 0, \dots, k)$, and let d_k $(k \in N)$ be given by the equalities (0.8)–(0.11). Let, moreover,*

$$v(t) \not\equiv Const.$$

Then

$$\triangle\left(v^{(i)}\right) < \frac{\omega^{k-i}}{d_{k-i}} \triangle\left(v^{(k)}\right) \qquad (i = 0, \dots, k-1),$$

where

$$\triangle\left(v^{(i)}\right) = \max\left\{v^{(i)}(t) : t \in [0,\omega]\right\} - \min\left\{v^{(i)}(t) : t \in [0,\omega]\right\} \tag{0.12}$$

for $i = 0, \dots, k$.

Remark 0.1. In [6], it was shown that

$$d_4 = \frac{2^{11}\cdot 3}{5}, \quad d_5 = 2^9\cdot 3\cdot 5, \quad d_6 = \frac{2^{16}\cdot 3^2\cdot 5}{61}, \quad d_7 = \frac{2^{14}\cdot 3^2\cdot 5\cdot 7}{17}.$$

1. Main Results

Theorem 1.1. *Let $j \in \{0,1\}$, the operator ℓ_0 admit the representation $\ell_0 = \ell_{0,1} - \ell_{0,2}$, where $\ell_{0,1}, \ell_{0,2} \in \mathcal{P}_\omega$, and let ℓ_i $(i = 1,\dots,n-1)$ be bounded linear operators. Let, moreover, the conditions*

$$\|\ell_{0,1}\| + \|\ell_{0,2}\| \neq 0 \tag{1.1}$$

$$\frac{\omega^{n-1}}{d_{n-1}}\|\ell_{0,1+j}\| + \Omega < 1, \tag{1.2}$$

$$\frac{\|\ell_{0,1+j}\|}{1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\|\ell_{0,1+j}\|} \leq \|\ell_{0,2-j}\|, \tag{1.3}$$

$$\|\ell_{0,2-j}\| \leq \frac{2d_{n-1}}{\omega^{n-1}}\left(1-\Omega+\sqrt{(1-\Omega)\left(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\|\ell_{0,1+j}\|\right)}\,\right) \tag{1.4}$$

hold with

$$\Omega = \sum_{i=1}^{n-1}\frac{\omega^{n-1-i}}{d_{n-1-i}}\|\ell_i\| \tag{1.5}$$

and d_i $(i = 0,\dots,n-1)$ defined by (0.8)–(0.11). *Then problem* (0.1), (0.2) *has a unique solution.*

In the case, where all the operators ℓ_i $(i = 0,\dots,n-1)$ admit the representation

$$\ell_i = \ell_{i,1} - \ell_{i,2} \tag{1.6}$$

with $\ell_{i,1}, \ell_{i,2} \in \mathcal{P}_\omega$, i.e., they are strongly bounded, the following assertion improves Theorem 1.1.

Theorem 1.2. *Let $j \in \{0,1\}$ and the operators ℓ_i $(i = 0,\dots,n-1)$ admit representations* (1.6) *where $\ell_{i,1}, \ell_{i,2} \in \mathcal{P}_\omega$. Let, moreover, conditions* (1.1)–(1.4) *hold with*

$$\Omega = \sum_{i=1}^{n-1}\frac{\omega^{n-1-i}}{d_{n-1-i}}\max\{\|\ell_{i,1}\|, \|\ell_{i,2}\|\}$$

and d_i $(i = 0,\dots,n-1)$ defined by (0.8)–(0.11). *Then problem* (0.1), (0.2) *has a unique solution.*

Remark 1.1. It is clear that if $\ell_i \equiv 0$ $(i = 1,\dots,n-1)$, then $\Omega = 0$ in Theorems 1.1 and 1.2.

Corollary 1.1. *Let $\sigma \in \{-1,1\}$ and $\sigma\ell_0 \in \mathcal{P}_\omega$. Let, moreover, the conditions*

$$\|\ell_0\| \neq 0, \tag{1.7}$$

$$\sum_{i=1}^{n-1} \frac{\omega^{n-1-i}}{d_{n-1-i}} \|\ell_i\| < 1, \tag{1.8}$$

and

$$\|\ell_0\| \leq \frac{4d_{n-1}}{\omega^{n-1}} \left(1 - \sum_{i=1}^{n-1} \frac{\omega^{n-1-i}}{d_{n-1-i}} \|\ell_i\|\right) \tag{1.9}$$

hold. Then problem (0.1), (0.2) *has a unique solution.*

For the equation

$$u^{(n)}(t) = \ell_0(u)(t) + q(t), \quad 0 \leq t \leq \omega, \tag{1.10}$$

with $\sigma\ell_0 \in \mathcal{P}_\omega$, and $\sigma \in \{-1, 1\}$, from Theorem 1.1 we immediately obtain

Corollary 1.2. *Let $\sigma \in \{-1, 1\}$, $\sigma\ell_0 \in \mathcal{P}_\omega$. Let, moreover, conditions* (1.7) *and*

$$\|\ell_0\| \leq \frac{4d_{n-1}}{\omega^{n-1}} \tag{1.11}$$

hold. Then problem (1.10), (0.2) *has a unique solution.*

The special case of equation (0.1) is an equation with deviating argument of the form

$$u^{(n)}(t) = p(t)u(\tau(t)) + q(t), \tag{1.12}$$

where $p, q \in L\big([0, \omega]; R\big)$ and $\tau : [0, \omega] \to [0, \omega]$ is a measurable function. The following assertion immediately follows from Theorem 1.1.

Corollary 1.3. *Let $\int\limits_0^\omega |p(s)|ds \neq 0$ and let the conditions*

$$\int_0^\omega [\sigma p(s)]_+ ds < \frac{d_{n-1}}{\omega^{n-1}}\,,$$

$$\frac{\int\limits_0^\omega [\sigma p(s)]_+ ds}{1 - \frac{\omega^{n-1}}{d_{n-1}} \int\limits_0^\omega [\sigma p(s)]_+ ds} \leq \int_0^\omega [\sigma p(s)]_- ds \leq \frac{2d_{n-1}}{\omega^{n-1}} \left(1 + \sqrt{1 - \frac{\omega^{n-1}}{d_{n-1}} \int_0^\omega [\sigma p(s)]_+ ds}\right),$$

hold with $\sigma = 1$ or $\sigma = -1$. Then problem (1.12), (0.2) *has a unique solution.*

If $\ell_0(x)(t) = p(t)x(t)$, then Corollary 1.2 also improves Proposition 0.2. In particular we get

Corollary 1.4. *Let either the assumptions of Proposition 0.1 be fulfilled or let $n = 2m$, $(-1)^m p(t) \geq 0$ for $t \in [0, \omega]$, $p(t) \not\equiv 0$, and let the inequality*

$$\int_0^\omega |p(s)|ds \leq \frac{4d_{n-1}}{\omega^{n-1}} \tag{1.13}$$

hold. Then problem (0.5), (0.2) *has a unique solution.*

Remark 1.2. It is not difficult to verify that condition (1.13) improves (0.7) for $n \le 7$.

Remark 1.3. Let $l_0 = 1$ and the numbers l_n $(n \in N)$ be defined by the equalities

$$l_{2p-1} = \frac{(-1)^{p+1}4^{2p-1}}{\sum\limits_{i=0}^{p-1} \frac{(-1)^i 16^i}{(2p-2i-1)!l_{2i}}}\,, \qquad l_{2p} = \frac{(-1)^{p+1}4^{2p}}{\sum\limits_{i=0}^{p-1} \frac{(-1)^i 16^i}{(2p-2i)!l_{2i}}} \qquad \text{for } p \in N. \tag{1.14}$$

Then the equality

$$d_{n-1} = l_{n-1} \tag{1.15}$$

guarantees the optimality of condition (1.11) (and, consequently, also the optimality of (1.4)) in a sense that it cannot be replaced by the condition

$$\|\ell_0\| \le \frac{4d_{n-1}}{\omega} + \varepsilon \tag{1.11$_\varepsilon$}$$

no matter how small $\varepsilon \in]0,1]$ is.

Remark 1.4. According to [6, On Remark 1.3] it follows that the equality

$$d_i = l_i \tag{1.16$_i$}$$

is true for $i \le 7$, i.e., in view of Remark 1.3, condition (1.11) (and also condition (1.4)) is optimal for $n \le 7$.

In [6], it is also proved (see On Remark 1.4 therein) that if (1.16$_i$) holds for $i = 1, \dots, n-1$ and

$$\max\left\{h_p(t)h_p(1-t) : 0 \le t \le 1\right\} = h_p^2(1/2), \tag{1.16}$$

$$\max\left\{f_p(s,t)f_p(1-s,1-t) : 0 \le s \le 1,\ 0 \le t \le 1\right\} = f_p^2(1/2,1/2) \tag{1.17}$$

for $p \le [\frac{n-2}{2}]$, where the functions f_p and h_p are defined by (0.10), then equality (1.16$_n$) holds.

However, in a general case (starting with $p = 3$, i.e., for $n \ge 8$), the proof of (1.16) and (1.17) is not known to the authors. One can find more details about this problem in [6].

2. Proofs

To prove the main theorems we need two auxiliary propositions. The first is rather trivial and we omit the proof.

Lemma 2.1. *Let $\ell \in \mathcal{P}_\omega$. Then for an arbitrary $v \in C\big([0,\omega]; R\big)$ the inequalities*

$$-m\ell(1)(t) \le \ell(v)(t) \le M\ell(1)(t) \qquad \textit{for } 0 \le t \le \omega$$

hold, where $m = -\min\{v(t) : 0 \le t \le \omega\}$, $M = \max\{v(t) : 0 \le t \le \omega\}$.

Lemma 2.2. *Let $v \in \widetilde{C}^{n-1}\big([0,\omega];R\big)$ and*

$$v(t) \not\equiv Const, \qquad v^{(i)}(0) = v^{(i)}(\omega) \qquad (i = 0,\dots,n-1). \tag{2.1}$$

Then each of the functions $v^{(i)}$ $(i = 1,\dots,n-1)$ changes its sign and therefore

$$\|v^{(i)}\|_C \le \Delta\left(v^{(i)}\right) \qquad (i = 1,\dots,n-1).$$

Proof. It is clear that if $v^{(k)} \not\equiv Const$ and $v^{(k)}(0) = v^{(k)}(\omega)$ then $v^{(k+1)} \not\equiv 0$ and $\int_0^\omega v^{(k+1)}(s)ds = 0$ for any fixed $k \in \{0,\dots,n-1\}$. Thus $v^{(k+1)}$ changes its sign. From this fact and (2.1) it follows by mathematical induction that the functions $v^{(i)}$ $(i = 1,\dots,n-1)$ change their signs. From this fact and (0.12), the second part of the lemma immediately follows. □

Proof of Theorem 1.1. We will prove the theorem case when conditions (1.2)–(1.4) are fulfilled with $j = 0$. The case where $j = 1$ can be proved analogously.

According to Theorem 0.1 it is sufficient to show that problem (0.3), (0.4) has only a trivial solution. Assume to the contrary that problem (0.3), (0.4) has a nontrivial solution v and put

$$\begin{aligned} M_i &= \max\left\{v^{(i)}(t) : t \in [0,\omega]\right\}, \\ m_i &= -\min\left\{v^{(i)}(t) : t \in [0,\omega]\right\} \end{aligned} \qquad (i = 0,\dots,n-1). \tag{2.2}$$

First assume that v is still non-negative or still non-positive. Without loss of generality we can assume that $v(t) \ge 0$ for $t \in [0,\omega]$. Obviously, $M_0 > 0$, $m_0 \le 0$. If $v \equiv Const$, from (0.3) we get $\|\ell_{0,1}\| = \|\ell_{0,2}\|$, which contradicts (1.1)–(1.3). Thus $v \not\equiv Const$ and from Lemma 2.2 it follows that

$$M_i > 0, \qquad m_i > 0 \qquad (i = 1,\dots,n-1). \tag{2.3}$$

Choose $t_1, t_2 \in [0,\omega]$ such that

$$v^{(n-1)}(t_1) = -m_{n-1}, \qquad v^{(n-1)}(t_2) = M_{n-1}. \tag{2.4}$$

Obviously, either

$$t_1 < t_2 \tag{2.5}$$

or

$$t_1 > t_2. \tag{2.6}$$

Let (2.5) be fulfilled. Then, in view of (0.12), (2.2), (2.4), and Lemmas 2.1, 2.2, the integration of (0.3) from t_1 to t_2 yields

$$\begin{aligned} \Delta\left(v^{(n-1)}\right) &= \int_{t_1}^{t_2} \ell_{0,1}(v)(s)ds - \int_{t_1}^{t_2} \ell_{0,2}(v)(s)ds + \sum_{i=1}^{n-1} \int_{t_1}^{t_2} \ell_i(v^{(i)})(s)ds \\ &\le M_0\|\ell_{0,1}\| + \sum_{i=1}^{n-1} \Delta\left(v^{(i)}\right)\|\ell_i\|. \end{aligned} \tag{2.7}$$

If (2.6) is fulfilled, then analogously to (2.7) the integration of (0.3) from 0 to t_2 and from t_1 to ω results in

$$M_{n-1} - v^{(n-1)}(0) \le M_0 \int_0^{t_2} \ell_{0,1}(1)(s)ds + \sum_{i=1}^{n-1} \int_0^{t_2} |\ell_i(v^{(i)})(s)|ds,$$

$$v^{(n-1)}(\omega) + m_{n-1} \le M_0 \int_{t_1}^{\omega} \ell_{0,1}(1)(s)ds + \sum_{i=1}^{n-1} \int_{t_1}^{\omega} |\ell_i(v^{(i)})(s)|ds.$$

Summing the last two inequalities, on account of (0.4), (0.12), and (2.2) we get

$$\Delta\left(v^{(n-1)}\right) \le M_0\|\ell_{0,1}\| + \sum_{i=1}^{n-1} \Delta\left(v^{(i)}\right)\|\ell_i\|. \tag{2.8}$$

Thus for both (2.5) and (2.6) inequality (2.8) is fulfilled.

Furthermore, from (2.8), according to Theorem 0.2, we get

$$\Delta\left(v^{(n-1)}\right)(1-\Omega) \le M_0\|\ell_{0,1}\|, \tag{2.9}$$

where Ω is defined by (1.5). Now from (2.9), again using Theorem 0.2, we obtain

$$\frac{d_{n-1}}{\omega^{n-1}}(M_0+m_0)(1-\Omega) < M_0\|\ell_{0,1}\|,$$

whence we get

$$M_0\left(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\|\ell_{0,1}\|\right) < -m_0(1-\Omega). \tag{2.10}$$

On the other hand, in view of (0.4), (2.2), and Lemmas 2.1, 2.2, the integration of (0.3) from 0 to ω yields

$$-m_0\|\ell_{0,2}\| \le M_0\|\ell_{0,1}\| + \sum_{i=1}^{n-1} \Delta\left(v^{(i)}\right)\|\ell_i\|. \tag{2.11}$$

According to Theorem 0.2, from (2.11) we obtain

$$-m_0\|\ell_{0,2}\| \le M_0\|\ell_{0,1}\| + \Delta\left(v^{(n-1)}\right)\Omega, \tag{2.12}$$

where Ω is defined by (1.5). Now, (2.12) and (2.9) result in

$$-m_0(1-\Omega)\|\ell_{0,2}\| \le M_0\|\ell_{0,1}\|. \tag{2.13}$$

Multiplying the corresponding sides of the inequalities (2.10) and (2.13), in view of (1.1), (1.2) and the fact that $-m_0 > 0$ (see (2.10)), we obtain

$$\left(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\|\ell_{0,1}\|\right)\|\ell_{0,2}\| < \|\ell_{0,1}\|,$$

which contradicts (1.3) with $j = 0$.

Now suppose that v assumes both positive and negative values. Then according to Lemma 2.2 it follows that $M_i > 0$, $m_i > 0$ $(i = 0, \dots, n-1)$. Choose $t_1, t_2 \in [0, \omega]$ such that (2.4) holds and without loss of generality we can assume that (2.5) is fulfilled.

In view of (2.2), and Lemmas 2.1 and 2.2, the integration of (0.3) from 0 to t_1, from t_1 to t_2, and from t_2 to ω, respectively, yields

$$m_{n-1} + v^{(n-1)}(0) \le M_0 \int_0^{t_1} \ell_{0,2}(1)(s)ds + m_0 \int_0^{t_1} \ell_{0,1}(1)(s)ds + \sum_{i=1}^{n-1} \int_0^{t_1} |\ell_i(v^{(i)})(s)|ds, \tag{2.14}$$

$$M_{n-1} + m_{n-1} \le M_0 \int_{t_1}^{t_2} \ell_{0,1}(1)(s)ds + m_0 \int_{t_1}^{t_2} \ell_{0,2}(1)(s)ds + \sum_{i=1}^{n-1} \Delta(v^{(i)})\|\ell_i\|, \tag{2.15}$$

$$M_{n-1} - v^{(n-1)}(\omega) \le M_0 \int_{t_2}^{\omega} \ell_{0,2}(1)(s)ds + m_0 \int_{t_2}^{\omega} \ell_{0,1}(1)(s)ds + \sum_{i=1}^{n-1} \int_{t_2}^{\omega} |\ell_i(v^{(i)})(s)|ds. \tag{2.16}$$

Summing (2.14) and (2.16), on account of (0.4), (0.12), and (2.2) we get

$$\Delta\left(v^{(n-1)}\right) \le M_0 \int_I \ell_{0,2}(1)(s)ds + m_0 \int_I \ell_{0,1}(1)(s)ds + \sum_{i=1}^{n-1} \Delta(v^{(i)})\|\ell_i\|, \tag{2.17}$$

where $I = [0, t_1] \cup [t_2, \omega]$. However, according to Theorem 0.2, (2.17) and (2.15) result in

$$\Delta\left(v^{(n-1)}\right)(1-\Omega) \le M_0 \int_I \ell_{0,2}(1)(s)ds + m_0 \int_I \ell_{0,1}(1)(s)ds, \tag{2.18}$$

$$\Delta\left(v^{(n-1)}\right)(1-\Omega) \le M_0 \int_{t_1}^{t_2} \ell_{0,1}(1)(s)ds + m_0 \int_{t_1}^{t_2} \ell_{0,2}(1)(s)ds. \tag{2.19}$$

Now we note that (1.2) and Theorem 0.2 imply $\frac{d_{n-1}}{\omega^{n-1}}(M_0 + m_0)(1-\Omega) < \triangle\left(v^{(n-1)}\right)(1-\Omega)$. In view of (1.2) and the latter inequality, from (2.18) and (2.19) we get

$$0 < m_0\biggl(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\int\limits_I \ell_{0,1}(1)(s)ds\biggr) < M_0\biggl(\frac{\omega^{n-1}}{d_{n-1}}\int\limits_I \ell_{0,2}(1)(s)ds-(1-\Omega)\biggr), \quad (2.20)$$

$$0 < M_0\biggl(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\int\limits_{t_1}^{t_2} \ell_{0,1}(1)(s)ds\biggr) < m_0\biggl(\frac{\omega^{n-1}}{d_{n-1}}\int\limits_{t_1}^{t_2} \ell_{0,2}(1)(s)ds-(1-\Omega)\biggr), \quad (2.21)$$

which immediately imply the inequalities $\frac{\omega^{n-1}}{d_{n-1}}\int_I \ell_{0,2}(1)(s)ds > 1-\Omega$, $\frac{\omega^{n-1}}{d_{n-1}} \times \int_{t_1}^{t_2} \ell_{0,2}(1)(s)ds > 1-\Omega$, and thus

$$\frac{\omega^{n-1}}{d_{n-1}}\|\ell_{0,2}\| > 2(1-\Omega). \quad (2.22)$$

Multiplying the corresponding sides of the inequalities (2.20) and (2.21) we obtain

$$\biggl(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\int\limits_I \ell_{0,1}(1)(s)ds\biggr)\biggl(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\int\limits_{t_1}^{t_2} \ell_{0,1}(1)(s)ds\biggr) < \biggl(\frac{\omega^{n-1}}{d_{n-1}}\int\limits_I \ell_{0,2}(1)(s)ds-(1-\Omega)\biggr)\biggl(\frac{\omega^{n-1}}{d_{n-1}}\int\limits_{t_1}^{t_2} \ell_{0,2}(1)(s)ds-(1-\Omega)\biggr). \quad (2.23)$$

On the other hand, since $(\alpha-\beta)(\alpha-\gamma) \geq \alpha(\alpha-(\beta+\gamma))$ if $\beta\gamma \in R_+$, we have

$$\biggl(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\int\limits_I \ell_{0,1}(1)(s)ds\biggr)\biggl(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\int\limits_{t_1}^{t_2} \ell_{0,1}(1)(s)ds\biggr) \geq (1-\Omega)\biggl(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\|\ell_{0,1}\|\biggr), \quad (2.24)$$

and, furthermore, in view of the inequality $4\alpha\beta \leq (\alpha+\beta)^2$, we have

$$\biggl(\frac{\omega^{n-1}}{d_{n-1}}\int\limits_I \ell_{0,2}(1)(s)ds-(1-\Omega)\biggr)\biggl(\frac{\omega^{n-1}}{d_{n-1}}\int\limits_{t_1}^{t_2} \ell_{0,2}(1)(s)ds-(1-\Omega)\biggr) \leq \frac{1}{4}\biggl(\frac{\omega^{n-1}}{d_{n-1}}\|\ell_{0,2}\|-2(1-\Omega)\biggr)^2. \quad (2.25)$$

Then, using (2.24) and (2.25) in (2.23), in view of (2.22) we get

$$\sqrt{(1-\Omega)\left(1-\Omega-\frac{\omega^{n-1}}{d_{n-1}}\|\ell_{0,1}\|\right)} < \frac{\omega^{n-1}}{2d_{n-1}}\|\ell_{0,2}\| - (1-\Omega),$$

which contradicts (1.4) with $j = 0$. Consequently, our assumptions fail, and so $v \equiv 0$. $\square$

Proof of Theorem 1.2. First note that if conditions (2.3) with M_i and m_i defined by (2.2)are fulfilled, then, in view of (0.12), for any measurable set $A \subset [0,\omega]$ the estimates

$$(-1)^j \int\limits_A \ell_i(v^{(i)})(s)ds = \int\limits_A \ell_{i,1+j}(v^{(i)})(s)ds - \int\limits_A \ell_{i,2-j}(v^{(i)})(s)ds$$

$$M_i \int\limits_A \ell_{i,1+j}(1)(s)ds + m_i \int\limits_A \ell_{i,2-j}(1)(s)ds \le \Delta\left(v^{(i)}\right) \max\{\|\ell_{i,1}\|, \|\ell_{i,2}\|\}$$

hold for $j = 0, 1$, $i = 1, \dots, n-1$. Consequently, all the arguments hold true in the proof of the inequalities (2.9), (2.8), (2.12), (2.18), and (2.19) using the above estimates, and thus Theorem 1.2 can be proved in the same way as Theorem 1.1. More precisely, at the end of the proof of Theorem 1.2 we obtain the contradiction with the assumptions for $\Omega = \sum\limits_{i=1}^{n-1} \frac{\omega^{n-1-i}}{d_{n-1-i}} \max\{\|\ell_{i,1}\|, \|\ell_{i,2}\|\}$. $\square$

Proof of Corollary 1.1 immediately follows from Theorem 1.1 with $j = \frac{1+\sigma}{2}$. $\square$

Proof of Corollary 1.2 immediately follows from Corollary 1.1 with $\ell_i \equiv 0$ ($i = 1, \dots, n-1$). $\square$

On Remark 1.3. Define the functions $W_{0,k}$, $W_{i,k} : [0,1] \to [0,1]$ and the numbers $l_{i,k}$ $(i, k \in N)$ by

$$W_{0,k}(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{1}{4} - \frac{1}{8k} \\ \sin k\pi(1-4t) & \text{for } \frac{1}{4} - \frac{1}{8k} < t < \frac{1}{4} + \frac{1}{8k} \\ -1 & \frac{1}{4} + \frac{1}{8k} \le t \le \frac{1}{2} \end{cases}, \tag{2.26}$$

$$W_{0,k}\left(\frac{1}{2}+t\right) = W_{0,k}\left(\frac{1}{2}-t\right) \qquad \text{for } 0 \le t \le \frac{1}{2}, \tag{2.27}$$

$$W_{m,k}(t) = \int\limits_0^t W_{m-1,k}(s)ds - \delta_m \int\limits_0^{1/4} W_{m-1,k}(s)ds \qquad \text{for } t \in [0,\omega], \qquad m \in N,$$

where

$$\delta_m = \begin{cases} 0 & \text{if } m = 2\mu - 1 \\ 1 & \text{if } m = 2\mu \end{cases}, \qquad \mu \in N,$$

$$l_{2p-1,k} = \frac{1}{|W_{2p-1,k}(1/4)|}, \qquad l_{2p,k} = \frac{1}{|W_{2p,k}(1/2)|}, \qquad p \in N. \tag{2.28}$$

To show the validity of Remark 1.3 we use the properties of $W_{0,k}$, $W_{i,k}$, and $l_{i,k}$ which are proved in [6]. In particular, the following equalities are valid for $i,k \in N$ (see Lemmas 2.3 and 2.4 in [6]):

$$W_{i,k}(0) = W_{i,k}(1), \tag{2.29}$$

$$W_{i,k}^{(j)}(t) = W_{i-j,k}(t) \qquad \text{for } t \in [0,1] \qquad j \le i, \tag{2.30}$$

$$\lim_{k \to +\infty} l_{i,k} = l_i, \tag{2.31}$$

$$\Delta(W_{i,k}) = \frac{1}{l_{i,k}} \Delta(W_{0,k}), \tag{2.32}$$

$$W_{i,k}\left(\frac{1}{2} - t\right) = (-1)^i W_{i,k}\left(\frac{1}{2} + t\right) \qquad \text{for } 0 \le t \le \frac{1}{2}, \tag{2.33}$$

$$W_{i,k}\left(\frac{1}{4} - t\right) = (-1)^{i-1} W_{i,k}\left(\frac{1}{4} + t\right) \qquad \text{for } 0 \le t \le \frac{1}{4}. \tag{2.34}$$

According to Theorem 0.2, in view of (2.29) and (2.30), we have

$$\Delta(W_{i,k}) < \frac{1}{d_i} \Delta(W_{i,k}^{(i)}) = \frac{1}{d_i} \Delta(W_{0,k}),$$

whence, with respect to (2.32), we obtain

$$l_{i,k} > d_i.$$

Now, assuming that (1.15) holds, on account of (2.31), it follows that for every $\varepsilon > 0$ there exists $k_0 \in N$ such that

$$l_{n-1} = d_{n-1} < l_{n-1,k} \le d_{n-1} + \frac{\varepsilon}{4} \qquad \text{for } k \ge k_0. \tag{2.35}$$

Put

$$v_0(t) = \left(d_{n-1} + \frac{\varepsilon}{4}\right) W_{n-1,k_0}(t) \qquad \text{for } t \in [0,1].$$

According to (2.28) and (2.35) we get

$$\|v_0\|_C > l_{n-1,k_0} \|W_{n-1,k_0}\|_C \ge 1.$$

Thus in view of (2.33) and (2.34) we have

$$\{t \in [0,1] : v_0(t) \ge 1\} \ne \emptyset, \qquad \{t \in [0,1] : v_0(t) \le -1\} \ne \emptyset,$$

which implies the existence of $t_1, t_2 \in [0,1]$ such that

$$v_0(t_1) = 1, \qquad v_0(t_2) = -1. \tag{2.36}$$

Now let $\omega = 1$, $\ell_0(x)(t) = |v_0^{(n)}(t)| x(\tau(t))$ with

$$\tau(t) = \begin{cases} t_1 & \text{for } W'_{0,k_0}(t) \ge 0 \\ t_2 & \text{for } W'_{0,k_0}(t) < 0 \end{cases}.$$

Then $\ell_0 \in \mathcal{P}_\omega$, $\|\ell_0\| \neq 0$, in view of (2.36) we have $v_0(\tau(t)) = \operatorname{sgn} W'_{0,k_0}(t)$, and from the definition of the function $W_{0,k}$ (see (2.26) and (2.27)) we have

$$\|\ell_0\| = \int_0^1 |v_0^{(n)}(s)|ds = 4\left(d_{n-1} + \frac{\varepsilon}{4}\right) \int_{\frac{1}{4}-\frac{1}{8k_0}}^{\frac{1}{4}} |\sin' \pi k_0(1-4s)|ds$$
$$= 4d_{n-1} + \varepsilon. \quad (2.37)$$

Thus, all the assumptions of Corollary 1.2 are satisfied except (1.11), instead of which condition (1.11_ε) is fulfilled with $\omega = 1$. On the other hand, by (2.30) we get

$$v_0^{(n)}(t) = W'_{0,k_0}(t) = |W'_{0,k_0}(t)| \operatorname{sgn} W'_{0,k_0}(t) = |v_0^{(n)}| v_0(\tau(t)) = \ell_0(v_0)(t).$$

Therefore v_0 is a nontrivial solution to the homogeneous problem

$$v^{(n)}(t) = \ell_0(v)(t), \qquad v^{(i)}(0) = v^{(i)}(1) \qquad (i = 0, \dots, n-1),$$

which contradicts the conclusion of Corollary 1.2.

Acknowledgement

The research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. A V0Z10190503, by the Grant Agency of the Czech Republic, grant No. 201/06/0254 (first author), and by the Georgian National Scientific Foundation, grant # GNSF /ST06/3 – 002 (second author).

References

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, Introduction to the theory of functional-differential equations. *Nauka, Moscow,* 1991.
2. E. Bravyi, A note on the Fredholm property of boundary value problems for linear functional differential equations. *Mem. Differential Equations Math. Phys.* **20**(2000), 133–135.
3. R. Hakl, Periodic boundary-value problem for third order linear functional differential equations. *Ukrain. Mat. Zh.* **60**(2008), No. 3, 413–425.
4. R. Hakl, A. Lomtatidze, and B. Půža, On a periodic boundary value problem for first-order functional-differential equations. (Russian) *Differ. Uravn.* **39**(2003), No. 3, 320–327, 428; English transl.: *Differ. Equ.* **39**(2003), No. 3, 344–352.
5. R. Hakl, A. Lomtatidze, and I. P. Satvroulakis, On a boundary value problem for scalar linear functional differential equations. *Abstr. Appl. Anal.* **2004**, No. 1, 45–67.
6. R. Hakl and S. Mukhigulashvili, On one estimate for periodic functions. *Georgian Math. J.* **12**(2005), No. 1, 97–114.
7. R. Hakl and S. Mukhigulashvili, On a boundary value problem for n-th order linear functional differential systems. *Georgian Math. J.* **12**(2005), No. 2, 229–236.
8. I. T. Kiguradze and T. Kusano, On periodic solutions of higher-order nonautonomous ordinary differential equations. (Russian) *Differ. Uravn.* **35**(1999), No. 1, 72–78; English transl.: *Differential Equations* **35**(1999), No. 1, 71–77.

9. I. Kiguradze and T. Kusano, On periodic solutions of even-order ordinary differential equations. *Ann. Mat. Pura Appl. (4)* **180**(2001), No. 3, 285–301.
10. I. Kiguradze and B. Půža, On boundary value problems for systems of linear functional-differential equations. *Czechoslovak Math. J.* **47(122)**(1997), No. 2, 341–373.
11. A. Lasota and Z. Opial, Sur les solutions périodiques des équations différentielles ordinaires. *Ann. Polon. Math.* **16**(1964), 69–94.
12. A. Lasota and F. H. Szafraniec, Sur les solutions périodiques d'une équation différentielle ordinaire d'ordre *n*. *Ann. Polon. Math.* **18**(1966), 339–344.
13. A. Lomtatidze and S. Mukhigulashvili, On periodic solutions of second order functional differential equations. *Mem. Differential Equations Math. Phys.* **5**(1995), 125–126.
14. S. Mukhigulashvili, Two-point boundary value problems for second order functional differential equations. *Mem. Differential Equations Math. Phys.* **20**(2000), 1–112.
15. S. Mukhigulashvili, On the solvability of a periodic problem for second-order nonlinear functional-differential equations. (Russian) *Differ. Uravn.* **42**(2006), No. 3, 356–365; English transl.: *Differ. Equ.* **42**(2006), No. 3, 380–390.
16. S. Mukhigulashvili, On a periodic boundary value problem for third order linear functional differential equations. *Nonlinear Anal.* **66**(2007), No. 2, 527–535.
17. H. H. Schaefer, Normed tensor products of Banach lattices. *Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces* (*Jerusalem*, 1972). *Israel J. Math.* **13**(1972), 400–415 (1973).
18. Š. Schwabik, M. Tvrdý, and O. Vejvoda, Differential and integral equations. Boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht–Boston, Mass.–London,* 1979.

(Received 17.07.2008)

Authors' addresses:

R. Hakl
Institute of Mathematics AS CR
Žižkova 22, 616 62 Brno
Czech Republic
E-mail: hakl@ipm.cz

S. Mukhigulashvili
Institute of Mathematics AS CR
Žižkova 22, 616 62 Brno
Czech Republic
E-mail: mukhig@ipm.cz
and
I. Chavchavadze State University
Faculty of physics and mathematics
32, I. Chavchavadze Ave., Tbilisi 0179
Georgia

Memoirs on Differential Equations and Mathematical Physics
Volume 20, 2000, 1–112

S. Mukhigulashvili

TWO-POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. In the paper effective sufficient conditions are obtained for unique solvability and correctness of the mixed problem and of the Dirichlet problem for second order linear singular functional differential equations. Some of these conditions are nonimprovable and some of them generalize results which are well known for ardinary differential equations.

2000 Mathematics Subject Classification. 34K10.

Key words and phrases: Functional deffirential equations, singular, unique solvability, correctness.

Main Notation

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}^+ =]0, +\infty[\,$.

Let $\alpha \in \mathbb{R}$.

$[\alpha]$ is the integral part of the number α,

$$[\alpha]_+ = \frac{|\alpha| + \alpha}{2}, \quad [\alpha]_- = \frac{|\alpha| - \alpha}{2}.$$

$C(\,]a, b[)$ is the space of continuous and bounded functions $u :]a, b[\to \mathbb{R}$ with the norm

$$\|u\|_C = \sup\{|u(t)| : a < t < b\}.$$

$\widetilde{C}_{\text{loc}}(]a, b[)$ is the set of the functions $u :]a, b[\to \mathbb{R}$ absolutely continuous on each subsegment of $]a, b[\,$.

$\widetilde{C}'_{\text{loc}}(\,]a, b[)$ is the set of the functions $u :]a, b[\to \mathbb{R}$ absolutely continuous on each subsegment of $]a, b[$ along with their first order derivatives.

$L([a, b])$ is the space of summable functions $u : [a, b] \to \mathbb{R}$ with the norm

$$\|u\|_L = \int_a^b |u(s)|ds.$$

$L_\infty(]a, b])$ is the space of essentially bounded functions $u :]a, b[\to \mathbb{R}$ with the norm

$$\|u\| = \operatorname*{ess\,sup}_{t \in [a,b]} |u(t)|.$$

$L_{\text{loc}}(]a, b[)$ $(L_{\text{loc}}(]a, b]))$ is the set of the measurable functions $u :]a, b[\to \mathbb{R}$ $(u :]a, b] \to \mathbb{R})$, summable on each subsegment of $]a, b[$ $(]a, b])$.

Let $x, y :]a, b[\to]0, +\infty[$ be continuous functions.

$C_x(]a, b[)$ is the space of functions $u \in C(]a, b[)$ such that

$$\|u\|_{C,x} = \sup\left\{\frac{|u(t)|}{x(t)} : \quad a < t < b\right\} < +\infty.$$

$L_y([a, b])$ is the space of the functions $u \in L(]a, b[)$ such that

$$\|u\|_{L,y} = \int_a^b y(s)|u(s)|ds < +\infty.$$

$\mathcal{L}(C_x; L_y)$ is the set of the linear operators $h : C_x(]a, b[) \to L_y([a, b])$ such that

$$\sup\left\{|h(u)(\cdot)| : \|u\|_{C,x} \le 1\right\} \in L_y([a, b]).$$

$\sigma : L_{\text{loc}}(]a, b[) \to \widetilde{C}_{\text{loc}}(]a, b[)$ is the operator defined by

$$\sigma(p)(t) = \exp\left(\int_{\frac{a+b}{2}}^t p(s)ds\right) \quad \text{for} \quad a \le t \le b,$$

where $p \in L_{\rm loc}(]a,b[)$.

If $\sigma(p) \in L([a,b])$, then we define the operators σ_1 and σ_2 by

$$\sigma_1(p)(t) = \frac{1}{\sigma(p)(t)} \int\limits_a^t \sigma(p)(s)ds \int\limits_t^b \sigma(p)(s)ds,$$

$$\sigma_2(p)(t) = \frac{1}{\sigma(p)(t)} \int\limits_a^t \sigma(p)(s)ds \quad \text{for} \quad a \le t \le b.$$

Let $f,\, g \in C(]a,b[)$ and $c \in [a,b]$. Then we write

$$f(t) = O(g(t)) \quad \big(f(t) = O^*(g(t))\big) \quad \text{as} \quad t \to c,$$

if

$$\limsup_{t \to c} \frac{|f(t)|}{|g(t)|} < +\infty \quad \Big(0 < \liminf_{t \to c} \frac{|f(t)|}{|g(t)|} \ \text{ and } \ \limsup_{t \to c} \frac{|f(t)|}{|g(t)|} < +\infty\Big).$$

Let A and B be normed spaces and let $\mathbb{U} : A \to \mathbb{B}$ be a linear operator. Then we denote the norm of the operator $\mathbb{U}$ as follows:

$$\|\mathbb{U}\|_{A \to \mathbb{B}}.$$

Introduction

During the last two decades the boundary value problems for functional differential equations attract the attention of many mathematicians and are intensively studied. At present the foundations of the general theory of such kind of problems are already laid and many of them are investigated in detail (see [1], [2], [19]–[23], [44] and references therein). Despite this fact, there remains a wide class of boundary value problems on the solvability of which not much is known. Among them are the two-point boundary value problems for linear singular functional differential equations of second order, and we devote our work to the investigation of these problems.

It should be noted that the present monograph is tightly connection with the works of I. T. Kiguradze [17], L. B. Shekhter [23] and A. G. Lomtatidze [27] in which for singular ordinary differential equations we developed the method of upper and lower Nagumo's functions in the case of boundary value problems and found the conditions under which Fredholm's alternative is valid in the case of linear equations. We introduced and described the set $\mathbb{V}_{0,i}$ (see Definition 1.1.2).

In the present work we consider the equation

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + g(u)(t) + p_2(t) \tag{0.0.1}$$

under the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2 \tag{0.0.2$_1$}$$

or

$$u(a) = c_1, \quad u'(b-) = c_2, \tag{0.0.2$_2$}$$

and separately for the case of homogeneous conditions

$$u(a) = 0, \quad u(b) = 0,$$
$$u(a) = 0, \quad u'(b-) = 0,$$

where c_1, $c_2 \in \mathbb{R}$, $p_j \in L_{\mathrm{loc}}(]a,b[)$ $(j = 0,1,2)$ and $g : C(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ is a continuous linear operator. In studying these problems the use is made of the auxiliary equation

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) - h(u)(t),$$

where $h : C(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ is the nonnegative linear operator.

The question of the unique solvability of problems (0.0.1), $(0.0.2_i)$ is studied in Chapter I. We introduced sets of two-dimensional vector functions $(p_0,p_1) :]a,b[\to \mathbb{R}^2$, $\mathbb{V}_{i,\beta}(]a,b[;h)$, $\beta \in [0,1]$ (see Definitions 1.1.3 and 1.1.4), which were found to be useful for our investigation. In Section 1.1, in terms of the sets $\mathbb{V}_{i,\beta}(]a,b[;h)$ we established theorems for the unique solvability of problems (0.0.1), $(0.0.23_i)$. The question on the unique solvability of problems (0.0.1), $(0.0.2_{i0})$ in the space with weight $C_\lambda(]a,b[)$ is studied separately. In the same chapter we can find corollaries of basic theorems and and also the effective sufficient conditions for the unique solvability of the above-mentioned problems. Among them there occur unimprovable conditions and those which generalize the well-known results for ordinary differential equations.

In Chapter II we consider the question dealing with the correctness of problems (0.0.1), $(0.0.2_i)$ under the assumption that $(p_0, p_1) \in \mathbb{V}_{i,\beta}(]a,b[;h)$. The effective sufficient conditions guaranteeing the correctness of the above-mentioned problems are presented.

Everywhere in our work, special attention is given to the case, when the operator g in equation (0.0.1) is defined by the equality

$$g(u)(t) = \sum_{k=1}^{n} g_k(t) u(\tau_k(t)),$$

where $g_k \in L_{\mathrm{loc}}(]a,b[)$, $\tau_k : [a,b] \to [a,b]$ $(k = 1, \dots, n)$ are measurable functions.

CHAPTER I

UNIQUE SOLVABILITY OF TWO-POINT BOUNDARY VALUE PROBLEMS FOR LINEAR SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

§ 1.1. Statement of the Problem and Formulation of Basic Results

In this chapter we consider the linear equation

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + g(u)(t) + p_2(t) \tag{1.1.1}$$

under the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2 \tag{1.1.2$_1$}$$

or

$$u(a) = c_1, \quad u'(b-) = c_2, \tag{1.1.2$_2$}$$

where p_0, $p_j \in L_{\mathrm{loc}}(]a,b[)$, $c_j \in \mathbb{R}$ $(j = 1,2)$ and $g : C(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ is a continuous linear operator.

The equation (1.1.1) will also be studied separately in the weighted space $C_{x^\beta}(]a,b[)$ under the homogeneous boundary conditions

$$u(a) = 0, \quad u(b) = 0 \tag{1.1.2$_{10}$}$$

or

$$u(a) = 0, \quad u'(b-) = 0, \tag{1.1.2$_{20}$}$$

where $\beta \in]0,1]$ and

$$x(t) = \int_a^t \sigma(p_1)(s)\,ds \bigg(\int_t^b \sigma(p_1)(s)\,ds \bigg)^{2-i} \quad \text{for} \quad a \le t \le b.$$

When considering the problems (1.1.1), (1.1.2$_1$) and (1.1.1), (1.1.2$_{10}$), it will always be assumed that

$$\begin{gathered} p_j \in L_{\mathrm{loc}}(]a,b[) \;\; (j = 0,1,2), \\ \sigma(p_1) \in L([a,b]), \quad p_0 \in L_{\sigma_1(p_1)}([a,b]), \end{gathered} \tag{1.1.3$_1$}$$

and when considering the problems (1.1.1), (1.1.2$_2$) and (1.1.1), (1.1.2$_{20}$) we will assume that

$$\begin{gathered} p_j \in L_{\mathrm{loc}}(]a,b]) \;\; (j = 0,1,2), \\ \sigma(p_1) \in L([a,b]), \quad p_0 \in L_{\sigma_2(p_1)}([a,b]). \end{gathered} \tag{1.1.3$_2$}$$

Introduce the following definitions.

Definition 1.1.1. Let $i \in \{1,2\}$. We will say that $w \in C(]a,b[)$ is the lower (upper) function of the problem (1.1.1), (1.1.2$_i$) if:

(a) w' is of the form $w'(t) = w_0(t) + w_1(t)$, where $w_0 :]a,b[\to \mathbb{R}$ is absolutely continuous on each segment from $]a,b[$, the function $w_1 :]a,b[\to \mathbb{R}$ is nondecreasing (nonincreasing) and its derivative is almost everywhere equal to zero;

(b) almost everywhere on $]a,b[$ the inequality

$$w''(t) \geq p_0(t)w(t) + p_1(t)w'(t) + g(w)(t) + p_2(t)$$
$$\big(w''(t) \leq p_0(t)w(t) + p_1(t)w'(t) + g(w)(t) + p_2(t)\big)$$

is satisfied:

(c) there exists the limit $w'(b-)$ and

$$w(a) \leq c_1, \quad w^{(i-1)}(b-) \leq c_2 \quad \big(w(a) \geq c_1, \quad w^{(i-1)}(b-) \geq c_2\big).$$

Definition 1.1.2. Let $i \in \{1,2\}$. We will say that a two-dimensional vector function $(p_0, p_1) :]a,b[\to \mathbb{R}^2$ belongs to the set $\mathbb{V}_{i,0}(]a,b[)$ if the conditions (1.1.3$_i$) are fulfilled, the solution of the problem

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t), \tag{1.1.4}$$
$$u(a) = 0, \quad \lim_{t \to a} \frac{u'(t)}{\sigma(p_1)(t)} = 1$$

has no zeros in the interval $]a,b[$ and $u^{(i-1)}(b-) > 0$.

Note that this definition is in a full agreement with that of the set $\mathbb{V}_{i,0}(]a,b[)$ given in [23] as the set of three-dimensional vector functions $(p_0, p_{11}, p_{12}) :]a,b[\to \mathbb{R}^3$ if $p_{11}(t) = p_{12}(t) = p_1(t)$ almost everywhere on $]a,b[$.

Definition 1.1.3. Let $i \in \{1,2\}$ and $h : C(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ be a continuous linear operator. We will say that a two-dimensional vector function $(p_0, p_1) :]a,b[\to \mathbb{R}^2$ belongs to the set $\mathbb{V}_{i,0}(\,]a,b[\,;h)$ if the conditions (1.1.3$_i$) are satisfied and the problem

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) - h(u)(t)$$
$$u(a) = 0, \quad u^{(i-1)}(b-) = 0$$

has a positive upper function w on the segment $[a,b]$.

Definition 1.1.4. Let $i \in \{1,2\}$, $\beta \in]0,1]$ and $h : C(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ be a continuous linear operator. We will say that a two-dimensional vector function $(p_0, p_1) :]a,b[\to \mathbb{R}^2$ belongs to the set $\mathbb{V}_{i,\beta}(\,]a,b[\,;h)$ if

$$(p_0, p_1) \in \mathbb{V}_{i,0}(]a,b[),$$

there exists a measurable function $q_\beta :]a,b[\to [0,+\infty[$ such that

$$\int_a^b |G(t,s)| q_\beta(s)\,ds = O^*(x^\beta(t))$$

as $t \to a$, $t \to b$ if $i = 1$, and as $t \to b$ if $i = 2$, where G is Green's function of the problem (1.1.4), $(1.1.2_{i0})$ and

$$x(t) = \int_a^t \sigma(p_1)(s)\,ds \bigg(\int_t^b \sigma(p_1)(s)\,ds \bigg)^{2-i} \quad \text{for} \quad a \le t \le b,$$

and the problem

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) - h(u)(t) - q_\beta(t),$$
$$u(a) = 0, \quad u^{(i-1)}(b-) = 0$$

on the interval $]a,b[$ has a positive upper function w such that

$$w(t) = O^*(x^\beta(t))$$

as $t \to a$, $t \to b$ if $i = 1$ and as $t \to a$ if $i = 2$.

1.1.1. Theorems on the Unique Solvability of the Problems (1.1.1), $(1.1.2_i)$ $(i = 1, 2)$.

Theorem $1.1.1_i$. *Let* $i \in \{1,2\}$,

$$p_2 \in L_{\sigma_i(p_1)}([a,b]) \tag{$1.1.5_i$}$$

and let the constants α, $\beta \in [0,1]$ *connected by the inequality*

$$\alpha + \beta \le 1 \tag{1.1.6}$$

be such that

$$(p_0, p_1) \in \mathbb{V}_{i,\beta}(]a,b[\,; h), \tag{$1.1.7_i$}$$

where

$$h \in \mathcal{L}\Big(C_{x^\beta}; L_{\frac{x^\alpha}{\sigma(p_1)}}\Big) \cap \mathcal{L}\big(C; L_{\sigma_i(p_1)}\big) \tag{$1.1.8_i$}$$

is a nonnegative operator and

$$x(t) = \int_a^t \sigma(p_1)(s)\,ds \bigg(\int_t^b \sigma(p_1)(s)\,ds \bigg)^{2-i} \quad \textit{for} \quad a \le t \le b. \tag{$1.1.9_i$}$$

Let, moreover, a continuous linear operator $g : C(]a,b[) \to L_{\sigma_i(p_1)}([a,b])$ *be such that for any function* $u \in C(]a,b[)$ *almost everywhere in the interval* $]a,b[$ *the inequality*

$$|g(u)(t)| \le h(|u|)(t) \tag{1.1.10}$$

is satisfied. Then the problem (1.1.1), (1.1.2$_i$) *has one and only one solution.*

Theorem 1.1.1$_{i0}$. *Let* $i \in \{1,2\}$ *and let the constants* $\alpha \in [0,1[$, $\beta \in]0,1]$ *connected by the inequality* (1.1.6) *be such that*

$$p_2 \in L_{\frac{x^{1-\beta}}{\sigma(p_1)}}\big([a,b]\big) \tag{1.1.11}$$

and the functions p_0, $p_1 :]a,b[\to \mathbb{R}$ *satisfy the inclusion* (1.1.7$_i$), *where*

$$h \in \mathcal{L}\big(C_{x^\beta}; L_{\frac{x^\alpha}{\sigma(p_1)}}\big) \tag{1.1.12}$$

is a nonnegative operator and the function $x :]a,b[\to \mathbb{R}^+$ *is defined by the equality* (1.1.9$_i$). *Let, moreover, a continuous linear operator* $g : C_{x^\beta}(]a,b[) \to L_{\frac{x^\alpha}{\sigma(p_1)}}([a,b])$ *be such that for any function* $u \in C_{x^\beta}(]a,b[)$ *almost everywhere in the interval* $]a,b[$ *the inequality* (1.1.10) *is satisfied. Then the problem* (1.1.1), (1.1.2$_{i0}$) *has one and only one solution in the space* $C_{x^\beta}(]a,b[)$.

Remark 1.1.1$_i$. Let $i \in \{1,2\}$ and all the requirements of Theorem 1.1.1$_i$ be satisfied. Then for any function $v_0 \in C(]a,b[)$ there exists a unique sequence $v_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, v_n is a solution of the problem

$$\begin{gathered} v''(t) = p_0(t)v_1(t) + p_1(t)v'(t) + g(v_{n-1})(t) + p_2(t), \\ v(a) = c_1, \quad v^{i-1}(b-) = c_2, \end{gathered} \tag{1.1.13$_i$}$$

and uniformly on $]a,b[$

$$\lim_{n\to\infty}(v_n(t) - u(t)) = 0, \quad \lim_{n\to\infty}\sigma_i(p_1)(t)(v_n'(t) - u'(t)) = 0, \tag{1.1.14}$$

where u is a solution of the problem (1.1.1), (1.1.2$_i$).

Remark 1.1.1$_{i0}$. Let $i \in \{1,2\}$ and all the requirements of Theorem 1.1.1$_{i0}$ be satisfied. Then for any function $v_0 \in C_{x^\beta}(]a,b[)$ there exists a unique sequence $v_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, such that for every $n \in \mathbb{N}$, v_n is a solution of the problem

$$\begin{gathered} v''(t) = p_0(t)v(t) + p_1(t)v'(t) + g(v_{n-1})(t) + p_2(t), \\ v(a) = 0, \quad v^{i-1}(b-) = 0, \end{gathered} \tag{1.1.13$_{i0}$}$$

and uniformly on $]a,b[$

$$\lim_{n\to\infty}\frac{v_n(t) - u(t)}{x^\beta(t)} = 0, \quad \lim_{n\to\infty}\frac{x^\alpha(t)}{\sigma(p_1)(t)}(v_n'(t) - u'(t)) = 0, \tag{1.1.15}$$

where u is a solution of the problem (1.1.1), (1.1.2$_{i0}$).

We can easily give examples of the operator h and the function p_1 such that $h \in \mathcal{L}(C_{x^\beta}; L_{\frac{x^\alpha}{\sigma(p_1)}})$ and $h \notin \mathcal{L}(C; L_{\sigma_i(p_1)})$.

Example 1.1.1. Let $\varepsilon > 0$, $p_1(t) \equiv 0$, $h(u)(t) = [(b-t)(t-a)]^{-2-\varepsilon}$ for $a \le t \le b$ and let $\tau : [a,b] \to \{a,b\}$ be a measurable function.

Example 1.1.2. Let $a = -1$, $b = 1$, $\alpha = \beta = \frac{1}{5}$, $p_1(t) \equiv 0$ and $h(u)(t) = (1-t^2)^{-3}u(\tau(t))$, $\tau(t) = \sqrt{1-(1-t^2)^{10}}$ for $-1 \le t \le 1$. Then it is clear that

$$\sigma(p_1)(t) = 1, \quad x(t) = 1-t^2, \quad x^{1/5}(\tau(t)) = (1-t^2)^2 \quad \text{for} \quad -1 \le t \le 1$$

and

$$\alpha + \beta < \frac{1}{2}.$$

In such a case if $u_1 \in C_{x^{\frac{1}{5}}}([-1,1])$ it follows from the inequality

$$|u_1(\tau(t))| \le \delta x^{1/5}(\tau(t)) \quad \text{for} \quad -1 \le t \le 1,$$

where

$$\delta = \sup\left\{\left|\frac{u_1(\tau(t))}{x^{1/5}(\tau(t))}\right| : \ -1 < t < 1\right\},$$

that

$$\int_{-1}^{1} x^{\alpha}(s)h(u_1)(s)\,ds \le \delta \int_{-1}^{1} (1-s^2)^{-4/5}\,ds < +\infty,$$

i.e., the condition (1.1.11$_i$) is satisfied.

Let now $u_2(t) \equiv 1$. Then $u_2 \in C(\,]-1,1[\,)$ and

$$\int_{-1}^{1} x(s)h(u_2)(s)\,ds = \int_{-1}^{1} (1-s^2)^{-2}\,ds,$$

i.e., owing to the fact that the last integral does not exist, the condition (1.1.8$_1$) is violated.

Consider the case where $p_0(t) \equiv 0$, $p_1(t) \equiv 0$, i.e., when the equation (1.1.1) has the form

$$u''(t) = g(u)(t) + p_2(t). \tag{1.1.16}$$

Then the following theorem is valid.

Theorem 1.1.2$_1$. *Let* $\gamma \in [0,1[$,

$$p_2 \in L_x([a,b]) \tag{1.1.17}$$

and

$$g \in \mathcal{L}(C; L_{x^\gamma}) \tag{1.1.18}$$

be a nonnegative operator, where

$$x(t) = (t-a)(t-b) \quad \textit{for} \quad a \le t \le b. \tag{1.1.19$_1$}$$

Let, moreover, there exist constants α, $\beta \in [0, \frac{1}{2}]$ *such that*

$$0 \le \beta < 1 - \gamma, \tag{1.1.20}$$

$$\alpha + \beta \le \frac{1}{2} \tag{1.1.21}$$

and

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s)\, ds < 2^\beta \, \frac{16}{b-a} \Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \tag{1.1.22}$$

Then the problem (1.1.16), (1.1.2$_1$) *has one and only one solution.*

Remark 1.1.2. Theorem 1.2.2$_1$ will remain valid if we replace the conditions (1.1.20) and (1.1.22) respectively by

$$0 < \beta < 1 - \gamma, \tag{1.1.23}$$

and

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s)\, ds \le 2^\beta \frac{16}{b-a} \Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \tag{1.1.24$_1$}$$

Theorem 1.1.2$_2$. *Let* $\gamma \in [0, 1[$ *and let a function* p_2 *and a nonnegative operator* g *satisfy respectively the inclusions* (1.1.17) *and* (1.1.18)*, where*

$$x(t) = t - a \quad \textit{for} \quad a \le t \le b. \tag{1.1.19$_2$}$$

Let, moreover, there exist constants α, $\beta \in [0, \frac{1}{2}]$ *such that the conditions* (1.1.20), (1.1.21) *are fulfilled and*

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s)\, ds \le \frac{8}{b-a} \Big(\frac{b-a}{4}\Big)^{\alpha+\beta}. \tag{1.1.24$_2$}$$

Then the problem (1.1.16), (1.1.2$_2$) *has one and only one solution.*

Theorem 1.1.2$_{i0}$. *Let* $i \in \{1, 2\}$, $\gamma \in [0, 1[$, $\delta \in]0, 1-\gamma[$,

$$p_2 \in L_{x^\gamma}([a, b]) \tag{1.1.25}$$

and let

$$g \in \mathcal{L}(C_{x^\delta}; L_{x^\gamma}) \tag{1.1.26}$$

be a nonnegative operator, where the function x is defined by the equality $(1.1.19_i)$. *Let, moreover, there exist constants $\alpha \in [0, \frac{1}{2}]$, $\beta \in]0, \frac{1}{2}]$, such that*

$$\delta \le \beta < 1 - \gamma \tag{1.1.27}$$

and the conditions (1.1.21), $(1.1.24_i)$ *are satisfied. Then the problem* (1.1.16), $(1.1.2_{i0})$ *has in the space $C_{x^\delta}(]a, b[)$ one and only one solution.*

Remark 1.1.3. The condition (1.1.22) is unimprovable in the sense that it cannot be replaced by the condition

$$\int_a^b x^\alpha(s) g(x^\beta)(s)\, ds < 2^\beta \frac{16}{b-a}\Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)} + \varepsilon \tag{1.1.28}$$

with no matter how small $\varepsilon > 0$.

Indeed, let

$$\alpha = 0, \quad \beta = 0, \quad a = -\frac{1}{2}, \quad b = \frac{1}{2},$$
$$\lambda = \frac{\varepsilon}{4(16+\varepsilon)}, \qquad \mu = 16\lambda\sqrt{1 + \frac{1}{(16+\varepsilon)^2}},$$
$$g_0(t) = \begin{cases} 64\mu^2(16\mu^2 - (1+4t)^2)^{-\frac{3}{2}} & \text{for } \ t \in \Big] - \dfrac{1}{4} - \lambda, \, -\dfrac{1}{4} + \lambda\Big[\\ 64\mu^2(16\mu^2 - (1-4t)^2)^{-\frac{3}{2}} & \text{for } \ t \in \Big]\dfrac{1}{4} - \lambda, \, \dfrac{1}{4} + \lambda\Big[\\ 0 \ \text{ for } \ \Big[-\dfrac{1}{2}, -\dfrac{1}{4} - \lambda\Big] \cup \Big[-\dfrac{1}{4} + \lambda, \, \dfrac{1}{4} - \lambda\Big] \cup \Big[\dfrac{1}{4} + \lambda, \, \dfrac{1}{2}\Big] \end{cases},$$
$$p_2(t) = 0, \quad \tau(t) = -\frac{4}{16+\varepsilon}\operatorname{sign} t \quad \text{for} \quad -\frac{1}{2} \le t \le \frac{1}{2},$$

and

$$g(u)(t) = g_0(t) u(\tau(t)).$$

Then the problem (1.1.16), $(1.1.2_{10})$ can be rewritten as

$$u''(t) = g_0(t) u(\tau(t)), \tag{1.1.29}$$
$$u\Big(-\frac{1}{2}\Big) = 0, \quad u\Big(\frac{1}{2}\Big) = 0. \tag{1.1.30}$$

Note that for the operator g defined in such a way the condition (1.1.18) is satisfied for $\gamma = 0$ and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g(1)(s)\, ds = \int_{-\frac{1}{2}}^{\frac{1}{2}} g_0(s)\, ds = 16 + \varepsilon,$$

i.e., instead of (1.1.22) the condition (1.1.28) is satisfied. In spite of this fact we can check directly that the function

$$u(t) = c\Big[\int\limits_{-\frac{1}{2}}^{t}\int\limits_{-\frac{1}{2}}^{s} g_0(\eta)\,\mathrm{sign}(-\eta)d\eta\,ds - \Big(4+\frac{\varepsilon}{4}\Big)\Big(t+\frac{1}{2}\Big)\Big]$$

is for any $c \in \mathbb{R}$ a solution of the problem (1.1.29), (1.1.30), i.e., the unique solvability is violated.

1.1.2. Effective Sufficient Conditions for the Unique Solvability of the Problem (1.1.1), (1.1.2$_i$) ($i = 1, 2$).

Corollary 1.1.1$_1$. *Let the function x be defined by* (1.1.9$_1$)*, the constants α, $\beta \in [0,1]$ be connected by* (1.1.6)*, the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy* (1.1.3$_1$)*,* (1.1.5$_1$)*,*

$$[p_0]_- \in L_{\frac{x^\alpha}{\sigma(p_1)}}([a,b]) \tag{1.1.31}$$

and for every function $u \in C(]a,b[)$ almost everywhere on interval $]a,b[$ the inequality (1.1.10) *is satisfied, where a nonnegative operator h satisfies the inclusion* (1.1.8$_1$)*. Let, moreover,*

$$\begin{aligned}&\Big[\Big(\int\limits_t^b \sigma(p_1)(\eta)d\eta\Big)^\alpha \int\limits_a^t \frac{([p_0(s)]_- x^\beta(s) + h(x^\beta)(s))}{\sigma(p_1)(s)}\Big(\int\limits_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds + \\ &+\Big(\int\limits_a^t \sigma(p_1)(\eta)d\eta\Big)^\alpha \int\limits_t^b \frac{([p_0(s)]_- x^\beta(s) + h(x^\beta)(s))}{\sigma(p_1)(s)}\Big(\int\limits_s^b \sigma(p_1)(\eta)d\eta\Big)^\alpha ds\Big] < \\ &< \frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta}\Bigg(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2}\Bigg)^{2(\alpha+\beta)} \quad \text{for} \quad a \le t \le b\end{aligned} \tag{1.1.32$_1$}$$

Then the problem (1.1.1)*,* (1.1.2$_1$) *has one and only one solution.*

Corollary 1.1.1$_2$. *Let the function x be defined by* (1.1.9$_2$)*, the constants α, $\beta \in [0,1]$ be connected by* (1.1.6)*, the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy* (1.1.3$_2$)*,* (1.1.5$_2$)*,* (1.1.31) *and for every function $u \in C(]a,b[)$ almost everywhere in the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where a* *nonnegative operator h satisfies* (1.1.8$_2$). *Let, moreover,*

$$\int_a^t \frac{([p_0(s)]_- x^\beta(s) + h(x^\beta)(s))}{\sigma(p_1)(s)} \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds +$$
$$+\Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^\alpha \int_t^b \frac{([p_0(s)]_- x^\beta(s) + h(x^\beta)(s))}{\sigma(p_1)(s)} ds <$$
$$< \Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha+\beta-1} \quad \textit{for} \quad a \le t \le b. \tag{1.1.32$_2$}$$

Then the problem (1.1.1), (1.1.2$_2$) *has one and only one solution.*

Corollary 1.1.1$_{i0}$. *Let $i \in \{1,2\}$, the function x be defined by* (1.1.9$_i$), *the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by* (1.1.6), *the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy* (1.1.3$_i$), (1.1.11), (1.1.31) *and for any function $u \in C_{x^\beta}(]a,b[)$ almost everywhere in the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where the nonnegative operator h satisfies the inclusion* (1.1.12). *Let, moreover,* (1.1.32$_i$) *be satisfied. Then the problem* (1.1.1), (1.1.2$_{i0}$) *has in the space $C_{x^\beta}(]a,b[)$ one and only one solution.*

Remark 1.1.4. Corollary 1.1.1$_i$ remains valid if we replace the conditions (1.1.8$_i$) and (1.1.32$_i$) respectively by the conditions

$$h \in \mathcal{L}(C; L_{\sigma_i(p_1)}), \tag{1.1.33}$$

and

$$\int_a^b \frac{([p_0(s)]_- x^{\alpha+\beta}(s) + x^\alpha(s)h(x^\beta)(s))}{\sigma(p_1)(s)} ds <$$
$$< \frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta} \left(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2}\right)^{2(\alpha+\beta)} \tag{1.1.34$_1$}$$

for $i = 1$ or by

$$\int_a^b \frac{([p_0(s)]_- x^{\alpha+\beta}(s) + x^\alpha(s)h(x^\beta)(s))}{\sigma(p_1)(s)} ds < \Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha+\beta-1} \tag{1.1.34$_2$}$$

for $i = 2$, where the function x is defined by (1.1.9$_i$).

Remark 1.1.4$_0$. Corollary 1.1.1$_{i0}$ remains valid if we replace (1.1.32$_i$) by (1.1.34$_i$) and reject the condition (1.1.12) at all.

