MASARYK UNIVERSITY FACULTY OF SCIENCE

HABILITATION THESIS

Solution spaces of almost periodic homogeneous linear difference and differential systems

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Preface

In this work, a problem from the qualitative theory of almost periodic difference and differential equations is solved. More precisely, using special constructions of almost periodic (and limit periodic) sequences and functions, non-almost periodic solutions of almost periodic homogeneous linear difference and differential systems are studied. The aim is to find systems, whose all solutions can be almost periodic, and to prove that, in any neighbourhood of such a system, there exists a system which does not possess an almost periodic solution other than the trivial one.

All results presented in this work are due to the author and are taken from papers denoted as (2)–(5), (7), (9), (13), (18), and (21) on pages 168–169. Note that four of the papers have a co-author, namely P. Hasil. In all cases, the contributions of the both authors are equivalent. Certain parts of this work are also taken from the Ph.D. thesis of the author (see (PhD) or directly papers (2)–(5)).

The history and basic motivation of the treated topic are included at the beginnings of chapters and sections. For reader's convenience, the end of each proof, example, and remark is identified by symbol $\Box$, $\Diamond$, and $\triangle$, respectively. The used notations are collected in sections titled Preliminaries. Definitions, theorems, lemmas, corollaries, examples, and remarks are numbered consecutively within each chapter.

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Contents

Abstracts

Chapter 1: We introduce the notions of almost periodic and limit periodic sequences in pseudometric spaces. Especially, we modify the Bochner definition of almost periodicity so that it remains equivalent with the Bohr one. We present an easily modifiable method for constructing almost periodic and limit periodic sequences with prescribed properties. We apply the method to construct an almost periodic homogeneous linear difference system which does not have any non-trivial almost periodic solution.

Chapter 2: Almost periodic homogeneous linear difference systems are analysed, where the coefficient matrices belong to a group. The goal is to find such groups that the systems having no non-trivial almost periodic solution form a dense subset of the set of all considered systems. A closer examination of the used methods reveals that the problem can be treated in such a generality that the entries of the coefficient matrices are allowed to belong to any complete metric field. The concepts of transformable and weakly transformable groups of matrices are introduced and these concepts enable us to derive efficient conditions for determining matrix groups with the required property.

Chapter 3: Limit periodic sequences with values in pseudometric spaces are considered. We construct limit periodic sequences with given values. For any totally bounded and countable set, we find a limit periodic sequence which attains each value from this set periodically. For any totally bounded countable set which is dense in itself, we construct a limit periodic bijective map from the integers into this set. The corresponding results about almost periodic sequences are explicitly formulated as well. As corollaries, we obtain new results about non-almost periodic solutions of complex almost periodic transformable difference systems.

Chapter 4: As in Chapter 2, we study homogeneous linear difference systems, where the coefficient matrices belong to a bounded group. Now we consider limit periodic systems and we find groups of matrices with the property that the systems, which do not possess non-zero asymptotically almost periodic solutions, form a dense subset in the space of all considered systems. Since the used method is substantially different from the processes applied in Chapter 2, we obtain new results also for almost periodic systems.

Chapter 5: We introduce almost periodic and limit periodic functions with values in a pseudometric space $\mathcal{X}$. We mention the Bohr and the Bochner definition of almost periodicity and the fundamental properties of almost periodic functions. In particular, we prove the equivalence of the Bohr and the Bochner concept and we briefly describe the connection between almost periodic functions and sequences. Similarly as in Chapter 1, we present a modifiable method for constructing almost periodic functions in $\mathcal{X}$.

Chapter 6: We analyse solutions of almost periodic skew-Hermitian and skew-symmetric homogeneous linear differential systems. It is known that the systems, whose all solutions are almost periodic, form an everywhere dense subset in space of all almost periodic skew-Hermitian or skew-symmetric systems (in the uniform topology). Applying a construction from Chapter 5, we prove that, in any neighbourhood of an almost periodic skew-Hermitian system, there exists a different almost periodic skew-Hermitian system which does not possess a non-trivial almost periodic solution. In addition, using a modification of the iterative process presented in Chapter 5, we obtain the same result for almost periodic skew-symmetric systems as well.

Chapter 7: In pseudometric spaces, limit periodic and almost periodic functions with given values are constructed. More precisely, for an arbitrary uniformly continuous function which attains finitely many values on $\mathbb{Z}$ and whose range is totally bounded, we construct an almost periodic function with the same range on $\mathbb{Z}$ and $\mathbb{R}$ which attains all value periodically. In addition, if the uniformly continuous function with a totally bounded range attains a value periodically, then we prove that the resulting function can be constructed as limit periodic.

Chapter 1

Almost periodic and limit periodic sequences in pseudometric spaces

In this chapter, we introduce the notions of limit periodicity and (asymptotic) almost periodicity for sequences in a general pseudometric space. At first, we mention the article [65] by K. Fan which considers asymptotically almost periodic sequences of elements of a metric space (based on the Fréchet concept from [74, 75]) and the article [177] by H. Tornehave about almost periodic functions of the real variable with values in a metric space. In these papers, it is shown that many theorems which are valid for complex valued sequences and functions are no longer true. For continuous functions, it is observed that the important property is the local connection by arcs of the space of values and also its completeness.

However, we do not use their results or other theorems and we introduce the considered generalizations of periodic sequences in pseudometric spaces without any additional restrictions; i.e., the definitions are similar to the classical ones, only the modulus being replaced by the distance. We can also refer to [88, 131, 132, 147, 186, 189]. We add that the concept of almost periodic functions of several variables with respect to Hausdorff metrics can be found in [165] which is an extension of [60] (see also [61], [149]).

In Banach spaces, a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if any sequence of translates of $\{\varphi_k\}$ has a subsequence which converges and its convergence is uniform with respect to k in the sense of the norm. In 1933, the continuous case of the previous result was proved by S. Bochner in [22], where the fundamental theorems of the theory of almost periodic functions with values in Banach spaces are proved as well (see, e.g., [6], [7, pp. 3–25] or [111], where the theorems are redemonstrated by the methods of the functional analysis). We remark that the discrete version of this result can be proved similarly as in [22] (or see directly papers [156, 179]).

In pseudometric spaces, it is easy to show that the above result is not generally true. Nevertheless, by a simple modification of the Bochner proof of this result, one can verify that a necessary and sufficient condition for a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ to be almost periodic is that any sequence of translates of $\{\varphi_k\}$ has a subsequence satisfying the Cauchy condition uniformly with respect to k.

In this chapter, we also analyse systems of the form

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z} \text{ (or } k \in \mathbb{N}_0),$$

where $\{A_k\}$ is almost periodic. The aim is to prove that there exists a system of the above form which does not have an almost periodic solution other than the trivial one (see Theorem 1.38 below). A closer examination of the method used in our construction reveals that the problem can be treated in possibly the most general setting:

almost periodic sequences attain values in a pseudometric space;

the entries of almost periodic matrices are elements of an infinite ring with a unit.

Note that many theorems about the existence of almost periodic solutions of general almost periodic difference systems are published in [21, 83, 87, 163, 182, 186, 187, 189] and several these existence theorems are proved there in terms of discrete Lyapunov functions. Here, we can also refer to the monograph [183] and [190, Theorems 3.6, 3.7, 3.8]. For linear systems with $k \in \mathbb{N}_0$, see [5, 168].

This chapter is organized as follows. In the first section, we mention the notation which is used throughout the whole chapter. Section 1.2 presents the definitions of (asymptotically) almost periodic and limit periodic sequences in a pseudometric space, the above necessary and sufficient condition for the almost periodicity of a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$, and some basic properties of almost periodic sequences in pseudometric spaces.

In Section 1.3, we show a way one can construct limit periodic and almost periodic sequences with given properties. We remark that our process is comprehensible and easily modifiable and that methods of generating almost periodic sequences are mentioned in [138, Section 4] as well.

Finally, in Section 1.4, we use results from the second and the third section to obtain a theorem which plays an important role in Chapter 2, where it is proved that the almost periodic homogeneous linear difference systems which do not have any non-zero almost periodic solution form a dense subset of the set of all considered systems. Using our constructions, we obtain generalizations of results from the paper of the author denoted as (1) and from [176], where unitary and orthogonal systems are studied.

1.1 Preliminaries

As usual, $\mathbb{R}^+$ denotes the set of all positive reals, $\mathbb{R}_0^+$ the set of all non-negative real numbers, and $\mathbb{N}_0$ the set of all natural numbers including the zero. Let $\mathcal{X} \neq \emptyset$ be an arbitrary set and let $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ be a pseudometric on $\mathcal{X}$. For given $\varepsilon > 0$ and $x \in \mathcal{X}$, in the same way as in metric spaces, we define the ε-neighbourhood of x in $\mathcal{X}$ as the set $\{y \in \mathcal{X};\ d(x, y) < \varepsilon\}$. The ε-neighbourhood is denoted by $\mathcal{O}_\varepsilon(x)$.

We consider sequences in $\mathcal{X}$. The scalar (and vector) valued sequences are denoted by the lower-case letters, the matrix valued sequences by the capital letters ($\mathcal{X}$ is a set of matrices in this case), and each one of the scalar and matrix valued sequences by symbols $\{\varphi_k\}$, $\{\psi_k\}$, $\{\chi_k\}$.

1.2 Generalizations of pure periodicity

Now we introduce the concept of (asymptotically) almost periodic and limit periodic sequences in the pseudometric space $\mathcal{X}$. We remark that the approach is very general and that the presented theory of almost periodic sequences does not distinguish between $x \in \mathcal{X}$ and $y \in \mathcal{X}$ if $d(x, y) = 0$.

1.2.1 Almost periodic sequences

We begin with a "natural" generalization of almost periodicity.

Definition 1.1. A sequence $\{\varphi_k\}$ is called *almost periodic* if for any $\varepsilon > 0$, there exists a positive integer $p(\varepsilon)$ such that any set consisting of $p(\varepsilon)$ consecutive integers (non-negative integers if $k \in \mathbb{N}_0$) contains at least one integer l with the property that

$$d(\varphi_{k+l}, \varphi_k) < \varepsilon, \quad k \in \mathbb{Z} \text{ (or } k \in \mathbb{N}_0).$$

The number l is called an *ε-translation number* of $\{\varphi_k\}$. For any $\varepsilon > 0$, the set of all ε-translation numbers of a sequence $\{\varphi_k\}$ is denoted by $T(\{\varphi_k\}, \varepsilon)$.

Remark 1.2. If $\mathcal{X}$ is a Banach space, then a necessary and sufficient condition for a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ to be almost periodic is it to be *normal*; i.e., $\{\varphi_k\}$ is almost periodic if and only if any sequence of translates of $\{\varphi_k\}$ has a subsequence, uniformly convergent for $k \in \mathbb{Z}$ in the sense of the norm. This result and Theorem 1.3 below are not valid if $\{\varphi_k\}$ is defined for $k \in \mathbb{N}_0$ and if we consider only translates to the right (consider $\mathcal{X} = \mathbb{R}$, $\varphi_0 = 1$, and $\varphi_k = 0$, $k \in \mathbb{N}$). But, if we consider translates to the left, then the both results are valid for $k \in \mathbb{N}_0$ as well. $\triangle$

It is seen that the result mentioned in Remark 1.2 is no longer valid if the space of values fails to be complete. Especially, in a pseudometric space $(\mathcal{X}, d)$, it is not generally true that a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if it is normal. Nevertheless, applying the methods from any one of the proofs of the results [7, Statement (ζ)], [46, Theorem 1.10], and [72, Theorem 1.14], one can easily prove that every normal sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic. Further, we can prove the next theorem (a generalization of the theorem called the Bochner concept) which we will need later. We add that its proof is a modification of the proof of [46, Theorem 1.26].

Theorem 1.3. *Let $\{\varphi_k\}_{k\in\mathbb{Z}}$ be given. For an arbitrary sequence $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$, there exists a subsequence $\{\tilde{h}_n\}_{n\in\mathbb{N}} \subseteq \{h_n\}_{n\in\mathbb{N}}$ with the Cauchy property with respect to $\{\varphi_k\}$, i.e., for any $\varepsilon > 0$, there exists $M = M(\varepsilon) \in \mathbb{N}$ for which the inequality*

$$d\big(\varphi_{k+\tilde{h}_i}, \varphi_{k+\tilde{h}_j}\big) < \varepsilon$$

holds for all $i, j, k \in \mathbb{Z}$, $i, j > M$, if and only if $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic.

Proof. If any sequence of translates of $\{\varphi_k\}$ has a subsequence which has the Cauchy property, then $\{\varphi_k\}$ is almost periodic. It can be proved similarly as [46, Theorem 1.10], where it is not used that $\mathcal{X}$ is complete. To prove the opposite implication, we assume that $\{\varphi_k\}$ is almost periodic and we use the well-known method of the diagonal extraction.

Let $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ and $\vartheta > 0$ be arbitrary. By Definition 1.1, there exists a positive integer p such that, in any set $\{h_n - p, h_n - p + 1, \dots, h_n\}$, there exists a ϑ-translation number l_n. We know that $0 \le h_n - l_n \le p$ for all $n \in \mathbb{N}$. We put $k_n := h_n - l_n$, $n \in \mathbb{N}$. Clearly, $k_n = c = \text{const.}$ (a constant value from $\{0, 1, \dots, p\}$) for infinitely many values of n. Since

$$d\left(\varphi_{k+h_n}, \varphi_{k+k_n}\right) = d\left(\varphi_{(k+h_n-l_n)+l_n}, \varphi_{k+h_n-l_n}\right) < \vartheta, \quad k \in \mathbb{Z},$$

there exists a subsequence $\{h_n^1\}$ of $\{h_n\}$ and an integer c_1 such that

$$d\left(\varphi_{k+h_n^1}, \varphi_{k+c_1}\right) < \vartheta, \quad k \in \mathbb{Z},\ n \in \mathbb{N}. \tag{1.1}$$

Consider now a sequence of positive numbers $\vartheta_1 > \vartheta_2 > \cdots > \vartheta_n > \cdots$ converging to 0. From the sequence $\{\varphi_{k+h_n}\}$, we extract a subsequence $\{\varphi_{k+h_n^1}\}$ which satisfies (1.1) for $\vartheta = \vartheta_1$. From this sequence, we extract a subsequence $\{\varphi_{k+h_n^2}\}$ for which an inequality analogous to (1.1) is valid. Of course, c is not same, but it depends on the subsequence. We proceed further in the same way. Next, we form the sequence $\{\varphi_{k+h_n^n}\}_{n\in\mathbb{N}}$. Assume that $\varepsilon > 0$ is given and that we have $2\vartheta_m < \varepsilon$ for $m \in \mathbb{N}$. As a result, for $i, j > m$, $i, j \in \mathbb{N}$, we obtain

$$d\left(\varphi_{k+h_i^i}, \varphi_{k+h_j^j}\right) \le d\left(\varphi_{k+h_i^i}, \varphi_{k+c_m}\right) + d\left(\varphi_{k+c_m}, \varphi_{k+h_j^j}\right) < \varepsilon, \quad k \in \mathbb{Z},$$

where c_m is the number corresponding to the sequence $\{\varphi_{k+h_n^m}\}_{n\in\mathbb{N}}$ and ϑ_m. □

Corollary 1.4. *Let $p \in \mathbb{N}$ be arbitrarily given and let $\{\varphi_k\}_{k\in\mathbb{Z}}$ be almost periodic. For any $\varepsilon > 0$, the set $\{lp;\, l \in \mathbb{N}\} \cap T(\{\varphi_k\}, \varepsilon)$ is infinite.*

Proof. It suffices to apply Theorem 1.3 for $h_n := pn$, $n \in \mathbb{N}$. Indeed, it holds

$$\sup_{k\in\mathbb{Z}} d\left(\varphi_{k+h_i}, \varphi_{k+h_j}\right) = \sup_{k\in\mathbb{Z}} d\left(\varphi_{k+h_i-h_j}, \varphi_k\right), \quad i, j \in \mathbb{N}.$$

□

In Chapter 2, we consider almost periodic sequences in complete metric spaces. Thus, we also need the following consequence of the so-called Bochner theorem.

Corollary 1.5. *An arbitrary sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ in a complete metric space is almost periodic if and only if, from any sequence of the form $\{\{\varphi_{k+h_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$, where $\{h_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}$, one can extract a subsequence converging uniformly for $k \in \mathbb{Z}$.*

Many results about almost periodic sequences with values in $\mathbb{C}$ are extendable to sequences with values in a complete metric space (or in a pseudometric space). We mention the following results which we will need later and which can be easily proved using methods from the classical theory of almost periodic functions (see [7, 46] for the classical cases and, e.g., [13] for generalizations). We also refer to [134, 189].

Theorem 1.6. *Let $\mathcal{X}_1, \mathcal{X}_2$ be arbitrary pseudometric spaces and $\Phi : \mathcal{X}_1 \to \mathcal{X}_2$ be a uniformly continuous map. If $\{\varphi_k\} \subseteq \mathcal{X}_1$ is almost periodic, then the sequence $\{\Phi(\varphi_k)\}$ is almost periodic as well.*

Proof. Taking $\varepsilon > 0$ arbitrarily, let $\delta(\varepsilon) > 0$ be the number corresponding to ε from the definition of the uniform continuity of Φ. Now, Theorem 1.6 follows from the fact that the set of all ε-translation numbers of $\{\Phi(\varphi_k)\}$ contains the set of all $\delta(\varepsilon)$-translation numbers of $\{\varphi_k\}$, i.e., from the inclusion

$$T\left(\{\varphi_k\}, \delta(\varepsilon)\right) \subseteq T\left(\{\Phi(\varphi_k)\}, \varepsilon\right).$$

□

Theorem 1.7. *For every sequence of almost periodic sequences*

$$\{\varphi_k^1\}, \ldots, \{\varphi_k^i\}, \ldots,$$

the sequence of $\lim\limits_{i\to\infty} \varphi_k^i$ is almost periodic if the convergence is uniform with respect to k.

Proof. The proof can be easily obtained by a modification of the proof of [46, Theorem 6.4]. □

Theorem 1.8. *Let $(\mathcal{X}, d)$ be a complete metric space. For an almost periodic sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ and an arbitrary sequence of integers $h_1, \ldots, h_i, \ldots,$ there exists a subsequence $\{\tilde{h}_i\}_{i\in\mathbb{N}}$ of $\{h_i\}_{i\in\mathbb{N}}$ such that*

$$\lim_{j\to\infty}\left(\lim_{i\to\infty} \varphi_{k+\tilde{h}_i-\tilde{h}_j}\right) = \varphi_k.$$

Proof. Since $\{\varphi_k\}$ is normal, we know that there exists $\{\bar{h}_i\}_{i\in\mathbb{N}} \subseteq \{h_i\}_{i\in\mathbb{N}}$ for which the sequence $\{\{\varphi_{k+\bar{h}_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$ converges uniformly to an almost periodic sequence (see Theorem 1.7), denoted as $\{\psi_k\}$. Applying Corollary 1.5 again, we obtain a subsequence $\{\tilde{h}_i\}_{i\in\mathbb{N}} \subseteq \{\bar{h}_i\}_{i\in\mathbb{N}}$ with the property that the sequence $\{\{\psi_{k-\tilde{h}_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$ is uniformly convergent. We denote the limit as $\{\chi_k\}$.

Now we choose $\varepsilon > 0$ arbitrarily. We have

$$\varrho\left(\psi_{k-\tilde{h}_i}, \chi_k\right) < \frac{\varepsilon}{2}, \quad \varrho\left(\psi_k, \varphi_{k+\tilde{h}_j}\right) < \frac{\varepsilon}{2}, \quad k \in \mathbb{Z},$$

if $i, j > n$ $(i, j \in \mathbb{N})$ for some sufficiently large $n = n(\varepsilon) \in \mathbb{N}$. Thus, for all $k \in \mathbb{Z}$, it is true

$$\varrho\left(\varphi_k, \chi_k\right) \leq \varrho\left(\varphi_k, \psi_{k-\tilde{h}_i}\right) + \varrho\left(\psi_{k-\tilde{h}_i}, \chi_k\right) < \varepsilon.$$

Because of the arbitrariness of $\varepsilon > 0$, we get the identity $\{\varphi_k\} \equiv \{\chi_k\}$. □

Remark 1.9. It is possible to prove that a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if every pair of sequences $\{h_i\}_{i\in\mathbb{N}}, \{l_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}$ have common subsequences $\{\tilde{h}_i\}_{i\in\mathbb{N}}, \{\tilde{l}_i\}_{i\in\mathbb{N}}$ with the property that

$$\lim_{j\to\infty}\left(\lim_{i\to\infty} \varphi_{k+\tilde{h}_i+\tilde{l}_j}\right) = \lim_{i\to\infty} \varphi_{k+\tilde{h}_i+\tilde{l}_i} \quad \text{pointwise for } k \in \mathbb{Z}. \tag{1.2}$$

In fact, condition (1.2) is necessary and sufficient in each one of the two modes of the convergence; i.e., in the strongest version, this condition is necessary in the uniform sense and sufficient in the pointwise sense. For almost periodic functions defined on $\mathbb{R}$ with values in $\mathbb{C}$, the above result is due to S. Bochner and it can be found in [23]. The proof from the paper can be generalized for complete metric spaces (see [142, 143]). If this necessary and sufficient condition is applied only to the case $\{l_i\} \equiv \{-h_i\}$ (as in Theorem 1.8), then one gets a different class of sequences called almost automorphic sequences (for more details, see [56, 151]; concerning the linear systems treated in Chapter 2, see [35, 126]). △

Taking $n \in \mathbb{N}$ and using Theorem 1.3 (and Remark 1.2) n-times, one can easily prove the corollaries below.

Corollary 1.10. *Let sequences $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ be given. Then, the sequence $\{\psi_k\}$ which is defined by*

$$\psi_k := \varphi_j^{i+1} \quad \textit{for all considered } k,$$

where $k = jn + i$, $j \in \mathbb{Z}$, $i \in \{0, \dots, n-1\}$, is almost periodic if and only if all sequences $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ are almost periodic.

Corollary 1.11. *Let $(\mathcal{X}_1, d_1), \dots, (\mathcal{X}_n, d_n)$ be pseudometric spaces and $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ be arbitrary sequences with values in $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. The sequence $\{\psi_k\}$, with values in $\mathcal{X}_1 \times \dots \times \mathcal{X}_n$ given by*

$$\psi_k := \left(\varphi_k^1, \dots, \varphi_k^n\right) \quad \textit{for all considered } k,$$

is almost periodic if and only if each one of sequences $\{\varphi_k^1\}, \dots, \{\varphi_k^n\}$ is almost periodic.

Corollary 1.12. *Let $\varepsilon > 0$ be arbitrary and let the sequences $\{\varphi_k^1\}_{k\in\mathbb{Z}}, \dots, \{\varphi_k^n\}_{k\in\mathbb{Z}}$ be almost periodic. Then, the set*

$$T(\{\varphi_k^1\}, \varepsilon) \cap \dots \cap T(\{\varphi_k^n\}, \varepsilon)$$

is relatively dense in $\mathbb{Z}$.

We remark that it is possible to use Corollaries 1.10, 1.11, and 1.12 to obtain more general versions of Theorems 1.23, 1.27, and 1.29 below.

1.2.2 Asymptotically almost periodic sequences

Now we mention the definition of asymptotic almost periodicity in pseudometric spaces.

Definition 1.13. A sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ (or $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{X}$) is *asymptotically almost periodic* if for every $\varepsilon > 0$, there exist positive integers $r(\varepsilon)$ and $R(\varepsilon)$ such that any set consisting of $r(\varepsilon)$ consecutive integers contains at least one number l for which

$$d\left(\varphi_{k+l}, \varphi_k\right) < \varepsilon, \quad k, k+l \geq R(\varepsilon),\ k \in \mathbb{N}.$$

Directly from Definition 1.13, we obtain the following theorem.

Theorem 1.14. *The range of any asymptotically almost periodic sequence is totally bounded.*

The Bochner theorem for asymptotically almost periodic sequences reads as follows.

Theorem 1.15. *Let $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ or $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{X}$ be given. The sequence $\{\varphi_k\}$ is asymptotically almost periodic if and only if any sequence $\{l_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ satisfying $\lim_{n\to\infty} l_n = \infty$ has a subsequence $\{\bar{l}_n\}_{n\in\mathbb{N}} \subseteq \{l_n\}$ such that, for any $\varepsilon > 0$, there exists $L(\varepsilon) \in \mathbb{N}$ with the property that it holds*

$$d\left(\varphi_{k+\bar{l}_i}, \varphi_{k+\bar{l}_j}\right) < \varepsilon, \quad i, j > L(\varepsilon),\ k \in \mathbb{N}.$$

Proof. See [65, Part 2]. □

Remark 1.16. In Banach spaces, a sequence is asymptotically almost periodic if and only if it is the sum of an almost periodic sequence and a sequence vanishing at infinity (see, e.g., [184]). △

1.2.3 Limit periodic sequences

Finally, we define limit periodicity in $\mathcal{X}$.

Definition 1.17. A sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ (or $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{X}$) is called *limit periodic* if there exists a sequence of periodic sequences $\{\varphi_k^n\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ (or $\{\varphi_k^n\}_{k\in\mathbb{N}_0} \subseteq \mathcal{X}$) for $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \varphi_k^n = \varphi_k$ uniformly with respect to $k \in \mathbb{Z}$ (or $k \in \mathbb{N}_0$).

Remark 1.18. Of course, the periods of sequences $\{\varphi_k^n\}$ in Definition 1.17 do not need to be the same for considered n. △

Remark 1.19. In fact, limit periodic sequences coincide with the so-called semi-periodic sequences. We refer to [17] (and to [8] in the continuous case). △

Remark 1.20. In the literature, it is possible to find another definition of limit periodicity which leads to a larger class of sequences. See, e.g., [59, 130]. We consider Definition 1.17 because it is the standard one in the theory of almost periodic functions (and we obtain the strongest results in this case). See, e.g., [18, 46]. △

Theorem 1.21. *There exists an almost periodic sequence $\{f_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ (with respect to the usual metric) which is not limit periodic.*

Proof. It suffices to consider, e.g., the sequence $\{\mathrm{e}^{\mathrm{i}k}\}_{k\in\mathbb{Z}}$ (or also [46, Theorem 1.27]). □

Theorem 1.22. *Any limit periodic sequence is almost periodic and any almost periodic sequence is asymptotically almost periodic.*

Proof. It suffices to consider Theorems 1.3, 1.7, and 1.15 together with Definitions 1.1, 1.13, and 1.17. □

1.3 Constructions of almost periodic and limit periodic sequences

In this section, we prove several theorems which facilitate to find almost periodic and limit periodic sequences having certain specific properties. In Theorem 1.23, we consider sequences for $k \in \mathbb{N}_0$; in Theorem 1.25 and Corollary 1.26, sequences for $k \in \mathbb{Z}$ obtained from almost periodic sequences for $k \in \mathbb{N}_0$; and, in Theorems 1.27 and 1.29, sequences for $k \in \mathbb{Z}$.

Theorem 1.23. *Let $\varphi_0, \dots, \varphi_m \in \mathcal{X}$ and $j \in \mathbb{N}$ be arbitrarily given. Let $\{r_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of non-negative real numbers such that*

$$\sum_{n=1}^{\infty} r_n < \infty. \tag{1.3}$$

Then, any sequence $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{X}$, where

$$\begin{gathered}
\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-(m+1)}\right), \quad k \in \{m+1, \dots, 2m+1\}, \\
\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-2(m+1)}\right), \quad k \in \{2(m+1), \dots, 3(m+1)-1\}, \\
\vdots \\
\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-j(m+1)}\right), \quad k \in \{j(m+1), \dots, (j+1)(m+1)-1\}, \\
\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k-(j+1)(m+1)}\right), \quad k \in \{(j+1)(m+1), \dots, 2(j+1)(m+1)-1\}, \\
\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k-2(j+1)(m+1)}\right), \quad k \in \{2(j+1)(m+1), \dots, 3(j+1)(m+1)-1\}, \\
\vdots \\
\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k-j(j+1)(m+1)}\right), \quad k \in \{j(j+1)(m+1), \dots, (j+1)^2(m+1)-1\}, \\
\vdots \\
\varphi_k \in \mathcal{O}_{r_n}\left(\varphi_{k-(j+1)^{n-1}(m+1)}\right), \quad k \in \big\{(j+1)^{n-1}(m+1), \\
\dots, 2(j+1)^{n-1}(m+1)-1\big\}, \\
\varphi_k \in \mathcal{O}_{r_n}\left(\varphi_{k-2(j+1)^{n-1}(m+1)}\right), \quad k \in \big\{2(j+1)^{n-1}(m+1), \\
\dots, 3(j+1)^{n-1}(m+1)-1\big\}, \\
\vdots \\
\varphi_k \in \mathcal{O}_{r_n}\left(\varphi_{k-j(j+1)^{n-1}(m+1)}\right), \quad k \in \big\{j(j+1)^{n-1}(m+1), \\
\dots, (j+1)^{n}(m+1)-1\big\}, \\
\vdots
\end{gathered}$$

is almost periodic.

Proof. Consider an arbitrary $\varepsilon > 0$. We need to prove that the set of all ε-translation numbers of $\{\varphi_k\}$ is relatively dense in $\mathbb{N}_0$. Using (1.3), one can find $n(\varepsilon) \in \mathbb{N}$ for which

$$\sum_{n=n(\varepsilon)}^{\infty} r_n < \frac{\varepsilon}{2}. \tag{1.4}$$

We see that

$$\begin{aligned} \varphi_{k+(j+1)^{n(\varepsilon)-1}(m+1)} &\in \mathcal{O}_{r_{n(\varepsilon)}}(\varphi_k), \\ \varphi_{k+2(j+1)^{n(\varepsilon)-1}(m+1)} &\in \mathcal{O}_{r_{n(\varepsilon)}}(\varphi_k), \\ &\vdots \\ \varphi_{k+j(j+1)^{n(\varepsilon)-1}(m+1)} &\in \mathcal{O}_{r_{n(\varepsilon)}}(\varphi_k) \end{aligned} \tag{1.5}$$

if

$$0 \le k < (j+1)^{n(\varepsilon)-1}(m+1).$$

Next, from (1.5) it follows (consider $i \in \{(j+1)^n, \dots, (j+1)^{n+1} - 1\}$, $n \in \mathbb{N}$)

$$\begin{aligned} \varphi_{k+(j+1)(j+1)^{n(\varepsilon)-1}(m+1)} &\in \mathcal{O}_{r_{n(\varepsilon)}+r_{n(\varepsilon)+1}}(\varphi_k), \\ \varphi_{k+((j+1)^2-1)(j+1)^{n(\varepsilon)-1}(m+1)} &\in \mathcal{O}_{r_{n(\varepsilon)}+r_{n(\varepsilon)+1}}(\varphi_k), \\ &\vdots \\ \varphi_{k+i(j+1)^{n(\varepsilon)-1}(m+1)} &\in \mathcal{O}_{r_{n(\varepsilon)}+r_{n(\varepsilon)+1}+\dots+r_{n(\varepsilon)+n}}(\varphi_k), \\ &\vdots \end{aligned}$$

for $k \in \{0, \dots, (j+1)^{n(\varepsilon)-1}(m+1) - 1\}$. Therefore (consider (1.4)), we have

$$\varphi_{k+l(j+1)^{n(\varepsilon)-1}(m+1)} \in \mathcal{O}_{\frac{\varepsilon}{2}}(\varphi_k), \quad 0 \le k < (j+1)^{n(\varepsilon)-1}(m+1),\, l \in \mathbb{N}_0. \tag{1.6}$$

We put

$$q(\varepsilon) := (j+1)^{n(\varepsilon)-1}(m+1). \tag{1.7}$$

Any $p \in \mathbb{N}_0$ can be expressed uniquely in the form

$$p = k(p) + l(p)q(\varepsilon) \quad \text{for some } k(p) \in \{0, \dots, q(\varepsilon) - 1\} \text{ and } l(p) \in \mathbb{N}_0.$$

Applying (1.6), we obtain

$$\begin{aligned} d\left(\varphi_p, \varphi_{p+lq(\varepsilon)}\right) &= d\left(\varphi_{k(p)+l(p)q(\varepsilon)}, \varphi_{k(p)+l(p)q(\varepsilon)+lq(\varepsilon)}\right) \\ &\le d\left(\varphi_{k(p)+l(p)q(\varepsilon)}, \varphi_{k(p)}\right) + d\left(\varphi_{k(p)}, \varphi_{k(p)+(l+l(p))q(\varepsilon)}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned} \tag{1.8}$$

where $p, l \in \mathbb{N}_0$ are arbitrary; i.e., $lq(\varepsilon)$ is an ε-translation number of $\{\varphi_k\}$ for all $l \in \mathbb{N}_0$. The fact that the set $\{lq(\varepsilon); l \in \mathbb{N}_0\}$ is relatively dense in $\mathbb{N}_0$ proves the theorem. $\square$

Remark 1.24. From the proof of Theorem 1.23 (see (1.7) and (1.8)), for any $\varepsilon > 0$ and any sequence $\{\varphi_k\}$ considered there, we get the existence of $n(\varepsilon) \in \mathbb{N}$ such that the set of all ε-translation numbers of $\{\varphi_k\}$ contains $\{l(j+1)^{n(\varepsilon)-1}(m+1); l \in \mathbb{N}\}$; i.e., we have

$$T\left(\{\varphi_k\}, n(\varepsilon)\right) := \left\{l(j+1)^{n(\varepsilon)-1}(m+1);\, l \in \mathbb{N}\right\} \subseteq T\left(\{\varphi_k\}, \varepsilon\right) \tag{1.9}$$

for every $\varepsilon > 0$. △

From Remark 1.24 (see (1.9)), we immediately obtain that the resulting sequence $\{\varphi_k\}$ in Theorem 1.23 is actually limit periodic.

Theorem 1.25. *Let $\{\varphi_k\}_{k\in\mathbb{N}_0}$ be an almost periodic sequence and let $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$ and $\{l_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be such that*

$$r_n l_n \to 0 \quad as \quad n \to \infty. \tag{1.10}$$

If for any $n \in \mathbb{N}$, there exists a set $T(r_n)$ of some r_n-translation numbers of $\{\varphi_k\}$ which is relatively dense in $\mathbb{N}_0$ and, for every non-zero $l = l(r_n) \in T(r_n)$, there exists $i = i(l) \in \{1, \dots, l_n + 1\}$ with the property that

$$\varphi_{(i-1)l+k} \in \mathcal{O}_{r_n l_n}\left(\varphi_{il-k}\right), \quad k \in \{0, \dots, l\}, \tag{1.11}$$

then the sequence $\{\psi_k\}_{k\in\mathbb{Z}}$, given by the formula

$$\psi_k := \varphi_k \quad for\ k \in \mathbb{N}_0; \qquad \psi_k := \varphi_{-k} \quad for\ k \in \mathbb{Z} \setminus \mathbb{N}_0, \tag{1.12}$$

is almost periodic.

If for any $n \in \mathbb{N}$, there exists a set $\widetilde{T}(r_n)$ of some r_n-translation numbers of $\{\varphi_k\}$ which is relatively dense in $\mathbb{N}_0$ and, for every non-zero $m = m(r_n) \in \widetilde{T}(r_n)$, there exists $i = i\,(m) \in \{1, \dots, l_n + 1\}$ with the property that

$$\varphi_{(i-1)m+k} \in \mathcal{O}_{r_n l_n}\left(\varphi_{im-k-1}\right), \quad k \in \{0, \dots, m-1\}, \tag{1.13}$$

then the sequence $\{\chi_k\}_{k\in\mathbb{Z}}$, given by the formula

$$\chi_k := \varphi_k \quad for\ k \in \mathbb{N}_0; \qquad \chi_k := \varphi_{-(k+1)} \quad for\ k \in \mathbb{Z} \setminus \mathbb{N}_0, \tag{1.14}$$

is almost periodic.

Proof. We prove only the first part of Theorem 1.25. The proof of the second case (the almost periodicity of $\{\chi_k\}$) is analogous. Let $\varepsilon > 0$ be arbitrarily small. Consider $n \in \mathbb{N}$ satisfying (see (1.10))

$$r_n l_n < \frac{\varepsilon}{3}. \tag{1.15}$$

We want to prove that the set $T(\{\psi_k\}, \varepsilon)$ of all ε-translation numbers of $\{\psi_k\}$ contains the numbers $\{\pm l; l \in T(r_n)\}$; i.e., we want to obtain the inequality

$$d\left(\psi_k, \psi_{k\pm l}\right) < \varepsilon, \quad l \in T(r_n),\ k \in \mathbb{Z}, \tag{1.16}$$

which proves the theorem because $\{\pm l;\, l \in T(r_n)\}$ is relatively dense in $\mathbb{Z}$.

First of all we choose arbitrary $l \in T(r_n)$. From the theorem, we have $i = i(l)$. Without loss of generality, we can consider only $+l$. (For $-l$, we can proceed similarly.) Because of $l_n \in \mathbb{N}$ and $l \in T(r_n)$, from (1.12) and (1.15) it follows

$$d\left(\psi_k, \psi_{k+l}\right) < \frac{\varepsilon}{3}, \quad k \notin \{-l, \dots, -1\},\ k \in \mathbb{Z}. \tag{1.17}$$

Let $k \in \{-l, \dots, -1\}$ be also arbitrarily chosen. Evidently, we have

$$k + (1-i)l \in \{-il, \dots, -(i-1)l - 1\}$$

and

$$\begin{aligned} d\left(\psi_k, \psi_{k+l}\right) &\leq d\left(\psi_k, \psi_{k+(1-i)l}\right) + d\left(\psi_{k+(1-i)l}, \psi_{k+l}\right) \\ &= d\left(\varphi_{-k}, \varphi_{(i-1)l-k}\right) + d\left(\varphi_{(i-1)l-k}, \varphi_{l+k}\right). \end{aligned} \tag{1.18}$$

The number $(i-1)l$ is an $(\varepsilon/3)$-translation number of $\{\varphi_k\}$. It follows from (1.15) and from $i \leq l_n + 1$. Therefore, we have

$$d\left(\varphi_{-k}, \varphi_{(i-1)l-k}\right) < \frac{\varepsilon}{3}. \tag{1.19}$$

Using (1.11) and (1.15), we get

$$d\left(\varphi_{(i-1)l-k}, \varphi_{il+k}\right) < r_n l_n < \frac{\varepsilon}{3}.$$

Thus, it holds

$$d\left(\varphi_{(i-1)l-k}, \varphi_{l+k}\right) < \frac{2\varepsilon}{3}. \tag{1.20}$$

Indeed, $(i-1)l$ is an $(\varepsilon/3)$-translation number of $\{\varphi_k\}$ (consider again (1.15) and the inequality $i - 1 \leq l_n$).

Altogether, from (1.18), (1.19), and (1.20), we obtain

$$d\left(\psi_k, \psi_{k+l}\right) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon. \tag{1.21}$$

Since the choice of k, l is arbitrary (see (1.17)), (1.21) gives (1.16). □

Corollary 1.26. *Let $m \in \mathbb{N}_0$, $j \in \mathbb{N}$, the sequence $\{\varphi_k\}_{k\in\mathbb{N}_0}$ be from Theorem 1.23, and $M > 0$ be arbitrary.*

If for all $n > M$, $n \in \mathbb{N}$, there exists at least one $i \in \{1, \dots, j\}$ satisfying

$$\varphi_{i(j+1)^n(m+1)+k} = \varphi_{(i+1)(j+1)^n(m+1)-k}, \quad k \in \{0, \dots, (j+1)^n(m+1)\}, \tag{1.22}$$

then the sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ given by (1.12) is almost periodic.

If for all $n > M$, $n \in \mathbb{N}$, there exists at least one $i \in \{1, \dots, j\}$ satisfying

$$\varphi_{i(j+1)^n(m+1)+k} = \varphi_{(i+1)(j+1)^n(m+1)-k-1}, \quad k \in \{0, \dots, (j+1)^n(m+1) - 1\}, \tag{1.23}$$

then the sequence $\{\chi_k\}_{k\in\mathbb{Z}}$ given by (1.14) is almost periodic.

Proof. We put

$$r_n := \frac{1}{n}, \quad l_n := 1, \quad T(r_n) := T\left(\{\varphi_k\}, n\left(\frac{r_n}{2}\right)\right) \quad \text{for all } n \in \mathbb{N},$$

where $T(\{\varphi_k\}, n(\varepsilon))$ is defined in (1.9). Since we can assume that $n(1/2) > M - 1$, it suffices to consider Theorem 1.25 and Remark 1.24 (using (1.22) and (1.23), we get (1.11) and (1.13), respectively). □

Theorem 1.27. *Let $\varphi_0, \ldots, \varphi_n \in \mathcal{X}$ and $j \in \mathbb{N}$ be given and let $\{r_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$ satisfy*

$$\sum_{i=1}^{\infty} r_i < \infty. \tag{1.24}$$

Then, every sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ for which

$$\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-(n+1)}\right), \quad k \in \{n+1, \ldots, 2(n+1)-1\},$$

$$\vdots$$

$$\varphi_k \in \mathcal{O}_{r_1}\left(\varphi_{k-j(n+1)}\right), \quad k \in \{j(n+1), \ldots, (j+1)(n+1)-1\},$$
$$\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k+(j+1)(n+1)}\right), \quad k \in \{-(j+1)(n+1), \ldots, -1\},$$

$$\vdots$$

$$\varphi_k \in \mathcal{O}_{r_2}\left(\varphi_{k+j(j+1)(n+1)}\right),$$
$$k \in \{-j(j+1)(n+1), \ldots, -(j-1)(j+1)(n+1)-1\},$$

$$\varphi_k \in \mathcal{O}_{r_3}\left(\varphi_{k-(j+1)^2(n+1)}\right),$$
$$k \in \{(j+1)(n+1), \ldots, (j+1)(n+1)+(j+1)^2(n+1)-1\},$$

$$\vdots$$

$$\varphi_k \in \mathcal{O}_{r_3}\left(\varphi_{k-j(j+1)^2(n+1)}\right),$$
$$k \in \{(j+1)(n+1)+(j-1)(j+1)^2(n+1), \ldots,$$
$$(j+1)(n+1)+j(j+1)^2(n+1)-1\},$$

$$\varphi_k \in \mathcal{O}_{r_4}\left(\varphi_{k+(j+1)^3(n+1)}\right),$$
$$k \in \{-(j+1)^3(n+1)-j(j+1)(n+1), \ldots, -j(j+1)(n+1)-1\},$$

$$\vdots$$

$$\varphi_k \in \mathcal{O}_{r_4}\left(\varphi_{k+j(j+1)^3(n+1)}\right),$$
$$k \in \{-j(j+1)^3(n+1)-j(j+1)(n+1), \ldots,$$
$$-(j-1)(j+1)^3(n+1)-j(j+1)(n+1)-1\},$$

$$\vdots$$

$$\begin{aligned}
&\varphi_k \in \mathcal{O}_{r_{2i}}\left(\varphi_{k+(j+1)^{2i-1}(n+1)}\right), \\
&\quad k \in \{-((j+1)^{2i-1} + \cdots + j(j+1)^3 + j(j+1))(n+1), \ldots, \\
&\qquad - (j(j+1)^{2i-3} + \cdots + j(j+1)^3 + j(j+1))(n+1) - 1\},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
&\varphi_k \in \mathcal{O}_{r_{2i}}\left(\varphi_{k+j(j+1)^{2i-1}(n+1)}\right), \\
&\quad k \in \{-(j(j+1)^{2i-1} + \cdots + j(j+1)^3 + j(j+1))(n+1), \ldots, \\
&\qquad - ((j-1)(j+1)^{2i-1} + \cdots + j(j+1)^3 + j(j+1))(n+1) - 1\}, \\
&\varphi_k \in \mathcal{O}_{r_{2i+1}}\left(\varphi_{k-(j+1)^{2i}(n+1)}\right), \\
&\quad k \in \{(j+1)(n+1) + j(j+1)^2(n+1) + \cdots + j(j+1)^{2i-2}(n+1), \ldots, \\
&\qquad (j+1)(n+1) + j(j+1)^2(n+1) + \cdots \\
&\qquad\qquad + j(j+1)^{2i-2}(n+1) + (j+1)^{2i}(n+1) - 1\},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
&\varphi_k \in \mathcal{O}_{r_{2i+1}}\left(\varphi_{k-j(j+1)^{2i}(n+1)}\right), \\
&\quad k \in \{(j+1)(n+1) + j(j+1)^2(n+1) + \cdots \\
&\qquad + j(j+1)^{2i-2}(n+1) + (j-1)(j+1)^{2i}(n+1), \ldots, \\
&\qquad (j+1)(n+1) + j(j+1)^2(n+1) + \cdots + j(j+1)^{2i}(n+1) - 1\},
\end{aligned}$$

$$\vdots$$

is limit periodic.

Proof. Let $\varepsilon > 0$ be arbitrarily given and let the number $i(\varepsilon) \in \mathbb{N}$ satisfy the condition (see (1.24))

$$\sum_{i=i(\varepsilon)}^{\infty} r_i < \frac{\varepsilon}{2}.$$

One can easily show that

$$\left\{l(j+1)^{i(\varepsilon)-1}(n+1);\, l \in \mathbb{Z}\right\} \subseteq T\left(\{\varphi_k\}, \varepsilon\right).$$

Of course, this inclusion proves the theorem. □

For $n = 0$, $j = 1$, we get the most important case of Theorem 1.27 which reads as follows.

Corollary 1.28. *Let $\psi_0 \in \mathcal{X}$ and $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$ satisfying*

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty \tag{1.25}$$

be arbitrary. Then, every sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ for which

$$\psi_k \in \mathcal{O}_{\varepsilon_1}\left(\psi_{k-2^0}\right), \quad k \in \{1\} = \{2-1\},$$

$$\psi_k \in \mathcal{O}_{\varepsilon_2}\left(\psi_{k+2^1}\right), \quad k \in \{-2, -1\},$$
$$\psi_k \in \mathcal{O}_{\varepsilon_3}\left(\psi_{k-2^2}\right), \quad k \in \{2, \dots, 2 + 2^2 - 1\},$$
$$\psi_k \in \mathcal{O}_{\varepsilon_4}\left(\psi_{k+2^3}\right), \quad k \in \{-2^3 - 2, \dots, -2 - 1\},$$
$$\psi_k \in \mathcal{O}_{\varepsilon_5}\left(\psi_{k-2^4}\right), \quad k \in \{2 + 2^2, \dots, 2 + 2^2 + 2^4 - 1\},$$
$$\vdots$$
$$\psi_k \in \mathcal{O}_{\varepsilon_{2i}}\left(\psi_{k+2^{2i-1}}\right), \quad k \in \{-2^{2i-1} - \cdots - 2^3 - 2, \cdots, -2^{2i-3} - \cdots - 2^3 - 2 - 1\},$$
$$\psi_k \in \mathcal{O}_{\varepsilon_{2i+1}}\left(\psi_{k-2^{2i}}\right), \quad k \in \{2 + 2^2 + \cdots + 2^{2i-2}, \dots, 2 + 2^2 + \cdots + 2^{2i-2} + 2^{2i} - 1\},$$
$$\vdots$$

is limit periodic.

Theorem 1.29. *Let $\varphi_0, \dots, \varphi_m \in \mathcal{X}$ be given, $\{r_i\}_{i\in\mathbb{N}} \subset \mathbb{R}_0^+$, $\{j_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$, and $n \in \mathbb{N}_0$ be arbitrary such that $m + n$ is even and*

$$\sum_{i=1}^{\infty} r_i\, j_i < \infty. \tag{1.26}$$

For any $\varphi_{m+1}, \dots, \varphi_{m+n}$, if one puts

$$\psi_k := \varphi_{k+\frac{m+n}{2}}, \quad k \in \left\{-\frac{m+n}{2}, \dots, \frac{m+n}{2}\right\},$$
$$M := \frac{m+n}{2}, \qquad N := m + n$$

and arbitrarily chooses

$$\psi_k \in \mathcal{O}_{r_1}\left(\psi_{k+N+1}\right), \quad k \in \{-N - M - 1, \dots, -M - 1\},$$
$$\vdots$$
$$\psi_k \in \mathcal{O}_{r_1}\left(\psi_{k+N+1}\right), \quad k \in \{-j_1 N - M - 1, \dots, -(j_1 - 1)N - M - 1\},$$
$$\psi_k \in \mathcal{O}_{r_1}\left(\psi_{k-N-1}\right), \quad k \in \{M + 1, \dots, N + M + 1\},$$
$$\vdots$$
$$\psi_k \in \mathcal{O}_{r_1}\left(\psi_{k-N-1}\right), \quad k \in \{(j_1 - 1)N + M + 1, \dots, j_1 N + M + 1\},$$
$$\vdots$$
$$\psi_k \in \mathcal{O}_{r_i}\left(\psi_{k+p_i}\right), \quad k \in \{-p_i - p_{i-1} - \cdots - p_1, \dots, -p_{i-1} - \cdots - p_1\},$$
$$\vdots$$
$$\psi_k \in \mathcal{O}_{r_i}\left(\psi_{k+p_i}\right), \quad k \in \{-j_i p_i - p_{i-1} - \cdots - p_1, \dots, -(j_i - 1)p_i - p_{i-1} - \cdots - p_1\},$$
$$\psi_k \in \mathcal{O}_{r_i}\left(\psi_{k-p_i}\right), \quad k \in \{p_{i-1} + \cdots + p_1, \dots, p_i + p_{i-1} + \cdots + p_1\},$$

$$\vdots$$

$$\psi_k \in \mathcal{O}_{r_i}\left(\psi_{k-p_i}\right), \quad k \in \{(j_i - 1)p_i + p_{i-1} + \cdots + p_1, \dots, j_i p_i + p_{i-1} + \cdots + p_1\},$$

$$\vdots$$

where

$$p_1 := (j_1 N + M + 1) + 1, \qquad p_2 := 2(j_1 N + M + 1) + 1,$$

$$p_3 := (2j_2 + 1)p_2, \quad \dots \quad p_i := (2j_{i-1} + 1)p_{i-1}, \quad \dots,$$

then the resulting sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ *is limit periodic.*

Proof. Consider arbitrary $\varepsilon > 0$ and a positive integer $n(\varepsilon) \geq 2$ for which (see (1.26))

$$\sum_{i=n(\varepsilon)}^{\infty} r_i\, j_i < \frac{\varepsilon}{4}.$$

One can show that

$$\{l p_{n(\varepsilon)};\, l \in \mathbb{Z}\} \subseteq T(\{\psi_k\}, \varepsilon)$$

which completes the proof. □

1.4 Application related to almost periodic difference systems

Let $m \in \mathbb{N}$ be arbitrarily given. We analyse almost periodic systems of m homogeneous linear difference equations of the form

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z} \text{ (or } k \in \mathbb{N}_0), \tag{1.27}$$

where $\{A_k\}$ is almost periodic. Let $\mathfrak{X}$ denote the set of all systems (1.27).

An important characteristic property of linear difference systems, which makes them simple to treat, is the well-known superposition principle (see, e.g., [1, 108, 115]). In particular, since we study homogeneous systems, we obtain that every solution of a system $\mathfrak{S} \in \mathfrak{X}$ can be expressed as a right linear combination of m solutions of $\mathfrak{S}$; i.e., any solution $\{x_k\}$ of $\mathfrak{S}$ can be written as

$$x_k = P_k \cdot x_l, \quad k \in \mathbb{Z} \text{ (or } k \in \mathbb{N}_0), \tag{1.28}$$

for some matrix valued sequence $\{P_k\}$ and some $l \in \mathbb{Z}$ ($l \in \mathbb{N}_0$). Conversely, for any considered $r_1, \dots, r_m$, the sequence $\{x_k\}$ defined by the formula

$$x_k := P_k \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}, \quad k \in \mathbb{Z} \text{ (or } k \in \mathbb{N}_0),$$

is a solution of $\mathfrak{S}$. For given $\mathfrak{S} \in \mathfrak{X}$ determined by $\{A_k\}$, the sequence $\{P_k\}$ is called the principal fundamental matrix if P_0 is the identity matrix. We immediately obtain

$$\begin{aligned} P_k &= \prod_{i=0}^{k-1} A_{k-i-1} \qquad && \text{for all } k \in \mathbb{N}; \\ P_k &= \prod_{i=k}^{-1} A_i^{-1} \qquad && \text{for all } k \in \mathbb{Z} \smallsetminus \mathbb{N}. \end{aligned} \tag{1.29}$$

Our aim is to study the existence of $\widehat{\mathfrak{S}} \in \mathfrak{X}$ which does not have any non-trivial almost periodic solution. We treat this problem in a very general setting and this motivates our requirements on the set of values of matrices A_k.

We need the set of entries of A_k to be a subset of a set R with two operations and unit elements such that R with them is a ring, because the multiplication of matrices A_k has to be associative (consider the natural expression of solutions of (1.27), i.e., consider (1.28) and (1.29)). We also need the set of all considered A_k to form a set X which has the below given properties (1.33); and we need that there exists at least one of the below mentioned functions F_1, $F_2 : [-1, 1] \to X$ (see (1.34), (1.35)). The conditions (1.34) are common. However, the main theorem of this chapter (the existence of the above system $\widehat{\mathfrak{S}} \in \mathfrak{X}$) is true, e.g., for many subsets of the set of unitary or orthogonal matrices which contain matrices having eigenvalue $\lambda = 1$. Thus, we also consider the existence of F_2.

Let $R = (R, \oplus, \odot)$ be an infinite ring with a unit and a zero denoted as e_1 and e_0, respectively. Symbol $\mathcal{M}(R, m)$ denotes the set of all $m \times m$ matrices with elements from R. If we consider the i-th column of $U \in \mathcal{M}(R, m)$, then we write U_i; and R^m if we consider the set of all $m \times 1$ vectors with entries attaining values from R. As usual, we define the multiplication $\cdot$ of matrices from $\mathcal{M}(R, m)$ (and $U \cdot v$, $U \in \mathcal{M}(R, m)$, $v \in R^m$) by $\oplus$ and $\odot$. Let d be a pseudometric on R and assume that

$$\text{operations } \oplus \text{ and } \odot \text{ are continuous with respect to } d. \tag{1.30}$$

In particular, we have pseudometrics on R^m and $\mathcal{M}(R, m)$, because $\mathcal{M}(R, m)$ can be expressed as $R^{m \times m}$; i.e., d in R^m and $\mathcal{M}(R, m)$ is the sum of m and m^2 non-negative numbers given by d in R, respectively. For simplicity, we also denote these pseudometrics as d.

The vector $v \in R^m$ is called non-zero (or non-trivial) if $d\left(v, (e_0, \ldots, e_0)^T\right) > 0$. We say that a non-zero vector $(r_1, \ldots, r_m)^T$, where $r_1, \ldots, r_m \in R$, is an e_1-eigenvector of $U \in \mathcal{M}(R, m)$ if

$$d\left(U \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}, \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}\right) = 0,$$

and that $V \in \mathcal{M}(R, m)$ is regular for a non-zero vector $(r_1, \ldots, r_m)^T \in R^m$ if

$$d\left(V \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}, \begin{pmatrix} e_0 \\ \vdots \\ e_0 \end{pmatrix}\right) > 0. \tag{1.31}$$

Next, we denote

$$\mathcal{I} := \begin{pmatrix} e_1 & e_0 & \dots & e_0 \\ e_0 & e_1 & \dots & e_0 \\ \vdots & \vdots & \ddots & \vdots \\ e_0 & e_0 & \dots & e_1 \end{pmatrix} \in \mathcal{M}(R, m).$$

If for given $U \in \mathcal{M}(R, m)$ and $X \subseteq \mathcal{M}(R, m)$, there exists the unique matrix $V \in X$ (we put $V = W$ if $d(V, W) = 0$) for which

$$U \cdot V = V \cdot U = \mathcal{I},$$

then we define $U^{-1} := V$ and we say that V is the inverse matrix of U in X.

For any function $H : [a, b] \to X$ ($a \leq 0 < b$, $a, b \in \mathbb{R}$) and $s \in \mathbb{R}$, we extend its domain of definition by the formula

$$H(s) := \begin{cases} H(\sigma) \cdot (H(b))^l & \text{for } s \geq 0; \\ (H(a))^l \cdot H(\sigma) & \text{for } s < 0 \text{ if } a < 0, \end{cases} \tag{1.32}$$

where $s = lb + \sigma$ for $l \in \mathbb{N}_0$, $\sigma \in [0, b)$ or $s = la + \sigma$ for $l \in \mathbb{N}_0$, $\sigma \in (a, 0]$. Hereafter, we restrict coefficients A_k in (1.27) to be elements of an infinite set $X \subseteq \mathcal{M}(R, m)$ such that

$$\mathcal{I} \in X; \quad U, V \in X \implies U \cdot V \in X,\ U^{-1} \text{ exists in } X; \tag{1.33}$$

and either

> there exists a continuous function $F_1 : [-1, 1] \to X$ satisfying
>
> $$F_1(0) = \mathcal{I}; \qquad F_1(t) = F_1^{-1}(-t), \quad t \in [0, 1]; \tag{1.34}$$
>
> and matrix $F_1(1)$ has no e_1-eigenvector

or

> there exist continuous $F_2 : [-1, 1] \to X$, $t_1, \dots, t_q \in (0, 1]$, $\delta > 0$ such that
>
> $$F_2(0) = \mathcal{I}; \qquad F_2\left(\sum_{i=1}^{p} s_i\right) = \prod_{i=1}^{p} F_2(s_i), \quad s_1, \dots, s_p \in [-1, 1]; \tag{1.35}$$
>
> and, for any $v \in R^m$, one can find $j \in \{1, \dots, q\}$ for which v is not an e_1-eigenvector of $F_2(t)$, $t \in (\max\{0, t_j - \delta\}, \min\{t_j + \delta, 1\})$.

Remark 1.30. A function F_1 satisfying (1.34) exists, e.g., if the considered pseudometric $d : X \times X \to \mathbb{R}_0^+$ is such that the map $U \mapsto U^{-1}$ is continuous on X and if there exists a continuous function $G : [0, 1] \to X$ which satisfies that at least one of matrices $G^{-1}(0)\, G(1)$, $G(1)\, G^{-1}(0)$ does not have an e_1-eigenvector.

Similarly, the conditions in (1.35) are realized if the map $U \mapsto U^{-1}$ is continuous on X and if there exist continuous functions $G_1, \dots, G_q : [0, 1] \to X$ such that

$$G_j^{-1}(0) \cdot G_j\left(\sum_{i=1}^{p} s_i\right) = \prod_{i=1}^{p} G_j^{-1}(0) \cdot G_j(s_i)$$

and

$$\left(G_j \left(\sum_{i=1}^{p} s_i \right) \cdot G_j(0) \right)^{-1} = \prod_{i=1}^{p} G_j^{-1}(0) \cdot G_j^{-1}(s_i)$$

or

$$G_j \left(\sum_{i=1}^{p} s_i \right) \cdot G_j^{-1}(0) = \prod_{i=1}^{p} G_j(s_i) \cdot G_j^{-1}(0)$$

and

$$\left(G_j(0) \cdot G_j \left(\sum_{i=1}^{p} s_i \right) \right)^{-1} = \prod_{i=1}^{p} G_j^{-1}(s_i) \cdot G_j^{-1}(0),$$

where

$$j \in \{1, \ldots, q\}, \quad p \in \mathbb{N}, \quad s_1, \ldots, s_p \in [0, 1];$$

for all $j_1, j_2 \in \{1, \ldots, q\}$, one can find $r = r(j_1, j_2) \in (0, 1]$ with at least one property from

$$G_{j_1}^{-1}(0) \cdot G_{j_1}(1) = G_{j_2}^{-1}(0) \cdot G_{j_2}(r), \quad G_{j_2}^{-1}(0) \cdot G_{j_2}(1) = G_{j_1}^{-1}(0) \cdot G_{j_1}(r)$$

or

$$G_{j_1}(1) \cdot G_{j_1}^{-1}(0) = G_{j_2}(r) \cdot G_{j_2}^{-1}(0), \quad G_{j_2}(1) \cdot G_{j_2}^{-1}(0) = G_{j_1}(r) \cdot G_{j_1}^{-1}(0);$$

and the condition on arbitrary $v \in R^m$ is the same as in (1.35), where

$$F_2(t), \quad t \in (\max\{0, t_j - \delta\}, \min\{t_j + \delta, 1\}),$$

is replaced by

$$G_j^{-1}(0) \cdot G_j(1) \quad \text{or} \quad G_j(1) \cdot G_j^{-1}(0).$$

$\triangle$

We recall that, for $U_1, \ldots, U_p \in X$ ($p \in \mathbb{N}$), we define

$$\prod_{i=1}^{p} U_i := U_1 \cdot U_2 \cdots U_p, \qquad \prod_{i=p}^{1} U_i := U_p \cdot U_{p-1} \cdots U_1.$$

For the above function H, we also use the conventional notation

$$(H(s))^0 := \mathcal{I}, \quad H^{-1}(s) := (H(s))^{-1} \quad \text{for all considered } s \in \mathbb{R}.$$

Actually, a closer examination of our process reveals that the pseudometric d can be defined "only" on the set

$$\{F_j(s_1) \cdots F_j(s_n) \cdot v;\ v \in R^m, s_1, \ldots, s_n \in [-1, 1]\}$$

and the set $\{F_j(t);\ t \in [-1, 1]\}$ can be countable for each $j \in \{1, 2\}$.

Remark 1.31. Now we comment our assumptions on R and X. Because of (1.30), the requirement about the existence of $\delta > 0$ (in (1.35)) can be omitted. Note that R does not need to be commutative. Thus, the set of all solutions of (1.27) is not generally a modulus over R with the scalar multiplication given by

$$r \begin{pmatrix} x_k^1 \\ \vdots \\ x_k^m \end{pmatrix} := \begin{pmatrix} r \odot x_k^1 \\ \vdots \\ r \odot x_k^m \end{pmatrix},$$

where $\{(x_k^1, \dots, x_k^m)^T\}$ is a solution of (1.27), $r \in R$, $k \in \mathbb{Z}$ ($k \in \mathbb{N}_0$). Indeed, the expression

$$P_k \cdot \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^m \end{pmatrix} = x_0^1 \cdot (P_k)_1 + \cdots + x_0^m \cdot (P_k)_m$$

does not need to hold for considered k and a solution $\{x_k\}$ of (1.27) (see (1.28)).

For the main requirements, consider two results concerning the existence, the uniqueness (and the uniform asymptotic stability) of an almost periodic solution of the almost periodic real (non-)homogeneous linear system (1.27) for $k \in \mathbb{Z}$ in [188] or directly the following simple example. Let $R := \mathbb{R}$, $m := 2$, and

$$X_1 := \left\{ \begin{pmatrix} 0 & 10^l \\ 10^{-l} & 0 \end{pmatrix}; \, l \in \mathbb{Z} \right\},$$
$$X_2 := \left\{ \begin{pmatrix} 0 & 10^l \\ 10^{-l} & 0 \end{pmatrix}; \, l \in \mathbb{Z} \right\} \cup \left\{ \begin{pmatrix} 10^l & 0 \\ 0 & 10^{-l} \end{pmatrix}; \, l \in \mathbb{Z} \right\}$$

with the usual metric on $\mathbb{R}$. For X_1, every $\mathfrak{S} \in \mathfrak{X}$ has all solutions almost periodic; at the same time, for X_2, it is easy to find a system from $\mathfrak{X}$ which has only one almost periodic solution—the trivial one. △

To prove the announced result, we need a sequence $\{a_k\}_{k \in \mathbb{N}_0}$ of real numbers, which has special properties (mentioned in the below given Lemmas 1.32–1.35). We define the sequence $\{a_k\}_{k \in \mathbb{N}_0}$ by the recurrent formula

$$a_0 := 1, \quad a_1 := 0, \qquad a_{2^n + k} := a_k - \frac{1}{2^n}, \quad k = 0, \dots, 2^n - 1, \ n \in \mathbb{N}. \tag{1.36}$$

For this sequence, we have the following auxiliary results.

Lemma 1.32. *The sequence $\{a_k\}$ is almost periodic.*

Proof. The lemma follows from Theorem 1.23, where it suffices to put $\varphi_k = a_k$ ($k \in \mathbb{N}$) and

$$\mathcal{X} = \mathbb{R}, \quad m = 0, \quad j = 1, \quad \varphi_0 = 1, \quad r_n = \frac{4}{2^n}, \ n \in \mathbb{N}.$$

□

Lemma 1.33. *The identity*

$$a_{2^{n+2} - 1 - i} = -a_{2^{n+1} + i} \tag{1.37}$$

holds for any $n \in \mathbb{N}_0$ and $i \in \{0, \dots, 2^n - 1\}$.

Before proving this lemma, observe that (1.37) is equivalent to

$$\sum_{k=2^{n+1}+i}^{2^{n+2}-1-i} a_k = 0, \quad n \in \mathbb{N}_0,\ i \in \{0, \dots, 2^n - 1\};$$

i.e., to

$$\sum_{k=0}^{2^{n+1}-1+i} a_k = \sum_{k=0}^{2^{n+2}-1-i} a_k, \quad n \in \mathbb{N}_0,\ i \in \{0, \dots, 2^n - 1\}.$$

Proof of Lemma 1.33. Obviously, (1.37) is true for $n \in \{0, 1\}$, because

$$a_2 = -a_3 = \frac{1}{2}, \quad a_4 = -a_7 = \frac{3}{4}, \quad a_5 = -a_6 = -\frac{1}{4};$$

i.e.,

$$\sum_{k=0}^{1} a_k = \sum_{k=0}^{3} a_k = \sum_{k=0}^{7} a_k = 1, \quad \sum_{k=0}^{4} a_k = \sum_{k=0}^{6} a_k = \frac{7}{4}.$$

Suppose that (1.37) is true also for $2, \dots, n-1$. We choose $i \in \{0, \dots, 2^n - 1\}$ arbitrarily. (We have $2^{n+2} - 1 - i \geq 2^{n+1} + 2^n$.) From (1.36) and the induction hypothesis, it follows that

$$a_{2^{n+2}-1-i} + a_{2^{n+1}+2^n+i} = -\frac{1}{2^n}, \quad a_{2^{n+1}+i} - a_{2^{n+1}+2^n+i} = \frac{1}{2^n}.$$

Summing up the above equalities, we get (1.37). □

Lemma 1.34. *We have*

$$\sum_{k=0}^{n} a_k \geq 1, \quad n \in \mathbb{N}_0. \tag{1.38}$$

Proof. Evidently, $a_0 = a_0 + a_1 = 1$. It means that (1.38) is true for $n = 0$ and $n = 1 = 2^1 - 1$. Let it be valid for arbitrarily given $2^p - 1$ and all $n < 2^p - 1$, i.e., let

$$\sum_{k=0}^{n} a_k \geq 1, \quad n \leq 2^p - 1,\ n \in \mathbb{N}_0.$$

Considering the definition of $\{a_k\}$, we obtain

$$\sum_{k=0}^{2^p+j-1} a_k = \sum_{k=0}^{2^p-1} a_k + \sum_{k=2^p}^{2^p+j-1} a_k \geq 1 + \sum_{k=0}^{j-1} a_k - j\frac{1}{2^p} \geq 1 + 1 - 1 = 1$$

for any $j \in \{1, \dots, 2^p\}$. Lemma 1.34 follows from the induction principle. □

Lemma 1.35. *We have*

$$\sum_{k=0}^{2^n-1} a_k = 1, \tag{1.39}$$

$$\sum_{k=0}^{2^{n+i}+2^n-1} a_k = 2 - \frac{1}{2^i}, \tag{1.40}$$

where $n \in \mathbb{N}_0$*,* $i \in \mathbb{N}$*.*

Proof. It is possible to prove this result by means of Lemma 1.33, but we prove it directly using (1.36) and the induction principle. We have

$$a_0 = 1, \quad a_0 + a_1 = 1, \quad a_0 + a_1 + a_2 + a_3 = 1.$$

If we assume that

$$\sum_{k=0}^{2^{n-1}-1} a_k = 1,$$

then we get (see (1.36))

$$\begin{aligned} \sum_{k=0}^{2^n-1} a_k &= \sum_{k=0}^{2^{n-1}-1} a_k + \sum_{k=2^{n-1}}^{2^n-1} a_k \\ &= \sum_{k=0}^{2^{n-1}-1} a_k + \sum_{k=0}^{2^{n-1}-1} \left(a_k - \frac{1}{2^{n-1}} \right) = 2 \sum_{k=0}^{2^{n-1}-1} a_k - 1 = 1. \end{aligned}$$

Therefore, (1.39) is proved. Analogously, applying (1.36) and (1.39), one can compute

$$\begin{aligned} \sum_{k=0}^{2^{n+i}+2^n-1} a_k &= \sum_{k=0}^{2^{n+i}-1} a_k + \sum_{k=2^{n+i}}^{2^{n+i}+2^n-1} a_k \\ &= 1 + \sum_{k=0}^{2^n-1} \left(a_k - \frac{1}{2^{n+i}} \right) = 1 + \left(1 - \frac{1}{2^i} \right) \end{aligned}$$

which gives (1.40). □

Applying matrix valued functions F_1, F_2, we obtain the next lemma.

Lemma 1.36. *For each $j \in \{1, 2\}$, any $n \in \mathbb{N}_0$, and each $i \in \{0, \dots, 2^n - 1\}$, it holds*

$$F_j(a_{2^{n+2}-1-i}) = F_j^{-1}(a_{2^{n+1}+i})$$

and, consequently,

$$\prod_{k=2^{n+1}+i}^{2^{n+2}-1-i} F_j(a_k) = \prod_{k=2^{n+2}-1-i}^{2^{n+1}+i} F_j(a_k) = \mathcal{I}.$$

Proof. Obviously, this lemma is a corollary of Lemma 1.33. Consider (1.34) and (1.35) with the fact that the multiplication of matrices is associative. □

From Lemma 1.35 (see (1.35)), we obtain the following formulas for F_2.

Lemma 1.37. *The equalities*

$$\prod_{k=0}^{2^n-1} F_2\left(a_k\right) = F_2(1), \qquad \prod_{k=0}^{2^{n+i}+2^n-1} F_2(a_k) = F_2\left(2 - \frac{1}{2^i}\right)$$

hold for all $n \in \mathbb{N}_0$ and $i \in \mathbb{N}$.

Now we can prove the main result of Chapter 1.

Theorem 1.38. *There exists a system of the form* (1.27) *which does not possess a non-zero almost periodic solution.*

Proof. First we suppose that the coefficients A_k belong to X so that there exists a function F_1 from (1.34). Using Theorem 1.6, we get the almost periodicity of the sequence $\{F_1(a_k)\}_{k\in\mathbb{N}_0}$, where $\{a_k\}$ is given by (1.36). We want to show that all non-zero solutions of the system $\mathfrak{S}_1 \in \mathfrak{X}$ determined by $\{F_1(a_k)\}$ are not almost periodic.

By contradiction, suppose that there exist $c_1, \dots, c_m \in R$ such that the vector valued sequence

$$\{f_k\} := \left\{ P_k \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \right\}, \quad k \in \mathbb{N}_0, \tag{1.41}$$

where $\{P_k\}_{k\in\mathbb{N}_0}$ is the principal fundamental matrix of $\mathfrak{S}_1$, is non-trivial and almost periodic; i.e., suppose that $\mathfrak{S}_1$ has a non-trivial almost periodic solution $\{f_k\}$. Since $\{f_k\}$ is almost periodic, $(c_1, \dots, c_m)^T$ is non-zero, and it holds

$$f_i = U_i \cdot F_1(1) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \quad \text{for any } i \in \mathbb{N} \text{ and some } U_i \in X, \tag{1.42}$$

we know that (see (1.31))

$$F_1(1) \text{ is regular for } c := (c_1, \dots, c_m)^T. \tag{1.43}$$

Considering (1.36), the uniform continuity of F_1 and the continuity of the multiplication of matrices (see (1.30)), Lemma 1.36, and (1.41), from the first part of Theorem 1.25 (see the proof of Corollary 1.26 and again Lemma 1.36), one can obtain that the sequence $\{g_k\}_{k\in\mathbb{Z}}$, where

$$g_k := f_k, \quad k \in \mathbb{N}_0; \qquad g_k := f_{-k}, \quad k \in \mathbb{Z} \setminus \mathbb{N}_0, \tag{1.44}$$

is almost periodic as well. Now we use Theorem 1.3 for $\{\varphi_k\} \equiv \{g_k\}$ and $\{h_n\}_{n\in\mathbb{N}} \equiv \{2^n\}_{n\in\mathbb{N}}$ (we can also consider directly $\{\varphi_k\} \equiv \{f_k\}$ and use Remark 1.2). Theorem 1.3 implies that, for any $\varepsilon > 0$, there exists an infinite set $N(\varepsilon) \subseteq \mathbb{N}$ such that the inequality

$$d\left(g_{k+2^{n_1}}, g_{k+2^{n_2}}\right) < \varepsilon, \quad k \in \mathbb{Z}, \tag{1.45}$$

holds for all $n_1, n_2 \in N(\varepsilon)$.

Using (1.34), we get $d\,(c, F_1(1)\,c) > 0$ and (consider (1.43))

$$\vartheta := d\left(F_1(1) \cdot c, F_1(1) \cdot F_1(1) \cdot c\right) > 0. \tag{1.46}$$

From Lemma 1.36 (for $i = 0$), (1.41), and (1.44) (see also (1.42)), we have

$$g_0 = c, \quad g_1 = F_1(1) \cdot c, \quad \dots, \quad g_{2^n} = F_1(1) \cdot c, \tag{1.47}$$

where $n \in \mathbb{N}$ is arbitrary. Hence, considering (1.36), it holds

$$d\left(g_{2^i+2^n}, F_1(1) \cdot F_1(1) \cdot c\right) \to 0 \quad \text{as} \quad n \to \infty \tag{1.48}$$

for every $i \in \mathbb{N}$, because F_1 is uniformly continuous and the multiplication of matrices is continuous. We also have

$$d\left(g_{2^{n_2}+2^{n_1}}, F_1(1) \cdot c\right) < \frac{\vartheta}{2} \tag{1.49}$$

for all $n_1, n_2 \in N(\vartheta/2)$. Indeed, put $k = 2^{n_2}$ in (1.45) and consider (1.47) for $n = n_2 + 1$. If we choose $n_1 \in N(\vartheta/2)$ and put $i = n_1$ in (1.48), then there exists $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, it holds

$$d\left(g_{2^{n_1}+2^n}, F_1(1) \cdot F_1(1) \cdot c\right) < \frac{\vartheta}{2}.$$

Thus, for arbitrarily given $n_2 \geq n_0$, $n_2 \in N(\vartheta/2)$, we obtain

$$d\left(g_{2^{n_2}+2^{n_1}}, F_1(1) \cdot F_1(1) \cdot c\right) < \frac{\vartheta}{2}. \tag{1.50}$$

Finally, applying (1.46), (1.49), and (1.50), we have

$$\vartheta \leq d\left(F_1(1) \cdot c, g_{2^{n_2}+2^{n_1}}\right) + d\left(g_{2^{n_2}+2^{n_1}}, F_1(1) \cdot F_1(1) \cdot c\right) < \vartheta.$$

This contradiction gives the proof when we consider (1.34) for $k \in \mathbb{N}_0$.

Let $k \in \mathbb{Z}$. Then, we can consider the system $\widetilde{\mathfrak{S}}_1$ determined by the sequence

$$B_k := F_1(a_k), \quad k \in \mathbb{N}_0; \qquad B_k := F_1(-a_{-k-1}), \quad k \in \mathbb{Z} \setminus \mathbb{N}_0. \tag{1.51}$$

Since the sequence $\{|\, a_k \,|\}_{k \in \mathbb{N}_0}$ is almost periodic (see Theorem 1.6) and has the form of $\{\varphi_k\}_{k \in \mathbb{N}_0}$ from Theorem 1.23 and since it is valid (see (1.37))

$$|\, a_{2^{n+2}-1-i} \,| = |\, a_{2^{n+1}+i} \,|, \quad n \in \mathbb{N}_0,\ i \in \{0, \dots, 2^n - 1\},$$

the fact that $\{B_k\}$ is almost periodic follows from the second part of Corollary 1.26, from Corollary 1.10, and Theorem 1.6. Next, the process is same as for $k \in \mathbb{N}_0$. Let $\{P_k\}_{k \in \mathbb{Z}}$ be the principal fundamental matrix of $\widetilde{\mathfrak{S}}_1$ and $g_k := f_k$, $k \in \mathbb{Z}$. Also now we have (1.45) and, consequently, we get the same contradiction.

Let the coefficients A_k belong to X so that there exists a function F_2 from (1.35). Consider the numbers $t_1, \dots, t_q \in (0, 1]$ and $\delta > 0$ from (1.35). Without loss of generality, we can assume that

$$\delta < t_1 < \cdots < t_q \quad \text{and} \quad t_q < 1 - \delta. \tag{1.52}$$

Indeed, if $t_j = 1$, then we can put $t_j := 1 - \delta/2$ and redefine δ. We repeat that any vector $v \in R^m$ determines some $j \in \{1, \dots, q\}$ (see again (1.35)) such that v is not an e_1-eigenvector of $F_2(t)$ for $t \in (t_j - \delta, t_j + \delta)$.

From Theorem 1.6 it follows that the sequence $\{F_2(a_k)\}_{k \in \mathbb{N}_0}$ is almost periodic. Thus, it determines a system of the form (1.27). We denote it as $\mathfrak{S}_2$. Suppose that system $\mathfrak{S}_2$ has a non-trivial almost periodic solution $\{x_k\}_{k \in \mathbb{N}_0}$. For the principal fundamental matrix $\{P_k\}$ of $\mathfrak{S}_2$, we have

$$x_k = P_k \cdot x_0, \quad k \in \mathbb{N}_0,$$

where the vector x_0 is non-zero. Using this fact and taking into account Lemma 1.34 and (1.35), we obtain

$$x_n = F_2(t) \cdot F_2^i(1) \cdot x_0 \quad \text{for some } i \in \mathbb{N},\ t \in [0,1), \tag{1.53}$$

and for arbitrary $n \in \mathbb{N}$. From Lemma 1.37, we also get

$$x_{2^n} = F_2(1) \cdot x_0 \quad \text{for all } n \in \mathbb{N}_0 \tag{1.54}$$

and

$$x_{2^{n+i}+2^n} = F_2\left(1 - \frac{1}{2^i}\right) \cdot F_2(1) \cdot x_0 \quad \text{for all } n \in \mathbb{N}_0,\ i \in \mathbb{N}. \tag{1.55}$$

Analogously as for $\{f_k\}$, one can extend $\{x_k\}_{k\in\mathbb{N}_0}$ by the formula

$$x_k := x_{-k}, \quad k \in \mathbb{Z} \setminus \mathbb{N}_0,$$

for all $k \in \mathbb{Z}$ so that the sequence $\{x_k\}_{k\in\mathbb{Z}}$ is almost periodic as well. Now we apply Theorem 1.3 for the sequences $\{x_k\}_{k\in\mathbb{Z}}$ and $\{2^n\}_{n\in\mathbb{N}}$. For any $\varepsilon > 0$, there exists an infinite set $M(\varepsilon) \subseteq \mathbb{N}$ such that, for any $n_1, n_2 \in M(\varepsilon)$, we have

$$d\left(x_{k+2^{n_1}}, x_{k+2^{n_2}}\right) < \varepsilon, \quad k \in \mathbb{Z}. \tag{1.56}$$

Since F_2 is uniformly continuous and the multiplication of matrices is continuous, for arbitrary $i \in \mathbb{N}$ and $\varepsilon > 0$, we have from (1.36) and (1.54) that

$$d\left(x_{2^i+2^n}, F_2(1) \cdot F_2(1) \cdot x_0\right) < \varepsilon \quad \text{for sufficiently large } n \in \mathbb{N}. \tag{1.57}$$

Because of the almost periodicity of $\{x_k\}$ and (1.53), the matrix $F_2(1)$ has to be regular for x_0. Let $\varepsilon > 0$ be arbitrarily small and $n_1 \in M(\varepsilon)$ arbitrarily large. From (1.56) and (1.57), where we choose $k = 2^{n_1-j}$ and $i = 2^{n_1-j}$ for $j \in \{0, \ldots, n_1\}$, it follows that, for given n_1, there exists sufficiently large $n_2 \in M(\varepsilon)$ for which

$$\begin{aligned} d\left(x_{2^{n_1-j}+2^{n_1}}, F_2(1) \cdot F_2(1) \cdot x_0\right) &\le d\left(x_{2^{n_1-j}+2^{n_1}}, x_{2^{n_1-j}+2^{n_2}}\right) \\ &\quad + d\left(x_{2^{n_1-j}+2^{n_2}}, F_2(1) \cdot F_2(1) \cdot x_0\right) < 2\varepsilon. \end{aligned} \tag{1.58}$$

Since ε (in (1.58)) is arbitrarily small, choosing $j = 0$, we get

$$d\left(x_{2^{n_1+1}}, F_2(1) \cdot F_2(1) \cdot x_0\right) = 0$$

which gives (see (1.54)) that $F_2(1)\, x_0$ is an e_1-eigenvector of $F_2(1)$, i.e., we have

$$d\left(F_2(1) \cdot x_0, F_2(1) \cdot F_2(1) \cdot x_0\right) = 0. \tag{1.59}$$

If we choose $j = 1$, then we obtain (consider (1.55))

$$d\left(F_2\left(\frac{1}{2}\right) \cdot F_2(1) \cdot x_0, F_2(1) \cdot F_2(1) \cdot x_0\right) = 0.$$

Analogously, for any j (the number n_1 is arbitrarily large), we get

$$d\left(F_2\left(1 - \frac{1}{2^j}\right) \cdot F_2(1) \cdot x_0, F_2(1) \cdot F_2(1) \cdot x_0\right) = 0.$$

Thus,

$$d\left(F_2\left(2-\frac{1}{2^j}\right)\cdot x_0, F_2(2)\cdot x_0\right)=0, \quad j\in\mathbb{N}, \tag{1.60}$$

and we have

$$d\left(F_2\left(2-\frac{1}{2^j}\right)\cdot x_0, F_2\left(2-\frac{1}{2^{j-1}}\right)\cdot x_0\right)=0, \quad j\in\mathbb{N}. \tag{1.61}$$

Because of

$$F_2\left(2-\frac{1}{2^j}\right)=F_2\left(\frac{1}{2^j}+2-\frac{1}{2^{j-1}}\right)=F_2\left(\frac{1}{2^j}\right)\cdot F_2\left(2-\frac{1}{2^{j-1}}\right)$$

and (see (1.59) and (1.60))

$$d\left(F_2\left(2-\frac{1}{2^{j-1}}\right)\cdot x_0, F_2(1)\cdot x_0\right)=0,$$

from (1.61) it follows

$$d\left(F_2\left(\frac{1}{2^j}\right)\cdot F_2(1)\cdot x_0, F_2(1)\cdot x_0\right)=0 \quad \text{for all } j\in\mathbb{N},$$

i.e., $F_2(1)\,x_0$ is an e_1-eigenvector of $F_2(2^{-j})$ for all $j\in\mathbb{N}$.

Since any number $t\in[0,1]$ can be expressed in the form

$$\sum_{i=1}^{\infty}\frac{a_i}{2^i}, \quad \text{where } a_i\in\{0,1\},$$

for considered $\delta>0$, there exists $n\in\mathbb{N}$ such that, for every $t\in[0,1]$, there exist $a_1,\dots,a_n\in\{0,1\}$ satisfying

$$\left| t-\sum_{i=1}^{n}\frac{a_i}{2^i}\right|<\delta.$$

Thus, $F_2(1)\,x_0$ is an e_1-eigenvector of $F_2(t_j+s_j)$ for some $|\,s_j\,|<\delta$ and any $j\in\{1,\dots,q\}$ which cannot be true. This contradiction shows that $\{x_k\}_{k\in\mathbb{N}_0}$ is not almost periodic.

If one considers the system $\widetilde{\mathfrak{S}}_2$ obtained from $\mathfrak{S}_2$ as in (1.51) (after replacing $\mathfrak{S}_1$ by $\mathfrak{S}_2$), then, similarly as for F_1 and $k\in\mathbb{N}_0$, one can prove that $\widetilde{\mathfrak{S}}_2\in\mathfrak{X}$ and that any its non-trivial solution $\{x_k\}_{k\in\mathbb{Z}}$ is not almost periodic. □

Remark 1.39. Let a non-zero $F_1(1)\,v\in R^m$ not be an e_1-eigenvector of matrix $F_1(1)$ from (1.34); i.e., the condition (1.34) be weakened in this way. Then, from the first part of the proof of Theorem 1.38, we obtain that the sequence $\{f_k\}$, given by (1.41), is not almost periodic for $(c_1,\dots,c_m)^T=v$. It means that there exists a system $\mathfrak{S}^1\in\mathfrak{X}$ with the principal fundamental matrix $\{P^1_k\}$ such that the sequence $\{P^1_k\,v\}_{k\in\mathbb{N}_0}$ or $\{P^1_k\,v\}_{k\in\mathbb{Z}}$ is not almost periodic.

Analogously, if one requires in (1.35) only that, for a non-zero vector $v\in R^m$, there exists $t\in(0,1]$ for which $F_2(1)\,v$ is not an e_1-eigenvector of $F_2(t)$, then there exists a system $\mathfrak{S}^2\in\mathfrak{X}$ satisfying that the sequence $\{P^2_k\,v\}_{k\in\mathbb{Z}}$ (or $\{P^2_k\,v\}_{k\in\mathbb{N}_0}$), where $\{P^2_k\}_{k\in\mathbb{Z}}$ (or $\{P^2_k\}_{k\in\mathbb{N}_0}$) is the principal fundamental matrix of $\mathfrak{S}^2$, is not almost periodic. △

The condition

$$F_2\left(\sum_{i=1}^{p} s_i\right) = \prod_{i=1}^{p} F_2(s_i), \quad s_1, \dots, s_p \in [-1,1],\ p \in \mathbb{N}, \tag{1.62}$$

in (1.35) is "strong". For example, from it follows that the multiplication of matrices from the set $\{F_2(t);\, t \in \mathbb{R}\}$ is commutative. We point out that, for many subsets of unitary or orthogonal matrices, it is not a limitation and that the method in the proof of Theorem 1.38 can be simplified in many cases. We show it in two important special cases.

Example 1.40. If for any non-trivial vector $v \in R^m$, there exists $\varepsilon(v) > 0$ with the property that

$$F_2(t) \cdot v \notin \mathcal{O}_{\varepsilon(v)}(v) \quad \text{for all } t \geq 1 \text{ (see (1.32))},$$

then the fact, that the systems $\mathfrak{S}_2$ and $\widetilde{\mathfrak{S}}_2$ from the proof of Theorem 1.38 do not have non-trivial almost periodic solutions, follows directly from Lemma 1.34 and (1.62). Indeed, the set $T(\{x_k\}, \varepsilon(x_0)) \setminus \{0\}$ is empty for any non-zero solution $\{x_k\}$. ◊

Example 1.41. Let function F_2, in addition to (1.35), satisfy

$$F_2(s) = F_2(0) = \mathcal{I} \tag{1.63}$$

for some positive irrational number s, (1.52) hold, and $p \in \mathbb{N}$ be arbitrary. Then, the system $\mathfrak{S}$ determined by the sequence $\{A_k\} := \{F_2(k/p)\}$, where $k \in \mathbb{N}_0$ or $k \in \mathbb{Z}$, has no non-trivial almost periodic solution.

The function $F_2(t/p)$, $t \in \mathbb{R}$, is continuous and periodic with period ps (see (1.62), (1.63)). Using the compactness of the interval $[0, ps]$, (1.62), and Theorem 1.3, we get that $\{F_2(k/p)\}_{k\in\mathbb{Z}}$ is almost periodic. Then, the almost periodicity of $\{F_2(k/p)\}_{k\in\mathbb{N}_0}$ is obvious.

Suppose, by contradiction, that $\{x_k\} \equiv \{P_k\, x_0\}$ is a non-trivial almost periodic solution of $\mathfrak{S}$. We mention that there exists $\delta > 0$ satisfying that, for any non-zero $v \in R^m$, one can find $j \in \mathbb{N}$ such that there exists a positive number $\vartheta(v)$ for which

$$\vartheta(v) \leq d\left(F_2\left(\frac{j}{p} + t\right) \cdot v, v\right), \quad t \in (-\delta, \delta), \tag{1.64}$$

because

$$\{F_2(k/p); k \in \mathbb{N}\} \text{ is dense in } \{F_2(t); t \in \mathbb{R}\}. \tag{1.65}$$

Evidently, (1.65) gives that

$$\{F_2(k/p); k \in N\} \text{ is dense in } \{F_2(t); t \in \mathbb{R}\} \tag{1.66}$$

for any set N which is relatively dense in $\mathbb{N}$.

Since the multiplication of matrices is continuous, there exists $\varepsilon > 0$ which satisfies that every vector u with the property $d(u, x_0) < \varepsilon$ determines the same j in (1.64) as x_0 and one can find

$$\vartheta(u) \geq \frac{\vartheta(x_0)}{2}. \tag{1.67}$$

From (1.62), we see that

$$x_k = F_2 \left(\sum_{i=0}^{k-1} \frac{i}{p} \right) \cdot x_0, \quad k \in \mathbb{N}. \tag{1.68}$$

Let l be an arbitrary positive $(\vartheta(x_0)/2)$-translation number of $\{x_k\}$. Thus, let

$$d(x_{k+l}, x_k) < \frac{\vartheta(x_0)}{2} \quad \text{for all } k \in \mathbb{N} \tag{1.69}$$

and let N be the set of all positive ε-translation numbers of $\{x_k\}$. Since

$$\sum_{i=0}^{k+l-1} \frac{i}{p} = \sum_{i=0}^{k-1} \frac{i}{p} + \frac{kl}{p} + \frac{l(l-1)}{2p}, \quad k \in \mathbb{N},$$

for all $k \in N$, we have (see again (1.62))

$$d(x_{k+l}, x_k) = d\left(F_2 \left(\frac{kl}{p} + \frac{l(l-1)}{2p} \right) \cdot F_2 \left(\sum_{i=0}^{k-1} \frac{i}{p} \right) \cdot x_0, F_2 \left(\sum_{i=0}^{k-1} \frac{i}{p} \right) \cdot x_0 \right). \tag{1.70}$$

From (1.66), where we replace $1/p$ by l/p, we get the choice of $k \in N$ such that

$$\left| \frac{j}{p} - \frac{kl}{p} - \frac{l(l-1)}{2p} \right| < \delta \pmod{s} \tag{1.71}$$

for j in (1.64) determined by x_0. From (1.64), (1.67) (consider the definition of ε), (1.68), (1.70), and (1.71), we have

$$d(x_{k+l}, x_k) \geq \frac{\vartheta(x_0)}{2}$$

for at least one $k \in \mathbb{N}$. But, at the same time, we have (1.69). This contradiction gives that $\{x_k\}$ cannot be almost periodic. See also the proof of the first part of [176, Proposition 2], where almost periodic unitary systems are studied. $\diamondsuit$

Chapter 2

Solutions of almost periodic difference systems

In this chapter, we study almost periodic solutions of the almost periodic homogeneous linear difference systems

$$x_{k+1} = A_k \cdot x_k. \tag{2.1}$$

Our aim is to analyse the systems (2.1) which have no non-trivial almost periodic solution. We are motivated by paper [176], where unitary systems (determined by unitary matrices A_k) are studied. One of the main results of [176] says that the systems whose solutions are not almost periodic form an everywhere dense subset in the space of all considered unitary systems. We remark that important partial cases of the theorem and the process are mentioned in [169] and [63, 104, 172], respectively.

In the proof of this result, it is substantially used that the group of considered matrices is not commutative. Thus, e.g., the dimension of the systems has to be at least two. We use methods based on our general constructions, because we want to generalize the result also for commutative groups of matrices (especially, for the scalar case). It implies that we can treat the problem in a general setting. Scalar sequences attain values in a metric space on an infinite field with continuous operations with respect to the metric similarly as scalar discrete processes in [45], where the main results are proved only for real or complex entries.

The almost periodicity of solutions of almost periodic linear difference equations is studied in [5] and [188] (non-homogeneous systems). We can also refer again to [45]. Explicit almost periodic solutions are obtained for a class of these equations in [83]. For other properties of (complex) almost periodic linear difference systems, see [15, 110, 148]. For difference systems of general forms, criteria of the existence of almost periodic solutions are presented in [26, 162, 186, 189]. The existence of an almost periodic sequence of solutions for an almost periodic difference equation is discussed in [93] (and [87] as in [189]). Concerning the existence theorems for almost periodic solutions of almost periodic delay difference systems, see [70] or [190] (methods and techniques from that paper are similarly used and developed in [191]).

This chapter is organized as follows. We begin with notations which are used throughout this chapter. Then we introduce general homogeneous linear difference systems and a metric in the space of all considered almost periodic systems. To formulate our results in a simple and consistent form, we introduce the concepts of transformable and weakly transformable groups of matrices. One of the conditions in the definition of transformable groups means that it is possible to transform any matrix into any other using finitely many arbitrarily small “jumps” in the complex case (for usual metrics on the group of considered matrices). We show that the group of all unitary matrices, the group of all orthogonal matrices of a dimension at least two with determinant 1, and some their subgroups are transformable. In addition, examples of weakly transformable groups (which are not transformable) are given.

In Section 2.3, we present a condition on weakly transformable groups ensuring that, in any neighbourhood of every considered system, there exists a system which does not possess an almost periodic solution other than the trivial one. Analogously, the corresponding Cauchy problem is explicitly solved as well.

2.1 Preliminaries

We use the following notations: $\mathbb{N}_0$ for the set of positive integers including the zero; $\mathbb{R}^+$ for the set of all positive reals; $\mathbb{R}_0^+$ for the set of all non-negative real numbers; and symbol i for the imaginary unit. Let $F = (F, \oplus, \odot)$ be an infinite field with a unit and a zero denoted as e_1 and e_0, respectively; and let $m \in \mathbb{N}$ be arbitrarily given. Henceforth, we consider m as the dimension of difference systems under consideration.

Symbol $\mathcal{M}at(F, m)$ denotes the set of all $m \times m$ matrices with elements from F and F^m the set of all $m \times 1$ vectors with entries attaining values from F. As usual, we define the identity matrix I and the zero matrix O. Analogously, for the trivial vector, we put $o := [e_0, e_0, \dots, e_0]^T \in F^m$. Since F is a field, we have the notion of the non-singular matrices from $\mathcal{M}at(F, m)$. For any invertible matrix U, we denote the inverse matrix as U^{-1}. For arbitrary $U_j, \dots, U_{j+n} \in \mathcal{M}at(F, m)$, $j \in \mathbb{Z}$, $n \in \mathbb{N}$, we define

$$\prod_{i=j}^{j+n} U_i := U_j \cdot U_{j+1} \cdots U_{j+n},$$

$$\prod_{i=j+n}^{j} U_i := U_{j+n} \cdot U_{j+n-1} \cdots U_j.$$

Let ϱ be a metric on F and assume that the operations $\oplus$ and $\odot$ are continuous with respect to ϱ and that the metric space (F, ϱ) is complete. The metric ϱ induces the metric in F^m and $\mathcal{M}at(F, m)$ as the sum of m and m^2 non-negative numbers given by ϱ in F, respectively. We also denote these metrics as ϱ. For any $\varepsilon > 0$ and α from a metric space, the ε-neighbourhood of α is denoted by $\mathcal{O}_\varepsilon^\varrho(\alpha)$. Note that the continuity of $\oplus$ and $\odot$ implies that the multiplication $\cdot$ of matrices from $\mathcal{M}at(F, m)$ (and $U \cdot v$, $U \in \mathcal{M}at(F, m)$, $v \in F^m$) is continuous. All considered sequences are defined for $k \in \mathbb{Z}$ (or $i, j \in \mathbb{N}$) and attain values in one of the metric spaces F, F^m, $\mathcal{M}at(F, m)$ (or $\mathbb{C}$).

2.2 General homogeneous linear difference systems

We consider m-dimensional homogeneous linear difference equations of the form

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z}, \tag{2.2}$$

where $\{A_k\}$ is an almost periodic sequence of non-singular matrices from a given infinite group $\mathcal{X} \subset \mathcal{M}at(F, m)$. We need the set of all considered A_k to form the group $\mathcal{X}$ which has the below given properties (see Definitions 2.1 and 2.14 below). The set of all these almost periodic systems is denoted by symbol $\mathcal{AP}\,(\mathcal{X})$.

We identify the sequence $\{A_k\}$ with the system (2.2) which is determined by $\{A_k\}$. In the space $\mathcal{AP}\,(\mathcal{X})$, we introduce the metric

$$\sigma\left(\{A_k\}, \{B_k\}\right) := \sup_{k \in \mathbb{Z}} \varrho\left(A_k, B_k\right), \quad \{A_k\}, \{B_k\} \in \mathcal{AP}\,(\mathcal{X}).$$

For $\varepsilon > 0$, symbol $\mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ denotes the ε-neighbourhood of $\{A_k\}$ in $\mathcal{AP}\,(\mathcal{X})$.