Consider the case where the equation (1.1.1) has the form

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + \sum_{k=1}^{n} g_k(t)u(\tau_k(t)) + p_2(t). \quad (1.1.35)$$

Corollary 1.1.2$_1$. *Let the function x be defined by* (1.1.9$_1$), *the constants α, $\beta \in [0,1]$ be defined by the inequality* (1.1.6), *the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy the conditions* (1.1.3$_1$), (1.1.5$_1$), (1.1.31), *$\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ be measurable functions and*

$$g_k x^\beta(\tau_k) \in L_{\frac{x^\alpha}{\sigma(p_1)}}([a,b]), \quad g_k \in L_{\sigma_1(p_1)}([a,b]) \quad (k = 1,\dots,n). \quad (1.1.36_1)$$

Let, moreover,

$$\Big(\int_t^b \sigma(p_1)(\eta)d\eta\Big)^\alpha \int_a^t \frac{([p_0(s)]_- x^\beta(s) + \sum\limits_{k=1}^{n} |g_k(s)|x^\beta(\tau_k(s)))}{\sigma(p_1)(s)} \times$$
$$\times \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds + \Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^\alpha \times$$
$$\times \int_t^b \frac{([p_0(s)]_- x^\beta(s) + \sum\limits_{k=1}^{n} |g_k(s)|x^\beta(\tau_k(s)))}{\sigma(p_1)(s)} \Big(\int_s^b \sigma(p_1)(\eta)d\eta\Big)^\alpha ds <$$
$$< \frac{4}{\int\limits_a^b \sigma(p_1)(\eta)d\eta} \left(\frac{\int\limits_a^b \sigma(p_1)(\eta)d\eta}{2}\right)^{2(\alpha+\beta)} \quad \textit{for} \quad a \le t \le b. \quad (1.1.37_1)$$

Then the problem (1.1.35), (1.1.2$_1$) *has one and only one solution.*

Corollary 1.1.2$_2$. *Let the function x be defined by* (1.1.9$_2$), *the constants α, $\beta \in [0,1]$ be connected by* (1.1.6), *the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy* (1.13$_2$), (1.1.5$_2$), (1.1.31), *$\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ be measurable functions and*

$$g_k x^\beta(\tau_k) \in L_{\frac{x^\alpha}{\sigma(p_1)}}([0,b]), \quad g_k \in L_{\sigma_2(p_1)}([a,b]) \quad (k = 1,\dots,n). \quad (1.1.36_2)$$

Let, moreover,

$$\int_0^t \frac{[p_0(s)]_- x^\beta(s) + \sum\limits_{k=1}^{n} |g_k(s)|x^\beta(\tau_k(s))}{\sigma(p_1)(s)} \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds +$$

$$+\Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^{\alpha} \int_t^b \frac{[p_0(s)]_- x^{\beta}(s) + \sum\limits_{k=1}^{n} |g_k(s)| x^{\beta}(\tau_k(s))}{\sigma(p_1)(s)}\, ds <$$

$$< \Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha+\beta-1} \quad \textit{for} \quad a \le t \le b. \tag{1.1.37$_2$}$$

Then the problem (1.1.35), (1.1.2$_2$) *has one and only one solution.*

Corollary 1.1.2$_{i0}$. *Let $i \in \{1,2\}$, the function x be defined by* (1.1.9$_i$), *the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by the inequality* (1.1.6), *the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy the conditions* (1.1.3$_i$), (1.1.11), (1.1.31), *$\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ be measurable functions and*

$$g_k x^{\beta}(\tau_k) \in L_{\frac{x^{\alpha}}{\sigma(p_1)}}([a,b]) \quad (k = 1,\dots,n). \tag{1.1.38}$$

Let, moreover, the conditions (1.1.37$_i$) *be satisfied. Then the problem* (1.1.35), (1.1.2$_{i0}$) *has in the space $C_{x^{\beta}}(]a,b[)$ one and only one solution.*

Remark 1.1.5. Corollary 1.1.2$_i$ remains valid if we replace the conditions (1.1.36$_i$) and (1.1.37$_i$) respectively by the conditions

$$g_k \in L_{\sigma_i(p_1)}([a,b]) \quad (k = 1,\dots,n) \tag{1.1.39}$$

and

$$\int_a^b \frac{[p_0(s)]_- x^{\alpha+\beta}(s) + x^{\alpha} \sum\limits_{k=1}^{n} |g_k(s)| x^{\beta}(\tau_k(s))}{\sigma(p_1)(s)}\, ds <$$

$$< \frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta} \left(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2}\right)^{2(\alpha+\beta)} \tag{1.1.40$_1$}$$

for $i = 1$ or by

$$\int_a^b \frac{[p_0(s)]_- x^{\alpha+\beta}(s) + x^{\alpha}(s) \sum\limits_{k=1}^{n} |g_k(s)| x^{\beta}(\tau_k(s))}{\sigma(p_1)(s)}\, ds <$$

$$< \Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha+\beta-1} \tag{1.1.40$_2$}$$

for $i = 2$, where the function x is defined by (1.1.9$_i$).

Remark 1.1.5$_0$. Corollary 1.1.2$_{i0}$ remains valid if we replace (1.1.37$_i$) by (1.1.40$_i$) and reject the condition (1.1.38) at all.

Corollary 1.1.3$_1$. *Let the function x be defined by* (1.1.9$_1$), *the constants α, $\beta \in [0,1]$ be connected by* (1.1.6), *the functions g_k, $p_j :]a,b[\to \mathbb{R}$ $(k = 1,\dots,n;\ j = 0,1,2)$ satisfy* (1.1.3$_1$), (1.1.5$_1$), (1.1.36$_1$), *where $\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ are measurable functions and*

$$p_0(t) \geq 0 \quad \textit{for} \quad a < t < b. \tag{1.1.41}$$

Let, moreover, for any $m \in \{1,\dots,n\}$ the condition

$$\begin{aligned}
&\sum_{k=1}^{n} \int_{a}^{\tau_m(t)} \frac{|g_k(s)|}{\sigma(p_1)(s)} \bigg(\int_{a}^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \int_{\tau_k(s)}^{b} \sigma(p_1)(\eta)d\eta \bigg)^{\beta} \times \\
&\qquad \times \bigg(\int_{a}^{s} \sigma(p_1)(\eta)d\eta \bigg)^{\alpha} ds \bigg(\int_{\tau_m(t)}^{b} \sigma(p_1)(\eta)d\eta \bigg)^{\alpha} + \\
&+ \sum_{k=1}^{n} \int_{\tau_m(t)}^{b} \frac{|g_k(s)|}{\sigma(p_1)(s)} \bigg(\int_{a}^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \int_{\tau_k(s)}^{b} \sigma(p_1)(\eta)d\eta \bigg)^{\beta} \times \\
&\qquad \times \bigg(\int_{s}^{b} \sigma(p_1)(\eta)d\eta \bigg)^{\alpha} ds \bigg(\int_{a}^{\tau_m(t)} \sigma(p_1)(\eta)d\eta \bigg)^{\alpha} < \\
&< \frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta} \left(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2} \right)^{2(\alpha+\beta)}, \quad a \leq t \leq b,
\end{aligned} \tag{1.1.42$_1$}$$

be valid. Then the problem (1.1.35), (1.1.2$_1$) *has one and only one solution.*

Corollary 1.1.3$_2$. *Let the function x be defined by the equality* (1.1.9$_2$), *the constants α, $\beta \in [0,1]$ be connected by* (1.1.6), *the functions $g_k, p_j :]a,b[\to \mathbb{R}$ $(k = 1,\dots,n;\ j = 0,1,2)$ satisfy the conditions* (1.1.3$_2$), (1.1.5$_2$), (1.1.36$_2$), (1.1.41), *where $\tau_k : [a,b] \to [a,b]$ $(k = 1,\dots,n)$ are measurable functions. Let, moreover, for any $m \in \{1,\dots,n\}$ the condition*

$$\begin{aligned}
&\sum_{k=1}^{n} \int_{a}^{\tau_m(t)} \frac{|g_k(s)|}{\sigma(p_1)(s)} \bigg(\int_{a}^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \bigg)^{\beta} \bigg(\int_{a}^{s} \sigma(p_1)(\eta)d\eta \bigg)^{\alpha} ds + \\
&+ \sum_{k=1}^{n} \int_{\tau_m(t)}^{b} \frac{|g_k(s)|}{\sigma(p_1)(s)} \bigg(\int_{a}^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \bigg)^{b} ds \bigg(\int_{a}^{\tau_m(t)} \sigma(p_1)(s)\, ds \bigg)^{\alpha} <
\end{aligned}$$

$$< \Big(\int_a^b \sigma(p_1)(\eta)d\eta\Big)^{\alpha+\beta-1}, \quad a \le t \le b, \tag{1.1.42$_2$}$$

be valid. Then the problem (1.1.35), (1.1.2$_2$) *has one and only one solution.*

Corollary 1.1.3$_{i0}$. *Let* $i \in \{1,2\}$, *the function* x *be defined by* (1.1.9$_i$), *the constants* $\alpha \in [0,1[$, $\beta \in]0,1]$ *be connected by* (1.1.6), *the functions* g_k, $p_j :]a,b[\to \mathbb{R}$ ($k = 1,\dots,n$; $j = 0,1,2$) *satisfy* (1.1.3$_i$), (1.1.11), (1.1.38), (1.1.41), *where* $\tau_k : [a,b] \to [a,b]$ ($k = 1,\dots,n$) *are measurable functions. Let, moreover, for any* $m \in \{1,\dots,n\}$ *the condition* (1.1.42$_i$) *be valid. Then the problem* (1.1.35), (1.1.2$_{i0}$) *has in the space* $C_{x^\beta}(]a,b[)$ *one and only one solution.*

Remark 1.1.6. The condition (1.1.42$_i$) consisting of n separate inequalities can be replaced by one inequality

$$\sum_{k=1}^n \int_a^t \frac{|g_k(s)|}{\sigma(p_1)(s)} \Big(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \int_{\tau_k(s)}^b \sigma(p_1)(\eta)d\eta\Big)^\beta \times$$
$$\times \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds \Big(\int_t^b \sigma(p_1)(\eta)d\eta\Big)^\alpha +$$
$$+ \sum_{k=1}^n \int_t^b \frac{|g_k(s)|}{\sigma(p_1)(s)} \Big(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta \int_{\tau_k(s)}^b \sigma(p_1)(\eta)d\eta\Big)^\beta \times$$
$$\times \Big(\int_s^b \sigma(p_1)(\eta)d\eta\Big)^\alpha ds \Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^\alpha < \frac{4}{\int_a^b \sigma(p_1)(\eta)d\eta} \times$$
$$\times \left(\frac{\int_a^b \sigma(p_1)(\eta)d\eta}{2}\right)^{2(\alpha+\beta)} \quad \text{for} \quad t \in \Theta_{\tau_1,\dots,\tau_n} \tag{1.1.43$_1$}$$

if $i = 1$ and

$$\sum_{k=1}^n \int_a^t \frac{|g_k(s)|}{\sigma(p_1)(s)} \Big(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta\Big)^\beta \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds +$$
$$+ \sum_{k=1}^n \int_t^b \frac{|g_k(s)|}{\sigma(p_1)(s)} \Big(\int_a^{\tau_k(s)} \sigma(p_1)(\eta)d\eta\Big)^\beta ds \Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^\alpha <$$

$$< \Bigg(\int\limits_a^b \sigma(p_1)(\eta)\Bigg)^{\alpha+\beta-1} \quad \text{for} \quad t \in \Theta_{\tau_1,\dots,\tau_n} \tag{1.1.43$_2$}$$

if $i = 2$, where

$$\Theta_{\tau_1,\dots,\tau_n} = \bigcup_{k=1}^{n} \big\{\tau_k(t) |\ \ a \le t \le b\big\}.$$

For clearness we will give one corollary for the equation

$$u''(t) = g_0(t)u(\tau(t)) + p_2(t). \tag{1.1.44}$$

Corollary 1.1.4$_i$. *Let $i \in \{1,2\}$, the constants α, $\beta \in [0,1]$ be connected by the inequality* (1.1.6), *$\tau : [a,b] \to [a,b]$ be a measurable function and*

$$p_2, g_0 \in L_x([a,b]), \tag{1.1.45}$$

where

$$x(t) = (a-t)(b-t)^{2-i} \quad \textit{for} \quad a \le t \le b. \tag{1.1.46}$$

Let, moreover,

$$\int\limits_a^b |g(s)| \big[(\tau(s)-a)(b-\tau(s))^{2-i}\big]^\beta \big[(s-a)(b-s)^{2-i}\big]^\alpha ds <$$

$$< \Big(\frac{2}{i}\Big)^{2(1-\alpha-\beta)} (b-a)^{\frac{2}{i}(\alpha+\beta)-1}. \tag{1.1.47$_i$}$$

Then the problem (1.1.44), (1.1.2$_i$) *has one and only one solution.*

Corollary 1.1.4$_{i0}$. *Let $i \in \{1,2\}$, the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by* (1.1.6), *$\tau : [a,b] \to [a,b]$ a be measurable function,*

$$p_2 \in L_{x^{1-\beta}}([a,b]), \tag{1.1.48}$$

where the function x is defined by (1.1.46). *Let, moreover, the condition* (1.1.47$_i$) *be satisfied. Then the problem* (1.1.44), (1.1.2$_{i0}$) *has one and only one solution in the space $C_{x^\beta}(]a,b[)$.*

Remark 1.1.7. In the case of the equation

$$u''(t) = g_0(t)u(t) + p_2(t) \tag{1.1.49}$$

the conditions (1.32$_1$), (1.1.34$_1$), (1.1.40$_1$), (1.1.42$_1$), (1.1.47$_1$) will take for $\alpha = \beta = 0$ the form

$$\int\limits_a^b |g_0(s)|\, ds < \frac{4}{b-a}.$$

As is known, this condition is unimprovable in the sense that no matter how small $\varepsilon > 0$ is, the inequality

$$\int_a^b |g_0(s)|\,ds \le \frac{4}{b-a} + \varepsilon$$

does not guarantee the unique solvability of the problem (1.1.49), $(1.1.2_1)$. This implies that the corollaries corresponding to the above conditions are unimprovable in the above-mentioned sense.

Corollary 1.1.5$_1$. *Let the function x be defined by* $(1.1.9_1)$*, the constants α, $\beta \in [0,1]$ be connected by the inequality* (1.1.6)*, the functions $p_j :]a,b[\to \mathbb{R}$ $(j = 0,1,2)$ satisfy the conditions* $(1.1.3_1)$, $(1.1.5_1)$ *and for any function $u \in C(]a,b[)$ almost everywhere in the interval $]a,b[$* (1.1.10) *be satisfied, where the nonnegative operator h satisfies the inclusion* $(1.1.8_1)$*. Let, moreover, in case $\beta < 1$,*

$$\frac{x(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x^\beta)(t)}{x^\beta(t)} - p_0(t)\Big) \le 2\beta^2 \quad \textit{for} \quad a < t < b, \qquad (1.1.50_1)$$

and in case $\beta = 1$,

$$\operatorname*{ess\,sup}_{t\in]a,b[}\Big[\frac{x(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x)(t)}{x(t)} - p_0(t)\Big)\Big] < 2 \qquad (1.1.51_1)$$

be satisfied. Then the problem (1.1.1), $(1.1.2_1)$ *has one and only one solution.*

Remark 1.1.8. The condition (1.1.51) is unimprovable in the sense that the validity of Corollary 1.1.5$_1$ is violated if we replace it by the condition

$$\operatorname*{ess\,sup}_{t\in]a,b[}\Big[\frac{x(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x)(t)}{x(t)} - p_0(t)\Big)\Big] \le 2\beta^2. \qquad (1.1.52)$$

Indeed, let $h(u) \equiv 0$, $p_1 \equiv 0$, $p_2 \equiv 0$. Then

$$\sigma(p_1)(t) = 1 \text{ and } x(t) = (b-t)(t-a) \text{ for } a \le t \le b$$

and the condition (1.1.52) will take the form

$$\operatorname*{ess\,sup}_{t\in]a,b[}\big(-(b-t)(t-a)p_0(t)\big) \le 2. \qquad (1.1.53)$$

If

$$p_0(t) = -\frac{2}{(b-t)(t-a)},$$

then the condition (1.1.53) is satisfied in the form of the equality, and at the same time, for any $c \in \mathbb{R}$ the function $c(b-t)(t-a)$ is a solution of the equation

$$u''(t) = -\frac{2}{(b-t)(t-a)}u(t), \qquad (1.1.54)$$

that is, the uniqueness of solution of the problem (1.1.54), ($1.1.2_{i0}$) is violated although the condition (1.1.52) along with the other requirements of Corollary $1.1.5_1$ is satisfied.

Corollary $\mathbf{1.1.5_2}$. *Let the function x be defined by* ($1.1.9_2$)*, the constants α, $\beta \in [0,1]$ be connected by* (1.1.6)*, the functions $p_j :]a,b[\to \mathbb{R}$ ($j = 0,1,2$) satisfy* ($1.1.3_2$)*,* ($1.1.5_2$) *and for any function $u \in C(]a,b[)$ almost everywhere in the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where a nonnegative operator h satisfies the inclusion* ($1.1.8_2$)*. Let, moreover,*

$$\operatorname*{ess\,sup}_{t\in]a,b[}\left[\frac{x^{2-[\beta]}(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x^\beta)(t)}{x^\beta(t)}-p_0(t)\Big)\right] < \beta(1-\beta), \qquad (1.1.50_2)$$

$$\frac{x^{2-\beta}}{\sigma^2(p_1)}[p_0]_- \in L_\infty([a,b]) \qquad (1.1.55)$$

if $0 < \beta \le 1$ and

$$0 \le p_0(t) - h(1)(t) \quad for \quad a < t < b \qquad (1.1.51_2)$$

if $\beta = 0$ be satisfied. Then the problem (1.1.1)*,* ($1.1.2_2$) *has one and only one solution.*

Remark 1.1.9. In the case $\beta = 1$, the condition (1.1.55) follows automatically from the condition ($1.1.50_2$).

Corollary $\mathbf{1.1.5_{10}}$. *Let the function x be defined by* ($1.1.9_1$)*, the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by* (1.1.6)*, the functions $p_j :]a,b[\to \mathbb{R}$ ($j = 0,1,2$) satisfy* ($1.1.3_1$)*,* (1.1.11) *and for any function $u \in C_{x^\beta}(]a,b[)$ almost everywhere on the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where the nonnegative operator h satisfies the inclusion* (1.1.12)*. Let, moreover, in case $0 < \beta < 1$ the condition* ($1.1.50_1$) *and in case $\beta = 1$ the condition* ($1.1.51_1$) *be satisfied. Then the problem* (1.1.1)*,* ($1.1.2_{10}$) *has in the space $C_{x^\beta}(]a,b[)$ one and only one solution.*

Corollary $\mathbf{1.1.5_{20}}$. *Let the function x be defined by* ($1.1.9_2$)*, the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by* (1.1.6)*, the functions $p_j :]a,b[\to \mathbb{R}$ ($j = 0,1,2$) satisfy* ($1.1.3_2$)*,* (1.1.11) *and for any function $u \in C_{x^\beta}(]a,b[)$ almost everywhere on the interval $]a,b[$ the inequality* (1.1.10) *be satisfied, where the nonnegative operator h satisfies the inclusion* (1.1.12)*. Let, moreover, the conditions* ($1.1.50_2$) *and* (1.1.55) *be satisfied. Then the problem* (1.1.1)*,* ($1.1.2_{20}$) *has one and only one solution in the space $C_{x^\beta}(]a,b[)$.*

Corollary $\mathbf{1.1.6_1}$. *Let the functions $\tau_k : [a,b] \to [a,b]$ ($k = 1,\dots,n$) be measurable and the functions p_j, $p_k \in L_{\rm loc}(]a,b[)$ ($k = 1,\dots,n$; $j = 0,1,2$) as well as the constants $\lambda_{l,m} \in]0,+\infty[$, $\beta_m \in [0,1]$ ($l,m = 1,2$), $c \in]a,b[$ be such that the conditions* ($1.1.3_1$)*,* ($1.1.5_1$) *are satisfied,*

$$g_k \in L_{\sigma_1(p_1)}([a,b]) \qquad (1.1.56_1)$$

and

$$\int_0^{+\infty} \frac{ds}{\lambda_{11}+\lambda_{12}s+s^2} > \frac{(c-a)^{1-\beta_1}}{1-\beta_1},$$
$$\int_0^{+\infty} \frac{ds}{\lambda_{21}+\lambda_{22}s+s^2} > \frac{(b-c)^{1-\beta_2}}{1-\beta_2}. \tag{1.1.57$_1$}$$

Let, moreover,

$$\begin{gathered}
(t-a)^{2\beta_2}\Big[p_0(t)-\sum_{k=1}^{n}|g_k(t)|\Big] \geq -\lambda_{11},\\
(t-a)^{\beta_1}\Big[p_1(t)+\frac{\beta_1}{t-a}-\sum_{k=1}^{n}|g_k(t)|(\tau_k(t)-t)\Big] \geq -\lambda_{12}\\
\textit{for} \quad a<t<c,\\
(b-t)^{2\beta_2}\Big[p_0(t)-\sum_{k=1}^{n}|g_k(t)|\Big] \geq -\lambda_{12},\\
(b-t)^{\beta_2}\Big[p_1(t)-\frac{\beta_2}{b-t}-\sum_{k=1}^{n}|g_k(t)|(\tau_k(t)-t)\Big] \leq \lambda_{22}\\
\textit{for} \quad c \leq t<b.
\end{gathered} \tag{1.1.58$_1$}$$

Then the problem (1.1.35), (1.1.2$_1$) *has one and only one solution.*

Corollary 1.1.6$_2$. *Let the functions* $\tau_k : [a,b] \to [a,b]$ $(k=1,\dots,n)$ *be measurable and the functions* $\widetilde{p_1}$, p_j, $g_k \in L_{\mathrm{loc}}(]a,b])$ $(k=1,\dots,n;\ j=0,1,2)$ *as well as the constants* $\lambda_{l,m} \in]0,+\infty[$, $(l,m=1,2)$, $\beta_r \in [0,1]$ $(r=1,2,3)$, $c \in]\max(a,b-1);b]$, $\varepsilon>0$ *and the dependent on them constant* $\alpha \in [0,1[$ *be such that the conditions*

$$\begin{gathered}
\sigma(\widetilde{p_1}) \in L([a,b]), \quad p_j\sigma_2(\widetilde{p_1}) \in L([a,b]) \quad (j=0,2),\\
g_k\sigma_2(\widetilde{p_1}) \in L([a,b]) \quad (k=1,\dots,n)
\end{gathered} \tag{1.1.56$_2$}$$

and

$$\int_\varepsilon^{+\infty} \frac{ds}{\lambda_{11}+\lambda_{12}s+s^2} > \frac{(c-a)^{1-\beta_1}}{1-\beta_1},$$
$$\int_0^{+\infty} \frac{ds}{\lambda_{21}+\lambda_{22}s+s^2} > \frac{(b-c)^{1-\beta_2}}{1-\beta_2} \tag{1.1.57$_2$}$$

are satisfied. Let, moreover,

$$
\begin{gathered}
(t-a)^{2\beta_2}\Big[p_0(t)-\sum_{k=1}^{n}|g_k(t)|\Big]\geq -\lambda_{11},\\
(t-a)^{\beta_1}\Big[\widetilde{p_1}(t)+\frac{\beta_1}{t-a}-\sum_{k=1}^{n}|g_k(t)|(\tau_k(t)-t)\Big]\geq -\lambda_{12}\\
\text{for}\quad a<t<c,\\
(b-t)^{\beta_2-\beta_3}\Big[p_0(t)-\sum_{k=1}^{n}|g_k(t)|\Big]\geq -\alpha\lambda_{21},\\
(b-t)^{\beta_2}\Big[\widetilde{p_1}(t)+\frac{\beta_3}{b-t}-\sum_{k=1}^{n}|g_k(t)|(\tau_k(t)-t)\Big]\geq \lambda_{22}\\
\text{for}\quad c\leq t<b.
\end{gathered}
\tag{1.1.58$_2$}
$$

Then for any function $p_1\in L_{\rm loc}(]a,b])$ *such that*

$$
p_1(t)\geq \widetilde{p_1}(t)\quad \text{for}\quad a<t<b, \tag{1.1.59}
$$

the problem (1.1.35), (1.1.2$_2$) *has one and only one solution.*

Consider now corollaries of Theorems 1.1.2$_i$ and 1.1.2$_{i0}$ for the equation

$$
u''(t)=\sum_{k=1}^{n}g_k(t)u(\tau_k(t))+p_2(t). \tag{1.1.60}
$$

Corollary 1.1.7$_1$. *Let* $\gamma\in[0,1[$, *the function* $p_2:]a,b[\to\mathbb{R}$ *satisfy the inclusion* (1.1.17),

$$
g_k\in L_{x^\gamma}([a,b])\quad (k=1,\dots,n) \tag{1.1.61}
$$

and

$$
g_k(t)\geq 0\quad (k=1,\dots,n)\quad \text{for}\quad a<t<b, \tag{1.1.62}
$$

where

$$
x(t)=(b-t)(t-a)\quad a\leq t\leq b. \tag{1.1.63$_1$}
$$

Let, moreover, there exist constants α, $\beta\in[0,\frac{1}{2}]$ *such that*

$$
0\leq\beta<1-\gamma,\quad \alpha+\beta\leq\frac{1}{2} \tag{1.1.64}
$$

and

$$
\begin{gathered}
\sum_{k=1}^{n}\int_a^b g_k(s)(b-\tau_k(s))^{\beta}(\tau_k(s)-a)^{\beta}(b-s)^{\alpha}(s-a)^{\alpha}\,ds<\\
<2^{\beta}\frac{16}{b-a}\Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}.
\end{gathered}
\tag{1.1.65}
$$

Then the problem (1.1.60), $(1.1.2_1)$ *has one and only one solution.*

Remark 1.1.10. Corollary $1.1.7_1$ remains valid if for $\beta \in]0, 1-\gamma[$ we replace the condition (1.1.65) by the following one:

$$\sum_{k=1}^{n} \int_a^b g_k(s)(b-\tau_k(s))^{\beta}(\tau_k(s)-a)^{\beta}(b-s)^{\alpha}(s-a)^{\alpha}\,ds \leq$$
$$\leq 2^{\beta}\frac{16}{b-a}\Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \tag{1.1.66$_1$}$$

Corollary 1.1.7$_2$. *Let* $\gamma \in [0,1[$, *the functions* p_2, $p_k :]a,b[\to \mathbb{R}$ $(k = 1,\dots,n)$ *satisfy the conditions* (1.1.17), (1.1.61), *and* (1.1.62), *where*

$$x(t) = t-a \quad \textit{for} \quad a \leq t \leq b. \tag{1.1.63$_2$}$$

Let, moreover, there exist constants α, $\beta \in [0, \frac{1}{2}]$ *such that the conditions* (1.1.64) *and*

$$\sum_{k=1}^{n} \int_a^b g_k(s)(\tau_k(s)-a)^{\beta}(s-a)^{\beta}\,ds \leq \frac{8}{b-a}\Big(\frac{b-a}{4}\Big)^{\alpha+\beta} \tag{1.1.66$_2$}$$

are satisfied. Then the problem (1.1.60), $(1.1.2_2)$ *has one and only one solution.*

Corollary 1.1.7$_{i0}$. *Let* $i \in \{1,2\}$, $\gamma \in [0,1[$, $\delta \in]0, 1-\gamma[$,

$$p_2 \in L_{x^\gamma}([a,b]), \quad g_k x^{\delta}(\tau_k) \in L_{x^\gamma}([a,b]) \ \ (k = 1,\dots,n),$$

and the condition (1.1.62) *be satisfied, where the function* x *is defined by* $(1.1.63_i)$. *Let, moreover, there exist constants* $\alpha \in [0, \frac{1}{2}]$, $\beta \in]0, \frac{1}{2}]$ *such that the conditions*

$$\delta \leq \beta < 1-\gamma, \quad \alpha+\beta \leq \frac{1}{2}$$

and $(1.1.66_i)$ *are satisfied. Then the problem* (1.1.60), $(1.1.2_{i0})$ *has in the space* $C_{x^\delta}(]a,b[)$ *one and only one solution.*

§ 1.2. Auxiliary Propositions

1.2.1. Statement of Auxiliary Problems and Some of Their Properties. Let us consider the linear equations

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(v)(t) + p_2(t), \tag{1.2.1}$$
$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(v)(t) \tag{1.2.1$_0$}$$

under the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2, \tag{1.2.2$_1$}$$

or

$$u(a) = c_1, \quad u'(b-) = c_2, \tag{1.2.2$_2$}$$

as well as under the conditions

$$v(a) = 0, \quad v(b) = 0, \tag{1.2.2$_{10}$}$$

$$v(a) = 0, \quad v'(b-) = 0, \tag{1.2.2$_{20}$}$$

where $c_1,\ c_2 \in \mathbb{R}$ and $h : C(\,]a,b[\,) \to L_{\mathrm{loc}}(]a,b[)$ is a continuous linear operator and

$$p_j \in L_{\mathrm{loc}}(]a,b[)\ (j = 0,1,2),\ \ \sigma(p_1) \in L([a,b]),\ \ p_0 \in L_{\sigma_1(p_1)}([a,b]) \tag{1.2.3$_1$}$$

or

$$p_j \in L_{\mathrm{loc}}(]a,b])\ (j = 0,1,2),\ \ \sigma(p_1) \in L([a,b]),\ \ p_0 \in L_{\sigma_2(p_1)}([a,b]). \tag{1.2.3$_2$}$$

For this purpose we will need the homogeneous equation

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) \tag{1.2.4}$$

under the initial conditions

$$v(a) = 0, \quad \lim_{t\to a} \frac{v'(t)}{\sigma(p_1)(t)} = 1, \tag{1.2.5}$$

$$v(b) = 0, \quad \lim_{t\to b} \frac{v'(t)}{\sigma(p_1)(t)} = -1, \tag{1.2.5$_1$}$$

or

$$v(b) = 1, \quad v'(b-) = 0. \tag{1.2.5$_2$}$$

The facts mentioned in the remarks below or their analogues have been proved in [23], pp. 110–158.

Remark 1.2.1. Let measurable functions $p_0,\ p_1\ :]a,b[\to \mathbb{R}$ satisfy the conditions (1.2.3$_1$) and the functions v_1 and v_2 be respectively solutions of the problems (1.2.4), (1.2.5) and (1.2.4), (1.2.5$_1$). Then any linearly independent with v_j, $(j = 1,2)$ solution $\widetilde{v}$ of the equation (1.2.4) satisfies the condition

$$\widetilde{v}(a) \neq 0 \quad \text{for} \quad j = 1$$

and

$$\widetilde{v}(b) \neq 0 \quad \text{for} \quad j = 2.$$

Remark 1.2.2. Let $i \in \{1,2\}$ and

$$(p_0, p_1) \in \mathbb{V}_{i,0}(]a,b[). \tag{1.2.6$_i$}$$

Then the problem (1.2.4), (1.2.2$_{i0}$) has only the trivial solution and the unique Green's function G can be represented as:

$$G(t,s) = \begin{cases} -\dfrac{v_2(t)v_1(s)}{v_2(a)\sigma(p_1)(s)} & \text{for } a \le s < t \le b, \\ -\dfrac{v_2(s)v_1(t)}{v_2(a)\sigma(p_1)(s)} & \text{for } a \le t < s \le b, \end{cases} \tag{1.2.7}$$

where v_1 and v_2 are respectively the solutions of the problems (1.2.4), (1.2.5) and (1.2.4), (1.2.5$_i$), and

$$G(t,s) < 0 \quad \text{for} \quad (t,s) \in]a,b[\times]a,b[\,, \tag{1.2.8}$$

$$G(a,s) = 0, \quad G(b,s) = i-1 \quad \text{for} \quad a \le s \le b. \tag{1.2.9$_i$}$$

Remark 1.2.3. Let $i \in \{1,2\}$ and the inclusion (1.2.6$_i$) be satisfied. Then there exist constants c_*, $d_* \in \mathbb{R}^+$ such that the estimates

$$d_* \le \frac{v_1(t)}{\int\limits_a^t \sigma(p_1)(s)\,ds} \le c_*, \quad d_* \le \frac{v_2(t)}{(\int\limits_t^b \sigma(p_1)(s)\,ds)^{2-i}} \le c_* \quad \text{for} \quad a < t < b, \tag{1.2.10$_i$}$$

$$\begin{gathered} \frac{|v_1'(t)|}{\sigma(p_1)(t)} \le 1 + c_* \int\limits_a^t |p_0(s)|\sigma_2(p_1)(s)\,ds, \\ \frac{|v_2'(t)|}{\sigma(p_1)(t)} \le 2 - i + c_* \int\limits_t^b \frac{|p_0(s)|}{\sigma(p_1)(s)} \Bigg(\int\limits_s^b \sigma(p_1)(\eta)\,d\eta \Bigg)^{2-i} ds \\ \text{for} \quad a \le t < b \end{gathered} \tag{1.2.11$_i$}$$

are valid, where v_1 and v_2 are respectively the solutions of the problems (1.2.4), (1.2.5) and (1.2.4), (1.2.5$_i$), and

$$\Big| \frac{\partial^{j-1} G(t,s)}{\partial t^{j-1}} \Big| \le$$

$$\le c_* \frac{\sigma_i(p_1)(s)}{[\sigma_i(p_1)(t)]^{j-1}} \;\; (j = 1,2) \;\; \text{for} \;\; (t,s) \in]a,b[\times]a,b[\;\; (t \ne s). \tag{1.2.12$_i$}$$

Remark 1.2.4. Let $i \in \{1,2\}$, the conditions (1.2.3$_i$) be satisfied and the problem (1.2.4), (1.2.2$_i$) have lower w_1 and upper w_2 functions such that

$$w_1(t) \le w_2(t) \quad \text{for} \quad a \le t \le b.$$

Then the problem (1.2.4), (1.2.2$_i$) has at least one solution v such that

$$w_1(t) \le v(t) \le w_2(t) \quad \text{for} \quad a \le t \le b.$$

Remark 1.2.5. Let $i \in \{1,2\}$ and the inclusion $(1.2.6_i)$ be satisfied. Then every upper function w of the problem (1.2.4), $(1.2.2_{i0})$ is nonnegative in the interval $]a,b[$; moreover, if

$$w(a) + w^{(i-1)}(b-) \neq 0,$$

then w is positive on the interval $]a,b[$.

Remark 1.2.6. Let $i \in \{1,2\}$, the functions p_0, $p_1 :]a,b[\to \mathbb{R}$ satisfy the conditions $(1.2.3_i)$ and

$$p_0(t) \geq 0 \quad \text{for} \quad a < t < b.$$

Then the inclusion $(1.2.6_i)$ is valid.

Lemma 1.2.1. *Let $i \in \{1,2\}$ and*

$$h \in \mathcal{L}(C; L_{\sigma_i(p_1)}) \tag{$1.2.13_i$}$$

where h is a nonnegative operator. Then

$$\mathbb{V}_{i,0}(]a,b[;h) \subset \mathbb{V}_{i,0}(]a,b[).$$

Proof. Let $(p_0,p_1) \in \mathbb{V}_{i,0}(]a,b[;h)$. Then the problem $(1.2.1_0)$, $(1.2.2_{i0})$ has a positive upper function w which because of the nonnegativeness of the operator h will at the same time be an upper function of the problem (1.2.4), $(1.2.2_{i0})$.

Consider first the case $i = 1$. For the equation (1.2.4) we pose the problem

$$v(a) = 0, \quad v(b) = w(b), \tag{1.2.14}$$

for which $\beta(t) \equiv 0$ and w are respectively lower and upper functions. Then by virtue of Remark 1.2.4, the problem (1.2.4), (1.2.14) has a solution v_0 such that

$$0 \leq v_0(t) \leq w(t) \quad \text{for} \quad a \leq t \leq b.$$

If we assume that $v_0(t_0) = 0$ for some $t_0 \in]a,b[$, then we will get the contradiction with the unique solvability of the Cauchy problem, i.e.,

$$v_0(t) > 0 \quad \text{for} \quad a < t \leq b. \tag{1.2.15}$$

As is seen from Remark 1.2.1 and the conditions (1.2.14) that v_1 a solution of the problem (1.2.4), $(1.2.5_1)$, and v_0 are linearly dependent, hence by virtue of (1.2.15),

$$v_1(t) > 0 \quad \text{for} \quad a < t \leq b,$$

i.e., as is seen from Definition 1.1.2, $(p_0,p_1) \in \mathbb{V}_{1,0}(]a,b[)$.

Let now $i = 2$, and for the equation (1.2.4) we pose the initial problem

$$v(b) = 0, \quad v'(b-) = -1$$

which, with regard for the conditions ($1.2.3_2$), has a unique solution $\widetilde{v}$ defined on the whole interval $[a, b]$. Then we choose $\varepsilon > 0$ such that the inequality

$$\varepsilon v(t) < w(t) \quad \text{for} \quad a < t < b \tag{1.2.16}$$

is satisfied; this is possible because the function w is positive. It is clear from (1.2.16) that

$$w_1(t) = w(t) - \varepsilon v(t)$$

is an upper function of the problem (1.2.4), ($1.2.2_{20}$) and

$$w_1'(b-) > 0, \quad w_1(t) > 0 \quad \text{for} \quad a \le t \le b.$$

We consider now for the equation (1.2.4) the problem

$$v(a) = 0, \quad v'(b-) = w_1'(b-), \tag{1.2.17}$$

for which $\beta(t) \equiv 0$ and w_1 are respectively lower and upper functions. Hence by virtue of Remark 1.2.4, the problem (1.2.4), (1.2.17) has a solution v_0 such that

$$0 \le v_0(t) \le w_1(t) \quad \text{for} \quad a < t < b$$

and

$$v_0(a) = 0, \quad v_0(b) > 0, \quad v_0'(b-) > 0.$$

Reasoning in the same way as for $i = 1$, we see that $(p_0, p_1) \in \mathbb{V}_{2,0}(]a, b[)$. □

Along with Lemma 1.2.1 we have proved the following

Lemma 1.2.2. *Let $i \in \{1, 2\}$, the functions p_0, p_1 $:]a, b[\to \mathbb{R}$ satisfy the conditions ($1.2.3_i$) and, moreover, let the problem (1.2.4), ($1.2.2_{i0}$) have a positive upper function. Then the inclusion ($1.2.6_i$) is satisfied.*

Lemma 1.2.3. *Let $i \in \{1, 2\}$, the functions p_0, p_1 $:]a, b[\to \mathbb{R}$ satisfy the inclusion ($1.2.6_i$) and the nonnegative operator h satisfy the inclusion ($1.2.13_i$). Let, moreover, $\rho_0 \in C(]a, b[)$ such that*

$$\rho_0(t) > 0 \quad \textit{for} \quad a < t < b \tag{1.2.18}$$

and

$$\sup\left\{\frac{1}{\rho_0(t)} \int_a^b |G(t,s)| h(\rho_0)(s)ds : \quad a < t < b\right\} < 1, \tag{1.2.19}$$

where G is Green's function of the problem (1.2.4), ($1.2.2_{i0}$). Then there exists a continuous function $\rho : [a, b] \to \mathbb{R}^+$ such that

$$\max\left\{\frac{1}{\rho(t)} \int_a^b |G(t,s)| h(\rho)(s)ds : \quad a \le t \le b\right\} < 1. \tag{1.2.20}$$

Proof. First of all we note that the existence of Green's function G of the problem (1.2.4), $(1.2.2_{i0})$ follows from Remark 1.2.2, and the boundedness of the integrals in the inequalities (1.2.19) and (1.2.20) for any continuous function ρ follows from the estimates $(1.2.12_i)$ and the inclusion $(1.2.13_i)$.

Consider now separately the case $i = 2$. By virtue of the equalities $(1.2.9_2)$, the inequality (1.2.19) can be satisfied only under the conditions

$$\rho_0(a) \geq 0, \quad \rho_0(b) > 0. \tag{1.2.21}$$

Then (1.2.19) can be rewritten as

$$\int\limits_a^b |G(t,s)|h(\rho_0)(s)ds < \rho_0(t) \quad \text{for} \quad a < t \leq b. \tag{1.2.22}$$

As is seen from the equalities $(1.2.9_2)$, there exist positive constants r_1 and δ such that

$$\int\limits_a^b |G(t,s)|h(1)(s)ds - 1 < 0 \quad \text{for} \quad a \leq t \leq a+\delta \tag{1.2.23}$$

and

$$\int\limits_a^b |G(t,s)|h(1)(s)ds - 1 < r_1 \quad \text{for} \quad a \leq t \leq b. \tag{1.2.24}$$

On the other hand, from (1.2.22) it follows the existence of a constant $r_2 > 0$ such that

$$r_2 < \rho_0(t) - \int\limits_a^b |G(t,s)|h(\rho_0)(s)\,ds \quad \text{for} \quad a+\delta \leq t \leq b. \tag{1.2.25}$$

Then from (1.2.22)–(1.2.25) we obtain

$$\frac{r_2}{r_1}\bigg(\int\limits_a^b |G(t,s)|h(1)(s)\,ds - 1\bigg) \leq \rho_0(t) - \int\limits_a^b |G(t,s)|h(\rho_0)(s)\,ds \;\; \text{for} \;\; a \leq t \leq b,$$

which implies the validity of the inequality (1.2.20) for the function $\rho(t) = \varepsilon + \rho_0(t)$, where $\varepsilon = \frac{r_2}{r_1}$.

To complete the proof of the lemma we note that for $i = 1$, unlike the case $i = 2$, the inequality (1.2.19) by virtue of $(1.2.9_i)$ can be satisfied also for

$$\rho(a) > 0, \quad \rho(b) \geq 0$$

and for

$$\rho(a) \geq 0, \quad \rho(b) \geq 0$$

as well.

In these cases the above lemma can be proved similarly to the case of the conditions (1.2.21) with the only difference that the inequality (1.2.22) will be valid for $t \in [a,b[$ or $t \in]a,b[$, the inequality (1.2.23) for $t \in [b-\delta,b]$ or $t \in [a+\delta;\, b-\delta]$, and the inequality (1.2.25) will be considered for $t \in [a,b-\delta[$ or $t \in]a+\delta,\, b-\delta[$. $\square$

Lemma 1.2.4. *Let $i \in \{1,2\}$,*

$$(p_0,p_1) \in \mathbb{V}_{i,0}(]a,b[;h), \tag{1.2.26$_i$}$$

where the nonnegative operator h satisfies the inclusion (1.2.13$_i$). *Then there exists a continuous function $\rho : [a,b] \to \mathbb{R}^+$ such that the inequality* (1.2.20) *holds, where G is Green's function of the problem* (1.2.4), (1.2.2$_{i0}$).

Proof. As is seen from the definition of the set $\mathbb{V}_{i,0}(\,]a,b[\,;h)$, the problem (1.2.1$_0$), (1.2.2$_{i0}$) has on the interval $[a,b]$ a positive upper function w. Then we introduce a continuous operator $\chi : C(]a,b[) \to C(]a,b[)$ by the equality

$$\chi(y)(t) = \frac{1}{2}\Big[|y(x)| - |w(t)-y(t)| + w(t)\Big] \quad \text{for} \quad a \le t \le b \tag{1.2.27}$$

which for any $v \in C(]a,b[)$ satisfies

$$0 \le \chi(v)(t) \le w(t) \quad \text{for} \quad a \le t \le b, \tag{1.2.28}$$

and consider the problem

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(\chi(v))(t), \tag{1.2.29}$$

$$v(a) = w(a), \quad v^{(i-1)}(b-) = w^{(i-1)}(b-). \tag{1.2.30$_i$}$$

Note that from Lemma 1.2.1 and Remark 1.2.2 it follows the existence of Green's function of the problem (1.2.4), (1.2.2$_i$). Introduce the operator $H : C(]a,b[) \to C(]a,b[)$ by the equality

$$H(g)(t) = v_0(t) + \int\limits_a^b |G(t,s)|h(\chi(y))(s)\,ds,$$

where v_0 is a solution of the problem (1.2.4), (1.2.30$_i$), and consider the equation

$$v(t) = H(v)(t) \tag{1.2.31}$$

which is equivalent to the problem (1.2.29), (1.2.30$_i$). Let us show that the operator H is compact. Let c_* be a constant mentioned in Remark 1.2.3,

$$r = c_* \int\limits_a^b \sigma_i(p_1)(s)h(w)(s)ds,$$

$$\mathbb{B}_r = \big\{z \in C(]a,b[) : \ \|z - v_0\|_C \le r\big\},$$

and $(x_n)_{n=1}^{\infty}$ be any sequence from $\mathbb{B}_r$. Then from the estimate $(1.2.12_i)$ for the sequence $y_n(t) = H(x_n)(t)$, $n \in \mathbb{N}$, we have

$$\|v_0 - y_n\|_C \le r, \quad n \in \mathbb{N}. \tag{1.2.32}$$

Consider separately the case $i = 1$. By virtue of $(1.2.9_1)$, (1.2.28) and the fact that the function v_0 is continuous, for any constant $\varepsilon > 0$ there exist a_1, $b_1 \in]a, b[$, $a_1 < b_1$ such that

$$\max\big\{|v_0(t_1) - v_0(t_2)| : \quad a \le t_1 \le t_2 \le a_1,\, b_1 \le t_1 \le t_2 \le b\big\} \le \frac{\varepsilon}{4}$$

and

$$\varepsilon^* \equiv \max\Bigg\{ \int\limits_a^b |G(t,s)| h(\chi(x_n))(s)\, ds : \quad a \le t \le a_1,\, b_1 \le t \le b \Bigg\} \le \frac{\varepsilon}{8}.$$

Then for any $n \in \mathbb{N}$ the estimate

$$|y_n(t_1) - y_n(t_2)| \le \frac{\varepsilon}{4} + 2\varepsilon^* \le \frac{\varepsilon}{2},$$
$$\text{for} \quad a \le t_1 \le t_2 \le a_1, \quad b_1 \le t_1 \le t_2 \le b,$$

is valid.

In the same way, by virtue of the estimates $(1.2.12_i)$, there exists a constant δ, $0 < \delta < \min(a_1 - a,\, b - b_1)$, such that for any $n \in \mathbb{N}$

$$|y_n(t_1) - y_n(t_2)| \le$$
$$\le (1 + r) \max\big\{|v_0'(t)| + \sigma_1^{-1}(p_1)(t) : \quad a_1 - \delta < t < b + \delta\big\} |t_2 - t_1| \le \frac{\varepsilon}{2}$$
$$\text{for} \quad |t_1 - t_2| \le \delta, \quad a_1 - \delta \le t_j \le b_1 + \delta \quad (j = 1, 2).$$

It follows from the last two estimates that if $t_j \in [a, b]$ $(j = 1, 2)$ and

$$|t_1 - t_2| \le \delta,$$

then

$$|y_n(t_1) - y_n(t_2)| \le \varepsilon, \quad n \in \mathbb{N}.$$

From this and from the inequality (1.2.32) we obtain that the sequence $(y_n)_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. In case $i = 2$, the same follows from the possibility to choose for any $\varepsilon > 0$, owing to $(1.1.9_2)$, (1.2.28), $a_1 \in]a, b[$ and $0 < \delta < a_1 - a$ such that

$$\max\big\{|v_0(t_1) - v_0(t_2)| : \quad a \le t_1 \le t_2 \le a_1\big\} \le \frac{\varepsilon}{4},$$
$$\max\Bigg\{ \int\limits_a^b |G(t,s)| h(w)(s)\, ds : \quad a \le t \le a_1 \Bigg\} \le \frac{\varepsilon}{4},$$

and

$$|y_n(t_1) - y_n(t_2)| \le$$
$$\le (1+r) \max\big\{|v_0'(t)| + \sigma_2^{-1}(p_1)(t): \ \ a_1 - \delta \le t \le b\big\}|t_1 - t_2| \le \frac{\varepsilon}{2}$$
$$\text{for} \quad |t_1 - t_2| \le \delta, \ \ a_1 - \delta \le t_j \le b \ \ (j = 1, 2).$$

Then according to the Arzella–Ascoli lemma, the operator H which is, as it is not difficult to show, continuous, transforms the ball $\mathbb{B}_r$ into its compact subset. In this case the equation (1.2.31), i.e., the problem (1.2.29), (1.2.30$_i$) has at least one solution, say v. Show that

$$0 < v(t) \le w(t) \quad \text{for} \quad a \le t \le b.$$

Let

$$v_1(t) = w(t) - v(t).$$

Then from the nonnegativeness of the operator h and also from the inequality (1.2.28) we have

$$v_1''(t) \le p_0(t)v_1(t) + p_1(t)v_1'(t) - h(w - \chi(v))(t) \le p_0(t)v_1(t) + p_1(t)v_1'(t)$$

and

$$v_1(a) = 0, \quad v_1^{(i-1)}(b-) = 0.$$

Hence v_1 is an upper function of the problem (1.2.4), (1.2.2$_{i0}$), and due to Remark 1.2.5,

$$v_1(t) \ge 0 \quad \text{for} \quad a < t < b,$$

i.e.,

$$v(t) \ge w(t) \quad \text{for} \quad a < t < b. \tag{1.2.33}$$

On the other hand, taking into account the inequality (1.2.28) and the fact that the operator h is nonnegative, from (1.2.29) and (1.2.30$_i$) we conclude that v is an upper function of the problem (1.2.4), (1.2.2$_{i0}$), i.e., by virtue of Remark 1.2.5,

$$v(t) > 0 \quad \text{for} \quad a \le t \le b. \tag{1.2.34}$$

It follows from (1.2.33) and (1.2.34) that the inequality $0 < v(t) \le w(t)$ is valid and hence

$$\chi(v)(t) = v(t) \quad \text{for} \quad a \le t \le b,$$

i.e., v as a solution of the equation (1.2.31) has the form

$$v(t) = v_0(t) + \int\limits_a^b |G(t,s)| h(v)(s)\, ds \quad \text{for} \quad a \le t \le b, \tag{1.2.35}$$

where by Remark 1.2.5,

$$v_0(t) > 0 \quad \text{for} \quad a \le t \le b. \tag{1.2.36}$$

If we introduce the notation $\rho(t) = v(t)$ and take into consideration (1.2.36), then in view of (1.2.35) we can see that our lemma is valid. $\square$

Lemma 1.2.5. *Let $i \in \{1,2\}$, the constants $\alpha \in [0,1[$ and $\beta \in]0,1]$ be connected by the inequality*

$$\alpha + \beta \le 1, \tag{1.2.37}$$

$$(p_0, p_1) \in \mathbb{V}_{i,\beta}(]a,b[;\, h), \tag{1.2.38$_i$}$$

where

$$h \in \mathcal{L}\Big(C_{x^\beta}; L_{\frac{x^\alpha}{\sigma(p_1)}}\Big) \tag{1.2.39$_i$}$$

is a nonnegative operator and

$$x(t) = \int_a^t \sigma(p_1)(s)\,ds \bigg(\int_t^b \sigma(p_1)(s)\,ds \bigg)^{2-i} \quad \textit{for} \quad a \le t \le b. \tag{1.2.40$_i$}$$

Then there exists a positive function $\rho \in C(]a,b[)$ such that the inequality (1.2.20) *is satisfied, where G is Green's function of the problem* (1.2.4), (1.2.2$_i$) *and*

$$\rho(t) = O^*(x^\beta(t)) \tag{1.2.41}$$

as $t \to a$, $t \to b$ if $i = 1$, and as $t \to a$ if $i = 2$.

Proof. As is seen from the definition of the set $\mathbb{V}_{i,\beta}(]a,b[;\, h)$, the functions $p_0,\ p_1 :]a,b[\to \mathbb{R}$ satisfy the inclusion (1.2.6$_i$) from which by virtue of Remark 1.2.2 it follows the existence of Green's function of the problem (1.2.4), (1.2.2$_{i0}$), and there exists a measurable function $q_\beta :]a,b[\to [0,+\infty[$ such that the problem

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(v)(t) - q_\beta(t), \tag{1.2.42}$$

$$v(a) = 0, \quad v^{(i-1)}(b-) = 0 \tag{1.2.43$_i$}$$

has in the interval $]a,b[$ a positive upper function w, where

$$w(t) = O^*(x^\beta(t)) \quad \text{and} \quad \int_a^b |G(t,s)| q_\beta(s)\,ds = O^*(x^\beta(t)) \tag{1.2.44}$$

as $t \to a$, $t \to b$ if $i = 1$, and as $t \to a$ if $i = 2$.

Introduce the operator χ as in the previous proof and let

$$H(y)(t) = \int_a^b |G(t,s)|(q_\beta(s) + h(\chi(y))(s))\,ds.$$

As we can see from the conditions $(1.2.39_i)$, (1.2.44), the operator χ transforms the space $C(]a,b[)$ into $C_{x^\beta}(]a,b[)$. Consider now the equations

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) - h(\chi(v))(t) - q_\beta(t), \tag{1.2.45}$$

$$v(t) = H(v)(t) \tag{1.2.46}$$

and note that the problem (1.2.45), $(1.2.43_i)$ is equivalent to the equation (1.2.46).

From the equality (1.2.7) by means of which Green's function is expressed, as well as from the estimates $(1.2.10_i)$ and the conditions (1.2.44), for any $y \in C(]a,b[)$ we have

$$|H(y)(t)| \le r_0 x^{1-\alpha}(t) \int_a^t \frac{x^\alpha(s)}{\sigma(p_1)(s)} h(x^\beta)(s)\,ds +$$

$$+ \int_a^b |G(t,s)| q_\beta(s)\,ds < +\infty \quad \text{for} \quad a \le t \le b, \tag{1.2.47}$$

where

$$r_0 = \frac{c_*^2}{d_*} \sup\left\{\frac{w(t)}{x^\beta(t)} : \quad a < t < b\right\}.$$

It follows from (1.2.37), (1.2.44) that the operator H transforms the space $C(]a,b[)$ into $C_{x^\beta}(]a,b[)$. Noticing that the right-hand side of the estimate (1.2.47) is independent of the function y, we make sure that a constant r exists such that for any $y \in C(]a,b[)$

$$\|H(y)\|_{C,x^\beta} \le r.$$

It is clear that this estimate is the more so valid if y belongs to the ball

$$\mathbb{B}_r = \big\{z \in C_{x^\beta}(]a,b[) : \quad \|z\|_{C,x^\beta} \le r\big\}.$$

Repeating now the reasoning of the previous proof, we can see that the operator $H : C_{x^\beta}(]a,b[) \to C_{x^\beta}(]a,b[)$ is compact and hence there exists a solution v of the equation (1.2.46) such that

$$v \in C_{x^\beta}(]a,b[), \tag{1.2.48}$$

$$\chi(v)(t) = v(t) \quad \text{for} \quad a \le t \le b,$$

and

$$v(t) > 0 \quad \text{for} \quad a < t < b. \tag{1.2.49}$$

Then the following representation is valid:

$$v(t) = \int_a^b |G(t,s)| \big(h(v)(s) + q_\beta(s)\big)\,ds, \tag{1.2.50}$$

whence with regard for (1.2.49) we obtain the inequality

$$v(t) \geq \int_a^b |G(t,s)| q_\beta(s)\, ds \quad \text{for} \quad a \leq t \leq b$$

which together with the conditions (1.2.44) and (1.2.48) implies that

$$v(t) = O^*(x^\beta(t)) \tag{1.2.51}$$

for $t \to a$, $t \to b$, if $i = 1$, and for $t \to a$ if $i = 2$. If we now take into consideration that owing to the conditions (1.2.44) and (1.2.51) we have

$$\inf\left\{\frac{1}{v(t)} \int_a^b |G(t,s)| q_\beta(s)\, ds : \quad a < t < b\right\} > 0,$$

then from (1.2.50) we obtain

$$\sup\left\{\frac{1}{v(t)} \int_a^b |G(t,s)| h(v)(s)\, ds : \quad a < t < b\right\} < 1. \tag{1.2.52}$$

Introducing the notation $\rho(t) = v(t)$, from (1.2.49), (1.2.51) and (1.2.52) we see that our lemma is valid. $\square$

Lemma 1.2.6. *Let $i \in \{1,2\}$, the function x be defined by* (1.2.40$_i$)*, the constants $\alpha \in [0,1[$, $\beta \in]0,1]$ be connected by* (1.2.37) *and the functions p_0, $p_1 :]a,b[\to \mathbb{R}$ satisfy* (1.2.38$_i$)*, where*

$$h \in \mathcal{L}\Big(C_{x^\beta};\, L_{\frac{x^\alpha}{\sigma(p_1)}}\Big) \cap \mathcal{L}\Big(C;\, L_{\sigma_i(p_1)}\Big) \tag{1.2.53$_i$}$$

is a nonnegative operator. Then there exists a continuous function $\rho : [a,b] \to \mathbb{R}^+$ such that the inequality (1.2.20) *is satisfied, where G is Green's function of the problem* (1.2.4)*,* (1.2.2$_{i0}$)*.*

Proof. By Lemma 1.2.5, from the fact that $h \in \mathcal{L}(C_{x^\beta};\, L_{\frac{x^\alpha}{\sigma(p_1)}})$ it follows the existence of the function $\rho_0 \in C(]a,b[)$ such that

$$\rho_0(t) > 0 \quad \text{for} \quad a < t < b$$

and

$$\sup\left\{\frac{1}{\rho_0(t)} \int_a^b |G(t,s)| h(\rho_0)(s)\, ds : \quad a < t < b\right\} < 1.$$

Then, taking into account that the operator h also belongs to $\mathcal{L}(C; L_{\sigma_i(p_1)})$, we can see by Lemma 1.2.3 that our lemma is valid. $\square$

Lemma 1.2.7. *Let $i \in \{1,2\}$, the function $x :]a,b[\to \mathbb{R}^+$ be defined by* $(1.2.40_i)$ *and the functions p_0, $p_1 :]a,b[\to \mathbb{R}$ satisfy the inclusion* $(1.2.6_i)$. *Then for any $\beta \in]0,1]$ we have*

$$\int_a^b |G(t,s)| \frac{\sigma^2(p_1)(s)}{x^{2-\beta-[\beta]}(s)}\, ds = O^*(x^\beta(s)) \tag{1.2.54}$$

as $t \to a$, $t \to b$ if $i = 1$, and as $t \to a$ if $i = 2$, where G is Green's function of the problem (1.2.4), $(1.2.2_{i0})$.

Proof. By Remark 1.2.2 and the inclusion $(1.2.6_i)$ there exists Green's function G of the problem (1.2.4), $(1.2.2_{i0})$ which is expressed by the equality (1.2.7).

Consider the case $i = 1$ separately and note that

$$\int_t^b \sigma(p_1)(s)\, ds \ge \int_{\frac{a+b}{2}}^b \sigma(p_1)(s)\, ds \quad \text{for} \quad a \le t \le \frac{a+b}{2}. \tag{1.2.55}$$

Then, taking into consideration (1.2.7), $(1.2.10_i)$ and (1.2.55), for any $\beta \in]0,1[$ we obtain for $t \in [a, \frac{a+b}{2}]$ the estimates

$$\int_a^b |G(t,s)| \frac{\sigma^2(p_1)(s)}{x^{2-\beta}(s)}\, ds \le \frac{c_*^2}{v_2(a)} \Bigg[\frac{x^\beta(t)}{\beta \int\limits_{\frac{a+b}{2}}^b \sigma(p_1)(s)\, ds} +$$

$$+ \frac{\big(\int_a^t \sigma(p_1)(s)\, ds\big)^\beta}{(1-\beta)\big(\int\limits_{\frac{a+b}{2}}^b \sigma(p_1)(s)\, ds\big)^{1-\beta}} + \frac{\big(\int_a^b \sigma(p_1)(s)\, ds\big)^{1-\beta}}{\beta\big(\int\limits_a^{\frac{a+b}{2}} \sigma(p_1)(s)\, ds\big)^{2-\beta}} x^\beta(t) \Bigg] \le$$

$$\le \frac{c_*^2}{\beta v_2(a)} \Bigg(\frac{1}{1-\beta} + \Big(\int_a^b \sigma(p_1)(s)\, ds\Big)^{1-\beta} \Big(\int_a^{\frac{a+b}{2}} \sigma(p_1)(s)\, ds\Big)^{\beta-2} \Bigg) x^\beta(t)$$

and

$$\int_a^b |G(t,s)| \frac{\sigma^2(p_1)(s)}{x^{2-\beta}(s)}\, ds \ge \frac{d_*^2}{v_2(a)} \Big(\int_t^b \sigma(p_1)(s)\, ds\Big)^\beta \Big(\int_{\frac{a+b}{2}}^b \sigma(p_1)(s)\, ds\Big)^{1-\beta} \times$$

$$\times \int_a^t \frac{\sigma(p_1)(s)\, ds}{\big(\int_a^s \sigma(p_1)(\eta)\, d\eta\big)^{1-\beta} \big(\int_s^b \sigma(p_1)(\eta)\, d\eta\big)^{2-\beta}} \ge$$

$$\geq \frac{d_*^2}{\beta v_2(a)}\,\frac{\Big(\int\limits_{\frac{a+b}{2}}^{b}\sigma(p_1)(s)\,ds\Big)^{1-\beta}}{\Big(\int\limits_{a}^{b}\sigma(p_1)(s)\,ds\Big)^{2-\beta}}\,x^\beta(t).$$

The last two estimates imply the validity of (1.2.54) as $t \to a$. Reasoning analogously for $t \in [\frac{a+b}{2}, b]$, we can see that this equality is also valid as $t \to b$. Consider the case $\beta = 1$. With regard for the equalities (1.2.7) and the estimates ($1.2.10_1$) we obtain

$$\frac{d_*^2}{2C_*} \leq \int\limits_a^b |G(t,s)|\sigma^2(p_1)(s)\,ds\,x^{-1}(t) \leq \frac{C_*^2}{2d_*} \quad \text{for} \quad a < t < b. \tag{1.2.56}$$

It follows from (1.2.56) that our lemma is valid in the case $\beta = 1$ as well.

Reasoning similarly, we can prove the lemma for $i = 2$. $\square$

1.2.2. Auxiliary Propositions to Theorems ($1.1.2_i$), ($1.1.2_{i0}$) ($i = 1, 2$). Consider in the interval $]a, b[$ the equation

$$v''(t) = g(v)(t), \tag{1.2.57}$$

where $g : C(]a, b[) \to L_{\rm loc}(]a, b[)$ is a continuous linear operator. We will also need the equation

$$v''(t) = 0 \quad \text{for} \quad a \leq t \leq b. \tag{1.2.58}$$

Note that Green's function of the problem (1.2.58), ($1.2.2_{i0}$) has the form

$$G(t,s) = \begin{cases} -(s-a)\Big(\dfrac{b-t}{b-a}\Big)^{2-i} & \text{for} \quad a \leq s < t \leq b, \\ -(t-a)\Big(\dfrac{b-s}{b-a}\Big)^{2-i} & \text{for} \quad a \leq t < s \leq b. \end{cases} \tag{$1.2.59_i$}$$

Lemma $1.2.8_1$. *Let $\gamma \in [0, 1[$, $\lambda \in [0; 1-\gamma[$ and*

$$g \in \mathcal{L}(C_{x^\lambda};\, L_{x^\gamma}) \tag{1.2.60}$$

be a nonnegative operator, where

$$x(t) = (b-t)(t-a) \quad \textit{for} \quad a \leq t \leq b. \tag{$1.2.61_1$}$$

Let, moreover, there exist constants α, $\beta \in [0, \frac{1}{2}]$ such that

$$\lambda \leq \beta < 1 - \gamma, \tag{1.2.62}$$

$$\alpha + \beta \leq \frac{1}{2}, \tag{1.2.63}$$

and

$$\int\limits_a^b x^\alpha(s)g(x^\beta)(s)\,ds < 2^\beta \frac{16}{b-a}\Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \qquad (1.2.64_1)$$

Then the problem (1.2.57), $(1.2.2_{10})$ *has only the zero solution in the space* $C_{x^\lambda}(]a,b[)$.

Proof. Suppose to the contrary that the problem (1.2.57), $(1.2.2_{i0})$ has a nonzero solution $v_0 \in C_{x^\lambda}(]a,b[)$.

If v_0 is a function of constant signs, then from the nonnegativeness of the operator g we obtain

$$v_0''(t)\operatorname{sign} v_0(t) \geq 0 \quad \text{for} \quad a < t < b,$$

which together with the conditions $(1.2.2_{i0})$ contradicts the assumption $v_0(t_0) \not\equiv 0$, i.e., v_0 is a function of constant signs.

Using Green's function of the problem (1.2.58), $(1.2.2_{i0})$, v_0 can be represented as follows:

$$v_0(t) = -\frac{1}{b-a}\bigg((b-t)\int\limits_a^t (s-a)g(v_0)(s)\,ds + (t-a)\int\limits_t^b (b-s)g(v_0)(s)\,ds\bigg)$$
$$\text{for} \quad a \leq t \leq b$$

and hence for any β the estimate

$$\frac{v_0(t)}{[(b-t)(t-a)]^\beta} \leq$$
$$\leq \frac{[(b-t)(t-a)]^{1-(\gamma+\beta)}}{b-a}\int\limits_a^b [(b-s)(s-a)]^\gamma g(x^\lambda)(s)\,ds \|v_0\|_{C,x^\lambda}$$
$$\text{for} \ a < t < b$$

is valid.

In the above estimate, taking into account the condition (1.2.60), if β satisfies the inequality (1.2.62), we get

$$\lim_{t\to a}\frac{v_0(t)}{[(b-t)(t-a)]^\beta} = 0, \quad \lim_{t\to b}\frac{v_0(t)}{[(b-t)(t-a)]^\beta} = 0.$$

These equalities imply the existence of points $t_1,\, t_2 \in]a,b[$ such that

$$\frac{v_0(t_1)}{(b-t_1)^\beta(t_1-a)^\beta} = \sup\Big\{\frac{v_0(t)}{(b-t)^\beta(t-a)^\beta} : \ a < t < b\Big\},$$
$$\frac{v_0(t_2)}{(b-t_2)^\beta(t_2-a)^\beta} = \inf\Big\{\frac{v_0(t)}{(b-t)^\beta(t-a)^\beta} : \ a < t < b\Big\}.$$

Without loss of generality we assume $t_1 < t_2$ and notice that by $(1.2.61_1)$ which defines the function x, we have

$$-g(x^\beta)(t)\frac{|v_0(t_2)|}{(b-t_2)^\beta(t_2-a)^\beta} \le$$
$$\le g(v_0)(t) \le g(x^\beta)(t)\frac{|v_0(t_1)|}{(b-t_1)^\beta(t_1-a)^\beta} \quad \text{for} \quad a < t < b. \quad (1.2.65)$$

Recall also one simple numerical inequality

$$A \cdot B \le \frac{(A+B)^2}{4}, \quad (1.2.66)$$

where $A \ge 0$ and $B \ge 0$.