2.2.1 Transformable groups

In this subsection, we introduce the concept of transformable groups (cf. [141, Definition 2.1]) and give illustrative examples of such groups.

Definition 2.1. We say that a group $\mathcal{X} \subset \mathcal{M}at(F, m)$ is *transformable* if the following conditions are fulfilled.

(i) For any $L \in \mathbb{R}^+$ and $\varepsilon > 0$, there exists $p = p(L, \varepsilon) \in \mathbb{N}$ with the property that, for any $n \geq p$ ($n \in \mathbb{N}$) and any sequence $\{C_0, C_1, \dots, C_n\} \subset \mathcal{X}$, $L \leq \varrho\,(C_i, O)$, $i \in \{0, 1, \dots, n\}$, one can find a sequence $\{D_1, \dots, D_n\} \subset \mathcal{X}$ for which

$$D_i \in \mathcal{O}^{\varrho}_{\varepsilon}(C_i), \quad i \in \{1, \dots, n\}, \qquad D_n \cdots D_2 \cdot D_1 = C_0.$$

(ii) The multiplication of matrices is uniformly continuous on $\mathcal{X}$ and has the Lipschitz property on a neighbourhood of I in $\mathcal{X}$. Especially, for every $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that

$$C \cdot D,\, D \cdot C \in \mathcal{O}^{\varrho}_{\varepsilon}(C) \quad \text{if} \quad C \in \mathcal{X},\, D \in \mathcal{O}^{\varrho}_{\eta}(I) \cap \mathcal{X};$$

and there exist $\zeta > 0$ and $P \in \mathbb{R}^+$ such that

$$C \cdot D,\, D \cdot C \in \mathcal{O}^{\varrho}_{\varepsilon P}(C) \quad \text{if} \quad C \in \mathcal{O}^{\varrho}_{\zeta}(I) \cap \mathcal{X},\, D \in \mathcal{O}^{\varrho}_{\varepsilon}(I) \cap \mathcal{X},\, \varepsilon \in (0, \zeta).$$

(iii) For any $L \in \mathbb{R}^+$, there exists $Q = Q(L) \in \mathbb{R}^+$ with the property that, for every $\varepsilon > 0$ and $C, D \in \mathcal{X} \smallsetminus \mathcal{O}^{\varrho}_{L}(O)$ satisfying $C \in \mathcal{O}^{\varrho}_{\varepsilon}(D)$, it is valid that

$$C^{-1} \cdot D,\, D \cdot C^{-1} \in \mathcal{O}^{\varrho}_{\varepsilon Q}(I).$$

For simplicity, in the below mentioned examples, we consider only the complex or real case and we speak about the classical case. Henceforth, we use known results of matrix analysis which can be found, e.g., in [80, 82, 94, 106].

Let us consider

$$((F,\oplus,\odot),\varrho(\cdot,\cdot)) = ((\mathbb{C},+,\cdot),|\cdot-\cdot|) \quad \text{or} \quad ((F,\oplus,\odot),\varrho(\cdot,\cdot)) = ((\mathbb{R},+,\cdot),|\cdot-\cdot|)).$$

Evidently, the multiplication of matrices satisfies the Lipschitz condition on any set $\mathcal{O}^{\varrho}_K(O)$. For an arbitrary matrix norm (especially, for the l_1-norm) denoted by $||\cdot||$, we have

$$||\,A^{-1}-(A+E)^{-1}\,|| \leq \frac{||\,A^{-1}\cdot E\,||}{1-||\,A^{-1}\cdot E\,||}\,||\,A^{-1}\,|| \tag{2.3}$$

for any matrices A, E such that A is invertible and $||\,A^{-1}\,E\,|| < 1$. If we have a bounded group $\mathcal{X} \subset \mathcal{M}at(\mathbb{C},m)$, then from (2.3) it follows that the map $C \mapsto C^{-1}$, $C \in \mathcal{X}$, has the Lipschitz property as well. Hence, the condition (iii) is satisfied.

Thus, the conditions (ii) and (iii) are fulfilled for any bounded group $\mathcal{X} \subset \mathcal{M}at(\mathbb{C},m)$. Further, for any bounded group $\mathcal{X}$, there exists $\varepsilon > 0$ for which $\mathcal{X} \cap \mathcal{O}^{\varrho}_{\varepsilon}(O) = \emptyset$. At the same time, from the condition (i), we know that $\mathcal{X} \cap \mathcal{O}^{\varrho}_1(O) = \emptyset$ for any transformable group $\mathcal{X} \subset \mathcal{M}at(\mathbb{C},m)$. Indeed, it suffices to consider $C_0 = I$ and the constant sequence $\{C_1,\dots,C_n\}$ given by a matrix C such that $||\,C\,|| < 1$.

Let $\varepsilon > 0$, a bounded group $\mathcal{X} \subset \mathcal{M}at(\mathbb{C},m)$, and $C_0, C_1, \dots, C_n \in \mathcal{X}$ be arbitrarily given. The uniform continuity of the multiplication of matrices on $\mathcal{X}$ implies the existence of $\eta = \eta(\varepsilon) > 0$ such that $C\,D, D\,C \in \mathcal{O}^{\varrho}_{\varepsilon}(C)$ if $D \in \mathcal{O}^{\varrho}_{\eta}(I) \cap \mathcal{X}$, $C \in \mathcal{X}$. We define the maps H_1, H_2 on $\mathcal{X} \times \mathcal{X}$ by

$$H_1\left((C,D)\right) := C\cdot D\cdot C^{-1}, \quad H_2\left((C,D)\right) := C^{-1}\cdot D\cdot C.$$

Since H_1, H_2 satisfy the Lipschitz condition, there exists $R \in \mathbb{R}^+$ such that the ranges of $\{C\} \times \mathcal{O}^{\varrho}_{\eta/R}(I) \cap \mathcal{X}$ in both of H_1 and H_2 are subsets of $\mathcal{O}^{\varrho}_{\eta}(I)$ for all $C \in \mathcal{X}$.

If we replace

$$\prod_{i_1=n}^{1} F_{i_1}\cdot\prod_{i_2=1}^{n} C_{i_2} \quad \text{by} \quad \prod_{i=1}^{n} E_i\cdot C_i,$$

where

$$F_1 = H_1\left((I,E_1)\right), \qquad F_2 = H_1\left((E_1\cdot C_1, E_2)\right),$$

$$\dots \quad F_n = H_1\left((E_1\cdot C_1\cdots E_{n-1}\cdot C_{n-1}, E_n)\right),$$

we see that $F_i \in \mathcal{O}^{\varrho}_{\eta/R}(I) \cap \mathcal{X}$, $i \in \{1,\dots,n\}$, implies $E_i \in \mathcal{O}^{\varrho}_{\eta}(I)$, $i \in \{1,\dots,n\}$. Thus, from the existence of matrices $F_1,\dots,F_n \in \mathcal{O}^{\varrho}_{\eta/R}(I) \cap \mathcal{X}$ for which

$$\prod_{i_1=n}^{1} F_{i_1}\cdot\prod_{i_2=1}^{n} C_{i_2} = C_0,$$

it follows the existence of matrices $D_1,\dots,D_n \in \mathcal{X}$ satisfying

$$D_i \in \mathcal{O}^{\varrho}_{\varepsilon}(C_i), \quad i \in \{1,\dots,n\}, \qquad D_n\cdots D_2\cdot D_1 = C_0.$$

It means that a bounded group $\mathcal{X} \subset \mathcal{M}at(\mathbb{C}, m)$ is transformable if, for any sufficiently small $\varepsilon > 0$, there exists $p(\varepsilon) \in \mathbb{N}$ such that, for all $n \geq p(\varepsilon)$ $(n \in \mathbb{N})$, any matrix from $\mathcal{X}$ can be expressed as a product of n matrices from $\mathcal{O}^{\varrho}_{\varepsilon}(I) \cap \mathcal{X}$.

We point out that several processes in the proofs of the below given results can be simplified in the classical case. For example, one can use that, for any $\varepsilon > 0$, $K \in \mathbb{R}^+$, and $n \in \mathbb{N}$, there exists $\xi = \xi(\varepsilon, K, n) > 0$ for which

$$\varrho\left(M_1 \cdot M_2 \cdots M_n, O\right) < \varepsilon, \quad M_1, M_2, \ldots, M_n \in \mathcal{O}^{\varrho}_K(O),$$

and

$$\varrho\left(M_1 \cdots M_n \cdot u, o\right) < \varepsilon, \quad M_1, \ldots, M_n \in \mathcal{O}^{\varrho}_K(O),\ u \in \mathcal{O}^{\varrho}_K(o),$$

if we have $M_i \in \mathcal{O}^{\varrho}_{\xi}(O)$ for at least one $i \in \{1, \ldots, n\}$ and $u \in \mathcal{O}^{\varrho}_{\xi}(o)$, respectively.

Now we mention important examples of transformable groups.

Example 2.2. The group of all unitary matrices is transformable. Obviously, it suffices to show that, for every $\varepsilon > 0$, any unitary matrix can be obtained as the n-th power of some unitary matrix from the ε-neighbourhood of I for all sufficiently large $n \in \mathbb{N}$. To show this, let $\varepsilon > 0$, $n \in \mathbb{N}$, and a $m \times m$ unitary matrix U with eigenvalues $\exp\left(\mathrm{i}\lambda_1\right), \ldots, \exp\left(\mathrm{i}\lambda_m\right)$, where $\lambda_1, \ldots, \lambda_m \in [-\pi, \pi)$, be arbitrarily given. We have

$$U = W \cdot J \cdot W^* \quad \text{for some unitary matrix } W = W(U),$$

where $J = \operatorname{diag}\left(\exp\left(\mathrm{i}\lambda_1\right), \ldots, \exp\left(\mathrm{i}\lambda_m\right)\right)$ and W^* denotes the conjugate transpose of W. We find a unitary matrix V for which $V^n = U$. By

$$W^* \cdot V^n \cdot W = \left(W^* \cdot V \cdot W\right)^n = J,$$

we obtain

$$V = W \cdot \operatorname{diag}\left(\exp\left(\mathrm{i}\lambda_1/n\right), \ldots, \exp\left(\mathrm{i}\lambda_m/n\right)\right) \cdot W^*.$$

Since the multiplication of matrices is uniformly continuous on the set of all unitary matrices, it remains to consider sufficiently large $n \in \mathbb{N}$. ◊

Example 2.3. Let $m \geq 2$ and $F = \mathbb{R}$. In this example, we show that the group $SO(m)$ of $m \times m$ orthogonal matrices with determinant 1 is transformable. Analogously as for unitary matrices, it is enough to prove that any orthogonal matrix U for which $\det U = 1$ is products of $n \geq p(\varepsilon)$, $n \in \mathbb{N}$, orthogonal matrices from the ε-neighbourhood of I for arbitrary $\varepsilon > 0$ and some $p(\varepsilon) \in \mathbb{N}$. Indeed, it is seen that there exists a neighbourhood of I which contains only orthogonal matrices with determinant 1.

Let $m = 2$. Observe that a two-dimensional orthogonal matrix has the form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where $\alpha \in [-\pi, \pi)$, if and only if its determinant is 1. It can be easily computed that

$$\begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_2 & \cos\alpha_2 \end{pmatrix} = \begin{pmatrix} \cos\left(\alpha_1+\alpha_2\right) & -\sin\left(\alpha_1+\alpha_2\right) \\ \sin\left(\alpha_1+\alpha_2\right) & \cos\left(\alpha_1+\alpha_2\right) \end{pmatrix}$$

for $\alpha_1, \alpha_2 \in \mathbb{R}$ and that, consequently, any this matrix (for some $\alpha \in [-\pi, \pi)$) can be obtained as the n-th power of the orthogonal matrix of this type given by the argument α/n for all $n \in \mathbb{N}$.

Now we use the induction principle with respect to m. Assume that the statement is true for $m-1 \geq 2$ and prove it for m. Let U be an orthogonal $m \times m$ matrix which is not in any one of the forms

$$\begin{pmatrix} 1 & o^T \\ o & V \end{pmatrix}, \qquad \begin{pmatrix} V & o \\ o^T & 1 \end{pmatrix}, \tag{2.4}$$

where V is an orthogonal matrix of dimension $m-1$, $o \in F^{m-1}$, and suppose that U has the element on the position $(1, m)$ different from 0 (in the opposite case, we put $U_2 := U$ in the below given process). We multiply U from the left by an orthogonal matrix U_1 which is in the second form from (2.4) and satisfies that $U_2 := U_1 \cdot U$ has 0 on the position $(1, m)$. For U_2, we define an orthogonal matrix U_3 so that the m-th row of U_3 is the last column of U_2 and so that the first column and the first row of U_3 are zero except the number 1 on the position $(1, 1)$. Obviously, the product $U_4 := U_3 \cdot U_2$ is equal to a matrix which has the second form from (2.4). Summarizing, we get $U = U_1^T \cdot U_3^T \cdot U_4$. Thus, one can express any orthogonal matrix U as a product of at most three matrices of the forms given in (2.4). Further, the matrices of this product can be evidently chosen so that the determinant of all of them is 1 if the determinant of the given matrix is 1 as well. Now the induction hypothesis gives the validity of the above statement. $\diamondsuit$

Example 2.4. Let a unitary matrix S be given. Let $\mathcal{X}_S$ be the set of the unitary matrices which are simultaneously diagonalizable for the single similarity matrix S, i.e., let

$$\mathcal{X}_S = \left\{ S^{-1} \cdot \text{diag}\, (\exp(\mathrm{i}\lambda_1), \ldots, \exp(\mathrm{i}\lambda_m)) \cdot S;\ \lambda_1, \ldots, \lambda_m \in [-\pi, \pi) \right\}.$$

Obviously, $\mathcal{X}_S$ is a subgroup of the $m \times m$ unitary group (different from the group if $m \geq 2$). Since diagonalizable (normal) matrices are simultaneously (unitarily) diagonalizable if and only if they commute under multiplication, $\mathcal{X}_S$ is a commutative group. Analogously as in Example 2.2, one can show that $\mathcal{X}_S$ is transformable. Further, $\mathcal{X}_S$ is transformable also for arbitrary non-singular matrix S. Especially, a transformable set does not need to be a subgroup of the $m \times m$ unitary group. $\diamondsuit$

Example 2.5. Now we consider the set of the unitary matrices with the determinant in the form $\exp(\mathrm{i}r)$, $r \in \mathbb{Q}$ or $r \in \mathbb{Z}$. Evidently, these matrices form a group as well. Considering diagonalizations of unitary matrices and the uniform continuity of the multiplication of unitary matrices, we get that this group is dense in the group of all unitary matrices. Thus (see Example 2.2), it satisfies (i). Finally, it is transformable. In general, any dense subgroup of a transformable set is transformable as well. $\diamondsuit$

Example 2.6. Let a unitary matrix S be given. Analogously as in Examples 2.4 and 2.5, we can show that the group

$$\{S^* \cdot \text{diag}\, (\exp(\mathrm{i}\lambda_1), \ldots, \exp(\mathrm{i}\lambda_m)) \cdot S;\ \lambda_1, \ldots, \lambda_m \in \mathbb{Q}\}$$

is transformable. In general, the matrices with eigenvalues in the form $\exp(\mathrm{i}r)$, where $r \in \mathbb{Q}$ or $r \in \mathbb{Z}$, from a given commutative transformable subgroup of the $m \times m$ unitary group form a transformable group if it is infinite. Indeed, if complex matrices A, B commute and have eigenvalues $\lambda_1, \dots, \lambda_m$ and $\mu_1, \dots, \mu_m$, respectively, then the eigenvalues of $A\,B$ are $\lambda_1 \mu_{j_1}, \lambda_2 \mu_{j_2}, \dots, \lambda_m \mu_{j_m}$ for some permutation $j_1, \dots, j_m$ of the indices $1, \dots, m$. $\Diamond$

Now we consider a bounded group $\mathcal{X} \subset \mathcal{M}at\,(\mathbb{C}, m)$ or $\mathcal{X} \subset \mathcal{M}at\,(\mathbb{R}, m)$ satisfying $\mathcal{X} \cap \mathcal{O}^{\varrho}_{L}(O) = \emptyset$ for some $L > 0$ and $q(\delta) \in \mathbb{N}$ for any $\delta > 0$ such that, for all $C \in \mathcal{X}$ and $l \geq q(\delta)$, $l \in \mathbb{N}$, there exist matrices $C_1, \dots, C_l \in \mathcal{X}$ with the property that

$$C_1 \in \mathcal{O}^{\varrho}_{\delta}(I), \quad C_j \in \mathcal{O}^{\varrho}_{\delta}\left(C_{j+1}\right), \ j \in \{1, \dots, l-1\}, \quad C_l \in \mathcal{O}^{\varrho}_{\delta}(C). \tag{2.5}$$

This implies (see (iii))

$$C_j^{-1} \cdot C_{j+1} \in \mathcal{O}^{\varrho}_{\delta Q(L)}(I) \cap \mathcal{X}, \ j \in \{1, \dots, l-1\}, \quad C_l^{-1} \cdot C \in \mathcal{O}^{\varrho}_{\delta Q(L)}(I) \cap \mathcal{X}.$$

For $\delta = \varepsilon / Q(L) > 0$, we have

$$C_1 \in \mathcal{O}^{\varrho}_{\delta}(I) \subseteq \mathcal{O}^{\varrho}_{\varepsilon}(I), \quad C_j^{-1} \cdot C_{j+1} \in \mathcal{O}^{\varrho}_{\varepsilon}(I), \ j \in \{1, \dots, l-1\}, \quad C_l^{-1} \cdot C \in \mathcal{O}^{\varrho}_{\varepsilon}(I).$$

Finally, since

$$C_1 \cdot \left(C_1^{-1} \cdot C_2\right) \cdot \left(C_2^{-1} \cdot C_3\right) \cdots \left(C_{l-1}^{-1} \cdot C_l\right) \cdot \left(C_l^{-1} \cdot C\right) = C,$$

the above mentioned condition is fulfilled (if one puts $p(\varepsilon) = q(\varepsilon / Q(L)) + 1$). Therefore, the group $\mathcal{X}$ is transformable.

Example 2.7. Let us show that the special unitary group $SU(m)$ (the group of all $m \times m$ unitary matrices with determinant 1) is transformable for $m \geq 2$, applying the implication mentioned above (see (2.5)) for arbitrarily given $C \neq I$, $C \in SU(m)$.

There exists an unitary matrix U such that

$$C = U^* \cdot D \cdot U, \quad D = \operatorname{diag}\left(\lambda_1, \dots, \lambda_m\right),$$

where $\lambda_1 \cdots \lambda_m = 1$. Let $\varphi_1, \dots, \varphi_m \in [0, 2\pi]$ be such that $\lambda_j = \mathrm{e}^{\mathrm{i}\varphi_j}, j \in \{1, \dots, m\}$. Since

$$\prod_{j=1}^{m} \mathrm{e}^{\mathrm{i}\varphi_j} = \mathrm{e}^{\mathrm{i}\sum\limits_{j=1}^{m} \varphi_j} = 1,$$

we have $\varphi_1 + \cdots + \varphi_m \equiv 0 \pmod{2\pi}$, i.e., $\varphi_1 + \cdots + \varphi_m = 2k\pi$ for some $k \in \{1, \dots, m-1\}$. Evidently, for an arbitrarily given $\delta > 0$, there exists $\varepsilon > 0$ with the property that we can change φ_j, $j \in \{1, \dots, m\}$, into $\varphi_j(\varepsilon_j) := \varphi_j + \varepsilon_j$, where

$$\sum_{j=1}^{m} \varepsilon_j = 0, \quad \varepsilon_1, \dots, \varepsilon_k \in [0, \varepsilon), \ \varepsilon_{k+1}, \dots, \varepsilon_m \in (-\varepsilon, 0],$$

so that

$$\varrho\left(U^* \cdot D \cdot U, U^* \cdot \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_1(\varepsilon_1)}, \dots, \mathrm{e}^{\mathrm{i}\varphi_m(\varepsilon_m)}\right) \cdot U\right) < \delta. \tag{2.6}$$

Indeed, it suffices to consider that the multiplication of matrices is uniformly continuous on the $m \times m$ unitary group. We obtain the matrix

$$U^* \cdot \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_1(\varepsilon_1)}, \dots, \mathrm{e}^{\mathrm{i}\varphi_m(\varepsilon_m)}\right) \cdot U \in SU(m),$$

because $\varphi_1(\varepsilon_1) + \cdots + \varphi_m(\varepsilon_m) = 2k\pi$.

Let $n \in \mathbb{N}$ be such that $n > 2\pi/\varepsilon$. We put

$$\varepsilon_1^1 := \frac{1}{n}\left(2\pi - \varphi_1\right), \quad \dots \quad \varepsilon_k^1 := \frac{1}{n}\left(2\pi - \varphi_k\right), \ \varepsilon_{k+1}^1 := -\frac{1}{n}\,\varphi_{k+1}, \quad \dots \quad \varepsilon_m^1 := -\frac{1}{n}\,\varphi_m,$$

$$\varepsilon_1^2 := \frac{2}{n}\left(2\pi - \varphi_1\right), \quad \dots \quad \varepsilon_k^2 := \frac{2}{n}\left(2\pi - \varphi_k\right), \ \varepsilon_{k+1}^2 := -\frac{2}{n}\,\varphi_{k+1}, \quad \dots \quad \varepsilon_m^2 := -\frac{2}{n}\,\varphi_m,$$

$$\vdots$$

$$\varepsilon_1^{n-1} := \frac{n-1}{n}\left(2\pi - \varphi_1\right), \quad \dots \quad \varepsilon_k^{n-1} := \frac{n-1}{n}\left(2\pi - \varphi_k\right),$$

$$\varepsilon_{k+1}^{n-1} := -\frac{n-1}{n}\,\varphi_{k+1}, \quad \dots \quad \varepsilon_m^{n-1} := -\frac{n-1}{n}\,\varphi_m,$$

$$\varepsilon_1^n := 2\pi - \varphi_1, \quad \dots \quad \varepsilon_k^n := 2\pi - \varphi_k, \ \varepsilon_{k+1}^n := -\varphi_{k+1}, \quad \dots \quad \varepsilon_m^n := -\varphi_m.$$

Let us consider the matrices

$$C_1 := U^* \cdot \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_1\left(\varepsilon_1^1\right)}, \dots, \mathrm{e}^{\mathrm{i}\varphi_m\left(\varepsilon_m^1\right)}\right) \cdot U \in SU(m),$$

$$C_2 := U^* \cdot \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_1\left(\varepsilon_1^2\right)}, \dots, \mathrm{e}^{\mathrm{i}\varphi_m\left(\varepsilon_m^2\right)}\right) \cdot U \in SU(m),$$

$$\vdots$$

$$C_{n-1} := U^* \cdot \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_1\left(\varepsilon_1^{n-1}\right)}, \dots, \mathrm{e}^{\mathrm{i}\varphi_m\left(\varepsilon_m^{n-1}\right)}\right) \cdot U \in SU(m),$$

$$C_n := U^* \cdot \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}\varphi_1\left(\varepsilon_1^n\right)}, \dots, \mathrm{e}^{\mathrm{i}\varphi_m\left(\varepsilon_m^n\right)}\right) \cdot U = I.$$

We have (see (2.6))

$$C_1 \in \mathcal{O}_\delta^\varrho(C), \quad C_2 \in \mathcal{O}_\delta^\varrho\left(C_1\right), \quad \dots \quad C_{n-1} \in \mathcal{O}_\delta^\varrho\left(I\right),$$

i.e., we can choose $q(\delta) > 2\pi/\varepsilon$.

To show the transformability of $SU(m)$, it suffices to consider that the group $SU(m)$ is connected (path-connected) and compact (totally bounded) for each m (see, e.g., [76]). Nevertheless, using the above mentioned process, one can show the transformability of other matrix groups. For example, the group of the unitary matrices with determinant $\mathrm{e}^{\mathrm{i}r}$ for some $r \in \mathbb{Q}$ is transformable as well. Indeed, for any $\delta > 0$ and an arbitrary unitary matrix $C \neq I$ with determinant $\mathrm{e}^{\mathrm{i}r}$, where $r \in \mathbb{Q}$, there exists a matrix C_0 satisfying $C_0 \in \mathcal{O}_\delta^\varrho(C)$, $|\,C_0\,| = \mathrm{e}^{\mathrm{i}r(C)}$ for some $r(C) \in \mathbb{Q} \cap [0, 2\pi)$ and one can analogously find $n \in \mathbb{N}$ and unitary matrices $C_1, \dots, C_n$ such that

$$C_1 \in \mathcal{O}_\delta^\varrho(C_0), \quad C_2 \in \mathcal{O}_\delta^\varrho\left(C_1\right), \quad \dots \quad C_n \in \mathcal{O}_\delta^\varrho\left(C_{n-1}\right)$$

and that

$$|\,C_1\,| = \mathrm{e}^{\mathrm{i}r(C)\frac{n-1}{n}}, \quad |\,C_2\,| = \mathrm{e}^{\mathrm{i}r(C)\frac{n-2}{n}}, \quad \dots \quad |\,C_n\,| = 1.$$

$\diamondsuit$

Example 2.8. We consider Hermitian symplectic matrices of dimension $m = 2n$, $n \in \mathbb{N}$. At first, we recall that a complex matrix S is said to be symplectic provided

$$S^* \cdot J \cdot S = J, \quad \text{where} \quad J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}.$$

The set $Sp(n)$ of the Hermitian symplectic matrices is the intersection of the set of all $2n \times 2n$ symplectic matrices and the set of all $2n \times 2n$ unitary matrices. In fact (see [91]), the Hermitian symplectic matrices are the matrices of the form

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

with the property that

$$A^*A + B^*B = I, \qquad A^*B = B^*A.$$

It is known (see, e.g., [76]) that $Sp(n)$ is a compact and simply connected group. This fact implies (2.5), i.e., $Sp(n)$ is a transformable group. ♢

Example 2.9. Let $\mathcal{X}_j \subset \mathcal{M}at\,(\mathbb{F}, m_j)$ for $j \in \{1, \dots, n\}$ be transformable groups. The direct sum

$$\bigoplus_{j=1}^{n} \mathcal{X}_j = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_n,$$

where $M \in \bigoplus_{j=1}^{n} \mathcal{X}_j$ if it is of the form

$$M = \begin{pmatrix} M_1 & O & \cdots & O \\ O & M_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & M_n \end{pmatrix}, \quad M_1 \in \mathcal{X}_1, M_2 \in \mathcal{X}_2, \dots, M_n \in \mathcal{X}_n,$$

is a transformable group as well. Condition (i) is fulfilled, because we can choose the same $p(L, \varepsilon)$ for all $\mathcal{X}_j$; condition (ii) is obviously satisfied; to verify condition (iii), it suffices to take into account that $\mathcal{X}_j \cap \mathcal{O}_1^{\varrho}(O) = \emptyset$, $j \in \{1, \dots, n\}$.

The importance of the direct sum of transformable groups lies, i.a., in many isomorphisms of groups with applications in physics. For example, the spin group $Spin(4)$ (see, e.g., [178]) is isomorphic to $SU(2) \oplus SU(2)$, and H. Georgi and S. Glashow use the isomorphism of $SU(3) \oplus SU(2) \oplus U(1)$ to a subgroup of $SU(5)$ for the Georgi–Glashow model in [81]. ♢

Example 2.10. Now we show that the intersection of circulant matrices and unitary matrices form a transformable group. A complex matrix is called circulant if it has the form of

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_m \\ a_m & a_1 & a_2 & \cdots & a_{m-1} \\ a_{m-1} & a_m & a_1 & \cdots & a_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}.$$

A matrix is circulant if and only if it can be written in the form of $\sum_{j=1}^{m} a_j B^{j-1}$, where the permutation matrix $B = (b_{kl})$ is given by $b_{12} = b_{23} = \cdots = b_{(m-1)m} = b_{m1} = 1$. Because of $B^m = I$, the product of two circulant matrices is a circulant matrix. The multiplication of circulant matrices is commutative and the Hermitian adjoint of any circulant matrix is circulant. Therefore, every circulant matrix is normal. A normal matrix is unitary if and only if all its eigenvalues have the absolute value of 1. Hence, the set of the circulant matrices, whose all eigenvalues have absolute value 1, is a group. We denote this group by $CU(m)$.

It is known (see, e.g., [3]) that all circulant matrices have the same eigenvectors. Thus, there exists a unitary matrix U with the property that any circulant matrix can be expressed as the product $U^* D\, U$, where D is a diagonal matrix. It means that circulant matrices are simultaneously diagonalizable for the single similarity matrix U. (In fact, normal matrices are simultaneously diagonalizable if and only if they commute.) Furthermore, for every diagonal matrix D, the matrix $U^* D\, U$ is circulant. Altogether, we have that $A \in CU(m)$ if and only if $A = U^* D\, U$ for some

$$D = \operatorname{diag}(d_1, d_2, \dots, d_m), \quad |\, d_j \,| = 1,\ j \in \{1, \dots, m\}.$$

The fact that $CU(m)$ is transformable comes from Example 2.4. Indeed, $CU(m) = \mathcal{X}_U$.

Evidently, $CU(1) = U(1)$. The group $CU(2)$ is just formed by symmetric unitary matrices given by complex numbers $a = a_1 + a_2\mathrm{i}, b = b_1 + b_2\mathrm{i}$ $(a_1, a_2, b_1, b_2 \in \mathbb{R})$ in the first row for which $|a|^2 + |b|^2 = 1$, $a_1 b_1 = -a_2 b_2$. Obviously, for any $a \in \mathbb{C}$ satisfying $|a| \le 1$, one can find b so that the above equalities are fulfilled. It is seen that b can be chosen, in addition, so that the function $a \mapsto b$ is continuous. Directly from this observation, we get condition (i). We remark that, for $m > 2$, the symmetric unitary matrices do not form a group, because the product of symmetric matrices is a symmetric matrix if and only if they commute under multiplication. $\Diamond$

2.2.2 Strongly and weakly transformable groups

In this subsection, we define modifications of transformable groups—strongly and weakly transformable groups. Before introducing strongly transformable groups, we mention an auxiliary result.

Lemma 2.11. *If $\mathcal{X}$ is transformable, then, for any $\{L_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ and $j \ge 2$, $j \in \mathbb{N}$, one can find $\{\varepsilon_i\}_{i\in\mathbb{N}} \equiv \{\varepsilon_i(\{L_i\}, j)\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ satisfying*

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty,$$

$$j^i \ge p\left(L_{g(i)}, \varepsilon_i\right) \quad \text{for infinitely many } i \in \mathbb{N},\ \text{some } g(i) \in \mathbb{N}, \tag{2.7}$$

where $g(i) \to \infty$ as $i \to \infty$.

Proof. The lemma follows directly from (i). Indeed, one can put

$$\varepsilon_i := 2^{-k}, \quad i = f(k) \text{ for some } k \in \mathbb{N};$$
$$\varepsilon_i := 2^{-i}, \quad i \notin \{f(k);\, k \in \mathbb{N}\},\ i \in \mathbb{N}$$

for arbitrarily given increasing discrete function $f : \mathbb{N} \to \mathbb{N}$ with the property that

$$j^{f(k)} \geq p\left(L_k, 2^{-k}\right), \quad k \in \mathbb{N},$$

whose inverse function is considered in (2.7) as g. □

Of course, inequality (2.7) does not need to be true for all $i \in \mathbb{N}$ or for a set of i which is relatively dense in $\mathbb{N}$. This fact motivates the following definition.

Definition 2.12. A group $\mathcal{X}$ is *strongly transformable* if it is transformable and if for any $L \in \mathbb{R}^+$, there exist $j = j(L) \in \mathbb{N}$ and a sequence $\{\varepsilon_i\}_{i\in\mathbb{N}} \equiv \{\varepsilon_i(L)\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ such that

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty, \tag{2.8}$$

$$j^i \geq p\left(L, \varepsilon_i\right) \quad \text{for all } i \in \mathbb{N}. \tag{2.9}$$

Example 2.13. Since we consider only maps which satisfy the Lipschitz condition in the above examples (in the classical case) and since we can choose

$$p\left(L, \varepsilon, m+1\right) \leq 3p\left(L, \varepsilon, m\right)$$

in (i) when we use the induction principle with respect to m (see Example 2.3), all concrete transformable groups of matrices mentioned in Examples 2.2–2.10 are actually strongly transformable. ◊

Example 2.13 shows that several transformable groups used in applications are strongly transformable. Nevertheless, if we change the metric in the examples above in a neighbourhood of -1 (in $\mathbb{R}$ or $\mathbb{C}$) so that the l^{-l}-neighbourhood of -1 becomes the l^{-1}-neighbourhood of -1 for all sufficiently large $l \in \mathbb{N}$ (and the rest remains unchanged), then all mentioned groups are still transformable and none of them is strongly transformable.

Since certain matrix groups (important in applications) are not transformable, although they possess transformable subgroups, we introduce the following generalization of the transformability.

Definition 2.14. A matrix group $\mathcal{X} \subset \mathcal{M}at\left(F, m\right)$ is *weakly transformable* if there exist a transformable group $\mathcal{X}_0 \subseteq \mathcal{X}$, matrices $X_1, \ldots, X_l \in \mathcal{X}$, and $\delta_{\mathcal{X}} > 0$ such that

(I) any $U \in \mathcal{X}$ can be expressed as $U = C(U) \cdot X_j$ for some $C(U) \in \mathcal{X}_0$, $j \in \{1, \ldots, l\}$,

and that

(II) $\varrho\left(C \cdot X_i, D \cdot X_j\right) > \delta_{\mathcal{X}}$ for all $C, D \in \mathcal{X}_0$, $i \neq j$, $i, j \in \{1, \ldots, l\}$.

Again, we give important examples of the considered type of matrix groups in the complex and real case.

Example 2.15. We use the fact that the group $SO(m)$ of all real orthogonal matrices of a dimension $m \geq 2$ with determinant 1 is transformable to prove that the group $O(m)$ of all real orthogonal matrices is weakly transformable. We put

$$\mathcal{X}_0 := SO(m), \quad X_1 := I, \quad X_2 := \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $U \in O(m) \smallsetminus SO(m)$, then $U = C(U)\, X_2$, where $C(U) = U\, X_2^{-1} \in SO(m)$. Indeed, $|\, U\,| = -1$, $|\, X_2^{-1}\,| = -1$. This gives us condition (I). The mapping $C \mapsto |\, C\,|$ is continuous on $O(m)$. Thus, there exists $\delta > 0$ such that

$$\varrho\,(C_1, C_2) > \delta \quad \text{if} \quad C_1, C_2 \in O(m),\ |\, C_1\,| = 1,\ |\, C_2\,| = -1,$$

i.e., $\varrho\,(C\,X_1, D\,X_2) > \delta$ for any $C, D \in \mathcal{X}_0$ (consider $|\, C\,X_1\,| = 1$, $|\, D\,X_2\,| = -1$). ◊

Example 2.16. Let $\mathcal{X}_1$ be a weakly transformable group and let $\mathcal{X}_2$ be an arbitrary finite matrix group. We can directly see that the direct sum $\mathcal{X}_1 \oplus \mathcal{X}_2$ is weakly transformable. Thus, the group $O(2) \oplus O_k(2)$, where

$$O_k(2) = \{I, T, \dots, T^{k-1}, S, ST, \dots, ST^{k-1}\}$$

and

$$T := \begin{pmatrix} \cos\frac{2\pi}{k} & \sin\frac{2\pi}{k} \\ -\sin\frac{2\pi}{k} & \cos\frac{2\pi}{k} \end{pmatrix}, \quad S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

is weakly transformable for all $k \in \mathbb{N}$. For the representation of this group, see [76]; for collections of finite groups, we refer to [16, 43]. ◊

Example 2.17. Evidently, any direct sum $\mathcal{X}_1 \oplus \mathcal{X}_2$ is weakly transformable if both $\mathcal{X}_1$ and $\mathcal{X}_2$ are weakly transformable. Especially, the group $O(m) \oplus O(m)$ is weakly transformable for $m \geq 2$. This group is isomorphic to the perplectic orthogonal group which is denoted by $PO(2m)$ and defined as

$$PO(2m) = \{P \in O(2m);\ R \cdot P = P \cdot R\},$$

where

$$R := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & .\cdot^{\cdot} & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that $PO(2m)$ is exactly the set of all centro-symmetric orthogonal matrices.

Let us show that the group $PO(3)$ is weakly transformable. In [58], there is shown that $PO(3)$ consists of $\{W(\varphi); \varphi \in [0, 2\pi]\}$ and the components

$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot W(\varphi); \varphi \in [0, 2\pi] \right\}, \qquad \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot W(\varphi); \varphi \in [0, 2\pi] \right\},$$

$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot W(\varphi); \varphi \in [0, 2\pi] \right\},$$

where

$$W(\varphi) := \frac{1}{2} \begin{pmatrix} \cos\varphi + 1 & \sqrt{2}\sin\varphi & \cos\varphi - 1 \\ -\sqrt{2}\sin\varphi & 2\cos\varphi & -\sqrt{2}\sin\varphi \\ \cos\varphi - 1 & \sqrt{2}\sin\varphi & \cos\varphi + 1 \end{pmatrix}.$$

Since $(W(\varphi))^{-1} = W(-\varphi)$, $\varphi \in [0, 2\pi]$, we can express $PO(3)$ by the following components

$$\begin{aligned}
PO_1(3) &= \{W(\varphi); \varphi \in [0, 2\pi]\}, \\
PO_2(3) &= \left\{ W(\varphi) \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}; \varphi \in [0, 2\pi] \right\}, \\
PO_3(3) &= \left\{ W(\varphi) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}; \varphi \in [0, 2\pi] \right\}, \\
PO_4(3) &= \left\{ W(\varphi) \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}; \varphi \in [0, 2\pi] \right\}.
\end{aligned}$$

We put (see Definition 2.14)

$$\mathcal{X}_0 := PO_1(3), \quad X_1 := I, \quad X_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = X_2^{-1},$$

$$X_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = X_3^{-1}, \quad X_4 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = X_4^{-1}.$$

The transformability of $PO_1(3)$ follows from the fact that $PO_1(3)$ is a connected matrix subgroup of $O(3)$ (see [58]). Let us explain it in detail. For any $\varphi \in [0, 2\pi]$ and sufficiently large $n \in \mathbb{N}$, we consider the sequence

$$\frac{\varphi}{n}, \quad 2\frac{\varphi}{n}, \quad \dots \quad (n-1)\frac{\varphi}{n}, \quad \varphi.$$

Obviously, for every $\delta > 0$, there exists $q \in \mathbb{N}$ satisfying

$$W\left(\frac{\varphi}{l}\right) \in \mathcal{O}^{\varrho}_{\delta}(I), \quad W\left(2\frac{\varphi}{l}\right) \in \mathcal{O}^{\varrho}_{\delta}\left(W\left(\frac{\varphi}{l}\right)\right), \quad \dots$$

$$W\left((l-1)\frac{\varphi}{l}\right) \in \mathcal{O}^{\varrho}_{\delta}\left(W\left((l-2)\frac{\varphi}{l}\right)\right), \quad W\left((l-1)\frac{\varphi}{l}\right) \in \mathcal{O}^{\varrho}_{\delta}\left(W\left(\varphi\right)\right)$$

for all $\varphi \in [0, 2\pi]$ and $l \geq q$. Now it suffices to realize (2.5).

It remains to show that the components have a positive distance. Let $\varphi, \psi \in [0, 2\pi]$ and $W_j(\cdot) \in PO_j(3)$, $j \in \{1, 2, 3, 4\}$. Since the entries of these matrices are continuous functions defined on the compact set $[0, 2\pi]$, the sets $PO_j(3)$, $j \in \{1, 2, 3, 4\}$, are also compact. Therefore, it suffices to show that these components are disjoint.

(1 & 2) Suppose that there exist $\varphi, \psi \in [0, 2\pi]$ such that $W_1(\varphi) = W_2(\psi)$, i.e.,

$$\begin{pmatrix} \cos\varphi + 1 & \sqrt{2}\sin\varphi & \cos\varphi - 1 \\ -\sqrt{2}\sin\varphi & 2\cos\varphi & -\sqrt{2}\sin\varphi \\ \cos\varphi - 1 & \sqrt{2}\sin\varphi & \cos\varphi + 1 \end{pmatrix} = \begin{pmatrix} \cos\psi - 1 & -\sqrt{2}\sin\psi & \cos\psi + 1 \\ -\sqrt{2}\sin\psi & -2\cos\psi & -\sqrt{2}\sin\psi \\ \cos\psi + 1 & -\sqrt{2}\sin\psi & \cos\psi - 1 \end{pmatrix}.$$

Considering the entries in the first row and the first column, we have $\varphi = \pi$ and $\psi \in \{0, 2\pi\}$. On the other hand, from entries in the first row and the third column, we obtain $\varphi \in \{0, 2\pi\}$ and $\psi = \pi$. This gives a contradiction.

(1 & 3) Again by contradiction, we suppose that there exist $\varphi, \psi \in [0, 2\pi]$ such that $W_1(\varphi) = W_3(\psi)$, i.e.,

$$\begin{pmatrix} \cos\varphi + 1 & \sqrt{2}\sin\varphi & \cos\varphi - 1 \\ -\sqrt{2}\sin\varphi & 2\cos\varphi & -\sqrt{2}\sin\varphi \\ \cos\varphi - 1 & \sqrt{2}\sin\varphi & \cos\varphi + 1 \end{pmatrix} = \begin{pmatrix} \cos\psi + 1 & -\sqrt{2}\sin\psi & \cos\psi - 1 \\ -\sqrt{2}\sin\psi & -2\cos\psi & -\sqrt{2}\sin\psi \\ \cos\psi - 1 & -\sqrt{2}\sin\psi & \cos\psi + 1 \end{pmatrix}.$$

At first, we pay our attention to the entries on the positions $(1, 2)$ and $(2, 1)$. We obtain

$$\sin\varphi = -\sin\psi = -\sin\varphi, \quad \text{i.e.,} \quad \varphi, \psi \in \{0, \pi, 2\pi\}.$$

Next, the entries $(3, 1)$ and $(2, 2)$ give

$$\cos\varphi = \cos\psi = -\cos\varphi, \quad \text{i.e.,} \quad \varphi, \psi \in \{\pi/2, 3\pi/2\}.$$

Hence, the components $PO_1(3)$ and $PO_3(3)$ are disjoint as well.

Finally, the cases (1 & 4), (2 & 3), and (3 & 4) are analogous to (1 & 2) and (2 & 4) is analogous to (1 & 3). ♢

2.3 Systems without almost periodic solutions

Now we can consider the gist of this chapter. At first, we prove an auxiliary result and a result concerning strongly transformable groups which is generalized later in this work.

Lemma 2.18. *If an almost periodic sequence of non-singular $A_k \in \mathcal{M}at(F, m)$ is such that, for any $\varepsilon > 0$, there exists $i = i(\varepsilon) \in \mathbb{Z}$ for which $\varrho(O, A_i) < \varepsilon$, then the system $x_{k+1} = A_k\, x_k$, $k \in \mathbb{Z}$, does not have a non-trivial almost periodic solution.*

Proof. By contradiction, suppose that we have an almost periodic sequence $\{A_k\}$, a sequence $\{h_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}$ satisfying

$$\varrho(O, A_{h_i}) < \frac{1}{i}, \quad i \in \mathbb{N}, \tag{2.10}$$

and a non-trivial almost periodic solution $\{x_k\}$ of the system $x_{k+1} = A_k\, x_k$. Using Corollary 1.5, we obtain uniformly convergent common subsequences

$$\{\{A_{k+\tilde{h}_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}} \quad \text{and} \quad \{\{x_{k+\tilde{h}_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$$

of the sequences $\{\{A_{k+h_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$ and $\{\{x_{k+h_i}\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}}$. The limits are denoted as $\{B_k\}$ and $\{y_k\}$.

We put $\varepsilon := \varrho\,(x_0, o)\,/2 > 0$. Because of the almost periodicity of $\{x_k\}$, there exists some $p\,(\varepsilon)$ from Definition 1.1. We consider the sets $N_i := \{i+1, i+2, \ldots, i+p\,(\varepsilon)\}$ for $i \in \mathbb{Z}$. Any one of the sets N_i contains a number $l \in T(\{x_k\}, \varepsilon)$. Thus,

$$x_l \notin \mathcal{O}^{\varrho}_{\varepsilon}(o). \tag{2.11}$$

From (2.10) it follows that $B_0 = O$. Since the multiplication of matrices is continuous, one can find $\vartheta > 0$ for which

$$C_j \cdots C_0 \cdot y \in \mathcal{O}^{\varrho}_{\varepsilon}(o), \quad j = 0, 1, \ldots, p\,(\varepsilon) - 1,$$

if $y \in \mathcal{O}^{\varrho}_{\vartheta}(y_0)$ and $C_i \in \mathcal{O}^{\varrho}_{\vartheta}(B_i)$, $i \in \{0, \ldots, j\}$. There exists $i \in \mathbb{N}$ such that

$$\varrho\left(A_{k+\tilde{h}_i}, B_k\right) < \vartheta, \quad \varrho\left(x_{k+\tilde{h}_i}, y_k\right) < \vartheta, \quad k \in \mathbb{Z}.$$

Therefore,

$$x_{j+\tilde{h}_i} \in \mathcal{O}^{\varrho}_{\varepsilon}(o), \quad j = 1, \ldots, p\,(\varepsilon)\,. \tag{2.12}$$

Indeed, it is valid

$$x_{j+\tilde{h}_i} = A_{j+\tilde{h}_i-1} \cdots A_{\tilde{h}_i} \cdot x_{\tilde{h}_i}, \quad j = 1, \ldots, p\,(\varepsilon)\,.$$

This contradiction (compare (2.11) with (2.12)) gives the proof. □

Theorem 2.19. *Let $\mathcal{X}$ be strongly transformable. Let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrarily given. If there exist $L \in \mathbb{R}^+$ and $\{M_i\}_{i\in\mathbb{N}}$ such that*

$$M_i,\ M_i^{-1} \in \mathcal{X} \setminus \mathcal{O}^{\varrho}_{L}(O), \quad i \in \mathbb{N}, \tag{2.13}$$

and that, for any non-zero vector $u \in F^m$, one can find $i = i(u) \in \mathbb{N}$ with the property that $M_i\, u \neq u$, then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ which does not possess a non-trivial almost periodic solution.

Proof. If $\{A_k\}$ has a non-trivial almost periodic solution, then there exists $K \in \mathbb{R}^+$ such that $\varrho\,(A_k, O) > K$ for all k. Indeed, it follows from Lemma 2.18. Since it suffices to consider only very small $\varepsilon > 0$, we can assume without loss of generality that

$$L + \varepsilon < \varrho\,(I, O)\,, \qquad L + \varepsilon < \varrho\,(A_k, O)\,, \quad k \in \mathbb{Z}; \tag{2.14}$$

otherwise we can put $B_k := A_k$, $k \in \mathbb{Z}$.

Let $\eta = \eta(\varepsilon/2)$, ζ, P and $Q = Q(L)$ be from (ii) and (iii), respectively. Further, let $\eta < \varepsilon < \zeta$ and let $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$, $n \in \mathbb{N}$, and $j \geq 2$ ($j \in \mathbb{N}$) satisfy

$$\sum_{i=1}^{\infty} \varepsilon_i < \frac{\eta}{PQ}, \tag{2.15}$$

$$j^i(n+1) \geq p\,(L, \varepsilon_i) \quad \text{for all } i \in \mathbb{N}. \tag{2.16}$$

The inequality (2.15) follows from (2.8) if we omit finitely many values of ε_i, and (2.16) from the fact that n and j can be arbitrarily large and from (2.9). We remark that $P, Q \geq 1$.

We put

$$B_k := A_k, \quad C_k := I \quad \text{for } k \in \{0, 1, \dots, n\}$$

and we choose

$$B_k = A_k \cdot C_k \quad \text{for some } C_k \in \mathcal{O}^{\varrho}_{\varepsilon_2 Q}\left(C_{k-(n+1)}\right) \cap \mathcal{X},\ k \in \{n+1, \dots, 2(n+1)-1\},$$

$$\vdots$$

$$B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{\varepsilon_2 Q}\left(C_{k-j^4(n+1)}\right) \cap \mathcal{X},\ k \in \{j^4(n+1), \dots, (j^4+1)(n+1)-1\},$$

arbitrarily such that

$$\prod_{k=(j^2+1)(n+1)-1}^{n+1} B_k = M_1, \qquad \prod_{k=(j^3+j^2)(n+1)-1}^{n+1} B_k = \prod_{k=(j^4+1)(n+1)-1}^{n+1} B_k = I.$$

For

$$C_1, \dots, C_n, D_1, \dots, D_n \in \mathcal{X}, \quad D_i \in \mathcal{O}^{\varrho}_{\vartheta}(C_i),$$

where $\vartheta > 0$, $L + \vartheta \leq \varrho\,(C_i, O)$, $i \in \{1, \dots, n\}$, and $n \geq p\,(L, \vartheta)$, we can express

$$D_n \cdots D_2 \cdot D_1 = C_n \cdot \left(C_n^{-1} \cdot D_n\right) \cdots C_1 \cdot \left(C_1^{-1} \cdot D_1\right),$$

where

$$(C_i^{-1} \cdot D_i) \in \mathcal{O}^{\varrho}_{\vartheta Q}\,(I)\,, \quad i \in \{1, \dots, n\}.$$

Using this fact and considering (2.13), (2.14), and (2.16), we get the existence of the above matrices C_k.

In the second step, we put

$$B_k := A_k \cdot C_{k+(j^4+1)(n+1)}, \quad k \in \{-(j^4+1)(n+1), \dots, -1\},$$

$$\vdots$$

$$B_k := A_k \cdot C_{k+j^4(j^4+1)(n+1)}, \quad k \in \{-j^4(j^4+1)(n+1), \dots, -(j^4-1)(j^4+1)(n+1)-1\},$$

and we denote

$$C_k := A_k^{-1} \cdot B_k, \quad k \in \{-j^4(j^4+1)(n+1), \dots, -1\}.$$

Now we choose

$$B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 PQ}\left(C_{k-(j^4+1)^2(n+1)}\right) \cap \mathcal{X},$$
$$k \in \{(j^4+1)(n+1), \dots, (j^4+1)(n+1) + (j^4+1)^2(n+1) - 1\},$$

$$\vdots$$

$$B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 PQ}\left(C_{k-j^4(j^4+1)^2(n+1)}\right) \cap \mathcal{X},$$
$$k \in \{(j^4+1+(j^4-1)(j^4+1)^2)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1)-1\},$$

arbitrarily such that

$$\prod_{k=j^7(n+1)-1}^{(j^4+1)(n+1)} B_k = M_1, \qquad \prod_{k=(j^7+j^6-j^0)(n+1)-1}^{(j^4+1)(n+1)} B_k = I,$$

$$\prod_{k=j^8\,(n+1)-1}^{(j^4+1)(n+1)} B_k = M_1, \qquad \prod_{k=(j^8+j^6-j^3)(n+1)-1}^{(j^4+1)(n+1)} B_k = I,$$

$$\prod_{k=j^9\,(n+1)-1}^{(j^4+1)(n+1)} B_k = M_2, \qquad \prod_{k=(j^9+j^6-j^0)(n+1)-1}^{(j^4+1)(n+1)} B_k = I,$$

$$\prod_{k=j^{10}(n+1)-1}^{(j^4+1)(n+1)} B_k = M_2, \qquad \prod_{k=(j^{10}+j^6-j^3)(n+1)-1}^{(j^4+1)(n+1)} B_k = I,$$

$$\prod_{k=(j^4+1)(n+1)+j^4(j^4+1)^2(n+1)-1}^{(j^4+1)(n+1)} B_k = I.$$

Such matrices C_k exist. Indeed, we can transform

$$\tilde{B}_k := A_k \cdot C_{k-(j^4+1)^2(n+1)},$$
$$k \in \{(j^4+1)(n+1), \dots, (j^4+1)(n+1)+(j^4+1)^2(n+1)-1\},$$

$$\vdots$$

$$\tilde{B}_k := A_k \cdot C_{k-j^4(j^4+1)^2(n+1)},$$
$$k \in \{(j^4+1+(j^4-1)(j^4+1)^2)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1)-1\},$$

into B_k by

$$B_k = A_k \cdot C_{k-(j^4+1)^2(n+1)} \cdot \tilde{C}_k,$$
$$k \in \{(j^4+1)(n+1), \dots, (j^4+1)(n+1)+(j^4+1)^2(n+1)-1\},$$

$$\vdots$$

$$B_k = A_k \cdot C_{k-j^4(j^4+1)^2(n+1)} \cdot \tilde{C}_k,$$
$$k \in \{(j^4+1+(j^4-1)(j^4+1)^2)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1)-1\},$$

where $\tilde{C}_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 Q}(I)$ for all considered k (see (iii)). Hence, we have (see (ii) and also the below given (2.20) and (2.21))

$$C_{k-(j^4+1)^2(n+1)} \cdot \tilde{C}_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 PQ}\left(C_{k-(j^4+1)^2(n+1)}\right),$$
$$k \in \{(j^4+1)(n+1), \dots, (j^4+1)(n+1)+(j^4+1)^2(n+1)-1\},$$

$$\vdots$$

$$C_{k-j^4(j^4+1)^2(n+1)} \cdot \tilde{C}_k \in \mathcal{O}^{\varrho}_{\varepsilon_4 PQ}\left(C_{k-j^4(j^4+1)^2(n+1)}\right),$$
$$k \in \{(j^4+1+(j^4-1)(j^4+1)^2)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1)-1\}.$$

Thus, we can obtain C_k from the previous step and from the above $\tilde{C}_k$.

We put

$$B_k := A_k \cdot C_{k+(j^4+1)^3(n+1)},$$
$$k \in \{-(j^4+1)^3(n+1) - j^4(j^4+1)(n+1), \dots, -j^4(j^4+1)(n+1)-1\},$$

$$\vdots$$

$$B_k := A_k \cdot C_{k+j^4(j^4+1)^3(n+1)},$$
$$\begin{aligned} k \in \{&-j^4(j^4+1)^3(n+1) - j^4(j^4+1)(n+1), \dots, \\ &-(j^4-1)(j^4+1)^3(n+1) - j^4(j^4+1)(n+1)-1\}, \end{aligned}$$

and we denote

$$C_k := A_k^{-1} \cdot B_k,$$
$$k \in \{-j^4(j^4+1)^3(n+1) - j^4(j^4+1)(n+1), \dots, -j^4(j^4+1)(n+1)-1\}.$$

We proceed further in the same way. In the $(2i-1)$-th step, we choose

$$B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{\varepsilon_{2i} PQ}\left(C_{k-(j^4+1)^{2i-2}(n+1)}\right) \cap \mathcal{X},$$
$$\begin{aligned} k \in \{&(j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-4}(n+1), \\ &\dots, (j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \cdots \\ &+ \dots + j^4(j^4+1)^{2i-4}(n+1) + (j^4+1)^{2i-2}(n+1)-1\}, \end{aligned}$$

$$\vdots$$

$$B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{\varepsilon_{2i} PQ}\left(C_{k-j^4(j^4+1)^{2i-2}(n+1)}\right) \cap \mathcal{X},$$
$$\begin{aligned} k \in \{&(j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-4}(n+1) \\ &+ (j^4-1)(j^4+1)^{2i-2}(n+1), \dots, (j^4+1)(n+1) + j^4(j^4+1)^2(n+1) \\ &+ \dots + j^4(j^4+1)^{2i-4}(n+1) + j^4(j^4+1)^{2i-2}(n+1)-1\}, \end{aligned}$$

such that

$$\prod_{k=p_1^0-1}^{q(i)} B_k = M_1, \qquad \prod_{k=p_1^0+(j^{3i}-j^0)(n+1)-1}^{q(i)} B_k = I,$$

$$\prod_{k=p_1^1-1}^{q(i)} B_k = M_1, \qquad \prod_{k=p_1^1+(j^{3i}-j^3)(n+1)-1}^{q(i)} B_k = I,$$

$$\vdots$$

$$\prod_{k=p_1^{i-1}-1}^{q(i)} B_k = M_1, \qquad \prod_{k=p_1^{i-1}+(j^{3i}-j^{3(i-1)})(n+1)-1}^{q(i)} B_k = I,$$

$$\vdots$$

$$\prod_{k=p_i^0-1}^{q(i)} B_k = M_i, \qquad \prod_{k=p_i^0+(j^{3i}-j^0)(n+1)-1}^{q(i)} B_k = I,$$

$$\prod_{k=p_i^1-1}^{q(i)} B_k = M_i, \qquad \prod_{k=p_i^1+(j^{3i}-j^3)(n+1)-1}^{q(i)} B_k = I,$$

$$\vdots$$

$$\prod_{k=p_i^{i-1}-1}^{q(i)} B_k = M_i, \qquad \prod_{k=p_i^{i-1}+(j^{3i}-j^{3(i-1)})(n+1)-1}^{q(i)} B_k = I,$$

and

$$\prod_{k=p(i)}^{q(i)} B_k = I,$$

where $p_1^0, \dots, p_1^{i-1}, \dots, p_i^0, \dots, p_i^{i-1}$ are arbitrary positive integers for which

$$q(i) + j^{2i}\,(n+1) \le p_1^0$$

and

$$p_1^0 + (j^{2i} + j^{3i})(n+1) \le p_1^1, \quad \dots \quad p_1^{i-2} + (j^{2i} + j^{3i})(n+1) \le p_1^{i-1},$$
$$p_1^{i-1} + (j^{2i} + j^{3i})(n+1) \le p_2^0,$$

$$\vdots$$

$$p_i^0 + (j^{2i} + j^{3i})(n+1) \le p_i^1, \quad \dots \quad p_i^{i-2} + (j^{2i} + j^{3i})(n+1) \le p_i^{i-1},$$
$$p_i^{i-1} + (j^{2i} + j^{3i})(n+1) \le p(i)$$

if

$$q(i) = (j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-4}(n+1),$$
$$p(i) = (j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-2}(n+1) - 1.$$

The existence of these numbers follows from

$$p(i) - q(i) = j^4(j^4+1)^{2i-2}(n+1) - 1 \ge (j^{2i} + j^{3i})(i^2+1)(n+1),$$
$$i, j \ge 2\,(i, j \in \mathbb{N}), \ n \in \mathbb{N},$$

and the existence of the above matrices B_k follows from (2.16) and from

$$j^{3i} - j^{3(i-k)} \ge j^{2i}, \quad k \in \{1, \dots, i\},\ i \in \mathbb{N},\ j \ge 2\,(j \in \mathbb{N})\,.$$

In the $2i$-th step, we put

$$\begin{aligned}
&B_k := A_k \cdot C_{k+(j^4+1)^{2i-1}(n+1)},\\
&k \in \{-((j^4+1)^{2i-1} + \cdots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1),\\
&\qquad \ldots, -(j^4(j^4+1)^{2i-3} + \cdots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1) - 1\},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
&B_k := A_k \cdot C_{k+j^4(j^4+1)^{2i-1}(n+1)},\\
&k \in \{-j^4((j^4+1)^{2i-1} + \cdots + (j^4+1)^3 + (j^4+1))(n+1),\\
&\qquad \ldots, -((j^4-1)(j^4+1)^{2i-1} + \cdots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1) - 1\},
\end{aligned}$$

and we denote

$$\begin{aligned}
C_k := A_k^{-1} \cdot B_k, \quad k \in \{&-(j^4(j^4+1)^{2i-1} + \cdots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1),\\
&\ldots, -(j^4(j^4+1)^{2i-3} + \cdots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1) - 1\}.
\end{aligned}$$

Using this construction, we obtain the sequence $\{B_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$.

We consider the system

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}. \tag{2.17}$$

Suppose that there exists a non-zero vector $u \in F^m$ for which the solution $\{x_k\}$ of (2.17) satisfying $x_{n+1} = u$ is almost periodic. We know that

$$x_k = \prod_{i=k-1}^{n+1} B_i \cdot u \quad \text{for } k > n+1,\ k \in \mathbb{N}. \tag{2.18}$$

If we choose $\{h_i\}_{i\in\mathbb{N}} \equiv \{j^{3(i-1)}(n+1)\}_{i\in\mathbb{N}}$ for $\{\varphi_k\} \equiv \{x_k\}$ in Corollary 1.5 (see also Theorem 1.3), then, for any $\vartheta > 0$, we get the existence of an infinite set $N = N(\vartheta) \subseteq \mathbb{N}_0$ such that

$$\varrho\left(x_{k+j^{3i_1}(n+1)}, x_{k+j^{3i_2}(n+1)}\right) < \vartheta, \quad k \in \mathbb{Z},\ i_1, i_2 \in N. \tag{2.19}$$

Thus, for every $\vartheta > 0$, there exist infinitely many ϑ-translation numbers in the form $(j^{3i_1} - j^{3i_2})(n+1)$, where $i_1 > i_2$ ($i_1, i_2 \in \mathbb{N}$). For some $i \in \mathbb{N}$ with the property that $M_i\, u \neq u$, we choose $\vartheta < \varrho\,(M_i\, u, u)$ and the above $i_1 > i_2 > i$ ($i_1, i_2 \in \mathbb{N}$) arbitrarily. We have (see (2.19))

$$\varrho\left(x_{k+(j^{3i_1}-j^{3i_2})(n+1)}, x_k\right) < \vartheta, \quad k \in \mathbb{Z}.$$

From (2.18) and the construction of $\{B_k\}$, we obtain

$$\varrho\left(x_{k+(j^{3i_1}-j^{3i_2})(n+1)}, x_k\right) > \vartheta \quad \text{for at least one } k \in \mathbb{N}.$$

This contradiction gives that $\{x_k\}$ cannot be almost periodic. It means that system (2.17) does not have a non-trivial almost periodic solution.

Now it suffices to show that $\{B_k\} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\})$; i.e., that $B_k \in \mathcal{O}^\varrho_{\tilde{\varepsilon}}(A_k)$ for all k and some $\tilde{\varepsilon} \in (0, \varepsilon)$ and that $\{B_k\}$ is almost periodic. It is seen that

$$C_k \in \mathcal{O}^\varrho_{\varepsilon_2 Q}(I), \quad k \in \{-j^4(j^4+1)(n+1), \ldots, 0, \ldots, (j^4+1)(n+1) - 1\},$$

$$C_k \in \mathcal{O}^{\varrho}_{(\varepsilon_2+\varepsilon_4)PQ}(I),$$
$$k \in \{(j^4+1)(n+1), \dots, (j^4+1+j^4(j^4+1)^2)(n+1)-1\},$$

$$C_k \in \mathcal{O}^{\varrho}_{(\varepsilon_2+\varepsilon_4)PQ}(I),$$
$$k \in \{-j^4(j^4+1)^3(n+1) - j^4(j^4+1)(n+1), \dots, -j^4(j^4+1)(n+1)-1\},$$

and that, for all $i \geq 3$ ($i \in \mathbb{N}$), it is valid

$$C_k \in \mathcal{O}^{\varrho}_{(\varepsilon_2+\varepsilon_4+\dots+\varepsilon_{2i})PQ}(I),$$
$$\begin{aligned} k \in \{&(j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-4}(n+1), \dots, \\ &(j^4+1)(n+1) + j^4(j^4+1)^2(n+1) + \dots + j^4(j^4+1)^{2i-2}(n+1) - 1\}, \end{aligned}$$

$$C_k \in \mathcal{O}^{\varrho}_{(\varepsilon_2+\varepsilon_4+\dots+\varepsilon_{2i})PQ}(I),$$
$$\begin{aligned} k \in \{&-(j^4(j^4+1)^{2i-1} + \dots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1), \dots, \\ &-(j^4(j^4+1)^{2i-3} + \dots + j^4(j^4+1)^3 + j^4(j^4+1))(n+1) - 1\}. \end{aligned}$$

Thus, we have (see (2.15))

$$C_k \in \mathcal{O}^{\varrho}_{\eta}(I), \quad k \in \mathbb{Z}, \tag{2.20}$$

and (see (iii))

$$B_k \in \mathcal{O}^{\varrho}_{\varepsilon/2}(A_k), \quad k \in \mathbb{Z}. \tag{2.21}$$

Indeed,

$$B_k = A_k \cdot C_k \quad \text{for all } k \in \mathbb{Z}. \tag{2.22}$$

From Theorem 1.27 it follows that the sequence $\{C_k\}$ is almost periodic. Using Corollary 1.12 and the almost periodicity of $\{A_k\}$, we see that the set

$$T(\{A_k\}, \delta) \cap T(\{C_k\}, \delta) \text{ is relatively dense in } \mathbb{Z} \tag{2.23}$$

for any $\delta > 0$. Since the multiplication of matrices is uniformly continuous on $\mathcal{X}$, considering (2.22), we have

$$T(\{A_k\}, \delta(\vartheta)) \cap T(\{C_k\}, \delta(\vartheta)) \subseteq T(\{B_k\}, \vartheta) \tag{2.24}$$

for arbitrary $\vartheta > 0$, where $\delta(\vartheta) > 0$ is the number corresponding to ϑ from the definition of the uniform continuity of the matrix multiplication. Finally, (2.23) and (2.24) give the almost periodicity of $\{B_k\}$ which completes the proof. □

Note that, in Theorem 2.19, condition (2.13) can be omitted (i.e., one can put $L = 0$). This fact follows from the below given Lemma 2.27. Now, for the later comparison, let us recall a result from [45] (see also [1, Theorem 2.10.1]) and one of its consequences.

Theorem 2.20. *Let $(F, \varrho(\cdot,\cdot)) = (\mathbb{C}, |\cdot - \cdot|)$. If a vector valued sequence $\{b_k\}$ is almost periodic and a matrix $A \in \mathcal{M}at(\mathbb{C}, m)$ non-singular, then a solution of*

$$x_{k+1} = A \cdot x_k + b_k, \quad k \in \mathbb{Z},$$

is almost periodic if and only if it is bounded.

Remark 2.21. Several modifications and generalizations of Theorem 2.20 are known. The first theorem of the type as Theorem 2.20 was established by E. Esclangon (in [64]) for quasiperiodic solutions of linear differential equations of higher orders. It was extended by H. Bohr and O. Neugebauer (in [29]) to the form mentioned in Remark 2.23 below. In [152], Theorem 2.20 is proved if $A \in \mathcal{M}at(\mathbb{R}, m)$ and $\{b_k\}$ is almost periodic in various metrics. $\triangle$

Corollary 2.22. *Let $(F, \varrho(\cdot,\cdot)) = (\mathbb{C}, |\cdot - \cdot|)$. Let a periodic sequence $\{A_k\}$ of $m \times m$ non-singular matrices with complex elements be given. Then, a solution of the system $x_{k+1} = A_k\, x_k$, $k \in \mathbb{Z}$, is almost periodic if and only if it is bounded.*

Proof. Every almost periodic sequence is bounded. Hence, we need to show only that the boundedness of a solution implies its almost periodicity. Assume that we have a periodic system $x_{k+1} = A_k\, x_k$, $k \in \mathbb{Z}$, and its bounded solution $\{x_k\}$. Let $n \in \mathbb{N}$ be a period of $\{A_k\}$. Applying Theorem 2.20, we get that the sequence $\{y_k^1\} \equiv \{x_{nk}\}$; i.e.,

$$y_0^1 = x_0, \qquad y_k^1 = \prod_{i=nk-1}^{0} A_i \cdot x_0, \quad k \in \mathbb{N}, \qquad y_k^1 = \prod_{i=nk}^{-1} A_i^{-1} \cdot x_0, \quad k \in \mathbb{Z} \setminus \mathbb{N};$$

is almost periodic. Indeed, $\{y_k^1\}$ is a bounded solution of the constant system

$$y_{k+1} = A_{n-1} \cdots A_1 \cdot A_0 \cdot y_k, \quad k \in \mathbb{Z}.$$

Analogously, one can show that the sequences $\{y_k^j\} \equiv \{x_{nk+j-1}\}$, $j \in \{2, 3, \dots, n\}$, are almost periodic as well. The almost periodicity of $\{x_k\}$ follows from Corollary 1.10. $\square$

Remark 2.23. We add that it is possible to obtain several modifications of Corollary 2.22 for non-homogeneous systems if the non-homogeneousness is almost periodic. We mention at least the most important one—the continuous version for differential systems. If a complex matrix valued function $A(t)$, $t \in \mathbb{R}$, is periodic and a complex vector valued function $b(t)$, $t \in \mathbb{R}$, is almost periodic, then any solution of $x'(t) = A(t)\, x(t) + b(t)$, $t \in \mathbb{R}$, is almost periodic if and only if it is bounded. See the introduction of Chapter 6 or, e.g., [72, Corollary 6.5]; for generalizations and supplements, see [129]. $\triangle$

Example 2.24. Consider again $(F, \varrho(\cdot,\cdot)) = (\mathbb{C}, |\cdot - \cdot|)$. We want to document that Corollary 2.22 is no longer true if $\{A_k\}$ is only almost periodic. We know (see Lemma 1.32 and consider the second part of Corollary 1.26) that the real sequence $\{a_k\}$ defined by the recurrent formula (1.36) on $\mathbb{N}_0$ and by the prescription

$$a_k := -a_{-k-1} \quad \text{for } k \in \mathbb{Z} \setminus \mathbb{N}_0 \tag{2.25}$$

is almost periodic and that it satisfies (see Lemma 1.35)

$$\sum_{k=0}^{2^n-1} a_k = 1, \qquad \sum_{k=0}^{2^{n+j}+2^n-1} a_k = 2 - \frac{1}{2^j}, \quad n \in \mathbb{N}_0,\ j \in \mathbb{N}. \tag{2.26}$$

Let $\mathcal{X}$ be the set of all $m \times m$ diagonal matrices with numbers on the diagonal which has absolute value 1. (It is easily seen that, in this case, $\mathcal{X}$ is strongly transformable.

See also Examples 2.4 and 2.13.) All solutions of the system of the form (2.2) given by the almost periodic (see Theorem 1.6) sequence $\{A_k\} \equiv \operatorname{diag}(\exp(\mathrm{i}a_k), \dots, \exp(\mathrm{i}a_k))$ are obviously bounded, but we will show that they are not almost periodic (except the trivial one).

It suffices to consider the scalar case, i.e., $m = 1$. We suppose that the system has an almost periodic solution $\{x_k\}$. We have

$$x_k = \exp\left(\mathrm{i}\sum_{j=0}^{k-1} a_j\right) \cdot x_0, \quad k \in \mathbb{N}.$$

Especially (see (2.26)),

$$x_{2^n} = \exp(\mathrm{i}) \cdot x_0, \quad x_{2^{n+j}+2^n} = \exp\left(2\mathrm{i} - 2^{-j}\mathrm{i}\right) \cdot x_0, \quad n \in \mathbb{N}_0,\ j \in \mathbb{N}. \tag{2.27}$$

Using Corollary 1.5 (or Theorem 1.3) for the sequence $\{2^j\}_{j\in\mathbb{N}}$, for any $\varepsilon > 0$, we get an infinite set $N = N(\varepsilon) \subseteq \mathbb{N}$ such that

$$|\, x_{k+2^{j(1)}} - x_{k+2^{j(2)}} \,| < \varepsilon \quad \text{for all } k \in \mathbb{Z},\ j(1), j(2) \in N.$$

For some $j(1) \in N$, the choice $k = 2^{j(1)}$ and (2.27) give

$$\left|\, \exp(\mathrm{i}) - \exp\left(2\mathrm{i} - 2^{j(1)-j(2)}\mathrm{i}\right) \right| \cdot |\, x_0 \,| < \varepsilon, \quad j(1) < j(2),\ j(1), j(2) \in N.$$

Since ε can be arbitrarily small and $j(2) > j(1)$ can be found for every $\varepsilon > 0$, we obtain $x_0 = 0$. Thus, the system does not have a non-trivial almost periodic solution. ◊

In the above example, we see that the boundedness is necessary to the almost periodicity of solutions but not sufficient. Now we prove a more important necessary (also not sufficient, see again Example 2.24) condition about the limitation of almost periodic solutions.

Lemma 2.25. *Let an almost periodic sequence of non-singular $A_k \in \mathcal{M}at(F, m)$ be given. Let $\{x_k\}$ be an almost periodic solution of the system $x_{k+1} = A_k\, x_k$, $k \in \mathbb{Z}$. Then, it is valid either $x_k = o$, $k \in \mathbb{Z}$, or*

$$\inf_{k\in\mathbb{Z}} \varrho(x_k, o) > 0.$$

Proof. Suppose that an almost periodic solution $\{x_k\}$ of a system satisfies

$$\inf_{k\in\mathbb{Z}} \varrho(x_k, o) = 0.$$

Let $\{h_i\}_{i\in\mathbb{N}} \subseteq \mathbb{Z}$ be such that

$$\lim_{i\to\infty} \varrho(x_{h_i}, o) = 0. \tag{2.28}$$

Considering Corollary 1.5 and Theorems 1.7 and 1.8, we get a subsequence $\{\tilde{h}_i\}$ of $\{h_i\}$ for which there exist almost periodic sequences $\{B_k\}$, $\{y_k\}$ satisfying

$$\lim_{i\to\infty} A_{k+\tilde{h}_i} = B_k, \quad \lim_{i\to\infty} B_{k-\tilde{h}_i} = A_k, \quad \lim_{i\to\infty} x_{k+\tilde{h}_i} = y_k, \quad \lim_{i\to\infty} y_{k-\tilde{h}_i} = x_k,$$

where the convergences can be uniform with respect to $k \in \mathbb{Z}$ (see Remark 1.9). We have $y_{k+1} = B_k\, y_k$, $k \in \mathbb{Z}$, and $y_0 = o$ (see (2.28)). Thus, $\{y_k\} \equiv \{o\}$. Consequently, $x_k = \lim_{i\to\infty} y_{k-\tilde{h}_i} = o$ for $k \in \mathbb{Z}$. □

Example 2.26. Applying Theorem 1.27 for $n = 0$, $\varphi_0 = 2$, $j = 1$, and $r_i = 3/2^i$, $i \in \mathbb{N}$, we construct the everywhere non-zero almost periodic sequence

$$b_0 := 2, \quad b_1 := 2 - 1, \quad b_{-2} := 2 - \frac{1}{2}, \quad b_{-1} := 1 - \frac{1}{2},$$

$$\vdots$$

$$b_k := b_{k+2^{2i-1}} - \frac{1}{2^{2i-1}}, \quad k \in \{-2^{2i-1} - \cdots - 2^3 - 2, \dots, -2^{2i-3} - \cdots - 2^3 - 2 - 1\},$$
$$b_k := b_{k-2^{2i}} - \frac{1}{2^{2i}}, \quad k \in \{2 + 2^2 + \cdots + 2^{2i-2}, \dots, 2 + 2^2 + \cdots + 2^{2i-2} + 2^{2i} - 1\},$$

$$\vdots$$

in the space $(\mathbb{R}, |\cdot - \cdot|)$. Since

$$\lim_{i\to\infty} b_{2^0-2^1+2^2-2^3+\cdots+(-2)^i} = 0,$$

the equation $x_{k+1} = b_k\, x_k$, $k \in \mathbb{Z}$, does not have a non-trivial almost periodic solution (see Lemma 2.18) and the vector valued sequence $\{b_k\, u\}$, where $u \neq o$, $u \in \mathbb{R}^m$, is not a solution of an almost periodic homogeneous linear difference system (see Lemma 2.25).

Moreover, for any bounded countable set of real numbers, it is shown in [73] that there exists an almost periodic sequence whose range is the set. (For details, we refer to the next chapter.) It means that there exists a large class of almost periodic sequences which cannot be solutions of any almost periodic system (2.2). $\Diamond$

To improve Theorem 2.19, we need also the next lemma.

Lemma 2.27. *Let $\mathcal{X}$ be transformable and let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrary. If there exists a matrix $M(\vartheta) \in \mathcal{O}^{\varrho}_{\vartheta}(O) \cap \mathcal{X}$ for any $\vartheta > 0$, then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ which does not have an almost periodic solution other than the trivial one.*

Proof. We put $L_i := \varrho\,(M_i, O)$ for matrices $M_i \in \mathcal{X}$, $i \in \mathbb{N}$, such that

$$\lim_{i\to\infty} L_i = 0, \qquad L_{i+1} < L_i, \quad i \in \mathbb{N}. \tag{2.29}$$

Let $\eta = \eta(\varepsilon/2)$, ζ, and P and $Q = Q(L_1)$ be from (ii) and (iii), respectively. We can assume (or choose $\{B_k\} \equiv \{A_k\}$) that

$$\eta < \varepsilon < \zeta, \qquad L_1 + \varepsilon < \varrho\,(A_k, O)\,, \quad k \in \mathbb{Z},$$

and that (see also Lemma 2.11) we have $\{\varepsilon_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$, $j \geq 2$ ($j \in \mathbb{N}$), and $n \in \mathbb{N}$ satisfying

$$\sum_{i=1}^{\infty} \varepsilon_i < \frac{\eta}{PQ},$$

$$j^i(n+1) \geq p\left(L_{g(i)}, \varepsilon_i\right) \quad \text{for infinitely many odd } i \in \mathbb{N}, \tag{2.30}$$

where $g(i)$ is from Lemma 2.11 as well.

The set of all $i \neq 1$ ($i \in \mathbb{N}$), which are not divisible by 2 and for which (2.30) is valid, is denoted by N. Let $N = \{i_1, i_2, \dots, i_l, \dots\}$, where $i_l < i_{l+1}$, $l \in \mathbb{N}$. Since we can redefine L_i (choose other M_i), we can also assume that $g(i_l) \geq l$, $l \in \mathbb{N}$. We will construct sequences $\{B_k\}$ and $\{C_k\}$ as in the proof of Theorem 2.19 for j^4 replaced by j. First of all we put

$$\begin{aligned} p_l &:= (j + 1 + j(j+1)^2 + \cdots + j(j+1)^{i_l-3})(n+1), \quad l \in \mathbb{N}, \\ q_l &:= (j + 1 + j(j+1)^2 + \cdots + j(j+1)^{i_l-1})(n+1) - 1, \quad l \in \mathbb{N}. \end{aligned}$$

Before the i_1-th step (for $k \leq p_1 - 1$), we choose the matrices C_k (consequently B_k) arbitrarily. We will obtain B_k and define

$$J_1 := \prod_{k=p_1}^{0} B_k, \qquad J_2 := \prod_{k=p_2}^{0} B_k, \qquad \dots$$

In the i_1-th step, we choose the matrices C_k arbitrarily if $J_1 \in \mathcal{O}^{\varrho}_{L_1}(O)$, and so that

$$\prod_{k=q_1}^{0} B_k = M_1 \quad \text{if} \quad J_1 \notin \mathcal{O}^{\varrho}_{L_1}(O).$$

Between the i_1-th step and the i_2-th step, we choose them again arbitrarily. In the i_2-th step, we choose them arbitrarily if $J_2 \in \mathcal{O}^{\varrho}_{L_2}(O)$, and so that

$$\prod_{k=q_2}^{0} B_k = M_2 \quad \text{if} \quad J_2 \notin \mathcal{O}^{\varrho}_{L_2}(O).$$

If we proceed further in the same way, then we get matrices C_k, B_k for all $k \in \mathbb{Z}$. Analogously as in the proof of Theorem 2.19, we can prove that $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$. We have

$$\varrho\left(\prod_{k=r_l}^{0} B_k, O\right) \leq L_l \quad \text{for } r_l \in \{p_l, q_l\},\ l \in \mathbb{N}. \tag{2.31}$$

Evidently, for any $u \in F^m$ and $\mu > 0$, there exists $\delta = \delta(u, \mu) > 0$ with the property that $\varrho(C\,u, o) < \mu$ if $C \in \mathcal{O}^{\varrho}_{\delta}(O)$. Using this, from (2.29), (2.31), and Lemma 2.25, we get that all non-trivial solutions of the system of the form (2.2) given by $\{B_k\}$ are not almost periodic. □

Now we can generalize Theorem 2.19 into the case of general transformable groups.

Theorem 2.28. *Let $\mathcal{X}$ be transformable. Let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrarily given. If there exists a sequence $\{M_i\}_{i\in\mathbb{N}} \subseteq \mathcal{X}$ such that, for any non-zero vector $u \in F^m$, one can find $i = i(u) \in \mathbb{N}$ with the property that $M_i\,u \neq u$, then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ which does not possess a non-trivial almost periodic solution.*

Proof. We obtain from Lemma 2.27 that it suffices to consider the case when

$$\mathcal{O}^{\varrho}_{\varepsilon}(O) \cap \mathcal{X} = \emptyset, \tag{2.32}$$

because we can assume that the number ε is small enough. Let $\eta = \eta(\varepsilon/2)$, ζ, P, and $Q = Q(\varepsilon)$ be from Definition 2.1. Note that Q does not depend on ε (consider (2.32)) and that $P, Q \geq 1$, $\eta \leq \varepsilon/2$ (consider $C = I$ in Definition 2.1). For simplicity, we also assume that $\varepsilon \leq \zeta$ (the number ζ is given for $\mathcal{X}$). Let $n \in \mathbb{N}$ be arbitrary such that

$$\frac{PQ}{2^{2n}} \leq \eta. \tag{2.33}$$

We will construct an almost periodic sequence of matrices $C_k \in \mathcal{X}$, $k \in \mathbb{Z}$, applying Corollary 1.28. Let us denote (see Definition 2.1 and also (2.32))

$$p_i := p\left(\varepsilon, \frac{1}{2^{2n+i}}\right), \quad i \in \mathbb{N}, \tag{2.34}$$

and

$$s_i := -2^{2i-1} - 2^{2i-3} - \cdots - 2^3 - 2, \quad i \in \mathbb{N},$$
$$t_i := 2 + 2^2 + \cdots + 2^{2i-2} + 2^{2i}, \quad i \in \mathbb{N}.$$

Evidently, there exists an increasing sequence $\{l(i)\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ for which

$$i^2 \leq 2^{l(i)-1}, \quad p_i \leq 2^{l(i)-1}, \quad i \in \mathbb{N}. \tag{2.35}$$

In the first step of the construction of $\{C_k\}$, we put $C_0 := I$, $C_1 := I$,

$$C_k := I, \quad k \in \{-2, -1\}, \qquad C_k := I, \quad k \in \{2, \dots, 2 + 2^2 - 1\},$$
$$\vdots$$
$$C_k := I, \quad k \in \{t_{l(1)-2}, \dots, t_{l(1)-1} - 1\}, \qquad C_k := I, \quad k \in \{s_{l(1)}, \dots, s_{l(1)-1} - 1\},$$

and define

$$B_k := A_k, \quad k \in \{s_{l(1)}, \dots, t_{l(1)-1} - 1\}.$$

In the second step, we choose matrices

$$B_k \in \mathcal{O}^{\varrho}_{2^{-2n-1}}(A_k), \quad k \in \{t_{l(1)-1}, \dots, t_{l(1)} - 1\}, \tag{2.36}$$

arbitrarily so that

$$\prod_{k=t_{l(1)-1}+p_1-1}^{0} B_k = I, \qquad \prod_{k=t_{l(1)-1}+p_1+(2^{l(1)}-1)-1}^{t_{l(1)-1}+p_1} B_k = M_1.$$

The existence of such matrices B_k follows directly from Definition 2.1 (see (2.34), (2.35)). We define

$$C_k := A_k^{-1} \cdot B_k, \quad k \in \{t_{l(1)-1}, \dots, t_{l(1)} - 1\}.$$

From (iii) in Definition 2.1 and from (2.36), it follows that

$$C_k \in \mathcal{O}^{\varrho}_{2^{-2n-1}Q}(I), \quad k \in \{t_{l(1)-1}, \dots, t_{l(1)} - 1\},$$

i.e.,

$$C_k \in \mathcal{O}^{\varrho}_{2^{-2n-1}Q}(C_{k-2^{2l(1)}}) \cap \mathcal{X}, \quad k \in \{t_{l(1)-1}, \dots, t_{l(1)} - 1\}.$$

In the third step, we define

$$C_k := C_{k+2^{2l(1)+1}}, \quad k \in \{s_{l(1)+1}, \dots, s_{l(1)} - 1\},$$

$$C_k := C_{k-2^{2l(1)+2}}, \quad k \in \{t_{l(1)}, \dots, t_{l(1)+1} - 1\},$$

$$\vdots$$

$$C_k := C_{k-2^{2l(2)-2}}, \quad k \in \{t_{l(2)-2}, \dots, t_{l(2)-1} - 1\},$$

$$C_k := C_{k+2^{2l(2)-1}}, \quad k \in \{s_{l(2)}, \dots, s_{l(2)-1} - 1\},$$

and put

$$B_k := A_k \cdot C_k, \quad k \in \{s_{l(2)}, \dots, t_{l(2)-1} - 1\}.$$

We remark that, before the third step, we have

$$B_k = A_k \cdot C_k, \quad k \in \{s_{l(1)}, \dots, t_{l(1)} - 1\}.$$

In the fourth step, we choose

$$B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{2^{-2n-2}PQ}(C_{k-2^{2l(2)}}) \cap \mathcal{X},\ k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\},$$

arbitrarily so that

$$\prod_{k=t_{l(2)-1}+p_2-1}^{0} B_k = I, \qquad \prod_{k=t_{l(2)-1}+p_2+(2^{l(2)}-1)-1}^{t_{l(2)-1}+p_2} B_k = M_1,$$

$$\prod_{k=t_{l(2)-1}+2p_2+2^{l(2)}-1}^{0} B_k = I, \qquad \prod_{k=t_{l(2)-1}+2p_2+2^{l(2)}+(2^{l(2)}-2^{l(1)})-1}^{t_{l(2)-1}+2p_2+2^{l(2)}} B_k = M_1,$$

$$\prod_{k=t_{l(2)-1}+3p_2+2\cdot 2^{l(2)}-1}^{0} B_k = I, \qquad \prod_{k=t_{l(2)-1}+3p_2+2\cdot 2^{l(2)}+(2^{l(2)}-1)-1}^{t_{l(2)-1}+3p_2+2\cdot 2^{l(2)}} B_k = M_2,$$

$$\prod_{k=t_{l(2)-1}+4p_2+3\cdot 2^{l(2)}-1}^{0} B_k = I, \qquad \prod_{k=t_{l(2)-1}+4p_2+3\cdot 2^{l(2)}+(2^{l(2)}-2^{l(1)})-1}^{t_{l(2)-1}+4p_2+3\cdot 2^{l(2)}} B_k = M_2.$$

Note that it is valid

$$t_{l(2)-1} + 4p_2 + 3 \cdot 2^{l(2)} + 2^{l(2)} - 2^{l(1)} < t_{l(2)},$$

because $t_{l(2)} - t_{l(2)-1} = 2^{2l(2)}$ and (see (2.35))

$$4p_2 + 3 \cdot 2^{l(2)} + 2^{l(2)} - 2^{l(1)} < 2^2 2^{l(2)-1} + 2^2 2^{l(2)} < 2 \cdot 2^2 \cdot 2^{l(2)} \leq 2^{2l(2)}.$$

Now we show that the above matrices B_k, C_k for $k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\}$ exist. First, we put

$$\tilde{B}_k := A_k \cdot C_{k-2^{2l(2)}}, \quad k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\}. \tag{2.37}$$

We use (i) in Definition 2.1 to transform $\tilde{B}_k$ into $B_k \in \mathcal{O}^{\varrho}_{2^{-2n-2}}\left(\tilde{B}_k\right) \cap \mathcal{X}$ arbitrarily as $U_1, \dots, U_l$ into $V_1, \dots, V_l$ for

$$l = p_2, \quad l = 2^{l(2)} - 1, \quad l = p_2 + 2^{l(2)} - \left(2^{l(2)} - 1\right), \quad l = 2^{l(2)} - 2^{l(1)},$$

$$l = p_2 + 2^{l(2)} - \left(2^{l(2)} - 2^{l(1)}\right), \quad l = 2^{l(2)} - 1,$$
$$l = p_2 + 2^{l(2)} - \left(2^{l(2)} - 1\right), \quad l = 2^{l(2)} - 2^{l(1)}$$

and

$$U_0 = \prod_{k=t_{l(2)-1}+p_2-1}^{t_{l(2)-1}} B_k = \left(\prod_{k=t_{l(2)-1}-1}^{0} B_k \right)^{-1}, \quad U_0 = \prod_{k=t_{l(2)-1}+p_2+(2^{l(2)}-1)-1}^{t_{l(2)-1}+p_2} B_k = M_1,$$

$$U_0 = \prod_{k=t_{l(2)-1}+2p_2+2^{l(2)}-1}^{t_{l(2)-1}+p_2+(2^{l(2)}-1)} B_k = M_1^{-1}, \quad U_0 = \prod_{k=t_{l(2)-1}+2p_2+2^{l(2)}+(2^{l(2)}-2^{l(1)})-1}^{t_{l(2)-1}+2p_2+2^{l(2)}} B_k = M_1,$$

$$U_0 = \prod_{k=t_{l(2)-1}+3p_2+2\cdot 2^{l(2)}-1}^{t_{l(2)-1}+2p_2+2^{l(2)}+(2^{l(2)}-2^{l(1)})} B_k = M_1^{-1},$$

$$U_0 = \prod_{k=t_{l(2)-1}+3p_2+2\cdot 2^{l(2)}+(2^{l(2)}-1)-1}^{t_{l(2)-1}+3p_2+2\cdot 2^{l(2)}} B_k = M_2,$$

$$U_0 = \prod_{k=t_{l(2)-1}+4p_2+3\cdot 2^{l(2)}-1}^{t_{l(2)-1}+3p_2+2\cdot 2^{l(2)}+(2^{l(2)}-1)} B_k = M_2^{-1},$$

$$U_0 = \prod_{k=t_{l(2)-1}+4p_2+3\cdot 2^{l(2)}+(2^{l(2)}-2^{l(1)})-1}^{t_{l(2)-1}+4p_2+3\cdot 2^{l(2)}} B_k = M_2,$$

respectively. It suffices to consider (2.32), (2.34), and the inequalities (the sequence $\{l(i)\}$ is increasing)

$$p_2 \le 2^{l(2)-1} \le 2^{l(2)} - 2^{l(1)} \le 2^{l(2)} - 1,$$
$$p_2 \le p_2 + 2^{l(2)} - \left(2^{l(2)} - 1\right) \le p_2 + 2^{l(2)} - \left(2^{l(2)} - 2^{l(1)}\right).$$

To prove the existence of the above B_k, $C_k = A_k^{-1} B_k$, it remains to show that

$$C_k \in \mathcal{O}^{\varrho}_{2^{-2n-2}PQ}\left(C_{k-2^{2l(2)}}\right) \cap \mathcal{X}, \quad k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\}. \tag{2.38}$$

We can express

$$B_k = A_k \cdot C_{k-2^{2l(2)}} \cdot \tilde{C}_k, \quad k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\}, \tag{2.39}$$

where $\tilde{C}_k = C^{-1}_{k-2^{2l(2)}} \, A^{-1}_k \, B_k \in \mathcal{X}$, i.e., it is valid (consider $B_k = A_k \, C_k$) that

$$C_k = C_{k-2^{2l(2)}} \cdot \tilde{C}_k \in \mathcal{X}, \quad k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\}. \tag{2.40}$$

Further, we have (see (2.37) and (2.39))

$$\tilde{C}_k = \tilde{B}^{-1}_k \cdot B_k, \quad k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\}.$$

We repeat that

$$B_k \in \mathcal{O}^{\varrho}_{2^{-2n-2}}\left(\tilde{B}_k\right), \quad k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\}.$$

Using (iii) in Definition 2.1, we obtain

$$\tilde{C}_k \in \mathcal{O}^{\varrho}_{2^{-2n-2}Q}\left(I\right), \quad k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\},$$

and, using condition (ii) (consider (2.33) and $\eta \le \varepsilon/2 < \zeta$), we obtain

$$C_{k-2^{2l(2)}} \cdot \tilde{C}_k \in \mathcal{O}^{\varrho}_{2^{-2n-2}PQ}\left(C_{k-2^{2l(2)}}\right), \quad k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\}. \tag{2.41}$$

Now (2.40) and (2.41) give (2.38).

Continuing in the same manner, in the $2i$-th step, we choose arbitrary matrices

$$B_k = A_k \cdot C_k, \quad C_k \in \mathcal{O}^{\varrho}_{2^{-2n-i}PQ}\left(C_{k-2^{2l(i)}}\right) \cap \mathcal{X},\ k \in \{t_{l(i)-1}, \dots, t_{l(i)} - 1\},$$

for which

$$\prod_{k=t_{l(i)-1}+p_i-1}^{0} B_k = I, \qquad \prod_{k=t_{l(i)-1}+p_i+(2^{l(i)}-1)-1}^{t_{l(i)-1}+p_i} B_k = M_1,$$

$$\vdots$$

$$\prod_{k=t_{l(i)-1}+ip_i+(i-1)2^{l(i)}-1}^{0} B_k = I, \qquad \prod_{k=t_{l(i)-1}+ip_i+(i-1)2^{l(i)}+(2^{l(i)}-2^{l(i-1)})-1}^{t_{l(i)-1}+ip_i+(i-1)2^{l(i)}} B_k = M_1,$$

$$\prod_{k=t_{l(i)-1}+(i+1)p_i+i2^{l(i)}-1}^{0} B_k = I, \qquad \prod_{k=t_{l(i)-1}+(i+1)p_i+i2^{l(i)}+(2^{l(i)}-1)-1}^{t_{l(i)-1}+(i+1)p_i+i2^{l(i)}} B_k = M_2,$$

$$\vdots$$

$$\prod_{k=t_{l(i)-1}+2ip_i+(2i-1)2^{l(i)}-1}^{0} B_k = I, \qquad \prod_{k=t_{l(i)-1}+2ip_i+(2i-1)2^{l(i)}+(2^{l(i)}-2^{l(i-1)})-1}^{t_{l(i)-1}+2ip_i+(2i-1)2^{l(i)}} B_k = M_2,$$

$$\vdots$$

$$\prod_{k=t_{l(i)-1}+((i-1)i+1)p_i+(i-1)i2^{l(i)}-1}^{0} B_k = I,$$

$$\prod_{k=t_{l(i)-1}+((i-1)i+1)p_i+(i-1)i2^{l(i)}+(2^{l(i)}-1)-1}^{t_{l(i)-1}+((i-1)i+1)p_i+(i-1)i2^{l(i)}} B_k = M_i,$$

$$\vdots$$

$$\prod_{k=t_{l(i)-1}+i^2p_i+(i^2-1)2^{l(i)}-1}^{0} B_k = I,$$

$$\prod_{k=t_{l(i)-1}+i^2p_i+(i^2-1)2^{l(i)}+(2^{l(i)}-2^{l(i-1)})-1}^{t_{l(i)-1}+i^2p_i+(i^2-1)2^{l(i)}} B_k = M_i.$$

We remark again that it is true (see (2.35))

$$i^2p_i + \left(i^2-1\right)2^{l(i)} + 2^{l(i)} - 2^{l(i-1)} < i^2 2^{l(i)-1} + i^2 2^{l(i)} < 2^{2l(i)}.$$

Thus, we have

$$t_{l(i)-1} + i^2p_i + \left(i^2-1\right)2^{l(i)} + 2^{l(i)} - 2^{l(i-1)} < t_{l(i)}.$$

The existence of the above matrices $B_k, C_k \in \mathcal{X}$ can be shown analogously to the fourth step.