Suppose $c \in]t_1, t_2[$ and $v_0(c) = 0$. Then the following representations are valid:

$$v_0(t_1) = \frac{c-t_1}{c-a}\int_a^{t_1}(s-a)g(-v_0)(s)\,ds + \frac{t_1-a}{c-a}\int_{t_1}^{c}(c-s)g(-v_0)(s)\,ds$$

and

$$|v_0(t_2)| = \frac{b-t_2}{b-c}\int_c^{t_2}(s-c)g(v_0)(s)\,ds + \frac{t_2-a}{b-c}\int_{t_2}^{b}(b-s)g(v_0)(s)\,ds.$$

These representations with regard for the inequality (1.2.65), for any α, β satisfying the conditions of the lemma, result in

$$v_0(t_1) \le \frac{[(c-t_1)(t_1-a)]^{1-\alpha}}{(c-a)[(b-t_2)(t_2-a)]^\beta}\int_a^c x^\alpha(s)g(x^\beta)(s)\,ds \cdot |v_0(t_2)| < +\infty$$

and

$$v_0(t_2) \le \frac{[(b-t_2)(t_2-c)]^{1-\alpha}}{(b-c)[(b-t_1)(t_1-a)]^\beta}\int_c^b x^\alpha(s)g(x^\beta)(s)\,ds \cdot |v_0(t_1)| < +\infty.$$

Multiplying the above inequalities, by means of (1.2.66) we obtain

$$\lambda\int_a^b x^\alpha(s)g(x^\beta)(s)\,ds \ge 1, \quad (1.2.67)$$

where

$$\lambda = \frac{1}{2}\sqrt{\frac{[(b-t_2)(t_2-c)(c-t_1)(t_1-a)]^{1-(\alpha+\beta)}[(t_2-c)(c-t_1)]^\beta}{(b-c)(c-a)(b-t_1)^\beta(t_2-a)^\beta}}.$$

Then by (1.2.66) we get the estimate

$$\lambda \leq \frac{1}{2}\sqrt{\frac{[(b-c)(c-a)]^{1-2(\alpha+\beta)}(t_2-t_1)^{2\beta}}{4^{2-2(\alpha+\beta)+\beta}[(b-t_1)(t_2-a)]^{\beta}}},$$

whence using once more the inequality (1.2.66) and taking into consideration the fact that

$$(t_2-t_1)^{2\beta} \leq [(b-t_1)(t_2-a)]^{\beta}, \tag{1.2.68}$$

we arrive at

$$\lambda \leq \frac{b-a}{16 \cdot 2^{\beta}}\Big(\frac{4}{b-a}\Big)^{2(\alpha+\beta)}. \tag{1.2.69}$$

Substituting the last inequality in (1.2.67), we obtain the contradiction with the condition ($1.2.64_1$), i.e., our assumption is invalid and $v_0(t) \equiv 0$. $\square$

Lemma 1.2.8$_2$. *Let* $\gamma \in [0,1[$, $\lambda \in [0, 1-\gamma[$ *and the nonnegative operator* g *satisfy the inclusion* (1.2.60), *where*

$$x(t) = t-a \quad \textit{for} \quad a \leq t \leq b. \tag{$1.2.61_2$}$$

Let, moreover, there exist constants α, $\beta \in [0, \frac{1}{2}]$ *such that the conditions* (1.2.62), (1.2.63) *are satisfied and*

$$\int\limits_a^b x^{\alpha}(s)g(x^{\beta})(s)\,ds \leq \frac{8}{b-a}\Big(\frac{b-a}{4}\Big)^{\alpha+\beta}. \tag{$1.2.64_2$}$$

Then the problem (1.2.57), ($1.2.2_{20}$) *has only the zero solution in the space* $C_{x^{\lambda}}(]a,b[)$.

Proof. Suppose to the contrary that the problem (1.2.57), ($1.2.2_{20}$) has a nonzero solution $v_0 \in C_{x^{\lambda}}(]a,b[)$. Similarly to the previous lemma we make sure that v_0 is of constant signs and the equality

$$\lim_{t\to a}\frac{v_0(t)}{(t-a)^{\beta}} = 0$$

is valid for any $\beta \in [\lambda, 1-\gamma[$. On the other hand, in any sufficiently small neighborhood of the point b, since $v_0'(b-) = 0$, the equality

$$\operatorname{sign}\Big(\frac{v_0(t)}{(t-a)^{\beta}}\Big)' = -\operatorname{sign} v_0(t)$$

is satisfied. It follows from the last two equalities that the function $\frac{v_0(t)}{(t-a)^{\beta}}$ attains neither its minimum nor its maximum at the points a and b. Let

$$\max\Big\{\frac{v_0(t)}{(t-a)^{\beta}}: \ a \leq t \leq b\Big\} = \frac{v_0(t_1)}{(t_1-a)^{\beta}}$$

and

$$\min\left\{\frac{v_0(t)}{(t-a)^\beta}:\quad a\le t\le b\right\}=\frac{v_0(t_2)}{(t_2-a)^\beta}.$$

Then from the above-said it is clear that $t_1,\ t_2\ \in]a,b[$. Without loss of generality we assume $t_1 < t_2$ and let the point $c\in]t_1,t_2[$ be such that $v_0(c)=0$. Then from the inequality

$$-g(x^\beta)(t)\frac{|v_0(t_2)|}{(t_2-a)^\beta}\le g(v_0)(t)\le g(x^\beta)(t)\frac{|v_0(t_1)|}{(t_1-a)^\beta}\quad\text{for}\ \ a<t<b$$

and from the equalities

$$v_0(t_1)=\frac{c-t_1}{c-a}\int_a^{t_1}(s-a)g(-v_0)(s)\,ds+\frac{t_1-a}{c-a}\int_{t_1}^{c}(c-s)g(-v_0)(s)\,ds,$$

$$|v_0(t_2)|=\int_c^{t_2}(s-c)g(v_0)(s)\,ds+(t_2-c)\int_{t_2}^{b}g(v_0)(s)\,ds$$

we obtain

$$v_0(t_1)\le\frac{(c-t_1)(t_1-a)^{1-\alpha}}{(c-a)(t_2-a)^\beta}\int_a^c x^\alpha(s)g(x^\beta)(s)\,ds\cdot|v_0(t_2)|$$

$$|v_0(t_2)|\le\frac{(t_2-c)^{1-\alpha}}{(t_1-a)^\beta}\int_c^b x^\alpha(s)g(x^\beta)(s)\,ds\cdot v_0(t_1).$$

Multiplying these inequalities, with regard for (1.2.66) we get

$$\lambda\int_a^b x^\alpha(s)g(x^\alpha)(s)\,ds\ge 1,\tag{1.2.70}$$

where

$$\lambda=\frac12\sqrt{\frac{[(t_1-a)(c-t_1)]^{1-(\alpha+\beta)}(t_2-c)^{1-\alpha}(c-t_1)^{\alpha+\beta}}{(c-a)(t_2-a)^\beta}}.$$

Then by (1.2.66) and $t_2-a>t_2-c$ we have

$$\lambda\le\frac12\sqrt{\frac{(c-a)^{1-2(\alpha+\beta)}(t_2-c)^{1-2(\alpha+\beta)}[(c-t_1)(t_2-c)]^{\alpha+\beta}}{4^{1-(\alpha+\beta)}}}.$$

Applying once more (1.2.66), we can see that

$$\lambda\le\frac{(t_2-a)^{1-2(\alpha+\beta)}(t_2-t_1)^{\alpha+\beta}}{2\cdot 4^{1-(\alpha+\beta)}}.\tag{1.2.71}$$

Notice that from the conditions $t_1,\ t_2 \in]a,b[$ as well as from the fact that for none of $\alpha,\ \beta \in [0, \frac{1}{2}]$ the expressions $\alpha+\beta$ and $1-2(\alpha+\beta)$ vanish simultaneously, we obtain the estimate

$$(t_2-a)^{1-2(\alpha+\beta)} \cdot (t_2-t_1)^{\alpha+\beta} < (b-a)^{1-(\alpha+\beta)},$$

with regard for which in (1.2.71) we get

$$\lambda < \frac{(b-a)}{8}\Big(\frac{4}{b-a}\Big)^{\alpha+\beta}.$$

Substituting the latter inequality in (1.2.70), we obtain the contradiction with the condition $(1.2.64_2)$, i.e., our assumption is invalid and $v_0(t) \equiv 0$. □

Remark 1.2.7. Lemma $1.2.8_1$ remains valid if for $\beta \neq 0$ we replace the condition $(1.2.64_1)$ by

$$\int\limits_a^b x^\alpha(s) g(x^\beta)(s)\, ds \leq 2^\beta \frac{16}{b-a}\Big(\frac{b-a}{4}\Big)^{2(\alpha+\beta)}. \tag{1.2.72}$$

Proof. If $\beta \neq 0$, then the inequality (1.2.68) will be strictly satisfied and hence the estimate (1.2.69) will take the form

$$\lambda < \frac{b-a}{16 \cdot 2^\beta}\Big(\frac{4}{b-a}\Big)^{2(\alpha+\beta)}.$$

Taking into consideration the last inequality in (1.2.67), we obtain the contradiction with the condition (1.2.72) which indicates the possibility to replace in case $\beta \neq 0$ the condition $(1.2.64_1)$ by (1.2.72). □

§ 1.3. Proof of Propositions on Existence and Uniqueness

1.3.1. Proof of Basic Theorems on Existence and Uniqueness of Solution of Two-Point Problems.

Proof of Theorem $1.1.1_i$. From the inclusions $(1.1.7_i)$ and $(1.1.8_i)$ and also from the fact that the operator h is nonnegative, for $\beta = 0$ by virtue of Lemma 1.2.4 and for $\beta > 0$ by virtue of Lemma 1.2.6 it follows that there exists a function $\rho \in C(]a,b[)$ such that

$$\rho(t) > 0 \quad \text{for} \quad a \leq t \leq b \tag{1.3.1}$$

and

$$\sup\bigg\{\frac{1}{\rho(t)}\int\limits_a^b |G(t,s)| h(\rho)(s)\, ds : \ \ a < t < b\bigg\} < 1, \tag{1.3.2}$$

where G is Green's function of the problem (1.2.4), $(1.2.2_{i0})$. Note that for any function $y \in C_\rho(]a,b[)$ the inequality

$$|y(t)| \leq \rho(t)\|y\|_{C,\rho} \quad \text{for} \quad a \leq t \leq b \tag{1.3.3}$$

is valid and, owing to the estimates ($1.2.10_i$), the representation (1.2.7) of Green's function and the conditions (1.1.5)–($1.1.8_i$) and (1.1.10), we have

$$\Big|\int_a^b G(t,s)p_2(s)\,ds\Big| < +\infty, \quad \Big|\int_a^b G(t,s)g(y)(s)\,ds\Big| < +\infty,$$

$$\Big|\int_a^b G(t,s)h(y)(s)\,ds\Big| < +\infty.$$

Introduce the continuous operators $\mathbb{U}_0$, $\mathbb{U} : C_\rho(]a,b[) \to C_\rho(]a,b[)$ by the equalities

$$\begin{aligned} \mathbb{U}_0(y)(t) &= \int_a^b G(t,s)g(y)(s)\,ds, \\ \mathbb{U}(g)(t) &= u_0(t) + \mathbb{U}_0(y)(t) + \int_a^b G(t,s)p_2(s)\,ds, \end{aligned} \tag{1.3.4}$$

where u_0 is a solution of the problem (1.2.4), ($1.2.2_i$). Clearly every solution of the problem (1.1.1), ($1.1.2_i$) is a solution of the equation

$$u(t) = \mathbb{U}(u)(t) \tag{1.3.5}$$

and vice versa.

From the definition of the norm of the operator it follows that

$$\|\mathbb{U}_0\|_{C_\rho \to C_\rho} =$$
$$= \sup\bigg\{\Big\|\int_a^b G(t,s)g(y)(s)\,ds\Big\|_{C,\rho} : \ x \in C_\rho(]a,b[), \ \ \|y\|_{C,\rho} = 1\bigg\}$$

which with regard for (1.1.10), (1.3.1)–(1.3.3) implies

$$\|\mathbb{U}_0\|_{C_\rho \to C_\rho} < 1, \tag{1.3.6}$$

i.e., the operator $\mathbb{U}$ contracts the space $C_\rho(]a,b[)$ into itself for any $p_2 \in L_{\sigma_i(p_1)}([a,b])$ and any operator g satisfying (1.1.10). Then by virtue of the theorem on contracting map the equation (1.3.5) has in the space $C_\rho(]a,b[)$ and hence in $C(]a,b[)$ a unique solution because, by (1.3.1), any function from $C(]a,b[)$ belongs to the space $C_\rho(]a,b[)$ as well. It remains to notice that the unique solvability of the problem (1.1.1), ($1.1.2_i$) follows from the equivalence of that problem and the equation (1.3.5). $\square$

Proof of Theorem $1.1.1_{i0}$. The inclusions ($1.1.7_i$), ($1.1.8_i$) and the nonnegativeness of the operator h imply by virtue of Lemma 1.2.5 the existence of a positive function $\rho \in C(]a,b[)$ such that

$$\rho(t) = O^*(x^\beta(t)) \tag{1.3.7}$$

as $t \to a$, $t \to b$, if $i = 1$, and as $t \to a$ if $i = 2$. Moreover, the condition (1.3.2) is satisfied, where G is Green's function of the problem (1.2.4), $(1.2.2_{i0})$. It is also clear that for any $y \in C_\rho(]a,b[)$ the inequality (1.3.3) is satisfied, and due to the estimates $(1.2.10_i)$ and the representation (1.2.7) of Green's functions we have

$$\begin{gathered}
\Big|\int\limits_a^b G(t,s)h(y)(s)\,ds\Big| \le r_1 x^{1-\alpha}(t)\int\limits_a^b \frac{x^\alpha(s)}{\sigma(p_1)(s)}\,h(x^\beta)(s)\,ds\,\|y\|_{C,x^\beta}, \\
\Big|\int\limits_a^b G(t,s)p_2(s)\,ds\Big| \le r_1 x^\beta(t)\int\limits_a^b \frac{x^{1-\beta}(s)}{\sigma(p_1)(s)}\,|p_2(s)|\,ds \quad \text{for} \quad a \le t \le b,
\end{gathered} \tag{1.3.8}$$

where

$$r_1 = \frac{c_*^2}{v_2(a)},$$

and the existence of integrals follows from the conditions (1.1.6), (1.1.11), (1.1.12). From (1.3.8) and (1.1.6), (1.1.10), (1.3.7) we also have that the operators

$$\mathbb{U}_0(y)(t) = \int\limits_a^b G(t,s)g(y)(s)\,ds$$

and

$$\mathbb{U}(y)(t) = \mathbb{U}_0(y)(t) + \int\limits_a^b G(t,s)p_2(s)\,ds$$

transform continuously the space $C_\rho(]a,b[)$ into itself. Repeating word by word the previous proof, we can see that the problem (1.1.1) $(1.1.2_{i0})$ has a unique solution u in the space $C_\rho(]a,b[)$. But as is seen from (1.3.7), u will be a unique solution in the space $C_{x^\beta}(]a,b[)$ as well. $\square$

Proof of Remark $1.1.1_i$. Under the conditions of Theorem $1.1.1_i$, as is seen from its proof, the operator $\mathbb{U}$ contracts the space $C_\rho([a,b])$ into itself. Then from the theorem on contracting map it follows that for any function $v_0 \in C_\rho(]a,b[)$ the sequence $v_n : [a,b] \to \mathbb{R}$, where v_n is the unique solution of the equation

$$v_n(t) = \mathbb{U}(v_{n-1})(t) \tag{1.3.9}$$

tends to the unique solution u of the equation (1.3.5) with respect to the norm $\|\cdot\|_{C,\rho}$. We introduce the notation

$$\|\mathbb{U}_0\|_{C_\rho \to C_\rho} = \mu \quad \text{and} \quad \|u - v_1\|_{C,\rho} = \omega,$$

and notice that by virtue of (1.3.6), we have $\mu < 1$. Then, as is known, the estimate

$$\|u - v_n\|_{C,\rho} \le \omega \, \frac{\mu^n}{1-\mu}\,, \quad n \in \mathbb{N}, \tag{1.3.10}$$

is valid and for any $n \in \mathbb{N}$ with regard for (1.3.3) we obtain

$$|u(t) - v_n(t)| \le \omega \, \frac{\mu^n}{1-\mu} \, \|\rho\|_C \quad \text{for} \quad a \le t \le b. \tag{1.3.11}$$

Differentiating the difference of the equations (1.3.5) and (1.3.9) and taking into account the inequalities (1.1.10), (1.3.11) and the estimates ($1.2.12_i$) of Green's function, we obtain

$$\sup\big\{\sigma_i(p_1)(t)|v_n'(t) - u'(t)| : \ \ a < t < b\big\} \le \omega' \, \frac{\mu^n}{1-\mu}\,, \quad n \in \mathbb{N}, \tag{1.3.12}$$

where

$$\omega' = \omega c_* \|\rho\|_C \int\limits_a^b \sigma_i(p_1)(s) h(1)(s)\, ds.$$

The inequalities (1.3.11), (1.3.12) imply the validity of the estimates (1.1.14), and after differentiating twice the equality (1.3.9) we see that v_n is a solution of the problem ($1.1.13_i$). □

Proof of Remark $1.1.1_{i0}$. Let ρ be the function appearing in the proof of Theorem $1.1.1_{i0}$. Introduce the constants μ and ω and the functions $v_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, as in the previous proof. Reasoning as above, we make sure that the estimate (1.3.10) is valid, and by virtue of the condition (1.3.7) for any $n \in \mathbb{N}$ we have

$$\frac{|u(t) - v_n(t)|}{x^\beta(t)} \le \omega \, \frac{\mu^n}{1-\mu} \, \sup\Big\{\frac{\rho(t)}{x^\beta(t)} : \ \ a < t < b\Big\}. \tag{1.3.13}$$

On the other hand, differentiating the difference of the equations (1.3.5) and (1.3.9), with regard for the equality (1.2.7) and the estimates ($1.2.10_i$), ($1.2.11_i$), for any $n \in \mathbb{N}$ we obtain

$$\frac{x^\alpha(t)}{\sigma(p_1)(t)} \, |u'(t) - v_n'(t)| \le r\|u - v_n\|_{C,\rho} \quad \text{for} \quad a \le t \le b, \tag{1.3.14}$$

where

$$r = (1+c^*)^2 \int\limits_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)} \, x(s) + \sigma(p_1)(s)\, ds \int\limits_a^b \frac{x^\alpha(s)}{\sigma(p_1)(s)} \, h(x^\beta)(s)\, ds.$$

The inequalities (1.3.10), (1.3.13) and (1.3.14) imply the validity of the estimates (1.1.15), and having differentiated twice the equality (1.3.9) we see that v_0 is a solution of the problem ($1.1.13_{i0}$). □

Proof of Theorem 1.1.2$_i$. Let G be Green's function of the problem (1.2.58), (1.2.2$_{i0}$). Introduce the operator $\mathbb{U}_0$ and the function q by the equalities

$$\mathbb{U}_0(y)(t) = \int_a^b G(t,s)g(y)(s)\,ds, \quad q(t) = \int_a^b G(t,s)p_2(s)\,ds. \tag{1.3.15}$$

From the representation (1.2.59$_i$) of Green's function and from the conditions (1.1.17), (1.1.18) it follows that the operator $\mathbb{U}_0$ transforms continuously the space $C(]a,b[)$ into itself and $q \in C(]a,b[)$.

Consider now the equation

$$u(t) = \mathbb{U}_0(u)(t) + u_0(t) + q(t), \tag{1.3.16}$$

where $u_0(t)$ is a solution of the problem (1.2.58), (1.1.2$_i$). Every its solution is a solution of the problem (1.1.16), (1.1.2$_i$), and vice versa.

Let $r > 0$, $\mathbb{B}_r = \big\{y \in C(]a,b[) : \ \|y\|_C \le r\big\}$ and choose any sequence $(x_n)_{n=1}^{\infty}$ from $\mathbb{B}_r$. Let, moreover, $y_n(t) = \mathbb{U}_0(x_n)(t)$, $n \in \mathbb{N}$. Then

$$\|y_n\|_C \le r_1, \quad n \in \mathbb{N}, \tag{1.3.17}$$

where

$$r_1 = r\int_a^b \Big(\frac{b-s}{b-a}\Big)^{2-i}(s-a)g(1)(s)\,ds.$$

Consider the case $i = 1$ separately. From the definition of Green's function G, for any $\varepsilon > 0$ it follows the existence of $a_1,\ b_1 \in]a,b[$, where $a_1 < b_1$, such that

$$\max\bigg\{\int_a^b |G(t,s)|g(1)(s)\,ds : \ a \le t \le a_1, \ \ b_1 \le t \le b\bigg\} \le \frac{\varepsilon}{4}\,,$$

which implies the validity of the estimate

$$|y_n(t_1) - y_n(t_2)| \le \frac{\varepsilon}{2}\,, \ \ n \in \mathbb{N}, \quad \text{for} \quad a \le t_1 \le t_2 \le a_1, \ \ b_1 \le t_1 \le t_2 \le b.$$

It is also clear that there exists a constant δ, $0 < \delta < \min(a_1 - a, b - b_1)$ for which the following inequality is valid:

$$\begin{gathered} |y_n(t_1) - y_n(t_2)| \le \\ \le r_1 \max\bigg\{\frac{1}{(b-t)(t-a)} \ : \ a_1 - \delta \le t \le b_1 + \delta\bigg\}|t_1 - t_2| \le \frac{\varepsilon}{2} \\ \text{for} \ \ |t_1 - t_2| \le \delta, \ \ a_1 - \delta \le t_j \le b_1 + \delta \ \ (j = 1,2). \end{gathered}$$

From the last two estimates we obtain that if $t_j \in [a,b]$ $(j = 1,2)$ and

$$|t_1 - t_2| \le \delta,$$

then

$$|y_n(t_1) - y_n(t_2)| \le \varepsilon, \quad n \in \mathbb{N}.$$

This and the inequality (1.3.17) imply that the sequence $(y_n)_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. In case $i = 2$ the same follows from the possibility of choosing for any $\varepsilon > 0$, $a_1 \in]a, b[$ and $0 < \delta < a_1 - a$ such that

$$\max\left\{ \int_a^b |G(t,s)| g(1)(s)\, ds : \ a \le t \le a_1 \right\} < \frac{\varepsilon}{4}\,,$$

$$|y_n(t_1) - y_n(t_2)| \le r_1 \max\left\{1 + \frac{1}{t-a} \ : \ a_1 - \delta \le t \le b\right\} |t_1 - t_2| \le \frac{\varepsilon}{2}$$
$$\text{for} \quad |t_1 - t_2| \le \delta, \ \ a_1 - \delta \le t_j \le b \ \ (j = 1, 2).$$

Then by the Arzella–Ascoli lemma we obtain that $\mathbb{U}_0$ is a compact operator. Consequently, taking into account Fredholm's alternatives, the equation (1.3.16) is uniquely solvable if the homogeneous equation

$$u(t) = \mathbb{U}_0(u)(t) \tag{1.3.16$_0$}$$

has only the trivial solution in the space $C(]a, b[)$.

It remains to note that by virtue of the conditions (1.1.18)–(1.1.21) and (1.1.22) if $i = 1$ and (1.1.24$_2$) if $i = 2$, all the requirement of Lemma 1.2.8$_i$ are satisfied for $\lambda = 0$, whence it follows that the problem (1.2.57), (1.2.2$_{i0}$), i.e., the equation (1.3.16$_0$) has only the trivial solution in the space $C(]a, b[)$. $\square$

Proof of Remark 1.1.2 follows directly from Remark 1.2.7.

Proof of Theorem 1.1.2$_{i0}$. Let x be a function defined by (1.1.19$_i$) and let G be Green's function of the problem (1.1.58), (1.1.2$_{i0}$) which is expressed by (1.2.59$_i$). Introduce the operator $\mathbb{U}_0$ and the function q by the equality (1.3.15). Then for any $y \in C_{x^\lambda}(]a, b[)$ the estimates

$$|\mathbb{U}_0(y)(t)| \le \frac{x^{1-\gamma}(t)}{(b-a)^{2-i}} \int_a^b x^\gamma(s) g(x^\lambda)(s)\, ds\, \|y\|_{C,x^\lambda},$$

$$|q(t)| \le x^{1-\gamma}(t) \int_a^b x^\gamma(s) |p_2(s)|\, ds \quad \text{for} \quad a \le t \le b$$

are valid, from which by the conditions $\lambda \in]0, 1-\gamma[$ and (1.1.25), (1.1.26) it follows that $\mathbb{U}_0$ transforms continuously the space $C_{x^\lambda}(]a, b[)$ into itself and $q \in C_{x^\lambda}(]a, b[)$.

Consider now the equation

$$u(t) = \mathbb{U}_0(u)(t) + q(t) \tag{1.3.18}$$

which is equivalent to the problem (1.1.16), ($1.1.2_{i0}$), and the corresponding homogeneous equation ($1.3.16_0$).

As is seen from Lemma $1.2.8_i$ and Remark 1.2.7, by virtue of the conditions $\lambda \in]0, 1-\gamma[$, (1.1.21), ($1.1.24_i$) and (1.1.25)–(1.1.27) the problem (1.2.57), ($1.1.2_{i0}$), i.e., the equation ($1.3.16_0$), has in the space $C_{x^\lambda}(]a,b[)$ only the trivial solution. Then according to Fredholm's alternatives, to prove the validity of our theorem it remains to show that the operator $\mathbb{U}_0$ is compact. Let $r > 0$,

$$\mathbb{B}_r = \big\{ z \in C_{x^\lambda}(]a,b[): \ \ \|z\|_{C,x^\lambda} \le r \big\}$$

$(x_n)_{n=1}^{\infty}$ be a sequence from $\mathbb{B}_r$ and $y_n(t) = \mathbb{U}_0(x_n)(t)$ for $n \in \mathbb{N}$.

Then as is seen from the definition of G, for any $n \in \mathbb{N}$ the estimate

$$|y_n^{(j)}(t)| \le r \, \frac{x^{1-j-\gamma}(t)}{(b-a)^{(1-j)(2-i)}} \int\limits_a^b x^{\gamma}(s) g(x^{\lambda})(s)\, ds \ \ (j = 0,1) \quad (1.3.19_i)$$
$$\text{for} \quad a < t < b$$

is valid, which by virtue of the condition $\lambda \in]0, 1-\gamma[$ yields

$$\|y_n(t)\|_{C,x^\lambda} \le r_1, \tag{1.3.20}$$

where

$$r_1 = \frac{r}{(b-a)^{2-i}} \int\limits_a^b x^{\gamma}(s) g(x^{\lambda})(s)\, ds \, \max\big\{ x^{1-(\lambda+\gamma)}(t): \ \ a \le t \le b \big\}.$$

Consider now the case $i = 1$ separately. From ($1.3.19_1$) for $j = 0$ and for any $\varepsilon > 0$ follows the existence of a_1, $b_1 \in]a,b[$, where $a_1 < b_1$, such that

$$|y_n(t)| \le \frac{\varepsilon}{4}\,, \ \ n \in \mathbb{N}, \quad \text{for} \quad a \le t \le a_1, \ \ b_1 \le t \le b,$$

which implies the estimate

$$|y_n(t_1) - y_n(t_2)| \le \frac{\varepsilon}{2}\,, \ \ n \in \mathbb{N},$$
$$\text{for} \quad a \le t_1 < t_2 \le a_1, \ \ b_1 \le t_1 < t_2 \le b.$$

Moreover, from ($1.3.19_1$) for $j = 1$ it follows the existence of a constant δ such that

$$|y_n(t_1) - y_n(t_2)| \le r_2 |t_1 - t_2| \le \frac{\varepsilon}{2}\,, \ \ n \in \mathbb{N},$$
$$\text{for} \quad a_1 - \delta \le t_l \le b_1 + \delta \ \ (l = 1,2),$$

where

$$r_2 = r \int\limits_a^b x^{\gamma}(s) g(x^{\lambda})(s)\, ds \, \max\big\{ x^{-\gamma}(t): \ \ a_1 - \delta \le t \le b_1 + \delta \big\}.$$

It is clear from the last two estimates that if $t_l \in [a,b]$ $(l=1,2)$ and

$$|t_1 - t_2| \le \delta,$$

then for any $n \in \mathbb{N}$

$$|y_n(t_1) - y_n(t_2)| \le \varepsilon.$$

This and the estimate (1.3.20) imply that the sequence $(y_n)_{n=1}^{\infty}$ is uniformly bounded and equicontinuous. In case $i=2$, by virtue of the estimates $(1.3.19_2)$ the same follows from the possibility of choosing, for any $\varepsilon > 0$, of $a_1 \in]a,b[$ and $0 < \delta < a_1 - a$ such that

$$|y_n(t)| \le \frac{\varepsilon}{4}\,, \quad n \in \mathbb{N} \quad \text{for} \quad a \le t \le b,$$

and

$$|y_n(t_1) - y_n(t_2)| \le r_2 |t_1 - t_2| \le \frac{\varepsilon}{2}\,, \quad n \in \mathbb{N}, \\ \text{for} \quad a_1 - \delta \le t_j \le b \ \ (j=1,2),$$

where

$$r_2 = r \int\limits_a^b x^{\gamma}(s) g(\rho)(s)\, ds \, \max\big\{x^{-\gamma}(t): \ \ a_1 - \delta \le t \le b\big\}.$$

Then by the Arzella–Ascoli lemma we have that $\mathbb{U}_0$ is a compact operator. □

1.3.2. Proof of Effective Sufficient Conditions for Solvability of the Problems (1.1.1), ($1.1.2_i$) and (1.1.1), ($1.1.2_{i0}$) ($i = 1, 2$). Before we proceed to proving the corollaries, we note that Green's function of the problem

$$v''(t) = p_1(t) v'(t), \tag{1.3.21}$$

$$v(a) = 0, \quad v^{(i-1)}(b-) = 0 \tag{$1.3.22_i$}$$

has the form

$$G_0(t,s) = \\ = \begin{cases} -\dfrac{1}{\sigma(p_1)(s)} \displaystyle\int\limits_a^s \sigma(p_1)(\eta)\, d\eta \Bigg(\dfrac{1}{\int\limits_a^b \sigma(p_1)(\eta) d\eta} \displaystyle\int\limits_t^b \sigma(p_1)(\eta) d\eta \Bigg)^{2-i} \\ \qquad\qquad \text{for} \quad a \le s < t \le b, \\ -\dfrac{1}{\sigma(p_1)(s)} \displaystyle\int\limits_a^t \sigma(p_1)(\eta)\, d\eta \Bigg(\dfrac{1}{\int\limits_a^b \sigma(p_1)(\eta) d\eta} \displaystyle\int\limits_s^b \sigma(p_1)(\eta)\, d\eta \Bigg)^{2-i} \\ \qquad\qquad \text{for} \quad a \le t < s \le b. \end{cases} \tag{$1.3.23_i$}$$

Proof of Corollary 1.1.1$_1$. It is clear that all the requirements of Theorem 1.1.1$_1$, except (1.1.7$_1$), follow directly from the conditions of our corollary. It remains only to show that the conditions (1.1.31), (1.1.32$_1$) imply the inclusion (1.1.7$_1$) as well.

Indeed, let $\beta > 0$ and

$$z_\lambda(t) = \Bigg[\bigg(\int\limits_t^b \sigma(p_1)(\eta)d\eta\bigg)^\alpha \times$$

$$\times \int\limits_a^t \frac{[p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta(s))}{\sigma(p_1)(s)}\bigg(\int\limits_a^s \sigma(p_1)(\eta)d\eta\bigg)^\alpha ds +$$

$$+\bigg(\int\limits_a^t \sigma(p_1)(\eta)d\eta\bigg)^\alpha \int\limits_t^b \frac{[p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta(s))}{\sigma(p_1)(s)}\bigg(\int\limits_s^b \sigma(p_1)(\eta)d\eta\bigg) ds \times$$

$$\times \frac{\big(\int_a^b \sigma(p_1)(s)\,ds\big)^{1-2(\alpha+\beta)}}{2^{2-2(\alpha+\beta)}}. \tag{1.3.24}$$

Then, as is seen from the conditions (1.1.31), (1.1.32$_1$), we can choose $\lambda > 0$ such that

$$z_\lambda(t) < 1 \quad \text{for} \quad a \le t \le b \tag{1.3.25}$$

be satisfied.

Introduce also the notation

$$q_\beta(t) = \frac{\sigma^2(p_1)(t)}{x^{2-\beta-[\beta]}(t)}, \quad w_\varepsilon(t) = \varepsilon \int\limits_a^b |G_0(t,s)| q_\beta(s)\,ds,$$

$$w(t) = \int\limits_a^b |G_0(t,s)|\big([p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta)(s)\big)\,ds + w_\varepsilon(t),$$

where $\varepsilon \in \mathbb{R}^+$, G_0 is Green's function of the problem (1.3.21), (1.3.22$_1$) which is defined by the equality (1.3.23$_1$), and by Lemma 1.2.7,

$$w_\varepsilon(t) = O^*(x^\beta(t)) \ \text{ as } \ t \to a, \ \ t \to b \tag{1.3.26}$$

for any $\varepsilon > 0$. From the conditions (1.3.25), (1.3.26) we have the possibility of choosing the constant $\varepsilon > 0$ such that

$$z_\lambda(t) + \sup\Big\{\frac{w_\varepsilon(t)}{x^\beta(t)} \ : \ a < t < b\Big\} < 1 \quad \text{for} \quad a \le t \le b. \tag{1.3.27}$$

By virtue of (1.3.23$_1$) we easily get the estimate

$$0 < w(t) \le z_\lambda(t) x^\beta(t) + w_\varepsilon(t) \quad \text{for} \quad a < t < b$$

which with regard for (1.3.27) results in

$$0 < w(t) \le x^\beta(t) \quad \text{for} \quad a < t < b. \tag{1.3.28}$$

The last inequality together with (1.3.26) means that

$$w(t) = O^*(x^\beta(t)) \quad \text{as} \quad t \to a, \quad t \to b. \tag{1.3.29}$$

On the other hand, it is clear that

$$w''(t) = -[p_0(t)]_-(\lambda + x^\beta(t)) + p_1(t)w'(t) - h(x^\beta)(s) - q_\beta(t).$$

Taking into account the inequality (1.3.28) and the fact that the operator h and the constant λ are nonnegative, the above equality results in

$$w(t)'' \le p_0(t)w(t) + p_1(t)w'(t) - h(w)(t) - q_\beta(t). \tag{1.3.30}$$

If we introduce the notation $\widetilde{w}(t) = \lambda + w(t)$, then

$$\widetilde{w}''(t) \le p_0(t)\widetilde{w}(t) + p_1(t)\widetilde{w}'(t), \tag{1.3.31}$$

where

$$\widetilde{w}(t) > 0 \quad \text{for} \quad a \le t \le b. \tag{1.3.32}$$

From the inequalities (1.3.31) and (1.3.32), by Lemma 1.2.2 we obtain the inclusion

$$(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[). \tag{1.3.33$_1$}$$

Then, as is seen from Remark 1.2.2, the problem (1.2.4), (1.2.2$_{i0}$) has Green's function G which is expressed by the equality (1.2.7). Using now the inequalities (1.2.10$_1$), we arrive at

$$\frac{d_*^2}{c_*} \le \varepsilon w_\varepsilon^{-1}(t) \int\limits_a^b |G(t,s)| q_\beta(s)\, ds \le \frac{c_*^2}{d_*} \quad \text{for} \quad a \le t \le b$$

which with regard for the equality (1.3.26) yields

$$\int\limits_a^b |G(t,s)| q_\beta(s)\, ds = O^*(x^\beta(s)) \quad \text{as} \quad t \to a, \;\; t \to b. \tag{1.3.34}$$

It remains to note that the conditions (1.2.28), (1.3.29), (1.3.33$_1$), (1.3.34) and the inequality (1.3.30), owing to Definition 1.1.4, ensure the inclusion (1.2.7$_1$) for $\beta > 0$.

Assume now that $\beta = 0$ and

$$w(t) = \int\limits_a^b |G_0(t,s)| \big([p_0(s)]_- + h(1)(s)\big)\, ds + \varepsilon v(t), \tag{1.3.35}$$

where v is a solution of the equation (1.3.21) under the boundary conditions

$$v(a) = 1, \quad v(b) = 1,$$

and

$$z_0(t) = \Bigg[\bigg(\int\limits_t^b \sigma(p_1)(\eta)d\eta\bigg)^\alpha \int\limits_a^t \frac{([p_0(s)]_- + h(1)(s))}{\sigma(p_1)(s)} \bigg(\int\limits_a^s \sigma(p_1)(\eta)d\eta\bigg)^\alpha ds +$$

$$+\bigg(\int\limits_a^t \sigma(p_1)(\eta)d\eta\bigg)^\alpha \int\limits_t^b \frac{([p_0(s)]_- + h(1)(s))}{\sigma(p_1)(s)} \bigg(\int\limits_s^b \sigma(p_1)(\eta)d\eta\bigg)^\alpha ds\Bigg] \times$$

$$\times \frac{(\int\limits_a^b \sigma(p_1)(s)\,ds)^{1-2\alpha}}{4^{1-\alpha}}.$$

Then, as is seen from the condition $(1.1.32_1)$,

$$z_0(t) < 1 \quad \text{for} \quad a \le t \le b,$$

and hence we can choose $\varepsilon > 0$ small enough for the inequality

$$z_0(t) + \varepsilon v(t) < 1 \tag{1.3.36}$$

to be fulfilled for $a \le t \le b$. Notice that by virtue of the equalities $(1.3.23_1)$, we obtain the estimate

$$0 < w(t) \le z_0(t) + \varepsilon v(t) \quad \text{for} \quad a \le t \le b$$

which with regard for (1.3.36) implies

$$0 < w(t) \le 1 \quad \text{for} \quad a \le t \le b. \tag{1.3.37}$$

On the other hand,

$$w''(t) = -[p_0(t)] + p_1(t)w'(t) - h(1)(t),$$

whence, taking into account (1.3.37) and the fact that the operator h is nonnegative, we obtain

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - h(w)(t).$$

Consequently, owing to Definition 1.1.3, the inclusion $(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[\,; h)$ is valid. $\square$

Proof of Corollary $1.1.1_2$. It is clear that all the requirements of Theorem $1.1.1_2$, except $(1.1.7_2)$ follow directly from the conditions of our corollary. It remains to show that the conditions (1.1.31), $(1.1.32_1)$ imply the inclusion $(1.1.7_2)$ as well.

To this end, we introduce for $\beta > 0$ the functions z_λ and w by the equalities

$$z_\lambda(t) = \Bigg[\int\limits_a^t \frac{([p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta)(s))}{\sigma(p_1)(s)} \Bigg(\int\limits_a^s \sigma(p_1)(\eta)d\eta \Bigg)^\alpha ds +$$

$$+ \Bigg(\int\limits_a^t \sigma(p_1)(\eta)d\eta \Bigg)^\alpha \int\limits_t^b \frac{([p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta)(s)}{\sigma(p_1)(s)} ds \Bigg] \times$$

$$\times \Bigg(\int\limits_a^b \sigma(p_1)(\eta)d\eta \Bigg)^{1-(\alpha+\beta)}$$

and

$$w(t) = \int\limits_a^b |G_0(t,s)|([p_0(s)]_-(\lambda + x^\beta(s)) + h(x^\beta)(s))\, ds + w_\varepsilon(t),$$

where G_0 is Green's function of the problem (1.3.21), $(1.3.22_2)$, and w_ε is defined just as in the previous proof. Then reasoning in the same manner as when proving Corollary $1.1.1_1$, we make sure that the inclusion $(1.1.7_2)$ is valid for $\beta > 0$.

In the case $\beta = 0$, we consider the function z_λ for $\lambda = 0$ and the function w defined by (1.3.35), where v is a solution of the equation (1.3.21) under the boundary conditions

$$v(a) = 1, \qquad v'(b-) = 1.$$

Then reasoning just in the same way as in proving Corollary $1.1.1_1$ for $\beta = 0$, we can see that the inclusion $(p_0, p_1) \in \mathbb{V}_{2,0}(]a,b[\,;h)$ is valid. $\square$

Proof of Corollary $1.1.1_{i0}$. Coincides completely with that of Corollary $1.1.1_i$ for $\beta > 0$. $\square$

Proof of Remark 1.1.4. Denote the left-hand side of $(1.1.32_i)$ by w. Then it is obvious that

$$w(t) \le \int\limits_a^b \frac{[p_0(s)]_- x^{\alpha+\beta}(s) + x^\alpha(s) h(x^\beta)(s)}{\sigma(p_1)(s)} ds \quad \text{for} \quad a \le t \le b,$$

i.e., it follows from $(1.1.34_i)$ that the condition $(1.1.32_i)$ is valid. On the other hand, $(1.1.34_i)$ implies the inclusion

$$h \in \mathcal{L}\Big(C_{x^\beta}; L_{\frac{x^\alpha}{\sigma(p_1)}}\Big)$$

which together with (1.1.33) means that $(1.1.8_i)$ is satisfied. $\square$

Proof of Remark $1.1.4_0$. As is seen from the proof of Remark 1.1.4, the conditions ($1.1.32_i$) and (1.1.12) follow simultaneously from ($1.1.34_i$). □

Proof of Corollary $1.1.2_i$. Introduce the notation

$$g(u)(t) = \sum_{k=1}^{n} g_k(t)u(\tau_k(t)) \tag{1.3.38}$$

and

$$h(u)(t) = \sum_{k=1}^{n} |g_k(t)|u(\tau_k(t)). \tag{1.3.39}$$

Then for any $u \in C(]a,b[)$ almost everywhere on the interval $]a,b[$ the inequality (1.1.10) is satisfied, and as is seen from ($1.1.36_i$), the inclusion ($1.1.8_i$) is valid. It is also clear that the condition ($1.1.37_i$) in our notation can be rewritten as ($1.1.32_i$). Hence all the requirements of Corollary $1.1.1_i$ are fulfilled and our corollary is valid. □

Proof of Corollary $1.1.2_{i0}$. Define the operators g and h by the equalities (1.3.38) and (1.3.39) and note that from the condition (1.3.38) it follows the inclusion (1.1.12). Reasoning similarly as when proving the above corollary, we can see that our corollary is valid. □

Proof of Remark 1.1.5. Denote the left-hand side of ($1.1.37_i$) by w. Then it is evident that

$$w(t) \leq \int\limits_a^b \frac{[p_0(s)]_- x^{\alpha+\beta}(s) + x^\alpha(s) \sum\limits_{k=1}^{n} |g_k(s)| x^\beta(\tau_k(s))}{\sigma(p_1)(s)}\, ds \quad \text{for} \quad a \leq t \leq b,$$

i.e., ($1.1.40_i$) implies the validity of the condition ($1.1.37_i$). On the other hand, ($1.1.40_i$) implies the inclusion

$$g_k x^\beta(\tau_k) \in L_{\frac{x^\alpha}{\sigma(p_1)}}([a,b])$$

which together with (1.1.39) means that ($1.1.36_i$) is satisfied. □

Proof of Remark $1.1.5_0$. As is seen from the proof of Remark 1.1.5, the conditions ($1.1.37_i$) and (1.1.38) follow simultaneously from ($1.1.40_i$). □

Proof of Corollary $1.1.3_1$. It is clear that all the requirements of Theorem $1.1.1_i$, except ($1.1.7_i$), follow directly from the conditions of our corollary. It remains to show that the conditions (1.1.41), ($1.1.42_1$) imply the inclusion ($1.1.7_1$) as well, where $h(u)(t) = \sum\limits_{k=1}^{n} |g_k(t)|u(\tau_k(t))$.

Indeed, let $\beta > 0$ and

$$z(t) = \Big[\sum_{k=1}^{n} \int_a^t \frac{|g_k(s)|}{\sigma(p_1)(s)} x^\beta(\tau_k(s)) \Big(\int_a^s \sigma(p_1)(\eta)d\eta\Big)^\alpha ds \Big(\int_t^b \sigma(p_1)(\eta)d\eta\Big)^\alpha +$$

$$+ \sum_{k=1}^{n} \int_t^b \frac{|g_k(s)|}{\sigma(p_1)(s)} x^\beta(\tau_k(s)) \Big(\int_s^b \sigma(p_1)(\eta)d\eta\Big)^\alpha ds \Big(\int_a^t \sigma(p_1)(\eta)d\eta\Big)^\alpha\Big] \times$$

$$\times \frac{\big(\int_a^b \sigma(p_1)(\eta)d\eta\big)^{1-2(\alpha+\beta)}}{2^{2-2(\alpha+\beta)}}.$$

Then as is seen from $(1.1.42_1)$, for every $m \in \{1, \dots, n\}$

$$z(\tau_m(t)) < 1 \quad \text{for} \quad a \le t \le b. \tag{1.3.40}$$

Moreover, let

$$w(t) = \sum_{k=1}^{n} \int_a^b |G_0(t,s)| g_k(s) x^\beta(\tau_k(s))\, ds + w_\varepsilon(t),$$

where the function w_ε is defined in the same way as in proving Corollary $1.1.1_1$, $\varepsilon > 0$, G_0 is Green's function of the problem (1.3.21), $(1.3.22_1)$ defined by the equality $(1.3.23_1)$ and by Lemma 1.2.7,

$$w_\varepsilon(t) = O^*(x^\beta(t)) \quad \text{as} \quad t \to a, \quad t \to b, \tag{1.3.41}$$

for any $\varepsilon > 0$. From the conditions (1.3.40), (1.3.41) it follows that we can choose a constant $\varepsilon > 0$ such that for every $m \in \{1, \dots, n\}$

$$z(\tau_m(t)) + \sup\Big\{\frac{w_\varepsilon(\tau_m(t))}{x^\beta(\tau_m(t))} : \quad a < t < b\Big\} < 1 \quad \text{for} \quad a \le t \le b. \tag{1.3.42}$$

Using the equality $(1.3.23_1)$ we can easily obtain the estimate

$$0 \le w(t) \le z(t) x^\beta(t) + w_\varepsilon(t) \quad \text{for} \quad a \le t \le b, \tag{1.3.43}$$

whence by virtue of (1.3.42) for every $m \in \{1, \dots, n\}$ the inequality

$$0 \le w(\tau_m(t)) \le x^\beta(\tau_m(t)) \quad \text{for} \quad a < t < b \tag{1.3.44}$$

is valid. Analogously, from (1.3.41) and (1.3.43) it follows the estimate

$$0 < w(t) \le r_0 x^\beta(t) \quad \text{for} \quad a < t < b, \tag{1.3.45}$$

where

$$r_0 = \sup\Big\{z(t) + \frac{w_\varepsilon(t)}{x^\beta(t)} : \quad a < t < b\Big\} < +\infty,$$

and according to (1.3.41) we get

$$w(t) = O^*(x^\beta(t)) \quad \text{as} \quad t \to a, \quad t \to b. \tag{1.3.46}$$

On the other hand, it is clear that

$$w''(t) = p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)|x^\beta(\tau_k(t)) - q_\beta(t),$$

which with regard for the conditions (1.1.41) and (1.3.44) results in

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)|w(\tau_k(t)) - q_\beta(t), \quad (1.3.47)$$

where, as is seen from Remark 1.2.6,

$$(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[). \quad (1.3.48)$$

Then, as we have shown in proving Corollary $1.1.1_1$,

$$\int_a^b |G(t,s)|q_\beta(s)\,ds = O^*(x^\beta(t)) \ \text{ as } \ t \to a, \ \ t \to b, \quad (1.3.49)$$

where G is Green's function of the problem (1.2.4), $(1.2.2_{i0})$. It remains to notice that the conditions (1.3.45), (1.3.46), (1.3.48), (1.3.49) and the inequality (1.3.47) by virtue of Definition 1.1.4 imply the inclusion $(1.1.7_1)$ for $\beta > 1$.

Suppose now that $\beta = 0$ and

$$w(t) = \sum_{k=1}^{n} \int_a^b |G_0(t,s)||g_k(s)|\,ds + \varepsilon v(t), \quad (1.3.50)$$

where v is a solution of the equation (1.3.21) under the boundary conditions

$$v(a) = 1 \quad \text{and} \quad v(b) = 1.$$

Then, as is seen from the condition $(1.1.42_1)$, for every $m \in \{1, \dots, n\}$

$$z(\tau_m(t)) < 1 \quad \text{for} \quad a \le t \le b$$

and hence for every $m \in \{1, \dots, n\}$ we can choose $\varepsilon > 0$ small enough for the inequality

$$z(\tau_m(t)) + \varepsilon v(\tau_m(t)) \le 1 \quad \text{for} \quad a \le t \le b. \quad (1.3.51)$$

to be fulfilled. Note that from the positiveness of v and also from $(1.3.23_1)$ we have the estimate

$$0 < w(t) \le z(t) + \varepsilon v(t) \quad \text{for} \quad a \le t \le b$$

which by virtue of (1.3.51) for every $m \in \{1, \dots, n\}$ yields

$$0 < w(\tau_m(t)) \le 1 \quad \text{for} \quad a \le t \le b. \quad (1.3.52)$$

On the other hand,

$$w''(t) = p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)|$$

which with regard for (1.1.41) and (1.3.52) gives

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)| w(\tau_k(t)).$$

Hence, owing to Definition 1.1.3, the inclusion $(p_0, p_1) \in \mathbb{V}_{1,0}(\,]a,b[\,;h)$, is valid, where $h(u)(t) = \sum\limits_{k=1}^{n} |g_k(t)| u(\tau_k(t))$. $\square$

Proof of Corollary 1.1.3$_2$. It is clear that all the requirements of Theorem 1.1.1$_2$, except (1.1.7$_2$), follow directly from the conditions of our corollary. It remains to show that the inclusion (1.1.7$_2$) follows from the condition (1.1.41), (1.1.42$_1$) as well.

To this end, we introduce for $\beta > 0$ the functions z and w by the equalities

$$z(t) = \Bigg[\sum_{k=1}^{n} \int_a^t \frac{|g_k(s)|}{\sigma(p_1)(s)} x^{\beta}(\tau_k(s)) \bigg(\int_a^t \sigma(p_1)(\eta)d\eta \bigg)^{\alpha} ds +$$

$$+ \sum_{k=1}^{n} \int_t^b \frac{|g_k(s)|}{\sigma(p_1)(s)} x^{\beta}(\tau_k(s))\, ds \bigg(\int_a^t \sigma(p_1)(\eta)d\eta \bigg)^{\alpha} \Bigg] \bigg(\int_a^b \sigma(p_1)(\eta)d\eta \bigg)^{1-(\alpha+\beta)}$$

and

$$w(t) = \sum_{k=1}^{n} \int_a^b |G_0(t,s)|\, |g_k(s)| x^{\beta}(\tau_k(s))\, ds + w_{\varepsilon}(t),$$

where G_0 is Green's function of the problem (1.3.21), (1.3.22$_2$) and w_ε is defined in the same way as in proving Corollary 1.1.1$_1$. Reasoning just as in proving Corollary 1.1.3$_1$, we make sure that the inclusion (1.1.7$_2$) is valid for $\beta > 0$.

In the case $\beta = 0$ we consider the function w defined by the equality (1.3.50), where v is a solution of the equation (1.3.21) for the boundary conditions

$$v(a) = 1, \qquad v'(b-) = 1.$$

Then, reasoning analogously as in proving Corollary 1.1.3$_1$ for $\beta = 0$, we can see that the inclusion $(p_0, p_1) \in \mathbb{V}_{2,0}(]a,b[\,;h)$ is valid. $\square$

Proof of Corollary 1.1.3$_{i0}$. Coincides completely with that of Corollary 1.1.3$_i$ for $\beta > 0$. $\square$

Proof of Remark 1.1.6. If the inequality (1.1.43$_i$) is satisfied for $t \in \theta_{\tau_1,\dots,\tau_n}$, then it will especially be satisfied on each of the sets θ_{τ_m}, where $m \in \{1,\dots,n\}$, i.e., each of the n inequalities of (1.1.42$_i$) will be satisfied. $\square$

Proof of Corollary 1.1.4$_i$ (1.1.4$_{i0}$). It is sufficient to substitute $p_0 \equiv 0$, $p_1 \equiv 0$, $k = 1$ in Remark 1.1.5$_i$ (1.1.5$_{i0}$). $\square$

Proof of Corollary 1.1.5$_1$. It is clear that all the requirements of Theorem 1.1.1$_1$, except (1.1.7$_1$), follow directly from the conditions of our corollary. It remains to show that the inclusion (1.1.7$_1$) follows from the conditions (1.1.50$_1$) for $0 \le \beta < 1$ and (1.1.51$_1$) for $\beta = 1$ as well.

Consider first the case $0 < \beta < 1$. Let x be a function defined by the equality (1.1.9$_1$). Then

$$(x^\beta(t))'' = p_1(t)(x^\beta(t))' - 2\beta^2 \frac{\sigma^2(p_1)(t)}{x^{1-\beta}(t)} -$$

$$-\beta(1-\beta)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}\bigg(\Big(\int\limits_a^b \sigma(p_1)(\eta)d\eta\Big)^2 + \Big(\int\limits_t^b \sigma(p_1)(\eta)d\eta\Big)^2\bigg). \tag{1.3.53}$$

From the condition (1.1.50$_1$) and the fact that the operator h is nonnegative it follows that

$$-\frac{x^{2-\beta}(t)}{\sigma^2(p_1)(t)}p_0(t) \le 2\beta^2\Big(\int\limits_a^b \sigma(p_1)(\eta)d\eta\Big)^{2(1-\beta)} \quad \text{for } a < t < b.$$

Moreover,

$$0 \le \lambda p_0(t) + \beta(1-\beta)\min\bigg\{\Big(\int\limits_a^s \sigma(p_1)(\eta)d\eta\Big)^2 +$$

$$+\Big(\int\limits_s^b \sigma(p_1)(\eta)d\eta\Big)^2 : \ a \le s \le b\bigg\}\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}, \tag{1.3.54}$$

where

$$\lambda = \frac{1-\beta}{2\beta}\Big(\int\limits_a^b \sigma(p_1)(\eta)d\eta\Big)^{-2(1-\beta)} \times$$

$$\times \min\bigg\{\Big(\int\limits_a^s \sigma(p_1)(\eta)d\eta\Big)^2 + \Big(\int\limits_s^b \sigma(p_1)(\eta)d\eta\Big)^2 : \ a \le s \le b\bigg\}.$$

Let $w(t) = x^\beta(t) + \lambda$, and rewrite the identity (1.3.53) as

$$w''(t) = p_0(t)w(t) + p_1(t)w'(t) - \Big(p_0(t)x^\beta(t) + 2\beta^2\frac{\sigma^2(p_1)(t)}{x^{1-\beta}(t)}\Big) -$$

$$-\bigg[\lambda p_0(t) + \beta(1-\beta)\bigg(\Big(\int\limits_a^t \sigma(p_1)(\eta)d\eta\Big)^2 + \Big(\int\limits_t^b \sigma(p_1)(\eta)d\eta\Big)^2\bigg)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}\bigg].$$

Then, taking into account the fact that the operator h is nonnegative, from the condition ($1.1.50_1$) and the inequality (1.3.54) we obtain

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t), \tag{1.3.55}$$

i.e., owing to Lemma 1.2.2 the inclusion

$$(p_0, p_1) \in \mathbb{V}_{1,0}(]a,b[) \tag{1.3.56}$$

is satisfied. Then, as is seen from Remark 1.2.2, there exists Green's function G of the problem (1.2.4), ($1.2.2_{i0}$), and by Lemma 1.2.6,

$$\int\limits_a^b |G(t,s)|q_\beta(s)\,ds = O^*(x^\beta(t)) \quad \text{for} \quad t \to a, \quad t \to b, \tag{1.3.57}$$

where

$$q_\beta(t) = \frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}.$$

Let now

$$\varepsilon = \beta(1-\beta)\min\bigg\{\Big(\int\limits_a^s \sigma(p_1)(\eta)d\eta\Big)^2 + \\ +\Big(\int\limits_s^b \sigma(p_1)(\eta)d\eta\Big)^2 : \ a \le t \le b\bigg\} \tag{1.3.58}$$

and rewrite (1.3.53) in the form

$$(x^\beta(t))'' = p_0(t)x^\beta(t) + p_1(t)(x^\beta(t))' - h(x^\beta)(t) - \varepsilon q_\beta(t) - \\ -\Big(p_0(t)x^\beta(t) - h(x^\beta)(t) + 2\beta^2\frac{\sigma^2(p_1)(t)}{x^{1-\beta}(t)}\Big) - \bigg[\beta(1-\beta)\bigg(\Big(\int\limits_a^t \sigma(p_1)(\eta)d\eta\Big)^2 + \\ +\Big(\int\limits_t^b \sigma(p_1)(\eta)d\eta\Big)^2\bigg) - \varepsilon\bigg]\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}. \tag{1.3.59}$$

Taking into account ($1.1.50_1$) and (1.3.58), we obtain

$$(x^\beta(t))'' \le p_0(t)x^\beta(t) + p_1(t)(x^\beta(t))' - h(x^\beta)(t) - \varepsilon q_\beta(t) \\ \text{for} \quad a < t < b. \tag{1.3.60}$$

From (1.3.56), (1.3.57), and (1.3.60), by virtue of Definition 1.1.4 we conclude that the inclusion ($1.1.7_1$) is satisfied for $0 < \beta < 1$.

Assume now that $\beta = 0$. Then the condition ($1.1.50_1$) takes the form

$$0 \le p_0(t) - h(1)(t) \quad \text{for} \quad a < t < b,$$

from which we can see that the function $w(t) \equiv 1$ satisfies the inequality

$$w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - h(w)(t),$$

i.e., owing to Definition 1.1.3 we can conclude that the inclusion $(1.1.7_1)$ is satisfied for $\beta = 0$.

Finally we consider the case $\beta = 1$ and note that

$$x''(t) = p_1(t)x'(t) - 2\sigma^2(p_1)(t). \tag{1.3.61}$$

It follows from $(1.1.51_1)$ that there exist constants ε, $\mu \in]0,1[$ such that

$$\operatorname*{ess\,sup}_{t\in\,]a,b[} \left(\frac{x(t)}{\sigma^2(p_1)(t)} \Big(\frac{h(x)(t)}{x(t)} - p_0(t) \Big) \right) < 2\mu^2 \tag{1.3.62}$$

and

$$\operatorname*{ess\,sup}_{t\in\,]a,b[} \left(\frac{x(t)}{\sigma^2(p_1)(t)} \Big(\frac{h(x)(t)}{x(t)} - p_0(t) \Big) \right) < 2 - \varepsilon. \tag{1.3.63}$$

Taking into account the fact that the operator h is nonnegative, from the condition (1.3.62) we get

$$-\frac{x^{2-\mu}(t)}{\sigma^2(p_1)(t)} p_0(t) \le 2\mu^2 \bigg(\int\limits_a^b \sigma(p_1)(\eta)d\eta \bigg)^{2(1-\mu)} \quad \text{for} \quad a < t < b.$$

Reasoning in the same way as for $0 < \beta < 1$, from the last inequality as well as from (1.3.62) we can see that the function $w(t) = x^\mu(t) + \lambda$, where

$$\lambda = \frac{1-\mu}{2\mu} \bigg(\int\limits_a^b \sigma(p_1)(\eta)d\eta \bigg)^{-2(1-\mu)} \times$$
$$\times \min \bigg\{ \Big(\int\limits_a^s \sigma(p_1)(\eta)d\eta \Big)^2 + \Big(\int\limits_s^b \sigma(p_1)(\eta)d\eta \Big)^2 : \ a \le s \le b \bigg\},$$

satisfies (1.3.55), i.e., the inclusion (1.3.56) is satisfied and there exists Green's function G of the problem (1.2.4), $(1.2.2_{i0})$. As is seen from Lemma 1.2.7, if $q_1(t) = \sigma^2(p_1)(t)$, then

$$\int\limits_a^b |G(t,s)| q_1(s)\,ds = O^*(x(s)) \quad \text{as} \quad t \to a, \ \ t \to b. \tag{1.3.64}$$

We rewrite now the identity (1.3.61) as follows:

$$x''(t) = p_0(t)x(t) + p_1(t)x'(t) - h(x)(t) - \varepsilon q_1(t) +$$
$$+ \big(h(x)(t) - p_0(t)x(t) - (2-\varepsilon)\sigma^2(p_1)(t) \big).$$

The latter with regard for (1.3.63) yields

$$x''(t) \le p_0(t)x(t) + p_1(t)x'(t) - h(x)(t) - \varepsilon q_1(t) \quad \text{for} \quad a < t < b. \tag{1.3.65}$$

From (1.3.56), (1.3.64), and (1.3.65), according to Definition 1.1.4 we conclude that the inclusion $(1.1.7_1)$ is satisfied for $\beta = 1$. $\square$

Proof of Corollary $1.1.5_2$. It is clear that all the requirements of Theorem $1.1.1_2$, except $(1.1.7_2)$, follow directly from the conditions of our corollary. It remains to show that the inclusion $(1.1.7_2)$ follows from the conditions $(1.1.50_2)$, (1.1.56) for $0 < \beta \le 1$ and from $(1.1.51_2)$ for $\beta = 1$.

First we consider the case $0 < \beta < 1$. Let x be the function defined by $(1.1.9_2)$. Then

$$(x^\beta(t))'' = p_1(t)(x^\beta(t))' - \beta(1-\beta)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}. \tag{1.3.66}$$

From $(1.1.50_2)$ it follows the existence of a constant $\varepsilon > 0$ such that

$$\operatorname*{ess\,sup}_{t \in \,]a,b[} \Big[\frac{x^2(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x^\beta)(t)}{x^\beta(t)} - p_0(t)\Big)\Big] < \beta(1-\beta) - \varepsilon \tag{1.3.67}$$

and likewise from the inclusion (1.1.55) it follows the existence of a constant λ such that

$$-\lambda\frac{x^{2-\beta}(t)}{\sigma^2(p_1)(t)}p_0(t) < \varepsilon \quad \text{for} \quad a < t < b. \tag{1.3.68}$$

Let $w(t) = x^\beta(t) + \lambda$, and rewrite the identity (1.3.66) in the form

$$w''(t) = p_0(t)w(t) + p_1(t)w'(t) - \Big(p_0(t)x^\beta(t) + \lambda p_0(t) + \beta(1-\beta)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}\Big),$$

whence with regard for (1.3.67), (1.3.68) and the fact that the operator h is nonnegative we can see that the inequality (1.3.55) is valid, i.e., by virtue of Lemma 1.2.2 the inclusion

$$(p_0, p_1) \in \mathbb{V}_{2,0}(]a, b[) \tag{1.3.69}$$

is satisfied. Then, as is seen from Remark 1.2.2, there exists Green's function G of the problem (1.2.4), $(1.2.2_{20})$, and by Lemma 1.2.7,

$$\int_a^b |G(t,s)| q_\beta(s)\, ds = O^*(x^\beta(s)) \quad \text{as} \quad t \to a, \tag{1.3.70}$$

where

$$q_\beta(t) = \frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}.$$

Rewrite now (1.3.66) as

$$(x^\beta(t))'' = p_0(t)x^\beta(t) + p_1(t)(x^\beta(t))' - h(x^\beta) - \varepsilon q_\beta(t) +$$

$$+\Big(h(x^\beta)(t) - p_0(t)x^\beta(t) - (\beta(1-\beta)-\varepsilon)\frac{\sigma^2(p_1)(t)}{x^{2-\beta}(t)}\Big).$$

This equality by virtue of the condition (1.3.67) enables us to see that (1.3.60) is satisfied. From the conditions (1.3.60), (1.3.69), (1.3.70) and according to Definition 1.1.4, we can conclude that the inclusion $(1.1.7_2)$ is satisfied for $0 < \beta < 1$.

Assume now that $\beta = 1$. From the condition $(1.1.50_2)$ for $\beta = 1$ it follows the existence of a constant $\varepsilon > 0$ such that

$$\operatorname*{ess\,sup}_{t\in\,]a,b[}\left[\frac{x(t)}{\sigma^2(p_1)(t)}\Big(\frac{h(x)(t)}{x(t)} - p_0(t)\Big)\right] < -\varepsilon. \tag{1.3.71}$$

Then it is clear from the negativeness of the operator h that

$$p_0(t) \geq 0 \quad \text{for} \quad a < t < b,$$

i.e., by virtue of Remark 1.2.6, the inclusion (1.3.69) is satisfied and hence there exists Green's function G of the problem (1.2.4), $(1.2.2_{20})$. As is seen from lemma 1.2.7, if $q_1(t) = \sigma^2(p_1)(t)$, then

$$\int_a^b |G(t,s)|q_1(s)\,ds = O^*(x(t)) \quad \text{as} \quad t \to a. \tag{1.3.72}$$

Note that

$$\begin{aligned} x''(t) = p_0(t)x(t) + p_1(t)x'(t) - h(x)(t) - \varepsilon q_1(t) + \\ + \big(h(x)(t) - p_0(t)x(t) + \varepsilon\sigma^2(p_1)(t)\big), \end{aligned}$$

whence with regard for (1.3.71) we see that (1.3.65) is satisfied.

From the conditions (1.3.65), (1.3.69), (1.3.72), owing to Definition 1.1.4 we conclude that the inclusion $(1.1.7_2)$ is satisfied for $\beta = 1$ as well.

The proof of the given and of the previous corollary is identical for the case $\beta = 0$. $\square$

Proof of Corollary $1.1.5_{i0}$. Coincides completely with that of Corollary $1.1.5_i$ for $0 < \beta \leq 1$. $\square$

Proof of Corollary $1.1.6_1$. Let

$$h(u)(t) = \sum_{k=1}^n |g_k(t)|u(\tau_k(t)). \tag{1.3.73}$$

Then we can see from $(1.1.56_1)$ that the inclusion $(1.1.8_1)$ is satisfied for $\beta = 0$. It is also clear that all the requirements of Theorem $1.1.1_1$ for $\alpha = 1$, $\beta = 0$, except $(1.1.7_1)$, follow directly from the conditions of our corollary. It remains to show that the conditions $(1.1.57_1)$, $(1.1.58_1)$ imply the inclusion $(1.1.7_1)$ as well.