In the $(2i+1)$-th step, we define

$$C_k := C_{k+2^{2l(i)+1}}, \quad k \in \{s_{l(i)+1}, \dots, s_{l(i)} - 1\},$$

$$C_k := C_{k-2^{2l(i)+2}}, \quad k \in \{t_{l(i)}, \dots, t_{l(i)+1} - 1\},$$

$$\vdots$$

$$C_k := C_{k-2^{2l(i+1)-2}}, \quad k \in \{t_{l(i+1)-2}, \dots, t_{l(i+1)-1} - 1\},$$

$$C_k := C_{k+2^{2l(i+1)-1}}, \quad k \in \{s_{l(i+1)}, \dots, s_{l(i+1)-1} - 1\},$$

and put

$$B_k := A_k \cdot C_k, \quad k \in \{s_{l(i+1)}, \dots, t_{l(i+1)-1} - 1\}.$$

We can construct $\{C_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ and $\{B_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ so that

$$B_k = A_k \cdot C_k, \quad k \in \mathbb{Z}. \tag{2.42}$$

We know that

$$C_k = I, \quad k \in \{s_{l(1)}, \dots, t_{l(1)-1} - 1\},$$

$$C_k \in \mathcal{O}^{\varrho}_{2^{-2n-1}Q}\left(C_{k-2^{2l(1)}}\right) \subseteq \mathcal{O}^{\varrho}_{2^{-2n-1}PQ}\left(I\right), \quad k \in \{t_{l(1)-1}, \dots, t_{l(1)} - 1\},$$

$$C_k = C_{k+2^{2l(1)+1}} \in \mathcal{O}^{\varrho}_{2^{-2n-1}PQ}\left(I\right), \quad k \in \{s_{l(1)+1}, \dots, s_{l(1)} - 1\},$$

$$C_k = C_{k-2^{2l(1)+2}} \in \mathcal{O}^{\varrho}_{2^{-2n-1}PQ}\left(I\right), \quad k \in \{t_{l(1)}, \dots, t_{l(1)+1} - 1\},$$

$$\vdots$$

$$C_k = C_{k-2^{2l(2)-2}} \in \mathcal{O}^{\varrho}_{2^{-2n-1}PQ}(I), \quad k \in \{t_{l(2)-2}, \dots, t_{l(2)-1} - 1\},$$

$$C_k = C_{k+2^{2l(2)-1}} \in \mathcal{O}^{\varrho}_{2^{-2n-1}PQ}(I), \quad k \in \{s_{l(2)}, \dots, s_{l(2)-1} - 1\},$$

$$C_k \in \mathcal{O}^{\varrho}_{2^{-2n-2}PQ}(C_{k-2^{2l(2)}}) \subseteq \mathcal{O}^{\varrho}_{(2^{-2n-1}+2^{-2n-2})PQ}(I), \quad k \in \{t_{l(2)-1}, \dots, t_{l(2)} - 1\},$$

$$\vdots$$

$$C_k \in \mathcal{O}^{\varrho}_{2^{-2n-i}PQ}(C_{k-2^{2l(i)}}) \subseteq \mathcal{O}^{\varrho}_{(2^{-2n-1}+\dots+2^{-2n-i})PQ}(I), \quad k \in \{t_{l(i)-1}, \dots, t_{l(i)} - 1\},$$

$$C_k = C_{k+2^{2l(i)+1}} \in \mathcal{O}^{\varrho}_{(2^{-2n-1}+\dots+2^{-2n-i})PQ}(I), \quad k \in \{s_{l(i)+1}, \dots, s_{l(i)} - 1\},$$

$$\vdots$$

Especially, we see that we can construct $\{C_k\}$ as in Corollary 1.28 for the sequence of numbers

$$\varepsilon_j := 2^{-i}PQ \quad \text{if} \quad j = f(i) \text{ for some } i \in \mathbb{N} \setminus \{1, \dots, 2n\};$$
$$\varepsilon_j := 0 \quad \text{if} \quad j \in \mathbb{N} \setminus \{f(i);\, i \in \mathbb{N} \setminus \{1, \dots, 2n\}\},$$

where $f : \mathbb{N} \setminus \{1, \dots, 2n\} \to \mathbb{N} \setminus \{1, \dots, 2n\}$ is a given increasing discrete function. Since the sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ satisfies (1.25), $\{C_k\}$ is almost periodic.

Further, we see that

$$C_k \in \mathcal{O}^{\varrho}_{(2^{-2n-1}+2^{-2n-2}+\dots+2^{-2n-j(k)})PQ}(I)$$

for all $k \in \mathbb{Z}$ and some $j(k) \in \mathbb{N}$ (which depends on k); i.e., we have

$$C_k \in \mathcal{O}^{\varrho}_{2^{-2n}PQ}(I), \quad k \in \mathbb{Z}.$$

From (2.33) it follows $C_k \in \mathcal{O}^{\varrho}_{\eta}(I)$ for $k \in \mathbb{Z}$ and, consequently, from condition (ii) and (2.42) it follows

$$B_k \in \mathcal{O}^{\varrho}_{\varepsilon/2}(A_k), \quad k \in \mathbb{Z}, \qquad \text{i.e.,} \qquad \sup_{k\in\mathbb{Z}} \varrho(A_k, B_k) \le \frac{\varepsilon}{2} < \varepsilon. \tag{2.43}$$

Now we prove the almost periodicity of $\{B_k\}_{k\in\mathbb{Z}}$ applying the almost periodicity of $\{A_k\}$ and $\{C_k\}$. Let $\vartheta > 0$ be arbitrarily small and let $\delta = \delta(\vartheta) > 0$ be such that (see (ii) in Definition 2.1)

$$U_2 \cdot V_2 \in \mathcal{O}^{\varrho}_{\vartheta}(U_1 \cdot V_1) \quad \text{if} \quad U_1, V_1, U_2, V_2 \in \mathcal{X} \text{ and } U_1 \in \mathcal{O}^{\varrho}_{\delta}(U_2), V_1 \in \mathcal{O}^{\varrho}_{\delta}(V_2). \tag{2.44}$$

Corollary 1.11 implies that the sequence $\{[A_k, C_k]\}_{k\in\mathbb{Z}}$ in $\mathcal{M}at(F, m) \times \mathcal{M}at(F, m)$ is almost periodic. Thus (or see directly Corollary 1.12), the set $T(\{A_k\}, \delta) \cap T(\{C_k\}, \delta)$ is relatively dense in $\mathbb{Z}$ and, from (2.42) and (2.44), we have

$$T(\{A_k\}, \delta) \cap T(\{C_k\}, \delta) \subseteq T(\{B_k\}, \vartheta).$$

Because of the arbitrariness of $\vartheta > 0$, we obtain the fact that $\{B_k\}$ is almost periodic.

The almost periodicity of $\{B_k\}$ together with (2.43) give that $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$. Suppose that there exists a vector $u \neq o$, $u \in F^m$, for which the solution $\{x_k\}_{k\in\mathbb{Z}}$ of

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is almost periodic. Let $i(u) \in \mathbb{N}$ and $\vartheta = \vartheta(u) > 0$ satisfy $\vartheta < \varrho\left(M_{i(u)}\, u, u\right)$. From Theorem 1.3 for $h_1 = 1$ and $h_i = 2^{l(i-1)}$, $i \geq 2$, $i \in \mathbb{N}$, it follows that the inequality

$$\varrho\left(x_{k+2^{l(i(1))}}, x_{k+2^{l(i(2))}}\right) < \vartheta, \quad k \in \mathbb{Z}, \tag{2.45}$$

is satisfied for infinitely many $i(1) < i(2)$, $i(1), i(2) \in \mathbb{N}$.

It is easily seen that

$$x_k = \prod_{i=k-1}^{0} B_i \cdot u, \quad k \in \mathbb{N}, \tag{2.46}$$

and that

$$x_{k+2^{l(i(2))}-2^{l(i(1))}} = \prod_{i=k+2^{l(i(2))}-2^{l(i(1))}-1}^{k} B_i \cdot x_k, \quad i(2) > i(1) \geq i(u),\ k, i(1), i(2) \in \mathbb{N}. \tag{2.47}$$

If we choose

$$k = t_{l(i(2))-1} + ((i(u)-1)i(2) + i(1) + 1)\, p_{i(2)} + ((i(u)-1)i(2) + i(1))\, 2^{l(i(2))}, \tag{2.48}$$

we obtain (from (2.46) and from the construction of $\{B_k\}$)

$$x_k = \prod_{i=k-1}^{0} B_i \cdot u = I \cdot u = u$$

and (from (2.47) and the construction)

$$x_{k+2^{l(i(2))}-2^{l(i(1))}} = \prod_{i=k+2^{l(i(2))}-2^{l(i(1))}-1}^{k} B_i \cdot x_k = M_{i(u)} \cdot x_k = M_{i(u)} \cdot u$$

for $i(2) > i(1) \geq i(u)$, $i(1), i(2) \in \mathbb{N}$.

Finally, we have

$$\varrho\left(x_k, x_{k+2^{l(i(2))}-2^{l(i(1))}}\right) = \varrho\left(u, M_{i(u)} \cdot u\right) > \vartheta, \quad i(2) > i(1) \geq i(u),\ i(1), i(2) \in \mathbb{N},$$

where k is given in (2.48). Of course, we can rewrite (2.45) into the form (k is replaced by $k - 2^{l(i(1))}$) of

$$\varrho\left(x_k, x_{k+2^{l(i(2))}-2^{l(i(1))}}\right) < \vartheta, \quad k \in \mathbb{Z},$$

which is valid for infinitely many $i(1), i(2) \in \mathbb{N}$ such that $i(1) < i(2)$. This contradiction proves that the system $\{B_k\} \in \mathcal{AP}(\mathcal{X})$ does not have any non-trivial almost periodic solution. $\square$

Now we mention preliminary results, which we need to prove our main result concerning weakly transformable groups (the main generalization of Theorem 2.19).

Lemma 2.29. *Let $\mathcal{X}$ be transformable. For any $\varepsilon > 0$ and $C, D \in \mathcal{X}$, there exist matrices $C_1, \ldots, C_j \in \mathcal{X}$ satisfying*

$$C_1 \in \mathcal{O}^{\varrho}_{\varepsilon}(C), \quad C_i \in \mathcal{O}^{\varrho}_{\varepsilon}(C_{i+1}) \text{ for } i \in \{1, \ldots, j-1\}, \quad C_j \in \mathcal{O}^{\varrho}_{\varepsilon}(D).$$

Proof. Let $\varepsilon > 0$ and $C, D \in \mathcal{X}$ be given. To prove the lemma, it suffices to use (i) in Definition 2.1 for $\eta = \eta(\varepsilon) \leq \varepsilon$ mentioned in (ii). Let $L > 0$ be such that $L < \varrho(I, O)$, $L < \varrho(C, O)$, and $L < \varrho(D, O)$ and let $p = p(L, \eta)$ have the property mentioned in (i). Let us choose $U_0 = C$ and $U_i = I$, $i \in \{1, \ldots, p\}$, in (i). There exist matrices $V_i \in \mathcal{O}^{\varrho}_{\eta}(I) \cap \mathcal{X}$, $i \in \{1, \ldots, p\}$, for which $V_p \cdots V_2 \cdot V_1 = C$. We put

$$M_1 := V_1, \quad M_2 := V_2 \cdot V_1, \quad \ldots \quad M_p := V_p \cdots V_2 \cdot V_1.$$

Since $V_i \in \mathcal{O}^{\varrho}_{\eta}(I) \cap \mathcal{X}$, $i \in \{1, \ldots, p\}$, and

$$M_1 = V_1 \in \mathcal{X}, \quad M_2 = V_2 \cdot M_1 \in \mathcal{X}, \quad \ldots \quad M_p = V_p \cdot M_{p-1} = C \in \mathcal{X},$$

we have

$$M_1 \in \mathcal{O}^{\varrho}_{\varepsilon}(I), \quad M_i \in \mathcal{O}^{\varrho}_{\varepsilon}(M_{i+1}) \text{ for } i \in \{1, \ldots, p-1\}.$$

Especially, $M_{p-1} \in \mathcal{O}^{\varrho}_{\varepsilon}(C)$. Analogously, we can find matrices $M_{p+1}, \ldots, M_{2p-1} \in \mathcal{X}$ with the property that

$$M_{p+1} \in \mathcal{O}^{\varrho}_{\varepsilon}(I), \quad M_{p+i} \in \mathcal{O}^{\varrho}_{\varepsilon}(M_{p+i+1}) \text{ for } i \in \{1, \ldots, p-2\}, \quad M_{2p-1} \in \mathcal{O}^{\varrho}_{\varepsilon}(D).$$

For the sequence $M_{p-1}, \ldots, M_1, I, M_{p+1}, \ldots, M_{2p-1}$ as $C_1, \ldots, C_j$, we obtain the statement of the lemma. □

Lemma 2.30. *Let $\mathcal{X}$ be weakly transformable and let $\mathcal{X}_0 \subseteq \mathcal{X}$, $X_1, \ldots, X_l \in \mathcal{X}$ be from Definition 2.14. If for some $C_0, D_0 \in \mathcal{X}_0$ and $i(1), i(2) \in \{1, \ldots, l\}$, the product $C_0\, X_{i(1)}\, D_0\, X_{i(2)}$ can be expressed as $M_0\, X_j$ for some $M_0 \in \mathcal{X}_0$ and $j \in \{1, \ldots, l\}$, then, for all $C, D \in \mathcal{X}_0$, matrix $C\, X_{i(1)}\, D\, X_{i(2)}$ can be expressed in the form $M\, X_j$ for some $M \in \mathcal{X}_0$.*

Proof. Since the multiplication of matrices is uniformly continuous on $\mathcal{X}_0$ (see (ii) in Definition 2.1) and since the multiplication of matrices is continuous on $\mathcal{M}at\,(F, m)$, the multiplication of matrices is uniformly continuous on $\mathcal{X}$ (the set of X_j is finite). Hence, there exists $\delta > 0$ such that

$$C_1\, X_{i(1)}\, D_1\, X_{i(2)} \in \mathcal{O}^{\varrho}_{\delta_{\mathcal{X}}}(C_0\, X_{i(1)}\, D_0\, X_{i(2)}) \quad \text{if} \quad C_1 \in \mathcal{O}^{\varrho}_{\delta}(C_0) \cap \mathcal{X},\ D_1 \in \mathcal{O}^{\varrho}_{\delta}(D_0) \cap \mathcal{X},$$

for each $i(1), i(2) \in \{1, \ldots, l\}$, where $\delta_{\mathcal{X}}$ is from Definition 2.14. This fact implies the lemma for matrices C, D (as C_1, D_1) from neighbourhoods of C_0, D_0. The radius δ of the neighbourhoods does not depend on the choices of C_0, D_0. Thus, it is valid

$$C_2\, X_{i(1)}\, D_2\, X_{i(2)} \in \mathcal{O}^{\varrho}_{\delta_{\mathcal{X}}}(C_1\, X_{i(1)}\, D_1\, X_{i(2)}) \quad \text{if} \quad C_2 \in \mathcal{O}^{\varrho}_{\delta}(C_1) \cap \mathcal{X},\ D_2 \in \mathcal{O}^{\varrho}_{\delta}(D_1) \cap \mathcal{X},$$

$$\vdots$$

$$C_j\, X_{i(1)}\, D_j\, X_{i(2)} \in \mathcal{O}^{\varrho}_{\delta_{\mathcal{X}}}(C_{j-1}\, X_{i(1)}\, D_{j-1}\, X_{i(2)}) \text{ if } C_j \in \mathcal{O}^{\varrho}_{\delta}(C_{j-1}) \cap \mathcal{X},\ D_j \in \mathcal{O}^{\varrho}_{\delta}(D_{j-1}) \cap \mathcal{X}.$$

Now, from Definition 2.1 and Lemma 2.29, it follows the statement for all $C, D \in \mathcal{X}_0$. □

Lemma 2.31. *Let $\mathcal{X}$ be weakly transformable and let $\mathcal{X}_0 \subseteq \mathcal{X}$, $X_1, \dots, X_l \in \mathcal{X}$ be from Definition 2.14. There exists $r \in \mathbb{N}$ with the property that*

$$C_1 \cdot X_i \cdot C_2 \cdot X_i \cdots C_r \cdot X_i \in \mathcal{X}_0 \quad \text{for all } C_1, C_2, \dots, C_r \in \mathcal{X}_0,\ i \in \{1, \dots, l\}.$$

Proof. Let $i \in \{1, \dots, l\}$ be arbitrarily given. Using Lemma 2.30 $(j-1)$-times (for arbitrary $j \in \mathbb{N}$), we obtain that

$$C_1 \cdot X_i \cdot C_2 \cdot X_i \cdots C_j \cdot X_i \in \mathcal{X}_0 \quad \text{for all } C_1, C_2, \dots, C_j \in \mathcal{X}_0$$

if and only if $X_i^j \in \mathcal{X}_0$. Indeed, it suffices to replace $C_1, \dots, C_j$ by $I, \dots, I$. For all $j \in \mathbb{N}$, one can express $X_i^j = C(j) \cdot X_{l(j)}$, where $l(j) \in \{1, \dots, l\}$ and $C(j) \in \mathcal{X}_0$. Evidently, there exist $j(1) > j(2)$ $(j(1), j(2) \in \mathbb{N})$ for which $l(j(1)) = l(j(2))$, i.e.,

$$X_i^{j(1)} = C(j(1)) \cdot X_{l(j(1))}, \quad X_i^{j(2)} = C(j(2)) \cdot X_{l(j(1))}.$$

Hence, we have

$$X_i^{j(1)-j(2)} = C(j(1)) \cdot (C(j(2)))^{-1} \in \mathcal{X}_0.$$

We see that, for each $i \in \{1, \dots, l\}$, there exists $r(i)$ such that $X_i^{r(i)} \in \mathcal{X}_0$. If $r \in \mathbb{N}$ is divisible by all $r(i)$, $i \in \{1, \dots, l\}$, then $X_i^r \in \mathcal{X}_0$, $i \in \{1, \dots, l\}$. The above equivalence completes the proof. □

Theorem 2.32. *Let $\mathcal{X}$ be weakly transformable and let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$, $\varepsilon > 0$ be arbitrarily given. If there exists a sequence $\{M_i\}_{i \in \mathbb{N}} \subseteq \mathcal{X}_0$ such that, for any non-zero vector $u \in F^m$, one can find $i = i(u) \in \mathbb{N}$ with the property that $M_i u \neq u$, then there exists $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ which does not have an almost periodic solution other than the trivial one.*

Proof. Let $\varepsilon < \delta_{\mathcal{X}}$ (see Definition 2.14) and let $n \in \mathbb{N}$ be an ε-translation number of $\{A_k\}$. Let us express (see again Definition 2.14)

$$A_k = C(A_k) \cdot X_{i(k)}, \quad C(A_k) \in \mathcal{X}_0,\ i(k) \in \{1, \dots, l\},$$

where $k \in \mathbb{Z}$. An arbitrary system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ has the feature of

$$B_k = C(B_k) \cdot X_{i(k)}, \quad C(B_k) \in \mathcal{X}_0,\ k \in \mathbb{Z}, \tag{2.49}$$

because $\varrho(A_k, B_k) < \varepsilon < \delta_{\mathcal{X}}$, $k \in \mathbb{Z}$. We also know that $i(k) = i(k + jn)$ for $k, j \in \mathbb{Z}$. Indeed, we have

$$\varrho(A_k, A_{k+n}) < \varepsilon < \delta_{\mathcal{X}}, \quad k \in \mathbb{Z},$$

i.e.,

$$\varrho\left(A_{k+jn}, A_{k+(j+1)n}\right) < \varepsilon < \delta_{\mathcal{X}}, \quad k, j \in \mathbb{Z}.$$

Thus, Lemma 2.30 and (2.49) imply that there exists $i \in \{1, \dots, l\}$ for which

$$B_{(j+1)n-1} \cdots B_{jn} = C(B_{(j+1)n-1} \cdots B_{jn}) \cdot X_i, \quad C(B_{(j+1)n-1} \cdots B_{jn}) \in \mathcal{X}_0,$$

where $j \in \mathbb{Z}$ and $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$. Furthermore, Lemma 2.31 gives $r \in \mathbb{N}$ with the property that

$$B_{(j+1)rn-1} \cdots B_{jrn} \in \mathcal{X}_0, \quad j \in \mathbb{Z},\ \{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\}). \tag{2.50}$$

The multiplication of matrices is uniformly continuous on $\mathcal{X}$ (see also the beginning of the proof of Lemma 2.30). Hence, for any $\vartheta > 0$, there exists $\delta(\vartheta) > 0$ such that

$$\varrho\left(C_1 \cdots C_{rn}, D_1 \cdots D_{rn}\right) < \vartheta \quad \text{if} \quad C_i, D_i \in \mathcal{X},\, C_i \in \mathcal{O}^{\varrho}_{\delta(\vartheta)}\left(D_i\right),\, i \in \{1, \ldots, rn\}. \tag{2.51}$$

Using (2.51), we obtain the inclusion

$$T(\{A_k\}, \delta(\vartheta)) \subseteq T(\{A_{k+rn-1} \cdots A_k\}, \vartheta), \quad \vartheta > 0,$$

which proves that $\{A_{k+rn-1} \cdots A_k\}_{k\in\mathbb{Z}}$ is almost periodic. Thus, $\{A_{(j+1)rn-1} \cdots A_{jrn}\}_{j\in\mathbb{Z}}$ is almost periodic as well. From (2.50) it follows that $\{A_{(j+1)rn-1} \cdots A_{jrn}\} \in \mathcal{AP}\left(\mathcal{X}_0\right)$.

Theorem 2.28 says that, in any neighbourhood of $\{A_{(j+1)rn-1} \cdots A_{jrn}\}$, there exists a system $\{B_{(j+1)rn-1} \cdots B_{jrn}\}_{j\in\mathbb{Z}} \in \mathcal{AP}\left(\mathcal{X}_0\right)$ which does not possess non-trivial almost periodic solutions. Let

$$\{B_{(j+1)rn-1} \cdots B_{jrn}\} \in \mathcal{O}^{\sigma}_{\delta(\varepsilon/2)/Q}\left(\{A_{(j+1)rn-1} \cdots A_{jrn}\}\right),$$

where $\delta(\varepsilon/2) > 0$ is from (2.51) for $rn = 2$ and $Q > 0$ is from Definition 2.1 for $\mathcal{X}_0$. Especially,

$$B_{(j+1)rn-1} \cdots B_{jrn} \in \mathcal{O}^{\varrho}_{\delta(\varepsilon/2)/Q}\left(A_{(j+1)rn-1} \cdots A_{jrn}\right) \cap \mathcal{X}_0, \quad j \in \mathbb{Z}. \tag{2.52}$$

For all $j \in \mathbb{Z}$, we can express

$$B_{(j+1)rn-1} \cdots B_{jrn} = A_{(j+1)rn-1} \cdots A_{jrn} \cdot \tilde{A}_j,$$

i.e.,

$$\tilde{A}_j = \left(A_{(j+1)rn-1} \cdots A_{jrn}\right)^{-1} \cdot B_{(j+1)rn-1} \cdots B_{jrn} \in \mathcal{X}_0. \tag{2.53}$$

We have (see also (2.52) and (iii) in Definition 2.1)

$$\tilde{A}_j \in \mathcal{O}^{\varrho}_{\delta(\varepsilon/2)}(I), \qquad \text{i.e.,} \qquad A_{jrn} \cdot \tilde{A}_j \in \mathcal{O}^{\varrho}_{\varepsilon/2}\left(A_{jrn}\right), \quad j \in \mathbb{Z}. \tag{2.54}$$

Now we show that the sequence $\{B_k\}_{k\in\mathbb{Z}}$, given by

$$B_{jrn} = A_{jrn} \cdot \tilde{A}_j, \quad B_{jrn+i} = A_{jrn+i}, \quad j \in \mathbb{Z},\, i \in \{1, \ldots, rn-1\}, \tag{2.55}$$

is almost periodic. Applying Corollary 1.10, we get that $\{B_k\}$ is almost periodic if and only if all sequences $\{A_{jrn} \cdot \tilde{A}_j\}_{j\in\mathbb{Z}}$, $\{A_{jrn+i}\}_{j\in\mathbb{Z}}$, $i \in \{1, \ldots, rn-1\}$, are almost periodic. Of course, the almost periodicity of each $\{A_{jrn+i}\}$ and $\{A_{jrn}\}$ is obvious. The sequence $\{[A_{jrn}, \tilde{A}_j]\}_{j\in\mathbb{Z}}$ in $\mathcal{M}at\,(F, m) \times \mathcal{M}at\,(F, m)$ is almost periodic (consider Corollary 1.11) if $\{\tilde{A}_j\}$ is almost periodic. Since we have

$$T(\{[A_{jrn}, \tilde{A}_j]\}, \delta(\vartheta)) \subseteq T(\{A_{jrn} \cdot \tilde{A}_j\}, \vartheta) \quad \text{for } \vartheta > 0$$

from (2.51) if $rn \geq 2$, the sequence $\{A_{jrn} \cdot \tilde{A}_j\}$ is almost periodic if $\{\tilde{A}_j\}$ is almost periodic. The fact that we can assume the almost periodicity of $\{\tilde{A}_j\}_{j\in\mathbb{Z}}$ follows from the proof of Theorem 2.28. There is constructed $\{B_k\}_{k\in\mathbb{Z}}$ as $B_k = A_k \cdot C_k$, $k \in \mathbb{Z}$, for an almost periodic sequence $\{C_k\}_{k\in\mathbb{Z}}$. If (2.32) is fulfilled, then it suffices to consider (2.53) as $C_k = A_k^{-1} \cdot B_k$, i.e., $B_{(j+1)rn-1}\cdots B_{jrn}$ as B_k and $A_{(j+1)rn-1}\cdots A_{jrn}$ as A_k. If (2.32) is not valid for any $\varepsilon > 0$, then it suffices to use Lemma 2.27.

Let us consider the general system

$$x_{j+1} = C_{(j+1)rn-1}\cdots C_{jrn}\cdot x_j, \quad j \in \mathbb{Z}. \tag{2.56}$$

Clearly, if $\{x_k\}_{k\in\mathbb{Z}}$ is a solution of $x_{k+1} = C_k\, x_k$, $k \in \mathbb{Z}$, then $\{x_{jrn}\}_{j\in\mathbb{Z}}$ is a solution of (2.56). Indeed, it is valid that

$$x_{k+rn} = \prod_{i=k+rn-1}^{k} C_i \cdot x_k, \quad k \in \mathbb{Z},$$

i.e.,

$$x_{(j+1)rn} = C_{(j+1)rn-1}\cdots C_{jrn}\cdot x_{jrn}, \quad j \in \mathbb{Z}.$$

One can easily show that $\{x_k\}_{k\in\mathbb{Z}}$ cannot be almost periodic if $\{x_{jrn}\}_{j\in\mathbb{Z}}$ is not almost periodic. Non-trivial solutions of $\{A_{(j+1)rn-1}\cdots A_{jrn}\cdot \tilde{A}_j\}_{j\in\mathbb{Z}}$ are not almost periodic. Thus, the system $\{B_k\}$ (defined in (2.55)) does not have an almost periodic solution other than the trivial one. Note that $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ follows from (2.54). □

The most important case is covered by the following corollary.

Corollary 2.33. *Let $\mathcal{X}$ be weakly transformable and have a dense countable subset. Let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrarily given. If for any vector $u \neq o$, $u \in F^m$, there exists $M(u) \in \mathcal{X}_0$ for which $M(u)\, u \neq u$, then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ which does not possess almost periodic solutions.*

Proof. Let X be a dense countable subset of $\mathcal{X}$. Since the operations $\oplus$ and $\odot$ are continuous with respect to ϱ, the multiplication of matrices and vectors on $\mathcal{M}at\,(F, m)$ and F^m is continuous as well. Thus, if $\varrho\,(M\,u, u) > 0$ for some $M \in \mathcal{X}$ and $u \in F^m$, then $\varrho\,(M_X\, u, u) > \varrho\,(M\,u, u)\,/2$ for some $M_X \in X$ from a neighbourhood of M. Now it suffices to use Theorem 2.32. □

Example 2.34. All concrete transformable and weakly transformable matrix groups, mentioned in Examples 2.2–2.10 and 2.15–2.17 (except $O(2) \oplus O_k(2)$), satisfy the conditions of Corollary 2.33. ◊

Applying the process used in the proof of Theorem 2.28 and considering Theorem 2.32 (by simple modifications), one can analogously prove the following theorem.

Theorem 2.35. *Let $\mathcal{X}$ be weakly transformable and let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$, $\varepsilon > 0$, and $u \in F^m$ be arbitrarily given. If there exists a matrix $M \in \mathcal{X}_0$ such that $M\,u \neq u$, then there exists $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ for which the solution of*

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u \tag{2.57}$$

is not almost periodic.

Remark 2.36. Let us illustrate the significance of Theorem 2.35. Let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrary. We consider the group $\mathcal{X}$ of complex matrices (with the usual metric) in the form

$$\begin{pmatrix} I & O \\ O & U \end{pmatrix},$$

where I is the $m_1 \times m_1$ identity matrix and U is a $m_2 \times m_2$ unitary matrix. This group is transformable, but it does not satisfy the condition of Theorem 2.32. Clearly, a system $\{B_k\} \in \mathcal{AP}(\mathcal{X})$ without almost periodic solutions does not exist. Of course, using Theorem 2.35 for arbitrarily given non-zero $u \in F^m$ satisfying $M\,u \neq u$ for some $M \in \mathcal{X}$, we obtain $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ with such a property that the solution of (2.57) is not almost periodic. In fact, using the above process (see also Corollary 2.33), one can construct a system $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ for which the solution of (2.57) is not almost periodic for all $u \in F^m$ such that $M\,u \neq u$ for some $M \in \mathcal{X}$; i.e., it is possible to obtain the same $\{B_k\}$ for all considered u. △

At the end, we say that it is possible to obtain various generalizations and modifications of results presented in this section. For simplicity, we consider only sufficiently general and, at the same time, important cases. Especially, in Chapter 1, the constructions are used for a ring with a pseudometric. For almost periodic sequences defined for $k \in \mathbb{N}_0$, it suffices to replace Corollary 1.5 by Remark 1.2 (see also [100]) and Corollary 1.28 by Theorem 1.23. Note that the basic theory of almost periodic sequences on $\mathbb{N}_0$ is established, e.g., in [57].

Chapter 3

Values of almost periodic and limit periodic sequences

The goal of this chapter is to find limit periodic and almost periodic sequences whose ranges consist of arbitrarily given sets. To find them, we apply a construction from Chapter 1. In addition, using a different construction, we obtain another result concerning values of limit periodic sequences. The obtained results are also used in the study of complex almost periodic (weakly) transformable difference systems (introduced in Chapter 2).

3.1 Preliminaries

As in the first chapter, let $\mathcal{X} \neq \emptyset$ be an arbitrary set and let $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a pseudometric on $\mathcal{X}$. For given $\varepsilon > 0$ and $x \in \mathcal{X}$, we define the ε-neighbourhood of x in $\mathcal{X}$ as the set $\{y \in \mathcal{X};\, d(x, y) < \varepsilon\}$. The ε-neighbourhood of x is denoted by $\mathcal{O}_\varepsilon(x)$.

3.2 Sequences with given values

In this section, we construct limit periodic (and almost periodic) sequences whose ranges consist of arbitrarily given sets satisfying only necessary conditions. We are motivated by paper [73], where a similar problem is investigated for real valued sequences. In that paper, using an explicit construction, it is shown that, for any bounded countable set of real numbers, there exists an almost periodic sequence whose range is this set and which attains each value in this set periodically. We extend this result to limit periodic sequences attaining values in $\mathcal{X}$.

Concerning almost periodic sequences with indices $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ (or asymptotically almost periodic sequences), we refer to [100], where it is proved that, for any precompact sequence $\{x_k\}_{k\in\mathbb{N}_0}$ in a metric space $\mathcal{X}$, there exists a permutation P of the set of non-negative integers such that the sequence $\{x_{P(k)}\}_{k\in\mathbb{N}_0}$ is almost periodic. Let us point out that the definition of the almost periodicity (in fact, asymptotic almost periodicity) in

[100] is based on the Bochner concept; i.e., a bounded sequence $\{x_k\}_{k\in\mathbb{N}_0}$ in $\mathcal{X}$ is called almost periodic if the set of sequences $\{x_{k+p}\}_{k\in\mathbb{N}_0}$, $p \in \mathbb{N}$, is precompact in the space of all bounded sequences in $\mathcal{X}$. We repeat that, for sequences with values in complete metric spaces, the Bochner definition is equivalent with the Bohr one which we prefer. Moreover, we know that these definitions remain also equivalent in an arbitrary pseudometric space if one replaces the convergence in the Bochner definition by the Cauchy property (see Theorem 1.3 and Corollary 1.5). However, it can be shown that the result of [100] for the almost periodicity on $\mathbb{N}_0$ cannot be true for the almost periodicity on $\mathbb{Z}$ or $\mathbb{R}$ (see also Remark 1.2).

In Banach spaces, another important necessary and sufficient condition for a function to be almost periodic is that it has the so-called approximation property; i.e., a function is almost periodic if and only if there exists a sequence of trigonometric polynomials which converges uniformly to the function on the whole real line in the norm topology (see, e.g., [46, Theorems 6.8, 6.15]). There exist generalizations of this result (see [38, 177]). For example, it is proved in [13] that an almost periodic function with fuzzy real numbers as values can be uniformly approximated by a sequence of generalized trigonometric polynomials. We add that fuzzy real numbers form a complete metric space. One shows that the approximation theorem is generally invalid if one does not require the completeness of the space of values. Thus, we cannot use this idea in our constructions for general pseudometric spaces.

We prove that, for a countable subset of $\mathcal{X}$, there exists a limit periodic sequence whose range is exactly this set. Since the range of any limit periodic sequence is totally bounded (see Theorems 1.14 and 1.22), this condition on the set is necessary. Now we prove that this condition is sufficient as well.

Theorem 3.1. *For any countable and totally bounded set $X \subseteq \mathcal{X}$, there exists a limit periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ satisfying*

$$\{\psi_k;\ k \in \mathbb{Z}\} = X \tag{3.1}$$

with the property that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ such that

$$\psi_l = \psi_{l+jq(l)}, \quad j \in \mathbb{Z}. \tag{3.2}$$

Proof. Let us put $X = \{\varphi_k;\ k \in \mathbb{N}\}$. Without loss of generality, we can assume that the set $\{\varphi_k;\ k \in \mathbb{N}\}$ is infinite because, for only finitely many different φ_k, we can define $\{\psi_k\}$ as periodic. Since $\{\varphi_k;\ k \in \mathbb{N}\}$ is totally bounded, for any $\varepsilon > 0$, it can be embedded into a finite number of spheres of radius ε. Let us denote by $x_1^i, \dots, x_{m(i)}^i$ the centres of the spheres of radius 2^{-i} which cover the set for all $i \in \mathbb{N}$. Evidently, we can also assume that

$$x_1^i, \dots, x_{m(i)}^i \in \{\varphi_k;\ k \in \mathbb{N}\}, \quad i \in \mathbb{N}, \tag{3.3}$$

and that

$$x_1^i = \varphi_i, \quad i \in \mathbb{N}. \tag{3.4}$$

We will construct $\{\psi_k\}$ applying Corollary 1.28. We choose arbitrary $n(1) \in \mathbb{N}$ for which $2^{2n(1)} > m(1)$. We put

$$\psi_0 := x_1^1,\ \psi_1 := x_2^1,\ \dots,\ \psi_{m(1)-1} := x_{m(1)}^1,$$

$$\psi_k := x_1^1, \quad k \in \{-2^{2n(1)-1} - \cdots - 2^3 - 2, \ldots, -1\} \cup \{m(1), \ldots, 2 + 2^2 + \cdots + 2^{2n(1)} - 1\},$$

and

$$\varepsilon_k := L, \quad k \in \{1, \ldots, 2n(1)+1\}, \tag{3.5}$$

where

$$L := \max_{i,j \in \{1,\ldots,m(1)\}} d\left(x_i^1, x_j^1\right) + 1.$$

In the second step, we choose $n(2) > n(1) + m(2)$ ($n(2) \in \mathbb{N}$). We define

$$\psi_k := \psi_{k+2^{2n(1)+1}}, \quad k \in \{-2^{2n(1)+1} - \cdots - 2^3 - 2, \cdots, -2^{2n(1)-1} - \cdots - 2^3 - 2 - 1\},$$

$$\psi_k := \psi_{k-2^{2n(1)+2}}, \quad k \in \{2 + 2^2 + \cdots + 2^{2n(1)}, \ldots, 2 + 2^2 + \cdots + 2^{2n(1)+2} - 1\},$$

$$\vdots$$

$$\psi_k := \psi_{k+2^{2n(2)-1}}, \quad k \in \{-2^{2n(2)-1} - \cdots - 2^3 - 2, \cdots, -2^{2n(2)-3} - \cdots - 2^3 - 2 - 1\},$$

and we put

$$\varepsilon_k := 0, \quad k \in \{2n(1)+2, \ldots, 2n(2)\}, \qquad \varepsilon_{2n(2)+1} := 2^{-1}. \tag{3.6}$$

Since $n(2) > n(1) + m(2)$, from the above definition of ψ_k, it follows that, for each $j \in \{1, \ldots, m(1)\}$, there exist at least $2m(2) + 2$ integers

$$l \in \{-2^{2n(2)-1} - \cdots - 2^3 - 2, \ldots, 2^{2n(2)-2} + \cdots + 2^2 + 2 - 1\}$$

such that $\psi_l = x_j^1$. Thus, we can define

$$\psi_k \in \mathcal{O}_{\varepsilon_{2n(2)+1}}\left(\psi_{k-2^{2n(2)}}\right), \quad k \in \{2 + 2^2 + \cdots + 2^{2n(2)-2}, \ldots, 2 + 2^2 + \cdots + 2^{2n(2)} - 1\},$$

with the property that

$$\{\psi_k;\, k \in \{2 + 2^2 + \cdots + 2^{2n(2)-2}, \ldots, 2 + 2^2 + \cdots + 2^{2n(2)-2} + 2^{2n(2)} - 1\}\}$$
$$= \{x_1^1, \ldots, x_{m(1)}^1, x_1^2, \ldots, x_{m(2)}^2\}.$$

In addition, we can put

$$\psi_{2^{2n(2)}} := \psi_0 = x_1^1 \tag{3.7}$$

and we can assume that

$$\psi_k = x_1^1 \quad \text{for some } k \in \{2 + \cdots + 2^{2n(2)-2}, \ldots, 2 + \cdots + 2^{2n(2)} - 1\} \setminus \{2^{2n(2)}\}.$$

In the third step, we choose $n(3) > n(2) + m(3)$ ($n(3) \in \mathbb{N}$) and we proceed analogously. We construct $\{\psi_k\}$ for

$$k \in \{-2^{2n(2)+1} - \cdots - 2^3 - 2, \ldots, -2^{2n(2)-1} - \cdots - 2^3 - 2 - 1\},$$

$$\vdots$$

$$k \in \{-2^{2n(3)-1} - \cdots - 2^3 - 2, \ldots, -2^{2n(3)-3} - \cdots - 2^3 - 2 - 1\}$$

as in the $2(n(2)+1)$-th, $\dots$, $2n(3)$-th steps of the process (mentioned in Corollary 1.28) for

$$\varepsilon_k := 0, \quad k \in \{2n(2)+2, \dots, 2n(3)\}. \tag{3.8}$$

Especially, we have

$$\psi_k = x_1^1, \quad k \in J_0^3 := \{j\, 2^{2n(2)};\ j \in \mathbb{Z}\} \cap \{-2^{2n(3)-1} - \dots - 2, \dots, 2 + \dots + 2^{2n(3)-2} - 1\}. \tag{3.9}$$

As in the second step, for all $j(1) \in \{1, 2\}$ and $j(2) \in \{1, \dots, m(j(1))\}$, there exist at least $2m(3)+2$ integers

$$l \in \{-2^{2n(3)-1} - \dots - 2^3 - 2, \dots, 2 + 2^2 + \dots + 2^{2n(3)-2} - 1\} \setminus \{j\, 2^{2n(2)};\ j \in \mathbb{Z}\}$$

such that $\psi_l = x_{j(2)}^{j(1)}$. It is seen that, to obtain

$$\psi_k \in \mathcal{O}_{\varepsilon_{2n(3)+1}}\left(\psi_{k-2^{2n(3)}}\right), \quad k \in \{2 + 2^2 + \dots + 2^{2n(3)-2}, \dots, 2 + 2^2 + \dots + 2^{2n(3)} - 1\},$$

where

$$\varepsilon_{2n(3)+1} := 2^{-2}, \tag{3.10}$$

satisfying

$$\{\psi_k;\ k \in \{2 + 2^2 + \dots + 2^{2n(3)-2}, \dots, 2 + 2^2 + \dots + 2^{2n(3)-2} + 2^{2n(3)} - 1\}\}$$
$$= \{x_1^1, \dots, x_{m(1)}^1, \dots, x_1^3, \dots, x_{m(3)}^3\},$$

we need less than (or equal to) $m(3)+1$ such integers l. Thus, we can define these ψ_k so that

$$\psi_k = x_1^1, \quad k \in I_0^3 := \{j\, 2^{2n(2)};\ j \in \mathbb{Z}\} \cap \{2 + \dots + 2^{2n(3)-2}, \dots, 2 + \dots + 2^{2n(3)} - 1\}, \tag{3.11}$$

$$\psi_{2^{2n(3)}+1} = \psi_1 = x_2^1, \quad \psi_{2^{2n(3)}-1} = \psi_{-1} = x_1^1 \tag{3.12}$$

and

$$\psi_k = \psi_1 \quad \text{for some } k \in \{2 + \dots + 2^{2n(3)-2}, \dots, 2 + \dots + 2^{2n(3)} - 1\} \setminus \{2^{2n(3)} + 1\},$$

$$\psi_k = \psi_{-1} \quad \text{for some } k \in \{2 + \dots + 2^{2n(3)-2}, \dots, 2 + \dots + 2^{2n(3)} - 1\} \setminus \{2^{2n(3)} - 1\}.$$

We proceed further in the same way. In the i-th step, we have $n(i) > n(i-1) + m(i)$ ($n(i) \in \mathbb{N}$) and

$$\psi_k := \psi_{k+2^{2n(i-1)+1}}, \quad k \in \{-2^{2n(i-1)+1} - \dots - 2, \cdots, -2^{2n(i-1)-1} - \dots - 2 - 1\},$$

$$\vdots$$

$$\psi_k := \psi_{k+2^{2n(i)-1}}, \quad k \in \{-2^{2n(i)-1} - \dots - 2, \cdots, -2^{2n(i)-3} - \dots - 2 - 1\},$$

and we denote

$$\varepsilon_k := 0, \quad k \in \{2n(i-1)+2, \dots, 2n(i)\}, \qquad \varepsilon_{2n(i)+1} := 2^{-i+1}. \tag{3.13}$$

We also have

$$\psi_k = \psi_0, \quad k \in J_0^i := \{j\, 2^{2n(2)};\ j \in \mathbb{Z}\} \cap \{-2^{2n(i)-1} - \cdots - 2, \dots, 2 + \cdots + 2^{2n(i)-2} - 1\}, \tag{3.14}$$

$$\begin{aligned} &\psi_k = \psi_1, \quad k \in J_1^i, \\ &J_1^i := \{1 + j\, 2^{2n(3)};\ j \in \mathbb{Z}\} \cap \{-2^{2n(i)-1} - \cdots - 2, \dots, 2 + \cdots + 2^{2n(i)-2} - 1\}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\psi_k = \psi_{-1}, \quad k \in J_{-1}^i, \\ &J_{-1}^i := \{-1 + j\, 2^{2n(3)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}, \end{aligned} \tag{3.16}$$

$$\vdots$$

$$\begin{aligned} &\psi_k = \psi_{i-3}, \quad k \in J_{i-3}^i, \\ &J_{i-3}^i := \{i - 3 + j\, 2^{2n(i-1)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}, \\ &\psi_k = \psi_{-i+3}, \quad k \in J_{-i+3}^i, \\ &J_{-i+3}^i := \{-i + 3 + j\, 2^{2n(i-1)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}, \end{aligned}$$

if $i - 3 < 2^{2n(2)}$. If $2^{2n(2)} \le i - 3 < 2^{2n(2)+1}$, we have

$$\vdots$$

$$\begin{aligned} &\psi_k = \psi_{-2^{2n(2)}+1}, \quad k \in J_{-2^{2n(2)}+1}^i, \\ &J_{-2^{2n(2)}+1}^i := \{-2^{2n(2)} + 1 + j\, 2^{2n(2^{2n(2)}+1)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}, \\ &\psi_k = \psi_{2^{2n(2)}+1}, \quad k \in J_{2^{2n(2)}+1}^i, \\ &J_{2^{2n(2)}+1}^i := \{2^{2n(2)} + 1 + j\, 2^{2n(2^{2n(2)}+2)};\ j \in \mathbb{Z}\} \\ &\qquad\qquad \cap \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\}, \end{aligned}$$

$$\vdots$$

If $2^{2n(2)+1} \le i-3$, then we omit the values $\psi_{j\, 2^{2n(2)}}, \psi_{1+j\, 2^{2n(3)}}, \psi_{-1+j\, 2^{2n(3)}}, \dots$ For simplicity, let $i - 2 < 2^{2n(2)}$.

Considering the construction, for all $j(1) \in \{1, \dots, i-1\}$, $j(2) \in \{1, \dots, m(j(1))\}$, there exist at least $2m(i) + 2$ integers

$$l \in \{-2^{2n(i)-1} - \cdots - 2^3 - 2, \dots, 2 + 2^2 + \cdots + 2^{2n(i)-2} - 1\} \setminus (J_0^i \cup \cdots \cup J_{-i+3}^i)$$

such that $\psi_l = x_{j(2)}^{j(1)}$. Evidently (similarly as in the third step), we can obtain

$$\psi_k \in \mathcal{O}_{\varepsilon_{2n(i)+1}}(\psi_{k-2^{2n(i)}}), \quad k \in \{2 + \cdots + 2^{2n(i)-2}, \dots, 2 + \cdots + 2^{2n(i)} - 1\},$$

for which

$$\begin{aligned}\{\psi_k;\, k \in \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)-2} + 2^{2n(i)} - 1\}\} \\ = \{x_1^1, \ldots, x_{m(1)}^1, \ldots, x_1^i, \ldots, x_{m(i)}^i\}\end{aligned} \tag{3.17}$$

and, in addition, we have

$$\begin{aligned}&\psi_k = \psi_0, \quad k \in I_0^i, \\ &I_0^i := \{j\, 2^{2n(2)};\, j \in \mathbb{Z}\} \cap \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\},\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\psi_k = \psi_1, \quad k \in I_1^i, \\ &I_1^i := \{1 + j\, 2^{2n(3)};\, j \in \mathbb{Z}\} \cap \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\},\end{aligned} \tag{3.19}$$

$$\begin{aligned}&\psi_k = \psi_{-1}, \quad k \in I_{-1}^i, \\ &I_{-1}^i := \{-1 + j\, 2^{2n(3)};\, j \in \mathbb{Z}\} \cap \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\},\end{aligned} \tag{3.20}$$

$$\vdots$$

$$\begin{aligned}&\psi_k = \psi_{i-3}, \quad k \in I_{i-3}^i, \\ &I_{i-3}^i := \{i - 3 + j\, 2^{2n(i-1)};\, j \in \mathbb{Z}\} \cap \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\}, \\ &\psi_k = \psi_{-i+3}, \quad k \in I_{-i+3}^i, \\ &I_{-i+3}^i := \{-i + 3 + j\, 2^{2n(i-1)};\, j \in \mathbb{Z}\} \cap \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\},\end{aligned}$$

and

$$\psi_k = \psi_{i-2}, \quad k = 2^{2n(i)} + i - 2, \qquad \psi_k = \psi_{-i+2}, \quad k = 2^{2n(i)} - i + 2,$$

$$\psi_k = \psi_{i-2} \quad \text{for some } k \in \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\} \setminus \{2^{2n(i)} + i - 2\},$$

$$\psi_k = \psi_{-i+2} \quad \text{for some } k \in \{2 + \cdots + 2^{2n(i)-2}, \ldots, 2 + \cdots + 2^{2n(i)} - 1\} \setminus \{2^{2n(i)} - i + 2\}.$$

Using this construction, we get the sequence $\{\psi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ with the property that (see (3.7), (3.9), (3.11), (3.14), (3.18))

$$\psi_k = \psi_0, \quad k \in \{j\, 2^{2n(2)};\, j \in \mathbb{Z}\},$$

and that (see (3.12), (3.15), (3.16), (3.19), (3.20))

$$\psi_k = \psi_1, \quad k \in \{1 + j\, 2^{2n(3)};\, j \in \mathbb{Z}\},$$

$$\psi_k = \psi_{-1}, \quad k \in \{-1 + j\, 2^{2n(3)};\, j \in \mathbb{Z}\},$$

and so on; i.e., for any $l \in \mathbb{Z}$, there exists $i(l) \in \mathbb{N}$ satisfying

$$\psi_k = \psi_l, \quad k \in \{l + j\, 2^{2n(i(l))};\, j \in \mathbb{Z}\}. \tag{3.21}$$

Now it suffices to show that the sequence $\{\psi_k\}$ is limit periodic. Indeed, (3.1) follows from the process, (3.3), (3.4), and (3.17); (3.2) follows from (3.21) for $q(l) = 2^{2n(i(l))}$. Since we construct $\{\psi_k\}$ using Corollary 1.28, $\{\psi_k\}$ is limit periodic if (1.25) is satisfied. Considering (3.5), (3.6), (3.8), (3.10), and (3.13), we have

$$\sum_{i=1}^{\infty} \varepsilon_i = L\,(2n(1) + 1) + 1 \tag{3.22}$$

which completes the proof. $\square$

From Theorem 3.1, we immediately obtain the following result concerning almost periodic sequences.

Corollary 3.2. *For any countable and totally bounded set $X \subseteq \mathcal{X}$, there exists an almost periodic sequence $\{\psi_k\}_{k\in\mathbb{Z}}$ satisfying* (3.1) *with the property that, for any $l \in \mathbb{Z}$, there exists $q(l) \in \mathbb{N}$ such that* (3.2) *is valid.*

Remark 3.3. We can mention several other corollaries which follow from Theorem 3.1, when the limit periodicity of $\{\psi_k\}$ is replaced by a concrete type of almost periodicity in the statement. For example, one can consider almost automorphic sequences which generalize classical almost periodicity and which have totally bounded ranges as well (see, e.g., [79, Definition 3.2 and Theorem 3.3, (v)]). △

Before the formulation of the next theorem, we add that a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathcal{X}$ is dense in itself if for any $k \in \mathbb{N}$ and $\varepsilon > 0$, there exist infinitely many $x \in \{x_k;\, k \in \mathbb{N}\}$ with the property that $d(x_k, x) < \varepsilon$.