Without restriction of generality we assume that $c \in]a,b[$. Then by $(1.1.57_1)$ there exist γ_m, η_m $(m=1,2)$ such that

$$0 \le \gamma_m < \eta_m < +\infty \quad (m=1,2)$$

and

$$\begin{aligned}
\int_{\gamma_1}^{\eta_1} \frac{ds}{\lambda_{11}+\lambda_{12}s+s^2} &= \frac{(c-a)^{1-\beta_1}}{1-\beta_1},\\
\int_{\gamma_2}^{\eta_2} \frac{ds}{\lambda_{21}+\lambda_{22}s+s^2} &= \frac{(b-c)^{1-\beta_2}}{1-\beta_2}.
\end{aligned} \tag{1.3.74}$$

Introduce the functions φ_1 and φ_2 by

$$\int_{\varphi_1(t)}^{\eta_1} \frac{ds}{\lambda_{11}+\lambda_{12}s+s^2} = \frac{(t-a)^{1-\beta_1}}{1-\beta_1} \quad \text{for} \quad a \le t \le c$$

and

$$\int_{\varphi_2(t)}^{\eta_2} \frac{ds}{\lambda_{21}+\lambda_{22}s+s^2} = \frac{(b-t)^{1-\beta_2}}{1-\beta_2} \quad \text{for} \quad c \le t \le b.$$

From (1.3.74) we have

$$\gamma_1 < \varphi_1(t) < \eta_1 \ \text{ for } \ a<t<c, \ \ \gamma_2 < \varphi_2(t) < \eta_2 \ \text{ for } \ c<t<b \ \text{ and}$$
$$\varphi_m(c) = \gamma_m \quad (m=1,2).$$

Introduce also the function w by

$$w(t) = \exp\bigg(\int_c^t (s-a)^{-\beta_1}\varphi_1(s)\,ds\bigg) \quad \text{for} \quad a \le t < c,$$
$$w(t) = \exp\bigg(\int_t^c (b-s)^{-\beta_2}\varphi_2(s)\,ds\bigg) \quad \text{for} \quad c \le t \le b.$$

Then

$$\begin{gathered}
w'(t) > 0 \quad \text{for} \quad a<t<c, \quad w'(t)<0 \quad \text{for} \quad c \le t < b,\\
w(t) > 0 \quad \text{for} \quad a \le t \le b,
\end{gathered} \tag{1.3.75}$$

$$w \in \widetilde{C}'_{\mathrm{loc}}(\,]a,c[) \cap \widetilde{C}'_{\mathrm{loc}}(\,]c;b[), \ \ w(c-) \ge w(c+), \tag{1.3.76}$$

and the equalities

$$
\begin{aligned}
w''(t) &= -\frac{\lambda_{11}}{(t-a)^{2\beta_1}}\,w(t) - \Big[\frac{\lambda_{12}}{(t-a)^{\beta_1}} + \frac{\beta_1}{t-a}\Big]w'(t) \\
&\qquad\qquad\qquad\qquad \text{for} \quad a < t < c, \\
w''(t) &= -\frac{\lambda_{21}}{(b-t)^{2\beta_2}}\,w(t) + \Big[\frac{\lambda_{22}}{(b-t)^{\beta_2}} + \frac{\beta_2}{b-t}\Big]w'(t) \\
&\qquad\qquad\qquad\qquad \text{for} \quad c \le t < b
\end{aligned}
\tag{1.3.77}
$$

are valid.

From the above equalities, by virtue of (1.3.75) it follows that

$$
w''(t) \le 0 \quad \text{for} \quad a < t < b. \tag{1.3.78}
$$

On the other hand, taking into account the conditions ($1.1.58_1$) in the equalities (1.3.77), we obtain

$$
\begin{aligned}
w''(t) \le \Big(p_0(t) - \sum_{k=1}^{n} |g_k(t)|\Big)w(t) + p_1(t)w'(t) - \\
-w'(t)\sum_{k=1}^{n} |g_k(t)|\big(\tau_k(t) - t\big) \quad \text{for} \quad a < t < b.
\end{aligned}
\tag{1.3.79}
$$

Analogously, from (1.3.78) it follows

$$
\int\limits_{t}^{\tau_k(t)} w'(s)\,ds \le w'(t)\big(\tau_k(t) - t\big) \quad (k = 1, \dots, n) \;\; \text{for} \quad a < t < b.
$$

Taking this inequality into consideration, from (1.3.79) we can see that

$$
w''(t) \le p_0(t)w(t) + p_1(t)w'(t) - \sum_{k=1}^{n} |g_k(t)|w\big(\tau_k(t)\big) \;\; \text{for} \;\; a < t < b.
$$

The latter inequality together with (1.3.75), (1.3.76) and by virtue of Definition 1.1.3 shows that the inclusion $(p_0, p_1) \in \mathbb{V}_{1,0}(]a, b[; h)$ is satisfied. $\square$

Proof of Corollary $1.1.6_2$. We define the operator h by the equality (1.3.73). Note also that if $p_1 \in L_{\text{loc}}(]a, b])$, then from the conditions (1.1.56) and (1.1.59) we obtain

$$
\begin{gathered}
\sigma(p_1) \in L([a, b]), \;\; p_j\sigma_2(p_1) \in L([a, b]) \;\; (j = 0, 2), \\
g_k\sigma_2(p_1) \in L([a, b]) \;\; (k = 1, \dots, n),
\end{gathered}
$$

i.e., the conditions ($1.1.3_2$), ($1.1.5_2$), and ($1.1.8_2$), are satisfied where $\beta = 0$, $\alpha = 1$. Then just as in the previous proof it remains to show that from the conditions ($1.1.57_2$)–(1.1.59) it follows the inclusion ($1.1.7_2$) for $\beta = 0$.

Without restriction of generality we assume that $c \in]a,b[$. Then by virtue of $(1.1.57_2)$ there exist constants γ_m, η_m $(m=1,2)$ such that

$$\varepsilon \leq \gamma_1 < \eta_1 < +\infty, \quad 0 < \gamma_2 < \eta_2 < +\infty$$

and (1.3.74) is satisfied. Introduce the functions φ_1 and φ_2 by

$$\int\limits_{\varphi_1(t)}^{\eta} \frac{ds}{\lambda_{11}+\lambda_{12}s+s^2} = \frac{(t-a)^{1-\beta_1}}{1-\beta_1} \quad \text{for} \quad a \leq t < c,$$

$$\int\limits_{\gamma_2}^{\varphi_2(t)} \frac{ds}{\lambda_{21}+\lambda_{22}s+s^2} = \frac{(b-t)^{1-\beta_2}}{1-\beta_2} \quad \text{for} \quad c \leq t \leq b.$$

From (1.3.74) we have

$$\gamma_1 < \varphi_1(t) < \eta_1 \ \text{ for } \ a < t < c, \ \ \gamma_2 < \varphi_2(t) < \eta_2 \ \text{ for } \ c < t < b,$$
$$\varphi_1(c) = \gamma_1 \quad \varphi_2(c) = \eta_2.$$

Introduce likewise the function w by the equalities

$$w(t) = \exp\bigg(\int\limits_a^t (s-a)^{-\beta_1}\varphi_1(s)\,ds\bigg) \quad \text{for} \quad a \leq t < c,$$

$$w(t) = \exp\bigg(\alpha\int\limits_c^t (b-s)^{-\beta_3}\varphi_2(s)\,ds\bigg) \quad \text{for} \quad c \leq t \leq b,$$

where $0 < \alpha < \min\Big(1; \frac{\gamma_1}{\eta_2}(b-c)^{-\beta_3}(c-a)^{-\beta_1}\Big)$, i.e.,

$$\alpha \in]0,1[. \tag{1.3.80}$$

Then

$$w'(t) > 0 \ \text{ for } \ t \in]a,c[\cup]c,b[, \ \ w(t) > 0 \ \text{ for } \ a \leq t \leq b, \tag{1.3.81}$$

$$w \in \widetilde{C}'_{\mathrm{loc}}(]a,c[) \cap \widetilde{C}'_{\mathrm{loc}}(]c;b[), \ \ w(c-) \geq w(c+), \ \ w'(b-) \geq 0, \tag{1.3.82}$$

and the equalities

$$w''(t) = -\frac{\lambda_{11}}{(t-a)^{2\beta_1}}w(t) - \Big[\frac{\lambda_{12}}{(t-a)^{\beta_1}} + \frac{\beta_1}{t-a}\Big]w'(t) \quad \text{for} \quad a < t < c \tag{1.3.83}$$

and

$$w''(t) = -\frac{\alpha\lambda_{21}}{(b-t)^{\beta_2-\beta_3}}w(t) - \Big[\frac{\lambda_{22}}{(b-t)^{\beta_2}} + \frac{\beta_3}{b-t}\Big]w'(t) - \\ -\alpha\big[1-\alpha(b-t)^{\beta_2+\beta_3}\big](b-t)^{\beta_3-\beta_2}w(t)\varphi_2^2(t), \ \text{ for } \ c < t < b \tag{1.3.84}$$

are valid. Note also that the condition $c \in [\max(a, b-1); b]$ and (1.3.80) imply

$$1-\alpha(b-t)^{\beta_2+\beta_3} \geq 0 \quad \text{for} \quad c \leq t \leq b.$$

Taking this into account in the equality (1.3.84), we obtain

$$w''(t) \leq -\frac{\alpha\lambda_{21}}{(b-t)^{\beta_2-\beta_3}}\,w(t) - \Big[\frac{\lambda_{22}}{(b-t)^{\beta_2}} + \frac{\beta_3}{b-t}\Big]w'(t). \quad (1.3.85)$$
$$\text{for} \quad a \leq t < b.$$

From (1.3.83) and (1.3.85), according to the condition (1.3.81), it is clear that the inequality (1.3.78) is satisfied.

On the other hand, taking into account in (1.3.83) and (1.3.85) the conditions (1.1.58$_2$), we get

$$w''(t) \leq \Big(p_0(t) - \sum_{k=1}^{n} |g_k(t)|\Big)w(t) + \widetilde{p}_1(t)w'(t) -$$
$$- w'(t)\sum_{k=1}^{n} |g_k(t)|\big(\tau_k(t)-t\big) \quad \text{for} \quad a < t < b,$$

which with regard for (1.3.81) and (1.1.59) imply that (1.3.79) is satisfied. Reasoning in the same way as in the previous proof, we see that the inclusion $(p_0, p_1) \in \mathbb{V}_{2,0}(]a, b[\,; h)$ is valid. □

Proof of Corollary 1.1.7$_1$. It is not difficult to notice that if we introduce the notation

$$g(u)(t) = \sum_{k=1}^{n} g_k(t)u\big(\tau_k(t)\big),$$

then the inequality (1.1.22) will be satisfied, and from (1.1.61), (1.1.62) it follows that the conditions (1.1.17) and (1.1.18) are valid. That is, all the requirements of Theorem 1.1.2$_1$ are fulfilled and this implies that our corollary is valid. □

Proof of Remark 1.1.10. Follows directly from that of Remark 1.1.2. □

Corollaries 1.1.7$_2$ and 1.1.7$_{i0}$ are proved analogously to Corollary 1.1.7$_1$.

CHAPTER II
CORRECTNESS OF TWO-POINT PROBLEMS FOR LINEAR SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

§ 2.1. Statement of the Problem and Formulation of Main Results

2.1.1. Statement of the Problem.

Let us Consider the functional differential equations

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + g(u)(t) + p_2(t), \tag{2.1.1}$$

$$u''(t) = p_{0k}(t)u(t) + p_{1k}(t)u'(t) + g_k(u)(t) + p_{2k}(t), \quad k \in \mathbb{N}, \tag{2.1.1$_k$}$$

under one of the following the boundary conditions

$$u(a) = 0, \quad u(-b) = 0, \tag{2.1.2$_{10}$}$$

$$u(a) = 0, \quad u'(b-) = 0; \tag{2.1.2$_{20}$}$$

$$u(a) = c_1, \quad u(b) = c_2, \tag{2.1.2$_1$}$$

$$u(a) = c_1, \quad u'(b-) = c_2; \tag{2.1.2$_2$}$$

$$u(a) = c_{1k}, \quad u(b) = c_{2k}, \tag{2.1.2$_{1k}$}$$

$$u(a) = c_{1k}, \quad u'(b-) = c_{2k}, \tag{2.1.2$_{2k}$}$$

where c_l, $c_{l_k} \in \mathbb{R}$, $(l = 1, 2; k \in \mathbb{N})$, g, $g_k : C(]a, b[) \to L_{\mathrm{loc}}(]a, b[)$, $k \in \mathbb{N}$, are continuous operators,

$$\begin{gathered} p_1, p_j \in L_{\mathrm{loc}}(]a, b[) \quad \sigma(p_1) \in L([a, b]), \\ p_j \in L_{\sigma_1(p_1)}([a, b]) \ \ (j = 0, 2) \end{gathered} \tag{2.1.3$_1$}$$

if $i = 1$,

$$\begin{gathered} p_1, p_j \in L_{\mathrm{loc}}(]a, b]) \quad \sigma(p_1) \in L([a, b]), \\ p_j \in L_{\sigma_2(p_1)}([a, b]) \quad (j = 0, 2) \end{gathered} \tag{2.1.3$_2$}$$

if $i = 2$, and $p_{jk} :]a, b[\to \mathbb{R}$ $(j = 0, 1, 2;\ k \in \mathbb{N})$ are measurable functions.

The correctness of the problem (2.1.1), (2.1.2$_i$) will be studied under the assumption that the inclusion

$$(p_0, p_1) \in \mathbb{V}_{i,0}(]a, b[\,; h)$$

is satisfied. (Effective sufficient conditions for the above inclusion to be fulfilled are given in §1.1, where

$$|g(x)(t)| \le h(|x|)(t)$$

almost everywhere in the interval $]a, b[$ for every $x \in C(]a, b[)$.)

Consider also the following linear equation

$$u''(t) = p_{0k}(t)u(t) + p_{1k}(t)u'(t) + p_{2k}(t). \tag{2.1.4$_k$}$$

Let G_k be Green's function of the problem (2.1.4$_k$), (2.1.2$_{i0}$) and $r \in \mathbb{R}^+$. Then we denote the set

$$\Big\{y(t): \;\; y(t) = \alpha_1 \widetilde{v}_k(t) + \int\limits_a^b G_k(t,s) g_k(x)(s)\,ds, \;\; \alpha_1 \in [0,r], \quad \|x\|_C \le r\Big\}$$

by $\mathbb{B}_{r,k}$ if $\widetilde{v}_k$ is a solution of the problem (2.1.4$_k$), (2.1.2$_{i0}$), and by $\mathbb{B}'_{r,k}$ if $\widetilde{v}_k$ is a solution of the problem (2.1.4$_k$), (2.1.2$_{ik}$).

Throughout this chapter the use will also be made of the notation

$$I_i(x)(t) = \int\limits_a^t x(s)\,ds \Big(\int\limits_t^b x(s)\,ds\Big)^{2-i} \quad \text{for} \quad a \le t \le b,$$

where $x \in L([a,b])$.

2.1.2. Formulation of Main Results.

Theorem 2.1.1$_i$. *Let $i \in \{1,2\}$, the continuous linear operators g, $g_k, h : C(]a,b[) \to L_{\text{loc}}(]a,b[)$ $(k \in \mathbb{N})$, the measurable functions p_j, $p_{jk} :]a,b[\to \mathbb{R}$ $(j = 0,1,2;\; k \in \mathbb{N})$ and the constants $\alpha \in [a,b]$, $\gamma \in]1,+\infty[$, $\beta, \mu \in \mathbb{R}$ be such that*

$$0 \le \beta < \mu < \frac{\gamma - 1}{\gamma - \alpha}\,, \tag{2.1.5}$$

$$\begin{gathered}\sigma^\gamma(p_1) \in L([a,b]), \;\; \int\limits_a^b \frac{|p_j(s)|}{\sigma(p_1)(s)}\, I_i^\mu(\sigma^\alpha(p_1))(s)\,ds < +\infty \;\; (j=0,2), \\ \int\limits_a^b \frac{h(1)(s)}{\sigma(p_1)(s)}\, I_i^\mu(\sigma^\alpha(p_1))(s)\,ds < +\infty,\end{gathered} \tag{2.1.6}$$

where h is a non-negative operator and uniformly on the segment $[a,b]$

$$\begin{gathered}\lim_{k\to\infty} \int\limits_a^t |p_1(s) - p_{1k}(s)|\,ds = 0, \\ \lim_{k\to\infty} \int\limits_a^t \frac{p_j(s) - p_{jk}(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds = 0 \quad (j=0,2),\end{gathered} \tag{2.1.7}$$

$$\lim_{k\to\infty}\Big(\sup\Big\{\Big|\int\limits_a^t \frac{g(y)(s) - g_k(y(s))}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \; a \le t \le b, \;\; y \in \mathbb{B}_{1k}\Big\}\Big) = 0. \tag{2.1.8}$$

Moreover, let

$$(p_0, p_1) \in \mathbb{V}_{i,0}(\,]a,b[,\, h), \tag{2.1.9}$$

where for every $x \in C(]a,b[)$ almost everywhere in the interval $]a,b[$ the inequality

$$|g(x)(t)| \le h(|x|)(t) \tag{2.1.10}$$

is satisfied. Then there exists a number k_0 such that if $k > k_0$, then the problem $(2.1.1_k)$, $(2.1.2_{i0})$ has a unique solution u_k and uniformly in the interval $]a,b[$

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)(u(t)-u_k(t)) = 0, \tag{2.1.11}$$

$$\lim_{k\to\infty} \frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)}(u'(t)-u_k'(t)) = 0, \tag{2.1.12}$$

where u is the solution of the problem (2.1.1), $(2.1.2_{i0})$.

Theorem $2.1.2_i$. *Let $i \in \{1,2\}$, the continuous linear operators g, $g_k, h : C(]a,b[) \to L_{\rm loc}(]a,b[)$ $(k \in \mathbb{N})$, the measurable functions p_j, $p_{jk} : (]a,b[) \to \mathbb{R}$ $(j = 0,1,2;\ k \in \mathbb{N})$ and the constans $\alpha \in [a,b]$, $\gamma \in]1,+\infty[$, c_l, c_{lk}, β, $\mu \in \mathbb{R}$ $(l = 1,2;\ k \in \mathbb{N})$ be such that conditions (2.1.5)–(2.1.7), (2.1.9), (2.1.10) and also*

$$\lim_{k\to\infty}\left(\sup\left\{\Big|\int\limits_a^t \frac{g(y)(s)-g_k(y(s))}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds\Big| :\right.\right.$$
$$\left.\left. a \le t \le b,\ x \in \mathbb{B}_{1k}'\right\}\right) = 0 \tag{2.1.13}$$

and

$$\lim_{k\to\infty} c_{lk} = c_l \quad (l = 1,2) \tag{2.1.14}$$

are satisfied. Then there exists a number k_0 such that if $k > k_0$, the problem $(2.1.1_k)$, $(2.1.2_{i0})$ has a unique solution u_k, and uniformly on the interval $]a,b[$ the equalities (2.1.12) and

$$\lim_{k\to\infty}\big(u(t)-u_k(t)\big) = 0 \tag{2.1.15}$$

are satisfied, where u is the solution of the problem (2.1.1), $(2.1.2_{i0})$.

2.1.3. Corollaries of Theorems $(2.1.1_i)$ $(2.1.2_i)$ $(i = 1,2)$.

Corollary $2.1.1_i$. *Let $i \in \{1,2\}$, the continuous linear operators g, $g_k, h : C(]a,b[) \to L_{\rm loc}(]a,b[)$ $(k \in \mathbb{N})$, the measurable functions η, p_j, p_{jk} $:]a,b[\to \mathbb{R}$ $(j = 0,1,2;\ k \in \mathbb{N})$ and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}^+$*

be such that the conditions (2.1.5)–(2.1.7)*,* (2.1.9)*,* (2.1.10) *are satisfied and for every* $y \in \widetilde{C}(]a,b[)$ *almost everywhere on the interval* $]a,b[$

$$\big|g_k(y)(t) - g(y)(t)\big| \le \eta(t)\|y\|_C \quad (k \in \mathbb{N}) \tag{2.1.16}$$

and uniformly on the segment $[a,b]$

$$\lim_{k\to\infty} \int_a^t \frac{g_k(y)(s) - g(y)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds = 0, \tag{2.1.17}$$

where

$$\int_a^b \frac{\eta(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds < +\infty. \tag{2.1.18}$$

Then there exists a number k_0*, such that for* $k > k_0$ *the problem* (2.1.1$_k$)*,* (2.1.2$_{i0}$) *has a unique solution* u_k*, and uniformly on the interval* $]a,b[$ *the equalities* (2.1.11)*,* (2.1.12) *are satisfied, where* u *is the solution of the problem* (2.1.1)*,* (2.1.2$_{i0}$)*.*

Corollary 2.1.2$_i$. *Let* $i \in \{1,2\}$*, the continuous linear operators* g*,* g_k*,* $h : C(]a,b[) \to L_{\mathrm{loc}}(]a,b[)$ $(k \in \mathbb{N})$*, the measurable functions* η*,* p_j*,* $p_{jk} :]a,b[\to \mathbb{R}$*,* $(j = 0,1,2;\ k \in \mathbb{N})$ *and constants* $\alpha \in [0,1]$*,* $\gamma \in]1,+\infty[$*,* β*,* $\mu \in \mathbb{R}^+$ *be such that the conditions* (2.1.5)–(2.1.7)*,* (2.1.9)*,* (2.1.10)*,* (2.1.14)*, and* (2.1.16)–(2.1.18) *are satisfied. Then there exists a number* k_0 *such that for* $k > k_0$ *the problem* (2.1.1$_k$)*,* (2.1.2$_{ik}$) *has a unique solution* u_k*, and uniformly on the interval* $]a,b[$ *the equalities* (2.1.12)*,* (2.1.15) *are satisfied, where* u *is the solution of the problem* (2.1.1)*,* (2.1.2$_i$)*.*

Consider now the case where the equations (2.1.1) and (2.1.1$_k$) are of the form

$$u''(t) = p_0(t)u(t) + p_1(t)u'(t) + \sum_{m=1}^{n} g_{0m}(t)u(\tau_{0m}(t)) + p_2(t) \tag{2.1.19}$$

and

$$u''(t) = p_{0k}(t)u(t) + p_{1k}(t)u'(t) + \sum_{m=1}^{n} g_{km}(t)u(\tau_{km}(t)) + p_{2k}(t), \tag{2.1.19$_k$}$$

where $g_{0m},\ g_{km} :]a,b[\to \mathbb{R}$ and $\tau_{0m},\ \tau_{km} : [a,b] \to [a,b]$ $(m = 1,\dots,n,\ k \in \mathbb{N})$ are measurable functions.

Corollary 2.1.3$_i$. *Let* $i \in \{1,2\}$*, the measurable functions* η*,* g_{0m}*,* g_{km}*,* p_j*,* $p_{jk} :]a,b[\to \mathbb{R}$*,* τ_{0m}*,* $\tau_{km} : [a,b] \to [a,b]$*,* $(m = 1,\dots,n;\ j = 0,1,2;\ k \in$

$\mathbb{N}$) *and the constants* $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ *be such that conditions* (2.1.5), (2.1.7), (2.1.18) *as well as*

$$\sigma^{\gamma}(p_1) \in L([a,b]),$$

$$\int_a^b \Big[|p_j(s)| + \sum_{m=1}^{n} |g_{0m}(s)|\Big] \frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(s)}{\sigma(p_1)(s)}\, ds < +\infty \quad (j=0,2), \tag{2.1.20}$$

$$\Big| \sum_{m=1}^{n} \big(g_{0m}(t) - g_{km}(t)\big)\Big| \le \eta(t) \quad (k \in \mathbb{N}) \tag{2.1.21}$$

are satisfied, and uniformly on the segment $[a,b]$

$$\lim_{k\to\infty} \sum_{m=1}^{n} \Big| \int_a^t \frac{g_{km}(s) - g_{0m}(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\, ds\Big| = 0, \tag{2.1.22}$$

$$\operatorname{ess\,sup}\Big\{ I_i^{\beta-\mu}(\sigma^{\alpha}(p_1))(t) \sum_{m=1}^{n} \Big| \int_{\tau_{0m}(t)}^{\tau_{km}(t)} \frac{\sigma(p_1)(s)}{I_i^{\mu}(\sigma^{\alpha}(p_1))(s)}\, ds\Big| : a<t<b\Big\} \to 0$$
$$\text{as } k \to +\infty. \tag{2.1.23}$$

Let also the condition (2.1.9) *be satisfied, where*

$$h(x)(t) = \sum_{m=1}^{n} |g_{0m}(t)| x(\tau_{0m}(t)).$$

Then there exists a number k_0 *such that for* $k > k_0$ *the problem* (2.1.19$_k$), (2.1.2$_{i0}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the equalities* (2.1.11), (2.1.12) *are satisfied, where* u *is the solution of the problem* (2.1.19), (2.1.2$_{i0}$).

Corollary 2.1.4$_i$. *Let* $i \in \{1,2\}$, *the measurable functions* η, g_{0m}, g_{km}, p_j, $p_{jk} :]a,b[\to \mathbb{R}$, τ_{0m}, $\tau_{km} : [a,b] \to [a,b]$, $(m=1,\dots,n;\ j=0,1,2;\ k \in \mathbb{N})$ *and the constants* $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, c_l, c_{lk}, β, $\mu \in \mathbb{R}$ $(l=1,2;\ k \in \mathbb{N})$ *be such that the conditions* (2.1.5), (2.1.7), (2.1.9), (2.1.14), (2.1.18), (2.1.20)–(2.1.23) *are satisfied, where* $h(x)(t) = \sum_{m=1}^{n} |g_{0m}(t)| x(\tau_{0m}(t))$. *Then there exists a number* k_0 *such that for* $k > k_0$ *the problem* (2.1.19$_k$), (2.1.2$_{ik}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the equalities* (2.1.12), (2.1.15) *are satisfied, where* u *is the solution of the problem* (2.1.19), (2.1.2$_i$).

Corollary 2.1.5$_i$. *Let* $i \in \{1,2\}$, *the measurable functions* η, g_{0m}, g_{km}, p_j, $p_{jk} :]a,b[\to \mathbb{R}$, τ_{0m}, $\tau_{km} : [a,b] \to [a,b]$, $(m=1,\dots,n;\ j=0,1,2;\ k \in \mathbb{N})$ *and the constants* $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ *be such that the* *conditions* (2.1.5), (2.1.7), (2.1.18), (2.1.22) *as well as*

$$\sigma^{\gamma}(p_1) \in L([a,b]), \quad \int\limits_a^b \frac{|p_j(s)|}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\, ds < +\infty \ \ (j=0,2), \tag{2.1.24}$$

$$\sum_{m=1}^{n} \big(|g_{km}(t)| + |g_{0m}(t)|\big) \le \eta(t) \ \ (k \in \mathbb{N}) \ \ \textit{for} \ \ a < t < b \tag{2.1.25}$$

and

$$\operatorname{ess\,sup}\Big\{ \sum_{m=1}^{n} |\tau_{0m}(t) - \tau_{km}(t)| : \ \ a \le t \le b \Big\} \to 0 \ \ \textit{for} \ \ k \to +\infty \tag{2.1.26}$$

are satisfied. Let also the condition (2.1.9) *be satisfied, where* $h(x)(t) = \sum\limits_{m=1}^{n} |g_{0m}(t)| x(\tau_{0m}(t))$. *Then there exists a number* k_0 *such that for* $k > k_0$ *the problem* (2.1.19$_k$), (2.1.2$_{i0}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the equalities* (2.1.11), (2.1.12) *are satisfied, where* u *is the solution of the problem* (2.1.19), (2.1.2$_{i0}$).

Corollary 2.1.6$_i$. *Let* $i \in \{1,2\}$, *the measurable functions* η, g_{0m}, g_{km}, p_j, p_{jm} $]a,b[\, \to \mathbb{R}$ τ_{0m}, $\tau_{km} : [a,b] \to [a,b]$, $(m = 1,\dots,n;\ j = 0,1,2;\ k \in \mathbb{N})$ *and the constants* $\alpha \in [0,1]$, $\gamma \in\,]1,+\infty[$, c_l, c_{lk}, β, $\mu \in \mathbb{R}$ $(l = 1,2;\ k \in \mathbb{N})$ *be such that the conditions* (2.1.5), (2.1.7), (2.1.9), (2.1.14), (2.1.18), (2.1.22) *and* (2.1.24)–(2.1.26) *are satisfied, where* $h(x)(t) = \sum\limits_{m=1}^{n} |g_{0m}(t)| x(\tau_{0m}(t))$. *Then there exists a number* k_0 *such that for* $k > k_0$ *the problem* (2.1.19$_k$), (2.1.2$_{ik}$) *has a unique solution* u_k, *and uniformly on the interval* $]a,b[$ *the equalities* (2.1.12), (2.1.15) *are satisfied, where* u *is the solution of the problem* (2.1.19), (2.1.2$_{i0}$).

For more clearness, let us consider the equations

$$u''(t) = g_0(t) u(\tau_0(t)) + p_2(t), \tag{2.1.27}$$

$$u''(t) = g_{0k}(t) u(\tau_k(t)) + p_{2k}(t), \tag{2.1.27$_k$}$$

where g_0, g_{0k}, p_2, p_{2k}; $]a,b[\, \to \mathbb{R}$, and τ_0, τ_{0k}; $[a,b] \to [a,b]$ $(k \in \mathbb{N})$ are measurable functions.

Corollary 2.1.7$_i$. *Let* $i \in \{1,2\}$, *the measurable functions* η, g_0, g_{0k}, p_2, $p_{2k} :]a,b[\, \to \mathbb{R}$, τ_0, $\tau_k : [a,b] \to [a,b]$, $(k \in \mathbb{N})$ *and the constants* β, $\mu \in \mathbb{R}$ *be such that the conditions*

$$\beta < \mu < 1, \tag{2.1.28}$$

$$|g_0(t)| + |g_{0k}(t)| \le \eta(t) \quad \textit{for} \quad a < t < b, \tag{2.1.29}$$

$$
\begin{gathered}
\int\limits_a^b |p_2(s)|(s-a)^{\mu}(b-s)^{\mu(2-i)}\,ds < +\infty, \\
\int\limits_a^b \eta(s)(s-a)^{\beta}(b-s)^{\beta(2-i)}\,ds < +\infty
\end{gathered}
\tag{2.1.30}
$$

are satisfied, and uniformly on the segment $[a,b]$

$$
\begin{gathered}
\lim_{k\to\infty}\int\limits_a^t \big(p_2(s)-p_{2k}(s)\big)(s-a)^{\beta}(b-s)^{\beta(2-i)}\,ds = 0, \\
\lim_{k\to\infty}\int\limits_a^t \big(g_0(s)-g_{0k}(s)\big)(s-a)^{\beta}(b-s)^{\beta(2-i)}\,ds = 0
\end{gathered}
\tag{2.1.31}
$$

and

$$
\operatorname{ess\,sup}\big\{|\tau_0(t)-\tau_k(t)| : \ \ a\le t\le b\big\} \to 0 \ \ \textit{as} \ \ k\to+\infty. \tag{2.1.32}
$$

Let, moreover, the inclusion

$$
(0,0)\in\mathbb{V}_{i,0}(]a,b[;h) \tag{2.1.33}
$$

be satisfied, where $h(x)(t) = |g_0(t)|x(\tau_0(t))$. *Then there exists a number* k_0*, such that for* $k>k_0$*, the problem* (2.1.27$_k$)*,* (2.1.2$_{i0}$) *has a unique solution* u_k*, and uniformly on the interval* $]a,b[$ *the conditions* (2.1.11)*,* (2.1.12) *are satisfied, where* u *is a solution of the problem* (2.1.27)*,* (2.1.2$_{i0}$)*.*

Corollary 2.1.8$_i$. *Let* $i\in\{1,2\}$*, the measurable functions* η*,* g_{0m}*,* g_{0k}*,* p_2*,* p_{2k} $:]a,b[\to\mathbb{R}$*,* τ_0*,* $\tau_k : [a,b]\to[a,b]$*,* $(k\in\mathbb{N})$ *and the constants* c_l*,* c_{lk}*,* β*,* $\mu\in\mathbb{R}$ $(l=1,2;\ k\in\mathbb{N})$ *be such that the conditions* (2.1.14) *and* (2.1.28)–(2.1.33) *are satisfied, where* $h(x)(t)=|g_0(t)|x(\tau_0(t))$*. Then there exists a number* k_0 *such that for* $k>k_0$ *the problem* (2.1.27$_k$)*,* (2.1.2$_{ik}$) *has a unique solution* u_k*, and uniformly on the interval* $]a,b[$ *the equalities* (2.1.12)*,* (2.1.15) *are satisfied, where* u *is the solution of the problem* (2.1.27)*,* (2.1.2$_i$)*.*

§ 2.2. Auxiliary Propositions

2.2.1. Correctness of the Initial Problem for Linear Second Order Ordinary Differential Equations. Consider on the interval $]a,b[$ the equations

$$
v''(t) = p_0(t)v(t) + p_1(t)u'(t) \tag{2.2.1}
$$

and

$$
v''(t) = p_{0k}(t)v(t) + p_{1k}(t)v'(t), \quad k\in\mathbb{N}, \tag{2.2.1$_k$}
$$

where

$$p_0,\, p_1 \in L_{\mathrm{loc}}(]a,b[), \quad \sigma(p_1) \in L([a,b]), \quad p_0 \in L_{\sigma_1(p_1)},([a,b]) \qquad (2.2.2_1)$$

$$p_{0k}, p_{1k} \in L_{\mathrm{loc}}(]a,b[), \quad k \in \mathbb{N}, \qquad (2.2.3_1)$$

or

$$p_0,\, p_1 \in L_{\mathrm{loc}}(]a,b]), \quad \sigma(p_1) \in L(\,[a,b]\,), \quad p_0 \in L_{\sigma_2(p_1)}([a,b]), \qquad (2.2.2_2)$$

$$p_{0k},\, p_{1k} \in L_{\mathrm{loc}}(\,]a,b]), \quad k \in \mathbb{N}, \qquad (2.2.3_2)$$

and the following initial conditions:

$$v(a) = 0, \quad \lim_{t\to a} \frac{v'(t)}{\sigma(p_1)(t)} = 1, \qquad (2.2.4_1)$$

$$v(a) = 0, \quad \lim_{t\to a} \frac{v'(t)}{\sigma(p_{1k})(t)} = 1, \qquad (2.2.4_k)$$

$$v(b) = 0, \quad \lim_{t\to b} \frac{v'(t)}{\sigma(p_1)(t)} = -1, \qquad (2.2.5_1)$$

$$v(b) = 0, \quad \lim_{t\to b} \frac{v'(t)}{\sigma(p_{1k})(t)} = -1, \qquad (2.2.5_{1k})$$

$$v(b) = 1, \quad v'(b) = 0. \qquad (2.2.5_2)$$

Remark 2.2.1. It has been shown in [23] that for the conditions $(2.2.2_i)$ the problems (2.2.1), (2.2.4) and (2.2.1), $(2.2.5_i)$ are uniquely solvable. Analogously, if

$$p_{0k},\, p_{1k} \in L_{\mathrm{loc}}(]a,b[), \;\; \sigma(p_{1k}) \in L([a,b]), \;\; p_{0k} \in L_{\sigma_1(p_{1k})}([a,b]),$$

then the problems $(2.2.1_k)$, $(2.2.4_k)$ and $(2.2.1_k)$, $(2.2.5_{1k})$ are uniquely solvable, and if

$$p_{0k},\, p_{1k} \in L_{\mathrm{loc}}(]a,b]), \;\; \sigma(p_{1k}) \in L([a,b]), \;\; p_{0k} \in L_{\sigma_2(p_{1k})}([a,b]),$$

then the problems $(2.2.1_k)$, $(2.2.4_k)$ and $(2.2.1_k)$, $(2.2.5_2)$ are uniquely solvable as well.

For brevity we introduce the notation

$$\Delta p_{jk}(t) = p_j(t) - p_{jk}(t) \;\; (j = 0,1,2;\; k \in \mathbb{N}) \;\text{ for }\; a < t < b.$$

Lemma 2.2.1₁. *Let the measurable functions $p_j,\, p_{jk} :]a,b[\to \mathbb{R}$ $(j = 0,1;\, k \in \mathbb{N})$ and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, $\beta,\, \mu \in \mathbb{R}$ such that*

$$0 \le \beta < \mu \le \frac{\gamma - 1}{\gamma - \alpha}\,, \qquad (2.2.6)$$

$$\sigma^\gamma(p_1) \in L([a,b]), \quad \int\limits_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\, I_1^\mu(\sigma^\alpha(p_1))(s)\, ds < +\infty \qquad (2.2.7_1)$$

and uniformly on the segment $[a,b]$ *the conditions*

$$\lim_{k\to\infty}\int\limits_a^t \frac{\Delta p_{0k}(s)}{\sigma(p_1)(s)}\, I_1^\beta(\sigma^\alpha(p_1))(s)\,ds=0,\quad \lim_{k\to\infty}\int\limits_a^t |\Delta p_{1k}(s)|\,ds=0 \quad (2.2.8_1)$$

be satisfied. Then there exists a number k_0 *such that for* $k>k_0$ *the problem* $(2.2.1_k)$, $(2.2.4_{1k})$ *has a unique solution* v_{1k} *and the problem* $(2.2.1_k)$, $(2.2.5_{1k})$ *has a unique solution* v_{2k}, *and uniformly on the interval* $]a,b[$

$$\lim_{k\to\infty}\big(v_{1k}(t)-v_1(t)\big)\bigg(\int\limits_a^t \sigma(p_1)(s)\,ds\bigg)^{-1}=0, \qquad (2.2.9_{11})$$

$$\lim_{k\to\infty}\big(v_{2k}(t)-v_2(t)\big)\bigg(\int\limits_t^b \sigma(p_1)(s)\,ds\bigg)^{-1}=0 \qquad (2.2.9_{12})$$

and

$$\lim_{k\to\infty}\frac{v_{1k}'(t)-v_1'(t)}{\sigma(p_1)(t)}\bigg(\int\limits_t^b \sigma^\alpha(p_1)(s)\,ds\bigg)^{\mu}=0, \qquad (2.2.10_{11})$$

$$\lim_{k\to\infty}\frac{v_{2k}'(t)-v_2'(t)}{\sigma(p_1)(t)}\bigg(\int\limits_a^t \sigma^\alpha(p_1)(s)\,ds\bigg)^{\mu}=0, \qquad (2.2.10_{12})$$

where v_1 *and* v_2 *are the solutions of the problems* (2.2.1), $(2.2.4_1)$ *and* (2.2.1), $(2.2.5_1)$, *respectively.*

Proof. It is clear from the definition of the constants α, β, γ, μ that

$$\beta-\mu<0,\quad 0<\frac{1-\alpha\beta}{1-\beta}<\frac{1-\alpha\mu}{1-\mu}\le\gamma. \qquad (2.2.11)$$

Hence

$$\sigma^\alpha(p_1),\ \ \sigma^{\frac{1-\alpha\beta}{1-\beta}}(p_1),\ \ \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)\in L([a,b]). \qquad (2.2.12)$$

Using the Hölder inequality, we obtain

$$\int\limits_{t_1}^{t_2}\sigma(p_1)(s)\,ds\le\bigg(\int\limits_{t_1}^{t_2}\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\bigg)^{1-\mu}\times$$

$$\times\bigg(\int\limits_{t_1}^{t_2}\sigma^\alpha(p_1)(s)\,ds\bigg)^{\mu}\quad\text{for}\quad a\le t_1\le t_2\le b, \qquad (2.2.13)$$

$$\int\limits_a^b \frac{\sigma(p_1)(s)}{\Big(\int\limits_a^s \sigma^\alpha(p_1)(\eta)\,d\eta\Big)^\beta}\,ds \le$$

$$\le \Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big(\int\limits_a^b \frac{\sigma^\alpha(p_1)(s)}{\Big(\int\limits_a^s \sigma^\alpha(p_1)(\eta)\,d\eta\Big)^{\frac{\beta}{\mu}}}\,ds\Big)^\mu =$$

$$= \Big(\frac{\mu}{\mu-\beta}\Big)^\mu \Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu-\beta}, \tag{2.2.14}$$

$$\int\limits_a^b \frac{\sigma(p_1)(s)}{\Big(\int\limits_s^b \sigma^\alpha(p_1)(\eta)\,d\eta\Big)^\beta}\,ds \le$$

$$\le \Big(\frac{\mu}{\mu-\beta}\Big)^\mu \Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu-\beta}, \tag{2.2.15}$$

where the existence of the integrals follows from (2.2.12). By means of (2.2.14), (2.2.15) we easily get

$$\int\limits_a^b \frac{\sigma(p_1)(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\,ds \le 2\Big(\frac{\mu}{\mu-\beta}\Big)^\mu I_1^{-\beta}(\sigma^\alpha(p_1))\Big(\frac{a+b}{2}\Big)\times$$

$$\times\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu-\beta} < +\infty. \tag{2.2.16}$$

It is also evident that for every $\delta \in [0,1[$

$$\int\limits_a^b \frac{\sigma^\alpha(p_1)(s)}{I_i^\delta(\sigma^\alpha(p_1))(s)}\,ds < +\infty. \tag{2.2.17}$$

By virtue of condition $(2.2.8_1)$, for every $\varepsilon > 1$ there exists a number k_0 such that for $k > k_0$

$$\varepsilon^{-1} \le \sigma(\Delta p_{1k})(t) \le \varepsilon \quad \text{for} \quad a \le t \le b. \tag{2.2.18}$$

We now proceed to the proof of the lemma. Taking into account the conditions $(2.2.7_1)$, (2.2.12) and the inequality (2.2.13), the inequality

$$\int\limits_a^b |p_0(s)|\sigma_1(p_1)(s)\,ds \le \int\limits_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\,I_1^\mu(\sigma^\alpha(p_1))(s)\,ds\times$$

$$\times\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{2(1-\mu)} < +\infty \tag{2.2.19}$$

is valid, i.e. the conditions $(2.2.2_1)$ are satisfied. In this case, owing to Remark 2.2.1, the problems (2.2.1), (2.2.4) and (2.2.1), $(2.2.5_1)$ are uniquely solvable. Integrating by parts and using (2.2.18), we arrive at

$$\Big|\int\limits_a^b \frac{p_{0k}(s)}{\sigma(p_{1k})(s)} I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds\Big| \le$$

$$\le \Big|\int\limits_a^b \frac{\Delta p_{0k}(s)}{\sigma(p_{1k})(s)}\, I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds\Big| + \int\limits_a^b \frac{|p_0(s)|}{\sigma(p_{1k})(s)}\, I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds \le$$

$$\le A_k \int\limits_a^b \Big|\Big(\sigma(\Delta p_{1k})(s)\frac{I_1^\mu(\sigma^\alpha(p_{1k}))(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\Big)'\Big|\,ds +$$

$$+\varepsilon^3 \int\limits_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\, I_1^\mu(\sigma^\alpha(p_1))(s)\,ds \quad \text{for} \quad k > k_0, \tag{2.2.20}$$

where

$$A_k = \sup\Big\{\Big|\int\limits_{t_1}^{t_2} \frac{\Delta p_{0k}(s)}{\sigma(p_1)(s)}\, I_1^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \quad a \le t_1 < t_2 \le b\Big\}.$$

In view of $(2.2.8_1)$

$$\lim_{k\to\infty} A_k = 0, \tag{2.2.21}$$

and by virtue of (2.2.18) the estimate

$$\Big|\Big(\sigma(\Delta p_{1k})(t)\frac{I_1^\mu(\sigma^\alpha(p_{1k}))(t)}{I_1^\beta(\sigma^\alpha(p_1))(t)}\Big)'\Big| \le \varepsilon^3|\Delta p_{1k}(t)| I_1^{\mu-\beta}(\sigma^\alpha(p_1))(t) +$$

$$+(\mu+\beta)\varepsilon^3 \int\limits_a^b \sigma^\alpha(p_1)(s)\,ds \frac{\sigma^\alpha(p_1)(t)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(t)} \quad \text{for} \quad a < t < b$$

is valid. Substituting the latter in (2.2.20) and taking into account $(2.2.7_1)$, $(2.2.8_1)$, (2.2.17) and (2.2.21), we can see that a constant $r_0 \in \mathbb{R}^+$ exist, such that

$$\sup\Big\{\int\limits_a^b \frac{|p_{0k}(s)|}{\sigma(p_{1k})(s)}\, I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds : \quad k > k_0\Big\} < r_0. \tag{2.2.22}$$

In the same way we get

$$p_{0k} \in L_{\sigma_1(p_{1k})}([a,b]) \quad \text{for} \quad k > k_0,$$

where in view of (2.2.18)

$$\sigma(p_{1k}) \in L([a,b]) \quad \text{for} \quad k > k_0,$$

which together with the conditions ($2.2.3_i$) and Remark 2.2.1 imply that the problems ($2.2.1_k$), ($2.2.4_k$) and ($2.2.1_k$), ($2.2.5_{1k}$) are uniquely solvable for $k > k_0$.

Note that the function $w_{jk}(t) = v_j(t) - v_{jk}(t)$ ($j = 1,2;\ k > k_0$) is a solution of the equation

$$\begin{aligned} v''(t) &= p_{0k}(t)v(t) + p_{1k}(t)v'(t) + \\ &+ \Delta p_{0k}(t)v_j(t) + \Delta p_{1k}(t)v_j'(t) \ \ (j = 1,2) \end{aligned} \tag{2.2.23}$$

and

$$w_{1k}(a) = 0, \ \lim_{t\to a} \frac{w_{1k}'(t)}{\sigma(p_{1k})(t)} = \sigma(\Delta p_{1k})(a) - 1, \tag{$2.2.24_1$}$$

$$w_{2k}(b) = 0, \ \lim_{t\to b} \frac{w_{2k}'(t)}{\sigma(p_{1k})(t)} = 1 - \sigma(\Delta p_{1k})(b), \tag{$2.2.24_2$}$$

where in view of ($2.2.8_1$),

$$\lim_{k\to\infty} \big\|1 - \sigma(\Delta p_{1k})\big\|_C = 0. \tag{2.2.25}$$

Consider first the case $j = 1$. From (2.2.23), ($2.2.24_1$) we have

$$\begin{aligned} \frac{w_{1k}'(t)}{\sigma(p_{1k})(t)} &= \sigma(\Delta p_{1k})(t) - 1 + \int_a^t \Delta p_{0k}(s)\, \frac{v_1(s) - w_{1k}(s)}{\sigma(p_{1k})(s)}\, ds + \\ &+ \int_a^t \frac{p_0(s)w_{1k}(s) + \Delta p_{1k}(s)v_1'(s)}{\sigma(p_{1k})(s)}\, ds \quad \text{for} \quad a < t < b, \end{aligned} \tag{2.2.26}$$

where the existence of integrals follows from the estimate ($1.2.10_1$), ($1.2.11_1$) and the conditions ($2.2.7_1$), ($2.2.8_1$). From (2.2.26), integration by parts results in

$$\begin{aligned} \frac{|w_{1k}'(t)|}{\sigma(p_{1k})(t)} &\le \big|1 - \sigma(\Delta p_{1k})(a)\big| + A_k \int_a^t \Big|\Big(\frac{v_1(s) - w_{1k}(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\, \sigma(\Delta p_{1k})(s)\Big)'\Big|\, ds + \\ &+ \int_a^t \frac{|p_0(s)w_{1k}(s) + \Delta p_{1k}(s)v_1'(s)|}{\sigma(p_{1k})(s)}\, ds \quad \text{for} \quad a < t < b, \end{aligned} \tag{2.2.27}$$

where in view of (2.2.18),

$$\int_a^t \left|\Big(\frac{v_1(s)-w_{1k}(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\,\sigma(\Delta p_{1k})(s)\Big)'\right| ds \leq$$

$$\leq \varepsilon \int_a^t \frac{|w_{1k}'(s)|+|v_1'(s)|}{I_1^\beta(\sigma^\alpha(p_1))(s)} + \big(|w_{1k}(s)|+|v_1(s)|\big)h_k(s)\,ds$$

with

$$h_k(t) = \frac{|\Delta p_{1k}(t)|}{I_1^\beta(\sigma^\alpha(p_1))(t)} + \beta\int_a^b \sigma^\alpha(p_1)(s)\,ds\,\frac{\sigma^\alpha(p_1)(t)}{I_1^{1+\beta}(\sigma^\alpha(p_1))(t)} \quad \text{for } a<t<b.$$

Substituting the latter inequality in (2.2.27), with regard for (2.2.18) we get

$$\frac{|w_{1k}'(t)|}{\sigma(p_1)(t)} \leq \varepsilon^2 A_k \int_a^t \frac{|w_{1k}'(s)|}{I_1^\beta(\sigma^\alpha(p_1))(s)}\,ds +$$

$$+\varepsilon^2\bigg[\big\|1-\sigma(\Delta p_{1k})\big\|_C + \int_a^t f_k(s)|w_{1k}(s)| + q_k(s)\,ds\bigg], \qquad (2.2.28)$$

where

$$f_k(t) = \frac{|p_{0k}(t)|}{\sigma(p_1)(t)} + A_k h_k(t),$$

$$q_k(t) = \frac{|v_1'(t)|}{\sigma(p_1)(t)}\Big(|\Delta p_{1k}(t)| + A_k\frac{\sigma(p_1)(t)}{I_1^\beta(\sigma^\alpha(p_1))(t)}\Big) + A_k h_k(t)|v_1(t)|$$

$$\text{for} \quad a<t<b.$$

From (2.2.28), using Gronwall-Bellman's lemma, it follows that

$$|w_{1k}'(t)| \leq r_k\sigma(p_1)(t)\bigg(\big\|1-\sigma(\Delta p_{1k})\big\|_C +$$

$$+\int_a^t f_k(s)|w_{1k}(s)| + q_k(s)\,ds\bigg) \quad \text{for} \quad a<t<b, \qquad (2.2.29)$$

where

$$r_k = \varepsilon^2\bigg[1+\exp\Big(\varepsilon^2 A_k\int_a^b \frac{\sigma(p_1)(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\,ds\Big)\bigg] \quad \text{for} \quad k>k_0$$

and by virtue of (2.2.16), (2.2.21),

$$\sup\{r_k: \quad k>k_0\} < +\infty. \qquad (2.2.30)$$

Let us now introduce the notation

$$z_k = |w_{1k}(t)|\Big(\int\limits_a^t \sigma(p_1)(s)\,ds\Big)^{-1} \quad \text{for} \quad a<t<b.$$

Integrating (2.2.29) from a to t, dividing by $\int\limits_a^t \sigma(p_1)(s)\,ds$ and using integration by parts, by virtue of the inequalities (2.2.13) and

$$\int\limits_s^t \sigma(p_1)(s)\,ds\Big(\int\limits_a^t \sigma(p_1)(s)\,ds\Big)^{-1} \le$$

$$\le \int\limits_s^b \sigma(p_1)(s)\,ds\Big(\int\limits_a^b \sigma(p_1)(s)\,ds\Big)^{-1} \quad \text{for} \quad a<s\le t<b$$

we obtain

$$z_k(t) \le r\int\limits_a^t f_k(s)I_1^\mu(\sigma^\alpha(p_1))(s)z_k(s)\,ds + \widetilde{r}_k \ \text{ for } \quad a<t<b,$$

where

$$r = \sup\big\{r_k : \quad k>k_0\big\}\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{2(1-\mu)}\Big(\int\limits_a^b \sigma(p_1)(s)\,ds\Big)^{-1},$$

$$\widetilde{r}_k = r\Bigg[\frac{\big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\big)^{1-\mu}}{\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds}\int\limits_a^b q_k(s)\Big(\int\limits_s^b \sigma^\alpha(p_1)(\eta)\,d\eta\Big)^\mu ds +$$

$$+\big\|1+\sigma(\Delta p_{1k})\big\|_C\Bigg].$$

Applying Gronwall–Bellman's lemma, from the latter inequality we get

$$z_k(t) \le \widetilde{r}_k \exp\Bigg(r\int\limits_a^b f_k(s)I_1^\mu(\sigma^\alpha(p_1))(s)\,ds\Bigg) \ \text{ for } a<t<b. \tag{2.2.31}$$

By virtue of (2.2.18) we note that the estimate

$$\int\limits_a^b f_k(s)I_1^\mu(\sigma^\alpha(p_1))(s)\,ds \le \varepsilon^3\int\limits_a^b \frac{|p_{0k}(s)|}{\sigma(p_{1k})(s)}\,I_1^\mu(\sigma^\alpha(p_{1k}))(s)\,ds +$$

$$+A_k\Big[\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{2(\mu-\beta)}\int\limits_a^b |\Delta p_{1k}(s)|\,ds+$$

$$+\beta\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\int\limits_a^b \frac{\sigma^\alpha(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(s)}\,ds\Big]\quad\text{for}\quad k>k_0$$

is valid, which with regard for the conditions (2.2.8$_1$), (2.2.17) with $\delta = 1+\beta-\mu$ and the condition (2.2.22) results in

$$\sup\Big\{\int\limits_a^b f_k(s)I_1^\mu(\sigma^\alpha(p_1))(s)\,ds:\quad k>k_0\Big\}<+\infty. \tag{2.2.32}$$

Just in the same way, taking into account the estimates (1.2.10$_1$), (1.2.11$_1$) and the inequality (2.2.13), we obtain

$$\int\limits_a^b q_k(s)\Big(\int\limits_s^b \sigma^\alpha(p_1)(\eta)\,d\eta\Big)^\mu ds\le$$

$$\le\int\limits_a^b |\Delta p_{1k}(s)|+A_k\,\frac{\sigma(p_1)(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\,ds\Big[\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^\mu+$$

$$+c^*\int\limits_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\,I_1^\mu(\sigma^\alpha(p_1))(s)\,ds\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\Big]+$$

$$+c^*A_k\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)ds\Big)^{1-\mu}\Big[\beta\int\limits_a^b \sigma^\alpha(p_1)(s)ds\int\limits_a^b \frac{\sigma^\alpha(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(s)}\,ds+$$

$$+\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{2(\mu-\beta)}\int\limits_a^b |\Delta p_{1k}(s)|\,ds\Big]\quad\text{for}\quad k>k_0,$$

By virtue of the inequalities (2.2.16), (2.2.17) with $\delta = 1+\beta-\mu$ and the conditions (2.2.7$_1$), (2.2.8$_1$) and (2.2.21)

$$\lim_{k\to\infty}\int\limits_a^b q_k(s)\Big(\int\limits_s^b \sigma^\alpha(p_1)(\eta)\,d\eta\Big)^\mu ds=0 \tag{2.2.33}$$

which together with (2.2.25) implies

$$\lim_{k\to\infty}\widetilde{r}_k=0. \tag{2.2.34}$$

Substituting (2.2.32) and (2.2.34) in (2.2.31) we get

$$\lim_{k\to\infty}\|z_k\|_C=0, \tag{2.2.35}$$

i.e., the condition ($2.2.9_{11}$) is satisfied.

Applying (2.2.13), we see from (2.2.29) that

$$\frac{|w_{1k}'(t)|}{\sigma(p_1)(t)}\bigg(\int_t^b\sigma^\alpha(p_1)(s)\,ds\bigg)^\mu\le$$

$$\le\widetilde r\bigg(\int_a^b\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\bigg)^{1-\mu}\bigg[\|z_k\|_C\int_a^b f_k(s)I_1^\mu(\sigma^\alpha(p_1))(s)\,ds+$$

$$+\int_a^b q_k(s)\bigg(\int_s^b\sigma^\alpha(p_1)(\eta)\,d\eta\bigg)^\mu ds\bigg]+$$

$$+\widetilde r\big\|1-\sigma(\Delta p_{1k})\big\|_C\bigg(\int_a^b\sigma^\alpha(p_1)(s)ds\bigg)^\mu\quad\text{for }\ a<t<b,$$

where $\widetilde r=\sup\{r_k:\ k>k_0\}$. The above inequality with regard for (2.2.25), (2.2.32), (2.2.33) and (2.2.35) implies that the condition ($2.2.10_{11}$) is valid.

Consider now the case $j=2$. Let $k>k_0$. Then for w_{2k}, i.e., for a solution of the problem (2.2.23), ($2.2.24_2$) the representation

$$-\frac{w_{2k}'(t)}{\sigma(p_{1k})(t)}=\sigma(\Delta p_{1k})(t)-1+\int_t^b\Delta p_{0k}(s)\frac{v_2(s)-w_{2k}(s)}{\sigma(p_{1k})(s)}\,ds+$$

$$+\int_t^b\frac{p_{0k}(s)w_{2k}(s)+\Delta p_{1k}v_2'(s)}{\sigma(p_{1k})(s)}\,ds\quad\text{for}\quad a<t<b$$

is valid. Repeating the arguments presented for $j=1$, where f_k, h_k are defined as before,

$$q_k(t)=\Big(|\Delta p_{1k}(t)|+A_k\,\frac{\sigma(p_1)(t)}{I_1^\beta(\sigma^\alpha(p_1))(t)}\Big)\,\frac{|v_2'(t)|}{\sigma(p_1)(t)}+A_kh_k(t)|v_2(t)|,$$

$$z_k(t)=|w_{2k}(t)|\bigg(\int_t^b\sigma(p_1)(s)ds\bigg)^{-1}$$

and

$$\widetilde{r}_k = r\Bigg[\frac{\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}}{\int\limits_a^b \sigma^{\alpha}(p_1)(s)\,ds}\int\limits_a^b q_k(s)\Big(\int\limits_a^s \sigma^{\alpha}(p_1)(\eta)\,d\eta\Big)^{\mu} ds +$$
$$+\big\|1+\sigma(\Delta p_{1k})\big\|_C\Bigg],$$

we see that the conditions $(2.2.9_{12})$, $(2.2.10_{12})$ are valid. $\square$

Lemma 2.2.1$_2$. *Let the measurable functions p_j, $p_{jk} :]a,b[\to \mathbb{R}$ $(j=0,1;$ $k\in\mathbb{N})$ and the constants $\alpha\in[0,1]$, $\gamma\in]1,+\infty[$, β, $\mu\in\mathbb{R}$ be such that the conditions* (2.2.6) *are satisfied,*

$$\sigma^{\gamma}(p_1)\in L([a,b]),\quad \int\limits_a^b \frac{|p_0(s)|}{\sigma(p_1)(s)}\,I_2^{\mu}(\sigma^{\alpha}(p_1))(s)\,ds<+\infty \qquad (2.2.7_2)$$

and uniformly on the segment $[a,b]$ the conditions

$$\begin{gathered}\lim_{k\to\infty}\int\limits_a^t \frac{\Delta p_{0k}(s)}{\sigma(p_1)(s)}\,I_2^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds=0,\\ \lim_{k\to\infty}\int\limits_a^t |\Delta p_{1k}(s)|\,ds=0\end{gathered} \qquad (2.2.8_2)$$

are satisfied. Then there exists a number k_0 such that for $k>k_0$ the problem $(2.2.1_k)$, $(2.2.4_k)$ has a unique solution v_{1k} and the problem $(2.2.1_k)$, $(2.2.5_2)$ has a unique solution v_{2k}, and uniformly on the interval $]a,b[$

$$\lim_{k\to\infty}\big(v_{1k}(t)-v_1(t)\big)\Big(\int\limits_a^t \sigma(p_1)(s)\,ds\Big)^{-1}=0, \qquad (2.2.9_{21})$$

$$\lim_{k\to\infty}\big(v_{2k}(t)-v_2(t)\big)=0 \qquad (2.2.9_{22})$$

and

$$\lim_{k\to\infty}\frac{v_{1k}'(t)-v_1'(t)}{\sigma(p_1)(t)}=0, \qquad (2.2.10_{21})$$

$$\lim_{k\to\infty}\frac{v_{2k}'(t)-v_2'(t)}{\sigma(p_1)(t)}\Big(\int\limits_a^t \sigma^{\alpha}(p_1)(s)\,ds\Big)^{\mu}=0, \qquad (2.2.10_{22})$$

where v_1 and v_2 are the solutions of the problems (2.2.1), (2.2.4) *and* (2.2.1), $(2.2.5_2)$, *respectively.*

Proof. Repeating word by word the previous proof for the case $j = 1$ and replacing everywhere I_1 by I_2, we can see that the problems $(2.2.1_k)$, $(2.2.4_k)$ and $(2.2.1_k)$, $(2.2.5_2)$ are uniquely solvable, the condition $(2.2.9_{21})$ is satisfied and for the function $w_{1k}(t) = v_1(t) - v_{1k}(t)$ the representation

$$\frac{|w_{1k}'(t)|}{\sigma(p_1)(t)} \leq r_k \bigg(\|z_k\|_C \int_a^t f_k(s) \Big(\int_a^s \sigma^\alpha(p_1)(\eta)\, d\eta \Big)^\mu ds + \\ + \int_a^t q_k(s)\, ds + \|1 - \sigma(\Delta p_{1k})\|_C \bigg) \quad \text{for} \quad a < t \leq b \tag{2.2.36}$$

is valid, where the functions f_k, q_k and z_k are defined in the previous proof. Using the same technique as when proving the relations (2.2.25), (2.2.32), (2.2.33), we obtain

$$\sup \bigg\{ \int_a^b f_k(s) I_2^\mu(\sigma^\alpha(p_1))(s)\, ds : \quad k > k_0 \bigg\} < +\infty,$$

$$\lim_{k\to\infty} \int_a^b q_k(s)\, ds = 0, \quad \lim_{k\to\infty} \|1 - \sigma(\Delta p_{1k})\|_C = 0$$

and

$$\lim_{k\to\infty} \|z_k\|_C = 0,$$

from which it follows with regard for (2.2.36) that the condition $(2.2.10_{21})$ is valid.

Note that the function $w_{2k}(t) = v_2(t) - v_{2k}(t)$ satisfies the conditions

$$w_{2k}(b) = 0, \quad w_{2k}'(b) = 0,$$

i.e., the representation

$$\frac{|w_{2k}'(t)|}{\sigma(p_{1k})(t)} = -\int_t^b \Delta p_{0k}(s)\, \frac{w_{2k}(s)}{\sigma(p_{1k})(s)}\, ds - \int_t^b \Delta p_{0k}(s)\, \frac{v_2(s)}{\sigma(p_{1k})(s)}\, ds - \\ - \int_t^b \frac{p_0(s) w_{1k}(s) + \Delta p_{1k}(s) v_2'(s)}{\sigma(p_{1k})(s)}\, ds \quad \text{for} \quad a < t \leq b$$

is valid. Repeating the arguments taking place in the proof of Lemma 2.2.1 for $j = 2$, we come to the conclusion that the conditions $(2.2.9_{12})$ and $(2.2.10_{22})$ are valid. But owing to the condition $p_1 \in L_{loc}(]a, b])$, it follows from $(2.2.9_{12})$ that $(2.2.9_{22})$ is valid. $\square$

Lemma 2.2.2. *Let $i \in \{1,2\}$, the measurable functions p_j, $p_{jk} :]a,b[\to \mathbb{R}$ and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ be such that the conditions* (2.2.6), $(2.2.7_i)$, $(2.2.8_i)$ *and*

$$(p_0, p_1) \in \mathbb{V}_{i,0}(]a,b[) \tag{$2.2.37_i$}$$

are satisfied. Then there exists a number k_0 such that for $k > k_0$

$$(p_{0k}, p_{1k}) \in \mathbb{V}_{i,0}(]a,b[). \tag{$2.2.38_i$}$$

Proof. Let $i = 1$ and v_1, v_2, v_{1k}, v_{2k} be solutions of the problems (2.2.1), (2.2.4), (2.2.1), $(2.2.5_1)$, $(2.2.1_k)$, $(2.2.4_k)$, $(2.2.1_k)$, $(2.2.5_{1k})$ respectively, whose existence and uniqueness follow from Remark 2.2.1.

As is seen from Definition 1.1.2 of the set $\mathbb{V}_{1,0}(]a,b[)$ and Remark 1.2.1, $v_1(b) > 0$ and $v_1(a) > 0$. Then by virtue of Remark 1.2.5 and the inclusion $(2.2.37_i)$,

$$v_1(t) + v_2(t) > 0 \quad \text{for} \quad a \le t \le b,$$

hence if

$$c = \min\big\{v_1(t) + v_2(t) : \quad a \le t \le b\big\},$$

then

$$c > 0. \tag{2.2.39}$$

On the other hand, by Lemma $2.2.1_i$, there exists a number k_0 such that for any $k > k_0$

$$-\frac{c}{2} < v_{jk}(t) - v_j(t) \ \ (j = 1,2) \quad \text{for} \quad a \le t \le b. \tag{2.2.40}$$

Thus for the solution v_k of the equation $(2.2.1_k)$, where

$$v_k(t) = v_{1k}(t) + v_{2k}(t),$$

the estimate

$$v_k(t) = \big(v_{1k}(t) - v_1(t)\big) + \big(v_{2k}(t) - v_2(t)\big) + \big(v_1(t) + v_2(t)\big)$$

is valid from which with regard for (2.2.39) and (2.2.40) we obtain

$$v_k(t) > 0 \quad \text{for} \quad a \le t \le b.$$

This inequality by virtue of Lemma 1.2.2 means that the inclusion $(2.2.38_i)$ is true. $\square$

Consider now the boundary conditions

$$u(a) = 0, \quad u(b) = 0 \tag{$2.2.41_1$}$$

and

$$u(a) = 0, \quad u'(b-) = 0. \tag{$2.2.41_2$}$$

The following Lemma is valid.

Lemma 2.2.3. *Let $i \in \{1,2\}$, the measurable functions f, p_j, $p_{jk} :]a,b[\to \mathbb{R}$ and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ satisfy the conditions* (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$) *and*

$$\int_a^b \frac{|f(s)|}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\, ds < +\infty. \tag{2.2.42}$$

Then there exists a number k_0 such that for $k > k_0$ the problem (2.2.1$_k$), (2.2.41$_i$) *has a unique Green's function G_k, and uniformly in the interval $]a,b[$*

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \int_a^b |G(t,s) - G_k(t,s)|\, |f(s)|\, ds = 0, \tag{2.2.43}$$

$$\lim_{k\to\infty} \frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)} \int_a^b \Big|\frac{\partial(G(t,s) - G_k(t,s))}{\partial t}\Big|\, |f(s)|\, ds = 0, \tag{2.2.44}$$

where G is Green's function of the problem (2.2.1), (2.2.41$_i$).

Proof. By Lemma 2.2.2$_i$, for $k > k_0$ the inclusion (2.2.38$_i$) is satisfied. Then as is seen from Remark 1.2.2, the inclusions (2.2.37$_i$) and (2.2.38$_i$) imply the existence of the functions G and G_k, respectively, where G is defined by the equality (1.2.7), and

$$G_k(t,s) = \begin{cases} -\dfrac{v_{2k}(t)v_{1k}(s)}{v_{2k}(a)\sigma(p_{1k})(s)} & \text{for} \quad a \le s < t \le b, \\ -\dfrac{v_{1k}(t)v_{2k}(s)}{v_{2k}(a)\sigma(p_{1k})(s)} & \text{for} \quad a \le t < s \le b, \end{cases} \tag{2.2.45}$$

where v_{1k} is the solution of the problem (2.2.1$_k$), (2.2.4$_{ik}$) and v_{2k} is that of the problem (2.2.1$_k$), (2.2.5$_{1k}$) for $i = 1$ and of the problem (2.2.1$_k$), (2.2.5$_2$) for $i = 2$.

From the estimates (1.2.10$_i$), (1.2.11$_i$) and the equalities (2.2.9$_{i1}$), (2.2.9$_{i2}$), (2.2.10$_{i1}$), (2.2.10$_{i2}$) it follows the existence of constants d_1 and d_2, such that on the interval $]a,b[$ the estimates

$$\begin{gathered} v_{1k}(t)\Big(\int_a^t \sigma(p_1)(s)\,ds\Big)^{-1} \le d_1, \quad v_{2k}(t)\Big(\int_t^b \sigma(p_1)(s)\,ds\Big)^{i-2} \le d_1 \\ \text{for} \quad k > k_0, \\ v_1(t)\Big(\int_a^t \sigma(p_1)(s)\,ds\Big)^{-1} \le d_1, \quad v_2(t)\Big(\int_t^b \sigma(p_1)(s)\,ds\Big)^{i-2} \le d_1 \end{gathered} \tag{2.2.46}$$

and

$$\frac{|v_{1k}'(t)|}{\sigma(p_1)(t)}\Big(\int\limits_t^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu(2-i)} \le d_1, \quad \frac{|v_{2k}'(t)|}{\sigma(p_1)(t)}\Big(\int\limits_a^t \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu} \le d_1$$

$$\text{for} \quad k > k_0, \tag{2.2.47}$$

$$\frac{|v_1'(t)|}{\sigma(p_1)(t)}\Big(\int\limits_t^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu(2-i)} \le d_1, \quad \frac{|v_2'(t)|}{\sigma(p_1)(t)}\Big(\int\limits_a^t \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu} \le d_1,$$

as well as

$$v_{2k}(a) \ge d_2 \ \text{ for } \ k > k_0, \quad v_2(a) \ge d_2 \tag{2.2.48}$$

are valid.

Introduce now the notation $w_{lk}^{(j)}(t) = v_l^{(j)}(t) - v_{lk}^{(j)}(t)$ $(l = 1,2;\ j = 0,1;\ k \in \mathbb{N})$ and

$$\omega_{1k} = \sup\Big\{|w_{1k}(t)|\Big(\int\limits_a^t \sigma(p_1)(s)\,ds\Big)^{-1} : \ a < t \le b\Big\},$$

$$\omega_{2k} = \sup\Big\{|w_{2k}(t)|\Big(\int\limits_t^b \sigma(p_1)(s)\,ds\Big)^{i-2} : \ a \le t < b\Big\},$$

$$\omega_{1k}' = \sup\Big\{\frac{|w_{1k}'(t)|}{\sigma(p_1)(t)}\Big(\int\limits_t^b \sigma^\alpha(p_1)(s)\,ds\Big)^{(2-i)\mu} : \ a < t < b\Big\},$$

$$\omega_{2k}' = \sup\Big\{\frac{|w_{2k}'(t)|}{\sigma(p_1)(t)}\Big(\int\limits_a^t \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu} : \ a < t < b\Big\}.$$

Then as is seen from Lemma $2.2.1_i$,

$$\lim_{k\to\infty} \omega_{jk} = 0, \quad \lim_{k\to\infty} \omega_{jk}' = 0 \ \ (j = 1,2). \tag{2.2.49}$$

It is also clear that the equality

$$\Big(\frac{I_i^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\Big)^j \int\limits_a^b \Big|\frac{\partial^j}{\partial t^j}\,(G_k(t,s) - G(t,s))\Big|\,|f(s)|\,ds =$$

$$= \Big(\frac{I_i^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\Big)^j \int\limits_a^t \Big|\frac{v_{2k}^{(j)}(t)v_{1k}(s)}{v_{2k}(a)\sigma(p_{1k})(s)} - \frac{v_2^{(j)}(t)v_1(s)}{v_2(a)\sigma(p_1)(s)}\Big|\,|f(s)|\,ds +$$

$$+ \int\limits_t^b \Big|\frac{v_{1k}^{(j)}(t)v_{2k}(s)}{v_{2k}(a)\sigma(p_{1k})(s)} - \frac{v_1^{(j)}(t)v_2(s)}{v_2(a)\sigma(p_1)(s)}\Big|\,|f(s)|\,ds \ \ (j = 0,1) \tag{2.2.50}$$

$$\text{for} \quad a < t < b$$

is valid.

Let $j = 0$. With regard for the inequalities (2.2.18) and (2.2.46) we obtain the estimate

$$\int_a^t \Big| \frac{v_{2k}(t)v_{1k}(s)}{v_{2k}(a)\sigma(\Delta p_{1k})(s)} - \frac{v_2(t)v_1(s)}{v_2(a)\sigma(\Delta p_{1k})(s)} \Big| \, |f(s)| \, ds \leq$$

$$\leq \frac{\varepsilon}{v_{2k}(a)} \bigg[|w_2(t)| \int_a^t \frac{|f(s)|}{\sigma(p_1)(s)} \, |v_{1k}(s)| \, ds +$$

$$+ |v_2(t)| \bigg(\int_a^t \frac{|f(s)|}{\sigma(p_1)(s)} \, |w_{1k}(s)| \, ds + \frac{|w_{2k}(a)|}{v_2(a)} \int_a^t \frac{|f(s)|}{\sigma(p_1)(s)} \, |v_1(s)| \, ds \bigg) \bigg] +$$

$$+ \frac{\|1 - \sigma(\Delta p_{1k})\|_C}{v_2(a)} \, v_2(t) \int_a^t \frac{|f(s)|}{\sigma(p_1)(s)} \, |v_1(s)| \, ds \leq$$

$$\leq r_k I_i^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \quad \text{for} \quad a \leq t \leq b,$$

where

$$r_k = \varepsilon \, \frac{d_1}{d_2} \bigg[\omega_{1k} + \omega_{2k} \Big(1 + \frac{d_1}{d_2} \int_a^b \sigma(p_1)(s) \, ds \Big) + \frac{d_1}{\varepsilon} \, \big\| 1 - \sigma(\Delta p_{1k}) \big\|_C \bigg] \times$$

$$\times \int_a^b \frac{|f(s)|}{\sigma(p_1)(s)} \, I_i^{\mu}(\sigma^{\alpha}(p_1))(s) \, ds$$

and in view of the conditions (2.2.8$_i$), (2.2.42), and (2.2.49),

$$\lim_{k \to \infty} r_k = 0. \tag{2.2.51}$$

Having analogously estimated the second integral in (2.2.50) for $j = 0$, we obtain for any $k > k_0$

$$I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \int_a^b |G(t,s) - G_k(t,s)| \, |f(s)| \, ds \leq 2r_k \quad \text{for} \quad a < t < b$$

which in view of (2.2.51) implies the validity of the condition (2.2.43).

Similarly, from the equality (2.2.50) for $j = 1$, with regard for (2.2.18), (2.2.46) and (2.2.47), for any $k > k_0$ we get

$$\frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)} \int_a^b \Big| \frac{\partial (G(t,s) - G_k(t,s))}{\partial t} \Big| \, |f(s)| \, ds \leq \widetilde{r}_k \quad \text{for} \quad a < t < b,$$

where

$$\widetilde{r}_k = 2\varepsilon \frac{d_1}{d_2} \left(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds \right) \int\limits_a^b \frac{|f(s)|}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\,ds \times$$

$$\times \left[\omega'_{1k} + \omega'_{2k} + \omega_{1k} + \omega_{2k} \Big(1 + \frac{d_1}{d_2} \int\limits_a^b \sigma(p_1)(s)\,ds \Big) + \frac{d_1}{\varepsilon}\, \|1 - \sigma(\Delta p_{1k})\|_C \right].$$

By the conditions (2.2.8$_i$), (2.2.42), and (2.2.49),

$$\lim_{k\to\infty} \widetilde{r}_k = 0$$

which guarantees the validity of the condition (2.2.44). $\square$

Lemma 2.2.4. *Let $i \in \{1,2\}$, the measurable functions f, p_j, $p_{jk} :]a,b[\to \mathbb{R}$ $(j = 0,1;\ k \in \mathbb{N})$ and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, $\beta, \mu \in \mathbb{R}$ satisfy conditions* (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$) *and*

$$\int\limits_a^b \frac{|f(s)|}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds < +\infty. \tag{2.2.52}$$

Then there exist a constant $r_1 \in \mathbb{R}^+$ and a number k_0 such that for $k > k_0$ the problem (2.2.1$_k$), (2.2.42$_i$) *has a unique Green's function G_k, and*

$$\Big| \int\limits_a^b G_k(t,s) f(s)\,ds \Big| \le r_1 \max\left\{ \Big| \int\limits_a^t \frac{f(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds : \ a \le t \le b \right\} \times$$
$$\times I_i^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t) \quad \text{for} \quad a \le t \le b \tag{2.2.53}$$

and

$$\frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)} \Big| \int\limits_a^b \frac{\partial G_k(t,s)}{\partial t}\, f(s)\,ds \Big| \le$$
$$\le r_1 \max\left\{ \Big| \int\limits_a^t \frac{f(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds : \ a \le t \le b \right\} \tag{2.2.54}$$
$$\text{for} \quad a < t < b.$$

Proof. In the proof of the previous lemma it has been shown that under the conditions of that lemma the problem (2.2.1$_k$), (2.2.42$_i$) has a unique Green's function G_k which is represented by the equality (2.2.45).

Consider separately the case $i=1$. First we note that in view of (2.2.12) and (2.2.17) the inequality

$$\int\limits_{t_1}^{t_2}\frac{\sigma(p_1)(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\,ds\le\Big(\int\limits_{t_1}^{t_2}\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\times$$
$$\times\Big(\int\limits_{a}^{b}\frac{\sigma^{\alpha}(p_1)(s)}{I_1^{\frac{\beta}{\mu}}(\sigma^{\alpha}(p_1))(s)}\,ds\Big)^{\mu}<+\infty\quad\text{for}\quad a\le t_1<t_2\le b \tag{2.2.55}$$

is valid. Integrating by parts and applying (2.2.48), we get

$$\Big|\int\limits_{a}^{b}\frac{\partial^j G(t,s)}{\partial t^j}\,f(s)\,ds\Big|\le$$
$$\le\frac{2}{d_2}\,\max\Big\{\Big|\int\limits_{a}^{t}\frac{f(s)}{\sigma(p_1)(s)}\,I_1^{\beta}(\sigma^{\alpha}(p_1))(s)\,ds\Big|:\ \ a\le t\le b\Big\}\times$$
$$\times\Big[|v_{2k}^{(j)}(t)|\int\limits_{a}^{t}\Big|\Big(\frac{v_{1k}(s)\sigma(\Delta p_{1k})(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\Big)'\Big|\,ds+$$
$$+|v_{1k}^{(j)}(t)|\int\limits_{t}^{b}\Big|\Big(\frac{v_{2k}(s)\sigma(\Delta p_{1k})(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\Big)'\Big|\,ds\ \ (j=0,1)\ \ \text{for}\ \ a<t<b. \tag{2.2.56}$$

Using now the estimates (2.2.46), (2.2.55), we obtain

$$|v_{2k}(t)|\int\limits_{a}^{t}\Big|\Big(\frac{v_{1k}(s)\sigma(\Delta p_{1k})(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\Big)'\Big|\,ds\le\varepsilon d_1\Big(\int\limits_{t}^{b}\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\times$$
$$\times\int\limits_{a}^{t}\frac{\sigma(p_1)(s)}{I_1^{\beta}(\sigma^{\alpha}(p_1))(s)}\,\frac{v_{1k}'(s)}{\sigma(p_1)(s)}\Big(\int\limits_{s}^{b}\sigma^{\alpha}(p_1)(\eta)\,d\eta\Big)^{\mu}ds+$$
$$+\varepsilon d_1^2 I_1^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)\Big[\Big(\int\limits_{a}^{b}\sigma^{\alpha}(p_1)(s)\,ds\Big)^{2(\mu-\beta)}\int\limits_{a}^{b}|\Delta p_{1k}(s)|\,ds+$$
$$+\int\limits_{a}^{b}\sigma^{\alpha}(p_1)(s)\,ds\int\limits_{a}^{b}\frac{\sigma^{\alpha}(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^{\alpha}(p_1))(s)}\,ds\Big]\le$$
$$\le\widetilde{r}_1 I_1^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)\ \ \text{for}\ \ a\le t\le b, \tag{2.2.57}$$

where

$$\widetilde{r}_1=\varepsilon d_1^2\bigg(\Big(\int\limits_a^b\frac{\sigma^\alpha(p_1)(s)}{I_1^{\frac{\beta}{\mu}}(\sigma^\alpha(p_1))(s)}\,ds\Big)^\mu+$$
$$+\Big(\int\limits_a^b\sigma^\alpha(p_1)(s)\,ds\Big)^{2(\mu-\beta)}\sup\Big\{\int\limits_a^b|\Delta p_{1k}(s)|\,ds:\quad k>k_0\Big\}+$$
$$+\int\limits_a^b\sigma^\alpha(p_1)(s)\,ds\int\limits_a^b\frac{\sigma^\alpha(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(s)}\,ds\bigg).$$

Analogously we have

$$|v_{1k}(t)|\int\limits_t^b\bigg|\Big(\frac{v_{2k}(s)\sigma(\Delta p_{1k})(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\Big)'\bigg|\,ds\le$$
$$\le\widetilde{r}_1I_1^{1-\mu}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)\quad\text{for}\quad a\le t\le b,\tag{2.2.58}$$
$$\frac{I_1^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\,|v_{2k}'(t)|\int\limits_a^t\bigg|\Big(\frac{v_{1k}(s)\sigma(\Delta p_{1k})(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\Big)'\bigg|\,ds\le$$
$$\le\widetilde{r}_2\quad\text{for}\quad a<t<b\tag{2.2.59}$$

and

$$\frac{I_1^\mu(\sigma^\alpha(p_1))(t)}{\sigma(p_1)(t)}\,|v_{1k}'(t)|\int\limits_t^b\bigg|\Big(\frac{v_{2k}(s)\sigma(\Delta p_{1k})(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\Big)'\bigg|\,ds\le$$
$$\le\widetilde{r}_2\quad\text{for}\quad a<t<b,\tag{2.2.60}$$

where

$$\widetilde{r}_2=\varepsilon d_1^2\bigg[\int\limits_a^b\frac{\sigma_1(p_1)(s)}{I_1^\beta(\sigma^\alpha(p_1))(s)}\,ds+\Big(\int\limits_a^b\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)\,ds\Big)^{1-\mu}\times$$
$$\times\bigg(\sup\Big\{\int\limits_a^b|\Delta p_{1k}|\,ds:k>k_0\Big\}\Big(\int\limits_a^b\sigma^\alpha(p_1)(s)\,ds\Big)^{2(\mu-\beta)}+$$
$$+\int\limits_a^b\sigma^\alpha(p_1)(s)\,ds\int\limits_a^b\frac{\sigma^\alpha(p_1)(s)}{I_1^{1+\beta-\mu}(\sigma^\alpha(p_1))(s)}\,ds\bigg)\bigg].$$

Let us now introduce the notation

$$r_1=\frac{4}{d_2}\max(\widetilde{r}_1;\widetilde{r}_2).$$

Substituting the estimates (2.2.57), (2.2.58) in (2.2.56) for $j = 0$, we see that the condition (2.2.53) is valid. Taking then into account (2.2.59), (2.2.60) in (2.2.56) for $j = 1$, we are convinced of the validity of (2.2.54).

For $i = 2$ the lemma is proved analogously. □

Lemma 2.2.5. *Let $i \in \{1,2\}$, the measurable functions p_j, $p_{jk} :]a,b[\to \mathbb{R}$ $(j = 0,1;\ k \in \mathbb{N})$ and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ satisfy the conditions* (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$). *Then there exists a number k_0 such that for $k > k_0$ the problem* (2.2.1$_k$), (2.2.41$_k$) *has a unique Green's function G_k for which the estimate*

$$\left|\frac{d^j G_k(t,s)}{dt^j}\right| \le c' \,\frac{\sigma_i(p_1)(s)}{[\sigma_i(p_1)(t)]^j} \quad (j = 0,1) \quad \textit{for} \quad a < t, s < b, \quad t \ne s, \tag{2.2.61}$$

is valid, where c' is a constant.

Proof. The existence of Green's function under the given conditions has been shown in Lemma 2.2.3. Similarly, by virtue of the estimate (1.2.12$_i$) from Remark 1.2.3,

$$\left|\frac{d^j G_k(t,s)}{dt^j}\right| \le c^* \,\frac{\sigma_i(p_{1k})(s)}{[\sigma_i(p_{1k})(t)]^j} \quad (j = 0,1) \quad \text{for} \quad a < t, s < b, \quad t \ne s,$$

whence with regard for the inequalities (2.2.18) and (2.2.48) follows the validity of our lemma. □

Consider now the equations

$$v''(t) = p_0(t)v(t) + p_1(t)v'(t) + p_2(t), \tag{2.2.62}$$

$$v''(t) = p_{0k}(t)v(t) + p_{1k}(t)v'(t) + p_{2k}(t), \tag{2.2.62$_k$}$$

where p_2, $p_{2k} \in L_{loc}(]a,b[)$ $(k \in \mathbb{N})$ and the boundary conditions

$$u(a) = c_1, \quad u(b) = c_2 \tag{2.2.63$_1$}$$

or

$$u(a) = c_1, \quad u'(b-) = c_2, \tag{2.2.63$_2$}$$

and

$$u(a) = c_{1k}, \quad u(b) = c_{2k} \tag{2.2.63$_{1k}$}$$

or

$$u(a) = c_{1k}, \quad u'(b-) = c_{2k}, \tag{2.2.63$_{2k}$}$$

where c_l, $c_{lk} \in \mathbb{R}$ $(l = 1,2;\ k \in \mathbb{N})$. Then the following lemma is valid.

Lemma 2.2.6. *Let $i \in \{1,2\}$, the measurable functions p_j, $p_{jk} :]a,b[\to \mathbb{R}$ ($j = 0,1,2$; $k \in \mathbb{N}$) and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ satisfy the conditions* (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$),

$$\int_a^b \frac{|p_2(s)|}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\, ds < +\infty \tag{2.2.64}$$

and uniformly on the segment $[a,b]$

$$\lim_{k\to\infty} \int_a^t \frac{p_2(s) - p_{2k}(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\, ds = 0. \tag{2.2.65}$$

Then there exists a number k_0 such that for $k > k_0$:

(a) *the problem* (2.2.62$_k$), (2.2.41$_i$) *has a unique solution $\widetilde{v}_k$, and uniformly on the interval $]a,b[$*

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)\big(\widetilde{v}(t) - \widetilde{v}_k(t)\big) = 0, \tag{2.2.66}$$

$$\lim_{k\to\infty} \frac{\widetilde{v}'(t) - \widetilde{v}_k'(t)}{\sigma(p_1)(t)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(t) = 0, \tag{2.2.67}$$

where $\widetilde{v}$ is a solution of the problem (2.2.61), (2.2.41$_i$)*;*

(b) *the problem* (2.2.62$_k$), (2.2.63$_{ik}$) *has a unique solution $\widetilde{v}_k$, and if*

$$\lim_{k\to\infty} c_{lk} = c_l \quad (l = 1,2), \tag{2.2.68}$$

then uniformly on the interval $]a,b[$ the conditions (2.2.67) *and*

$$\lim_{k\to\infty} \big(\widetilde{v}(t) - \widetilde{v}_k(t)\big) = 0 \tag{2.2.69}$$

are satisfied, where $\widetilde{v}$ is a solution of the problem (2.2.62), (2.2.63$_i$)*;*

(c) *the sequence $(\widetilde{v}_k)_{k=1}^{\infty}$, where $\widetilde{v}_k$ is a solution of the problem* (2.2.62$_k$), (2.2.41$_i$), ((2.2.62$_k$), (2.2.63$_{ik}$))*, is uniformly bounded and equicontinuous.*

Proof. First we prove the validity of proposition (a). It has been mentioned in the proof of Lemma 2.2.3 that under the above-mentioned conditions the problems (2.2.1), (2.2.41$_i$), and (2.2.1$_k$), (2.2.41$_i$) for $k > k_0$ have a unique Green's function G and G_k, respectively.

Let

$$\widetilde{v}(t) = \int_a^b G(t,s)p_2(s)\, ds \quad \text{and} \quad \widetilde{v}_k(t) = \int_a^b G_k(t,s)p_{2k}(s)\, ds.$$

Then

$$\widetilde{v}^{(j)}(t) - \widetilde{v}_k^{(j)}(t) = \int_a^b \frac{\partial^j G_k(t,s)}{\partial t^j}\, \big(p_2(s) - p_{2k}(s)\big)\, ds +$$

$$+\int_a^b \frac{\partial^j \Delta G_k(t,s)}{\partial t^j}\, p_2(s)\, ds \quad (j=0,1) \quad \text{for} \quad a<t<b.$$

Taking into account the equalities (2.2.43), (2.2.44) of Lemma 2.2.3 and the equalities (2.2.53), (2.2.54) of Lemma 2.2.4, by means of the conditions (2.2.64), (2.2.65) we make sure that the equalities (2.2.66) and (2.2.67) are valid.

Now we proceed to proving proposition (b). Let v_0 and v_{0k} be solutions of the problems (2.2.1), ($2.2.63_i$) and ($2.2.1_k$), ($2.2.63_{ik}$), respectively. Then

$$\widetilde{v}(t) = v_0(t) + \int_a^b G(t,s) p_2(s)\, ds \quad \widetilde{v}_k(t) = v_{0k}(t) + \int_a^b G(t,s) p_{2k}(s)\, ds$$

and

$$\widetilde{v}^{(j)}(t) - \widetilde{v}_k^{(j)}(t) = v_0^{(j)}(t) - v_{0k}^{(j)}(t) + \int_a^b \frac{\partial^j G_k(t,s)}{\partial t^j} \big(p_2(s) - p_{2k}(s)\big)\, ds +$$

$$+\int_a^b \frac{\partial^j \Delta G_k(t,s)}{\partial t^j}\, p_2(s)\, ds \quad (j=0,1) \quad \text{for} \quad a<t<b,$$

where

$$v_0(t) - v_{0k}(t) =$$
$$= c_1 \frac{v_2(t)}{v_2(a)} - c_{1k} \frac{v_{2k}(t)}{v_{2k}(a)} + c_2 \frac{v_1(t)}{v_1(b)} - c_{2k} \frac{v_{1k}(t)}{v_{1k}(b)} \quad \text{for} \quad a \le t < b$$

and v_j, v_{jk} ($j=1,2$; $k \ge k_0$) are the solutions mentioned in Lemma $2.2.1_i$. It follows from the given representation, Lemma $2.2.1_i$ and the condition (2.2.68) that uniformly in the interval $]a,b[$

$$\lim_{k\to\infty} \big(v_0(t) - v_{0k}(t)\big) = 0$$

and

$$\lim_{k\to\infty} \frac{v_0'(t) - v_{0k}'(t)}{\sigma(p_1)(t)}\, I_i^\mu(\sigma^\alpha(p_1))(t) = 0.$$

Next, reasoning analogously as in proving proposition (a), we can see that the conditions (2.2.67), (2.2.69) are valid.

The validity of proposition (c) follows immediately from (2.2.66) ((2.2.69)) and also from

$$\big|\widetilde{v}_k(t_1) - \widetilde{v}_k(t_2)\big| \le \big|\widetilde{v}_k(t_1) - \widetilde{v}(t_1)\big| + \big|\widetilde{v}_k(t_2) - \widetilde{v}(t_2)\big| + \big|\widetilde{v}(t_1) - \widetilde{v}(t_2)\big| \le$$
$$\le 2\|\widetilde{v}_k - v\|_C + \big|\widetilde{v}(t_1) - \widetilde{v}(t_2)\big|,$$

where t_1, $t_2 \in [a,b]$. $\square$

Remark 2.2.2. It is not difficult to notice that if the condition (2.1.8) is satisfied, then for any fixed $r \in \mathbb{R}^+$ the equality

$$\lim_{k\to\infty}\left(\sup\left\{\Big|\int_a^t \frac{g_k(x)(s)-g(x)(s)}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\,ds\Big| : a \le t \le b,\ \ x \in \mathbb{B}_{r,k}\right\}\right) = 0 \tag{2.2.70}$$

is valid. The same is true for the set $\mathbb{B}'_{r,k}$.

Lemma 2.2.7. *Let $i \in \{1,2\}$, the measurable functions p_j, $p_{jk} :]a,b[\to \mathbb{R}$ $(j = 0,1,2;\ k \in \mathbb{N})$ and the constants $\alpha \in [0,1]$, $\gamma \in]1,+\infty[$, β, $\mu \in \mathbb{R}$ satisfy the conditions* (2.2.6), (2.2.7$_i$), (2.2.8$_i$), (2.2.37$_i$), (2.2.64) *and* (2.2.65). *Moreover, let continuous linear operators g, $g_k : C(]a,b[) \to L_{loc}(]a,b[)$, be such that the condition* (2.1.8) *is satisfied. Then for every fixed $r \in \mathbb{R}^+$ the sequence $(z_k)_{k=1}^{\infty}$*

$$z_k(t) = \alpha_k \widetilde{v}_k(t) + \int_a^b G_k(t,s) g_k(x_k)(s)\,ds,$$

is uniformly bounded and equicontinuous, where $\widetilde{v}_k$ is a solution of the problem (2.2.62$_k$), (2.2.41$_i$), *G_k is the Green's function of that problem, and for every $\alpha_k \in [0,r]$, $x_k \in \mathbb{B}_{r,k}$ $(k \in \mathbb{N})$.*

Proof. Introduce the notation

$$\widetilde{z}_k(t) = \int_a^b G_k(t,s) g_k(x_k)(s)\,ds, \quad w_k(t) = \int_a^b G(t,s) g(x_k)(s)\,ds,$$

where G is Green's function of the problem (2.2.62), (2.2.41$_i$).

Similarly to the proof of Lemma 1.2.4 we see that

$$\sup\big\{\|w_k\|_C : k \in \mathbb{N}\big\} < +\infty$$

and for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for every $k \in \mathbb{N}$

$$\big|w_k(t_1) - w_k(t_2)\big| < \varepsilon \quad \text{for} \quad |t_1 - t_2| < \delta. \tag{2.2.71}$$

On the other hand, from the inequality

$$\big|\widetilde{z}_k(t) - w_k(t)\big| \le \Big|\int_a^b \big(G_k(t,s) - G(t,s)\big) g(x_k)(s)\,ds\Big| + \\ + \Big|\int_a^b G_k(t,s)\big(g_k(x_k)(s) - g(x_k)(s)\big)\,ds\Big|$$

by virtue of Lemmas 2.2.3–2.2.4 and Remark 2.2.2 with all conditions satisfied, we obtain

$$\lim_{k\to\infty} \|\widetilde{z}_k - w_k\|_C = 0 \tag{2.2.72}$$

which, owing to the inequality

$$\begin{gathered}\big|\widetilde{z}_k(t_1)-\widetilde{z}_k(t_2)\big| \le \big|\widetilde{z}_k(t_1)-w_k(t_1)\big| + \big|\widetilde{z}_k(t_2)-w_k(t_2)\big| + \\ +|w_k(t_2)-w_k(t_1)| \le 2\|\widetilde{z}_k - w_k\|_C + |w_k(t_2)-w_k(t_1)|\end{gathered}$$

with regard for (2.2.71) and (2.2.72), implies the uniform boundedness and equicontinuity of the sequence $(\widetilde{z}_k)_{k=1}^{\infty}$. This together with proposition (c) of Lemma 2.2.5 proves our lemma. □

Remark 2.2.3. Lemma 2.2.7 remains valid if $\widetilde{v}_k$ is a solution of the problem (2.2.62$_k$), (2.2.63$_{ik}$), $x_k \in \mathbb{B}'_{r,k}$ $(k \in \mathbb{N})$ and

$$\lim_{k\to\infty} c_{lk} = c_l \quad (l = 1, 2).$$

Lemma 2.2.8. *Let functions $\mathbb{V}_k \in L_\infty(]a,b[)$ and $H_k \in L([a,b])$ $(k \in \mathbb{N})$ be such that uniformly on $[a,b]$*

$$\lim_{k\to\infty} \int_a^t H_k(s)\,ds = 0, \tag{2.2.73}$$

$$\operatorname{ess\,sup}\big\{|\mathbb{V}_k(t)-\mathbb{V}(t)|:\ \ a\le t\le b\big\} \to 0 \quad as \quad k\to+\infty, \tag{2.2.74}$$

and let there exist a function $\eta \in L([a,b])$ such that everywhere on the interval $]a,b[$

$$|H_k(t)| \le \eta(t) \quad (k \in \mathbb{N}). \tag{2.2.75}$$

Then uniformly on the segment $[a,b]$

$$\lim_{k\to\infty} \int_a^t H_k(s)\mathbb{V}_k(s)\,ds = 0.$$

This lemma is a particular case of Lemma 2.1 from [19].

§ 2.3. Proof of Main Results

2.3.1. Proof of Theorems 2.1.1$_i$, 2.1.2$_i$ $(i = 1, 2)$.

Proof of Theorem 2.1.1$_i$. From the inclusion (2.1.9), by Lemma 1.2.1 we obtain $(p_0, p_1) \in \mathbb{V}_{i,0}(]a,b[)$, which, owing to Lemma 2.2.2 for $k > k_0$, implies $(p_{0k}, p_{1k}) \in \mathbb{V}_{i,0}(]a,b[)$. From Remark 1.2.2 follows the unique solvability of the problems (2.2.61), (2.1.2$_{i0}$) and (2.2.61$_k$), (2.1.2$_{i0}$). Denote by $\widetilde{v}$, $\widetilde{v}_k$ and G, G_k, respectively, solutions and Green's functions of these problems.

Then the problems (2.1.1), ($2.1.2_{i0}$) and ($2.1.1_k$), ($2.1.2_{i0}$) are equivalent, respectively, to the equations

$$u(t) = \mathbb{U}_0(u)(t) + \widetilde{v}(t) \tag{2.3.1}$$

and

$$u(t) = \mathbb{U}_k(u)(t) + \widetilde{v}_k(t), \tag{$2.3.1_k$}$$

where the continuous linear operators $\mathbb{U}_k$, $\mathbb{U}_0 : C(]a,b[) \to C(]a,b[)$ are defined by the equalities

$$\mathbb{U}_0(x)(t) = \int\limits_a^b G(t,s)g(x)(s)\,ds \quad \text{and} \quad \mathbb{U}_k(x)(t) = \int\limits_a^b G_k(t,s)g_k(x)(s)\,ds.$$

If $\rho : [a,b] \to \mathbb{R}^+$ is the function mentioned in the proof of Theorem $1.1.1_i$, then as is seen from that proof, there exists a constant $\lambda_0 \in [0,1[$ such that

$$\|\mathbb{U}_0\|_{C_\rho \to C_\rho} < \lambda_0. \tag{2.3.2}$$

Suppose that the equation

$$u(t) = \mathbb{U}_k(u)(t) \tag{$2.3.1_{0k}$}$$

has a non-zero solution u_{0k}. Not restricting the generality, we assume that

$$\|u_{0k}\|_{C,\rho} = 1 \quad \text{for} \quad k > k_0, \tag{2.3.3}$$

in which case $\|u_{0k}\|_C \le \|\rho\|_C$, i.e., if we introduce the notation $r = \|\rho\|_C$, then

$$u_{0k} \in \mathbb{B}_{rk} \quad \text{for} \quad k > k_0. \tag{2.3.4}$$

Also, from ($2.3.1_{0k}$), (2.3.3), by Lemma 2.2.7 it follows that the sequence $(u_{0k})_{k=1}^{\infty}$ is uniformly bounded and equicontinuous. Hence by the Arzella–Ascoli lemma, not restricting the generality we can assume that there exists a function $u_0 \in C(]a,b[)$ such that uniformly on the segment $[a,b]$

$$\lim_{k\to\infty} u_{0k}(t) = u_0(t). \tag{2.3.5}$$

It is clear from the equations (2.3.3), (2.3.5) that

$$\|u_0\|_{C,\rho} = 1. \tag{2.3.6}$$

Let us now introduce the notation

$$\Delta p_{jk}(t) = p_j(t) - p_{jk}(t) \ \ (j=0,1,2), \quad \Delta G_k(t,s) = G(t,s) - G_k(t,s),$$
$$\Delta g_k(x)(t) = g(x)(t) - g_k(x)(t) \ \ (k \in \mathbb{N}).$$

For u_{0k}, when $k > k_0$, the representation

$$u_{0k}(t) = \mathbb{U}_0(u_{0k})(t) + \int\limits_a^b \Delta G_k(t,s) g(u_{0k})(s)\, ds +$$

$$+ \int\limits_a^b G_k(t,s) \Delta g_k(u_{0k})(s)\, ds \quad (k \in \mathbb{N}) \quad \text{for} \quad a \le t \le b \tag{2.3.7}$$

is valid. Taking into account (2.3.4), (2.3.5), Remark 2.2.2, equality the (2.2.43) of Lemma 2.2.3 and also the equality (2.2.53) of Lemma 2.2.4 with all conditions satisfied, and then passing in (2.3.7) to limit as $k \to +\infty$, we get

$$u_0(t) = \mathbb{U}_0(u_0)(t)$$

which, with regard for (2.3.2), (2.3.6), results in the estimate

$$\|u_0\|_{C,\rho} < 1.$$

But this contradicts (2.3.6). Hence our assumption is invalid and the equation $(2.3.1_{0k})$ has only the zero solution, and because of its Fredholm property the equation $(2.3.1_k)$ is uniquely solvable. The unique solvability of the equation (2.3.1) follows from Theorem $1.1.1_i$.

Let u and u_k be respectively solutions of the equations (2.3.1) and $(2.3.1_k)$,

$$\begin{aligned} w_k(t) &= u(t) - u_k(t) \quad \text{for} \quad k > k_0, \\ \lambda_k &= \begin{cases} \|u_k\|_{C,\rho} & \text{for} \quad \|u_k\|_{C,\rho} > 1, \\ 1 & \text{for} \quad \|u_k\|_{C,\rho} \le 1, \end{cases} \\ \widetilde{u}_k(t) &= \lambda_k^{-1} u_k(t) \end{aligned} \tag{2.3.8}$$

and

$$\rho_k(t) = \frac{\widetilde{v}(t) - \widetilde{v}_k(t)}{\lambda_k} + \int\limits_a^b \Delta G_k(t,s) g(\widetilde{u}_k)(s)\, ds + \int\limits_a^b G_k(t,s) \Delta g_k(\widetilde{u}_k)(s)\, ds.$$

Then for w_k the representation

$$w_k(t) = \mathbb{U}_0(w_k)(t) + \lambda_k \rho_k(t) \quad \text{for} \quad a \le t \le b \tag{2.3.9}$$

is valid, and if $r = \|\rho\|_C$, then

$$\widetilde{u}_k \in \mathbb{B}_{r,k}. \tag{2.3.10}$$

In such a case, taking into account proposition (a) of Lemma 1.2.6, Remark 2.2.2, the equation (2.2.43) of Lemma 2.2.3 and also the equation (2.2.53) of Lemma 2.2.4, we obtain

$$\lim_{k\to\infty} \|\rho_k\|_{C,\rho} = 0. \tag{2.3.11}$$

On the other hand, from (2.3.9), with regard for (2.3.2), we get the estimate

$$\|w_k\|_{C,\rho} \le \alpha_k \lambda_k \quad \text{for} \quad k > k_0, \tag{2.3.12}$$

where

$$\alpha_k = \frac{\|\rho_k\|_{C,\rho}}{1-\lambda_0}$$

and by virtue of (2.3.11),

$$\lim_{k\to\infty} \alpha_k = 0. \tag{2.3.13}$$

Suppose now that we can extract from the sequence $(\lambda_k)_{k=1}^{\infty}$ a sequence $(\lambda_{k_m})_{m=1}^{\infty}$ such that $\lambda_{k_m} \ge 1$ for $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \lambda_{k_m} = +\infty, \tag{2.3.14}$$

and note that by our definition of the function w_k the inequality

$$\lambda_{k_m} - \|u\|_{C,\rho} \le \|w_{k_m}\|_{C,\rho} \tag{2.3.15}$$

is valid. Substituting now the inequality (2.3.12) in (2.3.15) and taking into account (2.3.13), we can see that this contradicts (2.3.14), i.e., our assumption is invalid, and there exists a constant $\lambda \in \mathbb{R}^+$ such that

$$\lambda_k \le \lambda \quad \text{for} \quad k > k_0 \tag{2.3.16}$$

which, with regard for (2.3.12), yields

$$\lim_{k\to\infty} \|w_k\|_{C,\rho} = 0. \tag{2.3.17}$$

Now we notice that (2.3.9) and (2.3.16) imply

$$|w_k^{(j)}(t)| \le \frac{d^j}{dt^j}\, \mathbb{U}_0(w_k)(t) + \lambda|\rho_k^{(j)}(t)| \;\; (j=0,1) \quad \text{for} \quad a < t < b. \tag{2.3.18$_j$}$$

Applying the estimates (2.2.46)–(2.2.48) and the inequalities (2.2.13), (2.2.10), we arrive at

$$\big|\mathbb{U}_0(w_k)(t)\big| \le r'\|w_k\|_{C,\rho} I_i^{1-\mu}\big(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)\big)(t) \quad \text{for} \quad a \le t \le b, \tag{2.3.19}$$

$$\frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)} \Big|\frac{d}{dt}\, \mathbb{U}_0(w_k)(t)\Big| \le r'\|w_k\|_{C,\rho} \quad \text{for} \quad a < t < b, \tag{2.3.20}$$

where

$$r' = \frac{d_1^2}{d_2} \int\limits_a^b \frac{h(\rho)(s)}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\, ds.$$

By definition of the function $\widetilde{u}_k$, in view of the inequality (2.1.10) and the equalities (2.2.43), (2.2.44) of Lemma 2.2.3, we make sure that uniformly on the interval $]a,b[$

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1))(t)\left|\int\limits_a^b \Delta G_k(t,s) g(\widetilde{u}_k)(s)\, ds\right| = 0 \tag{2.3.21}$$

and

$$\lim_{k\to\infty} \frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)} \left|\int\limits_a^b \frac{d\Delta G_k(t,s)}{dt}\, g(\widetilde{u}_k)(s)\, ds\right| = 0. \tag{2.3.22}$$

Just in the same way, taking into account the inclusion (2.3.10) and the equalities (2.2.53), (2.2.54) of Lemma 2.2.4, we can see that

$$\left|\int\limits_a^b G_k(t,s)\Delta g_k(\widetilde{u}_k)(s)\, ds\right| \le$$

$$\le r_1 \sup\left\{\left|\int\limits_a^t \frac{\Delta g_k(x)(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\, ds\right| :\ a \le t \le b,\ \ x \in \mathbb{B}_{r,k}\right\} \times$$

$$\times I_i^{1-\mu}(\sigma^{\alpha}(p_1))(t) \quad \text{for} \quad a \le t \le b, \tag{2.3.23}$$

$$\frac{I_i^{\mu}(\sigma^{\alpha}(p_1))(t)}{\sigma(p_1)(t)} \left|\int\limits_a^b \frac{d}{dt}\, G_k(t,s)\Delta g_k(\widetilde{u}_k)(s)\, ds\right| \le$$

$$\le r_1 \sup\left\{\left|\int\limits_a^t \frac{\Delta g_k(x)(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s)\, ds\right| :\right.$$

$$\left. a \le t \le b,\ \ x \in \mathbb{B}_{r,k}\right\} \quad \text{for} \quad a < t < b. \tag{2.3.24}$$

It is clear from the equalities (2.3.21)–(2.3.24), proposition (a) of Lemma 2.2.5 and also from the condition (2.1.8) and Remark 2.2.2 that uniformly on the interval $]a,b[$

$$\lim_{k\to\infty} I_i^{\mu-1}(\sigma^{\alpha}(p_1))(t)\rho_k(t) = 0 \tag{2.3.25}$$

and

$$\lim_{k\to\infty} \frac{\rho_k(t)}{\sigma(p_1)(t)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(t) = 0. \tag{2.3.26}$$

Multiplying (2.3.18$_0$) by $I_i^{\mu-1}(\sigma^\alpha(p_1))(t)$ and taking into consideration (2.3.17), (2.3.19) and (2.3.25) we see that the condition (2.1.11) is valid. Analogously, multiplying (2.3.18$_1$) by $\sigma^{-1}(p_1)(t)I_i^\mu\sigma^\alpha(p_1)(t)$ and taking into account (2.3.17), (2.3.20) and (2.3.26), we make sure that the condition (2.1.12) is valid. $\square$

Proof of Theorem 2.1.2$_i$. Reasoning in the same way as in the previous proof for the function $w_k(t) = u(t) - u_k(t)$, where u_k is a solution of the problem (2.1.1$_k$), (2.1.2$_{ik}$), using Remark 2.2.3 and proposition (b) of Lemma 2.2.6, we get the equality (2.3.17) which is the same as the condition (2.1.15). The proof of the condition (2.1.12) coincides completely with its proof in Theorem 2.1.1$_i$. $\square$

2.3.2. Proof of Corollaries.

Proof of Corollary 2.1.1$_i$. It is sufficient to show that (2.1.8) follows from (2.1.16)–(2.1.18). Suppose to the contrary that the condition (2.1.18) is violated. Then there exist $\varepsilon > 0$, a sequence of positive numbers $(k_m)_{m=1}^\infty$ and a sequence of functions

$$y_m \in \mathbb{B}_{k_m} \tag{2.3.27}$$

such that

$$\max\left\{\Big|\int_a^t \frac{\Delta g_{k_m}(y_m)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \ a \le t \le b\right\} > \varepsilon. \tag{2.3.28}$$

From (2.3.27) it follows

$$y_m(t) = \alpha_{1m}\widetilde{v}_{k_m}(t) + \int_a^b G_{k_m}(t,s)g_{k_m}(x_m)(s)\,ds \ \ (m \in \mathbb{N}), \tag{2.3.29}$$

where $x_m \in C(]a,b[)$ $(m \in \mathbb{N})$ and

$$0 \le \alpha_{1m} \le 1 \quad (m \in \mathbb{N}), \tag{2.3.30}$$

$$\|x_m\|_C \le 1 \quad (m \in \mathbb{N}). \tag{2.3.31}$$

Introduce the notation

$$z_m(t) = \int_a^b G_{k_m}(t,s)g_{k_m}(x_m)(s)\,ds \quad (m \in \mathbb{N})$$

and rewrite z_m as follows:

$$z_m(t) = \int_a^b G_{k_m}(t,s)\Delta g_{k_m}(x_m)(s)\,ds + \int_a^b G_{k_m}(t,s)g(x_m)(s)\,ds.$$

Then according to (2.1.10), (2.1.16), and (2.1.31) the inequality

$$|z_m^{(j)}(t)| \le \int_a^b \Big|\frac{\partial^j}{\partial t^j}\,\Delta G_{k_m}(t,s)\Big|\big(\eta(s)+h(1)(s)\big)\,ds+$$

$$+\int_a^b \Big|\frac{\partial^j}{\partial t^j}\,G(t,s)\Big|\big(\eta(s)+h(1)(s)\big)\,ds \quad (j=0,1) \qquad (2.3.32_j)$$

is valid. By the conditions (2.1.6) and (2.1.18),

$$\int_a^b \frac{\eta(s)+h(1)(s)}{\sigma(p_1)(s)}\,I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\,ds < +\infty$$

owing to which from ($2.3.32_0$), in view of the equality (2.2.43) of Lemma 2.2.3 and by Lemma 2.2.5 we obtain the existence of a constant λ_1 such that

$$\|z_m\|_C < \lambda_1 \quad (m\in\mathbb{N}). \qquad (2.3.33)$$

Consider now the case $i=1$ separately. From ($2.3.32_j$) ($j=0,1$), by Lemmas 2.2.3 and 2.2.5 and the fact that

$$G(a,s)=G(b,s)=0 \quad \text{for} \quad a<s<b$$

we can choose for any $\varepsilon_0>0$ constants m_0, a_1, b_1, δ, where

$$a<a_1<b_1<b, \quad \delta<\min(a_1-a,b-b_1),$$

such that

$$|z_m(t)| \le \frac{\varepsilon_0}{4}, \quad m>m_0 \quad \text{for} \quad a\le t\le a_1, \quad b_1\le t\le b,$$

i.e.,

$$\big|z_m(t_1)-z_m(t_2)\big| \le \frac{\varepsilon_0}{2}, \quad m>m_0, \quad \text{for} \quad a\le t_1,t_2\le a_1, \quad b_1\le t_1,t_2\le b, \qquad (2.3.34)$$

and $A\delta<\frac{\varepsilon_0}{2}$, where

$$A=\sup\big\{|z_m'(t)|:\quad a_1-\delta<t<b_1+\delta, \quad m>m_0\big\}<+\infty,$$

i.e.,

$$\begin{gathered}\big|z_m(t_1)-z_m(t_2)| \le A|t_1-t_2| < \frac{\varepsilon_0}{2}\,, \quad m>m_0 \\ \text{for} \quad a_1-\delta<t_1,t_2<b_1+\delta, \quad |t_1-t_2|<\delta.\end{gathered} \qquad (2.3.35)$$

The uniform boundedness and equicontinuity of the sequence $(z_m)_{m=1}^{\infty}$ follows from (2.3.33)–(2.3.35). Then by the Arzella–Ascoli lemma, not restricting the generality, we assume that uniformly on the segment $[a,b]$

$$\lim_{m\to\infty} z_m(t)=z(t). \qquad (2.3.36)$$

Notice now that however close may be a_1 from a and b_1 from b, the inequality (2.3.35) remains valid if we choose δ sufficiently small. Therefore, passing in (2.3.35) to limit, we can see that z is absolutely continuous on any segment contained in $]a,b[$, i.e.,

$$z \in \widetilde{C}_{loc}(]a,b[) \cap C([a,b]). \tag{2.3.37}$$

On the other hand, in view of (2.3.30), not restricting the generality, we can assume that

$$\lim_{m\to\infty} \alpha_{1m} = \alpha_0,$$

which together with proposition (a) of Lemma 2.2.6 implies

$$\lim_{m\to\infty} \alpha_{1m}\widetilde{v}_{k_m}(t) = \alpha_0\widetilde{v}(t) \quad \text{uniformly on} \quad [a,b], \tag{2.3.38}$$

where $\widetilde{v}$ is a solution of the problem (2.2.62), ($2.1.2_{i0}$).

Further, taking into account (2.3.36)–(2.3.38) in (2.3.29), we conclude that uniformly on the segment $[a,b]$

$$\lim_{m\to\infty} y_m(t) = y(t), \tag{2.3.39}$$

where

$$y \in \widetilde{C}_{loc}(]a,b[) \cap C([a,b]). \tag{2.3.40}$$

The same takes place in the case $i=2$ owing to the fact that the relations

$$G(a,s) = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\, G(t,s)\Big|_{t=b} = 1 \quad \text{for} \quad a < s < b$$

follow from the inequalities

$$\big|z_m(t_1) - z_m(t_2)\big| \le \frac{\varepsilon_0}{2}, \quad m > m_0 \quad \text{for} \quad a \le t_1, t_2 \le a_1$$

and

$$\big|z_m(t_1) - z_m(t_2)\big| \le A_1|t_1 - t_2| \le \frac{\varepsilon_0}{2}, \quad m > m_0$$
$$\text{for} \quad a_1 - \delta < t_1, t_2 \le b, \quad |t_1 - t_2| < \delta$$

with

$$A_1 = \sup\big\{|z_m'(t)| : \quad a_1 - \delta < t < b, \quad m > m_0\big\} < +\infty,$$

and from the condition (2.3.38).

Finally, the conditions (2.1.16)–(2.1.18) and (2.3.39) imply

$$\max\bigg\{\Big|\int\limits_a^t \frac{\Delta g_{k_m}(y_m)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \quad a \le t \le b\bigg\} \le$$

$$\le \max\bigg\{\Big|\int\limits_a^t \frac{\Delta g_{k_m}(y_m - y)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \quad a \le t \le b\bigg\} +$$

$$+\max\Big\{\Big|\int\limits_a^t \frac{\Delta g_{k_m}(y)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \ a \le t \le b\Big\} \le$$

$$\le \int\limits_a^b \frac{\eta(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\,\|y_m - y\|_C +$$

$$+\max\Big\{\Big|\int\limits_a^t \frac{\Delta g_{k_m}(y)(s)}{\sigma(p_1)(s)}\, I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\Big| : \ a \le t \le b\Big\} \to 0$$

$$\text{as } m \to +\infty.$$

But this contradicts (2.3.28) and proves the validity of our corollary. □

Proof of Corollary $2.1.2_i$. Coincides completely with that of the previous corollary with the only difference that the functions $\widetilde{v}_k$ and $\widetilde{v}$ in (2.3.38) are solutions of the problems $(2.2.62_k)$, $(2.2.2_{ik})$ and (2.2.62), $(2.1.2_i)$, respectively, where the validity of the equality (2.3.38) follows from proposition (b) of Lemma 2.2.6. □

Proof of Corollary $2.1.3_i$. It can be easily verified that under the notation

$$\begin{aligned} g(x)(t) &= \sum_{m=1}^n g_{0m}(s)x(\tau_{0m}(t)),\\ g_k(x)(t) &= \sum_{m=1}^n g_{km}(t)x(\tau_{km}(t)) \end{aligned} \tag{2.3.41}$$

all the requirements of Theorem $2.1.1_i$, except for (2.1.8), are satisfied.

First we show the existence of a constant λ_1 such that

$$\sup\Big\{\Big\|\frac{y'}{\sigma(p_1)}\, I_i^\mu(\sigma^\alpha(p_1))\Big\|_C : \ y \in \mathbb{B}_{1k}, \ k > k_0\Big\} \le \lambda_1. \tag{2.3.42}$$

To this end we choose arbitrarily $k_1 > k_0$ and $y_1 \in \mathbb{B}_{k_1}$. Then there exist $\alpha_1 < 1$, $x_1 \in C(]a,b[)$, $\|x_1\|_C \le 1$ such that

$$y_1(t) = \alpha_1 \widetilde{v}_{k_1}(t) + \int\limits_a^b G_{k_1}(t,s) g_{k_1}(x_1)(s)\,ds,$$

where $\widetilde{v}_{k_1}$ is a solution of the problem $(2.2.62_k)$, $(2.1.2_{i0})$. Next,

$$|y_1'(t)| \le |\widetilde{v}_{k_1}'(t)| + \int\limits_a^b \Big|\frac{\partial G_{k_1}(t,s)}{\partial t}\Big|\eta(s)\,ds +$$

$$+ \int\limits_a^b \Big|\frac{\partial G(t,s)}{\partial t}\Big| h(1)(s)\,ds \quad \text{for} \quad a < t < b.$$

By virtue of the equality (2.2.67) of Lemma 2.2.6, there exists a constant λ_2 such that for any $k \geq k_0$

$$\Big\| \frac{\widetilde{v}_k'}{\sigma(p_1)}\, I_i^{\mu}(\sigma^{\alpha}(p_1)) \Big\|_C < \lambda_2. \tag{2.3.43}$$

Taking into account (2.3.43), the representation (2.2.45) of Green's function the estimates (2.2.46)–(2.2.48), the inequality (2.2.13) and the conditions (2.1.18), (2.2.20) and (2.2.21), we make sure that the estimate (2.3.42) is valid, where

$$\lambda_1 = \lambda_2 + \frac{d_1^2}{d_2^2} \Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu}{1-\mu}}(p_1)(s)ds \Big)^{1-\mu} \Big(\int\limits_a^b \frac{\eta(s)+h(1)(s)}{\sigma(p_1)(s)}\, I_i^{\mu}(\sigma^{\alpha}(p_1))(s)\,ds \Big).$$

We now notice that if

$$\lim_{k\to\infty} \bigg(\sup \bigg\{ \sum_{m=1}^{n} \Big| \int\limits_a^t \frac{g_{0m}(s)-g_{km}(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s) y(\tau_{km}(s))\,ds \Big| : \\ a \leq t \leq b,\ \ y \in \mathbb{B}_{1k} \bigg\} \bigg) = 0 \tag{2.3.44}$$

and

$$\lim_{k\to\infty} \bigg(\sup \bigg\{ \sum_{m=1}^{n} \Big| \int\limits_a^t \frac{g_{0m}(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s) \int\limits_{\tau_{km}(s)}^{\tau_{0m}(s)} y'(\eta)\ d\eta\,ds \Big| : \\ a \leq t \leq b,\ \ y \in \mathbb{B}_{1k} \bigg\} \bigg) = 0, \tag{2.3.45}$$

then the condition (2.1.8) is satisfied.

Reasoning analogously to the proof of Corollary $2.1.1_i$, we obtain that (2.3.44) is satisfied if for any $y \in \widetilde{C}_{loc}(]a,b[) \cap C([a,b])$

$$\lim_{k\to\infty} \bigg(\sum_{m=1}^{n} \Big| \int\limits_a^t \frac{g_{0m}(s)-g_{km}(s)}{\sigma(p_1)(s)}\, I_i^{\beta}(\sigma^{\alpha}(p_1))(s) y(\tau_{km}(s))\,ds \Big| \bigg) = 0. \tag{2.3.46}$$

On the other hand, from (2.1.23) it follows that

$$\operatorname{ess\,sup} \bigg\{ \sum_{m=1}^{n} \big| \tau_{0m}(t) - \tau_{km}(t) \big| : \ \ a \leq t \leq b \bigg\} \to 0 \quad \text{as} \quad k \to +\infty,$$

and hence for every $y \in \widetilde{C}_{loc}(]a,b[) \cap C([a,b])$

$$\operatorname{ess\,sup} \bigg\{ \sum_{m=1}^{n} \big| y(\tau_{km}(t)) - y(\tau_{0m}(t)) \big| : a \leq t \leq b \bigg\} \to 0 \\ \text{as}\ \ k \to +\infty. \tag{2.3.47}$$

Then (2.1.21), (2.1.22), and (2.3.47) and lemma 2.2.8 imply the validity of the equality (2.3.46).

The validity of the equality (2.3.45) follows from the estimate (2.3.42), the condition (2.1.23) and the inequalities

$$\Big|\int\limits_a^t \frac{g_{0m}(s)}{\sigma(p_1)(s)}\,I_i^{\beta}(\sigma^\alpha(p_1))(s)\int\limits_{\tau_{km}(s)}^{\tau_{0m}(s)} y'(\eta)\,d\eta\,ds\Big| \le$$

$$\le \int\limits_a^b \frac{|g_{0m}(s)|}{\sigma(p_1)(s)}\,I_i^{\mu}(\sigma^\alpha(p_1))(s)\,ds\,\times$$

$$\times\,\operatorname{ess\,sup}\left\{I_i^{\beta-\mu}(\sigma^\alpha(p_1))(t)\Big|\int\limits_{\tau_{km}(t)}^{\tau_{0m}(t)}\frac{\sigma(p_1)(s)ds}{I_i^{\mu}(\sigma(p_1))(s)}\Big|:\ \ a\le t\le b\right\}\times$$

$$\times\Big\|\frac{y'}{\sigma(p_1)}\,I_i^{\mu}(\sigma^\alpha(p_1))\Big\|_C\quad (m=1,\dots,n;\ \ k\in\mathbb{N})\quad\text{for}\quad a\le t\le b.\qquad\square$$

Proof of Corollary $2.1.4_i$. Coincides with the previous proof with the only difference that in the inequality (2.3.42) we will assume that $y\in\mathbb{B}'_{1k}$, i.e., the validity of (2.3.43) with $\widetilde{v}_k$ as a solution of the problem $(2.1.4_k)$, $(2.1.2_{ik})$ will be shown by means of proposition (b) of Lemma 2.2.6. $\square$

Proof of Corollary $2.1.5_i$. It is not difficult to notice that the conditions (2.1.18), (2.1.25) yield

$$\int\limits_a^b \frac{|g_{0m}(s)|}{\sigma(p_1)(s)}\,I_i^{\beta}(\sigma^\alpha(p_1))(s)ds<+\infty\quad(m=1,\dots,n),\qquad(2.3.48)$$

whence, owing to the fact that $\beta<\mu$, together with (2.1.24), we obtain the validity of the conditions (2.1.20), (2.1.21). That is, as it has been shown in the proof of Lemma $2.1.3_i$, all the requirements of Theorem $2.1.1_i$, except for (2.1.8), are satisfied.

On the other hand, the condition (2.1.8) under the notation (2.3.41) follows from the conditions (2.3.44), (2.3.45). Repeating now word by word the proof of Corollary $2.1.3_i$, by the condition (2.1.26) we can see that (2.3.42) and (2.3.44) are valid.

Choosing $\mu_1>\mu$ so as to satisfy

$$\mu_1<1,\quad \frac{1-\alpha\mu_1}{1-\mu_1}\le\delta,$$

analogously to the inequalities (2.2.15),(2.2.16) we obtain

$$\int\limits_a^b \frac{\sigma(p_1)(s)}{I_i^\mu(\sigma^\alpha(p_1))(s)}\,ds \le \Big(2I_1^\mu(\sigma^\alpha(p_1))\Big(\frac{a+b}{2}\Big)\Big)^{2-i}\Big(\frac{\mu_1}{\mu_1-\mu}\Big)\times$$
$$\times\Big(\int\limits_a^b \sigma^{\frac{1-\alpha\mu_1}{1-\mu_1}}(p_1)(s)\,ds\Big)^{1-\mu_1}\Big(\int\limits_a^b \sigma^\alpha(p_1)(s)\,ds\Big)^{\mu_1-\mu} < +\infty.$$

From this and also from the condition (2.1.26), owing to the absolute continuity of the Lebesgue integral it follows that

$$\operatorname{ess\,sup}\Big\{\sum_{m=1}^n\Big|\int\limits_{\tau_{km}(t)}^{\tau_{0m}(t)}\frac{\sigma(p_1)(s)}{I_i^\mu(\sigma^\alpha(p_1))(s)}\,ds\Big|:\ \ a\le t\le b\Big\}\to 0 \tag{2.3.49}$$
$$\text{for}\quad k\to+\infty.$$

Then the validity of the equality (2.3.45) follows from the conditions (2.3.48), (2.3.49) and also from the estimate (2.3.42) and the inequality

$$\Big|\int\limits_a^t \frac{g_{0m}(s)}{\sigma(p_1)(s)}\,I_i^\beta(\sigma^\alpha(p_1))(s)\int\limits_{\tau_{km}(s)}^{\tau_{0m}(s)} y'(\eta)\,d\eta\,ds\Big| \le$$
$$\le\Big|\int\limits_a^b \frac{|g_{0m}(s)|}{\sigma(p_1)(s)}\,I_i^\beta(\sigma^\alpha(p_1))(s)\,ds\,\times$$
$$\times\operatorname{ess\,sup}\Big\{\Big|\int\limits_{\tau_{km}(t)}^{\tau_{0m}(t)}\frac{\sigma(p_1)(s)}{I_i^\mu(\sigma^\alpha(p_1))(s)}\,ds\Big|:\ \ a\le t\le b\Big\}\times$$
$$\times\Big\|\frac{y'}{\sigma(p_1)}\,I_i^\mu(\sigma^\alpha(p_1))\Big\|_C\quad (m=1,\dots,n;\ \ k\in\mathbb{N}).\qquad\square$$

Proof of Corollary 2.1.6$_i$. Coincides with the previous proof with the only difference that in the inequality (2.3.42) it will be assumed that $y\in\mathbb{B}'_{1k}$, i.e., the validity of the inequality (2.3.43) with $\widetilde{v}_k$ as a solution of the problem (2.1.4$_k$), (2.1.2$_{ik}$) will be shown by means of proposition (b) of Lemma 2.2.6. $\square$

Proof of Corollary 2.1.7$_i$ (2.1.8$_i$). It is easily seen that for any $\alpha\in[0,1]$ and $\gamma>1$, by conditions (2.1.28)–(2.1.32) ((2.1.28)–(2.1.32), (2.1.14)), all the requirements of Corollary (2.1.5$_i$) ((2.1.6$_i$)) are satisfied for $p_j\equiv 0$, $p_{jk}\equiv 0$ ($j=0,1$; $k\in\mathbb{N}$), $n=1$, whence it follows that our corollary is valid. $\square$

References

1. N. B. Azbelev, V. P. Maximov, and L. F. Rakhmatulina, Introduction to the theory of functional differential equations. (Russian) *Nauka, Moscow*, 1991.

2. N. B. Azbelev, M. J. Alves, and E. I. Bravyii, On singular boundary value problems for linear functional differential equations of second order. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* **2**(1996), 3–11.

3. M. Zh. Alves, On the solvability of two-point boundary value problem for a singular nonlinear functional differential equation. *Izv. Vyssh. Uchebn. Zaved. Mat.* **2**(1996), 12–19.

4. M. Alves, About a problem arising in chemical reactor theory. *Mem. Differential Equations Math. Phys.* **19**(2000), 133–141.

5. P. B. Bailey, L. F. Shampine, and P. E. Waltman, Nonlinear two-point boundary value problems. *Academic Press, New York*, 1968.

6. S. N. Bernstein, On equations of variational calculus. (Russian) *Uspekhi Mat. Nauk* **8**(1940), No. 1, 32–74.

7. C. De la Vallée Poussin, Sur l'equation differentielle lineaire du second ordre. Determination d'une integral pur deux valeurs assignées. Extension aux equations d'ordren. *J. Math. Pures Appl.* (9) **8**(1929), No. 2, 125–144.

8. N. I. Vasil'ev and Yu. A. Klokov, Fundamentals of the theory of boundary value problems for ordinary differential equations. (Russian) *Zinatne, Riga*, 1978.

9. V. V. Gudkov, Yu. A. Klokov, A. Ja. Lepin, and V. D. Ponomarev, Two-point boundary value problems for ordinary differential equations. (Russian) *Zinatne, Riga*, 1973.

10. H. Epheser, Über die existenz der lösungen von randwertaufgaben mit gewöhnlichen nichtlinearen differentialgleichungen zweiter ordnung. *Math. Z.* **61**(1955), No. 4, 435–454.

11. L. V. Kantorovich and G. P. Akimov, Functional analysis. (Russian) *Nauka, Moscow*, 1990.

12. I. T. Kiguradze, On a priori estimates of bounded functions derivatives satisfying differential inequalities of second order. (Russian) *Differentsial'nye Uravneniya* **3**(1967), No. 7, 1043–1052.

13. I. T. Kiguradze, On some singular boundary value problems for nonlinear ordinary differential equations of second order. (Russian) *Differentsial'nye Uravneniya* **4**(1968), No. 10, 1753–1773.

14. I. T. Kiguradze, On a singular two-point boundary value problem. (Russian) *Differentsial'nye Uravneniya* **5**(1969), No. 11, 2002–2016.

15. I. T. Kiguradze, On oscillatory conditions for singular linear differential equations of second order. (Russian) *Mat. Zametki* **6**(1969), No. 5, 633–639.

16. I. T. Kiguradze, On a singular boundary value problem. *J. Math. Anal. Appl.* **30**(1970), No. 3, 475–489.

17. I. T. Kiguradze, Certain singular boundary value problems for ordinary differential equations. (Russian) *Tbilisi University Press, Tbilisi*, 1975.

18. I. T. Kiguradze and A. G. Lomtatidze, On certain boundary value problems for second order linear ordinary differential equations with singularities. *J. Math. Anal. Appl.* **10**(1984), No. 2, 325–347.

19. I. T. Kiguradze and B. Půža, On boundary value problems for systems of linear functional differential equations. *Czechoslovak Math. J.* **47**(122), 1997.

20. I. T. Kiguradze and B. Půža, On a certain singular boundary value problem for linear differential equations with deviating arguments. *Czechoslovak Math. J.* **47**(1997), No. 2, 233–244.

21. I. T. Kiguradze and B. Půža, On the solvability of boundary value problems for systems of nonlinear differential equations with deviating arguments. *Mem. Differential Equations Math. Phys.* **10**(1997), 157–161.

22. I. T. Kiguradze and B. Půža, On boundary value problems for functional differential equations. *Mem. Differential Equations Math. Phys.* **12**(1997), 106–113.

23. I. T. Kiguradze and B. L. Shekhter, Singular boundary value problems for ordinary differential equations of second order. (Russian) *Sovrem. Probl. Mat. Noveishie Dostizhenija* **30**(1987), 105–201.

24. Yu. A. Klokov, The method of solution of limiting boundary value problem for the ordinary differential equation of second order. (Russian) *Mat. Sb.* **53**(1961), No. 2, 219–232.

25. M. A. Krasnosel'sky, On one boundary value problem. (Russian) *Izv. Akad. Nauk SSSR, Ser. Mat.* **20**(1956), No. 2, 241–252.

26. A. G. Lomtatidze, On one singular three-point boundary value problem. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **17**(1986), 122–134.

27. A. G. Lomtatidze, On one boundary value problem for linear ordinary differential equations of second order with singularities. (Russian) *Differentsial'nye Uravneniya* **22**(1986), No. 3, 416–426.