Theorem 3.4. *Let a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathcal{X}$ be totally bounded and dense in itself. There exists an injective limit periodic sequence $\{f_k\}_{k\in\mathbb{Z}}$ satisfying*

$$\{f_k;\, k \in \mathbb{Z}\} = \{x_k;\, k \in \mathbb{N}\}\,. \tag{3.23}$$

Proof. Without loss of generality, we can assume that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is injective. We put $X := \{x_k;\, k \in \mathbb{N}\}$. We know that, for any $n \in \mathbb{N}$, there exists an odd integer $m = m(n) \geq n$ such that

$$\min_{i\in\{1,\dots,m\}} d\,(x_i, x_l) < \frac{1}{2^n}, \quad l \in \mathbb{N}. \tag{3.24}$$

In the first step, let us consider the periodic sequence $\{f^1_k\}_{k\in\mathbb{Z}}$ given by values

$$f^1_0 = x_1, f^1_1 = x_2, f^1_{-1} = x_3, \dots, f^1_{m(1)} = x_{2m(1)}, f^1_{-m(1)} = x_{2m(1)+1}$$

and period $2m(1) + 1$. In the second step, we define a sequence $\{f^2_k\}_{k\in\mathbb{Z}} \subset X$ with period $[2m(1) + 1]m(2)$ so that

$$f^2_{-m(1)} = f^1_{-m(1)}, \dots, f^2_0 = f^1_0, \dots, f^2_{m(1)} = f^1_{m(1)},$$

$$\left\{x_1, \dots, x_{m(2)}\right\} \subset \left\{f^2_l;\, l \in \{1, \dots, [2m(1) + 1]\, m(2)\}\right\}, \tag{3.25}$$

$$d\left(f^2_l, f^1_l\right) < \frac{1}{2}, \quad l \in \{1, \dots, [2m(1) + 1]\, m(2)\}, \tag{3.26}$$

$$f^2_i \neq f^2_j, \quad i \neq j,\, i, j \in \{1, \dots, [2m(1) + 1]\, m(2)\}. \tag{3.27}$$

We can find such a sequence $\{f^2_k\}_{k\in\mathbb{Z}}$. Conditions (3.25), (3.26) follow from (3.24) and (3.27) can be satisfied because, in any neighbourhood of each considered value x_i, there are infinitely many values from X.

We proceed further in the same way. In the n-th step, we define a sequence $\{f_k^n\}_{k\in\mathbb{Z}} \subset X$ with period $[2m(1)+1]\, m(2)\cdots m(n)$ arbitrarily so that

$$f^n_{\frac{-[2m(1)+1]m(2)\cdots m(n-1)+1}{2}} = f^{n-1}_{\frac{-[2m(1)+1]m(2)\cdots m(n-1)+1}{2}}, \dots, f_0^n = f_0^{n-1}, \dots$$
$$\dots, f^n_{\frac{[2m(1)+1]m(2)\cdots m(n-1)-1}{2}} = f^{n-1}_{\frac{[2m(1)+1]m(2)\cdots m(n-1)-1}{2}}$$

and

$$\left\{x_1, \dots, x_{m(n)}\right\} \subset \left\{f_l^n;\, l \in \{1, \dots, [2m(1)+1]\, m(2)\cdots m(n)\}\right\}, \tag{3.28}$$

$$d\left(f_l^n, f_l^{n-1}\right) < \frac{1}{2^{n-1}}, \quad l \in \{1, \dots, [2m(1)+1]\, m(2)\cdots m(n)\}, \tag{3.29}$$

$$f_i^n \neq f_j^n, \quad i \neq j,\, i, j \in \{1, \dots, [2m(1)+1]\, m(2)\cdots m(n)\}. \tag{3.30}$$

We put

$$f_k := \lim_{n\to\infty} f_k^n = f_k^{|k|+j}, \quad k \in \mathbb{Z},\, j \in \mathbb{N}.$$

We have (see (3.29))

$$d\left(f_k, f_k^n\right) \le d\left(f_k^n, f_k^{n+1}\right) + d\left(f_k^{n+1}, f_k^{n+2}\right) + \cdots < \frac{1}{2^{n-1}}, \quad k \in \mathbb{Z},\, n \in \mathbb{N}.$$

Thus, the sequence $\{f_k\}_{k\in\mathbb{Z}}$ is limit periodic. Since

$$f_l = f_l^n, \quad l \in \left\{\frac{-[2m(1)+1]\, m(2)\cdots m(n)+1}{2}, \dots, 0, \right.$$
$$\left. \dots, \frac{[2m(1)+1]\, m(2)\cdots m(n)-1}{2}\right\}, \quad n \in \mathbb{N} \setminus \{1\},$$

from (3.30), we obtain

$$f_i \neq f_j, \quad i \neq j,\, i, j \in \left\{\frac{-[2m(1)+1]\, m(2)\cdots m(n)+1}{2}, \dots, 0, \right.$$
$$\left. \dots, \frac{[2m(1)+1]\, m(2)\cdots m(n)-1}{2}\right\}, \quad n \in \mathbb{N} \setminus \{1\},$$

i.e., we have ($m(n) \to \infty$ for $n \to \infty$)

$$f_i \neq f_j, \quad i \neq j,\, i, j \in \mathbb{Z}.$$

It remains to prove that $\{f_k\}$ satisfies (3.23). Of course, this fact can be easily shown considering (3.28). □

Again, we obtain a new result in the almost periodic case.

Corollary 3.5. *Let a sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathcal{X}$ be totally bounded and dense in itself. There exists an injective almost periodic sequence $\{f_k\}_{k\in\mathbb{Z}}$ satisfying* (3.23).

Remark 3.6. For any sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathcal{X}$ which is not dense in itself, Theorem 3.4 cannot be valid. Indeed, if there exist $l \in \mathbb{N}$ and $\varepsilon > 0$ such that the intersection of the ε-neighbourhood of x_l and the set $\{x_k;\, k \in \mathbb{N}\}$ contains only a finite number of elements of $\{x_k;\, k \in \mathbb{N}\}$, then any almost periodic sequence $\{f_k\}_{k\in\mathbb{Z}}$ satisfying (3.23) attains a value from this neighbourhood infinitely many times (consider directly Definition 1.1). △

At the end of this section, we assume that $\mathcal{X}$ is a uniform space with a non-empty family $\mathcal{U}$ of entourages. We generalize the definition of limit periodicity from Chapter 1 as follows.

Definition 3.7. A sequence $\{f_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ is called *limit periodic* if it is the uniform limit of periodic sequences $\{f_k^n\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$, $n \in \mathbb{N}$, i.e., for any $U \in \mathcal{U}$, there exists $n_0 \in \mathbb{N}$ such that $(f_k, f_k^n) \in U$ for all $k \in \mathbb{Z}$, $n \geq n_0$, $n \in \mathbb{N}$.

We can also formulate the corresponding generalizations of almost periodicity and asymptotic almost periodicity. Especially (see, e.g., [10]), a continuous multivalued map $f : \mathbb{R} \to \mathcal{X}$ is said to be almost periodic if for any entourage $U \in \mathcal{U}$, there exists a positive number $p \in \mathbb{R}$ such that every real interval of length p contains a number s for which $f(x+s)$ is in the U-neighbourhood of $f(x)$ and intersects the U-neighbourhood of all $y \in f(x)$, $x \in \mathbb{R}$. For the fundamental properties of this type of almost periodic functions, we refer to [10, 11, 109, 158] (and also [50, 51, 139]).

If there exists a countable fundamental system of entourages of $\mathcal{X}$, then the uniform structure of $\mathcal{X}$ can be defined by a pseudometric (see, e.g., [107, Theorem 13 on p. 186] and references cited therein). Hence, we have:

Theorem 3.8. *The statement of Theorem 3.1 remains true if $\mathcal{X}$ is a uniform space with a countable fundamental system of entourages.*

Considering Theorem 3.4, we also obtain the following result.

Theorem 3.9. *Let $\mathcal{X}$ have a countable fundamental system of entourages and let a totally bounded sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathcal{X}$ be such that, for any $k \in \mathbb{N}$ and any entourage $U \in \mathcal{U}$, there exist infinitely many $x \in \{x_k;\, k \in \mathbb{N}\}$ with the property that $(x_k, x) \in U$. Then, there exists an injective limit periodic sequence $\{f_k\}_{k\in\mathbb{Z}}$ satisfying* (3.23).

3.3 Applications related to almost periodic difference systems

Let $m \in \mathbb{N}$ be arbitrarily given. We consider m-dimensional homogeneous linear difference systems of the form

$$y_{k+1} = A_k \cdot y_k, \quad k \in \mathbb{Z},$$

where $\{A_k\}$ is an almost periodic sequence of matrices from a given group X of $m \times m$ matrices with complex elements. Of course, in $\mathbb{C}$, we consider the usual metric d given by the absolute value of difference. This metric induces the metric on the set $\mathbb{C}^m$ of all $m \times 1$ complex vectors and on the set $\mathbb{C}^{m\times m}$ of all $m \times m$ complex matrices as the sum of m and m^2 non-negative numbers, respectively. For the sake of simplicity and convenience, these metrics are denoted by d as well. Let $I \in \mathbb{C}^{m\times m}$ be the identity matrix.

Lemma 3.10. *Let $X \subset \mathbb{C}^{m\times m}$ be a bounded group. If $\{f_k\}_{k\in\mathbb{Z}} \subseteq X$ is almost periodic, then the sequence $\{f_k^{-1}\}_{k\in\mathbb{Z}}$ of the inverse matrices is almost periodic as well.*

Proof. For an arbitrary matrix norm (especially, for the ℓ_1-norm which corresponds to d) denoted by $\|\cdot\|$, we have (2.3) for any matrices A, E such that A is non-singular and $\| A^{-1} E \| < 1$. For the bounded group X, (2.3) implies that the map $C \mapsto C^{-1}$ has the Lipschitz property on X. Hence, the almost periodicity of $\{f_k^{-1}\}_{k\in\mathbb{Z}}$ is guaranteed by Theorem 1.6. □

Theorem 3.11. *Let $X \subset \mathbb{C}^{m\times m}$ be a bounded group. There exists an almost periodic sequence $\{C_k\}_{k\in\mathbb{Z}} \subseteq X$ satisfying $\overline{\{C_k;\, k \in \mathbb{Z}\}} = X$ such that all solutions of the system*

$$y_{k+1} = C_k \cdot y_k, \quad k \in \mathbb{Z}, \tag{3.31}$$

are almost periodic.

Proof. It is seen that every bounded group of complex matrices has a dense countable subgroup. Thus, we can assume that $X = \{x_k;\, k \in \mathbb{N}\}$. We apply Corollary 3.2. There exists an almost periodic sequence $\{f_k\}_{k\in\mathbb{Z}}$ satisfying (3.23). From Corollary 1.10 and Lemma 3.10, it follows that the sequence

$$C_{2k} := f_k, \quad C_{2k+1} := f_k^{-1}, \quad k \in \mathbb{Z}, \tag{3.32}$$

is almost periodic as well. Note that f_k^{-1} denotes the inverse matrix of f_k and that $f_k^{-1} = f_j$ for some $j = j(k) \in \mathbb{Z}$. The principal fundamental matrix $\{\Phi_k\}_{k\in\mathbb{Z}}$ of the system (3.31) given by (3.32) attains values

$$\Phi(0) = I, \Phi(1) = f_0, \Phi(2) = I, \quad \dots \quad \Phi(2n) = I, \Phi(2n+1) = f_n, \quad \dots$$

$$\Phi(-1) = f_{-1}, \Phi(-2) = I, \quad \dots \quad \Phi(-2n) = I, \Phi(-2n-1) = f_{-n-1}, \quad \dots$$

The almost periodicity of $\{\Phi_k\}$ is equivalent to the almost periodicity of $\{f_k\}$ (see directly Definition 1.1). □

To obtain counterparts of Theorem 3.11 (the below given Theorems 3.12 and 3.13), we use Corollary 2.33 and Theorem 2.35.

Theorem 3.12. *Let $X \subset \mathbb{C}^{m\times m}$ be a weakly transformable group. If there exists a matrix $M(u) \in X_0$ for any non-zero vector $u \in \mathbb{C}^m$ such that $M(u)\, u \neq u$, then there exists an almost periodic sequence $\{D_k\}_{k\in\mathbb{Z}} \subseteq X$ satisfying $\overline{\{D_k;\, k \in \mathbb{Z}\}} = X$ for which the system*

$$y_{k+1} = D_k \cdot y_k, \quad k \in \mathbb{Z}, \tag{3.33}$$

does not have any non-trivial almost periodic solution.

Proof. Evidently, in the complex case, any weakly transformable group has a dense countable weakly transformable subgroup. Hence, as in the proof of Theorem 3.11, we can assume that $X = \{x_k;\, k \in \mathbb{N}\}$ and we can apply Corollary 3.2. We obtain an almost periodic sequence $\{f_k\}_{k\in\mathbb{Z}}$ with the property that $\{f_k;\, k \in \mathbb{Z}\} = \{x_k;\, k \in \mathbb{N}\}$.

Let $\{B_k\}_{k\in\mathbb{Z}} \subseteq \{x_k;\, k \in \mathbb{N}\}$ be an arbitrary almost periodic sequence mentioned in the statement of Corollary 2.33. We put

$$D_{3k} := f_k, \quad D_{3k+1} := f_k^{-1}, \quad D_{3k+2} := B_k, \quad k \in \mathbb{Z}. \tag{3.34}$$

Corollary 1.10 and Lemma 3.10 give the almost periodicity of $\{D_k\}_{k\in\mathbb{Z}}$. For the principal fundamental matrix $\{\Psi_k\}_{k\in\mathbb{Z}}$ of the system (3.33) determined by (3.34), we have

$$\Psi(0) = I, \quad \Psi(1) = f_0, \quad \Psi(2) = I, \quad \Psi(3) = B_0,$$

$$\vdots$$

$$\begin{aligned} \Psi(3n) &= B_{n-1} \cdots B_1 \cdot B_0, \\ \Psi(3n+1) &= f_n \cdot B_{n-1} \cdots B_1 \cdot B_0, \\ \Psi(3n+2) &= B_{n-1} \cdots B_1 \cdot B_0, \end{aligned}$$

$$\vdots$$

and

$$\Psi(-1) = B_{-1}^{-1}, \quad \Psi(-2) = f_{-1} \cdot B_{-1}^{-1}, \quad \Psi(-3) = B_{-1}^{-1},$$

$$\vdots$$

$$\begin{aligned} \Psi(-3n+2) &= \left(B_{-1} \cdot B_{-2} \cdots B_{-n}\right)^{-1}, \\ \Psi(-3n+1) &= f_{-n} \cdot \left(B_{-1} \cdot B_{-2} \cdots B_{-n}\right)^{-1}, \\ \Psi(-3n) &= \left(B_{-1} \cdot B_{-2} \cdots B_{-n}\right)^{-1}, \end{aligned}$$

$$\vdots$$

From Corollary 2.33, we know that the sequence $\{\Psi_{3k}\, u\}_{k\in\mathbb{Z}}$ is not almost periodic for any non-zero vector $u \in \mathbb{C}^m$. Thus, the sequence $\{\Psi_k\, u\}_{k\in\mathbb{Z}}$ cannot be almost periodic. □

Analogously, using Theorem 2.35, one can prove:

Theorem 3.13. *Let $X \subset \mathbb{C}^{m\times m}$ be a weakly transformable group and $u \in \mathbb{C}^m$ be an arbitrary non-zero vector. If there exists a matrix $M(u) \in X_0$ such that $M(u)\, u \neq u$, then there exists an almost periodic sequence $\{D_k\}_{k\in\mathbb{Z}} \subseteq X$ satisfying $\overline{\{D_k;\, k \in \mathbb{Z}\}} = X$ for which the solution of*

$$x_{k+1} = D_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is not almost periodic.

From the proofs of Theorems 3.11 and 3.12, we also get the following theorems which finish this chapter. Note that we partially study almost periodic systems in Chapter 4 as well.

Theorem 3.14. *For any countable bounded group $X \subset \mathbb{C}^{m\times m}$, there exists an almost periodic sequence $\{C_k\}_{k\in\mathbb{Z}}$ satisfying $\{C_k;\, k \in \mathbb{Z}\} = X$ such that all solutions of the system*

$$y_{k+1} = C_k \cdot y_k, \quad k \in \mathbb{Z},$$

are almost periodic.

Theorem 3.15. *Let $X \subset \mathbb{C}^{m\times m}$ be a countable weakly transformable group and let there exist a matrix $M(u) \in X_0$ for any non-zero vector $u \in \mathbb{C}^m$ such that $M(u)\, u \neq u$. There exists an almost periodic sequence $\{D_k\}_{k\in\mathbb{Z}}$ satisfying $\{D_k;\, k \in \mathbb{Z}\} = X$ for which the system*

$$y_{k+1} = D_k \cdot y_k, \quad k \in \mathbb{Z},$$

does not have any non-trivial almost periodic solution.

Theorem 3.16. *Let $X \subset \mathbb{C}^{m\times m}$ be a countable weakly transformable group and let $u \in \mathbb{C}^m$ be an arbitrary non-zero vector. If there exists a matrix $M(u) \in X_0$ such that $M(u)\, u \neq u$, then there exists an almost periodic sequence $\{D_k\}_{k\in\mathbb{Z}}$ satisfying $\{D_k;\, k \in \mathbb{Z}\} = X$ for which the solution of*

$$x_{k+1} = D_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is not almost periodic.

Chapter 4

Solutions of limit periodic difference systems

Now we consider limit periodic homogeneous linear difference systems of the form

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z}. \tag{4.1}$$

Limit periodic systems (4.1) form the smallest class of systems which generalize pure periodic systems in the form (4.1) and which can have at least one non-almost periodic solution (for complex matrices A_k from a bounded group, cf. Corollary 2.22). The basic motivation comes from Chapter 2, where we study non-almost periodic solutions of almost periodic systems. In this chapter, we improve the main results of Chapter 2 in a certain sense, because we analyse non-almost periodic solutions in the limit periodic case. In addition, we obtain results about non-asymptotically almost periodic solutions. Since the methods used in this chapter are substantially different from the process from Chapter 1 applied in Chapter 2, we also obtain new results for almost periodic systems.

The necessity of generalizations of periodic mathematical models is implied by various oscillatory phenomena in natural sciences. The models induce the research of limit periodic, almost periodic, and asymptotically almost periodic sequences in connection with difference equations. There are many significant books dealing with (asymptotically) almost periodic solutions of difference and differential equations, e.g., [32, 39, 46, 72, 183]. We also refer to references given in these books.

In this chapter, we use methods based on constructions of limit periodic sequences. A similar method was firstly applied in [78], where non-almost periodic solutions of homogeneous linear difference equations are found as classes of constructible sequences. A method of constructions of minimal cocycles, which are obtained as solutions of homogeneous linear differential systems, is described in [141] (see also [140]). Special constructions of homogeneous linear differential systems with almost periodic coefficients are used in [124, 125] as well.

This chapter is organized as follows. We begin with the used notations. In Section 4.2, we introduce properties (denoted by P and P^*) which allow us to improve the results of Chapter 2 for bounded groups of coefficient matrices in Section 4.3. In Section 4.4, the coefficient matrices of the considered systems belong to a given commutative group. We find a condition on the group under which the systems, whose fundamental matrices are not almost periodic, form an everywhere dense subset in the space of all considered systems. The treated problems are discussed for the elements of the coefficient matrices from an arbitrary infinite field with an absolute value. Nevertheless, the presented results are new even for the field of complex numbers.

4.1 Preliminaries

Let $(F, \oplus, \odot)$ be an infinite field with a zero e_0 and a unit e_1. Let $|\cdot| : F \to \mathbb{R}$ be an absolute value on F; i.e., let

(a) $|f| \geq 0$ and $|f| = 0$ if and only if $f = e_0$,

(b) $|f \odot g| = |f| \cdot |g|$,

(c) $|f \oplus g| \leq |f| + |g|$

for all $f, g \in F$. As $\mathbb{F}$, we understand each one of the fields $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ with the usual absolute value. For arbitrary $p \in \mathbb{N}$, we put $p\mathbb{N} := \{pj;\ j \in \mathbb{N}\}$. We remark that $\mathrm{i} \in \mathbb{C}$ stands for the imaginary unit.

Let $m \in \mathbb{N}$ be arbitrarily given (as the dimension of systems under consideration). Symbol $Mat(F, m)$ denotes the set of all $m \times m$ matrices with elements from F and F^m the set of all $m \times 1$ vectors with entries from F. Next, $\cdot$, $+$ stand for the multiplication and the addition on spaces $Mat(F, m)$, F^m. As usual, we denote the identity matrix $I \in Mat(F, m)$, the zero matrix $O \in Mat(F, m)$, and the zero vector $o \in F^m$.

The absolute value on F gives the norm on F^m and $Mat(F, m)$ as the sum of m and m^2 non-negative numbers which are the absolute values of the elements, respectively. For simplicity, both of the norms are denoted by $\|\cdot\|$. Especially (consider (a), (b), (c)), we have

(A) $\|M\| \geq 0$ and $\|M\| = 0$ if and only if $M = O$,

(B) $\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = o$,

(C) $\|M + N\| \leq \|M\| + \|N\|$,

(D) $\|u + v\| \leq \|u\| + \|v\|$,

(E) $\|M \cdot N\| \leq \|M\| \cdot \|N\|$,

(F) $\|M \cdot u\| \leq \|M\| \cdot \|u\|$

for all $M, N \in Mat(F, m)$, $u, v \in F^m$. Henceforth, we will use properties (A), (B), (C), (D), (E), (F) without emphasizing.

The absolute value on F and the norms on F^m, $Mat(F, m)$ induce the metrics. For the sake of convenience, we denote each one of these metrics by ϱ. The ε-neighbourhood of α is denoted by $\mathcal{O}^{\varrho}_{\varepsilon}(\alpha)$ in all above mentioned spaces with metric ϱ. In addition, we assume that the valued field F (with $|\cdot|$) is separable. Hence, all considered spaces are separable. We remark that metric space (F, ϱ) does not need to be complete.

4.2 General homogeneous linear difference systems

Let $\mathcal{X} \subset Mat(F, m)$ be a group. We study the homogeneous linear difference systems

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad \text{where} \quad \{A_k\} \subseteq \mathcal{X}. \tag{4.2}$$

Let $\mathcal{P}(\mathcal{X})$, $\mathcal{LP}(\mathcal{X})$, and $\mathcal{AP}(\mathcal{X})$ denote the set of all systems (4.2) for which the sequence of matrices A_k is periodic, limit periodic, and almost periodic, respectively. Note that we identify the sequence $\{A_k\}$ with the system (4.2) which is determined by $\{A_k\}$. In $\mathcal{AP}(\mathcal{X})$, we define the metric

$$\sigma(\{A_k\}, \{B_k\}) := \sup_{k \in \mathbb{Z}} \varrho(A_k, B_k), \quad \{A_k\}, \{B_k\} \in \mathcal{AP}(\mathcal{X}).$$

Symbol $\mathcal{O}^\sigma_\varepsilon(\{A_k\})$ stands for the ε-neighbourhood of $\{A_k\}$ in $\mathcal{AP}(\mathcal{X})$.

To study limit periodic systems of the form (4.2), we introduce properties of $\mathcal{X}$ denoted by P and P^* as follows.

Definition 4.1. We say that $\mathcal{X}$ has property P if for every $a > 0$ and $\delta > 0$, there exist $\zeta(a) > 0$ and $l = l(a, \delta) \in \mathbb{N}$ such that, for any vector $u \in F^m$ satisfying $\| u \| > a$, one can find matrices $M_1, \dots, M_l \in \mathcal{X}$ for which

$$M_1 \in \mathcal{O}^\varrho_\delta(I), \quad M_i \in \mathcal{O}^\varrho_\delta(M_{i+1}),\ i \in \{1, \dots, l-1\}, \quad \| M_l \cdot u - u \| > \zeta(a).$$

Definition 4.2. We say that $\mathcal{X}$ has property P^* if for any $a > 0$ and $\delta > 0$, there exist $M(a) \in \mathcal{X}$, $\zeta(a) > 0$, and $l = l(a, \delta) \in \mathbb{N}$ such that, for any $N \in \mathcal{X}$, one can find matrices $M_1, \dots, M_l \in \mathcal{X}$ satisfying

$$M_1 \in \mathcal{O}^\varrho_\delta(N), \quad M_i \in \mathcal{O}^\varrho_\delta(M_{i+1}),\ i \in \{1, \dots, l-1\}, \quad M_l = M(a)$$

and

$$\| M(a) \cdot u - u \| > \zeta(a), \quad u \in F^m,\ \| u \| > a.$$

We formulate Definitions 4.1 and 4.2 in the above form for general $a > 0$, because we apply symbols $\zeta(a), l(a, \delta)$ later (i.a., in the proofs of the main results of this chapter). Of course, we obtain the identical definitions if we consider only one number $a < 1$ as arbitrarily given. Indeed, we have

$$\left| f^l \right| = | f |^l, \quad \| f \odot u \| = | f | \cdot \| u \|, \quad f \in F,\ u \in F^m,\ l \in \mathbb{Z}.$$

Thus, we can simplify them into the following forms.

Definition 4.3. The group $\mathcal{X}$ has property P if there exists $\zeta > 0$ and if for all $\delta > 0$, there exists $l \in \mathbb{N}$ such that, for any vector $u \in F^m$ satisfying $\| u \| \geq 1$, one can find matrices $M_1, \dots, M_l \in \mathcal{X}$ with the property that

$$M_1 \in \mathcal{O}^\varrho_\delta(I), \quad M_i \in \mathcal{O}^\varrho_\delta(M_{i+1}),\ i \in \{1, \dots, l-1\}, \quad \| M_l \cdot u - u \| > \zeta.$$

Definition 4.4. The group $\mathcal{X}$ has property P^* if there exist $M \in \mathcal{X}$ and $\zeta > 0$ such that, for every $\delta > 0$, there exists $l \in \mathbb{N}$ with the property that, for any $N \in \mathcal{X}$, one can find matrices $M_1, \ldots, M_l \in \mathcal{X}$ satisfying

$$M_1 \in \mathcal{O}^{\varrho}_{\delta}(N), \quad M_i \in \mathcal{O}^{\varrho}_{\delta}(M_{i+1}),\ i \in \{1, \ldots, l-1\}, \quad M_l = M$$

and

$$\| M \cdot u - u \| > \zeta, \quad u \in F^m,\ \| u \| \geq 1.$$

Remark 4.5. Especially, $\mathcal{X}$ has property P if there exist $\zeta > 0$ and continuous functions $f_i : [0, 1] \to Mat(F, m)$ for $i \in \{1, \ldots, n\}$ with the properties that

$$f_i(r) \in \mathcal{X},\ r \in \mathbb{Q} \cap [0, 1], \quad f_i(0) = I, \quad i \in \{1, \ldots, n\},$$

and

$$\max_{i \in \{1,\ldots,n\}} \| f_i(1) \cdot u - u \| > \zeta, \quad u \in F^m,\ \| u \| \geq 1.$$

$\triangle$

Remark 4.6. Let $F = \mathbb{C}$ and let $\mathcal{X}$ be weakly transformable. If there exists a matrix $M \in \mathcal{X}_0$ such that $Mu \neq u$ for all non-zero vectors $u \in \mathbb{C}^m$, then $\mathcal{X}$ has property P. It follows directly from Definitions 2.14 and 4.3 and, considering the compactness of the set $C := \{u \in \mathbb{C}^m;\ \| u \| = 1\}$, from the inequality

$$\inf_{u \in C} \| M \cdot u - u \| > 0.$$

Note that the number $l \in \mathbb{N}$ considered in Definition 4.3 has to exist for any $\delta > 0$, because the set $\mathcal{X}$ is totally bounded.

Analogously, if for every non-zero vector $u \in \mathbb{C}^m$, there exists a matrix $M(u) \in \mathcal{X}_0$ satisfying $M(u)u \neq u$, then we can assume that

$$\inf_{u \in C} \| M(u) \cdot u - u \| > 0.$$

Indeed, there exists a finite number of matrices $M_1, \ldots, M_j \in \mathcal{X}_0$ satisfying

$$\max_{i \in \{1,\ldots,j\}} \| M_i \cdot u - u \| > 0, \quad u \in \mathbb{C}^m,\ \| u \| = 1. \tag{4.3}$$

Hence, in this case, $\mathcal{X}$ has property P as well. $\triangle$

Example 4.7. Based on Remark 4.6, we can mention many examples of weakly transformable groups $\mathcal{X} \subset Mat(\mathbb{F}, m)$ with property P. Since each one of concrete transformable and weakly transformable groups mentioned in Examples 2.2–2.10, 2.15, and 2.17 contains matrices $M_1, \ldots, M_j$ from a transformable subgroup such that (4.3) is satisfied, all these groups have property P (it also follows from Remark 4.5). $\lozenge$

Remark 4.8. It is seen that $\mathcal{X}$ has property P if it has property P^*. One can trivially show that the converse implication is not true. It suffices to consider the real case for $m = 3$ and the group $\mathcal{X} = SO(3)$ which consists of all orthogonal matrices with determinant 1. This group has property P (see Example 4.7). For any matrix $M \in SO(3)$, there exists a vector $u \in \mathbb{R}^3$ with the properties that $Mu = u$ and $\| u \| = 1$. This fact implies that $\mathcal{X} = SO(3)$ cannot have property P^*. $\triangle$

To formulate our results in a simple and consistent form, we also introduce the following definition concerning non-asymptotically almost periodic solutions of systems (4.2).

Definition 4.9. We say that a system $\{A_k\}_{k\in\mathbb{Z}} \in \mathcal{AP}(\mathcal{X})$ does not have any non-zero asymptotically almost periodic solution in the strong sense if there exists a sequence $\{l_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ satisfying $\lim_{n\to\infty} l_n = \infty$ with the property that, for any non-zero solution $\{x_k\}_{k\in\mathbb{Z}}$, there exist $\vartheta > 0$ and $Q \in \mathbb{N}$ such that the inequality

$$\left\| x_{k+l_i} - x_{k+l_j} \right\| > \vartheta$$

is valid for all $i > j \geq Q$ and for a set of k which is relatively dense in $\mathbb{N}$ and which depends on i and j.

Remark 4.10. Evidently, the systems considered in Definition 4.9 form a special class of systems whose non-zero solutions are not asymptotically almost periodic. For example, the scalar system $x_{k+1} = \mathrm{e}^{\mathrm{i}k} x_k$ (for $k \in \mathbb{Z}$) does not have non-zero asymptotically almost periodic solutions, but it is not true that this system does not have any non-zero asymptotically almost periodic solution in the strong sense. △

In Section 4.4, we intend to improve results of Section 4.3. To show how the results of Section 4.4 improve theorems from Section 4.3, we need to reformulate Definition 4.3 for bounded groups applying the following two lemmas (which we will need later as well).

Lemma 4.11. *Let $p \in \mathbb{N}$ be given. The multiplication of p matrices is continuous in the Lipschitz sense on any bounded subset of $Mat(F, m)$.*

Proof. Let $K > 0$ be given. Since the addition and the multiplication have the Lipschitz property on the set of $f \in F$ satisfying $|\, f \,| < K$, the statement of the lemma is true. □

Lemma 4.12. *Let a bounded group $X \subseteq Mat(F, m)$ be given. There exists $L > 1$ such that*

$$M \cdot N^{-1}, N^{-1} \cdot M \in \mathcal{O}^{\varrho}_{aL}(I) \quad \text{if} \quad M, N \in X,\ M \in \mathcal{O}^{\varrho}_{a}(N). \tag{4.4}$$

Proof. We know that the inequality

$$\| M \| < K, \quad M \in X, \qquad \text{i.e.,} \qquad \left\| M^{-1} \right\| < K, \quad M \in X, \tag{4.5}$$

holds for some $K > 0$. The map $f \mapsto -f$, the multiplication, and the addition have the Lipschitz property on the set of all $f \in F$ satisfying $|\, f \,| < K$. In addition, for any $M \in X$, we have (see (4.5))

$$\det M < m!K^m, \quad \det M = \frac{1}{\det M^{-1}} > \frac{1}{m!K^m}.$$

Hence, the map

$$M \mapsto \frac{1}{\det M}, \quad M \in X,$$

has the Lipschitz property as well. Let a matrix $M \in X$ be given. If we use the expression

$$m^{-1}_{i,j} = \frac{M_{j,i}}{\det M}, \quad i, j \in \{1, \dots, m\},$$

where $m_{i,j}^{-1}$ are elements of $M^{-1} \in X$ and $M_{j,i}$ are the algebraic complements of the elements $m_{j,i}$ of M, then it is seen that the map $M \mapsto M^{-1}$ is continuous in the Lipschitz sense on X.

Evidently, Lemma 4.11 and the Lipschitz continuity of $M \mapsto M^{-1}$ on X imply the existence of $L > 1$ for which (4.4) is valid. □

Using Lemmas 4.11 and 4.12 for bounded $\mathcal{X}$ and for M_2 replaced by $M_2 \cdot M_1, \dots, M_l$ by $M_l \cdots M_2 \cdot M_1$, we can rewrite Definition 4.3 as follows.

Definition 4.13. A bounded group $\mathcal{X} \subset Mat(F, m)$ has property P if there exists $\zeta > 0$ and if for all $\delta > 0$, there exists $l \in \mathbb{N}$ such that, for any vector $u \in F^m$ satisfying $\| u \| \geq 1$, one can find matrices $M_1, \dots, M_l \in \mathcal{X}$ with the property that

$$M_i \in \mathcal{O}_\delta^\varrho(I), \quad i \in \{1, \dots, l\}, \qquad \| M_l \cdots M_1 \cdot u - u \| > \zeta. \tag{4.6}$$

We introduce the following direct generalization of Definition 4.13.

Definition 4.14. Let a non-zero vector $u \in F^m$ be given. We say that $\mathcal{X}$ has property P with respect to u if there exists $\zeta > 0$ such that, for all $\delta > 0$, one can find matrices $M_1, \dots, M_l \in \mathcal{X}$ satisfying (4.6).

Remark 4.15. Since a group with property P has property P with respect to any non--zero vector u (consider $|| f \odot u || = | f | \cdot || u ||$, $f \in F$, $u \in F^m$), we can refer to a lot of examples recalled in Example 4.7. Furthermore, we point out that any group, which contains a subgroup having property P with respect to a vector u, has property P with respect to u as well. △

4.3 Systems without asymptotically almost periodic solutions

In this section, we consider that the given group $\mathcal{X}$ is bounded. We can directly prove the main result of this section which reads as follows.

Theorem 4.16. *Let $\mathcal{X}$ have property P. For any $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$ which does not have any non-zero asymptotically almost periodic solution.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset F^m$ be a sequence of non-zero vectors such that $\overline{\{u_n;\, n \in \mathbb{N}\}} = F^m$. We know that there exists $K > 1$ satisfying $\| M \| < K$ for all $M \in \mathcal{X}$. In addition, for any u_n, there exists $L_n > 0$ satisfying

$$\| M u_n \| > L_n, \quad M \in \mathcal{X}. \tag{4.7}$$

Indeed, $\lim_{j \to \infty} \| M_j u_n \| = 0$ for a sequence of $M_j \in \mathcal{X}$ implies the following contradiction

$$\| u_n \| = \lim_{j \to \infty} \left\| M_j^{-1} M_j u_n \right\| \leq \limsup_{j \to \infty} \left(\left\| M_j^{-1} \right\| \cdot \| M_j u_n \| \right) \leq K \lim_{j \to \infty} \| M_j u_n \| = 0.$$

Since $\{A_k\} \in \mathcal{LP}(\mathcal{X})$, there exist sequences $\{B_k^n\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ for $n \in \mathbb{N}$ with the property that

$$\| A_k - B_k^n \| < \frac{\varepsilon}{2^{n+1}K}, \quad k \in \mathbb{Z},\, n \in \mathbb{N}, \tag{4.8}$$

where $\{B_k^n\}$ has period $p_n \in \mathbb{N}$. We can assume that $p_n \geq 2$, $n \in \mathbb{N}$.

For $M, M_1, \ldots, M_l \in \mathcal{X}$, it is seen that

$$\begin{aligned} &\left\| (M_1 \cdots M_l)^{-1} \cdot M \cdot M_1 \cdots M_l - I \right\| \\ &\leq \left\| (M_1 \cdots M_l)^{-1} \right\| \cdot \| M - I \| \cdot \| M_1 \cdots M_l \| \leq K^2 \| M - I \| . \end{aligned} \tag{4.9}$$

Let matrices $A_k, \ldots, A_{k+i-1+l}, E_k, \ldots, E_{k+i-1} \in \mathcal{X}$ be arbitrary. The product

$$E_k \cdot E_{k+1} \cdots E_{k+i-1} \cdot A_k \cdot A_{k+1} \cdots A_{k+i-1} \cdots A_{k+i-1+l}$$

can be expressed in the form

$$A_k \cdot F_k \cdots A_{k+i-1} \cdot F_{k+i-1} \cdots A_{k+i-1+l} \cdot F_{k+i-1+l}, \tag{4.10}$$

where

$$F_k = (A_k)^{-1} \cdot E_k \cdot A_k,$$

$$\vdots$$

$$F_{k+i-1} = (A_k \cdot A_{k+1} \cdots A_{k+i-1})^{-1} \cdot E_{k+i-1} \cdot A_k \cdot A_{k+1} \cdots A_{k+i-1},$$

and

$$F_{k+i} = I, \quad \ldots \quad F_{k+i-1+l} = I.$$

Analogously, one can express

$$A_k \cdot A_{k+1} \cdots A_{k+i-1} \cdots A_{k+i-1+l} \cdot E_k \cdot E_{k+1} \cdots E_{k+i-1}$$

in the above given form (4.10) for

$$F_k = A_{k+1} \cdots A_{k+i-1+l} \cdot E_k \cdot (A_{k+1} \cdots A_{k+i-1+l})^{-1},$$

$$\vdots$$

$$F_{k+i-1} = A_{k+i} \cdots A_{k+i-1+l} \cdot E_{k+i-1} \cdot (A_{k+i} \cdots A_{k+i-1+l})^{-1},$$

and

$$F_{k+i} = I, \quad \ldots \quad F_{k+i-1+l} = I.$$

In the both cases, using (4.9), we have

$$F_k, \ldots, F_{k+i-1+l} \in \mathcal{O}^{\varrho}_{K^2 a}(I) \quad \text{if} \quad E_k, \ldots, E_{k+i-1} \in \mathcal{O}^{\varrho}_{a}(I). \tag{4.11}$$

Considering Lemma 4.12, let $L > 1$ be such that

$$M \cdot N^{-1}, N^{-1} \cdot M \in \mathcal{O}^{\varrho}_{aL}(I) \quad \text{if} \quad M, N \in \mathcal{X},\, M \in \mathcal{O}^{\varrho}_{a}(N). \tag{4.12}$$

For all u_i, there exists $\xi(L_i) = \xi(u_i) > 0$ such that

$$\| M \cdot v - u \| > \frac{\zeta(L_i)}{2} \quad \text{if} \quad \| M \cdot u - u \| > \zeta(L_i),\ \| v - u \| \leq \xi(L_i), \tag{4.13}$$

where $M \in \mathcal{X}$ is arbitrary. Indeed, we have

$$\| M \cdot u - u \| \leq \| M \cdot u - M \cdot v \| + \| M \cdot v - u \| \leq K \| v - u \| + \| M \cdot v - u \| .$$

Assume that $\xi(L_i) < \zeta(L_i)/2$. In fact, we can put $\xi(L_i) := \zeta(L_i)/(2K)$.

Let us consider $\zeta(L_1)$ and matrices $D_1^{1,1}, \dots, D_{l(1,1)}^{1,1} \in \mathcal{X}$ satisfying

$$D_1^{1,1} \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2K^3L}}(I), \qquad D_i^{1,1} \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2K^3L}}\left(D_{i+1}^{1,1}\right), \quad i \in \{1, \dots, l(1,1) - 1\},$$

$$\left\| D_{l(1,1)}^{1,1} \cdot u_1 - u_1 \right\| > \zeta(L_1).$$

Expressing

$$D_{l(1,1)}^{1,1} = D_{l(1,1)}^{1,1} \cdot \left(D_{l(1,1)-1}^{1,1}\right)^{-1} \cdots D_2^{1,1} \cdot \left(D_1^{1,1}\right)^{-1} \cdot D_1^{1,1},$$

where (see (4.12))

$$D_{l(1,1)}^{1,1} \cdot \left(D_{l(1,1)-1}^{1,1}\right)^{-1}, \dots, D_2^{1,1} \cdot \left(D_1^{1,1}\right)^{-1}, D_1^{1,1} \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2K^3}}(I),$$

we know (see also (4.11) and (4.13)) that there exist matrices $C_0^1, C_1^1, \dots, C_{2p_1 l(1,1)-1}^1 \in \mathcal{X}$ such that

$$\left\| A_{p_1 l(1,1)-2} \cdot C_{p_1 l(1,1)-2}^1 \cdots A_1 \cdot C_1^1 \cdot A_0 \cdot C_0^1 \cdot u_1 - u_1 \right\| > \xi(L_1)$$

and that

$$C_0^1, \dots, C_{p_1 l(1,1)-2}^1 \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2K}}(I), \quad C_{p_1 l(1,1)-1}^1 = \cdots = C_{2p_1 l(1,1)-1}^1 = I.$$

Indeed, one can put

$$C_0^1 = \cdots = C_{p_1 l(1,1)-2}^1 = C_{p_1 l(1,1)-1}^1 = \cdots = C_{2p_1 l(1,1)-1}^1 = I$$

if

$$\left\| A_{p_1 l(1,1)-2} \cdots A_1 \cdot A_0 \cdot u_1 - u_1 \right\| > \xi(L_1).$$

We define the periodic sequence $\{C_k^1\}_{k \in \mathbb{Z}} \subset \mathcal{X}$ with period $2p_1 l(1,1)$ by the matrices

$$C_0^1, \dots, C_{p_1 l(1,1)-2}^1, C_{p_1 l(1,1)-1}^1, \dots, C_{2p_1 l(1,1)-1}^1.$$

In the second step, we consider $\zeta(L_1)$, $\zeta(L_2)$ and the numbers (see Definition 4.1)

$$l(2,1) := l\left(L_1, \frac{\varepsilon}{2^2 K^3 L}\right), \quad l(2,2) := l\left(L_2, \frac{\varepsilon}{2^2 K^3 L}\right).$$

We recall that, for any $v_1, v_2 \in F^m$ satisfying $\| v_1 \| > L_1$, $\| v_2 \| > L_2$, there exist matrices $D_1^{2,1}, \dots, D_{l(2,1)}^{2,1} \in \mathcal{X}$, $D_1^{2,2}, \dots, D_{l(2,2)}^{2,2} \in \mathcal{X}$ with the property that

$$D_1^{2,1} \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^2K^3L}}(I), \qquad D_i^{2,1} \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^2K^3L}}\left(D_{i+1}^{2,1}\right), \quad i \in \{1, \dots, l(2,1) - 1\},$$

$$D_1^{2,2} \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^2K^3L}}(I), \qquad D_i^{2,2} \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^2K^3L}}\left(D_{i+1}^{2,2}\right), \quad i \in \{1, \dots, l(2,2)-1\},$$

and

$$\left\| D_{l(2,1)}^{2,1} \cdot v_1 - v_1 \right\| > \zeta(L_1), \quad \left\| D_{l(2,2)}^{2,2} \cdot v_2 - v_2 \right\| > \zeta(L_2).$$

Assume that $l(2,2) \geq l(2,1) \geq l(1,1) > 1$. Denote

$$h_1 := (0!)^2 p_1 l(1,1), \quad h_2 := (1!)^2 p_1 p_2 l(1,1) l(2,2).$$

Analogously as in the first step (consider also (4.7)), we can show that there exist matrices $C_0^2, C_1^2, \dots, C_{2^2 2h_2 - 1}^2 \in \mathcal{X}$ such that

$$\begin{aligned} \big\| A_{h_2+h_2-2} \cdot C^1_{h_2+h_2-2} \cdot C^2_{h_2+h_2-2} \cdots A_0 \cdot C_0^1 \cdot C_0^2 \cdot u_1 \\ - A_{h_2-1} \cdot C^1_{h_2-1} \cdot C^2_{h_2-1} \cdots A_0 \cdot C_0^1 \cdot C_0^2 \cdot u_1 \big\| > \xi(L_1), \end{aligned}$$

$$\begin{aligned} \big\| A_{2h_2+h_2-h_1-1} \cdot C^1_{2h_2+h_2-h_1-1} \cdot C^2_{2h_2+h_2-h_1-1} \cdots A_0 \cdot C_0^1 \cdot C_0^2 \cdot u_1 \\ - A_{2h_2-1} \cdot C^1_{2h_2-1} \cdot C^2_{2h_2-1} \cdots A_0 \cdot C_0^1 \cdot C_0^2 \cdot u_1 \big\| > \xi(L_1), \end{aligned}$$

$$\begin{aligned} \big\| A_{3h_2+h_2-2} \cdot C^1_{3h_2+h_2-2} \cdot C^2_{3h_2+h_2-2} \cdots A_0 \cdot C_0^1 \cdot C_0^2 \cdot u_2 \\ - A_{3h_2-1} \cdot C^1_{3h_2-1} \cdot C^2_{3h_2-1} \cdots A_0 \cdot C_0^1 \cdot C_0^2 \cdot u_2 \big\| > \xi(L_2), \end{aligned}$$

$$\begin{aligned} \big\| A_{4h_2+h_2-h_1-1} \cdot C^1_{4h_2+h_2-h_1-1} \cdot C^2_{4h_2+h_2-h_1-1} \cdots A_0 \cdot C_0^1 \cdot C_0^2 \cdot u_2 \\ - A_{4h_2-1} \cdot C^1_{4h_2-1} \cdot C^2_{4h_2-1} \cdots A_0 \cdot C_0^1 \cdot C_0^2 \cdot u_2 \big\| > \xi(L_2) \end{aligned}$$

and, at the same time, such that

$$C_0^2, C_1^2, \dots, C_{2^2 2h_2-1}^2 \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^2K}}(I),$$

$$C^2_{j2h_1+0} = C^2_{j2h_1+1} = \cdots = C^2_{j2h_1+h_1-1} = I, \quad j \in \{0, 1, \dots, 2^2 p_2 l(2,2) - 1\},$$

$$C^2_{j2^2h_1+h_1} = C^2_{j2^2h_1+h_1+1} = \cdots = C^2_{j2^2h_1+2h_1-1} = I, \quad j \in \{0, 1, \dots, 2p_2 l(2,2) - 1\},$$

i.e., only matrices

$$C^2_{j2^2h_1+3h_1}, C^2_{j2^2h_1+3h_1+1}, \dots, C^2_{j2^2h_1+4h_1-1} \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^2K}}(I), \quad j \in \{0, 1, \dots, 2p_2 l(2,2) - 1\},$$

do not need to be I. Especially, we have

$$C_j^2 = I \quad \text{if} \quad C_j^1 \neq I \text{ for some } j \in \{0, 1, \dots, 2^2 2h_2 - 1\}.$$

We consider the periodic sequence $\{C_k^2\}_{k \in \mathbb{Z}} \subset \mathcal{X}$ given by period $2^2 2h_2$ and the matrices $C_0^2, C_1^2, \dots, C_{2^2 2h_2-1}^2$.