28. A. G. Lomtatidze, On positive solutions of singular boundary value problems for nonlinear ordinary differential equations of second order. *Differentsial'nye Uravneniya* **22**(1986), No. 6, 1092.

29. A. G. Lomtatidze, On one boundary value problem for linear differential equations with nonintegrable singularities. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **14**(1983), 136–144.

30. A. G. Lomtatidze, On one singular boundary value problem for linear equations of second order. *In: Boundary value problems. Perm, Perm Polytechnical Inst.*, 1984, 46–50.

31. A. G. Lomtatidze, On the solvability of boundary value problems for nonlinear ordinary differential equations of second order with singularities. (Russian) *Reports of the Extended Session of a Seminar of the I. N.*

Vekua Institute of Applied Mathematics, Vol. 1, No. 3 (*Russian*) (*Tbilisi*, 1985), 85–92, *Tbiliss. Gos. Univ., Tbilisi*, 1985.

32. A. G. Lomtatidze, On oscillatory properties of solutions of linear differential equations of second order. (Russian) *Reports of the Extended Session of a Seminar of the I. N. Vekua Institute of Applied Mathematics, Vol.* 19, (*Russian*) (*Tbilisi*, 1985), 39–53, *Tbiliss. Gos. Univ., Tbilisi*, 1985.

33. A. G. Lomtatidze, On positive solutions of boundary value problems for second order ordinary differential equations with singularities. (Russian) *Differentsial'nye Uravneniya* **23**(1987), No. 10, 1685–1692.

34. A. Lomtatidze and L. Malaguti, On a nonlocal boundary value problem for second order nonlinear singular equations. *Georgian Math. J.* **7**(2000), No. 1, 133–154.

35. A. G. Lomtatidze and L. Malaguti, On a two-point boundary value problem for second order differential equations with singularities. *Nonlin. Anal.* (*Accepted for publication*).

36. A. G. Lomtatidze and P. Torres, On a two-point boundary value problem for second order singular equations. *Czechoslovak Math. J.* (*Accepted for publication*).

37. A. G. Lomtatidze and S. Mukhigulashvili, On periodic solutions of second order functional differential equations. *Mem. Differential Equations Math. Phys.* **5**(1995), 125–126.

38. A. G. Lomtatidze and S. Mukhigulashvili, On a two-point boundary value problem for second order differential equations. *Mem. Differential Equations Math. Phys.* **10**(1997), 125–128.

39. A. G. Lomtatidze and S. Mukhigulashvili, On a two-point boundary value problem for second order functional differential equations. *Mem. Differential Equations Math. Phys.* **10**(1997), 150–152.

40. A. G. Lomtatidze and S. Mukhigulashvili, Some two-point boundary value problems for second order functional differential equations. *Folia Fac. Sci. Natur. Univ. Masaryk Brun. Math.* (*Accepted for publication*).

41. Mukhigulashvili, On a two-point boundary value problem for second order functional differential equations. *Mem. Differential Equations Math. Phys.* **6**(1997), 124–126.

42. M. Nagumo, Über die differentialgleichung $y'' = f(t, y, y')$. *Proc. Phys. Math. Soc. Japan* **19**(1937), 861–866.

43. A. I. Perov, On a two-point boundary value problem. (Russian) *Dokl. Akad. Nauk SSSR* **122**(1958), No. 6, 982–985.

44. B. Půža, On a singular two-point boundary value problem for the nonlinear m-th order differential equations with deviating argument. *Georgian Math. J.* **4**(1997), No. 6, 557–566.

45. G. H. Ruder, Boundary value problems for a class of nonlinear differential equations. *Pacific J. Math.* **22**(1967), No. 3, 477–503.

46. J. Sansone, Ordinary differential equations. (Russian) *Inostr. Literat., Moscow,* **2**(1954).

Electronic Journal of Qualitative Theory of Differential Equations
2012, No. **38**, 1-34; **http://www.math.u-szeged.hu/ejqtde/**

Two-Point Boundary Value Problems For Strongly Singular Higher-Order Linear Differential Equations With Deviating Arguments

Sulkhan Mukhigulashvili* and Nino Partsvania

Abstract

For strongly singular higher-order differential equations with deviating arguments, under two-point conjugated and right-focal boundary conditions, Agarwal-Kiguradze type theorems are established, which guarantee the presence of Fredholm's property for the above mentioned problems. Also we provide easily verifiable best possible conditions that guarantee the existence of a unique solution of the studied problems.

2000 Mathematics Subject Classification: 34K06, 34K10

Key words and phrases: Higher order differential equation, linear, deviating argument, strong singularity, Fredholm's property.

1 Statement of the main results

1.1. Statement of the problems and the basic notations. Consider the differential equations with deviating arguments

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for} \quad a < t < b, \tag{1.1}$$

with the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \ (i = 1, \cdots, m), \quad u^{(j-1)}(b) = 0 \ (j = 1, \cdots, n-m), \tag{1.2}$$

$$u^{(i-1)}(a) = 0 \ (i = 1, \cdots, m), \quad u^{(j-1)}(b) = 0 \ (j = m+1, \cdots, n). \tag{1.3}$$

Here $n \geq 2$, m is the integer part of $n/2$, $-\infty < a < b < +\infty$, $p_j, q \in L_{loc}(]a, b[)$ $(j = 1, \cdots, m)$, and $\tau_j \, :]a, b[\to]a, b[$ are measurable functions. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we denote the right (the left) limit of the function $u^{(j-1)}$ at the point $a\,(b)$. Problems (1.1), (1.2), and (1.1), (1.3) are said to be singular if some or all the coefficients of (1.1) are non-integrable on $[a, b]$, having singularities at the end-points of this segment.

*Corresponding author.

The linear ordinary differential equations and differential equations with deviating arguments with boundary conditions (1.2) and (1.3), and with the conditions

$$\int_a^b (s-a)^{n-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+ds < +\infty,$$
$$\int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)|ds < +\infty \quad (j = 2, \cdots, m), \tag{1.4}$$
$$\int_a^b (s-a)^{n-m-1/2}(b-s)^{m-1/2}|q(s)|ds < +\infty,$$

and

$$\int_a^b (s-a)^{n-1}[(-1)^{n-m}p_1(s)]_+ds < +\infty,$$
$$\int_a^b (s-a)^{n-j}|p_j(s)|ds < +\infty \quad (j = 2, \cdots, m), \tag{1.5}$$
$$\int_a^b (s-a)^{n-m-1/2}|q(s)|ds < +\infty,$$

respectively, were studied by I. Kiguradze, R. P. Agarwal and some other authors (see [1], [2], [4] - [22]).

The first step in studying the linear ordinary differential equations under conditions (1.2) or (1.3), in the case when the functions p_j and q have strong singularities at the points a and b, i.e. when conditions (1.4) and (1.5) are not fulfilled, was made by R. P. Agarwal and I. Kiguradze in the article [3].

In this paper the Agarwal-Kiguradze type theorems are proved which guarantee Fredholm's property for problems (1.1), (1.2), and (1.1), (1.3) (see Definition 1.1). Moreover, we establish optimal, in some sense, sufficient conditions for the solvability of problems (1.1), (1.2), and (1.1), (1.3).

Throughout the paper we use the following notation.

$R^+ = [0, +\infty[$;

$[x]_+$ is the positive part of number x, that is $[x]_+ = \frac{x+|x|}{2}$;

$L_{loc}(]a, b[)$ $(L_{loc}(]a, b]))$ is the space of functions $y:]a, b[\to R$, which are integrable on $[a+\varepsilon, b-\varepsilon]$; $([a+\varepsilon, b])$ for arbitrary small $\varepsilon > 0$;

$L_{\alpha,\beta}(]a, b[)$ $(L^2_{\alpha,\beta}(]a, b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha(b-t)^\beta$ functions $y:]a, b[\to R$, with the norm

$$||y||_{L_{\alpha,\beta}} = \int_a^b (s-a)^\alpha(b-s)^\beta|y(s)|ds \quad \Big(||y||_{L^2_{\alpha,\beta}} = \Big(\int_a^b (s-a)^\alpha(b-s)^\beta y^2(s)ds\Big)^{1/2}\Big);$$

$L([a, b]) = L_{0,0}(]a, b[)$, $L^2([a, b]) = L^2_{0,0}(]a, b[)$;

$M(]a,b[)$ is the set of measurable functions $\tau :]a,b[\to]a,b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a,b[)$ $(\widetilde{L}^2_{\alpha}(]a,b]))$ is the Banach space of functions $y \in L_{loc}(]a,b[)$ $(L_{loc}(]a,b]))$, satisfying

$$\mu_1 \equiv \max\Big\{\Big[\int\limits_a^t (s-a)^\alpha \Big(\int\limits_s^t y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : a \le t \le \frac{a+b}{2}\Big\}+$$

$$+\max\Big\{\Big[\int\limits_t^b (b-s)^\beta \Big(\int\limits_t^s y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : \frac{a+b}{2} \le t \le b\Big\} < +\infty,$$

$$\mu_2 \equiv \max\Big\{\Big[\int\limits_a^t (s-a)^\alpha \Big(\int\limits_s^t y(\xi)d\xi\Big)^2 ds\Big]^{1/2} : a \le t \le b\Big\} < +\infty.$$

The norm in this space is defined by the equality $||\cdot||_{\widetilde{L}^2_{\alpha,\beta}} = \mu_1$ $(||\cdot||_{\widetilde{L}^2_{\alpha}} = \mu_2)$.

$\widetilde{C}^{n-1,\,m}(]a,\,b[)$ $(\widetilde{C}^{n-1,\,m}(]a,\,b]))$ is the space of functions $y \in \widetilde{C}^{n-1}_{loc}(]a,\,b[)$ $(y \in \widetilde{C}^{n-1}_{loc}(]a,\,b]))$, satisfying

$$\int\limits_a^b |y^{(m)}(s)|^2 ds < +\infty. \tag{1.6}$$

When problem (1.1), (1.2) is discussed, we assume that for $n = 2m$, the conditions

$$p_j \in L_{loc}(]a,\,b[) \ \ (j = 1, \cdots, m) \tag{1.7}$$

are fulfilled, and for $n = 2m + 1$, along with (1.7), the conditions

$$\limsup_{t\to b} \Big|(b-t)^{2m-1} \int\limits_{t_1}^t p_1(s)ds\Big| < +\infty \ \ (t_1 = \frac{a+b}{2}) \tag{1.8}$$

are fulfilled. Problem (1.1), (1.3) is discussed under the assumptions

$$p_j \in L_{loc}(]a,\,b]) \ \ (j = 1, \cdots, m). \tag{1.9}$$

A solution of problem (1.1), (1.2) ((1.1), (1.3)) is sought in the space $\widetilde{C}^{n-1,\,m}(]a,\,b[)$ $(\widetilde{C}^{n-1,\,m}(]a,\,b]))$.

By $h_j :]a,\,b[\times]a,\,b[\to R_+$ and $f_j : R \times M(]a,b[) \to C_{loc}(]a,b[\times]a,b[)$ $(j = 1,\ldots,m)$ we denote the functions and the operators, respectively, defined by the equalities

$$\begin{aligned} h_1(t,s) &= \Big|\int\limits_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+ d\xi\Big|, \\ h_j(t,s) &= \Big|\int\limits_s^t (\xi-a)^{n-2m}p_j(\xi)d\xi\Big| \quad (j = 2, \cdots, m), \end{aligned} \tag{1.10}$$

and,

$$f_j(c,\tau_j)(t,s)=\Big|\int\limits_s^t(\xi-a)^{n-2m}|p_j(\xi)|\Big|\int\limits_\xi^{\tau_j(\xi)}(\xi_1-c)^{2(m-j)}d\xi_1\Big|^{1/2}d\xi\Big| \quad (j=1,\cdots,m). \tag{1.11}$$

Let, moreover,

$$m!!=\begin{cases}1 & \text{for } \ m\le 0\\ 1\cdot 3\cdot 5\cdots m & \text{for } \ m\ge 1\end{cases},$$

if $m=2k+1$.

1.2. Fredholm type theorems.

Along with (1.1), we consider the homogeneous equation

$$v^{(n)}(t)=\sum_{j=1}^m p_j(t)v^{(j-1)}(\tau_j(t)) \quad \text{for} \quad a<t<b. \tag{1.1$_0$}$$

In the case where conditions (1.4) and (1.5) are violated, the question on the presence of the Fredholm's property for problem (1.1), (1.2) ((1.1), (1.3)) in some subspace of the space $\widetilde{C}_{loc}^{n-1,m}(]a,\,b[)$ ($\widetilde{C}_{loc}^{n-1,m}(]a,\,b])$) remains so far open. This question is answered in Theorem 1.1 (Theorem 1.2) formulated below which contains optimal in a certain sense conditions guaranteeing the Fredholm's property for problem (1.1), (1.2) ((1.1), (1.3)) in the space $\widetilde{C}^{n-1,\,m}(]a,\,b[)$ ($\widetilde{C}^{n-1,\,m}(]a,\,b])$).

Definition 1.1. We will say that problem (1.1), (1.2) ((1.1), (1.3)) has the Fredholm's property in the space $\widetilde{C}^{n-1,m}(]a,\,b[)$ ($\widetilde{C}^{n-1,m}(]a,\,b])$), if the unique solvability of the corresponding homogeneous problem (1.1$_0$), (1.2) ((1.1$_0$), (1.3)) in that space implies the unique solvability of problem (1.1), (1.2) ((1.1), (1.3)) for every $q\in\widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ ($q\in\widetilde{L}^2_{2n-2m-2}(]a,b])$).

Theorem 1.1. *Let there exist $a_0\in]a,\,b[$, $b_0\in]a_0,\,b[$, numbers $l_{kj}>0$, $\gamma_{kj}>0$, and functions $\tau_j\in M(]a,\,b[)$ $(k=0,1,\ j=1,\dots,m)$ such that*

$$\begin{gathered}(t-a)^{2m-j}h_j(t,s)\le l_{0j} \quad \textit{for} \quad a<t\le s\le a_0,\\ \limsup_{t\to a}(t-a)^{m-\frac12-\gamma_{0j}}f_j(a,\tau_j)(t,s)<+\infty,\end{gathered} \tag{1.12}$$

$$\begin{gathered}(b-t)^{2m-j}h_j(t,s)\le l_{1j} \quad \textit{for} \quad b_0\le s\le t<b,\\ \limsup_{t\to b}(b-t)^{m-\frac12-\gamma_{1j}}f_j(b,\tau_j)(t,s)<+\infty,\end{gathered} \tag{1.13}$$

and

$$\sum_{j=1}^m\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\,l_{kj}<1 \quad (k=0,1). \tag{1.14}$$

Let, moreover, (1.1_0), (1.2) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,\,b[)$. *Then problem* (1.1), (1.2) *has the unique solution* u *for every* $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\;b[)$, *and there exists a constant* r, *independent of* q, *such that*

$$||u^{(m)}||_{L^2} \le r||q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}. \tag{1.15}$$

Corollary 1.1. *Let numbers* $\kappa_{kj}, \nu_{kj} \in R^+$ *be such that*

$$\nu_{k1} > 2n+2-2k(2m-n), \quad \nu_{kj} > 2 \quad (k=0,1;\; j=2,\dots,m), \tag{1.16}$$

$$\limsup_{t\to a} \frac{|\tau_j(t)-t|}{(t-a)^{\nu_{0j}}} < +\infty, \quad \limsup_{t\to b} \frac{|\tau_j(t)-t|}{(b-t)^{\nu_{1j}}} < +\infty, \tag{1.17}$$

and

$$\sum_{j=1}^{m} \frac{2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\kappa_{kj} < 1 \; (k=0,1). \tag{1.18}$$

Moreover, let $\kappa \in R^+$, $p_{0j} \in L_{n-j,\,2m-j}(]a,b[;R^+)$, *and*

$$\begin{aligned} -\frac{\kappa}{[(t-a)(b-t)]^{2n}} - p_{01}(t) \le (-1)^{n-m}p_1(t) \le \\ \le \frac{\kappa_{01}}{(t-a)^n} + \frac{\kappa_{11}}{(t-a)^{n-2m}(b-t)^{2m}} + p_{01}(t), \end{aligned} \tag{1.19}$$

$$|p_j(t)| \le \frac{\kappa_{0j}}{(t-a)^{n-j+1}} + \frac{\kappa_{1j}}{(t-a)^{n-2m}(b-t)^{2m-j+1}} + p_{0j}(t) \quad (j=2,\dots,m). \tag{1.20}$$

Let, moreover, (1.1_0), (1.2) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,\,b[)$. *Then problem* (1.1), (1.2) *has the unique solution* u *for every* $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\;b[)$, *and there exists a constant* r, *independent of* q, *such that* (1.15) *holds.*

Theorem 1.2. *Let there exist* $a_0 \in]a,b[$, *numbers* $l_{0j} > 0$, $\gamma_{0j} > 0$, *and functions* $\tau_j \in M(]a,\,b[)$ *such that condition* (1.12) *is fulfilled and*

$$\sum_{j=1}^{m} \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} l_{0j} < 1. \tag{1.21}$$

Let, moreover, problem (1.1_0), (1.3) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,\,b])$. *Then problem* (1.1), (1.3) *has the unique solution* u *for every* $q \in \widetilde{L}^2_{2n-2m-2}(]a,\;b])$, *and there exists a constant* r, *independent of* q, *such that*

$$||u^{(m)}||_{L^2} \le r||q||_{\widetilde{L}^2_{2n-2m-2}}. \tag{1.22}$$

Corollary 1.2. *Let numbers* $\kappa_{0j}, \nu_{0j} \in R^+$ *be such that*

$$\nu_{01} > 2n+2, \qquad \nu_{0j} \ge 2 \quad (j=2,\dots,m), \tag{1.23}$$

$$\limsup_{t\to a} \frac{|\tau_j(t)-t|}{(t-a)^{\nu_{0j}}} < +\infty, \tag{1.24}$$

and

$$\sum_{j=1}^{m}\frac{2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\kappa_{0j}<1. \tag{1.25}$$

Let, moreover, $\kappa\in R^{+}$, $p_{0j}\in L_{n-j,0}(]a,b];R^{+})$, *and*

$$-\frac{\kappa}{(t-a)^{2n}}-p_{01}(t)\le(-1)^{n-m}p_1(t)\le\frac{\kappa_{01}}{(t-a)^{n}}+p_{01}(t), \tag{1.26}$$

$$|p_j(t)|\le\frac{\kappa_{0j}}{(t-a)^{n-j+1}}+p_{0j}(t)\quad(j=2,\dots,m). \tag{1.27}$$

Let, moreover, problem (1.1_0), (1.3) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,\,b])$. *Then problem* (1.1), (1.3) *has the unique solution* u *for every* $q\in\widetilde{L}^{2}_{2n-2m-2}(]a,\,b])$, *and there exists a constant* r, *independent of* q, *such that* (1.22) *holds.*

Theorem 1.3. *Let* $c_1=a, c_2=b$,

$$\operatorname*{ess\,sup}_{a<t<b}\frac{1}{|t-c_i|^{m+1-j}}\Big|\int_t^{\tau_j(t)}|\xi-c_i|^{m-j-1}d\xi\Big|<+\infty\ (j=1,\dots,m) \tag{1.28}$$

if $i=1,2$ (*if* $i=1$),

$$p_j\in L_{n-j,\,2m-j}(]a,\,b[)\quad\Big(p_j\in L_{n-j,0}(]a,\,b])\Big)\ (j=1,\dots,m), \tag{1.29}$$

and let problem (1.1), (1.2) ((1.1), (1.3)) *be uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,\,b[)$ (*in the space* $\widetilde{C}^{n-1,\,m}(]a,\,b])$. *Then this problem is uniquely solvable in the space* $\widetilde{C}^{n-1}(]a,\,b[)$ (*in the space* $\widetilde{C}^{n-1}(]a,\,b])$ *as well.*

Remark 1.1. In [3], an example is constructed which demonstrates that if condition (1.29) is violated, then problem (1.1), (1.2) (problem (1.1), (1.3)) with $\tau_j(t)\equiv t\,(j=1,\dots,m)$ may be uniquely solvable in the space $\widetilde{C}^{n-1,m}(]a,\,b[)$ (in the space $\widetilde{C}^{n-1,m}(]a,\,b])$) and this problem may have infinite set of solutions in the space $\widetilde{C}^{loc}(]a,\,b[)$ (in the space $\widetilde{C}^{loc}(]a,\,b])$).

Also, in [3] it is demonstrated that strict inequalities (1.14), (1.21), (1.18), (1.25) are sharp because they cannot be replaced by nonstrict ones.

1.2. Existence and uniqueness theorems.

Theorem 1.4. *Let there exist numbers* $t^*\in]a,\,b[$, $\ell_{kj}>0$, $\bar{l}_{kj}\ge0$, *and* $\gamma_{kj}>0$ ($k=0,1$; $j=1,\dots,m$) *such that along with*

$$\sum_{j=1}^{m}\Big(\frac{(2m-j)2^{2m-j+1}\,l_{0j}}{(2m-1)!!(2m-2j+1)!!}+\frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\,\bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}}\Big)<\frac{1}{2}, \tag{1.30}$$

$$\sum_{j=1}^{m}\Big(\frac{(2m-j)2^{2m-j+1}\,l_{1j}}{(2m-1)!!(2m-2j+1)!!}+\frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\,\bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}}\Big)<\frac{1}{2}, \tag{1.31}$$

the conditions

$$(t-a)^{2m-j}h_j(t,s) \le l_{0j}, \ \ (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s) \le \bar{l}_{0j} \ \ for \quad a<t\le s\le t^*, \qquad (1.32)$$

$$(b-t)^{2m-j}h_j(t,s) \le l_{1j}, \ \ (b-t)^{m-\gamma_{1j}-1/2}f_j(b,\tau_j)(t,s) \le \bar{l}_{1j} \quad for \quad t^*\le s\le t<b \qquad (1.33)$$

hold. Then for every $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ *problem* (1.1), (1.2) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,\,b[)$.

To illustrate this theorem, we consider the second order differential equation with a deviating argument

$$u''(t) = p(t)u(\tau(t)) + q(t), \qquad (1.34)$$

under the boundary conditions

$$u(a) = 0, \ \ u(b) = 0. \qquad (1.35)$$

From Theorem 1.4, with $n = 2, \ m = 1, \ t^* = (a+b)/2, \ \gamma_{01} = \gamma_{11} = 1/2, \ l_{01} = l_{11} = \kappa_0, \ \bar{l}_{01} = \bar{l}_{11} = \sqrt{2}\kappa_1/\sqrt{b-a}$, we get

Corollary 1.3. *Let function* $\tau \in M(]a,b[)$ *be such that*

$$\begin{aligned} 0 \le \tau(t) - t &\le \frac{2^6}{(b-a)^6}(t-a)^7 \quad for \quad a < t \le \frac{a+b}{2}, \\ -\frac{2^6}{(b-a)^6}(b-t)^7 &\le t - \tau(t) \le 0 \quad for \quad \frac{a+b}{2} \le t < b. \end{aligned} \qquad (1.36)$$

Moreover, let function $p :]a,b[\to R$ *and constants* $\kappa_0, \ \kappa_1$ *be such that*

$$-\frac{2^{-2}(b-a)^2\kappa_0}{[(b-t)(t-a)]^2} \le p_1(t) \le \frac{2^{-7}(b-a)^6\kappa_1}{[(b-t)(t-a)]^4} \quad for \quad a < t \le b \qquad (1.37)$$

and

$$4\kappa_0 + \kappa_1 < \frac{1}{2}. \qquad (1.38)$$

Then for every $q \in \widetilde{L}^2_{0,\,0}(]a,b[)$ *problem* (1.34), (1.35) *is uniquely solvable in the space* $\widetilde{C}^{1,\,1}(]a,\,b[)$.

Theorem 1.5. *Let there exist numbers* $t^* \in]a,\,b[, \ \ell_{0j} > 0, \ \bar{\ell}_{0j} \ge 0,$ *and* $\gamma_{0j} > 0$ $(j = 1,\dots,m)$ *such that conditions*

$$(t-a)^{2m-j}h_j(t,s) \le l_{0j}, \ \ (t-a)^{m-\gamma_{0j}-1/2}f_j(a,\tau_j)(t,s) \le \bar{l}_{0j} \ for \quad a<t\le s\le b, \qquad (1.39)$$

and

$$\sum_{j=1}^{m}\Big(\frac{(2m-j)2^{2m-j+1}\,l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\,\bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}}\Big) < 1 \qquad (1.40)$$

hold. Then for every $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$ *problem* (1.1), (1.3) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,\,b])$.

Theorem 1.6. *Let there exist numbers* $t^* \in]a, b[$, $\ell_{kj} > 0$, $\bar{l}_{kj} \geq 0$, *and* $\gamma_{kj} > 0$ $(k = 0, 1;\ j = 1, \dots, m)$ *such that along with* (1.40) *and*

$$\sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < 1, \tag{1.41}$$

conditions (1.32), (1.33) *hold. Moreover, let* $\tau_j \in M(]a, b[)$ $(j = 1, \dots, n)$ *and*

$$\operatorname{sign}[(\tau_j(t) - t^*)(t - t^*)] \geq 0 \quad \textit{for} \quad a < t < b. \tag{1.42}$$

Then for every $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a, b[)$ *problem* (1.1), (1.2) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\,m}(]a,\, b[)$.

Also, from Theorem 1.6, with $n = 2,\ m = 1,\ t^* = (a+b)/2,\ \gamma_{01} = \gamma_{11} = 1/2,\ l_{01} = l_{11} = \kappa_0,\ \bar{l}_{01} = \bar{l}_{11} = \sqrt{2}\kappa_1/\sqrt{b-a}$, we get

Corollary 1.4. *Let functions* $p :]a, b[\to R$, $\tau \in M(]a, b[)$ *and constants* $\kappa_0 > 0$, $\kappa_1 > 0$ *be such that along with* (1.36) *and* (1.37) *the inequalities*

$$\operatorname{sign}[(\tau(t) - \frac{a+b}{2})(t - \frac{a+b}{2})] \geq 0 \quad \textit{for} \quad a < t < b \tag{1.43}$$

and

$$4\kappa_0 + \kappa_1 < 1 \tag{1.44}$$

hold. Then for every $q \in \widetilde{L}^2_{0,\,0}(]a, b[)$ *problem* (1.34), (1.35) *is uniquely solvable in the space* $\widetilde{C}^{1,\,1}(]a,\, b[)$.

2 Auxiliary propositions

2.1. Lemmas on integral inequalities. Now we formulate two lemmas which are proved in [3].

Lemma 2.1. *Let* $\in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ *and*

$$u^{(j-1)}(t_0) = 0 \quad (j = 1, \dots, m), \qquad \int_{t_0}^{t_1} |u^{(m)}(s)|^2 ds < +\infty. \tag{2.1}$$

Then

$$\int_{t_0}^{t} \frac{(u^{(j-1)}(s))^2}{(s-t_0)^{2m-2j+2}} ds \leq \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!} \Big)^2 \int_{t_0}^{t} |u^{(m)}(s)|^2 ds \quad \textit{for} \quad t_0 \leq t \leq t_1. \tag{2.2}$$

Lemma 2.2. *Let* $u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$, *and*

$$u^{(j-1)}(t_1) = 0 \quad (j = 1, \dots, m), \qquad \int\limits_{t_0}^{t_1} |u^{(m)}(s)|^2 ds < +\infty. \tag{2.3}$$

Then

$$\int\limits_{t}^{t_1} \frac{(u^{(j-1)}(s))^2}{(t_1 - s)^{2m-2j+2}} ds \le \Big(\frac{2^{m-j+1}}{(2m-2j+1)!!} \Big)^2 \int\limits_{t}^{t_1} |u^{(m)}(s)|^2 ds \quad \textit{for} \quad t_0 \le t \le t_1. \tag{2.4}$$

Let $t_0, t_1 \in]a, b[,\ u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ and $\tau_j \in M(]a, b[)$ $(j = 1, \dots, m)$. Then we define the functions $\mu_j : [a,\ (a+b)/2] \times [(a+b)/2,\ b] \times [a, b] \to [a, b]$, $\rho_k : [t_0,\, t_1] \to R_+$ $(k = 0, 1)$, $\lambda_j : [a, b] \times]a,\ (a+b)/2] \times [(a+b)/2,\ b[\times]a, b[\to R_+$, by the equalities

$$\mu_j(t_0, t_1, t) = \begin{cases} \tau_j(t) & \text{for } \tau_j(t) \in [t_0,\ t_1] \\ t_0 & \text{for } \tau_j(t) < t_0 \\ t_1 & \text{for } \tau_j(t) > t_1 \end{cases},$$

$$\rho_k(t) = \Big| \int\limits_{t}^{t_k} |u^{(m)}(s)|^2 ds \Big|, \qquad \lambda_j(c, t_0, t_1, t) = \Big| \int\limits_{t}^{\mu_j(t_0,t_1,t)} (s - c)^{2(m-j)} ds \Big|^{1/2}. \tag{2.5}$$

Let also functions $\alpha_j : R^3_+ \times [0, 1[\to R_+$ and $\beta_j \in R_+ \times [0, 1[\to R_+$ $(j = 1, \cdots, m)$ be defined by the equalities

$$\alpha_j(x, y, z, \gamma) = x + \frac{2^{m-j}\, y\, z^{\gamma}}{(2m-2j-1)!!}, \ \beta_j(y, \gamma) = \frac{2^{2m-j-1}}{(2m-2j-1)!!(2m-3)!!} \frac{y^{\gamma}}{\sqrt{2\gamma}}. \tag{2.6}$$

Lemma 2.3. *Let* $a_0 \in]a, b[,\ t_0 \in]a,\ a_0[,\ t_1 \in]a_0,\ b[$, *and the function* $u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ *be such that conditions* (2.1) *hold. Moreover, let constants* $l_{0\,j} > 0$, $\overline{l}_{0\,j} \ge 0$, $\gamma_{0j} > 0$, *and functions* $\overline{p}_j \in L_{loc}(]t_0,\, t_1[)$, $\tau_j \in M(]a, b[)$ *be such that the inequalities*

$$(t - t_0)^{2m-1} \int\limits_{t}^{a_0} [\overline{p}_1(s)]_+ ds \le l_{0\,1}, \tag{2.7}$$

$$(t - t_0)^{2m-j} \Big| \int\limits_{t}^{a_0} \overline{p}_j(s) ds \Big| \le l_{0\,j} \ \ (j = 2, \dots, m), \tag{2.8}$$

$$(t - t_0)^{m-\frac{1}{2}-\gamma_{0j}} \Big| \int\limits_{t}^{a_0} \overline{p}_j(s) \lambda_j(t_0, t_0, t_1, s) ds \Big| \le \overline{l}_{0\,j} \ \ (j = 1, \dots, m) \tag{2.9}$$

hold for $t_0 < t \le a_0$. *Then*

$$\begin{aligned} &\int\limits_{t}^{a_0} \overline{p}_j(s) u(s) u^{(j-1)}(\mu_j(t_0, t_1, s)) ds \le \\ &\le \alpha_j(l_{0j}, \overline{l}_{0j}, a_0 - a, \gamma_{0j}) \rho_0^{1/2}(\tau^*) \rho_0^{1/2}(t) + \overline{l}_{0j} \beta_j(a_0 - a, \gamma_{0j}) \rho_0^{1/2}(\tau^*) \rho_0^{1/2}(a_0) + \\ &+ l_{0j} \frac{(2m-j) 2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \rho_0(a_0) \quad \textit{for} \quad t_0 < t \le a_0, \end{aligned} \tag{2.10}$$

where $\tau^* = \sup\{\mu_j(t_0,t_1,t) : t_0 \le t \le a_0, j = 1,\dots,m\} \le t_1$.

Proof. In view of the formula of integration by parts, for $t \in [t_0, a_0]$ we have

$$\int_t^{a_0} \overline{p}_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds = \int_t^{a_0} \overline{p}_j(s)u(s)u^{(j-1)}(s)ds +$$
$$+\int_t^{a_0} \overline{p}_j(s)u(s)\Big(\int_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi)d\xi\Big)ds = u(t)u^{(j-1)}(t)\int_t^{a_0} \overline{p}_j(s)ds+ \tag{2.11}$$
$$+\sum_{k=0}^{1}\int_t^{a_0}\Big(\int_s^{a_0}\overline{p}_j(\xi)d\xi\Big)u^{(k)}(s)u^{(j-k)}(s)ds + \int_t^{a_0} \overline{p}_j(s)u(s)\Big(\int_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi)d\xi\Big)ds$$

$(j = 2,\dots,m)$, and

$$\int_t^{a_0} \overline{p}_1(s)u(s)u(\mu_1(t_0,t_1,s))ds \le \int_t^{a_0}[\overline{p}_1(s)]_+u^2(s)ds+$$
$$+\int_t^{a_0}|\overline{p}_1(s)u(s)|\Big|\int_s^{\mu_1(t_0,t_1,s)} u'(\xi)d\xi\Big|ds \le u^2(t)\int_t^{a_0}[\overline{p}_1(s)]_+ds+ \tag{2.12}$$
$$+2\int_t^{a_0}\Big(\int_s^{a_0}[\overline{p}_1(\xi)]_+d\xi\Big)|u(s)u'(s)|ds + \int_t^{a_0}|\overline{p}_1(s)u(s)|\Big|\int_s^{\mu_1(t_0,t_1,s)} u'(\xi)d\xi\Big|ds.$$

On the other hand, by conditions (2.1), the Schwartz inequality and Lemma 2.1, we deduce that

$$|u^{(j-1)}(t)| = \frac{1}{(m-j)!}\Big|\int_{t_0}^{t}(t-s)^{m-j}u^{(m)}(s)ds\Big| \le (t-t_0)^{m-j+1/2}\,\rho_0^{1/2}(t) \tag{2.13}$$

for $t_0 \le t \le a_0$ $(j = 1,\dots,m)$. If along with this, in the case $j > 1$, we take into account inequality (2.8), and lemma 2.1, for $t \in [t_0, a_0]$, we obtain the estimates

$$\Big|u(t)u^{(j-1)}(t)\int_t^{a_0}\overline{p}_j(s)ds\Big| \le (t-t_0)^{2m-j}\Big|\int_t^{a_0}\overline{p}_j(s)ds\Big|\rho_0(t) \le l_{0j}\rho_0(t), \tag{2.14}$$

and

$$\sum_{k=0}^{1}\int_t^{a_0}\Big(\int_s^{a_0}\overline{p}_j(\xi)d\xi\Big)u^{(k)}(s)u^{(j-k)}(s)ds \le l_{0j}\sum_{k=0}^{1}\int_t^{a_0}\frac{|u^{(k)}(s)u^{(j-k)}(s)|}{(s-t_0)^{2m-j}}ds \le$$
$$\le l_{0j}\sum_{k=0}^{1}\Big(\int_t^{a_0}\frac{|u^{(k)}(s)|^2ds}{(s-t_0)^{2m-2k}}\Big)^{1/2}\Big(\int_t^{a_0}\frac{|u^{(j-k)}(s)|^2ds}{(s-t_0)^{2m+2k-2j}}\Big)^{1/2} \le \tag{2.15}$$
$$\le l_{0j}\rho_0(a_0)\sum_{k=0}^{1}\frac{2^{2m-j}}{(2m-2k-1)!!(2m+2k-2j-1)!!}.$$

Analogously, if $j=1$, by (2.7) we obtain

$$
\begin{gathered}
u^2(t)\int\limits_t^{a_0}[\overline{p}_1(s)]_+ds \le l_{01}\rho_0(t),\\
2\int\limits_t^{a_0}\Big(\int\limits_s^{a_0}[\overline{p}_1(\xi)]_+d\xi\Big)|u(s)u'(s)|ds \le l_{01}\rho_0(a_0)\frac{(2m-1)2^{2m}}{[(2m-1)!!]^2}
\end{gathered}
\tag{2.16}
$$

for $t_0 < t \le a_0$.

By the Schwartz inequality, Lemma 2.1, and the fact that ρ_0 is nondecreasing function, we get

$$
\Big|\int\limits_s^{\mu_j(t_0,t_1,s)} u^{(j)}(\xi)d\xi\Big| \le \frac{2^{m-j}}{(2m-2j-1)!!}\,\lambda_j(t_0,t_0,t_1,s)\,\rho_0^{1/2}(\tau^*)
\tag{2.17}
$$

for $t_0 < s \le a_0$. Also, due to (2.2), (2.9) and (2.13), we have

$$
\begin{gathered}
|u(t)|\int\limits_t^{a_0}|\overline{p}_j(s)|\lambda_j(t_0,t_0,t_1,s)ds = (t-t_0)^{m-1/2}\rho_0^{1/2}(t)\int\limits_t^{a_0}|\overline{p}_j(s)|\lambda_j(t_0,t_0,t_1,s)ds \le\\
\le \overline{l}_{0j}\,(t-t_0)^{\gamma_{0j}}\,\rho_0^{1/2}(t),\\
\int\limits_t^{a_0}|u'(s)|\Big(\int\limits_s^{a_0}|\overline{p}_j(\xi)|\lambda_j(t_0,t_0,t_1,\xi)d\xi\Big)ds \le \overline{l}_{0j}\int\limits_t^{a_0}\frac{|u'(s)|}{(s-t_0)^{m-\frac{1}{2}-\gamma_{0j}}}ds \le\\
\le \overline{l}_{0j}\frac{2^{m-1}(a_0-a)^{\gamma_{0j}}}{(2m-3)!!\sqrt{2\gamma_{0j}}}\rho_0^{1/2}(a_0)
\end{gathered}
$$

for $t_0 < t \le a_0$. From the last three inequalities it is clear that

$$
\begin{gathered}
\Big|\frac{(2m-2j-1)!!}{2^{m-j}\rho_0^{1/2}(\tau^*)}\int\limits_t^{a_0}\overline{p}_j(s)u(s)\Big(\int\limits_s^{\mu_j(t_0,t_1,s)}u^{(j)}(\xi)d\xi\Big)ds\Big| \le \int\limits_t^{a_0}|\overline{p}_j(s)u(s)|\lambda_j(t_0,t_0,t_1,s)ds \le\\
\le |u(t)|\int\limits_t^{a_0}|\overline{p}_j(s)|\lambda_j(t_0,t_0,t_1,s)ds + \int\limits_t^{a_0}|u'(s)|\Big(\int\limits_s^{a_0}|\overline{p}_j(\xi)|\lambda_j(t_0,t_0,t_1,\xi)d\xi\Big)ds \le\\
\le \overline{l}_{0j}\,(t-t_0)^{\gamma_{0j}}\rho_0^{1/2}(t) + \overline{l}_{0j}\frac{2^{m-1}(a_0-a)^{\gamma_{0j}}}{(2m-3)!!\sqrt{2\gamma_{0j}}}\rho_0^{1/2}(a_0)
\end{gathered}
\tag{2.18}
$$

for $t_0 < t \le a_0$. Now, note that from (2.11) and (2.12) by (2.14)-(2.16) and (2.18), it immediately follows inequality (2.10). $\square$

The following lemma can be proved similarly to Lemma 2.3.

Lemma 2.4. *Let $b_0 \in]a,\, b[,\ t_1 \in]b_0,\, b[,\ \ t_0 \in]a,\, b_0[,$ and the function $u \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ be such that conditions (2.3) hold. Moreover, let constants $l_{1\,j} > 0,\ \bar{l}_{1\,j} \geq 0,\ \gamma_{1j} > 0,$ and functions $\overline{p}_j \in L_{loc}(]t_0,\, t_1[),\ \tau_j \in M(]a, b[)$ be such that the inequalities*

$$(t_1 - t)^{2m-1} \int\limits_{b_0}^{t} [\overline{p}_1(s)]_+ ds \leq l_{1\,1}, \tag{2.19}$$

$$(t_1 - t)^{2m-j} \Big| \int\limits_{b_0}^{t} \overline{p}_j(s) ds \Big| \leq l_{1\,j} \ \ (j = 2, \dots, m), \tag{2.20}$$

$$(t_1 - t)^{m-\frac{1}{2}-\gamma_{1j}} \Big| \int\limits_{b_0}^{t} \overline{p}_j(s)\lambda_j(t_1, t_0, t_1, s) ds \Big| \leq \bar{l}_{1\,j} \ \ (j = 1, \dots, m) \tag{2.21}$$

hold for $b_0 < t \leq t_1$. Then

$$\begin{gathered}\int\limits_{b_0}^{t} \overline{p}_j(s) u(s) u^{(j-1)}(\mu_j(t_0, t_1, s)) ds \leq \\ \leq \alpha_j(l_{1j}, \bar{l}_{1j}, b - b_0, \gamma_{1j}) \rho_1^{1/2}(\tau_*) \rho_1^{1/2}(t) + \bar{l}_{1j} \beta_j(b - b_0, \gamma_{1j}) \rho_1^{1/2}(\tau_*) \rho_1^{1/2}(b_0) + \\ + l_{1j} \frac{(2m - j) 2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \rho_1(b_0) \quad \textit{for} \quad b_0 \leq t < t_1,\end{gathered} \tag{2.22}$$

where $\tau_* = \inf\{\mu_j(t_0, t_1, t) : b_0 \leq t \leq t_1, j = 1, \dots, m\} \geq t_0$.

2.2. Lemma on the property of functions from the space $\widetilde{C}^{n-1,m}(]a,\, b[)$.

Lemma 2.5. *Let*

$$w(t) = \sum_{i=1}^{n-m} \sum_{k=i}^{n-m} c_{ik}(t) u^{(n-k)}(t) u^{(i-1)}(t),$$

where $\widetilde{C}^{n-1,m}(]a,\, b[)$, and each $c_{ik} : [a, b] \to R$ is an $(n - k - i + 1)$ -times continuously differentiable function. Moreover, if

$$u^{(i-1)}(a) = 0 \ (i = 1, \dots, m), \quad \limsup_{t \to a} \frac{|c_{ii}(t)|}{(t-a)^{n-2m}} < +\infty \ \ (i = 1, \dots, n - m),$$

then

$$\liminf_{t \to a} |w(t)| = 0,$$

and if $u^{(i-1)}(b) = 0 \ \ (i = 1, \dots, n - m)$, then

$$\liminf_{t \to b} |w(t)| = 0.$$

The proof of this lemma is given in [9].

2.3. Lemmas on the sequences of solutions of auxiliary problems.
Now for every natural k we consider the auxiliary boundary problems

$$u^{(n)}(t)=\sum_{j=1}^{m}p_j(t)u^{(j-1)}(\mu_j(t_{0k},t_{1k},t))+q_k(t)\quad\text{for}\quad t_{0k}\le t\le t_{1k}, \tag{2.23}$$

$$u^{(i-1)}(t_{0k})=0\ (i=1,\dots,m),\quad u^{(j-1)}(t_{1k})=0\ (j=1,\dots,n-m), \tag{2.24}$$

where

$$a<t_{0k}<t_{1k}<b\ (k\in N),\qquad \lim_{k\to+\infty}t_{0k}=a,\ \lim_{k\to+\infty}t_{1k}=b, \tag{2.25}$$

and

$$u^{(n)}(t)=\sum_{j=1}^{m}p_j(t)u^{(j-1)}(\mu_j(t_{0k},b,t))+q_k(t)\quad\text{for}\quad t_{0k}\le t\le b, \tag{2.26}$$

$$u^{(i-1)}(t_{0k})=0\ (i=1,\dots,m),\quad u^{(j-1)}(b)=0\ (j=1,\dots,n-m), \tag{2.27}$$

where

$$a<t_{0k}<b\ (k\in N),\qquad \lim_{k\to+\infty}t_{0k}=a. \tag{2.28}$$

Throughout this section, when problems (1.1), (1.2) and (2.23), (2.24) are discussed we assume that

$$p_j\in L_{loc}(]a,b[)\ \ (j=1,...,m),\quad q,q_k\in\widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\,b[), \tag{2.29}$$

and for an arbitrary $(m-1)$-times continuously differentiable function $x:]a,b[\to R$, we set

$$\Lambda_k(x)(t)=\sum_{j=1}^{m}p_j(t)x^{(j-1)}(\mu_j(t_{0k},t_{1k},t)),\quad \Lambda(x)(t)=\sum_{j=1}^{m}p_j(t)x^{(j-1)}(\tau_j(t)). \tag{2.30}$$

Problems (1.1), (1.3) and (2.26), (2.27) are considered in the case

$$p_j\in L_{loc}(]a,b])\ \ (j=1,...,m),\quad q,q_k\in\widetilde{L}^2_{2n-2m-2,\,0}(]a,\,b[), \tag{2.31}$$

and for an arbitrary $(m-1)$-times continuously differentiable function $x:]a,b]\to R$, we set

$$\Lambda_k(x)(t)=\sum_{j=1}^{m}p_j(t)x^{(j-1)}(\mu_j(t_{0k},b,t)),\quad \Lambda(x)(t)=\sum_{j=1}^{m}p_j(t)x^{(j-1)}(\tau_j(t)). \tag{2.32}$$

Remark 2.1. From the definition of the functions μ_j $(j=1,\dots,m)$, the estimate

$$|\mu_j(t_{0k},t_{1k},t)-\tau_j(t)|\le\begin{cases}0 & \text{for}\ \ \tau_j(t)\in]t_{0k},\ t_{1k}[\\ \max\{b-t_{1k},\ \ t_{0k}-a\} & \text{for}\ \ \tau_j(t)\notin]t_{0k},\ t_{1k}[\end{cases}$$

follows. Thus, if conditions (2.25) hold, then

$$\lim_{k\to+\infty}\mu_j(t_{0k},t_{1k},t)=\tau_j(t)\quad(j=1,\dots,m)\quad\text{uniformly in}\quad]a,b[. \tag{2.33}$$

Lemma 2.6. *Let conditions* (2.25) *hold and the sequence of the* $(m-1)$*-times continuously differentiable functions* $x_k :]t_{0k}, t_{1k}[\to R$*, and functions* $x^{(j-1)} \in C([a,b])$ $(j = 1, \dots, m)$ *be such that*

$$\lim_{k\to+\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, \dots, m) \quad \textit{uniformly in} \quad]a,b[\;\; (]a,b]). \tag{2.34}$$

Then for any nonnegative function $w \in C([a,b])$ *and* $t^* \in]a,b[$,

$$\lim_{k\to+\infty} \int_{t^*}^{t} w(s)\Lambda_k(x_k)(s)ds = \int_{t^*}^{t} w(s)\Lambda(x)(s)ds \tag{2.35}$$

uniformly in $]a,b[$*, where* Λ_k *and* Λ *are defined by equalities* (2.30).

Proof. We have to prove that for any $\delta \in]0, \ \min\{b - t^*, \ t^* - a\}[$, and $\varepsilon > 0$, there exists a constant $n_0 \in N$ such that

$$\Big| \int_{t^*}^{t} w(s)(\Lambda_k(x_k)(s) - \Lambda(x)(s))ds \Big| \le \varepsilon \quad \text{for} \quad t \in [a+\delta, \, b-\delta], \ k > n_0. \tag{2.36}$$

Let, now $w(t_*) = \max\limits_{a\le t\le b} w(t)$, and $\varepsilon_1 = \varepsilon\Big(2w(t_*) \sum\limits_{j=1}^{m} \int_{a+\delta}^{b-\delta} |p_j(s)|ds\Big)^{-1}$. Then from the inclusions $x_k^{(j-1)} \in C([a+\delta, \, b-\delta])$, $x^{(j-1)} \in C([a,b])$ $(j = 1, \dots, m)$, conditions (2.33) and (2.34), it follows the existence of such constant $n_0 \in N$ that

$$|x_k^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s)) - x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s))| \le \varepsilon_1, |x^{(j-1)}(\mu_j(t_{0k}, t_{1k}, s)) - x^{(j-1)}(\tau_j(s))| \le \varepsilon_1$$

for $t \in [a+\delta, \, b-\delta], \ k > n_0, \ \ j = 1, \dots, m$. Thus from the inequality

$$|\Lambda_k(x_k)(s) - \Lambda(x)(s)| \le |\Lambda_k(x_k)(s) - \Lambda_k(x)(s)| + |\Lambda_k(x)(s) - \Lambda(x)(s)| \le 2\varepsilon_1 \sum_{j=1}^{m} |p_j(t)|,$$

we have (2.36). □

The proof of the following lemma is analogous to that of Lemma 2.6.

Lemma 2.7. *Let conditions* (2.28) *hold and the sequence of the* $(m-1)$*-times continuously differentiable functions* $x_k :]t_{0k}, b] \to R$*, and functions* $x^{(j-1)} \in C([a,b])$ $(j = 1, \dots, m)$ *be such that* $\lim\limits_{l\to+\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t)$ $(j = 1, \dots, m)$ *uniformly in* $]a,b]$. *Then for any nonnegative function* $w \in C([a,b])$*, and* $t^* \in]a,b]$*, condition* (2.35) *holds uniformly in* $]a,b]$*, where* Λ_k *and* Λ *are defined by equalities* (2.32).

Lemma 2.8. *Let condition* (2.25) *hold, and for every natural* k*, problem* (2.23), (2.24) *have a solution* $u_k \in \widetilde{C}_{loc}^{n-1}(]a,b[)$*, and there exist a constant* $r_0 > 0$ *such that*

$$\int_{t_{0k}}^{t_{1k}} |u_k^{(m)}(s)|ds \le r_0^2 \ \ (k \in N) \tag{2.37}$$

holds, and if $n = 2m+1$, *let there exist constants* $\rho_j \geq 0$, $\overline{\rho}_j \geq 0$, $\gamma_{1j} > 0$ *such that*

$$\rho_j = \sup\Big\{(b-t)^{2m-j}\Big|\int_{t_1}^{t}(s-a)p_j(s)ds\Big| : t_0 \leq t < b\Big\} < +\infty,$$
$$\overline{\rho}_j = \sup\Big\{(b-t)^{m-\gamma_{1j}-1/2}\int_{t_1}^{t}(s-a)\Big|p_j(s)\Big|\lambda_j(b,t_{0k},t_{1k},s)ds : t_0 \leq t < b\Big\} < +\infty, \tag{2.38}$$

for $t_1 = \frac{a+b}{2}$, $(j = 1,...,m)$. *Moreover, let*

$$\lim_{k\to+\infty} ||q_k - q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} = 0, \tag{2.39}$$

and the homogeneous problem (1.1$_0$), (1.2) *have only the trivial solution in the space* $\widetilde{C}^{n-1,m}(]a,b[)$. *Then nonhomogeneous problem* (1.1), (1.2) *has a unique solution* u *such that*

$$||u^{(m)}||_{L^2} \leq r_0, \tag{2.40}$$

and

$$\lim_{k\to+\infty} u_k^{(j-1)}(t) = u^{(j-1)}(t) \quad (j = 1,\dots,n) \quad \textit{uniformly in} \quad]a,b[\tag{2.41}$$

(that is, uniformly on $[a+\delta, b-\delta]$ *for an arbitrarily small* $\delta > 0$*).*

Proof. Suppose $t_1,\dots,t_n$ are the numbers such that

$$\frac{a+b}{2} = t_1 < \cdots < t_n < b, \tag{2.42}$$

and $g_i(t)$ are the polynomials of $(n-1)$-th degree, satisfying the conditions

$$g_j(t_j) = 1, \quad g_j(t_i) = 0 \ \ (i \neq j; \ \ i,j = 1,\dots,n). \tag{2.43}$$

Then for every natural k, for the solution u_k of problem (2.23), (2.24) the representation

$$u_k(t) = \sum_{j=1}^{n}\Big(u_k(t_j) - \frac{1}{(n-1)!}\int_{t_1}^{t_j}(t_j-s)^{n-1}(\Lambda_k(u_k)(s)+q_k(s))ds\Big)g_j(t)+$$
$$+\frac{1}{(n-1)!}\int_{t_1}^{t}(t-s)^{n-1}(\Lambda_k(u_k)(s)+q_k(s))ds \tag{2.44}$$

is valid. For an arbitrary $\delta \in]0, \frac{a+b}{2}[$, we have

$$\Big|\int_{t}^{t_1}(s-t)^{n-j}(q_k(s)-q(s))ds\Big| = (n-j)\Big|\int_{t}^{t_1}(s-t)^{n-j-1}\Big(\int_{s}^{t_1}(q_k(\xi)-q(\xi))d\xi\Big)ds\Big| \leq$$
$$\leq n\Big(\int_{t}^{t_1}(s-a)^{2m-2j}ds\Big)^{1/2}\Big(\int_{t}^{t_1}(s-a)^{2n-2m-2}\Big(\int_{s}^{t_1}(q_k(\xi)-q(\xi))d\xi\Big)^2 ds\Big)^{1/2} \leq$$

$$\leq n\Big|(t_1-a)^{2m-2j+1}-\delta^{2m-2j+1}\Big|^{1/2}||q_k-q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \quad \text{for } \ a+\delta\leq t\leq t_1,$$

$$\Big|\int\limits_{t_1}^{t}(t-s)^{n-j}(q_k(s)-q(s))ds\Big|\leq n\Big|(b-t_1)^{2n-2m-2j+1}-\delta^{2n-2m-2j+1}\Big|^{1/2}\times \tag{2.45}$$
$$\times||q_k-q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \qquad \text{for} \quad t_1\leq t\leq b-\delta \ \ (j=1,\dots,n-1).$$

Hence, by condition (2.39), we find

$$\lim_{k\to+\infty}\int\limits_{t}^{t_1}(s-t)^{n-j}(q_k(s)-q(s))ds=0 \quad \text{ uniformly in } \ ]a,b[\ \ (j=1,\dots,n-1). \tag{2.46}$$

Analogously one can show that if $t_0\in]a,b[$, then

$$\lim_{k\to+\infty}\int\limits_{t_0}^{t}(s-t_0)(q_k(s)-q(s))ds=0 \quad \text{ uniformly on } \ I(t_0), \tag{2.47}$$

where $I(t_0)=[t_0,(a+b)/2]$ for $t_0<(a+b)/2$ and $I(t_0)=[(a+b)/2,t_0]$ for $t_0>(a+b)/2$.

In view of inequalities (2.37), the identities

$$u_k^{(j-1)}(t)=\frac{1}{(m-j)!}\int\limits_{t_{ik}}^{t}(t-s)^{m-j}u_k^{(m)}(s)ds \tag{2.48}$$

for $i=0,1;\ j=1,\dots,m;\ \ k\in N$, yield

$$|u_k^{(j-1)}(t)|\leq r_j[(t-a)(b-t)]^{m-j+1/2} \tag{2.49}$$

for $t_{0k}\leq t\leq t_{1k}$ $(j=1,\dots,m;\ \ k\in N)$, where

$$r_j=\frac{r_0}{(m-j)!}(2m-2j+1)^{-1/2}\Big(\frac{2}{b-a}\Big)^{m-j+1/2} \qquad (j=1,\dots,m). \tag{2.50}$$

By virtue of the Arzela-Ascoli lemma and conditions (2.37) and (2.49), the sequence $\{u_k\}_{k=1}^{+\infty}$ contains a subsequence $\{u_{k_l}\}_{l=1}^{+\infty}$ such that $\{u_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ $(j=1,\dots,m)$ are uniformly convergent in $]a,b[$. Suppose

$$\lim_{l\to+\infty}u_{k_l}(t)=u(t). \tag{2.51}$$

Then in view of (2.49), $u^{(j-1)}\in C([a,b])$ $(j=1,\dots,m)$, and

$$\lim_{l\to+\infty}u_{k_l}^{(j-1)}(t)=u^{(j-1)}(t) \quad (j=1,\dots,m) \quad \text{uniformly in} \quad]a,b[. \tag{2.52}$$

If along with this we take into account conditions (2.25) and (2.46), from (2.44) by lemma 2.6 we find

$$u(t)=\sum_{j=1}^{n}\Big(u(t_j)-\frac{1}{(n-1)!}\int\limits_{t_1}^{t_j}(t_j-s)^{n-1}(\Lambda(u)(s)+q(s))ds\Big)g_j(t)+$$
$$+\frac{1}{(n-1)!}\int\limits_{t_1}^{t}(t-s)^{n-1}(\Lambda(u)(s)+q(s))ds \quad \text{for} \quad a<t<b, \tag{2.53}$$

$$|u^{(j-1)}(t)| \le r_j[(t-a)(b-t)]^{m-j+1/2} \quad \text{for} \quad a<t<b \ (j=1,\dots,m), \tag{2.54}$$

$u \in \widetilde{C}^{n-1}_{loc}(]a,b[)$, and

$$\lim_{l\to+\infty} u^{(j-1)}_{k_l}(t) = u^{(j-1)}(t) \quad (j=1,\dots,n-1) \quad \text{uniformly in} \quad]a,b[. \tag{2.55}$$

On the other hand, for any $t_0 \in]a,b[$ and natural l, we have

$$(t-t_0)u^{(n-1)}_{k_l}(t) = u^{(n-2)}_{k_l}(t) - u^{(n-2)}_{k_l}(t_0) + \int_{t_0}^{t} (s-t_0)(\Lambda_k(u_{k_l})(s) + q_{k_l}(s))ds. \tag{2.56}$$

Hence, due to (2.25), (2.47), (2.55), and Lemma 2.6 we get

$$\lim_{l\to+\infty} u^{(n-1)}_{k_l}(t) = u^{(n-1)}(t) \quad \text{uniformly in} \quad]a,b[. \tag{2.57}$$

Now it is clear that (2.55), (2.57), and (2.37) results in (2.40) and (2.41). Therefore, $u \in \widetilde{C}^{n-1,\,m}_{loc}(]a,b[)$. On the other hand, from (2.53) it is obvious that u is a solution of (1.1). In the case where $n=2m$, from (2.54) equalities (1.2) follow, that is, u is a solution of problem (1.1), (1.2).

Let us show that u is the solution of that problem in the case $n=2m+1$ as well. In view of (2.54), it suffice to prove that $u^{(m)}(b)=0$. First we find an estimate for the sequence $\{u_k\}_{k=1}^{+\infty}$. For this, without loss of generality we assume that

$$t_1 \le t_{1k} \qquad (k \in N). \tag{2.58}$$

From (2.44), by (2.39) and (2.49), it follows the existence of a positive constant ρ_0, independent of k, such that

$$\begin{aligned} &|u^{(m+1)}_k(t)| \le \\ &\le \rho_0 + \frac{1}{(m-1)!}\Big(\Big|\int_{t_1}^{t}(t-s)^{m-1}\Lambda_k(u_k)(s)ds\Big| + \Big|\int_{t_1}^{t}(t-s)^{m-1}q_k(s)ds\Big|\Big) \end{aligned} \tag{2.59}$$

for $t_1 \le t \le t_{1k}$, and

$$||q_k||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \le \rho_0, \tag{2.60}$$

for $k \in N$. On the other hand, it is evident that

$$\begin{aligned} \Big|\int_{t_1}^{t}(t-s)^{m-1}\Lambda_k(u_k)(s)ds\Big| \le \sum_{j=1}^{m}\Big|\int_{t_1}^{t}(t-s)^{m-1}p_j(s)u^{(j-1)}_k(s)ds\Big| + \\ + \sum_{j=1}^{m}\Big|\int_{t_1}^{t}(t-s)^{m-1}p_j(s)\Big(\int_{s}^{\mu_j(t_{0k},t_{1k},s)} u^{(j)}_k(\xi)d\xi\Big)ds\Big| \end{aligned} \tag{2.61}$$

for $t_1 \le t \le t_{1k}$ $(k \in N)$.

Let, now $m > 1$. From Lemma 2.2 and condition (2.37) we get the estimates

$$\int\limits_{t_1}^{t} \frac{|u_k^{(j)}(s)|^2}{(b-s)^{2m-2j}} ds \le \int\limits_{t_0}^{t_{1k}} \frac{|u_k^{(j)}(s)|^2}{(t_{1k}-s)^{2m-2j}} ds \le 2^{2m} r_0^2 \tag{2.62}$$

for $t_1 \le t \le t_{1k}$ $(j = 1, \dots, m)$. Then by conditions (2.38) we find

$$\Big|\int\limits_{t_1}^{t} (t-s)^{m-1} p_j(s) u_k^{(j-1)}(s) ds\Big| =$$

$$= \Big|\int\limits_{t_1}^{t} \frac{1}{(b-s)^{2m-j}} \Big(\frac{\partial}{\partial s} \frac{(t-s)^{m-1} u_k^{(j-1)}(s)}{s-a}\Big)\Big((b-s)^{2m-j} \int\limits_{t_1}^{s} (\xi - a) p_j(\xi) d\xi\Big) ds\Big| \le$$

$$\begin{aligned} &\le \frac{4m\rho_j}{b-a}\Big(\int\limits_{t_1}^{t} \frac{|u_k^{(j-1)}(s)|}{(b-s)^{m-j+2}} ds + \int\limits_{t_1}^{t} \frac{|u_k^{(j)}(s)|}{(b-s)^{m-j+1}} ds\Big) \le \\ &\le \frac{4m\rho_j}{b-a}\Big[\Big(\int\limits_{t_1}^{t} \frac{(u_k^{(j-1)}(s))^2}{(b-s)^{2m-2j+2}} ds\Big)^{1/2} + \Big(\int\limits_{t_1}^{t} \frac{(u_k^{(j)}(s))^2}{(b-s)^{2m-2j}} ds\Big)^{1/2}\Big] \times \\ &\times \Big(\int\limits_{t_1}^{t} (b-s)^{-2} ds\Big)^{1/2} \le \frac{2^m m r_0 \rho_j}{b-a} (b-t)^{-1/2} \end{aligned} \tag{2.63}$$

for $t_1 \le t \le t_{1k}$ $(j = 1, \dots, m)$. On the other hand, by the Schwartz inequality, the definition of the functions μ_j and (2.4) it is clear that

$$\begin{aligned} \Big|\int\limits_{s}^{\mu_j(t_{0k}, t_{1k}, s)} u_k^{(j)}(\xi) d\xi\Big| &\le \frac{2^{m-j}}{(2m-2j-1)!!} \lambda_j(b, t_{0k}, t_{1k}, s) \Big(\int\limits_{t_{0k}}^{t_{1k}} |u_k^{(m)}(\xi)|^2 d\xi\Big)^{1/2} \le \\ &\le 2^m r_0 \lambda_j(b, t_{0k}, t_{1k}, s) \end{aligned} \tag{2.64}$$

for $t_1 < s \le t_{1k}$ $(j = 1, \dots, m)$. Then by the integration by parts and (2.38), (2.64) we get

$$\Big|\int\limits_{t_1}^{t} (t-s)^{m-1} p_j(s) \Big(\int\limits_{s}^{\mu_j(t_{0k}, t_{1k}, s)} u_k^{(j)}(\xi) d\xi\Big) ds\Big| \le$$

$$\begin{aligned} &\le 2^m r_0 \Big|\int\limits_{t_1}^{t} \Big|\frac{\partial}{\partial s} \frac{(t-s)^{m-1}}{s-a}\Big| \Big(\int\limits_{t_1}^{s} (\xi - a) |p_j(\xi)| \lambda_j(b, t_{0k}, t_{1k}, \xi) d\xi\Big) ds\Big| \le 2^m r_0 \times \\ &\times \overline{\rho}_j \int\limits_{t_1}^{t} \Big|\frac{\partial}{\partial s} \frac{(t-s)^{m-1}}{s-a}\Big| (b-s)^{\gamma_{1j} - m + 1/2} ds \le 2^m r_0 \overline{\rho}_j \times \end{aligned} \tag{2.65}$$

$$\times\int\limits_{t_1}^{t}\Big(\frac{m-1}{s-a}+\frac{t-a}{(s-a)^2}\Big)(b-s)^{\gamma_{1j}-3/2}ds\leq\frac{(m+1)2^{m+1}r_0\overline{\rho}_j(b-a)^{\gamma_{1j}}}{b-a}\times$$

$$\times\int\limits_{t_1}^{t}(b-s)^{-3/2}ds\leq\frac{(m+1)2^{m+2}r_0(b-a)^{\gamma_{1j}}\overline{\rho}_j}{b-a}(b-t)^{-1/2}$$

for $t_1<s\leq t_{1k}$ $(j=1,\dots,m)$.

Thus from (2.61), by (2.63) and (2.65) we have

$$\Big|\int\limits_{t_1}^{t}(t-s)^{m-1}\Lambda_k(u_k)(s)ds\Big|\leq\kappa_0(b-t)^{-1/2} \tag{2.66}$$

for $t_1\leq t\leq t_{1k},\ m>1$, where $\kappa_0=\frac{r_0(m+1)2^{m+2}}{b-a}\sum\limits_{j=1}^{m}(\rho_j+\overline{\rho}_j(b-a)^{\gamma_{1j}})$.

Let, now $m=1$, then due to (2.37), (2.38), and (2.64) we obtain

$$\begin{aligned}&\Big|\int\limits_{t_1}^{t}(t-s)^{m-1}\Lambda_k(u_k)(s)ds\Big|=\Big|\int\limits_{t_1}^{t}p_1(s)u_k(s)ds+\\
&+\int\limits_{t_1}^{t}p_1(s)\Big(\int\limits_{s}^{\mu_1(t_{01},t_{1k},s)}u_k'(\xi)d\xi\Big)ds\Big|\leq\frac{|u_k(t)|}{(t-a)}\Big|\int\limits_{t_1}^{t}(s-a)p_1(s)ds\Big|+\\
&+\Big|\int\limits_{t_1}^{t}\Big(\frac{|u_k'(s)|}{(s-a)(b-s)}+\frac{|u_k(s)|}{(s-a)^2(b-s)}\Big)\Big((b-s)\int\limits_{t_1}^{s}(\xi-a)p_1(\xi)d\xi\Big)ds\Big|+\\
&+\frac{2r_0}{t_1-a}\int\limits_{t_1}^{t}(s-a)|p_1(s)|\lambda_1(b,t_{01},t_{1k},s)ds\leq\frac{2\rho_1}{b-a}\Big[\frac{|u_k(t)|}{b-t}+\\
&+r_0\Big(\int\limits_{t_1}^{t}\frac{1}{(b-s)^2}ds\Big)^{1/2}+\frac{2}{b-a}(t-t_1)^{1/2}\Big(\int\limits_{t_1}^{t}\frac{u_k^2(s)}{(b-s)^2}ds\Big)^{1/2}\Big]+\\
&+\frac{4r_0\,\overline{\rho}_1}{b-a}(b-t)^{\gamma_{11}-1/2}\quad\text{for}\quad t_1\leq t\leq t_{1k}.\end{aligned} \tag{2.67}$$

On the other hand, from (2.24), (2.37), and Lemma 2.2 it follow the estimates

$$|u_k(t)|=\Big|\int\limits_{t}^{t_{1k}}u_k'(s)ds\Big|\leq\Big((t_{1k}-t)\int\limits_{t}^{t_{1k}}u_k'^2(s)ds\Big)^{1/2}\leq r_0(b-t)^{1/2},$$

$$\int\limits_{t}^{t_{1k}}\frac{u_k^2(s)}{(b-s)^2}ds\leq\int\limits_{t}^{t_{1k}}\frac{u_k^2(s)}{(t_{1k}-s)^2}ds\leq 2r_0,$$

for $t_1 \le t \le t_{1k}$. Then from (2.67) by these inequalities we get

$$\begin{gathered}\Big|\int\limits_{t_1}^{t}(t-s)^{m-1}\Lambda_k(u_k)(s)ds\Big| \le \frac{2\rho_1}{b-a}\Big(\frac{2r_0}{(b-t)^{1/2}}+\frac{4r_0}{(b-a)^{1/2}}\Big)+\\ +\frac{4r_0\overline{\rho}_1}{(b-a)}(b-t)^{\gamma_{11}-1/2} \le \kappa_1((b-t)^{-1/2}+(b-t)^{\gamma_{11}-1/2})+\kappa_2,\end{gathered} \tag{2.68}$$

where $\kappa_1=\frac{4r_0}{b-a}(\rho_1+\overline{\rho}_1)$, $\kappa_2=\frac{8r_0}{(b-a)^{3/2}}\rho_1$.

If $m>1$, due to conditions (2.60) and the fact that $n=2m+1$, we have

$$\Big|\int\limits_{t_1}^{t}(t-s)^{m-1}q_k(s)ds\Big| = (m-1)\Big|\int\limits_{t_1}^{t}(t-s)^{2m-n-1}\Big((t-s)^{n-m-1}\int\limits_{t_1}^{s}|q_k(\xi)|d\xi\Big)ds\Big| \le$$

$$\le m(b-t)^{-1/2}||q_k||_{\widetilde{L}^2_{2n-2m-2,2m-2}} \le m\rho_0(b-t)^{-1/2} \quad \text{for} \quad t_1 \le t < b, \tag{2.69}$$

and if $m=1$,

$$\int\limits_{t_1}^{t}\Big|\int\limits_{t_1}^{s}q_k(\xi)d\xi\Big|ds \le (b-t)^{1/2}||q_k||_{\widetilde{L}^2_{0,0}} \le \rho_0(b-t)^{1/2} \quad \text{for} \quad t_1 \le t < b. \tag{2.70}$$

Also it is clear that

$$u_k^{(m)}(t) = \int\limits_{t_{1k}}^{t} u_k^{(m+1)}(s)ds, \tag{2.71}$$

since $u_k^{(m)}(t_{1k})=0$.

Now, from (2.59), by (2.66) and (2.69) if $m>1$, and by (2.68) if $m=1$, we have, respectively,

$$\begin{gathered}|u^{(m+1)}(t)| \le \rho_0+(m\rho_0+\kappa_0)(b-t)^{-1/2},\\ |u^{(m+1)}(t)| \le \rho_0+\kappa_2+\kappa_1[(b-t)^{-1/2}+(b-t)^{\gamma_{11}-1/2}]+\int\limits_{t_1}^{t}|q_k(s)|ds,\end{gathered} \tag{2.72}$$

for $t_1 \le t \le t_{1k}$. From (2.71), by (2.72), and (2.70), it follows the existence of a constant $\rho^*>0$ such that

$$|u_k^{(m)}(t)| \le \rho^*[(b-t)^{1/2}+(b-t)^{\gamma_{11}+1/2}] \quad \text{for} \quad t_1 \le t < t_{1k},\ m \ge 1,$$

from which, in view of (2.25), (2.55), and (2.57), it is evident that $u^{(m)}(b)=0$. Thus we have proved that u is the solution of problem (1.1), (1.2) also in the case $n=2m+1$.

To complete the proof of the lemma, it remains to show that equality (2.41) is satisfied. First note that in the space $\widetilde{C}^{n-1,m}(]a,b[)$ problem (1.1), (1.2) does not have another solution since in that space the homogeneous problem (1.1_0), (1.2) has only the trivial solution. Now assume the contrary. Then there exist $\delta \in]0, \frac{b-a}{2}[$, $\varepsilon > 0$, and an increasing sequence of natural numbers $\{k_l\}_{l=1}^{+\infty}$ such that

$$\max\Big\{ \sum_{j=1}^{n} |u_{k_l}^{(j-1)}(t) - u^{(j-1)}(t)| : a+\delta \le t \le b-\delta \Big\} > \varepsilon \quad (l \in N). \tag{2.73}$$

By virtue of the Arzela-Ascoli lemma and condition (2.37) the sequence $\{u_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ ($j = 1, \dots, m$), without loss of generality, can be assumed to be uniformly converging in $]a, b[$. Then, in view of what we have shown above, conditions (2.55) and (2.57) hold. But this contradicts condition (2.73). The obtained contradiction proves the validity of the lemma. □

Analogously we can prove the following lemma if we apply Lemma 2.7 instead of Lemma 2.6.

Lemma 2.9. *Let condition* (2.28) *hold, for every natural k problem* (2.26), (2.27) *have a solution $u_k \in \widetilde{C}_{loc}^{n-1}(]a, b])$, and let there exist a constant $r_0 > 0$ such that*

$$\int_{t_{0k}}^{b} |u_k^{(m)}(s)|ds \le r_0^2 \quad (k \in N), \tag{2.74}$$

$$\lim_{k\to+\infty} ||q_k - q||_{\widetilde{L}^2_{2n-2m-2}} = 0, \tag{2.75}$$

and the homogeneous problem (1.1_0), (1.3) *has only the trivial solution in the space $\widetilde{C}^{n-1,m}(]a, b])$. Then the nonhomogeneous problem* (1.1), (1.3) *has a unique solution u such that inequality* (2.40) *holds, and*

$$\lim_{k\to+\infty} u_k^{(j-1)}(t) = u^{(j-1)}(t) \quad (j = 1, \dots, n) \quad \textit{uniformly in} \quad]a, b] \tag{2.76}$$

(that is, uniformly on $[a + \delta, b]$ for an arbitrarily small $\delta > 0$).

To prove Lemma 2.11 we need the following proposition, which is a particular case of Lemma 4.1 in [8].

Lemma 2.10. *If $u \in C_{loc}^{n-1}(]a, b[)$, then for any $s, t \in]a, b[$ the equality*

$$(-1)^{n-m} \int_s^t (\xi - a)^{n-2m} u^{(n)}(\xi) u(\xi) d\xi = w_n(t) - w_n(s) + \nu_n \int_s^t |u^{(m)}(\xi)|^2 d\xi \tag{2.77}$$

is valid, where $\nu_{2m} = 1$, $\nu_{2m+1} = \frac{2m+1}{2}$, $w_{2m}(t) = \sum_{j=1}^{m} (-1)^{m+j-1} u^{(2m-j)}(t)u(t)$,

$$w_{2m+1}(t) = \sum_{j=1}^{m} (-1)^{m+j}[(t-a)u^{(2m+1-j)}(t) - ju^{(2m-j)}(t)]u^{(j-1)}(t) - \frac{t-a}{2}|u^{(m)}(t)|^2.$$

Lemma 2.11. *Let $a_0 \in]a, b[$, $b_0 \in]a_0, b[$, the functions h_j and the operators f_j be given by equalities* (1.10) *and* (1.11). *Let, moreover, $\tau_j \in M(]a, b[)$, and the constants $l_{k,j} > 0$, $\gamma_{kj} > 0$ $(k = 0, 1;\ j = 1, \dots, m)$ be such that conditions* (1.12)-(1.14) *are fulfilled. Then there exist positive constants δ and r_1 such that if $a_0 \in]a, a+\delta[$, $b_0 \in]b-\delta, b[$, $t_0 \in]a, a_0[$, $t_1 \in]b_0, b[$, and $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a, b[)$, an arbitrary solution $u \in C^{n-1}_{loc}(]a, b[)$ of the problem*

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t) u^{(j-1)}(\mu_j(t_0, t_1, t)) + q(t), \tag{2.78}$$

$$u^{(i-1)}(t_0) = 0 \ \ (i = 1, \dots, m), \qquad u^{(j-1)}(t_1) = 0 \ \ (j = 1, \dots, n-m) \tag{2.79}$$

satisfies the inequality

$$\begin{aligned} &\int_{t_0}^{t_1} |u^{(m)}(s)|^2 ds \le \\ &\le r_1 \Big(\Big| \sum_{j=1}^{m} \int_{a_0}^{b_0} (s-a)^{n-2m} p_j(s) u(s) u^{(j-1)}(\mu_j(t_0, t_1, s)) ds \Big| + ||q||^2_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \Big). \end{aligned} \tag{2.80}$$

Proof. From conditions (1.12) and (1.13) it follows the existence of constants $\overline{\ell}_{kj} \ge 0$ such that

$$(t-a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(a, \tau_j)(t, s) \le \overline{\ell}_{0j} \quad \text{for} \quad a < t \le s \le a_0,$$

$$(b-t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(b, \tau_j)(t, s) \le \overline{\ell}_{1j} \quad \text{for} \quad b_0 \le s \le t < b.$$

Consequently, all the requirements of Lemma 2.3 with $\overline{p}_j(t) = (t-a)^{n-2m}(-1)^{n-m} p_j(t)$, $a < t_0 < a_0$, and Lemma 2.4 with $\overline{p}_j(t) = (b-t)^{n-2m}(-1)^{n-m} p_j(t)$, $b_0 < t_1 < b$, are fulfilled. Also from condition (1.14) and the definition of a constant ν_n, it follows the existence of $\nu \in]0, 1[$ such that

$$\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} \ell_{kj} < \nu_n - 2\nu \ \ (k = 0, 1). \tag{2.81}$$

On the other hand, without loss of generality we can assume that $a_0 \in]a, a+\delta[$ and $b_0 \in]b-\delta, b[$, where δ is a constant such that

$$\sum_{j=1}^{m} (\overline{l}_{0j} \beta_j(\delta, \gamma_{0j}) + \overline{l}_{1j} \beta_j(\delta, \gamma_{1j})) < \nu, \tag{2.82}$$

where the functions β_j are defined by (2.6). Let now $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a, b[)$, u be a solution of problem (2.78), (2.79), and

$$r_1 = 2^{2m+1}(1 + b - a)^2 \nu^{-2}. \tag{2.83}$$

Multiplying both sides of (2.78) by $(-1)^{n-m}(t-a)^{n-2m}u(t)$ and then integrating from t_0 to t_1, by Lemma 2.10 we obtain

$$
\begin{aligned}
&(n-2m)\frac{t_0-a}{2}|u^{(m)}(t_0)|^2+\nu_n\int_{t_0}^{t_1}|u^{(m)}(s)|^2ds=\\
&=(-1)^{n-m}\sum_{j=1}^{m}\int_{t_0}^{t_1}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds+ \\
&+(-1)^{n-m}\int_{t_0}^{t_1}(s-a)^{n-2m}q(s)u(s)ds.
\end{aligned}
\tag{2.84}
$$

From Lemma 2.3 with $\overline{p}_j(t)=(t-a)^{n-2m}(-1)^{n-m}p_j(t)$, Lemma 2.4 with $\overline{p}_j(t)=(b-t)^{n-2m}(-1)^{n-m}p_j(t)$, and the equalities $\rho_0(t_0)=\rho_1(t_1)=0$, by (2.81) we get

$$
\begin{aligned}
&(-1)^{n-m}\sum_{j=1}^{m}\int_{t_0}^{a_0}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\le\\
&\le\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}l_{0j}\rho_0(a_0)+\sum_{j=1}^{m}\overline{l}_{0j}\beta_j(a-a_0,\gamma_{0j})\rho_0(\tau^*)\le\\
&\le(\nu_n-2\nu)\rho_0(a_0)+\sum_{j=1}^{m}\overline{l}_{0j}\beta_j(\delta,\gamma_{0j})\int_{t_0}^{t_1}|u^{(m)}(s)|^2ds,
\end{aligned}
\tag{2.85}
$$

$$
\begin{aligned}
&(-1)^{n-m}\sum_{j=1}^{m}\int_{b_0}^{t_1}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\le\\
&\le\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}l_{1j}\rho_1(b_0)+\sum_{j=1}^{m}\overline{l}_{1j}\beta_j(b_0-b,\gamma_{1j})\rho_1(\tau_*)\le\\
&\le(\nu_n-2\nu)\rho_1(b_0)+\sum_{j=1}^{m}\overline{l}_{1j}\beta_j(\delta,\gamma_{1j})\int_{t_0}^{t_1}|u^{(m)}(s)|^2ds.
\end{aligned}
\tag{2.86}
$$

If along with this we take into account inequalities (2.82) and $a_0\le b_0$, we find

$$
(-1)^{n-2m}\sum_{j=1}^{m}\int_{t_0}^{t_1}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\le
$$

$$\leq \Big|\sum_{j=1}^{m}\int_{a_0}^{b_0}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\Big|+$$

$$+(\nu_n-2\nu)\Big(\rho_0(a_0)+\rho_1(b_0)\Big)+\nu\int_{t_0}^{t_1}|u^{(m)}(s)|^2ds \leq (\nu_n-\nu)\int_{t_0}^{t_1}|u^{(m)}(s)|^2ds+ \tag{2.87}$$

$$+\Big|\sum_{j=1}^{m}\int_{a_0}^{b_0}(s-a)^{n-2m}p_j(s)u(s)u^{(j-1)}(\mu_j(t_0,t_1,s))ds\Big|.$$

On the other hand, if we put $c=(a+b)/2$, then again on the basis of Lemmas 2.1, 2.2, and Young's inequality we get

$$\Big|\int_{t_0}^{t_1}(s-a)^{n-2m}q(s)u(s)ds\Big| \leq \Big|\int_{t_0}^{c}(s-a)^{n-2m}q(s)u(s)ds\Big|+\Big|\int_{c}^{t_1}(s-a)^{n-2m}q(s)u(s)ds\Big|=$$

$$=\Big|\int_{t_0}^{c}[(n-2m)u(s)+(s-a)^{n-2m}u'(s)]\Big(\int_{s}^{c}q(\xi)d\xi\Big)ds\Big|+$$

$$+\Big|\int_{c}^{t_1}[(n-2m)u(s)+(s-a)^{n-2m}u'(s)]\Big(\int_{c}^{s}q(\xi)d\xi\Big)ds\Big| \leq$$

$$\leq \Big[(n-2m)\Big(\int_{t_0}^{c}\frac{u^2(s)}{(s-a)^{2m}}ds\Big)^{1/2}+\Big(\int_{t_0}^{c}\frac{u'^2(s)}{(s-a)^{2m-2}}ds\Big)^{1/2}\Big]\times$$

$$\times\Big(\int_{t_0}^{c}(s-a)^{2n-2m-2}\Big(\int_{s}^{c}q(\xi)d\xi\Big)^2ds\Big)^{1/2}+$$

$$+(1+b-a)\Big[(n-2m)\Big(\int_{c}^{t_1}\frac{u^2(s)}{(b-s)^{2m}}ds\Big)^{1/2}+\Big(\int_{c}^{t_1}\frac{u'^2(s)}{(b-s)^{2m-2}}ds\Big)^{1/2}\Big]\times$$

$$\times\Big(\int_{c}^{t_1}(b-s)^{2m-2}\Big(\int_{c}^{s}q(\xi)d\xi\Big)^2ds\Big)^{1/2} \leq 2^{m+1}(1+b-a)||q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}\times$$

$$\times\Big[\Big(\int_{t_0}^{c}|u^{(m)}(s)|^2ds\Big)^{1/2}+\Big(\int_{c}^{t_1}|u^{(m)}(s)|^2ds\Big)^{1/2}\Big] \leq$$

$$\leq \frac{\nu}{2}\int_{t_0}^{t_1}|u^{(m)}(s)|^2ds+2^{2m+3}(1+b-a)^2\nu^{-1}||q||^2_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}. \tag{2.88}$$

In view of inequalities (2.87), (2.88) and notation (2.83), equality (2.84) results in estimate (2.80). □

The proof of the following lemma is analogous to that of Lemma 2.11.

Lemma 2.12. *Let $a_0 \in]a,b[$, the functions h_j and the operators f_j be given by equalities* (1.10) *and* (1.11)*. Let, moreover, $\tau_j \in M(]a,b])$, constants $l_{0,j} > 0$, $\gamma_{0j} > 0$, $(j = 1,\dots,m)$ be such that conditions* (1.12) *and* (1.21) *are fulfilled. Then there exists a positive constant r_1 such that for any $t_0 \in]a, a_0[$, and $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$, an arbitrary solution $u \in C^{n-1}_{loc}(]a,b])$ of the problem*

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t) u^{(j-1)}(\mu_j(t_0,b,t)) + q(t), \tag{2.89}$$

$$u^{(i-1)}(t_0) = 0 \quad (i = 1,\dots,m), \qquad u^{(j-1)}(b) = 0 \quad (j = m+1,\dots,n) \tag{2.90}$$

satisfies the inequality

$$\int_{t_0}^{b} |u^{(m)}(s)|^2 ds \le r_1 \Big(\Big| \sum_{j=1}^{m} \int_{a_0}^{b} (s-a)^{n-2m} p_j(s) u(s) u^{(j-1)}(\mu_j(t_0,b,s)) ds \Big| + ||q||^2_{\widetilde{L}^2_{2n-2m-2}} \Big).$$

Lemma 2.13. *Let $\tau_j \in M(]a,b[)$, $a_0 \in]a,b[$, $b_0 \in]a_0,b[$, conditions* (1.7), (1.12)- (1.14), *hold, and let in the case when n is odd, in addition* (1.8) *be fulfilled, where the functions h_j, β_j and the operators f_j are given by equalities* (1.10)-(1.11)*, and l_{kj}, $\overline{l}_{kj}$, γ_{kj} $(k = 0,1;\ j = 1,\dots,m)$ are nonnegative numbers. Moreover, let the homogeneous problem* (1.1_0), (1.2) *in the space $\widetilde{C}^{n-1,m}(]a,b[)$ have only the trivial solution. Then there exist $\delta \in]0, \frac{b-a}{2}[$ and $r > 0$ such that for any $t_0 \in]a, a+\delta]$, $t_1 \in]b+\delta, b]$, and $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ problem* (2.78), (2.79) *is uniquely solvable in the space $\widetilde{C}^{n-1}(]a,b[)$, and its solution admits the estimate*

$$\Big(\int_{t_0}^{t_1} |u^{(m)}(s)|^2 ds \Big)^{1/2} \le r ||q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}. \tag{2.91}$$

Proof. First note that all the requirements of Lemma 2.11 are fulfilled, and in view of (1.8) and (1.13), conditions (2.38) of Lemma 2.8 hold.

Let, now $\delta \in]0, \min\{b-b_0, a_0-a\}]$ be such as in Lemma 2.11 and assume that estimate (2.91) is invalid. Then for an arbitrary natural k there exist

$$t_{0k} \in]a,\ a+\delta/k[, \qquad t_{1k} \in]b-\delta/k,\ b[, \tag{2.92}$$

and a function $q_k \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ such that problem (2.23), (2.24) has a solution $u_k \in \widetilde{C}^{n-1}(]a,b[)$, satisfying the inequality

$$\Big(\int_{t_{0k}}^{t_{1k}} |u_k^{(m)}(s)| ds \Big)^{1/2} > k ||q_k||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}. \tag{2.93}$$

In the case when the homogeneous equation

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t) u^{(j-1)}(\mu_j(t_{0k}, t_{1k}, t)) \tag{2.33_0}$$

under the boundary conditions (2.24) has a nontrivial solution, in (2.23) we put that $q_k(t) \equiv 0$ and assume that u_k is that nontrivial solution of problem (2.33_0), (2.24).

Let now

$$v_k(t) = \Big(\int\limits_{t_{0k}}^{t_{1k}} |u_k^{(m)}(s)|ds\Big)^{-1/2} u_k(t), \quad q_{0k}(t) = \Big(\int\limits_{t_{0k}}^{t_{1k}} |u_k^{(m)}(s)|ds\Big)^{-1/2} q_k(t). \tag{2.94}$$

Then v_k is a solution of the problem

$$\begin{gathered} v^{(n)}(t) = \sum_{i=1}^{m} p_i(t) v^{(i-1)}(\mu_i(t_{0k}, t_{1k}, t)) + q_{0k}(t) \quad \text{for} \quad t_{0k} \le t \le t_{1k}, \\ v^{(i-1)}(t_{0k}) = 0 \ (i = 1, \dots, m), \qquad v^{(i-1)}(t_{1k}) = 0 \ (i = 1, \dots, n-m). \end{gathered} \tag{2.95}$$

Moreover, in view of (2.93), it is clear that

$$\int\limits_{t_{0k}}^{t_{1k}} |v_k^{(m)}(s)|^2 ds = 1, \quad ||q_{0k}||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} < \frac{1}{k} \ \ (k \in N). \tag{2.96}$$

On the other hand, in view of the fact that problem (1.1_0), (1.2) has only the trivial solution in the space $\widetilde{C}^{n-1,m}(]a, b[)$, by Lemmas 2.8, 2.11, and (2.96) we have

$$\begin{gathered} \lim_{t\to+\infty} v_k^{(j-1)}(t) = 0 \quad \text{uniformly in} \quad]a, b[\ \ (j = 1, \dots n), \\ 1 < r_0 \Big(\Big| \int\limits_{a_0}^{b_0} (s-a)^{n-2m} \Lambda_k(v_k)(s) ds \Big| + k^{-2} \Big) \ \ (k \in N), \end{gathered} \tag{2.97}$$

where r_0 is a positive constant independent of k. Now, if we pass to the limit in (2.97) as $k \to +\infty$, by Lemma 2.6 we obtain the contradiction $1 < 0$. Consequently, for any solution of problem (2.78), (2.79), with arbitrary $q \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a, b[)$, estimate (2.91) holds. Thus the homogeneous equation

$$v^{(n)}(t) = \sum_{j=1}^{m} p_j(t) v^{(j-1)}(\mu_j(t_0, t_1, t)) \quad \text{for} \quad t_0 \le t \le t_1, \tag{2.82_0}$$

under conditions (2.79), has only the trivial solution. But for arbitrarily fixed $t_0 \in]a, a + \delta[$, $t_1 \in]b - \delta, b[$, and $q \in L([t_0, t_1])$ problem (2.78), (2.79) is regular and has the Fredholm property in the space $\widetilde{C}^{n-1}(]t_0, t_1[)$. Thus problem (2.78), (2.79) is uniquely solvable. $\square$

Analogously we can prove the following lemma if we apply Lemmas 2.7 and 2.12 instead of Lemmas 2.6 and 2.11.

Lemma 2.14. *Let $\tau_j \in M(]a, b[)$, $a_0 \in]a, b[$, conditions* (1.9), (1.12) *and* (1.21) *hold, where the functions h_j, β_j and the operators f_j are given by equalities* (1.10)-(1.11), *and l_{0j}, $\bar{l}_{0j}$ γ_{0j} $(j = 1, \dots, m)$ are nonnegative numbers. Let, moreover, the homogeneous problem* (1.1_0), (1.3) *in the space $\widetilde{C}^{n-1}(]a, b])$ have only the trivial solution. Then there exist* *positive constants* δ *and* r *such that if* $a_0 \in]a, a+\delta[$, *and* $q \in \widetilde{L}^2_{2n-2m-2}(]a,b])$, *problem* (2.89), (2.90) *is uniquely solvable in the space* $\widetilde{C}^{n-1}(]a,b])$, *and its solution admits the estimate* $\int\limits_{t_0}^{b} |u^{(m)}(s)|^2 ds \leq r||q||_{\widetilde{L}^2_{2n-2m-2}}$.

Lemma 2.15. *Let* $\tau_j \in M(]a,b[)$, $\alpha \geq 0$, $\beta \geq 0$, *and let there exist* $\delta \in]0, b-a[$ *such that*

$$|\tau_j(t) - t| \leq k_1(t-a)^{\beta} \quad \text{for} \quad a < t \leq a+\delta. \tag{2.98}$$

Then

$$\Big|\int\limits_{t}^{\tau(t)} (s-a)^{\alpha} ds\Big| \leq \begin{cases} k_1[1+k_1\delta^{\beta-1}]^{\alpha}(t-a)^{\alpha+\beta} & \text{for } \beta \geq 1 \\ k_1[\delta^{1-\beta}+k_1]^{\alpha}(t-a)^{\alpha\beta+\beta} & \text{for } 0 \leq \beta < 1 \end{cases},$$

for $a < t \leq a+\delta$.

Proof. First note that

$$\Big|\int\limits_{t}^{\tau(t)} (s-a)^{\alpha} ds\Big| \leq (\max\{\tau(t), t\} - a)^{\alpha} |\tau(t) - t| \quad \text{for} \quad a \leq t \leq a+\delta,$$

and $\max\{\tau(t), t\} \leq t + |\tau(t) - t|$ for $a \leq t \leq a+\delta$. Then in view of condition (2.98) we get

$$\Big|\int\limits_{t}^{\tau(t)} (s-a)^{\alpha} ds\Big| \leq k_1[(t-a) + k_1(t-a)^{\beta}]^{\alpha}(t-a)^{\beta} \quad \text{for} \quad a \leq t \leq a+\delta.$$

From this inequality it immediately follows the validity of the lemma. ☐

Analogously, one can prove

Lemma 2.16. *Let* $\tau_j \in M(]a,b[)$, $\alpha \geq 0$, $\beta \geq 0$ *and let there exist* $\delta \in]0, b-a[$ *such that*

$$|\tau_j(t) - t| \leq k_1(b-t)^{\beta} \quad \text{for} \quad b-\delta \leq t < b. \tag{2.99}$$

Then

$$\Big|\int\limits_{t}^{\tau(t)} (b-t)^{\alpha} ds\Big| \leq \begin{cases} k_1[1+k_1\delta^{\beta-1}]^{\alpha}(b-t)^{\alpha+\beta} & \text{for } \beta \geq 1 \\ k_1[\delta^{1-\beta}+k_1]^{\alpha}(b-t)^{\alpha\beta+\beta} & \text{for } 0 \leq \beta < 1 \end{cases},$$

for $b-\delta \leq t < b$.

3 Proofs

Proof of Theorem 1.1 (Theorem 1.2). Suppose problem (1.1_0), (1.2) (problem (1.1_0), (1.3)) has only the trivial solution, and r and δ are the numbers appearing in Lemma 2.13 (Lemma 2.14). Set

$$t_{0k} = a + \delta/k \qquad t_{1k} = b - \delta/k \quad (k \in N). \tag{3.1}$$

By Lemma 2.13 (Lemma 2.14), for every natural k, problem (2.78), (2.79) in the space $\widetilde{C}^{n-1}_{loc}(]a,\,b[)$ (problem (2.89), (2.90) in the space $\widetilde{C}^{n-1}_{loc}(]a,\,b]))$ has a unique solution u_k, and

$$\Big(\int\limits_{t_{0k}}^{t_{1k}}|u_k^{(m)}(s)|^2ds\Big)^{1/2}\le r||q||_{\widetilde{L}^2_{2n-2m-2,2m-2}}\ \Big(\Big(\int\limits_{t_{0k}}^{b}|u_k^{(m)}(s)|^2ds\Big)^{1/2}\le r||q||_{\widetilde{L}^2_{2n-2m-2}}\Big),\quad (3.2)$$

where the constant r does not depend on q. From (3.2), by Lemma 2.8 with $r_0=r||q||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}}$ (by Lemma 2.9 with $r_0=r||q||_{\widetilde{L}^2_{2n-2m-2}}$), it follows that problem (1.1), (1.2) (problem (1.1), (1.3)) in the space $\widetilde{C}^{n-1}_{loc}(]a,\,b[)$ $(\widetilde{C}^{n-1}_{loc}(]a,\,b]))$ is uniquely solvable for an arbitrary $q\in\widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\,b[)$ $(q\in\widetilde{L}^2_{2n-2m-2}(]a,\,b]))$. Thus that problem has Fredholm's property, and its solution admits estimate (1.15) (estimate (1.22)). □

Proof of Corollary 1.1. In view of conditions (1.18), there exists a number $\varepsilon>0$ such that

$$\sum_{j=1}^{m}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\Big(\frac{\kappa_{kj}}{2m-j}+\varepsilon\Big)<1\ (k=0,1).\qquad (3.3)$$

On the other hand, in view of conditions (1.19) and (1.20) we have

$$\begin{gathered}(t-a)^{2m-j}h_j(t,s)\le\frac{\kappa_{0j}}{2m-j}+\kappa_{1j}\int\limits_a^{a_0}\frac{(\xi-a)^{2m-j}}{(b-\xi)^{2m+1-j}}d\xi+\int\limits_a^{a_0}(\xi-a)^{n-j}p_{0j}(\xi)d\xi\\ \text{for}\quad a<t\le s\le a_0,\\ (b-t)^{2m-j}h_j(t,s)\le\frac{\kappa_{1j}}{2m-j}+\kappa_{0j}\int\limits_{b_0}^{b}\frac{(b-\xi)^{2m-j}}{(\xi-a)^{2m-j+1}}d\xi+\\ +(b-a)^{n-2m}\int\limits_{b_0}^{b}(b-\xi)^{2m-j}p_{0j}(\xi)d\xi\quad\text{for}\quad b_0\le s\le t<b.\end{gathered}\qquad (3.4)$$

Let δ be the constant defined in Lemmas 2.15, 2.16. From (1.19) it follows the existence of $a_0\in]a,a+\delta[$ and $b_0\in]b-\delta,b[$ such that

$$|p_1(t)|\le\frac{\kappa}{[(t-a)(b-t)]^{2n}}+p_{01}(t)\quad\text{for}\quad t\in[a,a_0]\cup[b_0,b].\qquad (3.5)$$

On the other hand, from lemmas 2.15, and 2.16 by the condition (1.17) it follows the existence of a constant k_0 such that

$$\begin{gathered}\Big|\int\limits_t^{\tau_j(t)}(s-a)^{2(m-j)}ds\Big|^{1/2}\le k_0^{1/2}(s-a)^{m-j+\nu_{0j}/2}\quad\text{for}\quad a\le t\le a_0,\\ \Big|\int\limits_t^{\tau_j(t)}(b-s)^{2(m-j)}ds\Big|^{1/2}\le k_0^{1/2}(b-s)^{m-j+\nu_{1j}/2}\quad\text{for}\quad b_0\le t\le b.\end{gathered}\qquad (3.6)$$

Consequently, if $p_{01} \in L_{n-j,\,2m-j}(]a, b[)$, then by (1.16) and (3.6), from (1.19) and (1.20) it follows the existence of a nonnegative constant k_2 such that

$$\begin{aligned} (t-a)^{m-1} f_j(a,\tau_1)(t,s) &\le k_2(a_0-a)^{\varepsilon_0} \quad \text{for} \quad a \le t < s \le a_0, \\ (b-t)^{m-1} f_j(b,\tau_1)(t,s) &\le k_2(b-b_0)^{\varepsilon_0} \quad \text{for} \quad b_0 \le s < t \le b, \end{aligned} \tag{3.7}$$

where $0 < \varepsilon_0 = \min\{\nu_{k1} - 2n - 2 + 2k(2m-n),\, \nu_{kj} - 2 : k = 0, 1;\ j = 2, \dots, m\}$. Now, from (3.4), and (3.7) it is clear that we can choose $\delta_1 \le \delta$ so that if $\max\{b - b_0,\, a_0 - a\} \le \delta_1$, then

$$\begin{aligned} (t-a)^{2m-j} h_j(t,s) &\le \frac{\kappa_{0j}}{2m-j} + \varepsilon \quad \text{for} \quad a < t \le s \le a_0, \\ (b-t)^{2m-j} h_j(t,s) &\le \frac{\kappa_{1j}}{2m-j} + \varepsilon \quad \text{for} \quad b_0 \le s \le t < b, \end{aligned}$$

$j \in \{1, \dots, m\}$. From (3.7), the last inequalities and (3.3), it is clear that all the assumptions of Theorem 1.1, with $\ell_{kj} = \frac{\kappa_{kj}}{2m-j} + \varepsilon$, $\gamma_{kj} = 1/2$, and $\max\{b - b_0,\, a_0 - a\} \le \delta_1$, are fulfilled, and thus the corollary is valid. ☐

Proof of Theorem 1.3. It suffice to show that if $u \in \widetilde{C}^{n-1}_{loc}(]a, b[)$ $(u \in \widetilde{C}^{n-1}_{loc}(]a, b]))$ is a solution of problem (1.1_0), (1.2) $((1.1_0), (1.3))$, then

$$\int_a^b |u^{(m)}(s)|^2 ds < +\infty. \tag{3.8}$$

For an arbitrary $t_0 \in]a, b[$ we have

$$\begin{aligned} u^{(m)}(t) = w(t_0) &+ \frac{1}{(n-m-1)!} \int_{t_0}^t (t-s)^{n-m-1} \Big(\sum_{j=1}^m p_j(s) u^{(j-1)}(s) \Big) ds, + \\ &+ \frac{1}{(n-m-1)!} \int_{t_0}^t (t-s)^{n-m-1} \Big(\sum_{j=1}^m p_j(s) \int_s^{\tau_j(s)} u^{(j)}(\xi) d\xi \Big) ds, \end{aligned} \tag{3.9}$$

where $w(t_0) = \sum\limits_{j=m+1}^n \frac{(t_0-a)^{j-m-1}}{(j-m-1)!} u^{(j-1)}(t_0)$. Now note that by the equalities

$$|u^{(i)}(t)| = \frac{1}{(k-i-1)!} \Big| \int_c^t (t-s)^{k-i-1} u^{(k)}(s) ds \Big| \quad \text{for} \quad a < t < b, \tag{3.10}$$

$k = 1, \dots, m,\ i = 0, \dots, k-1,$ with $c = a$, from (3.9) we get the estimate

$$\begin{aligned} |u^{(m)}(t)| \le |w(t_0)| &+ (1 - \delta_{1m}) ||u^{(m-1)}||_C \sum_{j=1}^{m-1} \Big(\int_t^{t_0} (s-a)^{n-j-1} |p_j(s)| ds + \\ &+ \int_t^{t_0} (s-a)^{n-m-1} |p_j(s)| \Big| \int_s^{\tau_j(s)} (\xi - a)^{m-j-1} d\xi \Big| ds \Big) + \\ &+ ||u^{(m-1)}||_C \int_t^{t_0} (s-a)^{n-m-1} |p_m(s)| ds \quad \text{for} \quad a < t < t_0, \end{aligned} \tag{3.11}$$

where δ_{ij} is Kronecker's delta. Then conditions (1.28) yield

$$|u^{(m)}(t)| \le |w(t_0)| + (1-\delta_{1m})||u^{(m-1)}||_C \int\limits_t^{t_0} (s-a)^{-1} p(s)ds +$$

$$+\gamma||u^{(m-1)}||_C \int\limits_t^{t_0} p(s)ds + ||u^{(m-1)}||_C \int\limits_t^{t_0} (s-a)^{n-m-1}|p_m(s)|ds \quad \text{for} \quad a<t<t_0,$$

where $p(t) = \sum\limits_{j=1}^{m} (t-a)^{n-j}|p_j(t)|$,

$$\gamma_j = \operatorname*{ess\,sup}_{a<t<b} \frac{1}{|t-a|^{m+1-j}} \Big| \int\limits_t^{\tau_j(t)} (\xi-a)^{m-j-1} d\xi \Big|, \quad \gamma = \max\{\gamma_1, \dots, \gamma_m\}.$$

Consequently, in view of condition (1.29), $u^{(m)} \in L([a,t_0])$. Analogously, by (3.10) with $c=b$, we can show that $u^{(m)} \in L([t_0,b])$. Finally $u^{(m)} \in L([a,b])$ and if we put $v(t) = \int\limits_a^t |u^{(m)}(s)|ds$, then

$$v \in C([a,b]), \tag{3.12}$$

and from (3.10) it is clear that

$$|u^{(i)}(t)| \le (t-a)^{m-i-1} v(t) \ (i=1,\dots,m-1) \quad \text{for} \quad a<t<t_0. \tag{3.13}$$

In view of condition (1.29) we can choose $\delta > 0$ such that

$$\int\limits_a^{a+\delta} p(s)ds < \frac{1}{2}. \tag{3.14}$$

From (3.9), by conditions (1.28), (3.12) and inequality (3.13), we get

$$|u^{(m)}(t)| \le |w(t_0)| + \int\limits_t^{t_0} \frac{p(s)v(s)}{s-a} ds + \sum_{j=1}^{m} \int\limits_t^{t_0} (s-a)^{n-m-1}|p_j(s)| \Big| \int\limits_s^{\tau_j(s)} (\xi-a)^{m-j-1} v(\xi) d\xi \Big| ds \le$$

$$\le |w(t_0)| + \int\limits_t^{t_0} \frac{p(s)v(s)}{s-a} ds + \gamma||v||_C \int\limits_a^{a_0} p(s)ds, \quad \text{for} \quad a<t<a+\delta.$$

Consequently, if $w_0 = |w(t_0)| + \gamma||v||_C \int\limits_a^{a_0} p(s)ds$, then

$$|u^{(m)}(t)| \le w_0 + \int\limits_t^{t_0} \frac{p(s)v(s)}{s-a} ds \quad \text{for} \quad a<t<a+\delta. \tag{3.15}$$

From the last inequality, by the integration by parts and (3.14), we get

$$v(t) \le w_0(t-a) + (t-a)\int\limits_t^{t_0} \frac{p(s)v(s)}{s-a}ds + \frac{1}{2}v(t) \quad \text{for} \quad a < t < a+\delta.$$

The last inequality, by the Gronwall-Bellman lemma, results in

$$\frac{v(t)}{t-a} \le 2w_0 e^{2\int_t^{t_0} p(s)ds} \le 2w_0 e \quad \text{for} \quad a < t < a+\delta.$$

Due to this inequality, from (3.15) by (3.14) we get $|u^{(m)}(t)| \le w_0(1+e)$ for $a < t < a+\delta$. Analogously we can show that $u^{(m)}$ is bounded in the neighborhood of the point b. Therefore, condition (3.8) is satisfied. □

Proof of Theorem 1.4. From Theorem 1.1 by conditions (1.30)-(1.33) it is obvious that problem (1.1), (1.2) has Fredholm's property. Thus to prove Theorem 1.4, it suffice to show that the homogeneous problem (1.1_0), (1.2) has only the trivial solution in the space $\widetilde{C}^{n-1,m}(]a,b[)$. Suppose $u \in \widetilde{C}^{n-1,m}(]a,b[)$ is a solution of problem (1.1_0), (1.2). Then from Theorem 1.1 it is clear that

$$\rho = \int\limits_a^b |u^{(m)}(s)|^2 ds < +\infty. \tag{3.16}$$

Multiplying both sides of (1.1_0) by $(-1)^{n-m}(t-a)^{n-2m}u(t)$ and integrating from t_0 to t_1, by Lemma 2.10 we obtain

$$w_n(t) - w_n(s) + \nu_n \int\limits_s^t |u^{(m)}(\xi)|^2 d\xi = (-1)^{n-m}\sum_{j=1}^m \int\limits_s^t (\xi-a)^{n-2m} p_j(\xi) u^{(j-1)}(\tau_j(\xi))u(\xi)d\xi.$$

Moreover, from Lemma 2.5 it is evident that

$$\liminf_{s\to a} |w_n(s)| = 0, \quad \liminf_{t\to b} |w_n(t)| = 0.$$

Then by (3.16) we get

$$\nu_n \rho = (-1)^{n-m}\sum_{j=1}^m \int\limits_a^b (\xi-a)^{n-2m} p_j(\xi) u^{(j-1)}(\tau_j(\xi))u(\xi)d\xi. \tag{3.17}$$

According to (1.32), (1.33) and (3.16), all the conditions of Lemmas 2.3 and 2.4 with $\overline{p}_j(t) = (-1)^{n-m}(t-a)^{n-2m}p_j(t)$, $a_0 = b_0 = t^*$, $t_0 = a$, $t_1 = b$ and $\mu_j(t_0,t_1,t) = \tau_j(t)$ hold. Consequently, due to equalities $\rho_0(a) = \rho_1(b) = 0$, we have

$$\begin{aligned} &(-1)^{n-m}\int\limits_a^b (\xi-a)^{n-2m} p_j(\xi) u^{(j-1)}(\tau_j(\xi))u(\xi)d\xi \le \\ &\le \bar{l}_{0j}\beta_j(t^*-a,\gamma_{0j})\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(t^*) + l_{0j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\rho_0(t^*) + \\ &+\bar{l}_{1j}\beta_j(b-t^*,\gamma_{1j})\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t^*) + l_{1j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}\rho_1(t^*) \end{aligned} \tag{3.18}$$

for $a < t^* < b$. On the other hand, due to conditions (1.30) and (1.31), the number $\nu \in]0,1[$ can be chosen such that inequalities

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(l_{0j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar{l}_{0j}\beta_j(t^*-a,\gamma_{0j})\Big)<\frac{\nu_n-\nu}{2},\\
&\sum_{j=1}^{m}\Big(l_{1j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar{l}_{1j}\beta_j(b-t^*,\gamma_{1j})\Big)<\frac{\nu_n-\nu}{2}
\end{aligned} \tag{3.19}$$

are satisfied. Thus according to (3.18), (3.19), and inequalities $\rho_0^{1/2}(\tau^*)\rho_0^{1/2}(t^*) \le \rho$, $\rho_1^{1/2}(\tau_*)\rho_1^{1/2}(t^*) \le \rho$, (3.17) implies the inequality $\nu_n\rho \le (\nu_n-\nu)\rho$, and consequently, $\rho = 0$. Hence, by

$$|u(t)| = \frac{1}{(k-1)!}\Big|\int_a^t (t-s)^{m-1}u^{(m)}(s)ds\Big| \le (t-a)^{m-1/2}\rho \quad \text{for} \quad a<t<b,$$

we have $u(t) \equiv 0$. □

Proof of Theorem 1.5. The proof is analogous to that of Theorem 1.4. The only difference is that instead of Theorem 1.1, Theorem 1.2 is applied. □

Proof of Theorem 1.6. Let u be a nonzero solution of the problem (1.1_0), (1.2). Then analogously to Theorem 1.4, from conditions (1.40),(1.41), (1.32) and (1.33) it follow the validity of relations (3.16), (3.17), (3.18) and the existence of $\nu \in]0,1[$ such that

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(l_{0j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar{l}_{0j}\beta_j(t^*-a,\gamma_{0j})\Big)<\nu_n-\nu,\\
&\sum_{j=1}^{m}\Big(l_{1j}\frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!}+\bar{l}_{1j}\beta_j(b-t^*,\gamma_{1j})\Big)<\nu_n-\nu.
\end{aligned} \tag{3.20}$$

For the constants τ^* and τ_*, appearing in inequality (3.18), which are defined in Lemmas 2.3 and 2.4 (with $t_0 = a$, $t_1 = b$, $a_0 = b_0 = t^*$, and $\mu_j(t_0,t_1,t) = \tau_j(t)$), from the condition (1.42) we have the estimates

$$\tau^* \le t^* \quad \text{for} \quad a<t\le t^*, \qquad t^* \le \tau_* \quad \text{for} \quad t^* \le t < b.$$

By the last estimates, from (3.18) it immediately follows the inequality $\nu_n\rho \le (\nu_n-\nu)\rho$. Thus $u \equiv 0$.

□

Acknowledgement

This work is supported by the Academy of Sciences of the Czech Republic (Institutional Research Plan # AV0Z10190503) and by the Shota Rustaveli National Science Foundation (Project # GNSF/ST09_175_3-101).

References

[1] R. P. Agarwal, *Focal boundary value problems for differential and difference equations,* Mathematics and Its applications, vol. 436, Kluwer Academic Publishers, Dordrecht, 1998.

[2] R. P. Agarwal and D. O'Regan, *Singular differential and integral equations with applications,* Kluwer Academic Publishers, Dordrecht, 2003.

[3] R. P. Agarwal, I. Kiguradze, *Two-point boundary value problems for higher-order linear differential equations with strong singularities,* Boundary Value Problems 2006, 1-32; Article ID 83910.

[4] E. Bravyi, *A not on the Fredholm property of boundary value problems for linear functional differential equations,* Mem. Differential Equations Math. Phys. **20** (2000), 133-135.

[5] S. A. Brykalov, *Problems for functional-differential equations with monotone bounadry conditions,* (Russian) Differential'nye Uravneniya **32** (1996), No. 6, 731-738; English transl.: Differential equations **32** (1996), No. 6, 740-747.

[6] I. T. Kiguradze, *On a singular multi-point boundary value problem,* Ann. Mat. Pura Appl. **86** (1970), 367-399.

[7] I. T. Kiguradze, *Some singular boundary value problems for ordinary differential equations,* (Russian) Tbilisi University Press, Tbilisi, 1975.

[8] I. T. Kiguradze and T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous or- dinary differential equations,* Mathematics and Its Applications (Soviet Series), vol. 89, Kluwer Academic Publishers, Dordrecht, 1993, Translated from the 1985 Russian original.

[9] I. Kiguradze, G. Tskhovrebadze, *On two-point boundary value problems for systems of higher order ordinary differential equations with singularities,* Georgian Mathematical Journal **1** (1994), no. 1, 31-45.

[10] I. Kiguradze, B. Půža, *On certain singular boundary value problem for linear differential equations with deviating arguments,* Czechoslovak Math. J **47** (1997), no. 2, 233-244.

[11] I. Kiguradze, B. Půža, *On the Vallee-Poussin problem for singular differential equations with deviating arguments,* Arch. Math. **33** (1997), No. 1-2, 127-138.

[12] I. Kiguradze, B. Půža, *On boundary value problems for systems of linear functional differential equations,* Czechoslovak Math. J. **47** (1997), No. 2, 341-37

[13] I. Kiguradze, B. Půža, and I. P. Stavroulakis, *On singular boundary value problems for functional differential equations of higher order*, Georgian Mathematical Journal **8** (2001), no. 4, 791-814.

[14] I. Kiguradze, *Some optimal conditions for the solvability of two-point singular boundary value problems,* Functional Differential Equations **10** (2003), no. 1-2, 259-281, Functional differential equations and applications (Beer-Sheva, 2002).

[15] T. Kiguradze, *On conditions for linear singular boundary value problems to be well posed,*(Russian) Differential'nye Uravneniya, **46** (2010), No. 2, pp. 183-190; English transl.: Differ. Equations, 46(2010), No. 2, pp. 187-194.

[16] I. Kiguradze, *On two-point boundary value problems for higher order singular ordinary differential equations,* Mem. Differential Equations Math. Phys. **31** (2004), 101-107.

[17] A. Lomtatidze, *On one boundary value problem for linear ordynary differential equations of second order with singularities,* Differential'nye Uravneniya **222** (1986), No. 3, 416-426.

[18] S. Mukhigulashvili, *Two-point boundary value problems for second order functional -differential equations,* Mem. Differential Equations Math. Phys. **20** (2000), 1-112.

[19] S. Mukhigulashvili, N. Partsvania, *On two-point boundary value problems for higher order functional differential equations with strong singularities,* Mem. Differential Equations Math. Phys. (accepted).

[20] B. Půža, *On a singular two-point boundary value problem for the nonlinear mth-order differential equation with deviating arguments,* Georgian Mathematical Journal **4** (1997), no. 6, 557-566.

[21] B. Půža and A. Rabbimov, *On a weighted boundary value problem for a system of singular functional-differential equations,* Mem. Differential Equations Math. Phys. **21** (2000), 125-130.

[22] Š. Schwabik, M. Tvrdy, and O. Vejvoda, *Differential and integral equations, boundary value problems and adjoints,* Academia, Praha,(1979).

(Received July 12, 2011)

Authors' addresses:

Sulkhan Mukhigulashvili

1. Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic.

2. Ilia State University, Faculty of Physics and Mathematics, 32 I. Chavchavadze St., Tbilisi 0179, Georgia.

E-mail: `mukhig@ipm.cz`

Nino Partsvania

1. A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 University St., Tbilisi 0186, Georgia.

2. International Black Sea University, 2 David Agmashenebeli Alley 13km, Tbilisi 0131, Georgia.

E-mail: `ninopa@rmi.ge`

The Dirichlet Boundary Value Problems For Strongly Singular Higher-Order Nonlinear Functional-Differential Equations

S. Mukhigulashvili, Brno

15.12.2011

Abstract

The a priori boundedness principle is proved for the Dirichlet boundary value problems for strongly singular higher-order nonlinear functional-differential equations. Several sufficient conditions of solvability of the Dirichlet problem under consideration are derived from the a priori boundedness principle. The proof of the a priori boundedness principle is based on the Agarwal–Kiguradze type theorems, which guarantee the existence of the Fredholm property for strongly singular higher-order linear differential equations with argument deviations under the two-point conjugate and right-focal boundary conditions.

Key words and phrases: Higher order functional-differential equations, Dirichlet boundary value problem, strong singularity, Fredholm property, a priori boundedness principle.

2000 Mathematics Subject Classification: 34K06, 34K10

1 Statement of the main results

1.1. Statement of the problem and a survey of the literature. Consider the functional differential equation

$$u^{(n)}(t) = F(u)(t) \tag{1.1}$$

with the two-point boundary conditions

$$u^{(i-1)}(a) = 0 \ (i = 1, \cdots, m), \quad u^{(i-1)}(b) = 0 \ (i = 1, \cdots, n-m). \tag{1.2}$$

Here $n \geq 2$, m is the integer part of $n/2$, $-\infty < a < b < +\infty$, and the operator F acting from the set of $(m-1)$-th time continuously differentiable on $]a,\ b[$ functions, to the set $L_{loc}(]a,\ b[)$. By $u^{(j-1)}(a)$ $(u^{(j-1)}(b))$ we denote the right (the left) limit of the function $u^{(j-1)}$ at the point $a\,(b)$.

The problem is singular in the sense that for an arbitrary x the right-hand side of equation (1.41) may have nonintegrable singularities at the points a and b.

Throughout the paper we use the following notations:

$R^+ = [0, +\infty[$;

$[x]_+$ the positive part of number x, that is $[x]_+ = \frac{x+|x|}{2}$;

$L_{loc}(]a, b[)$ $(L_{loc}(]a, b]))$ is the space of functions $y :]a,\ b[\to R$, which are integrable on $[a+\varepsilon,\ b-\varepsilon]$ for arbitrary small $\varepsilon > 0$;

$L_{\alpha,\beta}(]a, b[)$ $(L^2_{\alpha,\beta}(]a, b[))$ is the space of integrable (square integrable) with the weight $(t-a)^\alpha (b-t)^\beta$ functions $y :]a,\ b[\to R$, with the norm

$$||y||_{L_{\alpha,\beta}} = \int_a^b (s-a)^\alpha (b-s)^\beta |y(s)| ds \quad \Big(||y||_{L^2_{\alpha,\beta}} = \Big(\int_a^b (s-a)^\alpha (b-s)^\beta y^2(s) ds \Big)^{1/2} \Big);$$

$L([a, b]) = L_{0,0}(]a, b[)$, $L^2([a, b]) = L^2_{0,0}(]a, b[)$;

$M(]a, b[)$ is the set of the measurable functions $\tau :]a, b[\to]a, b[$;

$\widetilde{L}^2_{\alpha,\beta}(]a, b[)$ $(\widetilde{L}^2_\alpha(]a, b]))$ is the Banach space of $y \in L_{loc}(]a, b[)$ $(L_{loc}(]a, b]))$ functions, with the norm

$$||y||_{\widetilde{L}^2_{\alpha,\beta}} \equiv \max \Big\{ \Big[\int_a^t (s-a)^\alpha \Big(\int_s^t y(\xi) d\xi \Big)^2 ds \Big]^{1/2} : a \le t \le \frac{a+b}{2} \Big\} +$$

$$+ \max \Big\{ \Big[\int_t^b (b-s)^\beta \Big(\int_t^s y(\xi) d\xi \Big)^2 ds \Big]^{1/2} : \frac{a+b}{2} \le t \le b \Big\} < +\infty.$$

$L_n(]a, b[)$ is the Banach space of $y \in L_{loc}(]a, b[)$ functions, with the norm

$$||y||_{\widetilde{L}^2_{\alpha,\beta}} = \sup \Big\{ [(s-a)(b-t)]^{m-1/2} \int_s^t (\xi - a)^{n-2m} |y(\xi)| d\xi : \quad a < s \le t < b \Big\} < +\infty.$$

$C^{n-1}_{loc}(]a,\ b[)$, $(\widetilde{C}^{n-1}_{loc}(]a,\ b[))$ is the space of the functions $y :]a,\ b[\to R$, which are continuous (absolutely continuous) together with $y',\ y'',\ \cdots, y^{(n-1)}$ on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.

$\widetilde{C}^{n-1,\, m}(]a,\ b[)$ is the space of the functions $y \in \widetilde{C}^{n-1}_{loc}(]a,\ b[)$, such that

$$\int_a^b |x^{(m)}(s)|^2 ds < +\infty. \tag{1.3}$$

$C^{m-1}_1(]a,\ b[)$ is the Banach space of the functions $y \in C^{m-1}_{loc}(]a,\ b[)$, such that

$$\begin{aligned} \limsup_{t \to a} \frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \ (i = 1, \cdots, m), \\ \limsup_{t \to b} \frac{|x^{(i-1)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \ (i = 1, \cdots, n-m), \end{aligned} \tag{1.4}$$

with the norm:
$$||x||_{C_1^{m-1}} = \sum_{i=1}^{m} \sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \;:\; a<t<b\Big\},$$
where $\alpha_i(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2}$.

$\widetilde{C}_1^{m-1}(]a,\, b[)$ is the Banach space of the functions $y \in \widetilde{C}_{loc}^{m-1}(]a,\, b[)$, such that conditions (1.7) and (1.4) hold, with the norm:
$$||x||_{\widetilde{C}_1^{m-1}} = \sum_{i=1}^{m} \sup\Big\{\frac{|x^{(i-1)}(t)|}{\alpha_i(t)} \;:\; a<t<b\Big\} + \Big(\int_a^b |x^{(m)}(s)|^2 ds\Big)^{1/2}.$$

$D_n(]a,\, b[\times R^+)$ is the set of such functions $\delta :]a,\, b[\times R^+ \to L_n(]a,\, b[)$ that $\delta(t,\,\cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,\, b[$, and $\delta(\cdot,\, \rho) \in L_n(]a,\, b[)$ for any $\rho \in R^+$.

$D_{2n-2m-2,\, 2m-2}(]a,\, b[\times R^+)$ is the set of such functions $\delta :]a,\, b[\times R^+ \to \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\, b[)$ that $\delta(t,\,\cdot) : R^+ \to R^+$ is nondecreasing for every $t \in]a,\, b[$, and $\delta(\cdot,\, \rho) \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\, b[)$ for any $\rho \in R^+$.

A solution of problem (1.1), (1.2) is sought in the space $\widetilde{C}^{n-1,\, m}(]a,\, b[)$.

The singular ordinary differential and functional-differential equations, have been studied with sufficient completeness under different boundary conditions, see for example [1], [2], [4] – [12], [15], [21]- [25] and the references cited therein. But the equation (1.1), even under the boundary condition (1.2), is not studied in the case when the operator F has the form

$$F(x)(t) = \sum_{j=1}^{m} p_j(t) x^{(j-1)}(\tau_j(t)) + f(x)(t), \tag{1.5}$$

where the singularity of the functions $p_j : L_{loc}([a,\, b])$ be such that the inequalities

$$\begin{gathered}\int_a^b (s-a)^{n-1}(b-s)^{2m-1}[(-1)^{n-m}p_1(s)]_+ ds < +\infty,\\ \int_a^b (s-a)^{n-j}(b-s)^{2m-j}|p_j(s)| ds < +\infty \quad (j=2,\cdots,m),\end{gathered} \tag{1.6}$$

are not fulfilled (in this case we sad that the linear part of the operator F is a strongly singular), the operator f continuously acting from $C_1^{m-1}(]a,\, b[)$ to $L_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}(]a,\, b[)$, and the inclusion

$$\sup\{f(x)(t) : ||x||_{C_1^{m-1}} \le \rho\} \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\, b[). \tag{1.7}$$

holds. The first step in studying of the differential equations with strong singularities was made by R. P. Agarwal and I. Kiguradze in the article [3], where the linear ordinary differential equations under conditions (1.2), in the case when the functions p_j have strong singularities at the points a and b, are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles of I. Kiguradze [13], [14], and N. Partsvania [20]. In the papers [18], [19] these results are generalized for linear differential equation with deviating arguments i.e., are proved the Agarwal-Kiguradze type theorems, which guarantee Fredholm's property for linear differential equation with deviating arguments.

In this paper, on the bases of articles [3], and [17] we prove a priori boundedness principle for the problem (1.1), (1.2) in the case where the operator has form (1.5).

Now we introduce some results from the articles [18], [19], which we need for this work. Consider the equation

$$u^{(n)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \qquad \text{for} \qquad a < t < b. \tag{1.8}$$

For problem (1.8), (1.2) we assume, that when $n = 2m$, then the conditions

$$p_j \in L_{loc}(]a,\, b[) \;\; (j = 1, \cdots, m) \tag{1.9}$$

are fulfilled and when $n = 2m + 1$, along with (1.9), the condition

$$\limsup_{t\to b} \Big|(b-t)^{2m-1} \int_{t_1}^{t} p_1(s)ds\Big| < +\infty \;\; (t_1 = \frac{a+b}{2}) \tag{1.10}$$

holds.

By $h_j :]a,\, b[\times]a,\, b[\to R_+$ and $f_j : [a,\, b] \times M(]a, b[) \to C_{loc}(]a, b[\times]a, b[)$ $(j = 1, \dots, m)$ we denote the functions and operator, respectively defined by the equalities

$$\begin{aligned} h_1(t,s) &= \Big|\int_s^t (\xi - a)^{n-2m}[(-1)^{n-m}p_1(\xi)]_+ d\xi\Big|, \\ h_j(t,s) &= \Big|\int_s^t (\xi - a)^{n-2m}p_j(\xi)d\xi\Big| \quad (j = 2, \cdots, m), \end{aligned} \tag{1.11}$$

and

$$f_j(c,\tau_j)(t,s) = \Big|\int_s^t (\xi - a)^{n-2m}|p_j(\xi)|\Big|\int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)}d\xi_1\Big|^{1/2} d\xi\Big|. \tag{1.12}$$

Let also $k = 2k_1 + 1$ $(k_1 \in N)$, then

$$k!! = \begin{cases} 1 & \text{for} \;\; k \le 0 \\ 1 \cdot 3 \cdot 5 \cdots k & \text{for} \;\; k \ge 1 \end{cases}.$$

Now we can to introduce the main theorem of paper [18].

Theorem 1.1. *Let there exist the numbers* $t^* \in]a,\, b[,\ \ell_{kj} > 0,\ \bar{l}_{kj} \geq 0,$ *and* $\gamma_{kj} > 0$ $(k = 0, 1;\ j = 1, \dots, m)$ *such that along with*

$$B_0 \equiv \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{0j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}\, \bar{l}_{0j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \Big) < \frac{1}{2}, \tag{1.13}$$

$$B_1 \equiv \sum_{j=1}^{m} \Big(\frac{(2m-j)2^{2m-j+1}\, l_{1j}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{0j}}\, \bar{l}_{1j}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \Big) < \frac{1}{2}, \tag{1.14}$$

the conditions

$$(t-a)^{2m-j} h_j(t,s) \leq l_{0j}, \quad (t-a)^{m-\gamma_{0j}-1/2} f_j(a, \tau_j)(t,s) \leq \bar{l}_{0j} \tag{1.15}$$

for $a < t \leq s \leq t^*$, *and*

$$(b-t)^{2m-j} h_j(t,s) \leq l_{1j}, \quad (b-t)^{m-\gamma_{1j}-1/2} f_j(b, \tau_j)(t,s) \leq \bar{l}_{1j} \tag{1.16}$$

for $t^* \leq s \leq t < b$ *hold. Then problem* (1.8), (1.2) *is uniquely solvable in the space* $\widetilde{C}^{n-1,\, m}(]a,\, b[)$.

Also, in [19] is proved the following theorem:

Theorem 1.2. *Let all the conditions of Theorem 1.1 are satisfied. Then the unique solution* u *of problem* (1.8), (1.2) *for every* $q \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a, b[)$ *admit the estimate*

$$||u^{(m)}||_{L^2} \leq r||q||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}, \tag{1.17}$$

with

$$r = \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0,\, B_1\})(2m-1)!!}, \qquad \nu_{2m} = 1, \quad \nu_{2m+1} = \frac{2m+1}{2},$$

and thus constant $r > 0$ *dependent only on the numbers* $l_{kj},\, \bar{l}_{kj}, \gamma_{kj}\, (k = 1, 2;\ j = 1, \cdots, m)$, *and* $a,\ b,\ t^*,\ n$.

Remark 1.1. Under the conditions of Theorem 1.2, for every $q \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a, b[)$ the unique solution u of problem (1.8), (1.2) admits the estimate

$$||u^{(m)}||_{\widetilde{C}^{m-1}_1} \leq r_n||q||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}, \tag{1.18}$$

with

$$r_n = \Big(1 + \sum_{j=1}^{m} \frac{2^{m-j+1/2}}{(m-j)!(2m-2j+1)^{1/2}(b-a)^{m-j+1/2}}\Big) \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0,\, B_1\})(2m-1)!!}.$$

1.2. Theorems on a solvability of problem (1.1), (1.2).

Define the operator $P : C^{m-1}_1(]a,\, b[) \times C^{m-1}_1(]a,\, b[) \to L_{loc}(]a,\ b[)$, by the equality

$$P(x,y)(t) = \sum_{j=1}^{m} p_j(x)(t) y^{(j-1)}(\tau_j(t)) \qquad \text{for} \qquad a < t < b \tag{1.19}$$

where $p_j : C_1^{m-1}(]a,\, b[) \to L_{loc}(]a,\ b[)$, and $\tau_j \in M(]a,\, b[)$. Also for any $\gamma > 0$ define the set A_γ by the relation

$$A_\gamma = \{x \in \widetilde{C}_1^{m-1}(]a,\, b[) : ||x||_{\widetilde{C}_1^{m-1}} \leq \gamma\}. \tag{1.20}$$

For formulate this a priori boundedness principle we have to introduce

Definition 1.1. Let γ_0 and γ be the positive numbers. We said that the continuous operator $P : C_1^{m-1}(]a,\, b[) \times C_1^{m-1}(]a,\, b[) \to L_n(]a,\ b[)$ to be γ_0, γ consistent with boundary condition (1.2) if:

i. for any $x \in A_{\gamma_0}$ and almost all $t \in]a,\ b[$ the inequality

$$\sum_{j=1}^{m} |p_j(x)(t)x^{(j-1)}(\tau_j(t))| \leq \delta(t,\, ||x||_{\widetilde{C}_1^{m-1}})||x||_{\widetilde{C}_1^{m-1}} \tag{1.21}$$

holds, where $\delta \in D_n(]a,\, b[\times R^+)$.

ii. for any $x \in A_{\gamma_0}$ and $q \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\ b[)$ the equation

$$y^{(n)}(t) = \sum_{j=1}^{m} p_j(x)(t)y^{(j-1)}(\tau_j(t)) + q(t) \tag{1.22}$$

under boundary conditions (1.2), has the unique solution y in the space $\widetilde{C}^{n-1,\, m}(]a,\, b[)$ and

$$||y||_{\widetilde{C}_1^{m-1}} \leq \gamma ||q||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}. \tag{1.23}$$

Definition 1.2. We said that the operator P to be γ consistent with boundary condition (1.2), if the operator P be γ_0, γ consistent with boundary condition (1.2) for any $\gamma_0 > 0$.

In the sequel it will always be assumed that the operator F_p defined by equality

$$F_p(x)(t) = |F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)|,$$

continuously acting from $C_1^{m-1}(]a,\, b[)$ to $L_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}(]a,\ b[)$, and

$$\widetilde{F}_p(t, \rho) \equiv \sup\{F_p(x)(t) : ||x||_{C_1^{m-1}} \leq \rho\} \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\ b[) \tag{1.24}$$

for each $\rho \in [0,\, +\infty[$.

Then the following theorem is valid

Theorem 1.3. *Let the operator P be $\gamma_0,\, \gamma$ consistent with boundary condition* (1.2), *and there exist a positive number $\rho_0 \leq \gamma_0$, such that*

$$||\widetilde{F}_p(\,\cdot,\, \min\{2\rho_0, \gamma_0\})||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}} \leq \frac{\gamma_0}{\gamma}. \tag{1.25}$$

Let moreover, for any $\lambda \in]0,\,1[$, an arbitrary solution $x \in A_{\gamma_0}$ of the equation

$$x^{(n)}(t) = (1-\lambda)P(x,\,x)(t) + \lambda F(x)(t) \tag{1.26}$$

under the conditions (1.2), *admits the estimate*

$$||x||_{\widetilde{C}_1^{m-1}} \leq \rho_0. \tag{1.27}$$

Then problem (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,\,b[)$.