We continue in the same manner. Before the n-th step, we have

$$C^i_{j2^{n-1}h_1+h_1} = C^i_{j2^{n-1}h_1+h_1+1} = \cdots = C^i_{j2^{n-1}h_1+2h_1-1} = I, \quad j \in \mathbb{Z}, \, i \in \{1, \dots, n-1\},$$

for the resulting periodic sequences $\{C^i_k\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ with periods $2i^2h_i$, $i \in \{3,\dots,n-1\}$, where
$$h_i := ((i-1)!)^2 p_1p_2\cdots p_i l(1,1)l(2,2)\cdots l(i,i).$$

In the n-th step, we consider $\zeta(L_1),\dots,\zeta(L_{n-1}),\zeta(L_n)$ and the integers
$$l(n,1) := l\left(L_1, \frac{\varepsilon}{2^nK^3L}\right),\dots,l(n,n-1) := l\left(L_{n-1}, \frac{\varepsilon}{2^nK^3L}\right), l(n,n) := l\left(L_n, \frac{\varepsilon}{2^nK^3L}\right).$$
Let
$$l(n,n) \ge l(n,n-1) \ge \cdots \ge l(n,1) \ge \cdots \ge l(2,2) \ge l(2,1) \ge l(1,1) > 1.$$
We put
$$h_n := ((n-1)!)^2 p_1p_2\cdots p_n l(1,1)l(2,2)\cdots l(n,n).$$
For all $v_1,\dots,v_n \in F^m$ satisfying $\|\, v_1 \,\| > L_1,\dots,\|\, v_n \,\| > L_n$, there exist matrices
$$D^{n,1}_1,\dots,D^{n,1}_{l(n,1)},\dots,D^{n,n}_1,\dots,D^{n,n}_{l(n,n)} \in \mathcal{X}$$
with the property that
$$D^{n,1}_1 \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^nK^3L}}(I), \qquad D^{n,1}_i \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^nK^3L}}\left(D^{n,1}_{i+1}\right), \quad i \in \{1,\dots,l(n,1)-1\},$$
$$\vdots$$
$$D^{n,n}_1 \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^nK^3L}}(I), \qquad D^{n,n}_i \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^nK^3L}}\left(D^{n,n}_{i+1}\right), \quad i \in \{1,\dots,l(n,n)-1\},$$
and
$$\left\|\, D^{n,1}_{l(n,1)} \cdot v_1 - v_1 \,\right\| > \zeta(L_1),$$
$$\vdots$$
$$\left\|\, D^{n,n}_{l(n,n)} \cdot v_n - v_n \,\right\| > \zeta(L_n).$$
Thus, considering
$$h_n - 1 > \cdots > h_n - h_{n-1} = h_{n-1}\left[(n-1)^2 p_n l(n,n) - 1\right] > h_{n-1} l(n,n),$$
we know that there exist matrices $C^n_0, C^n_1,\dots,C^n_{2n^2h_n-1} \in \mathcal{X}$ such that
$$\begin{aligned}\|\, A_{h_n+h_n-2} \cdot C^1_{h_n+h_n-2} \cdots C^n_{h_n+h_n-2} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_1 \\ - A_{h_n-1} \cdot C^1_{h_n-1} \cdots C^n_{h_n-1} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_1 \,\| > \xi(L_1),\end{aligned}$$
$$\vdots$$
$$\begin{aligned}\|\, A_{nh_n+h_n-h_{n-1}-1} \cdot C^1_{nh_n+h_n-h_{n-1}-1} \cdots C^n_{nh_n+h_n-h_{n-1}-1} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_1 \\ - A_{nh_n-1} \cdot C^1_{nh_n-1} \cdots C^n_{nh_n-1} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_1 \,\| > \xi(L_1),\end{aligned}$$
$$\vdots$$

$$\begin{aligned}&\big\| A_{[(i-1)n+1]h_n+h_n-2} \cdot C^1_{[(i-1)n+1]h_n+h_n-2} \cdots C^n_{[(i-1)n+1]h_n+h_n-2} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_i \\ &- A_{[(i-1)n+1]h_n-1} \cdot C^1_{[(i-1)n+1]h_n-1} \cdots C^n_{[(i-1)n+1]h_n-1} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_i \big\| > \xi(L_i),\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\big\| A_{inh_n+h_n-h_{n-1}-1} \cdot C^1_{inh_n+h_n-h_{n-1}-1} \cdots C^n_{inh_n+h_n-h_{n-1}-1} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_i \\ &\qquad - A_{inh_n-1} \cdot C^1_{inh_n-1} \cdots C^n_{inh_n-1} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_i \big\| > \xi(L_i),\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\big\| A_{[(n-1)n+1]h_n+h_n-2} \cdot C^1_{[(n-1)n+1]h_n+h_n-2} \cdots C^n_{[(n-1)n+1]h_n+h_n-2} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_n \\ &- A_{[(n-1)n+1]h_n-1} \cdot C^1_{[(n-1)n+1]h_n-1} \cdots C^n_{[(n-1)n+1]h_n-1} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_n \big\| > \xi(L_n),\end{aligned}$$

$$\vdots$$

$$\begin{aligned}&\big\| A_{n^2h_n+h_n-h_{n-1}-1} \cdot C^1_{n^2h_n+h_n-h_{n-1}-1} \cdots C^n_{n^2h_n+h_n-h_{n-1}-1} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_n \\ &\qquad - A_{n^2h_n-1} \cdot C^1_{n^2h_n-1} \cdots C^n_{n^2h_n-1} \cdots A_0 \cdot C^1_0 \cdots C^n_0 \cdot u_n \big\| > \xi(L_n)\end{aligned}$$

and such that

$$C^n_0, C^n_1, \dots, C^n_{2n^2h_n-1} \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^nK}}(I),$$

where

$$C^n_j = I, \quad j \in \{0, 1, \dots, 2(n-1)^2h_{n-1} - 1\}. \tag{4.14}$$

In addition, we can assume that

$$C^n_j = I \quad \text{if} \quad C^i_j \neq I \text{ for some } j \in \{0, 1, \dots, 2n^2h_n - 1\},\ i \in \{1, \dots, n-1\}. \tag{4.15}$$

It follows from the inequalities

$$\begin{aligned}&h_n - 1 > \cdots > h_n - h_{n-1} > 2^2h_{n-1}l(n,n) > 2^4h_{n-2}l(n,n) > \cdots \\ &\qquad\qquad \cdots > 2^{2n-4}h_2l(n,n) \geq 2^{2n-2}h_1l(n,n) \geq 2^{2n}l(n,n)\end{aligned}$$

and from the fact that it suffices to choose only the matrices

$$C^n_{j2^nh_1+(2^{n-1}+1)h_1}, C^n_{j2^nh_1+(2^{n-1}+1)h_1+1}, \dots, C^n_{j2^nh_1+(2^{n-1}+1)h_1+h_1-1}$$

as different from I. We consider the periodic sequence $\{C^n_k\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ given by the above values $C^n_0, C^n_1, \dots, C^n_{2n^2h_n-1}$ and period $2n^2h_n$.

We define

$$B_k := A_k \cdot C^1_k \cdot C^2_k \cdots C^n_k \cdots, \quad k \in \mathbb{Z}. \tag{4.16}$$

For all $k \in \mathbb{Z}$, there exists $i \in \mathbb{N}$ such that $B_k = A_k \cdot C^i_k$ (consider (4.15) together with (4.14)). Especially, the definition of the sequence of B_k is correct and $B_k \in \mathcal{X}$, $k \in \mathbb{Z}$. We have to prove that $\{B_k\}_{k\in\mathbb{Z}}$ is limit periodic and that

$$\sup_{k\in\mathbb{Z}} \| B_k - A_k \| < \varepsilon. \tag{4.17}$$

Since

$$C^n_k \in \mathcal{O}^{\varrho}_{\frac{\varepsilon}{2^nK}}(I), \quad k \in \mathbb{Z},\ n \in \mathbb{N}, \tag{4.18}$$

we have

$$\| B_k - A_k \| = \left\| A_k \cdot C_k^i - A_k \right\| \leq \| A_k \| \cdot \left\| C_k^i - I \right\| < K \frac{\varepsilon}{2^i K} \leq \frac{\varepsilon}{2}$$

for some $i \in \mathbb{N}$. Thus, (4.17) is satisfied. Similarly, if we express

$$\begin{aligned} &\left\| B_k - B_k^n \cdot C_k^1 \cdots C_k^n \right\| \\ &\leq \left\| B_k - B_k^n \cdot C_k^1 \cdots C_k^n \cdots \right\| + \left\| B_k^n \cdot C_k^1 \cdots C_k^n \cdots - B_k^n \cdot C_k^1 \cdots C_k^n \right\| \\ &\leq \| A_k - B_k^n \| \cdot \left\| C_k^1 \cdots C_k^n \cdots \right\| + \left\| B_k^n \cdot C_k^1 \cdots C_k^n \right\| \cdot \left\| C_k^{n+1} \cdots C_k^{n+j} \cdots - I \right\|, \end{aligned}$$

then we obtain (see (4.8), (4.18))

$$\left\| B_k - B_k^n \cdot C_k^1 \cdots C_k^n \right\| < \frac{\varepsilon}{2^{n+1} K} \cdot K + K \frac{\varepsilon}{2^{n+1} K} = \frac{\varepsilon}{2^n}, \quad k \in \mathbb{Z}, \, n \in \mathbb{N}.$$

It means that $\{B_k\}$ is the uniform limit of the sequence of $B_k^n \cdot C_k^1 \cdots C_k^n$ as $n \to \infty$. Any sequence $\{B_k^n \cdot C_k^1 \cdots C_k^n\}_{k \in \mathbb{Z}}$ has period $2(n!)^2 p_1 \cdots p_n l(1,1) \cdots l(n,n)$. Hence, $\{B_k\}$ is limit periodic.

It remains to show that the system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ does not possess any non-zero asymptotically almost periodic solution. On contrary, suppose that the solution $\{x_k\}_{k \in \mathbb{Z}}$ of the Cauchy problem

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u_l$$

is asymptotically almost periodic. Applying Theorem 1.15 for $l_1 := 1$, $l_{n+1} := h_n$, $n \in \mathbb{N}$, we obtain

$$\left\| x_{k+h_{n_1}} - x_{k+h_{n_2}} \right\| < \xi(L_l), \quad k \in \mathbb{N}, \tag{4.19}$$

for infinitely many $n_1, n_2 \in \mathbb{N}$. Let integers $n_1 > n_2 \geq l$ be arbitrarily given. It holds

$$\begin{aligned} &\left\| x_{[(l-1)n_1+n_2+1]h_{n_1}+h_{n_1}-h_{n_2}} - x_{[(l-1)n_1+n_2+1]h_{n_1}} \right\| \\ &= \left\| B_{[(l-1)n_1+n_2+1]h_{n_1}+h_{n_1}-h_{n_2}-1} \cdots B_1 \cdot B_0 \cdot u_l - B_{[(l-1)n_1+n_2+1]h_{n_1}-1} \cdots B_1 \cdot B_0 \cdot u_l \right\|. \end{aligned}$$

From (consider (4.14) and (4.16))

$$\begin{aligned} B_{[(l-1)n_1+n_2+1]h_{n_1}+h_{n_1}-h_{n_2}-1} = A_{[(l-1)n_1+n_2+1]h_{n_1}+h_{n_1}-h_{n_2}-1} \times& \\ \times C^1_{[(l-1)n_1+n_2+1]h_{n_1}+h_{n_1}-h_{n_2}-1} \cdots C^{n_1}_{[(l-1)n_1+n_2+1]h_{n_1}+h_{n_1}-h_{n_2}-1}&, \end{aligned}$$

$$\vdots$$

$$B_{[(l-1)n_1+n_2+1]h_{n_1}-1} = A_{[(l-1)n_1+n_2+1]h_{n_1}-1} \cdot C^1_{[(l-1)n_1+n_2+1]h_{n_1}-1} \cdots C^{n_1}_{[(l-1)n_1+n_2+1]h_{n_1}-1},$$

$$\vdots$$

$$B_1 = A_1 \cdot C_1^1 \cdots C_1^{n_1}, \quad B_0 = A_0 \cdot C_0^1 \cdots C_0^{n_1},$$

we obtain

$$\left\| x_{[(l-1)n_1+n_2+1]h_{n_1}+h_{n_1}-h_{n_2}} - x_{[(l-1)n_1+n_2+1]h_{n_1}} \right\| > \xi(L_l).$$

Especially, it is valid that

$$\sup_{k\in\mathbb{N}} \left\| x_{k+h_{n_1}} - x_{k+h_{n_2}} \right\| > \xi(L_l), \quad n_1 > n_2 \geq l,\, n_1, n_2 \in \mathbb{N}. \tag{4.20}$$

This contradiction (see (4.19)) proves that the solution of

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u_n$$

is not asymptotically almost periodic for any u_n, $n \in \mathbb{N}$.

Let us consider an arbitrary non-zero vector $u \in F^m$. We know that there exists a sequence of $u_{i(n)}$ for which $\lim_{n\to\infty} u_{i(n)} = u$. For the solution $\{z_k\}_{k\in\mathbb{Z}}$ of

$$z_{k+1} = B_k \cdot z_k, \quad k \in \mathbb{Z}, \qquad z_0 = u_{i(n)},$$

we have (see (4.20))

$$\sup_{k\in\mathbb{N}} \left\| z_{k+h_{n_1}} - z_{k+h_{n_2}} \right\| > \xi(L_{i(n)}), \quad n_1 > n_2 \geq i(n),\, n_1, n_2 \in \mathbb{N}.$$

The value $\xi(L_{i(n)})$ can be chosen as $\xi(L_{i(n)}) := \zeta(L_{i(n)})/(2K)$ and $L_{i(n)}$ can be chosen as a constant value L^* for sufficiently large n, because there exist $\vartheta > 0$ and $L^* > 0$ satisfying

$$\| M \cdot v \| > L^*, \quad v \in \mathcal{O}^{\varrho}_{\vartheta}(u),\, M \in \mathcal{X}. \tag{4.21}$$

Note that (4.21) follows directly from (4.7) and from the Lipschitz continuity of multiplication. Without loss of generality, we assume that $u_{i(n)} \in \mathcal{O}^{\varrho}_{\vartheta}(u)$, $n \in \mathbb{N}$. In this case, it is valid

$$\sup_{k\in\mathbb{N}} \left\| z_{k+h_{n_1}} - z_{k+h_{n_2}} \right\| > \frac{\zeta\,(L^*)}{2K}, \quad n_1 > n_2 \geq i(n),\, n, n_1, n_2 \in \mathbb{N}.$$

Therefore, for the solution $\{y_k\}_{k\in\mathbb{Z}}$ of the following problem

$$y_{k+1} = B_k \cdot y_k, \quad k \in \mathbb{Z}, \qquad y_0 = u, \tag{4.22}$$

it holds

$$\sup_{k\in\mathbb{N}} \left\| y_{k+h_{n_1}} - y_{k+h_{n_2}} \right\| > \frac{\zeta\,(L^*)}{4K}, \quad n_1 > n_2 \geq i(n),\, n_1, n_2 \in \mathbb{N}, \tag{4.23}$$

where $n \in \mathbb{N}$ is sufficiently large. Indeed, it follows from

$$\| z_k - y_k \| = \left\| B_{k-1} \cdots B_0 \cdot u_{i(n)} - B_{k-1} \cdots B_0 \cdot u \right\| \leq K \left\| u_{i(n)} - u \right\|, \quad k \in \mathbb{N}.$$

Finally, (4.23) implies that the solution $\{y_k\}$ of (4.22) cannot be asymptotically almost periodic (consider again Theorem 1.15). □

Remark 4.17. Since $\mathcal{P}(\mathcal{X})$ is a dense subset of $\mathcal{LP}(\mathcal{X})$ (consider directly Definition 1.17), the statement of Theorem 4.16 does not change if one replaces the system $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ by $\{A_k\} \in \mathcal{P}(\mathcal{X})$. △

Example 4.18. Theorem 4.16 can be applied for all groups of matrices recalled in Example 4.7. In addition, considering Remark 4.6, other matrix groups with property P can be constructed using the direct sums, because the direct sum of a weakly transformable group and a finite group and the direct sum of two weakly transformable groups are weakly transformable as well (see Examples 2.16 and 2.17). ◊

Remark 4.19. Let us consider that the statement of Theorem 4.16 is true for a bounded matrix group $\mathcal{X} \subset Mat(\mathbb{F}, m)$. Especially, it is valid when $A_k := I$, $k \in \mathbb{Z}$. Since the multiplication of matrices is continuous in the Lipschitz sense on $\mathcal{X}$ (see Lemma 4.11), for any non-zero vector $u \in \mathbb{F}^m$ and $\delta > 0$, there exist matrices $M_1, \dots, M_l$ (where M_i is the product of i matrices from a neighbourhood of I) with the property that

$$M_1 \in \mathcal{O}_\delta^\varrho(I), \quad M_i \in \mathcal{O}_\delta^\varrho(M_{i+1}),\ i \in \{1, \dots, l-1\}, \quad \| M_l \cdot u - u \| > 0.$$

Similarly as in Remark 4.6, one can show that $\mathcal{X}$ has property P. Thus, the condition of Theorem 4.16 (that $\mathcal{X}$ has property P) is necessary if $F = \mathbb{F}$. △

Theorem 4.20. *Let $\mathcal{X}$ have property P^*. For any $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$ which does not have any non-zero asymptotically almost periodic solution in the strong sense.*

Proof. We can proceed as in the proof of Theorem 4.16 and construct the limit periodic system $\{B_k\}_{k\in\mathbb{Z}}$ for a decreasing sequence $\{L_i\}_{i\in\mathbb{N}}$ of positive numbers with the property that $\lim_{i\to\infty} L_i = 0$. The value $L_i > 0$ is given by a non-zero vector $u_i \in F^m$ in the sense that $\| M u_i \| > L_i$, $M \in \mathcal{X}$. Let the sequence $\{u_i\}_{i\in\mathbb{N}}$ be such that, for any non-zero vector $u \in F^m$, one can find $j \in \mathbb{N}$ for which $\| u_i \| \le \| u \|$, $i \ge j$.

Especially, from the construction, we obtain

$$\| B_{h_n+h_n-2} \cdots B_1 \cdot B_0 \cdot u_1 - B_{h_n-1} \cdots B_1 \cdot B_0 \cdot u_1 \| > \xi(L_1),$$

$$\vdots$$

$$\left\| B_{nh_n+h_n-h_{n-1}-1} \cdots B_1 \cdot B_0 \cdot u_1 - B_{nh_n-1} \cdots B_1 \cdot B_0 \cdot u_1 \right\| > \xi(L_1),$$

$$\vdots$$

$$\left\| B_{[(i-1)n+1]h_n+h_n-2} \cdots B_1 \cdot B_0 \cdot u_i - B_{[(i-1)n+1]h_n-1} \cdots B_1 \cdot B_0 \cdot u_i \right\| > \xi(L_i),$$

$$\vdots$$

$$\left\| B_{inh_n+h_n-h_{n-1}-1} \cdots B_1 \cdot B_0 \cdot u_i - B_{inh_n-1} \cdots B_1 \cdot B_0 \cdot u_i \right\| > \xi(L_i),$$

$$\vdots$$

$$\left\| B_{[(n-1)n+1]h_n+h_n-2} \cdots B_1 \cdot B_0 \cdot u_n - B_{[(n-1)n+1]h_n-1} \cdots B_1 \cdot B_0 \cdot u_n \right\| > \xi(L_n),$$

$$\vdots$$

$$\left\| B_{n^2h_n+h_n-h_{n-1}-1} \cdots B_1 \cdot B_0 \cdot u_n - B_{n^2h_n-1} \cdots B_1 \cdot B_0 \cdot u_n \right\| > \xi(L_n),$$

$$\vdots$$

Let a non-zero vector $u \in F^m$ be given. We consider the solution $\{x_k\}_{k\in\mathbb{Z}}$ of the Cauchy problem

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

From the proof of Theorem 4.16, we know that

$$\inf_{k\in\mathbb{Z}} \| x_k \| > L_j \quad \text{for some } j \in \mathbb{N} \tag{4.24}$$

and that (see (4.23))

$$\sup_{k\in\mathbb{N}} \left\| x_{k+h_{n_1}} - x_{k+h_{n_2}} \right\| > \vartheta$$

for all sufficiently large integers $n_1 > n_2$ and for some $\vartheta > 0$. Hence, for all $n_1 > n_2 \geq n_0$, $n_1, n_2 \in \mathbb{N}$, where $n_0 \in \mathbb{N}$ is sufficiently large, there exists an integer $l \geq h_{n_2}$ with the property that

$$\begin{aligned}\left\| x_{l+h_{n_1}-h_{n_2}+1} - x_{l+1} \right\| &= \left\| B_{l+h_{n_1}-h_{n_2}} \cdots B_l \cdots B_0 \cdot u - B_l \cdots B_0 \cdot u \right\| \\ &= \left\| \left[B_{l+h_{n_1}-h_{n_2}} \cdots B_{l+1} - I \right] B_l \cdots B_0 \cdot u \right\| > \vartheta.\end{aligned}$$

Considering (4.24), the construction, and the property P^* of $\mathcal{X}$, we can assume that

$$\left\| \left[B_{l+h_{n_1}-h_{n_2}} \cdots B_{l+1} - I \right] x_k \right\| > \vartheta, \quad k \in \mathbb{N}. \tag{4.25}$$

Since the multiplication and addition are continuous in the Lipschitz sense on bounded subsets of F and $\{B_k\}$ is limit periodic, Theorem 1.3 and Corollary 1.11 (together with Theorem 1.22) imply that the sequence $\{B_{k+h_{n_1}-h_{n_2}} \cdots B_{k+1}\}_{k\in\mathbb{Z}}$ is almost periodic for all given integers $n_1 > n_2 \geq n_0$. Thus, for any $\eta > 0$, we get the existence of a sequence $\{q_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ which is relatively dense in $\mathbb{N}$ and which has the property that

$$\left\| B_{l+h_{n_1}-h_{n_2}} \cdots B_{l+1} - B_{q_n+h_{n_1}-h_{n_2}} \cdots B_{q_n+1} \right\| < \eta, \quad n \in \mathbb{N}. \tag{4.26}$$

Combining (4.25) and (4.26) for a sufficiently small number η, we have

$$\left\| x_{k+h_{n_1}} - x_{k+h_{n_2}} \right\| = \left\| \left[B_{k+h_{n_1}-1} \cdots B_{k+h_{n_2}} - I \right] x_{k+h_{n_2}} \right\| > \frac{\vartheta}{2}$$

for all $k = q_n - h_{n_2} + 1$, $n \in \mathbb{N}$. □

Remark 4.21. Based on the proofs of Theorems 4.16 and 4.20, it is possible to prove that the group $\mathcal{X}$ is transformable if it has property P^*. △

Example 4.22. One can easily show that the transformable groups of matrices recalled in Example 4.7 have actually property P^* (except $SO(m)$ if m is odd); i.e., for these groups, one can apply Theorem 4.20 which improves Theorem 4.16. ♢

The process used in the proof of Theorem 4.16 can be applied for almost periodic systems as well.

Theorem 4.23. *Let $\mathcal{X}$ have property P. For any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ which does not have any non-zero asymptotically almost periodic solution.*

Proof. The statement of this theorem can be proved using the construction from the proof of Theorem 4.16, where it suffices to put $p_n := 2$ and $B^n_k := A_k$ for all $n \in \mathbb{N}$, $k \in \mathbb{Z}$. Indeed, the almost periodicity of the sequence $\{A_k \cdot C^1_k \cdots C^n_k\}_{k\in\mathbb{Z}}$ follows from Theorem 1.3 and Corollary 1.11 (the multiplication and addition have the Lipschitz property on any bounded subset of F) and the almost periodicity of $\{B_k\}_{k\in\mathbb{Z}}$ comes from Theorem 1.7. □

Corollary 4.24. *Let $\mathcal{X}$ have property P. For any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\})$ which does not have any non-zero almost periodic solution.*

Proof. See Theorems 1.22 and 4.23. □

In the complex case, Theorem 4.23 is a generalization of Corollary 2.33 (see also Corollary 4.24). It follows from Remark 4.6. Of course, in the general case, the results presented in Chapter 2 do not follow from ones presented here.

Remark 4.25. In fact, considering the proofs of the above theorems, we see that it is not necessary to introduce the map $|\cdot|$ on the whole field. It suffices to define it on a neighbourhood of e_0 and for all elements of matrices from the matrix group $\mathcal{X}$ in such a way that condition (a) is replaced by $|f| \geq 0$, $|e_0| = 0$, $|e_1| = 1$ and condition (b) by $|f \odot g| \leq |f| \cdot |g|$, $|f| = |-f|$ (consider also Definitions 4.1 and 4.2). Let $\mathcal{X}$ have property P with respect to a given non-zero vector u and let there exist a neighbourhood of u with the property that, for all vectors v from the neighbourhood and all $M \in \mathcal{X}$, there exist the norms $\| M\, v \|$. In this case, for any limit periodic (or almost periodic) sequence $\{A_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$, there exists a limit periodic (or almost periodic) system given by $\{B_k\}_{k\in\mathbb{Z}}$ in an arbitrarily small neighbourhood of $\{A_k\}$ for which the solution of

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is not asymptotically almost periodic. △

Example 4.26. Now we show that there exists a group $\mathcal{X}$ with property P which is not weakly transformable. Especially, we show that Corollary 4.24 does not follow from Corollary 2.33. For simplicity, we use Remark 4.25. Let F be the field of all meromorphic functions defined on an open connected set which contains the set

$$D = \{z \in \mathbb{C};\ \mathrm{Re}\ z \in [-\pi, \pi],\ \mathrm{Im}\ z = 0\}.$$

Note that, using the analytic continuation to eliminate removable singularities, meromorphic functions can be added, subtracted, multiplied, and the quotient can be formed. For the set $F_0 \subset F$ of all bounded functions on D, we put

$$|f| := \sup_{z\in D} |f(z)|, \quad f \in F_0.$$

Let $m = 1$, i.e., let us consider the scalar case. Especially, $\| f \| = |f|$, $f \in F_0$.

We put

$$\mathcal{X} := \{c\,[\sin(nz) \pm \mathrm{i}\cos(nz)];\ c \in \mathbb{C},\ |c| = 1,\ n \in \mathbb{Z}\}.$$

The set $\mathcal{X}$ forms a group. Indeed, the identity element

$$1 \equiv \mathrm{i}\,[\sin(0z) - \mathrm{i}\cos(0z)] \in \mathcal{X},$$

associativity is obvious, the inverse elements

$$(c\,[\sin(nz) \pm \mathrm{i}\cos(nz)])^{-1} = c^{-1}\,[\sin(nz) \mp \mathrm{i}\cos(nz)]\,, \quad c \in \mathbb{C},\ |c| = 1,\ n \in \mathbb{Z},$$

belong to $\mathcal{X}$, and the closure of multiplication follows from the formulas

$$[\sin(nz) + \mathrm{i}\cos(nz)] \cdot [\sin(lz) \pm \mathrm{i}\cos(lz)] = \mp \cos[(n \pm l)z] \pm \mathrm{i}\sin[(n \pm l)z],$$
$$[\sin(nz) - \mathrm{i}\cos(nz)] \cdot [\sin(lz) - \mathrm{i}\cos(lz)] = -\cos[(n + l)z] - \mathrm{i}\sin[(n + l)z].$$

Evidently, this group is bounded.

Firstly, we show that $\mathcal{X}$ is not weakly transformable. The subgroup

$$\mathcal{X}_0 := \{c\,[\sin 0 \pm \mathrm{i}\cos 0];\ c \in \mathbb{C},\ |\,c\,| = 1\} = \{f \equiv c \in \mathbb{C};\ |\,c\,| = 1\}$$

is transformable. It can be directly verified that

$$\|\, c - d\,[\sin(nz) \pm \mathrm{i}\cos(nz)]\,\| > 1, \quad c, d \in \mathbb{C},\ |\,c\,| = |\,d\,| = 1,\ n \in \mathbb{Z} \setminus \{0\}.$$

Thus, there does not exist a transformable subgroup of $\mathcal{X}$ which contains $\mathcal{X}_0$ and at least one other element. To prove that $\mathcal{X}$ is not weakly transformable, it suffices to consider that the distance of any two different components

$$\mathcal{X}_0 = \{f \equiv c \in \mathbb{C};\ |\,c\,| = 1\},$$
$$\mathcal{X}_n^+ := \{c\,[\sin(nz) + \mathrm{i}\cos(nz)];\ c \in \mathbb{C},\ |\,c\,| = 1\}, \quad n \in \mathbb{Z} \setminus \{0\},$$
$$\mathcal{X}_n^- := \{c\,[\sin(nz) - \mathrm{i}\cos(nz)];\ c \in \mathbb{C},\ |\,c\,| = 1\}, \quad n \in \mathbb{Z} \setminus \{0\},$$

is greater that 1.

Secondly, we show that $\mathcal{X}$ has property P. It suffices to put $\zeta(a) := 2a$ for all $a > 0$. For $\delta = \pi/l$, $l \in \mathbb{N}$, we can choose matrices

$$M_1 = \left(\mathrm{e}^{\mathrm{i}\delta}\right), M_2 = \left(\mathrm{e}^{\mathrm{i}2\delta}\right), \dots, M_{l-1} = \left(\mathrm{e}^{\mathrm{i}(l-1)\delta}\right), M_l = \left(\mathrm{e}^{\mathrm{i}l\delta}\right) = (-1)$$

for which we have

$$\|\, M_1 - I\,\| < \delta, \|\, M_2 - M_1\,\| < \delta, \dots, \|\, M_{l-1} - M_l\,\| < \delta.$$

Evidently, it is also valid that

$$\|\, M_l \cdot u - u\,\| = \|\, -u - u\,\| = 2\,\|\, u\,\| > \zeta(a) = 2a, \quad \|\, u\,\| > a,\ u \in F_0.$$

$\diamondsuit$

Analogously as Theorem 4.23 (see the proof of Theorem 4.20), one can obtain the following result.

Theorem 4.27. *Let $\mathcal{X}$ have property P^*. For any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ which does not have any non-zero asymptotically almost periodic solution in the strong sense.*

Remark 4.28. We recall that Theorems 4.23 and 4.27 do not follow from Theorems 4.16 and 4.20. See Theorem 1.21. $\triangle$

4.4 Systems with non-almost periodic solutions

Henceforth, we assume that $\mathcal{X}$ is commutative. To prove the announced result (the below given Theorem 4.31), we use Lemmas 4.29 and 4.30.

Lemma 4.29. *Let $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrarily given. Let $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a decreasing sequence satisfying*

$$\lim_{n\to\infty} \delta_n = 0 \tag{4.27}$$

and let $\{B_k^n\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ be periodic sequences for $n \in \mathbb{N}$ such that

$$B_k^n \in \mathcal{O}^{\varrho}_{\delta_n}(I), \quad k \in \mathbb{Z},\, n \in \mathbb{N}, \tag{4.28}$$

$$B_k^j = I \quad or \quad B_k^i = I, \quad k \in \mathbb{Z},\, i \neq j,\, i, j \in \mathbb{N}. \tag{4.29}$$

If one puts

$$B_k := A_k \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \cdots, \quad k \in \mathbb{Z},$$

then $\{B_k\} \in \mathcal{LP}(\mathcal{X})$. In addition, if

$$\delta_1 < \frac{\varepsilon}{\sup\limits_{l\in\mathbb{Z}} \| A_l \|}, \tag{4.30}$$

then $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$.

Proof. Condition (4.29) means that, for any $k \in \mathbb{Z}$, there exists $i \in \mathbb{N}$ such that

$$B_k = A_k \cdot B_k^i. \tag{4.31}$$

Especially, the definition of $\{B_k\}_{k\in\mathbb{Z}}$ is correct and $B_k \in \mathcal{X}$, $k \in \mathbb{Z}$.

We show that $\{B_k\}$ is limit periodic. Since $\{A_k\}$ is limit periodic and $A_k \in \mathcal{X}$, $k \in \mathbb{Z}$, there exist periodic sequences $\{C_k^n\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ for $n \in \mathbb{N}$ with the property that

$$\| A_k - C_k^n \| < \frac{1}{n}, \quad k \in \mathbb{Z},\, n \in \mathbb{N}. \tag{4.32}$$

Let $\{B_k^n\}$ and $\{C_k^n\}$ have period $p_n \in \mathbb{N}$ and $q_n \in \mathbb{N}$ for $n \in \mathbb{N}$, respectively. The sequence $\{C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ has period $q_n \cdot p_1 \cdot p_2 \cdots p_n$; i.e., it is periodic for all $n \in \mathbb{N}$. It is valid that

$$\begin{aligned}
&\left\| B_k - C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| \\
&\le \left\| B_k - C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \cdots \right\| + \left\| C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \cdots - C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| \\
&\le \| A_k - C_k^n \| \cdot \left\| B_k^1 \cdot B_k^2 \cdots B_k^n \cdots \right\| + \left\| C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| \cdot \left\| B_k^{n+1} \cdots B_k^{n+j} \cdots - I \right\|
\end{aligned}$$

and that

$$\begin{aligned}
\left\| C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| &\le \| C_k^n \| \cdot \left\| B_k^1 \cdot B_k^2 \cdots B_k^n \right\| \\
&\le (\| A_k \| + \| C_k^n - A_k \|) \cdot \left\| B_k^1 \cdot B_k^2 \cdots B_k^n \right\|.
\end{aligned}$$

Hence (see (4.28), (4.29), (4.32)), we have

$$\left\| B_k - C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| < \frac{1}{n}(m+\delta_1) + \left(\sup_{l\in\mathbb{Z}} \| A_l \| + \frac{1}{n} \right)(m+\delta_1)\,\delta_{n+1}$$

for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$. Considering (4.27), we get that $\{B_k\}$ is the uniform limit of the sequence of periodic sequences $\{C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n\}$. Especially, $\{B_k\} \in \mathcal{LP}(\mathcal{X})$.

Let (4.30) be true. We have to prove that $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$, i.e.,

$$\sup_{k\in\mathbb{Z}} \| A_k - B_k \| < \varepsilon. \tag{4.33}$$

Since

$$B_k^n \in \mathcal{O}_{\delta_1}^\varrho(I), \quad k \in \mathbb{Z},\, n \in \mathbb{N},$$

considering (4.31), we have

$$\| A_k - B_k \| \le \| A_k \| \cdot \left\| I - B_k^i \right\| \le \delta_1 \sup_{l\in\mathbb{Z}} \| A_l \|$$

for some $i \in \mathbb{N}$ and for all $k \in \mathbb{Z}$. Thus (see (4.30)), we obtain (4.33). □

Lemma 4.30. *If for any $\delta > 0$ and $K > 0$, there exist matrices $M_1, \dots, M_l \in \mathcal{X}$ such that*

$$M_i \in \mathcal{O}_\delta^\varrho(I), \quad i \in \{1,\dots,l\}, \qquad \| M_l \cdots M_1 \| > K, \tag{4.34}$$

then, for any $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$ whose fundamental matrix is not almost periodic.

Proof. We can assume that all solutions of $\{A_k\}$ are almost periodic. Especially (consider Corollary 1.11), for any $\vartheta > 0$, there exist infinitely many positive integers p with the property that

$$\| A_{p-1} \cdots A_1 \cdot A_0 - I \| < \vartheta. \tag{4.35}$$

Let $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a decreasing sequence satisfying (4.27) and (4.30). For δ_n and $K_n := n$, $n \in \mathbb{N}$, we consider matrices

$$\begin{gathered} M_1^1, M_2^1, \dots, M_{l_1}^1 \in \mathcal{X}, \\ M_1^2, M_2^2, \dots, M_{l_2}^2 \in \mathcal{X}, \\ \vdots \\ M_1^j, M_2^j, \dots, M_{l_j}^j \in \mathcal{X}, \\ \vdots \end{gathered}$$

such that

$$M_i^j \in \mathcal{O}_{\delta_j}^\varrho(I), \quad i \in \{1,2,\dots,l_j\},\, j \in \mathbb{N}, \tag{4.36}$$

and

$$\left\| M_{l_j}^j \cdots M_2^j \cdot M_1^j \right\| > K_j = j, \quad j \in \mathbb{N}. \tag{4.37}$$

Let a sequence of positive numbers ϑ_n for $n \in \mathbb{N}$ be given.

Let us consider $p_1^1, p_2^1 \in \mathbb{N}$ such that $p_2^1 - p_1^1 > 2l_1$ and that (see (4.35))

$$\left\| A_{p_2^1-1} \cdots A_1 \cdot A_0 - I \right\| < \vartheta_1. \tag{4.38}$$

In addition, let p_1^1 and p_2^1 be even (consider Corollary 1.4). We define the periodic sequence $\{B_k^1\}_{k\in\mathbb{Z}}$ with period p_2^1 by values

$$B_0^1 := I, B_1^1 := I, \dots, B_{p_1^1-2}^1 := I,$$

$$B_{p_1^1-1}^1 := I, B_{p_1^1}^1 := I, B_{p_1^1+1}^1 := M_1^1, B_{p_1^1+2}^1 := I, B_{p_1^1+3}^1 := M_2^1, B_{p_1^1+4}^1 := I,$$

$$\vdots$$

$$B_{p_1^1+2l_1-3}^1 := M_{l_1-1}^1, B_{p_1^1+2l_1-2}^1 := I, B_{p_1^1+2l_1-1}^1 := M_{l_1}^1,$$
$$B_{p_1^1+2l_1}^1 := I, B_{p_1^1+2l_1+1}^1 := I, B_{p_1^1+2l_1+2}^1 := I,$$

$$\vdots$$

$$B_{p_2^1-1}^1 := I.$$

We put

$$\tilde{B}_k^1 := A_k \cdot B_k^1, \quad k \in \mathbb{Z}.$$

We have

$$\left\| \tilde{B}_{p_2^1-1}^1 \cdots \tilde{B}_1^1 \cdot \tilde{B}_0^1 \right\| = \left\| M_{l_1}^1 \cdots M_2^1 \cdot M_1^1 \cdot A_{p_2^1-1} \cdots A_1 \cdot A_0 \right\|.$$

Again, we can assume that, for any $\vartheta > 0$, there exist infinitely many positive integers p with the property that

$$\left\| \tilde{B}_{p-1}^1 \cdots \tilde{B}_1^1 \cdot \tilde{B}_0^1 - I \right\| < \vartheta. \tag{4.39}$$

Otherwise, we obtain the system $\{B_k\} \equiv \{\tilde{B}_k^1\}$ with a non-almost periodic solution. Indeed, it suffices to consider Lemma 4.29 for $B_k^{n+1} = I$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$.

Analogously, let us consider $p_1^2, p_2^2 \in \mathbb{N}$ satisfying $p_2^2 - 4l_2 > p_1^2 > p_2^1$ and (see (4.39))

$$\left\| \tilde{B}_{p_2^2-1}^1 \cdots \tilde{B}_1^1 \cdot \tilde{B}_0^1 - I \right\| < \vartheta_2. \tag{4.40}$$

Let $p_1^2, p_2^2 \in 4\mathbb{N}$ (see Corollary 1.4). We define the periodic sequence $\{B_k^2\}_{k\in\mathbb{Z}}$ with period p_2^2 by values

$$B_0^2 := I, B_1^2 := I, \dots, B_{p_1^2-1}^2 := I,$$

$$B_{p_1^2}^2 := I, B_{p_1^2+1}^2 := I, B_{p_1^2+2}^2 := M_1^2, B_{p_1^2+3}^2 := I,$$

$$B_{p_1^2+4}^2 := I, B_{p_1^2+5}^2 := I, B_{p_1^2+6}^2 := M_2^2, B_{p_1^2+7}^2 := I,$$

$$\vdots$$

$$B_{p_1^2+4l_2-4}^2 := I, B_{p_1^2+4l_2-3}^2 := I, B_{p_1^2+4l_2-2}^2 := M_{l_2}^2, B_{p_1^2+4l_2-1}^2 := I,$$

$$B^2_{p_1^2+4l_2} := I, B^2_{p_1^2+4l_2+1} := I, B^2_{p_1^2+4l_2+2} := I, B^2_{p_1^2+4l_2+3} := I,$$

$$\vdots$$

$$B^2_{p_2^2-1} := I.$$

For

$$\tilde{B}^2_k := A_k \cdot B^1_k \cdot B^2_k, \quad k \in \mathbb{Z},$$

it holds

$$\left\| \tilde{B}^2_{p_2^2-1} \cdots \tilde{B}^2_1 \cdot \tilde{B}^2_0 \right\| = \left\| M^2_{l_2} \cdots M^2_2 \cdot M^2_1 \cdot \tilde{B}^1_{p_2^2-1} \cdots \tilde{B}^1_1 \cdot \tilde{B}^1_0 \right\|.$$

Especially, for all $k \in \mathbb{Z}$, there exists $i \in \{1, 2\}$ such that $\tilde{B}^2_k := A_k \cdot B^i_k$.

We continue in the same manner. Let us assume that all obtained systems $\{\tilde{B}^j_k\}_{k\in\mathbb{Z}}$ have only almost periodic solutions. Thus, for every $\vartheta > 0$ and $j \in \mathbb{N}$, one can find infinitely many $p \in \mathbb{N}$ such that

$$\left\| \tilde{B}^j_{p-1} \cdots \tilde{B}^j_1 \cdot \tilde{B}^j_0 - I \right\| < \vartheta.$$

In the n-th step, we consider $p_1^n, p_2^n \in 2^n\mathbb{N}$ such that $p_2^n - 2^n l_n > p_1^n > p_2^{n-1}$ and

$$\left\| \tilde{B}^{n-1}_{p_2^n-1} \cdots \tilde{B}^{n-1}_1 \cdot \tilde{B}^{n-1}_0 - I \right\| < \vartheta_n. \tag{4.41}$$

We define the periodic sequence $\{B^n_k\}_{k\in\mathbb{Z}}$ with period p_2^n by values

$$B^n_0 := I, B^n_1 := I, \dots, B^n_{p_1^n-1} := I,$$

$$B^n_{p_1^n} := I, B^n_{p_1^n+1} := I, \dots, B^n_{p_1^n+2^{n-1}-1} := I,$$
$$B^n_{p_1^n+2^{n-1}} := M^n_1, B^n_{p_1^n+2^{n-1}+1} := I, \dots, B^n_{p_1^n+2^n-1} := I,$$

$$B^n_{p_1^n+2^n} := I, B^n_{p_1^n+2^n+1} := I, \dots, B^n_{p_1^n+2^n+2^{n-1}-1} := I,$$
$$B^n_{p_1^n+2^n+2^{n-1}} := M^n_2, B^n_{p_1^n+2^n+2^{n-1}+1} := I, \dots, B^n_{p_1^n+2\cdot 2^n-1} := I,$$

$$\vdots$$

$$B^n_{p_1^n+(l_n-1)2^n} := I, B^n_{p_1^n+(l_n-1)2^n+1} := I, \dots, B^n_{p_1^n+(l_n-1)2^n+2^{n-1}-1} := I,$$
$$B^n_{p_1^n+(l_n-1)2^n+2^{n-1}} := M^n_{l_n}, B^n_{p_1^n+(l_n-1)2^n+2^{n-1}+1} := I, \dots, B^n_{p_1^n+l_n 2^n-1} := I,$$

$$B^n_{p_1^n+l_n 2^n} := I, B^n_{p_1^n+l_n 2^n+1} := I, \dots, B^n_{p_1^n+l_n 2^n+2^{n-1}-1} := I,$$
$$B^n_{p_1^n+l_n 2^n+2^{n-1}} := I, B^n_{p_1^n+l_n 2^n+2^{n-1}+1} := I, \dots, B^n_{p_1^n+(l_n+1)2^n-1} := I,$$

$$\vdots$$

$$B^n_{p_2^n-1} := I.$$

If we put
$$\tilde{B}_k^n := A_k \cdot B_k^1 \cdot B_k^2 \cdots B_k^n, \quad k \in \mathbb{Z},$$
then
$$\left\| \tilde{B}_{p_2^n-1}^n \cdots \tilde{B}_1^n \cdot \tilde{B}_0^n \right\| = \left\| M_{l_n}^n \cdots M_2^n \cdot M_1^n \cdot \tilde{B}_{p_2^n-1}^{n-1} \cdots \tilde{B}_1^{n-1} \cdot \tilde{B}_0^{n-1} \right\|. \tag{4.42}$$

Finally, we put
$$B_k := A_k \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \cdots, \quad k \in \mathbb{Z}.$$
From the construction, we obtain that, for any $k \in \mathbb{Z}$, there exists $i \in \mathbb{N}$ such that $B_k = A_k \cdot B_k^i$. It means that (4.29) is satisfied. Since (4.28) follows from the construction and from (4.36), we can use Lemma 4.29 which guarantees that $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$. It remains to prove that the fundamental matrix of $\{B_k\}$ is not almost periodic. On contrary, let us suppose its almost periodicity. Then, the fundamental matrix is bounded (see Theorem 1.14); i.e., there exists $K_0 > 0$ with the property that
$$\| B_k \cdots B_1 \cdot B_0 \| < K_0, \quad k \in \mathbb{N}. \tag{4.43}$$
Let us choose $n \in \mathbb{N}$ for which $n \geq K_0 + 1$. We repeat that the multiplication of matrices is continuous (see also Lemma 4.11). Hence, for given matrix
$$M_1^n \cdot M_2^n \cdots M_{l_n}^n = M_{l_n}^n \cdots M_2^n \cdot M_1^n \in \mathcal{X},$$
there exists $\theta_n > 0$ such that
$$\left\| M_{l_n}^n \cdots M_2^n \cdot M_1^n \right\| - 1 < \left\| M_{l_n}^n \cdots M_2^n \cdot M_1^n \cdot C \right\|, \quad C \in \mathcal{O}_{\theta_n}^\varrho(I). \tag{4.44}$$

We can assume that $\vartheta_n = \theta_n$ in (4.41) (see also (4.38), (4.40)). We construct sequences $\{B_k^j\}$ in such a way that
$$B_0^j = I, B_1^j = I, \ldots, B_{p_2^n-1}^j = I, \quad j > n,\ j, n \in \mathbb{N}.$$
Indeed, $p_1^{j+1} > p_2^j > p_1^j$, $j \in \mathbb{N}$. Thus, (4.37), (4.41), (4.42), and (4.44) imply
$$\begin{aligned} \left\| B_{p_2^n-1} \cdots B_1 \cdot B_0 \right\| &= \left\| \tilde{B}_{p_2^n-1}^n \cdots \tilde{B}_1^n \cdot \tilde{B}_0^n \right\| \\ &= \left\| M_{l_n}^n \cdots M_2^n \cdot M_1^n \cdot \tilde{B}_{p_2^n-1}^{n-1} \cdots \tilde{B}_1^{n-1} \cdot \tilde{B}_0^{n-1} \right\| \\ &> \left\| M_{l_n}^n \cdots M_2^n \cdot M_1^n \right\| - 1 > n - 1 \geq K_0. \end{aligned} \tag{4.45}$$
This contradiction (cf. (4.43) and (4.45)) completes the proof. □

Theorem 4.31. *Let $\mathcal{X}$ have property P with respect to a vector u. For any $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$ whose fundamental matrix is not almost periodic.*

Proof. Let us consider the solution $\{x_k^0\}_{k\in\mathbb{Z}}$ of the Cauchy problem
$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

If $\{x^0_k\}$ is not almost periodic, then the statement of the theorem is true for $B_k := A_k$, $k \in \mathbb{Z}$. Hence, we assume that $\{x^0_k\}$ is almost periodic.

We put

$$\delta_n := \frac{1}{n+1} \cdot \frac{\varepsilon}{\sup\limits_{l \in \mathbb{Z}} \| A_l \|}, \quad n \in \mathbb{N}. \tag{4.46}$$

We know that there exist $\zeta > 0$ and matrices

$$\begin{gathered} M^1_1, M^1_2, \dots, M^1_{l_1} \in \mathcal{X}, \\ M^2_1, M^2_2, \dots, M^2_{l_2} \in \mathcal{X}, \\ \vdots \\ M^j_1, M^j_2, \dots, M^j_{l_j} \in \mathcal{X}, \\ \vdots \end{gathered}$$

such that

$$M^j_i \in \mathcal{O}^{\varrho}_{\delta_j}(I), \quad i \in \{1, \dots, l_j\}, \tag{4.47}$$

$$\left\| M^j_{l_j} \cdots M^j_2 \cdot M^j_1 \cdot u - u \right\| > \zeta \tag{4.48}$$

for all $j \in \mathbb{N}$. Of course, we can consider l_j such that

$$l_j \geq \dots \geq l_2 \geq l_1 \geq 2. \tag{4.49}$$

Denote

$$K_j := \left\| M^j_{l_j} \cdots M^j_2 \cdot M^j_1 \right\|, \quad j \in \mathbb{N}. \tag{4.50}$$

For

$$\vartheta_j := \frac{\zeta}{2\,(K_j + 2)}, \quad j \in \mathbb{N}, \tag{4.51}$$

we have

$$\| M \cdot v - w \| > \frac{\zeta}{2} \quad \text{if} \quad \| M \cdot u - u \| > \zeta,\ M \in \mathcal{O}^{\varrho}_{K_j+1}(O) \cap \mathcal{X},\ v, w \in \mathcal{O}^{\varrho}_{\vartheta_j}(u). \tag{4.52}$$

Indeed, for considered $u, v, w \in F^m$ and $M \in \mathcal{X}$, it holds (see (4.51))

$$\begin{aligned} \| M \cdot u - u \| &\leq \| M \cdot u - M \cdot v \| + \| M \cdot v - w \| + \| w - u \| \\ &< (K_j + 1) \| u - v \| + \| w - u \| + \| M \cdot v - w \| < \frac{\zeta}{2} + \| M \cdot v - w \|. \end{aligned}$$

The almost periodicity of $\{x^0_k\}$ (see Corollary 1.4) implies that there exists an even positive integer $j_{(1,0)}$ such that

$$\left\| x^0_0 - x^0_{j_{(1,0)}} \right\| = \left\| u - x^0_{j_{(1,0)}} \right\| < \frac{\vartheta_1}{2}. \tag{4.53}$$

Let us define a periodic sequence $\{B_k^1\}$ with period $j(1,0)+r_1$, where $r_1 := 2l_1$. If

$$\left\| x^0_{j_{(1,0)}} - x^0_{j_{(1,0)}+r_1} \right\| \geq \frac{\vartheta_1}{2}, \tag{4.54}$$

then we put $B_k^1 := I$, $k \in \mathbb{Z}$; and if

$$\left\| x^0_{j_{(1,0)}} - x^0_{j_{(1,0)}+r_1} \right\| < \frac{\vartheta_1}{2}, \tag{4.55}$$

then

$$B_0^1 := I, B_1^1 := I, \dots, B^1_{j_{(1,0)}-1} := I,$$

$$B^1_{j_{(1,0)}} := I, B^1_{j_{(1,0)}+1} := M_1^1, B^1_{j_{(1,0)}+2} := I, B^1_{j_{(1,0)}+3} := M_2^1,$$

$$\vdots$$

$$B^1_{j_{(1,0)}+2l_1-4} := I, B^1_{j_{(1,0)}+2l_1-3} := M^1_{l_1-1}, B^1_{j_{(1,0)}+2l_1-2} := I, B^1_{j_{(1,0)}+2l_1-1} := M^1_{l_1}.$$

For $\tilde{B}_k^1 := A_k \cdot B_k^1$, $k \in \mathbb{Z}$, we consider the solution $\{x_k^1\}_{k\in\mathbb{Z}}$ of the initial problem

$$x_{k+1} = \tilde{B}_k^1 \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Lemma 4.29 gives that $\{\tilde{B}_k^1\} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$. In the case, when $\{x_k^1\}$ is not almost periodic, we can put $B_k := \tilde{B}_k^1$, $k \in \mathbb{Z}$. Thus, we have to consider the almost periodicity of $\{x_k^1\}$. Especially (see Corollary 1.4), there exist infinitely many numbers $j \in 4\mathbb{N}$ with the property that

$$\| x_0^1 - x_j^1 \| = \| u - x_j^1 \| < \frac{\vartheta_2}{2}. \tag{4.56}$$

Let us consider an integer $j_{(1,1)} \in 4\mathbb{N}$ satisfying (4.56) and the inequality

$$j_{(1,1)} \geq j_{(1,0)} + r_1. \tag{4.57}$$

For $r_2 := 8l_1l_2$, we define a sequence $\{B_k^{(1,2)}\}_{k\in\mathbb{Z}}$ with period $j_{(1,1)}+r_2$. We put $B_k^{(1,2)} := I$ for all $k \in \mathbb{Z}$ if

$$\left\| x^1_{j_{(1,1)}} - x^1_{j_{(1,1)}+r_2} \right\| \geq \frac{\vartheta_2}{2}. \tag{4.58}$$

In the second case, when (4.58) is not valid, we define

$$B_0^{(1,2)} := I, B_1^{(1,2)} := I, \dots, B^{(1,2)}_{j_{(1,1)}-1} := I,$$

$$B^{(1,2)}_{j_{(1,1)}} := I, B^{(1,2)}_{j_{(1,1)}+1} := I, B^{(1,2)}_{j_{(1,1)}+2} := M_1^2, B^{(1,2)}_{j_{(1,1)}+3} := I,$$

$$B^{(1,2)}_{j_{(1,1)}+4} := I, B^{(1,2)}_{j_{(1,1)}+5} := I, B^{(1,2)}_{j_{(1,1)}+6} := M_2^2, B^{(1,2)}_{j_{(1,1)}+7} := I,$$

$$\vdots$$

$$B^{(1,2)}_{j_{(1,1)}+4l_2-4} := I, B^{(1,2)}_{j_{(1,1)}+4l_2-3} := I, B^{(1,2)}_{j_{(1,1)}+4l_2-2} := M^2_{l_2}, B^{(1,2)}_{j_{(1,1)}+4l_2-1} := I,$$

$$B^{(1,2)}_{j_{(1,1)}+4l_2} := I, B^{(1,2)}_{j_{(1,1)}+4l_2+1} := I, B^{(1,2)}_{j_{(1,1)}+4l_2+2} := I, B^{(1,2)}_{j_{(1,1)}+4l_2+3} := I,$$

$$\vdots$$

$$B^{(1,2)}_{j_{(1,1)}+r_2-1} := I.$$

For $\tilde{B}^{(1,2)}_k := A_k \cdot B^1_k \cdot B^{(1,2)}_k$, $k \in \mathbb{Z}$, we consider the solution $\{x^{(1,2)}_k\}_{k\in\mathbb{Z}}$ of the initial problem

$$x_{k+1} = \tilde{B}^{(1,2)}_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Again, we can assume that $\{x^{(1,2)}_k\}_{k\in\mathbb{Z}}$ is almost periodic. Let an integer $j_{(2,1)} \in 8\mathbb{N}$ have the properties that

$$\left\| x^{(1,2)}_0 - x^{(1,2)}_{j_{(2,1)}} \right\| = \left\| u - x^{(1,2)}_{j_{(2,1)}} \right\| < \frac{\vartheta_2}{2} \tag{4.59}$$

and that

$$j_{(2,1)} \geq j_{(1,1)} + r_2. \tag{4.60}$$

We define a periodic sequence $\{B^{(2,2)}_k\}_{k\in\mathbb{Z}}$ with period $j_{(2,1)}(r_2 - r_1)$. If

$$\left\| x^{(1,2)}_{j_{(2,1)}} - x^{(1,2)}_{j_{(2,1)}+r_2-r_1} \right\| \geq \frac{\vartheta_2}{2}, \tag{4.61}$$

we put $B^{(2,2)}_k := I$ for all $k \in \mathbb{Z}$; and, in the other case, we define

$$B^{(2,2)}_0 := I, B^{(2,2)}_1 := I, \dots, B^{(2,2)}_{j_{(2,1)}-1} := I,$$

$$B^{(2,2)}_{j_{(2,1)}} := I, B^{(2,2)}_{j_{(2,1)}+1} := I, B^{(2,2)}_{j_{(2,1)}+2} := I, B^{(2,2)}_{j_{(2,1)}+3} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+4} := M^2_1, B^{(2,2)}_{j_{(2,1)}+5} := I, B^{(2,2)}_{j_{(2,1)}+6} := I, B^{(2,2)}_{j_{(2,1)}+7} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+8} := I, B^{(2,2)}_{j_{(2,1)}+9} := I, B^{(2,2)}_{j_{(2,1)}+10} := I, B^{(2,2)}_{j_{(2,1)}+11} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+12} := M^2_2, B^{(2,2)}_{j_{(2,1)}+13} := I, B^{(2,2)}_{j_{(2,1)}+14} := I, B^{(2,2)}_{j_{(2,1)}+15} := I,$$

$$\vdots$$

$$B^{(2,2)}_{j_{(2,1)}+8l_2-8} := I, B^{(2,2)}_{j_{(2,1)}+8l_2-7} := I, B^{(2,2)}_{j_{(2,1)}+8l_2-6} := I, B^{(2,2)}_{j_{(2,1)}+8l_2-5} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+8l_2-4} := M^2_{l_2}, B^{(2,2)}_{j_{(2,1)}+8l_2-3} := I, B^{(2,2)}_{j_{(2,1)}+8l_2-2} := M^2_{l_2}, B^{(2,2)}_{j_{(2,1)}+8l_2-1} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+8l_2} := I, B^{(2,2)}_{j_{(2,1)}+8l_2+1} := I, B^{(2,2)}_{j_{(2,1)}+8l_2+2} := I, B^{(2,2)}_{j_{(2,1)}+8l_2+3} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+8l_2+4} := I, B^{(2,2)}_{j_{(2,1)}+8l_2+5} := I, B^{(2,2)}_{j_{(2,1)}+8l_2+6} := I, B^{(2,2)}_{j_{(2,1)}+8l_2+7} := I,$$

$$\vdots$$

$$B^{(2,2)}_{j_{(2,1)}+r_2-r_1-1} := I, \dots, B^{(2,2)}_{j_{(2,1)}(r_2-r_1)-1} := I.$$

Finally, in the second step, we consider the periodic sequence of

$$B^2_k := B^{(1,2)}_k \cdot B^{(2,2)}_k, \quad k \in \mathbb{Z}. \tag{4.62}$$

Note that its period is $[j_{(1,1)} + r_2][j_{(2,1)}(r_2 - r_1)]$. Consequently, we consider

$$\tilde{B}_k^2 := A_k \cdot B_k^1 \cdot B_k^2, \quad k \in \mathbb{Z}, \tag{4.63}$$

and the solution $\{x_k^2\}_{k\in\mathbb{Z}}$ of

$$x_{k+1} = \tilde{B}_k^2 \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

In the case, when $\{x_k^2\}$ is not almost periodic, we can put $B_k := \tilde{B}_k^2$ for $k \in \mathbb{Z}$ and use Lemma 4.29 for $B_k^{j+2} = I$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$ (see also (4.63)). Thus, we have to assume that $\{x_k^2\}$ is almost periodic.

We continue in the same manner. Before the n-th step, we define

$$\tilde{B}_k^{n-1} := A_k \cdot B_k^1 \cdot B_k^2 \cdots B_k^{n-1}, \quad k \in \mathbb{Z}.$$

Let $\{\tilde{B}_k^{n-1}\}_{k\in\mathbb{Z}}$ have period q_{n-1}, e.g., let

$$\begin{aligned} q_{n-1} :=& [j_{(1,0)} + r_1][j_{(1,1)} + r_2][j_{(2,1)}(r_2 - r_1)] \times \cdots \\ & \cdots \times [j_{(1,n-2)} + r_{n-1}][j_{(2,n-2)}(r_{n-1} - r_1)] \cdots [j_{(n-1,n-2)}(r_{n-1} - r_{n-2})]. \end{aligned}$$

Consider the solution $\{x_k^{n-1}\}$ of

$$x_{k+1} = \tilde{B}_k^{n-1} \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Again, we consider that the sequence $\{x_k^{n-1}\}$ is almost periodic. Otherwise, we can put $B_k := \tilde{B}_k^{n-1}$, $k \in \mathbb{Z}$. Especially, for all $p \in \mathbb{N}$, there exist infinitely many numbers $j \in p\mathbb{N}$ with the property that

$$\left\| x_0^{n-1} - x_j^{n-1} \right\| = \left\| u - x_j^{n-1} \right\| < \frac{\vartheta_n}{2}. \tag{4.64}$$

Denote

$$p_n := 2^{1+\sum_{i=1}^{n-1} i}, \quad n \geq 2,\, n \in \mathbb{N}, \tag{4.65}$$

$$r_n := 2p_n l_1 l_2 \cdots l_n, \quad n \geq 2,\, n \in \mathbb{N}. \tag{4.66}$$

Let us consider an integer $j_{(1,n-1)} \in p_n\mathbb{N}$ satisfying (4.64) and

$$j_{(1,n-1)} \geq q_{n-1}. \tag{4.67}$$

We define $\{B_k^{(1,n)}\}_{k\in\mathbb{Z}}$ with period $j_{(1,n-1)} + r_n$. If

$$\left\| x_{j_{(1,n-1)}}^{n-1} - x_{j_{(1,n-1)}+r_n}^{n-1} \right\| \geq \frac{\vartheta_n}{2}, \tag{4.68}$$

we put $B_k^{(1,n)} := I$, $k \in \mathbb{Z}$. In the other case, we put

$$B_0^{(1,n)} := I, B_1^{(1,n)} := I, \ldots, B_{j_{(1,n-1)}-1}^{(1,n)} := I,$$

$$B_{j_{(1,n-1)}}^{(1,n)} := I, B_{j_{(1,n-1)}+1}^{(1,n)} := I, \ldots, B_{j_{(1,n-1)}+\frac{p_n}{2}-1}^{(1,n)} := I,$$

$$B^{(1,n)}_{j_{(1,n-1)}+\frac{p_n}{2}} := M^n_1, B^{(1,n)}_{j_{(1,n-1)}+\frac{p_n}{2}+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+p_n-1} := I,$$
$$B^{(1,n)}_{j_{(1,n-1)}+p_n} := I, B^{(1,n)}_{j_{(1,n-1)}+p_n+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+p_n+\frac{p_n}{2}-1} := I,$$
$$B^{(1,n)}_{j_{(1,n-1)}+p_n+\frac{p_n}{2}} := M^n_2, B^{(1,n)}_{j_{(1,n-1)}+p_n+\frac{p_n}{2}+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+2p_n-1} := I,$$
$$\vdots$$
$$B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n} := I, B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n+\frac{p_n}{2}-1} := I,$$
$$B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n+\frac{p_n}{2}} := M^n_{l_n}, B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n+\frac{p_n}{2}+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+l_np_n-1} := I,$$
$$B^{(1,n)}_{j_{(1,n-1)}+l_np_n} := I, B^{(1,n)}_{j_{(1,n-1)}+l_np_n+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+l_np_n+\frac{p_n}{2}-1} := I,$$
$$B^{(1,n)}_{j_{(1,n-1)}+l_np_n+\frac{p_n}{2}} := I, B^{(1,n)}_{j_{(1,n-1)}+l_np_n+\frac{p_n}{2}+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+(l_n+1)p_n-1} := I,$$
$$\vdots$$
$$B^{(1,n)}_{j_{(1,n-1)}+r_n-1} := I.$$

For
$$\tilde{B}^{(1,n)}_k := \tilde{B}^{n-1}_k \cdot B^{(1,n)}_k = A_k \cdot B^1_k \cdot B^2_k \cdots B^{n-1}_k \cdot B^{(1,n)}_k, \quad k \in \mathbb{Z},$$
we consider the solution $\{x^{(1,n)}_k\}$ of

$$x_{k+1} = \tilde{B}^{(1,n)}_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Again, we assume that $\{x^{(1,n)}_k\}$ is almost periodic. Let a number $j_{(2,n-1)} \in 2p_n\mathbb{N}$ have the properties that
$$\left\| x^{(1,n)}_0 - x^{(1,n)}_{j_{(2,n-1)}} \right\| = \left\| u - x^{(1,n)}_{j_{(2,n-1)}} \right\| < \frac{\vartheta_n}{2} \tag{4.69}$$
and
$$j_{(2,n-1)} \geq j_{(1,n-1)} + r_n. \tag{4.70}$$
We define the following periodic sequence $\{B^{(2,n)}_k\}_{k\in\mathbb{Z}}$ with period $j_{(2,n-1)}(r_n - r_1)$. If
$$\left\| x^{(1,n)}_{j_{(2,n-1)}} - x^{(1,n)}_{j_{(2,n-1)}+r_n-r_1} \right\| \geq \frac{\vartheta_n}{2}, \tag{4.71}$$
then $B^{(2,n)}_k := I$, $k \in \mathbb{Z}$. In the other case, we put
$$B^{(2,n)}_0 := I, B^{(2,n)}_1 := I, \dots, B^{(2,n)}_{j_{(2,n-1)}-1} := I,$$
$$B^{(2,n)}_{j_{(2,n-1)}} := I, B^{(2,n)}_{j_{(2,n-1)}+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+p_n-1} := I,$$
$$B^{(2,n)}_{j_{(2,n-1)}+p_n} := M^n_1, B^{(2,n)}_{j_{(2,n-1)}+p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2p_n-1} := I,$$
$$B^{(2,n)}_{j_{(2,n-1)}+2p_n} := I, B^{(2,n)}_{j_{(2,n-1)}+2p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2p_n+p_n-1} := I,$$
$$B^{(2,n)}_{j_{(2,n-1)}+2p_n+p_n} := M^n_2, B^{(2,n)}_{j_{(2,n-1)}+2p_n+p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+4p_n-1} := I,$$

$$\vdots$$

$$B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n} := I, B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n+p_n-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n+p_n} := M^n_{l_n}, B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n+p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2l_np_n-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}+2l_np_n} := I, B^{(2,n)}_{j_{(2,n-1)}+2l_np_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2l_np_n+p_n-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}+2l_np_n+p_n} := I, B^{(2,n)}_{j_{(2,n-1)}+2l_np_n+p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2(l_n+1)p_n-1} := I,$$

$$\vdots$$

$$B^{(2,n)}_{j_{(2,n-1)}+r_n-r_1-1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}(r_n-r_1)-1} := I.$$

We continue in the n-th step. We define

$$\tilde{B}^{(n-1,n)}_k := \tilde{B}^{n-1}_k \cdot B^{(1,n)}_k \cdot B^{(2,n)}_k \cdots B^{(n-1,n)}_k, \quad k \in \mathbb{Z}.$$

We consider the solution $\{x^{(n-1,n)}_k\}_{k\in\mathbb{Z}}$ of

$$x_{k+1} = \tilde{B}^{(n-1,n)}_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Again, we have to assume that $\{x^{(n-1,n)}_k\}$ is almost periodic. Let $j_{(n,n-1)} \in 2^{n-1}p_n\mathbb{N}$ satisfy

$$\left\| x^{(n-1,n)}_0 - x^{(n-1,n)}_{j_{(n,n-1)}} \right\| = \left\| u - x^{(n-1,n)}_{j_{(n,n-1)}} \right\| < \frac{\vartheta_n}{2} \tag{4.72}$$

and

$$j_{(n,n-1)} \geq j_{(n-1,n-1)}(r_n - r_{n-2}). \tag{4.73}$$

We define a periodic sequence $\{B^{(n,n)}_k\}_{k\in\mathbb{Z}}$ with period $j_{(n,n-1)}(r_n - r_{n-1})$. If

$$\left\| x^{(n-1,n)}_{j_{(n,n-1)}} - x^{(n-1,n)}_{j_{(n,n-1)}+r_n-r_{n-1}} \right\| \geq \frac{\vartheta_n}{2}, \tag{4.74}$$

we put $B^{(n,n)}_k := I$, $k \in \mathbb{Z}$. If inequality (4.74) is not valid, we put

$$B^{(n,n)}_0 := I, B^{(n,n)}_1 := I, \dots, B^{(n,n)}_{j_{(n,n-1)}-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}} := I, B^{(n,n)}_{j_{(n,n-1)}+1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}+2^{n-2}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+2^{n-2}p_n} := M^n_1, B^{(n,n)}_{j_{(n,n-1)}+2^{n-2}p_n+1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n} := I, B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n+1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n+2^{n-2}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n+2^{n-2}p_n} := M^n_2, B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n+2^{n-2}p_n+1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}+2^np_n-1} := I,$$

$$\vdots$$

$$B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n} := I, B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n+1} := I,$$
$$\dots, B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n+2^{n-2}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n+2^{n-2}p_n} := M^n_{l_n}, B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n+2^{n-2}p_n+1} := I,$$
$$\dots, B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n} := I, B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n+1} := I,$$
$$\dots, B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n+2^{n-2}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n+2^{n-2}p_n} := I, B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n+2^{n-2}p_n+1} := I,$$
$$\dots, B^{(n,n)}_{j_{(n,n-1)}+(l_n+1)2^{n-1}p_n-1} := I,$$

$$\vdots$$

$$B^{(n,n)}_{j_{(n,n-1)}+r_n-r_{n-1}-1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}(r_n-r_{n-1})-1} := I,$$

where (see (4.49))

$$r_n - r_{n-1} = r_{n-1}\left[2^{n-1}l_n - 1\right] \geq p_{n-1}2^n\left[2^{n-1}l_n-1\right] = 2p_n\left[2^{n-1}l_n-1\right] > l_n2^{n-1}p_n.$$

Finally, in the n-th step, we define

$$B^n_k := B^{(1,n)}_k \cdot B^{(2,n)}_k \cdots B^{(n,n)}_k, \quad k \in \mathbb{Z}, \tag{4.75}$$

and

$$\tilde{B}^n_k := A_k \cdot B^1_k \cdot B^2_k \cdots B^n_k, \quad k \in \mathbb{Z}.$$

Then, we consider the solution $\{x^n_k\}_{k\in\mathbb{Z}}$ of

$$x_{k+1} = \tilde{B}^n_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Applying Lemma 4.29 for $B^{n+j}_k = I$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$, it suffices to consider the case, when $\{x^n_k\}$ is almost periodic, and to continue in the construction.

All sequence $\{B^n_k\}_{k\in\mathbb{Z}}$ is periodic as the product of n periodic sequences. Let q_n be a period of $\{B^n_k\}$, $n \in \mathbb{N}$. In the construction, we can obtain matrices different from I only for

$$B^1_{2l+1}, B^{(1,2)}_{4l+2}, B^{(2,2)}_{8l+4}, \dots, B^{(1,n)}_{lp_n+\frac{p_n}{2}}, B^{(2,n)}_{2lp_n+p_n}, \dots, B^{(n,n)}_{2^{n-1}lp_n+2^{n-2}p_n}, \dots, \tag{4.76}$$

where $l \in \mathbb{Z}$. Considering (4.65) and (4.75) (see also (4.62)), the structure of the indices of matrices in (4.76) gives (4.29). It is seen that (4.27) and (4.30) follow from (4.46) and from the construction. Analogously, (4.28) follows from (4.47). Thus, applying Lemma 4.29 for the sequence of

$$B_k := A_k \cdot B^1_k \cdot B^2_k \cdots B^n_k \cdots, \quad k \in \mathbb{Z},$$

we have that $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$.

To complete the proof, it suffices to show that the solution $\{x_k\}_{k\in\mathbb{Z}}$ of the problem

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is not almost periodic. On contrary, let us suppose that $\{x_k\}$ is almost periodic. We use Theorem 1.3 for $h_1 = 0$, $h_{n+1} = r_n$, $n \in \mathbb{N}$ (see (4.66)). We know that, for any $\xi > 0$, there exist infinitely many $i, j \in \mathbb{N}$ satisfying

$$\left\| x_{k+l_i} - x_{k+l_j} \right\| < \xi, \quad k \in \mathbb{Z}. \tag{4.77}$$

From the construction (consider (4.57), (4.60), ..., (4.67), (4.70), ..., (4.73)), we obtain

$$B_k^{n+j} = I, \quad k \in \{0, 1, \dots, q_n - 1\},\ n, j \in \mathbb{N}. \tag{4.78}$$

Hence, we get

$$\begin{aligned}\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| &= \left\| B_{j(1,0)-1} \cdots B_1 \cdot B_0 \cdot u - B_{j(1,0)+r_1-1} \cdots B_1 \cdot B_0 \cdot u \right\| \\ &= \left\| B^1_{j(1,0)-1} \cdots B^1_1 \cdot B^1_0 \cdot A_{j(1,0)-1} \cdots A_1 \cdot A_0 \cdot u \right. \\ &\quad \left. - B^1_{j(1,0)+r_1-1} \cdots B^1_1 \cdot B^1_0 \cdot A_{j(1,0)+r_1-1} \cdots A_1 \cdot A_0 \cdot u \right\| \\ &= \left\| B^1_{j(1,0)-1} \cdots B^1_1 \cdot B^1_0 \cdot x^0_{j(1,0)} - B^1_{j(1,0)+r_1-1} \cdots B^1_1 \cdot B^1_0 \cdot x^0_{j(1,0)+r_1} \right\|,\end{aligned}$$

i.e.,

$$\begin{aligned}&\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| \\ &= \left\| B^1_{j(1,0)-1} \cdots B^1_1 \cdot B^1_0 \cdot x^0_{j(1,0)} - B^1_{j(1,0)+r_1-1} \cdots B^1_1 \cdot B^1_0 \cdot x^0_{j(1,0)+r_1} \right\|.\end{aligned} \tag{4.79}$$

If (4.54) is valid, then we can rewrite (4.79) into

$$\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| = \left\| I \cdots I \cdot I \cdot x^0_{j(1,0)} - I \cdots I \cdot I \cdot x^0_{j(1,0)+r_1} \right\| \geq \frac{\vartheta_1}{2}.$$

If (4.55) is true, then we have

$$\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| = \left\| I \cdots I \cdot I \cdot x^0_{j(1,0)} - M^1_{l_1} \cdots M^1_2 \cdot M^1_1 \cdot x^0_{j(1,0)+r_1} \right\| > \frac{\zeta}{2} \geq \frac{\vartheta_1}{2},$$

which follows from (4.48), (4.51), (4.52), and from (see (4.53), (4.55))

$$\left\| u - x^0_{j(1,0)+r_1} \right\| \leq \left\| u - x^0_{j(1,0)} \right\| + \left\| x^0_{j(1,0)} - x^0_{j(1,0)+r_1} \right\| < \frac{\vartheta_1}{2} + \frac{\vartheta_1}{2} = \vartheta_1.$$

In the both cases, we get

$$\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| \geq \frac{\vartheta_1}{2}. \tag{4.80}$$

Considering (4.78) and the construction, we can express

$$\begin{aligned}\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| &= \left\| B_{j(1,1)-1} \cdots B_1 \cdot B_0 \cdot u - B_{j(1,1)+r_2-1} \cdots B_1 \cdot B_0 \cdot u \right\| \\ &= \Big\| B^{(1,2)}_{j(1,1)-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot \tilde{B}^1_{j(1,1)-1} \cdots \tilde{B}^1_1 \cdot \tilde{B}^1_0 \cdot u \\ &\qquad - B^{(1,2)}_{j(1,1)+r_2-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot \tilde{B}^1_{j(1,1)+r_2-1} \cdots \tilde{B}^1_1 \cdot \tilde{B}^1_0 \cdot u \Big\| \\ &= \Big\| B^{(1,2)}_{j(1,1)-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot x^1_{j(1,1)} \\ &\qquad - B^{(1,2)}_{j(1,1)+r_2-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot x^1_{j(1,1)+r_2} \Big\|,\end{aligned}$$

i.e.,

$$\begin{aligned}&\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| \\ &= \left\| B^{(1,2)}_{j(1,1)-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot x^1_{j(1,1)} - B^{(1,2)}_{j(1,1)+r_2-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot x^1_{j(1,1)+r_2} \right\|.\end{aligned} \tag{4.81}$$

If (4.58) is valid, then (4.81) takes the form

$$\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| = \left\| I \cdots I \cdot I \cdot x^1_{j(1,1)} - I \cdots I \cdot I \cdot x^1_{j(1,1)+r_2} \right\| \geq \frac{\vartheta_2}{2}. \tag{4.82}$$

If (4.58) is not valid, then we have

$$\begin{aligned}&\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| \\ &\quad = \left\| I \cdots I \cdot I \cdot x^1_{j(1,1)} - M^2_{l_2} \cdots M^2_2 \cdot M^2_1 \cdot x^1_{j(1,1)+r_2} \right\| > \frac{\zeta}{2} \geq \frac{\vartheta_2}{2}.\end{aligned} \tag{4.83}$$

Indeed, it suffices to consider (4.48), (4.51), (4.52), and the inequality (see also (4.56))

$$\left\| u - x^1_{j(1,1)+r_2} \right\| \leq \left\| u - x^1_{j(1,1)} \right\| + \left\| x^1_{j(1,1)} - x^1_{j(1,1)+r_2} \right\| < \frac{\vartheta_2}{2} + \frac{\vartheta_2}{2} = \vartheta_2.$$

Again, one can express

$$\begin{aligned}\left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| &= \left\| B_{j(2,1)-1} \cdots B_1 \cdot B_0 \cdot u - B_{j(2,1)+r_2-r_1-1} \cdots B_1 \cdot B_0 \cdot u \right\| \\ &= \Big\| B^{(2,2)}_{j(2,1)-1} \cdots B^{(2,2)}_1 \cdot B^{(2,2)}_0 \cdot \tilde{B}^{(1,2)}_{j(2,1)-1} \cdots \tilde{B}^{(1,2)}_1 \cdot \tilde{B}^{(1,2)}_0 \cdot u \\ &\qquad - B^{(2,2)}_{j(2,1)+r_2-r_1-1} \cdots B^{(2,2)}_0 \cdot \tilde{B}^{(1,2)}_{j(2,1)+r_2-r_1-1} \cdots \tilde{B}^{(1,2)}_0 \cdot u \Big\| \\ &= \Big\| B^{(2,2)}_{j(2,1)-1} \cdots B^{(2,2)}_1 \cdot B^{(2,2)}_0 \cdot x^{(1,2)}_{j(2,1)} \\ &\qquad - B^{(2,2)}_{j(2,1)+r_2-r_1-1} \cdots B^{(2,2)}_1 \cdot B^{(2,2)}_0 \cdot x^{(1,2)}_{j(2,1)+r_2-r_1} \Big\|,\end{aligned}$$

i.e.,

$$\left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| = \left\| B^{(2,2)}_{j(2,1)-1} \cdots B^{(2,2)}_1 \cdot B^{(2,2)}_0 \cdot x^{(1,2)}_{j(2,1)} - B^{(2,2)}_{j(2,1)+r_2-r_1-1} \cdots B^{(2,2)}_1 \cdot B^{(2,2)}_0 \cdot x^{(1,2)}_{j(2,1)+r_2-r_1} \right\|. \tag{4.84}$$

If (4.61) is valid, then (4.84) gives

$$\left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| = \left\| I \cdots I \cdot I \cdot x^{(1,2)}_{j(2,1)} - I \cdots I \cdot I \cdot x^{(1,2)}_{j(2,1)+r_2-r_1} \right\| \geq \frac{\vartheta_2}{2}. \tag{4.85}$$

If (4.61) is not valid, then (4.84) gives

$$\begin{aligned} &\left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| \\ &\quad = \left\| I \cdots I \cdot I \cdot x^{(1,2)}_{j(2,1)} - M^2_{l_2} \cdots M^2_2 \cdot M^2_1 \cdot x^{(1,2)}_{j(2,1)+r_2-r_1} \right\| > \frac{\zeta}{2} \geq \frac{\vartheta_2}{2}, \end{aligned} \tag{4.86}$$

where (4.48), (4.51), (4.52), (4.59), and (4.61) are used.

Finally (see (4.82), (4.83), (4.85), and (4.86)), from the second step of the construction, we have

$$\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| \geq \frac{\vartheta_2}{2}, \quad \left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| \geq \frac{\vartheta_2}{2}. \tag{4.87}$$

Analogously as (4.80) and (4.87) (consider again (4.48), (4.51), (4.52) and the construction with (4.64), (4.68), (4.69), (4.71), ..., (4.72), (4.74)), one can obtain

$$\left\| x_{j_{(1,n-1)}} - x_{j_{(1,n-1)}+r_n} \right\| \geq \frac{\vartheta_n}{2},$$

$$\left\| x_{j_{(2,n-1)}} - x_{j_{(2,n-1)}+r_n-r_1} \right\| \geq \frac{\vartheta_n}{2},$$

$$\vdots$$

$$\left\| x_{j_{(n,n-1)}} - x_{j_{(n,n-1)}+r_n-r_{n-1}} \right\| \geq \frac{\vartheta_n}{2}$$

for all $n \in \mathbb{N}$.

Considering Lemma 4.30, we can assume that (see (4.50) and (4.51))

$$\sup_{j\in\mathbb{N}} K_j < \infty, \qquad \text{i.e.,} \qquad \vartheta := \inf_{j\in\mathbb{N}} \vartheta_j > 0. \tag{4.88}$$

Thus, for all $n \in \mathbb{N}$, we obtain

$$\left\| x_{j_{(1,n-1)}} - x_{j_{(1,n-1)}+r_n} \right\| \geq \frac{\vartheta}{2},$$

$$\left\| x_{j_{(2,n-1)}} - x_{j_{(2,n-1)}+r_n-r_1} \right\| \geq \frac{\vartheta}{2},$$

$$\vdots$$

$$\left\| x_{j_{(n,n-1)}} - x_{j_{(n,n-1)}+r_n-r_{n-1}} \right\| \geq \frac{\vartheta}{2}.$$

Especially, for all $i \neq j$, $i, j \in \mathbb{N}$, there exists $l \in \mathbb{Z}$ such that

$$\| x_{l+l_i} - x_{l+l_j} \| \geq \frac{\vartheta}{2}.$$

This contradiction (consider (4.77) for $2\xi \leq \vartheta$) proves that $\{x_k\}$ is not almost periodic. □

Remark 4.32. It is easy to see that the statement of Theorem 4.31 does not change if one replaces system $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ by a periodic one. Indeed, it follows directly from Definition 1.17. △

Remark 4.33. To illustrate Theorem 4.31, let us consider an arbitrary periodic system $\{M_k\}$ in the complex case (i.e., for $F = \mathbb{C}$ with the usual absolute value). It means that we have a system

$$x_{k+1} = M_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad \text{where} \qquad M_k = M_{k+p}, \quad k \in \mathbb{Z},$$

for a positive integer p and arbitrarily given non-singular complex matrices $M_0, \dots, M_{p-1}$. We know that a solution of $\{M_k\}$ is almost periodic if and only if it is bounded (see Corollary 2.22). The fundamental matrix $\Phi(k, 0)$ of $\{M_k\}$ satisfying $\Phi(0, 0) = I$ is given by

$$\Phi(lp + i, 0) = M_{i-1} \cdots M_1 \cdot M_0 \cdot (M_{p-1} \cdots M_1 \cdot M_0)^l, \quad l \in \mathbb{N} \cup \{0\}, \, i \in \{1, \dots, p\}.$$

Thus, to describe the structure of almost periodic solutions, it suffices to consider the multiples $(M_{p-1} \cdots M_1 \cdot M_0)^l$ and, in fact, the constant system

$$x_{k+1} = M_{p-1} \cdots M_1 \cdot M_0 \cdot x_k, \quad k \in \mathbb{Z}.$$

For any constant system given by a non-singular complex matrix M, one can easily find a commutative matrix group $\mathcal{X}$ containing M and having property P with respect to a vector (e.g., one can consider the group generated by matrices cM for all complex numbers $c = \sin l + \mathrm{i} \cos l$, $l \in \mathbb{Z}$). Applying Theorem 4.31, we know that, in any neighbourhood of the considered system, there exists a limit periodic system whose coefficient matrices are from the group and whose fundamental matrix is not almost periodic. In addition, such a limit periodic system can be found for any commutative group $\mathcal{X}$ which contains M and which has property P with respect to at least one vector. △

Remark 4.34. We repeat that the basic motivation comes from the previous section, where non-asymptotically almost periodic solutions of limit periodic systems are considered. Of course, systems with coefficient matrices from bounded groups are analysed in Section 4.3. For general groups, it is not possible to prove the main results of Section 4.3. It suffices to consider the constant system given by matrix $I/2$ in the complex case. Any solution $\{x_k\}_{k \in \mathbb{Z}}$ of this system has the property that

$$|| \, x_{l+1} \, || = \frac{|| \, x_l \, ||}{2}, \quad l \in \mathbb{N}.$$

Thus, there exists a neighbourhood of the system such that, for any solution $\{y_k\}_{k\in\mathbb{Z}}$ of an almost periodic system from the neighbourhood, we obtain $\lim_{k\to\infty} || \, y_k \, || = 0$, which gives the asymptotic almost periodicity of $\{y_k\}$ (see Remark 1.16).

At the same time, in Section 4.3, there is required that the studied matrix group has property P. Since the group $\mathcal{X}$ has property P only with respect to one vector in the statement of Theorem 4.31, we can apply this theorem for groups of matrices in the following form

$$\begin{pmatrix} X & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

where X is taken from a commutative matrix group having property P with respect to a concrete vector. In this sense, Theorem 4.31 generalizes Theorem 4.16 as well. △

The construction from the proof of Theorem 4.31 can be applied for the Cauchy (initial) problem. Especially, we immediately obtain the following result.

Theorem 4.35. *Let a non-zero vector $u \in F^m$ be given. Let $\mathcal{X}$ have the property that there exist $\zeta > 0$ and $K > 0$ such that, for all $\delta > 0$, one can find matrices $M_1, \ldots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}^{\varrho}_{\delta}(I), \ i \in \{1, \ldots, l\}, \quad \| \, M_l \cdots M_1 \cdot u - u \, \| > \zeta, \quad \| \, M_l \cdots M_1 \, \| < K. \tag{4.89}$$

For any $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$ for which the solution of

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is not almost periodic.

Proof. The theorem follows from the proof of Theorem 4.31, where (4.88) is satisfied (i.e., the case, which is covered by Lemma 4.30, does not happen). □

Remark 4.36. We point out that, in a certain sense, Theorem 4.35 has been improved in [41]. △

Similarly to Theorem 4.23 which is the almost periodic version of Theorem 4.16, we formulate the below given Theorem 4.39 as the almost periodic version of Theorem 4.31. We need the next two lemmas.

Lemma 4.37. *Let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrarily given. Let $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a decreasing sequence satisfying* (4.27) *and let $\{B^n_k\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ be periodic sequences for $n \in \mathbb{N}$ such that* (4.28) *and* (4.29) *are valid. Then, $\{B_k\} \in \mathcal{AP}(\mathcal{X})$ if*

$$B_k := A_k \cdot B^1_k \cdot B^2_k \cdots B^n_k \cdots, \quad k \in \mathbb{Z}.$$

In addition, $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ if (4.30) *is fulfilled.*

Proof. The lemma can be proved analogously as Lemma 4.29. In the proof of Lemma 4.29, it suffices to put $C_k^n = A_k$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$, to use Theorem 1.7, and to consider the almost periodicity of $\{C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ which follows from Theorems 1.3, 1.14, and 1.22 and from Lemma 4.11. □

Using the same way which is applied in the proof of Lemma 4.30, we can prove its almost periodic counterpart. Indeed, we do not use the limit periodicity of $\{A_k\}$ in the proof (consider also Lemma 4.37).

Lemma 4.38. *If for any $\delta > 0$ and $K > 0$, there exist matrices $M_1, \dots, M_l \in \mathcal{X}$ such that* (4.34) *is valid, then, for any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ whose fundamental matrix is not almost periodic.*

Theorem 4.39. *Let $\mathcal{X}$ have property P with respect to a vector. For any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ whose fundamental matrix is not almost periodic.*

Proof. The theorem can be proved using the same construction as Theorem 4.31. It suffices to replace Lemma 4.29 by Lemma 4.37 and Lemma 4.30 by Lemma 4.38. □

Analogously, we get the following result as well.

Theorem 4.40. *Let a non-zero vector $u \in F^m$ be given. Let $\mathcal{X}$ have the property that there exist $\zeta > 0$ and $K > 0$ such that, for all $\delta > 0$, one can find matrices $M_1, \dots, M_l \in \mathcal{X}$ satisfying* (4.89)*. For any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ for which the solution of*

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is not almost periodic.

Remark 4.41. We add that Theorems 4.39 and 4.40 do not follow from Theorems 4.31 and 4.35. Consider Theorem 1.21. △

At the end, we remark that all main results presented in this chapter remain true if one replaces $k \in \mathbb{Z}$ by $k \in \mathbb{N}$; and we repeat that the main results of Chapter 2 are not covered by results about almost periodic systems presented in this chapter.

Chapter 5

Almost periodic and limit periodic functions in pseudometric spaces

This chapter is analogous to Chapter 1, where almost periodic and limit periodic sequences are considered. Here we consider almost periodic and limit periodic functions. Our aim is to mention basic properties of considered functions and to show a way one can generate functions with several prescribed properties. Since our process can be used for generalizations of classical (complex valued) almost periodic (and limit periodic) functions, we introduce the almost and limit periodicity in pseudometric spaces and we present our method for functions with values in a pseudometric space $\mathcal{X}$ as in Chapter 1.

We point out that we obtain the most important case if $\mathcal{X}$ is a Banach space, and that the theory of almost periodic functions of the real variable with values in a Banach space, given by S. Bochner in [22], is in its essential lines similar to the theory of classical almost periodic functions which is due to H. Bohr in [27, 28]. We introduce almost periodic and limit periodic functions in pseudometric spaces using a trivial extension of the Bohr concept, where the modulus is replaced by the distance. In the classical case, we refer to the monographs [18, 72, 128]; for functions with values in Banach spaces, to [7, 46, 117]; for other extensions, to [9, 12, 19, 20, 31, 77, 84, 99, 181]; for modifications, to [46, 85] and references cited therein; for applications, to [32, 47, 160, 164].

Necessary and sufficient conditions for a continuous function with values in a Banach space to be almost periodic may be no longer valid for continuous functions in general metric spaces. For the approximation condition, it is seen that the completeness of the space of values is necessary and H. Tornehave (in [177]) also required the local connection by arcs of the space of values. In the Bochner condition, it suffices to replace the convergence by the Cauchy condition. Since we need the Bochner concept as well, we recall that the Bochner condition means that any sequence of translates of a given continuous function has a subsequence which converges uniformly on the domain of the function. The fact, that this condition is equivalent with the Bohr definition of almost periodicity in Banach spaces, was proved by S. Bochner in [22].

We begin with the used notation in Section 5.1. The above mentioned Bohr definition and Bochner condition are formulated in Section 5.2 (with some basic properties of considered functions). In Section 5.2, processes from [46] are generalized. Analogously, the theory of almost periodic functions of real variable with fuzzy real numbers as values is developed in [13] (see also [157]). In Section 5.3, we mention the way one can construct almost periodic functions with prescribed properties in a pseudometric space. We present it in Theorems 5.20, 5.22, and 5.24 below. Note that it is possible to obtain many modifications and generalizations of our method. A special construction of almost periodic functions with given properties is published (and applied) in [101] as well.

5.1 Preliminaries

Let $\mathcal{X}$ be an arbitrary pseudometric space with a pseudometric ϱ. Symbol $\mathcal{O}_\varepsilon(x)$ denotes the ε-neighbourhood of x in $\mathcal{X}$ for arbitrary $\varepsilon > 0$, $x \in \mathcal{X}$. The set of all non-negative real numbers is denoted by $\mathbb{R}_0^+$.

5.2 Generalizations of pure periodicity

As in the first chapter, we define the notion of almost and limit periodicity in pseudometric spaces.

5.2.1 Almost periodic functions

At first, we introduce the almost periodicity in $\mathcal{X}$. Observe that we are not able to distinguish between $x \in \mathcal{X}$ and $y \in \mathcal{X}$ if $\varrho\,(x, y) = 0$.

Definition 5.1. A continuous function $\psi : \mathbb{R} \to \mathcal{X}$ is *almost periodic* if for any $\varepsilon > 0$, there exists a number $p\,(\varepsilon) > 0$ with the property that any interval of length $p\,(\varepsilon)$ of the real line contains at least one point s such that

$$\varrho\,(\psi(t+s), \psi(t)) < \varepsilon, \quad t \in \mathbb{R}.$$

The number s is called an *ε-translation number* and the set of all ε-translation numbers of ψ is denoted by $T(\psi, \varepsilon)$.

Remark 5.2. It is possible to introduce almost periodic functions defined on various sets. For almost periodic functions defined on the torus (on the annuloid), see [136, 154]; on a tube, see [68]; on a circle, see [30]. $\triangle$

If $\mathcal{X}$ is a Banach space, then a continuous function ψ is almost periodic if and only if any set of translates of ψ has a subsequence, uniformly convergent on $\mathbb{R}$ in the sense of the norm. See, e.g., [46, Theorem 6.6]. Evidently, this result cannot be longer valid if the space of values is not complete. Nevertheless, we prove the below given Theorem 5.5, where the convergence is replaced by the Cauchy condition. Before proving this result, we mention two simple lemmas. Their proofs are easily obtained by modifying the proofs of [46, Theorem 6.2] and [46, Theorem 6.5].

Lemma 5.3. *An almost periodic function with values in $\mathcal{X}$ is uniformly continuous on the real line.*

Proof. Let an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ be given and let $p = p(\varepsilon/3)$, where $\varepsilon > 0$ is arbitrary, be from Definition 5.1. Since ψ is uniformly continuous on the interval $I := [-1, 1+p]$, there exists $\delta = \delta(\varepsilon) \in (0,1)$ such that

$$\varrho\left(\psi(t_1), \psi(t_2)\right) < \frac{\varepsilon}{3}, \quad t_1, t_2 \in I, \ |\, t_1 - t_2 \,| < \delta.$$

Let $t_1, t_2 \in \mathbb{R}$ satisfying $|\, t_1 - t_2 \,| < \delta$ be arbitrary and $s = s(t_1, \delta) \in [-t_1, -t_1 + p]$ be an $(\varepsilon/3)$-translation number of ψ. Evidently, $t_1 + s \in I$, $t_2 + s \in I$. Finally, we have

$$\begin{aligned}\varrho\left(\psi(t_1), \psi(t_2)\right) \leq \varrho\left(\psi(t_1), \psi(t_1+s)\right) + \varrho\left(\psi(t_1+s), \psi(t_2+s)\right)& \\ + \varrho\left(\psi(t_2+s), \psi(t_2)\right) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,&\end{aligned}$$

which terminates the proof. □

Lemma 5.4. *The set of all values of an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ is totally bounded in $\mathcal{X}$.*

Proof. Let $p = p(\varepsilon/2)$ be from Definition 5.1 for arbitrarily given $\varepsilon > 0$. Obviously, the set of all values of ψ on $[0, p]$ is a subset of a finite number of neighbourhoods of radius $\varepsilon/2$. Let us denote by $x_1, x_2, \ldots, x_q$ the centres of these neighbourhoods which cover the set $\{\psi(t);\, t \in [0,p]\}$. For an arbitrary $t \in \mathbb{R}$, we take an $(\varepsilon/2)$-translation number $s = s(t) \in [-t, -t+p]$ of ψ. Thus, $t + s \in [0, p]$. Let $x(t) \in \{x_1, x_2, \ldots, x_q\}$ be the centre of the neighbourhood of radius $\varepsilon/2$ which contains $\psi(t+s)$. We obtain

$$\varrho\left(x(t), \psi(t)\right) \leq \varrho\left(x(t), \psi(t+s)\right) + \varrho\left(\psi(t+s), \psi(t)\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It shows that, for any $\varepsilon > 0$, the set of all values of ψ is covered by a finite number of neighbourhoods of radius ε. □

Theorem 5.5. *Let $\psi : \mathbb{R} \to \mathcal{X}$ be a continuous function. Then, ψ is almost periodic if and only if, from any sequence of the form $\{\psi(t + s_n)\}_{n\in\mathbb{N}}$, where s_n are real numbers, one can extract a subsequence $\{\psi(t + r_n)\}_{n\in\mathbb{N}}$ satisfying the Cauchy uniform convergence condition on $\mathbb{R}$, i.e., for any $\varepsilon > 0$, there exists $l(\varepsilon) \in \mathbb{N}$ with the property that*

$$\varrho\left(\psi(t + r_i), \psi(t + r_j)\right) < \varepsilon, \quad t \in \mathbb{R},$$

for all $i, j > l(\varepsilon)$, $i, j \in \mathbb{N}$.

Proof. We prove the sufficiency of the condition using a simple extension of the argument used in the proof of [46, Theorem 1.10]. Suppose, on contrary, that ψ is not almost periodic. Then, there exists a number $\varepsilon > 0$ such that, for any $p \in \mathbb{N}$, one can find an interval of length p which does not contain any ε-translation number of ψ. Consider an arbitrary number $l_1 \in \mathbb{N}$ and an interval $(a_1, b_1) \subseteq \mathbb{R}$ of the length greater than $2(l_1 + 1)$ which contains no ε-translation number of ψ. We choose $l_2 \in \mathbb{Z}$ such that $l_2 - l_1 \in (a_1, b_1)$. Thus, $l_2 - l_1$ is not an ε-translation number of ψ. Next, there exists an interval $(a_2, b_2) \subseteq \mathbb{R}$ of the length greater than $2(l_1+l_2+1)$ such that there exists no ε-translation number of ψ in (a_2, b_2). We can also find $l_3 \in \mathbb{Z}$ for which $l_3 - l_1, l_3 - l_2 \in (a_2, b_2)$ and, hence, $l_3 - l_1, l_3 - l_2$ cannot be ε-translation numbers of ψ.

Proceeding in a similar way, we get a sequence $\{l_n\}_{n\in\mathbb{N}}$ satisfying that none of the numbers $l_{n_1} - l_{n_2}$, where $n_1 \neq n_2$ $(n_1, n_2 \in \mathbb{N})$, is an ε-translation number of ψ. Therefore, we obtain

$$\varrho\left(\psi(t + l_{n_1} - l_{n_2}), \psi(t)\right) \geq \varepsilon$$

for all $n_1 \neq n_2$ $(n_1, n_2 \in \mathbb{N})$ and at least one $t \in \