From theorem 1.3 with $\rho_0 = \gamma_0$ immediately follows

Corollary 1.1. *Let the operator P be γ_0, γ consistent with boundary condition* (1.2), *and*

$$|F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)| \leq \eta(t,\,||x||_{\widetilde{C}_1^{m-1}}) \tag{1.28}$$

for $x \in A_{\gamma_0}$ and almost all $t \in]a,\,b[$, and

$$||\eta(\cdot,\,\gamma_0)||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} \leq \frac{\gamma_0}{\gamma}, \tag{1.29}$$

where $\eta \in D_{2n-2m-2,\,2m-2}(]a,\,b[\times R^+)$. Then problem (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,\,b[)$.

Corollary 1.2. *Let the operator P be γ consistent with boundary condition* (1.2), *inequality* (1.28) *holds for $x \in \widetilde{C}_1^{m-1}(]a,\,b[)$ and almost all $t \in]a,\,b[$, where $\eta(\cdot,\,\rho) \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\,b[)$ for any $\rho \in R^+$, and*

$$\limsup_{\rho\to+\infty} \frac{1}{\rho}||\eta(\cdot,\ \rho)||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} < \frac{1}{\gamma}. \tag{1.30}$$

Then the problem (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,\,b[)$.

When we discuss problem (1.41), (1.2), and $n = 2m+1$, we assume that the continuous operator $p_1 : \widetilde{C}_1^{m-1}(]a,\,b[) \to L_{loc}(]a,\,b[)$, by such that

$$\limsup_{t\to b} \Big|(b-t)^{2m-1}\int_{t_1}^{t} p_1(x)(s)ds\Big| < +\infty \ \ (t_1 = \frac{a+b}{2}) \tag{1.31}$$

for any $x \in \widetilde{C}_1^{m-1}(]a,\,b[)$.

Now define the operators $h_j : C_1^{m-1}(]a,\,b[)\times]a,\,b[\times]a,\,b[\to L_{loc}(]a,\,b[\times]a,\,b[)$, $f_j : C_1^{m-1}(]a,\,b[) \times [a,\,b] \times M(]a,b[) \to C_{loc}(]a,\,b[\times]a,\,b[)$ $(j = 1,\ldots,m)$ by the equalities

$$\begin{gathered} h_1(x,t,s) = \Big|\int_s^t (\xi-a)^{n-2m}[(-1)^{n-m}p_1(x)(\xi)]_+d\xi\Big|, \\ h_j(x,t,s) = \Big|\int_s^t (\xi-a)^{n-2m}p_j(x)(\xi)d\xi\Big| \quad (j = 2,\cdots,m), \end{gathered} \tag{1.32}$$

and

$$(1.33) \qquad f_j(x,c,\tau_j)(t,s) = \Big| \int_s^t (\xi - a)^{n-2m} |p_j(x)(\xi)| \Big| \int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)} d\xi_1 \Big|^{1/2} d\xi \Big|.$$

Theorem 1.4. *Let the continuous operator* $P : C_1^{m-1}(]a,\, b[) \times C_1^{m-1}(]a,\, b[) \to L_n(]a,\, b[)$ *admits to the condition* (1.21) *where* $\delta \in D_n(]a,\, b[\times R^+)$, $\tau_j \in M(]a,\, b[)$ *and the numbers* $\gamma_0,\, t^* \in]a,\, b[,\ l_{kj} > 0,\, \bar{l}_{kj} > 0,\ \gamma_{kj} > 0\,(k = 1, 2;\ j = 1, \cdots, m)$, *be such that the inequalities*

$$(1.34) \qquad (t-a)^{2m-j} h_j(x,t,s) \le l_{0j}, \quad \limsup_{t\to a} (t-a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(x,a,\tau_j)(t,s) \le \bar{l}_{0j}$$

for $a < t \le s \le t^*,\ ||x||_{\widetilde{C}_1^{m-1}} \le \gamma_0$,

$$(1.35) \qquad (b-t)^{2m-j} h_j(x,t,s) \le l_{1j}, \quad \limsup_{t\to b} (b-t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(x,b,\tau_j)(t,s) \le \bar{l}_{1j}$$

for $t^* \le s \le t < b,\ ||x||_{\widetilde{C}_1^{m-1}} \le \gamma_0$, *and conditions* (1.13), (1.14) *hold. Let moreover the operator* F *and function* $\eta \in D_{2n-2m-2,\,2m-2}(]a,\, b[\times R^+)$ *be such that condition* (1.28) *and inequality*

$$(1.36) \qquad ||\eta(\cdot,\, \gamma_0)||_{\widetilde{L}^2_{2n-2m-2,\,2m-2}} < \frac{\gamma_0}{r_n},$$

be fulfilled, where

$$r_n = \Big(1 + \sum_{j=1}^m \frac{2^{m-j+1/2}}{(m-j)!(2m-2j+1)^{1/2}(b-a)^{m-j+1/2}}\Big) \frac{2^m(1+b-a)(2n-2m-1)}{(\nu_n - 2\max\{B_0,\, B_1\})(2m-1)!!}.$$

Then problem (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,\, b[)$.

Theorem 1.5. *Let the operator* F *and function* η *are such that condition* (1.28), (1.30) *hold and the continuous operator* $P : C_1^{m-1}(]a,\, b[) \times C_1^{m-1}(]a,\, b[) \to L_n(]a,\, b[)$ *admits condition* (1.21) *where* $\delta \in D_n(]a,\, b[\times R^+)$. *Let moreover the measurable functions* $\tau_j \in M(]a,\, b[)$ *and the numbers* $t^* \in]a,\, b[,\ l_{kj} > 0,\ \bar{l}_{kj} > 0,\ \gamma_{kj} > 0, (k = 0, 1;\ j = 1, \cdots, m)$ *be such that the inequalities*

$$(1.37) \qquad (t-a)^{2m-j} h_j(x,t,s) \le l_{0j}, \quad \limsup_{t\to a} (t-a)^{m-\frac{1}{2}-\gamma_{0j}} f_j(x,a,\tau_j)(t,s) \le \bar{l}_{0j}$$

for $a < t \le s \le t^*,\ x \in \widetilde{C}_1^{m-1}(]a,\, b[)$,

$$(1.38) \qquad (b-t)^{2m-j} h_j(x,t,s) \le l_{1j}, \quad \limsup_{t\to b} (b-t)^{m-\frac{1}{2}-\gamma_{1j}} f_j(x,b,\tau_j)(t,s) \le \bar{l}_{1j}$$

for $t^* \le s \le t < b,\ x \in \widetilde{C}_1^{m-1}(]a,\, b[)$, *and conditions* (1.13), (1.14) *hold. Then problem* (1.1), (1.2) *is solvable in the space* $\widetilde{C}^{n-1,m}(]a,\, b[)$.

Remark 1.2. Let $\gamma_0 > 0$, operators $\alpha_j(t)p_j(x)(t)$ $(j = 1, \cdots, m)$ continuously acting from the space $C_1^{m-1}(]a, b[)$ to the space $L_n(]a, b[)$, exist the function $\delta_j \in D_n(]a, b[)$ such that for any $x \in A_{\gamma_0}$

$$|p_j(x)(t)|\alpha_j(t) \leq \delta_j(t, ||x||_{\widetilde{C}_1^{m-1}}) \quad \text{for} \quad a < t < b, \tag{1.39}$$

and exists constants $\kappa > 0$, $\varepsilon > 0$ such that

$$\begin{aligned} |\tau_j(t) - t| &\leq \kappa(t-a) \ \ (j = 1, \cdots, m) \quad \text{for} \quad a < t < a + \varepsilon, \\ |\tau_j(t) - t| &\leq \kappa(b-t) \ \ (j = 1, \cdots, m) \quad \text{for} \quad b - \varepsilon < t < b, \end{aligned} \tag{1.40}$$

Then the operator P defined by equality (1.19), continuously acting from A_{γ_0} to the space $L_n(]a, b[)$, and there exists the function $\delta \in D_n(]a, b[)$ such that item i of definition 1.1 holds.

Now consider the equation with deviating arguments

$$u^{(n)}(t) = f(t, u(\tau_1(t)), u'(\tau_2(t)), \cdots, u^{(m-1)}(\tau_m(t))) \quad \text{for} \quad a < t < b, \tag{1.41}$$

where $-\infty < a < b < +\infty$, $f :]a, b[\times R^m \to R$ is a function, satisfying the local Caratheodory conditions and $\tau_j \in M(]a, b[)$ $(j = 0, \ldots, n-1)$ are measurable functions.

Corollary 1.3. *Let the functions $\tau_j \in M(]a, b[)$ and the numbers $t^* \in]a, b[$, $\kappa \geq 0$, $\varepsilon > 0$, $l_{kj} > 0$, $\bar{l}_{kj} > 0$, $\gamma_{kj} > 0$, $(k = 0, 1;\ j = 1, \cdots, m)$ be such that the conditions* (1.13)-(1.16), (1.40) *and the inclusions*

$$\alpha_j p_j \in L_n(]a, b[) \quad (j = 1, \cdots, m) \tag{1.42}$$

are fulfilled. Let moreover

$$\begin{aligned} &\Big| f(t, x(\tau_1(t)), x'(\tau_2(t)), \cdots, x^{(m-1)}(\tau_m(t))) - \sum_{j=1}^{m} p_j(t) x^{(j-1)}(\tau_j(t))(t) \Big| \leq \\ &\leq \eta(t, ||x||_{\widetilde{C}_1^{m-1}}) \end{aligned} \tag{1.43}$$

for $x \in \widetilde{C}_1^{m-1}(]a, b[)$ and almost all $t \in]a, b[$, where $\eta(\cdot, \rho) \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a, b[)$ for any $\rho \in R^+$, and condition (1.30) *holds. Then problem* (1.41), (1.2) *is solvable in the space $\widetilde{C}^{n-1,m}(]a, b[)$.*

Remark 1.3. From conditions (1.42) do not follow the conditions (1.6).

Now for illustration of our results consider on $]a, b[$ the second order functional-differential equations

$$u''(t) = -\frac{\lambda |u(t)|^k}{[(t-a)(b-t)]^{2+k/2}} u(\tau(t)) + q(x)(t), \tag{1.44}$$

$$u''(t) = -\frac{\lambda |\sin u^k(t)|}{[(t-a)(b-t)]^2} u(\tau(t)) + q(x)(t), \tag{1.45}$$

where λ, $k \in R^+$ the function $\tau \in M(]a, b[)$, the operator $q : C_1^{m-1}(]a, b[) \to \widetilde{L}^2_{0,0}(]a,\ b[)$ is continuous and

$$\eta(t,\ \rho) \equiv \sup\{|q(x)(t)| : ||x||_{\widetilde{C}_1^{m-1}} \leq \rho\} \in \widetilde{L}^2_{0,0}(]a,\ b[).$$

Than from Theorems 1.4 and 1.5 follows

Corollary 1.4. *Let the function* $\tau \in M(]a,\, b[)$, *the continuous operator* $q : C_1^{m-1}(]a,\, b[) \to \widetilde{L}^2_{0,0}(]a,\, b[)$, *and the numbers* $\gamma_0 > 0$, $\lambda \geq 0$, $k > 0$, *by such that*

$$(1.46)\qquad |\tau(t) - t| \leq \begin{cases} (t-a)^{3/2} & \textit{for} \;\; a < t \leq (a+b)/2 \\ (b-t)^{3/2} & \textit{for} \;\; (a+b)/2 \leq t < b \end{cases},$$

$$(1.47)\qquad ||\eta(t,\, \gamma_0)||_{\widetilde{L}^2_{0,\,0}} \leq \Big(1 + \sqrt{\frac{2}{b-a}}\Big)^{-1} \frac{(b-a)^2 - 16\lambda\gamma_0^k(1 + [2(b-a)]^{1/4})}{2(1+b-a)(b-a)^2},$$

and

$$(1.48)\qquad \lambda < \frac{(b-a)^2}{32\gamma_0^k(1 + [2(b-a)]^{1/4})}.$$

Then the problem (1.44), (1.2) *is solvable.*

Corollary 1.5. *Let the function* $\tau \in M(]a,\, b[)$, *continuous operator* $q : C_1^{m-1}(]a,\, b[) \to \widetilde{L}^2_{0,0}(]a,\, b[)$, *and the number* $\lambda \geq 0$ *by such, that inequalities* (1.30) *with* $n = 2$, (1.46) *and*

$$(1.49)\qquad \lambda < \frac{(b-a)^2}{32(1 + [2(b-a)]^{1/4})},$$

hold. Then the problem (1.45), (1.2) *is solvable.*

2 Auxiliary Propositions

2.1. Lemmas on some properties of the equation $x^{(n)}(t) = \lambda(t)$.

First, we introduce two lemmas without proofs. First Lemma is proved in [3].

Lemma 2.1. *Let* $i \in 1, 2,\;\; x \in \widetilde{C}^{m-1}_{loc}(]t_0,\, t_1[)$ *and*

$$(2.1)\qquad x^{(j-1)}(t_i) = 0 \quad (j = 1, \dots, m), \qquad \int_{t_0}^{t_1} |x^{(m)}(s)|^2 ds < +\infty.$$

Then

$$(2.2)\qquad \Big|\int_{t_i}^{t} \frac{(x^{(j-1)}(s))^2}{(s-t_i)^{2m-2j+2}} ds\Big|^{1/2} \leq \frac{2^{m-j+1}}{(2m-2j+1)!!}\Big|\int_{t_i}^{t} |x^{(m)}(s)|^2 ds\Big|^{1/2}$$

for $t_0 \leq t \leq t_1$.

This second lemma is a particular case of Lemma 4.1 in [7]

Lemma 2.2. *If $x \in C^{n-1}_{loc}(]a, a_1])$, then for any $s, t \in]a, a_1]$ the equality*

$$(-1)^{n-m}\int_s^t (\xi - a)^{n-2m} x^{(n)}(\xi)x(\xi)d\xi = w_n(x)(t) - w_n(x)(s) + \nu_n \int_s^t |x^{(m)}(\xi)|^2 d\xi$$

is valid, where $\nu_{2m} = 1, \quad \nu_{2m+1} = \frac{2m+1}{2}, \quad w_{2m}(x)(t) = \sum_{j=1}^{m}(-1)^{m+j-1}x^{(2m-j)}(t)x(t),$

$$w_{2m+1}(x)(t) = \sum_{j=1}^{m}(-1)^{m+j}[(t-a)x^{(2m+1-j)}(t) - jx^{(2m-j)}(t)]x^{(j-1)}(t) - \frac{t-a}{2}|x^{(m)}(t)|^2.$$

Lemma 2.3. *Let the numbers $a_1 \in]a, b[$, $t_{0k} \in]a, a_1[$, and ε_{ik}, ε_i, β_k, $\beta \in R^+$, $k \in N, i = 1, \cdots, n-m$ are such that*

$$\lim_{k\to+\infty} t_{0\,k} = a, \quad \lim_{k\to+\infty} \beta_k = \beta, \quad \lim_{k\to+\infty} \varepsilon_{i,k} = \varepsilon_i. \tag{2.3}$$

Let, moreover

$$\lambda \in \widetilde{L}^2_{2n-2m-2,0}(]a,\ a_1]), \tag{2.4}$$

is a nonnegative function, $x_k \in \widetilde{C}^{n-1,\,m}(]a,\ b[)$ be a solution of the problem

$$x^{(n)}(t) = \beta_k \lambda(t), \tag{2.5}$$

$$x^{(i-1)}(t_{0\,k}) = 0 \ \ (i = 1, \cdots, m), \qquad x^{(i-1)}(a_1) = \varepsilon_{i,k} \ \ (i = 1, \cdots, n-m), \tag{2.6}$$

and $x \in \widetilde{C}^{n-1,\,m}(]a,\ b[)$ be a solution of the problem

$$x^{(n)}(t) = \beta\lambda(t), \tag{2.7}$$

$$x^{(i-1)}(a) = 0 \ \ (i = 1, \cdots, m), \qquad x^{(i-1)}(a_1) = \varepsilon_i \ \ (i = 1, \cdots, n-m). \tag{2.8}$$

Then

$$\lim_{k\to+\infty} x_k^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, \dots, n) \quad \textit{uniformly in} \quad]a, a_1]. \tag{2.9}$$

Proof. First, prove our lemma under the assumption that there exists the number $r_1 > 0$ such that the estimates

$$\int_{t_{0k}}^{a_1} |x_k^{(m)}(s)|^2 ds \le r_1 \quad k \in N \tag{2.10}$$

hold. Now, suppose that $t_1, \dots, t_n$ are such numbers that $t_{0k} < t_1 < \cdots < t_n < a_1$ $(k \in N)$, and $g_i(t)$ are the polynomials of $(n-1)$-th degree, satisfying the conditions $g_j(t_j) = 1$, $g_j(t_i) = 0 \quad (i \neq j;\ i, j = 1, \dots, n)$. Then if x_k is a solution of the problem (2.5), (2.6), and x is a solution of the problem (2.7), (2.8). For the solution $x - x_k$ of the equation $\frac{d^n(x(t)-x_k(t))}{dt^n} = (\beta - \beta_k)\lambda(t)$, the representation

$$
\begin{aligned}
x(t) - x_k(t) = \sum_{j=1}^{n} \Big((x(t_j) - x_k(t_j)) - \frac{\beta - \beta_k}{(n-1)!} \int_{t_1}^{t_j} (t_j - s)^{n-1} \lambda(s) ds \Big) g_j(t) + \\
+ \frac{\beta - \beta_k}{(n-1)!} \int_{t_1}^{t} (t - s)^{n-1} \lambda(s) ds \quad k \in N \quad \text{for} \quad t_{0k} \le t \le a_1
\end{aligned} \tag{2.11}
$$

is valid. On the other hand in view of inequality (2.10), the identities

$$
x_k^{(i-1)}(t) = \frac{1}{(m-i)!} \int_{t_{0\,k}}^{t} (t-s)^{m-i} x_k^{(m)}(s) ds \qquad (i = 1, 2, \;\; k \in N)
$$

by Schwartz inequality yield

$$
|x_k^{(i-1)}(t)| \le r_2 (t - a)^{m-i-1/2} \quad \text{for} \quad t_{0k} \le t \le a_1 \;\; (i = 1, 2, \;\; k \in N), \tag{2.12}
$$

where $r_2 = \frac{r_1}{(m-i)!\sqrt{2m-2i+1}}$. By virtue of the Arzela-Ascoli lemma and (2.3), (2.12) the sequence $\{x_k\}_{k=1}^{+\infty}$ contains a subsequence $\{x_{k_l}\}_{l=1}^{+\infty}$ which is uniformly convergent in $]a, a_1]$. Suppose $\lim\limits_{l \to +\infty} x_{k_l}(t) = x_0(t)$. Thus from (2.11) by (2.3) it follows the existence of such $r_3 > 0$ that

$$
|x_{k_l}^{(j-1)}(t)| \le r_3 + |x^{(j-1)}(t)| \quad (j = 1, \cdots, n) \quad \text{for} \quad t_{0k_l} \le t \le a_1,
$$

and then without loss of generality we can assume that

$$
\lim_{l \to +\infty} x_{k_l}^{(j-1)}(t) = x_0^{(j-1)}(t) \quad (j = 1, \ldots, n) \quad \text{uniformly in} \quad]a, a_1]. \tag{2.13}
$$

Then in virtue of (2.3), (2.11), and (2.13) we have

$$
x(t) - x_0(t) = \sum_{j=1}^{n} \Big((x(t_j) - x_0(t_j)) \Big) g_j(t) \quad \text{for} \quad a \le t \le a_1.
$$

From the last two relation by (2.10) it is clear that $x^{(n)} = x_0^{(n)}$ and $x_0 \in \widetilde{C}^{n-1,\, m}(]a,\, b[)$. I.e., the function $x_0 \in \widetilde{C}^{n-1,\, m}(]a,\, b[)$ is a solution of problem (2.7), (2.8). In view of (2.4) all the conditions of Theorem 1.1 are fulfilled, thus problem (2.7), (2.8) is uniquely solvable in the space $\widetilde{C}^{n-1,\, m}(]a,\, b[)$ and $x = x_0$. Therefore from (2.13) follows

$$
\lim_{l \to +\infty} x_{k_l}^{(j-1)}(t) = x^{(j-1)}(t) \quad (j = 1, \ldots, n) \quad \text{uniformly in} \quad]a, a_1]. \tag{2.14}
$$

Now suppose that relations (2.9) are not fulfilled. Then there exist $\delta \in]0, \frac{a_1 - a}{2}[$, $\varepsilon > 0$, and the increasing sequence of natural numbers $\{k_l\}_{l=1}^{+\infty}$ such that

$$
\max \Big\{ \sum_{j=1}^{n} |x_{k_l}^{(j-1)}(t) - x^{(j-1)}(t)| : a + \delta \le t \le a_1 \Big\} > \varepsilon \quad (l \in N). \tag{2.15}
$$

By virtue of Arcela-Ascoli lemma and condition (2.10) the sequence $\{x_{k_l}^{(j-1)}\}_{l=1}^{+\infty}$ ($j = 1, \ldots, m$), without loss of generality, can be assumed to be uniformly converging in $]a+\delta,\, a_1]$. Then, in view of what we have shown above, equality (2.14) holds. But this contradicts condition (2.15). Thus (2.9) holds if the conditions (2.10) are fulfilled.

Let now the conditions (2.10) are not fulfilled. Then exists the subsequence $\{t_{0k_l}\}_{l=1}^{+\infty}$of the sequence $\{t_{0k}\}_{k=1}^{+\infty}$, such that

$$\int_{t_{0k}}^{a_1} |x_{k_l}^{(m)}(s)|^2 ds \geq l \ \ (l \in N). \tag{2.16}$$

Suppose that $\beta_l = \Big(\int\limits_{t_{0k}}^{a_1} |x_{k_l}^{(m)}(s)|^2 ds\Big)^{-1}$ and $v_l(t) = u_{k_l}(t)\beta_l$. Thus in view of (2.16) and our notations

$$\int\limits_{t_{0k_l}}^{a_1} |v_{k_l}^{(m)}(s)|^2 ds = 1 \ \ (l \in N), \qquad \lim_{l\to+\infty} \beta_l = 0, \tag{2.17}$$

$$v_l^{(n)}(t) = \beta_l \lambda(t), \tag{2.18}$$

$$v_l^{(i-1)}(t_{0\,k_l}) = 0 \ \ (i = 1, \cdots, m), \quad v_l^{(i-1)}(a_1) = \varepsilon_{i,k_l}\beta_l \ \ (i = 1, \cdots, n-m, \ \ l \in N). \tag{2.19}$$

From the first part of our lemma by (2.17) it follows that there exists limit $\lim\limits_{l\to+\infty} v_l(t) \equiv v_0(t)$, and v_0 is a solution of corresponding of (2.18), (2.19) homogeneous problem. thus $v_0 \equiv 0$. On the other hand from (2.17) it is clear that $\int\limits_{t_{0k_l}}^{a_1} |v_0^{(m)}(s)|^2 ds = 1$, which contradict with $v_0 \equiv 0$. Thus our assumption is invalid and (2.10) holds. □

Analogously one can prove

Lemma 2.4. *Let the numbers* $b_1 \in]a,\, b[,\ t_{0k} \in]b_1, b[$, *and* $\varepsilon_{ik},\ \varepsilon_i,\ \beta_k,\ \beta \in R^+,\ k \in N, i = 1, \cdots, n-m$ *are such that*

$$\lim_{k\to+\infty} t_{0\,k} = b, \quad \lim_{k\to+\infty} \beta_k = \beta, \quad \lim_{k\to+\infty} \varepsilon_{i,k} = \varepsilon_i.$$

Let moreover, $\lambda \in \widetilde{L}^2_{0,\,2m-2}(]b_1,\ b])$ *is a nonnegative function,* $x_k \in \widetilde{C}^{n-1,\,m}(]a,\, b[)$ *be a solution of the problem* (2.5) *under the conditions*

$$x^{(i-1)}(b_1) = \varepsilon_{i,k} \ \ (i = 1, \cdots, m), \qquad x^{(i-1)}(t_{0\,k}) = 0 \ \ (i = 1, \cdots, n-m),$$

and $x \in \widetilde{C}^{n-1,\,m}(]a,\, b[)$ *be a solution of the equation* (2.7) *under the conditions*

$$x^{(i-1)}(b_1) = \varepsilon_i \ \ (i = 1, \cdots, m), \qquad x^{(i-1)}(b) = 0 \ \ (i = 1, \cdots, n-m). \tag{2.20}$$

Then the equalities (2.9) *hold.*

Lemma 2.5. *Let $a < a_1 < b_1 < b$, $\varepsilon_i \in R^+$ and*

$$\lambda \in \widetilde{L}^2_{2n-2m-2,\,0}(]a,\ a_1]) \quad (\lambda \in \widetilde{L}^2_{0,\,2m-2}(]b_1, b]))$$

is nonnegative function. Then for the solution $x \in \widetilde{C}^{n-1,\,m}(]a,\,b[)$ of the problem (2.7), (2.8) ((2.7), (2.20)) with $\beta = 1$, the estimate

$$\int_a^{a_1} |x^{(m)}(s)|^2 ds \le \Theta_1(x, a_1, \lambda) \qquad \Big(\int_{b_1}^{b} |x^{(m)}(s)|^2 ds \le \Theta_2(x, b_1, \lambda)\Big) \quad (k \in N) \tag{2.21}$$

is valid, where

$$\begin{gathered}\Theta_1(x, a_1, \lambda) = 2|w_n(x)(a_1)| + \gamma_1 ||\lambda||^2_{\widetilde{L}^2_{2n-2m-2,0}(]a,a_1])},\\ \Big(\Theta_2(x, b_1, \lambda) = 2|w_n(x)(b_1)| + \gamma_2 ||\lambda||^2_{\widetilde{L}^2_{0,2m-2}(]b_1,b])}\Big),\end{gathered} \tag{2.22}$$

and

$$\gamma_1 = \Big(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\Big)^2, \quad \gamma_2 = \Big(\frac{2^{m-1}(2m+1)(b-a+1)}{(2m-1)!!}\Big)^2.$$

Proof. Suppose that x_k is a solution of problem (2.5), (2.6) with $\beta_k = 1, \varepsilon_{ik} = \varepsilon_i$. Then in view of Lemma 2.3, relations (2.9) hold. On the other hand by Lemma 2.2 we get

$$\nu_n \int_{t_{0k}}^{a_1} |x_k^{(m)}(s)|^2 ds \le -w_n(x_k)(a_1) + \int_{t_{0k}}^{a_1} (s-a)^{n-2m} \lambda(s) |x_k(s)| ds. \tag{2.23}$$

Now, on the basis of Lemma 2.1, Schwartz's and Young's inequalities we get

$$\Big|\int_{t_{0k}}^{a_1} (s-a)^{n-2m} \lambda(s) x_k(s) ds\Big| = \Big|\int_{t_{0k}}^{a_1} [(n-2m)x_k(s) + (s-a)^{n-2m} x_k'(s)]\Big(\int_s^{a_1} \lambda(\xi)d\xi\Big) ds\Big| \le$$

$$\le \Big[(n-2m)\Big(\int_{t_{0k}}^{a_1} \frac{x_k^2(s)}{(s-a)^{2m}} ds\Big)^{1/2} + \Big(\int_{t_{0k}}^{a_1} \frac{x_k'^2(s)}{(s-a)^{2m-2}} ds\Big)^{1/2}\Big] ||\lambda||_{\widetilde{L}_{2n-2m-2,0}(]a,a_1])}$$

$$\le \frac{2^{m-1}(2m+1)}{(2m-1)!!}\Big(\int_{t_{0k}}^{a_1} |x_k^{(m)}(s)|^2 ds\Big)^{1/2} ||\lambda||_{\widetilde{L}_{2n-2m-2,0}(]a,a_1])} \le$$

$$\le \frac{1}{2}\int_{t_{0k}}^{a_1} |x_k^{(m)}(s)|^2 ds + \frac{1}{2}\Big(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\Big)^2 ||\lambda||^2_{\widetilde{L}_{2n-2m-2,0}(]a,a_1])}.$$

Thus from (2.23) by the definition of the numbers ν_n immediately follows that estimate

$$\int_{t_{0k}}^{a_1} |x_k^{(m)}(s)| ds \le 2|w_n(x_k)(a_1)| + \Big(\frac{2^{m-1}(2m+1)}{(2m-1)!!}\Big)^2 ||\lambda||^2_{\widetilde{L}_{2n-2m-2,0}(]a,a_1])} \quad (k \in N).$$

By (2.9) from the last inequality (2.21) and (2.22) follows. Thus Lemma is proved for the problem (2.7), (2.8).

Analogously, by using Lemma 2.4 one can prove the case of problem (2.7), (2.20).

□

2.2. Lemmas on Banach space $\widetilde{C}_1^{m-1}(]a,\, b[)$.

Definition 2.1. Let $\rho \in R^+$ and the function $\eta \in L_{loc}(]a, b[)$ be nonnegative. Then $S(\rho, \eta)$ is a set of such $y \in C_{loc}^{n-1}(]a,\, b[)$ that

$$\Big|y^{(i-1)}\Big(\frac{a+b}{2}\Big)\Big| \le \rho \quad (i = 1, \ldots, n), \tag{2.24}$$

$$|y^{(n-1)}(t) - y^{(n-1)}(s)| \le \int\limits_s^t \eta(\xi)d\xi \qquad \text{for} \quad a < s \le t < b, \tag{2.25}$$

and

$$y^{(i-1)}(a) = 0 \ (i = 1, \cdots, m), \quad y^{(i-1)}(b) = 0 \ (i = 1, \cdots, n-m). \tag{2.26}$$

Lemma 2.6. *Let for the function $y \in \widetilde{C}^{n-1,m}(]a,\, b[)$, conditions* (2.26) *be satisfied. Then $y \in \widetilde{C}_1^{m-1}(]a,\, b[)$ and the estimates*

$$|y^{(i-1)}(t)| \le \frac{|t-c_k|^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}}\Big|\int\limits_{c_k}^t |y^{(m)}(s)|^2 ds\Big|^{1/2} \qquad \textit{for} \quad a < t < b, \tag{2.27}$$

$i = 1, \ldots, m,$ *hold for* $k = 1, 2,$ *where* $c_1 = a, c_2 = b.$

Proof. First not that in view of inclusion $y \in \widetilde{C}^{n-1,m}(]a,\, b[)$, the equality

$$y^{(i-1)}(t) = \sum_{j=i}^{l} \frac{(t-c)^{j-i}}{(j-i)!} y^{(j-1)}(c) + \frac{1}{(l-i)!}\int\limits_c^t (t-s)^{l-i} y^{(l)}(s)ds \quad \text{for} \quad a < t < b \tag{2.28}$$

for $i = 1, \cdots, l, \ \ l = 1, \cdots, n$, holds, where

$$1.\ c \in [a, b] \quad \text{if} \quad l \le m; \ \ 2.\ c \in]a, b] \quad \text{if} \quad l = m+1 \ \text{ and } \ n = 2m+1;$$

$$3.\ c \in]a, b[\quad \text{if} \quad l > m,$$

and exists $r > 0$ such that

$$\int\limits_a^b |y^{(m)}(s)|^2 ds \le r. \tag{2.29}$$

Equality (2.28), with $l = m, \ c = a$ and with $l = m, \ c = b$ by conditions (2.26), (2.29) and Schwartz inequality yields (2.27). From (2.27) and (2.29) it is clear that $y \in \widetilde{C}_1^{m}(]a,\, b[)$. $\square$

Lemma 2.7. *Let $\rho \in R^+$, and $\eta \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a, b[)$ is a nonnegative function. Then $S(\rho, \eta)$ is a compact subset of the space $\widetilde{C}_1^{m-1}(]a,\, b[)$.*

Proof. Condition (2.25) yields the inequality $|y^{(n)}(t)| \le \eta(t)$. Thus there exists such function $\eta_1 \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,b[)$ that

$$y^{(n)}(t) = \eta_1(t), \quad \text{for} \quad a < t < b \tag{2.30}$$

$$|\eta_1(t)| \le \eta(t) \quad \text{for} \quad a < t < b \tag{2.31}$$

From the Theorem 1.1, follows that problem (2.30), (2.26) has unique solution $y \in C^{n-1,\,m}(]a,b[)$, i.e. there exists $r > 0$ such that the inequality (2.29) holds.

For any $y \in S(\rho,\eta)$, from equality (2.28) with $l = n$, by (2.24), (2.30) and (2.31)we get

$$|y^{(i-1)}(t)| \le \gamma_i(t) \quad \text{for} \quad a < t < b, \quad (i = 1, \cdots, n), \tag{2.32}$$

where

$$\gamma_i(t) = \rho_i + \frac{1}{(n-i)!}\Big|\int_c^t (t-s)^{n-i}\eta(s)ds\Big| \quad (i = 1, \cdots, n).$$

Let, now $y_k \in S(\rho,\eta)$ $(k \in N)$. By virtue of the Arzela-Ascoli lemma and conditions (2.25), (2.32) the sequence $\{y_k\}_{k=1}^{+\infty}$ contains a subsequence $\{y_{k_\ell}\}_{\ell=1}^{+\infty}$ such that $\{y_{k_\ell}^{(i-1)}\}_{\ell=1}^{+\infty}$ $(i = 1, \cdots, n)$ are uniformly convergent on $]a,\ b[$. Thus without loss of generality we can assume that $\{y_k^{(i-1)}\}_{k=1}^{+\infty}$ $(i = 1, \cdots, n-1)$ are uniformly convergent on $]a,\ b[$. Let $\lim\limits_{k\to+\infty} y_k(t) = y_0(t)$, then $y_0 \in \widetilde{C}^{n-1}_{loc}(]a,\ b[)$ and

$$\lim_{k\to+\infty} y_k^{(i-1)}(t) = y_0^{(i-1)}(t) \quad (i = 1, \cdots, n) \quad \text{uniformly on} \quad]a,\ b[. \tag{2.33}$$

From (2.33) in view of the inclusions $y_k \in S(\rho,\eta)$ immediately follows that

$$\Big|y_0^{(i-1)}\Big(\frac{a+b}{2}\Big)\Big| \le \rho \quad (i = 1, \ldots, n), \tag{2.34}$$

$$y_0^{(i-1)}(a) = 0 \ (j = 1, \cdots, m), \quad y_0^{(i-1)}(b) = 0 \ (j = 1, \cdots, n-m). \tag{2.35}$$

and

$$|y_0^{(n-1)}(t) - y_0^{(n-1)}(s)| \le \int\limits_s^t \eta(\xi)d\xi \qquad \text{for} \quad a < s \le t < b. \tag{2.36}$$

From (2.34)-(2.36) it is clear that $y_0 \in S(\rho,\eta)$. To finish the proof we must shove that

$$\lim_{k\to+\infty} ||y_k(t) - y_0(t)||_{\widetilde{C}_1^{m-1}} = 0, \tag{2.37}$$

and

$$S(\rho,\eta) \subset \widetilde{C}_1^{m-1}(]a,\ b[). \tag{2.38}$$

Let, $x_k = y_0 - y_k$, and $a_1 \in]a,\ b[,\ b_1 \in]a_1, b[$. Then it is cleat that $x_k \in S(\rho', \eta')$ where $\rho' = 2\rho,\ \ \eta' = 2\eta$. Thus for any x_k exists $\eta_k \in \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a, b[)$ such that

$$x_k^{(n)}(t) = \eta_k(t), \tag{2.39}$$

$$x_k^{(i-1)}(a) = 0\ \ (i = 1, \cdots, n), \qquad x_k^{(i-1)}(b) = 0\ \ (i = 1, \cdots, n-m). \tag{2.40}$$

where

$$|\eta_k(t)| \le 2\eta(t) \quad \text{for} \quad a < t < b \quad (k \in N). \tag{2.41}$$

On the other hand, from (2.27) with $y = x_k$, in view of (2.40) we get

$$\begin{aligned} |x_k^{(i-1)}(t)| &\le \Big(\int_a^t |x_k^{(m)}(s)|^2 ds\Big)^{1/2} (t-a)^{m-i+1/2} \quad \text{for} \quad a < t < a_1, \\ |x_k^{(i-1)}(t)| &\le \Big(\int_t^b |x_k^{(m)}(s)|^2 ds\Big)^{1/2} (b-t)^{m-i+1/2} \quad \text{for} \quad b_1 < t < b, \end{aligned} \tag{2.42}$$

for $i = 1, \ldots, m$.

Let now w_n be the operator defined in Lemma 2.2 and Θ_1, Θ_2 are functions defined by (2.22) with $\lambda = \eta_k$. Then conditions (2.33) yields

$$\lim_{k\to+\infty} w_n(x_k)(a_1) = 0, \qquad \lim_{k\to+\infty} w_n(x_k)(b_1) = 0\ \ (k \in N), \tag{2.43}$$

and from definition of norm $||\cdot||_{\widetilde{L}^2_{\alpha,\beta}}$, (2.41) and (2.43), follows that for any $\varepsilon > 0$ we can choose $a_1 \in]a,\ \min\{a+1,\ b\}[,\ \ b_1 \in]\max\{b-1,\ b\},\ a_1[$ and $k_0 \in N$, such that

$$\begin{aligned} \Theta_1(x_k,\ a_1,\ 2\eta) &\le \frac{\varepsilon}{6}(b-b_1)^{m-1/2} \quad (k \ge k_0), \\ \Theta_2(x_k,\ b_1,\ 2\eta) &\le \frac{\varepsilon}{6}(a_1-a)^{m-1/2} \quad (k \ge k_0). \end{aligned} \tag{2.44}$$

By using lemma 2.5 for x_k, in view of (2.42) and (2.44)we get

$$\int_a^{a_1} |x_k^{(m)}(s)|^2 ds \le \frac{\varepsilon}{6} \qquad \int_{b_1}^b |x_k^{(m)}(s)|^2 ds \le \frac{\varepsilon}{6} \qquad (k \ge k_0), \tag{2.45}$$

$$\frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \le \frac{\varepsilon}{2m} \qquad \text{for} \quad t \in]a,\ a_1] \cup [b_1,\ b[,\ \ (1 \le i \le m,\ \ k \ge k_0). \tag{2.46}$$

Also, in view of (2.33) without loss of generality we can assume that

$$\frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \le \frac{\varepsilon}{2m} \quad \text{for} \quad a_1 \le t \le b_1, \quad (1 \le i \le m,\ k \ge k_0), \tag{2.47}$$

and

$$\int_{a_1}^{b_1} |x_k^{(m)}(s)|^2 ds \le \frac{\varepsilon}{6} \quad (k \ge k_0). \tag{2.48}$$

From (2.45)-(2.48), equality (2.37) immediately follows.

Let, now $y \in S(\rho,\ \eta)$ and $y_k = \delta_k y$, where $\lim\limits_{k\to+\infty} \delta_k = 0$. Then by (2.33) it is clear, that $y_0 \equiv 0$ and than from (2.37) it follows $y \in \widetilde{C}_1^{m-1}(]a,\ b[)$, i.e. the inclusion (2.38) holds. $\square$

Lemma 2.8. *Let $\tau_j \in M(]a,b[)$, $\alpha \geq 0$, $\beta \geq 0$ and exists $\delta \in]0, b-a[$ such that*

$$|\tau_j(t) - t| \leq k_1(t-a)^\beta \quad \textit{for} \quad a < t \leq a+\delta. \tag{2.49}$$

Then

$$\Big| \int\limits_t^{\tau(t)} (s-a)^\alpha ds \Big| \leq \begin{cases} k_1[1 + k_1\delta^{\beta-1}]^\alpha (t-a)^{\alpha+\beta} & \textit{for} \;\; \beta \geq 1 \\ k_1[\delta^{1-\beta} + k_1]^\alpha (t-a)^{\alpha\beta+\beta} & \textit{for} \;\; 0 \leq \beta < 1 \end{cases},$$

for $a < t \leq a+\delta$.

Proof. First note that

$$\Big| \int\limits_t^{\tau(t)} (s-a)^\alpha ds \Big| \leq (\max\{\tau(t), t\} - a)^\alpha |\tau(t) - t| \quad \text{for} \quad a \leq t \leq a+\delta,$$

and $\max\{\tau(t), t\} \leq t + |\tau(t) - t| \quad \text{for} \quad a \leq t \leq a+\delta$. Then in view of condition (2.49) we get

$$\Big| \int\limits_t^{\tau(t)} (s-a)^\alpha ds \Big| \leq k_1[(t-a) + k_1(t-a)^\beta]^\alpha (t-a)^\beta \quad \text{for} \quad a \leq t \leq a+\delta.$$

Last inequality yields the validity of our lemma. □

Analogously one can prove

Lemma 2.9. *Let $\tau_j \in M(]a,b[)$, $\alpha \geq 0$, $\beta \geq 0$ and exists $\delta \in]0, b-a[$ such that*

$$|\tau_j(t) - t| \leq k_1(b-t)^\beta \quad \textit{for} \quad b-\delta \leq t < b. \tag{2.50}$$

Then

$$\Big| \int\limits_t^{\tau(t)} (b-t)^\alpha ds \Big| \leq \begin{cases} k_1[1 + k_1\delta^{\beta-1}]^\alpha (b-t)^{\alpha+\beta} & \textit{for} \;\; \beta \geq 1 \\ k_1[\delta^{1-\beta} + k_1]^\alpha (b-t)^{\alpha\beta+\beta} & \textit{for} \;\; 0 \leq \beta < 1 \end{cases},$$

for $b-\delta \leq t < b$.

2.3. Lemmas on the solutions of auxiliary problems.

Throughout of this section we assume that the operator $P : C_1^{m-1}(]a,\ b[) \times C_1^{m-1}(]a,\ b[) \to L_n(]a,\ b[)$ be γ_0, γ consistent with boundary condition (1.2), and operator $q : C_1^{m-1}(]a,\ b[) \to \widetilde{L}^2_{2n-2m-2,\,2m-2}(]a,\ b[)$, be continuous.

Consider for any $x \in \widetilde{C}_1^{m-1}(]a,\ b[) \subset C_1^{m-1}(]a,\ b[)$ the nonhomogeneous equation

$$y^{(n)}(t) = \sum_{i=1}^m p_i(x)(t) y^{(i-1)}(\tau_i(t)) + q(x)(t), \tag{2.51}$$

and corresponding homogeneous equation

$$y^{(n)}(t) = \sum_{i=1}^{m} p_i(x)(t) y^{(i-1)}(\tau_i(t)), \tag{2.52}$$

and let, E^n be a set of the solutions of problem (2.51), (2.26).

From inequality (1.23) of item (ii) of definition 1.1, it follows that boundary problem (2.51), (2.26) has the unique solution y in the space $\widetilde{C}^{n-1,m}(]a,\, b[)$. But in view of Lemma 2.6 it is clear that $y \in \widetilde{C}_1^{m-1}(]a,\, b[)$. Thus $E^n \cap \widetilde{C}_1^{m-1}(]a,\, b[) \neq \emptyset$, and exists the operator $U : \widetilde{C}_1^{m-1}(]a,\, b[) \to E^n \cap \widetilde{C}_1^{m-1}(]a,\, b[)$ defined by the equality

$$U(x)(t) = y(t).$$

Lemma 2.10. *$U : \widetilde{C}_1^{m-1}(]a,\, b[) \to E^n \cap \widetilde{C}_1^{m-1}(]a,\, b[)$ is a continuous operator.*

Proof. Let $x_k \in \widetilde{C}_1^{m-1}(]a,\, b[)$ and $y_k(t) = U(x_k)(t) \quad (k = 1, 2)$, $y = y_2 - y_1$, and the operator P is defined by (1.19). Then

$$y^{(n)}(t) = P(x_2,\, y)(t) + q_0(x_1,\, x_2)(t)$$

where $q_0(x_1,\, x_2)(t) = P(x_2,\, y_1)(t) - P(x_1,\, y_1)(t) + q(x_2)(t) - q(x_1)(t)$. Hence, by item ii of definition 1.1 we have

$$||U(x_2) - U(x_1)||_{\widetilde{C}_1^{m-1}} \leq \gamma ||q_0(x_1,\, x_2)||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}.$$

Since the operators P and q are continuous, this estimate implies the continuity of the operator U. □

3 Proofs

Proof of remark 1.1. Let x be a solution of problem (1.8), (1.2), then from inequalities (2.27) it follows the estimate

$$|x^{(i-1)}(t)| \leq \frac{[(b-t)(t-a)]^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}} \Big(\frac{2}{b-a}\Big)^{m-i+1/2} ||x^{(m)}||_{L^2} \tag{3.1}$$

for $a \leq t \leq b$. From this estimate, by definition of norm in the space $\widetilde{C}^{m-1}(]a,\, b[)$, and estimate (1.17) immediately follows (1.18). □

Proof of theorem 1.3. Let δ and λ are the functions and numbers appearing in Definition 1.1. We set

$$\eta(t) = \delta(t,\ \gamma_0)\gamma_0 + \widetilde{F}_p(t,\, \min\{2\rho_0, \gamma_0\}), \tag{3.2}$$

$$\chi(s) = \begin{cases} 1 & \text{for } \ 0 \leq s \leq \rho_0 \\ 2 - s/\rho_0 & \text{for } \ \rho_0 < s < 2\rho_0 \\ 0 & \text{for } \ s \geq 2\rho_0 \end{cases}, \tag{3.3}$$

$$q(x)(t) = \chi(||x||_{\widetilde{C}_1^{m-1}}) F_p(x)(t). \tag{3.4}$$

From (1.24) it is clear that the nonnegative functions $\widetilde{F}_p$, η, admits the inclusion

$$\widetilde{F}_p(\cdot,\, \min\{2\rho_0, \gamma_0\}),\ \eta \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\ b[), \tag{3.5}$$

and for every $x \in A_{\gamma_0} \subset \widetilde{C}_1^{m-1}(]a,\, b[)$ and almost all $t \in]a,\, b[$ the inequality

$$|q(x)(t)| \leq \widetilde{F}_p(t,\ \min\{2\rho_0, \gamma_0\}) \quad \text{for} \quad a < t < b \tag{3.6}$$

holds.

Let $U : A_{\gamma_0} \to E^n \cap \widetilde{C}_1^{m-1}(]a,\, b[)$ is a operator appeared in Lemma 2.10, from which it follows that U is a continuous operator. On the other hand from items i and ii of Definition 1.1, (1.25) and (3.6) it is clear, that for each $x \in A_{\gamma_0}$, the conditions

$$||y||_{\widetilde{C}_1^{m-1}} \leq \gamma_0, \quad |y^{(n-1)}(t) - y^{(n-1)}(s)| \leq \int\limits_s^t \eta(\xi) d\xi \quad \text{for} \quad a < t < b$$

hold. Thus in view of definition 2.1 the operator U maps the ball A_{γ_0} into its own subset $S(\rho_1,\, \eta)$. From lemma 2.2 follows that $S(\rho_1,\, \eta)$ is the compact subset of the ball $A_{\gamma_0} \subset \widetilde{C}_1^{m-1}(]a,\, b[)$. i.e. the operator u maps the ball A_{γ_0} into its own compact subset. Therefore, owing to Schauders's principle, there exists $x \in S(\rho_1,\, \eta) \subset A_{\gamma_0}$, such that

$$x(t) = U(x)(t) \quad \text{for} \quad a < t < b.$$

Thus by (2.51) and notation (3.4), the function $x\,(x \in A_{\gamma_0})$ is a solution of problem $(1.26), (1.2)$, where

$$\lambda = \chi(||x||_{\widetilde{C}_1^{m-1}}). \tag{3.7}$$

If $\gamma_0 = \rho_0$ then in view of condition $x \in A_{\gamma_0}$, by (3.3) we have that $\lambda = 1$, and then in view of (2.51) and (3.4) the function x is a solution of problem $(1.1), (1.2)$ which admits to the estimate (1.27).

Let us show now, that x admits estimate (1.27) in the case when $\rho_0 < \gamma_0$. Assume the contrary. Then either

$$\rho_0 < ||x||_{\widetilde{C}_1^{m-1}} < 2\rho_0, \tag{3.8}$$

or

$$||x||_{\widetilde{C}_1^{m-1}} \geq 2\rho_0. \tag{3.9}$$

If condition (3.8) holds, then by virtue of (3.3) and (3.7) we have that $\lambda \in]0, 1[$, which by the conditions of our theorem guarantees the validity of estimate (1.27). But this contradict (3.8).

Assume now that (3.9) is fulfilled. Then by virtue of (3.3) and (3.7) we have that $\lambda = 0$. Therefore $x \in A_{\gamma_0}$ is a solution of problem $(2.52), (1.2)$. Thus from item ii of Definition 1.1 it is obvious that $x \equiv 0$, because problem $(2.52), (1.2)$ has only a trivial solution. But this contradict condition (3.9), i.e. estimate (1.27) is valid. From estimate (1.27) and (3.3) we have that $\lambda = 1$, and then in view of (2.51) and (3.4) the function x is a solution of problem $(1.1), (1.2)$ which admits to the estimate (1.27). $\square$

Proof of Corollary 1.2. First note that in view of condition (1.30) exists such $\gamma_0 > 2\rho_0$, that condition (1.25) holds, and in view of definition 1.2 the operator P is γ_0, γ consistent.

On the other hand from (1.30) follows the existence of the number ρ_0, such that

$$\gamma||\eta(\cdot,\, \rho)||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}} < \rho \quad \text{for} \qquad \rho > \rho_0. \tag{3.10}$$

Let x be a solution of problem (1.26), (1.2) for some $\lambda \in]0,\, 1[$. Then $y = x$ is also a solution of problem (1.22), (1.2) where $q(t) = \lambda\Big(F(x)(t) - P(x,\, x)(t)\Big)$. Let now $\rho = ||x||_{\widetilde{C}_1^{m-1}}$ and assume that

$$\rho > \rho_0. \tag{3.11}$$

holds. Then in view of the $\gamma-$consistency of operator p with boundary conditions (1.2), inequality (1.23) holds and thus by condition (1.28) we have

$$\rho = ||x||_{\widetilde{C}_1^{m-1}} \le \gamma||q(x)||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}} \le \gamma||\eta(\cdot,\, \rho)||_{\widetilde{L}^2_{2n-2m-2,\, 2m-2}}.$$

But the last inequality contradict (3.10). Thus assumption (3.11) is not valid and $\rho \le \rho_0$. Therefore for any $\lambda \in]0,\ 1[$ an arbitrary solution of the problem (1.26), (1.2) admits the estimate (1.27). Therefore all the conditions of Theorem 1.3 ar fulfilled, from which the solvability of problem (1.1), (1.2) follows. □

Proof of theorem 1.4. Let r_n be the constant defined in Remark 1.1. First prove that operator P is γ_0, r_n consistent with boundary conditions (1.2). From the conditions of our theorem it is obvious that the item (i) of definition 1.1 is satisfied. Let now x be an arbitrary fixed function from the set A_{γ_0} and let $p_j(t) \equiv p_j(x)(t)$. Thus in view of (1.34), (1.35) all the assumptions of Theorem 1.1 are satisfied, and then for any $q \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\, b[)$ the problem (1.22), (1.2) has unique solution y. Also in view of Remark 1.1 there exists the constant $r_n > 0$, (which depends only on the numbers $l_{kj},\, \overline{l}_{kj}, \gamma_{kj}\, (k = 0, 1;\ j = 1, \cdots, m)$, and $a,\, b,\, t^*,\, n$) such that estimate (1.23) holds with $\gamma = r_n$. I.e., the operator P is $\gamma_0,\, r_n$ consistent with boundary conditions (1.2). Therefore all the assumptions of Corollary 1.1 are fulfilled, from which the solvability of problem (1.1), (1.2) follows. □

Proof of theorem 1.5. Let r_n be the constant defined in Remark 1.1. First prove that operator P is r_n consistent with boundary conditions (1.2). From the conditions of our theorem it is obvious that the item (i) of definition 1.1 is satisfied. Let now γ_0 be an arbitrary nonnegative number, x be arbitrary fixed function from the space A_{γ_0} and let $p_j(t) \equiv p_j(x)(t)$. Then in view of (1.37), (1.38) all the assumptions of Theorem 1.1 are satisfied and then for any $q \in \widetilde{L}^2_{2n-2m-2,\, 2m-2}(]a,\, b[)$ the problem (1.22), (1.2) has unique solution y. Also in view of Remark 1.1 there exists the constant $r_n > 0$, (which depends only on the numbers $l_{kj},\, \overline{l}_{kj}, \gamma_{kj}\, (k = 0, 1;\ j = 1, \cdots, m)$, and $a,\, b,\, t^*,\, n,$) such that estimate (1.23) holds with $\gamma = r_n$. I.e., the operator P is $\gamma_0,\, r_n$ consistent with boundary conditions (1.2) for arbitrary $\gamma_0 > 0$. Thus by Definition 1.1, the operator P is r_n consistent with boundary conditions (1.2). Therefore all the assumptions of Corollary 1.2 are fulfilled, from which follows the solvability of problem (1.1), (1.2) follows. □

Proof of Remarc 1.2. By the Schwartz's inequality, definition of the norm $||y||_{\widetilde{C}_1^{m-1}}$ and inequalities (1.39), (2.2) for ani $x, y \in A_{\gamma_0}$ and $z = y - x$ we have

$$|p_j(y)(t)z^{(j-1)}(\tau_j(t))| = |p_j(y)(t)z^{(j-1)}(t)| + |p_j(y)(t)|\Big|\int\limits_t^{\tau_j(t)} z^{(j)}(\psi)d\psi\Big| \leq$$
$$\leq ||z||_{\widetilde{C}_1^{m-1}}|p_j(y)(t)|\alpha_j(t)\Big(1 + \frac{1}{\alpha_j(t)}\Big(\int\limits_t^{\tau_j(t)} (\psi - a)^{2m-2j}d\psi\Big)^{1/2}\Big) \tag{3.12}$$

for $a < t < b$. On the other hand, from the conditions (1.40) by Lemmas 2.8 and 2.9 it is cleat that

$$\alpha_j^{-1}(s)\Big(\int\limits_s^{\tau_j(s)} (\xi - a)^{2m-2j}d\xi\Big)^{1/2} \leq \frac{\sqrt{\kappa(1+\kappa)}}{\varepsilon^{m-j+1/2}} \quad \text{for} \quad s \in]a,\, a+\varepsilon] \cup [b-\varepsilon,\, b[,$$

$$\alpha_j^{-1}(s)\Big(\int\limits_s^{\tau_j(s)} (\xi - a)^{2m-2j}d\xi\Big)^{1/2} \leq \varepsilon^{-2m+2j-1}\Big(\int\limits_a^{b} (\xi - a)^{2m-2j}d\xi\Big)^{1/2} =$$
$$= \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\varepsilon^{2m-2j+1}} \quad \text{for} \quad s \in]a+\varepsilon, b-\varepsilon[.$$

Then if we put

$$\kappa_0 = \max_{1 \leq j \leq m}\Big\{\frac{\sqrt{\kappa(1+\kappa)}}{\varepsilon^{m-j+1/2}},\ \frac{(b-a)^{m-j+1/2}}{\sqrt{2m-2j+1}\varepsilon^{2m-2j+1}}\Big\}, \tag{3.13}$$

from (3.12) by the last estimates we get the inequality

$$|p_j(y)(t)z^{(j-1)}(\tau_j(t))| \leq ||z||_{\widetilde{C}_1^{m-1}}(1+\kappa_0)|p_j(y)(t)|\alpha_j(t) \leq$$
$$\leq ||z||_{\widetilde{C}_1^{m-1}}(1+\kappa_0)\delta_j(t,\, ||y||_{\widetilde{C}_1^{m-1}}) \tag{3.14}$$

for $a < t < b$. Analogously we get that

$$|(p_j(y)(t) - p_j(x)(t))x^{(j-1)}(\tau_j(t))| \leq ||x||_{\widetilde{C}_1^{m-1}}(1+\kappa_0)|p_j(y)(t) - p_j(x)(t)|\alpha_j(t)$$

for $a < t < b$. from (3.14) and the last inequality it is obvious that the operator P defined by equality (1.19) continuously acting from A_{γ_0} to the space $L_n(]a,\, b[)$, and the item (ii) of definition 1.1 holds, with $\delta(t,\, \rho) = (1+\kappa_0)\sum\limits_{j=1}^{m}\delta_j(t,\, \rho)$. □

Proof of Corollary 1.3. From conditions (1.42) and (1.40) by the Remark 1.2 we obtain that the operator P defined by equality (1.19) with $p_j(x)(t) = p_j(t)$, continuously acting from A_{γ_0} to the space $L_n(]a,\, b[)$, for any $\gamma_0 > 0$, i.e., continuously acting from $\widetilde{C}_1^{m-1}(]a,\, b[)$ to the space $L_n(]a,\, b[)$.

Therefore it is clear that all the conditions of Theorem 1.5 would be satisfied with

$$F(x)(t) = f(t, x(\tau_1(t)), x'(\tau_2(t)), \cdots, x^{(m-1)}(\tau_m(t))), \quad \delta(t,\rho) = (1+\kappa_0)\sum_{j=1}^{m} |p_j(t)|,$$

where the constant κ_0 is defined by equality (3.13). Thus problem (1.41), (1.2) is solvable. □

Proof of Corollary 1.4. Let the operators $F, p_1 : C^{m-1}(]a, b[) \to L_{loc}(]a, b[)$, and the function $\eta :]a, b[\times R^+ \to R^+$ be defined by equalities

$$F(x)(t) = -\frac{\lambda|x(t)|^k}{[(t-a)(b-t)]^{2+k/2}}x(\tau(t)) + q(x)(t), \quad p_1(x)(t) = -\frac{\lambda|x(t)|^k}{[(t-a)(b-t)]^{2+k/2}}.$$

Then it is easy to verify that in view of (1.46)-(1.48), conditions (1.13), (1.14), (1.28), (1.34)-(1.43) are satisfied with

$$\begin{gathered}\delta(t,\,\rho) = \frac{\rho^k\lambda}{[(t-a)(b-t)]^2}, \quad l_{01} = l_{11} = \frac{4\gamma_0^k\lambda}{(b-a)^2}, \quad \bar{l}_{01} = \bar{l}_{11} = \frac{16\gamma_0^k\lambda}{(b-a)^2},\\ r_2 = \left(1+\sqrt{\frac{2}{b-a}}\right)\frac{2(1+b-a)(b-a)^2}{(b-a)^2 - 16\lambda\gamma_0^k(1+[2(b-a)]^{1/4})},\\ B_0 = B_1 = \frac{16\lambda\gamma_0^k}{(b-a)^2}(1+[2(b-a)]^{1/4}), \quad t^* = (a+b)/2, \quad \gamma_{01} = \gamma_{11} = \frac{1}{4}.\end{gathered} \tag{3.15}$$

Thus all the condition of theorem 1.4 are satisfied, from which follows solvability of problem (1.44), (1.2). □

Proof of Corollary 1.5. Let the operators $F, p_1 : C^{m-1}(]a, b[) \to L_{loc}(]a, b[)$, and the function $\eta :]a, b[\times R^+ \to R^+$ be defined by equalities

$$F(x)(t) = -\frac{\lambda|\sin x^k(t)|}{[(t-a)(b-t)]^2}x(\tau(t)) + q(x)(t), \quad p_1(x)(t) = -\frac{\lambda|\sin x^k(t)|}{[(t-a)(b-t)]^2}.$$

Then it is easy to verify that in view of (1.30), (1.46), and (1.49), all the conditions of Theorem 1.5 follow, where δ, l_{11}, l_{01}, $\bar{l}_{11}$, $\bar{l}_{01}$, r_2, B_0, B_1, t^*, γ_{01}, γ_{11}, are defined by (3.15) with $\rho = 1$, $\gamma_0 = 1$, from which follows solvability of problem (1.44), (1.2). □

Acknowledgement

This work is supported by the Academy of Sciences of the Czech Republic (Institutional Research Plan # AV0Z10190503) and by the Shota Rustaveli National Science Foundation (Project # GNSF/ST09_175_3-101).

References

[1] R. P. Agarwal: *Focal boundary value problems for differential and difference equations.* Kluwer Academic Publishers, Dirdrecht, 1998. x, 289 pp. Zbl 0914.34001, MR 1619877

[2] R. P. Agarwal and D. O'Regan: *Singular differential and integral equations with applications,* Kluwer Academic Publishers, Dordrecht, 2003. xii, 402 pp. Zbl 1055.34001, MR 2011127

[3] R. P. Agarwal, I. Kiguradze: *Two-point boundary value problems for higher-order linear differential equations with strong singularities,* Bound. Value Probl. 2006, 32 p. Zbl 1137.34006, MR 2211396

[4] E. Bravyi: *A not on the Fredholm property of boundary value problems for linear functional differential equations,* Mem. Differential Equ. Math. Phys. **20** (2000), 133-135. Zbl 0968.34049

[5] I. T. Kiguradze: *On a singular multi-point boundary value problem,* Ann. Mat. Pura Appl., IV. Ser. 86, (1970), 367-399. Zbl 0251.34012

[6] I. T. Kiguradze: *Some singular boundary value problems for ordinary differential equations,* (Russian) Tbilisi University Press, Tbilisi, 1975.

[7] I. T. Kiguradze and T. A. Chanturia: *Asymptotic properties of solutions of nonautonomous or- dinary differential equations,* Mathematics and Its Applications. Soviet Series, 89. kluwer Academic Publishers group, Dortrecht, xiv, 1993, 331 pp. Zbl 0782.34002, MR 1220223

[8] I. Kiguradze, G. Tskhovrebadze: *On two-point boundary value problems for systems of higher order ordinary differential equations with singularities,* Georgian Math. J. **1** (1994), No. 1, 31-45. Zbl 0804.34023

[9] I. Kiguradze, B. Půža, and I. P. Stavroulakis: *On singular boundary value problems for functional differential equations of higher order*, Georgian Math. J. **8** (2001), No. 4, 791-814. MR 1884501

[10] I. Kiguradze: *Some optimal conditions for the solvability of two-point singular boundary value problems,* Funct. Differ. Equ. 10 (2003), No. 1-2, 259-281. Zbl 1062.34017, MR 2017411

[11] T. Kiguradze: *On conditions for linear singular boundary value problems to be well posed,*(English. Russian original) Differ. Equ. 46, No. 2, 187-194 (2010); translation from Differ. Uravn. 46 (2010), No. 2, 183-190. Zbl 1203.34037

[12] I. Kiguradze: *On two-point boundary value problems for higher order singular ordinary differential equations,* Mem. Differential Equ. Math. Phys. 31 (2004), 101-107. Zbl 1074.34020, MR 2061760

[13] I. Kiguradze: *The Dirichlet and focal boundary value problems for higher order quasi-halflinear singular differential equations,* Mem. Differential Equ. Math. Phys. **54** (2011), 126-133.

[14] I. Kiguradze: *On Extremal Solutions of Two-Point Boundary Value Problems for Second Order Nonlinear Singular Differential Equations,* Bull. Georgian National Acad. Sci., **5** (2011), No. 2, 31-35.

[15] A. Lomtatidze: *On one boundary value problem for linear ordynary differential equations of second order with singularities,* Differential'nye Uravneniya **222** (1986), No. 3, 416-426.

[16] S. Mukhigulashvili: *Two-point boundary value problems for second order functional -differential equations,* Mem. Differential Equ. Math. Phys. **20** (2000), 1-112. Zbl 0981.34057

[17] S. Mukhigulashvili, N. Partsvania: *On two-point boundary value problems for higher order functional differential equations with strong singularities,* Mem. Differential Equ. Math. Phys. **54** (2011), 134-138.

[18] S. Mukhigulashvili, N. Partsvania, *Two-Point Boundary Value Problems For Strongly Singular Higher-Order Linear Differential Equations With Deviating Arguments,* E. J. Qualitative Theory of Diff. Equ., 2012, No. 38, 1-34.

[19] S. Mukhigulashvili: *On One Estimates For Solutions of Two-Point Boundary Value Problems For Strongly Singular Higher-Order Linear Differential Equations,* Mem. Differential Equ. Math. Phys. (accepted).

[20] N. Partsvania: *On solvability and well-posedness of two-point weighted singular boundary value problems,* Mem. Differential Equ. Math. Phys. **54** (2011), 139-146.

[21] B. Půža: *On a singular two-point boundary value problem for the nonlinear mth-order differential equation with deviating arguments,* Georgian Mathematical J. **4** (1997), No. 6, 557-566. MR 1478554

[22] B. Půža and A. Rabbimov, *On a weighted boundary value problem for a system of singular functional-differential equations,* Mem. Differential Equ. Math. Phys. **21** (2000), 125-130.

[23] I. Rachunková, S. Staněk, and M. Tvrdý, *Singularities and Laplacians in boundary value problems for nonlinear ordinary differential equations,* Handbook of differential equations: ordinary differential equations. Vol. III, 607-722, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2006. MR2457638

[24] I. Rachunková, S. Staňek, and M. Tvrdý: *Solvability of nonlinear singular prob- lems for ordinary differential equations,* Contemporary Mathematics and Its Applications, 5. Hindawi Publishing Corporation. New York 2008. 268 pp. MR 2572243

[25] Š. Schwabik, M. Tvrdý, and O. Vejvoda: *Differential and integral equations, boundary value problems and adjoints,* D. Reidel Publishing Co., Dordrecht-Boston, mass.-London, 1979. 248 pp. MR 0542283

Author's addresses:

Sulkhan Mukhigulashvili

1. Mathematical Institute, Academy of Sciences of the czech Republic, Žižkova 22, 616 62 Brno, czech Republic.

2. I. chavchavadze State University, Faculty of physics and mathematics, I. chavchavadze St. No.32, Tbilisi 0179, Georgia.

E-mail: mukhig@ipm.cz

47. L. Tonelli, Sell'equazione differenziale $y'' = f(t, y, u')$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **8**(1939), 75–88.

48. F. Hartman, Ordinary differential equations. (Russian) *Mir, Moscow*, 1970.

49. B. L. Shekhter, On the unique solvability of one linear two-point boundary value problem. (Russian) *Differentsial'nye Uravneniya* **11**(1975), No. 4, 687–693.

50. B. L. Shekhter, On a two-point boundary value problem for ordinary differential equations of second order with discontinuous right-hand side. (Russian) *Trudy Tbiliss. Gos. Univ.* **A9**(1975), 19–31.

(Received 4.06.1999)

Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia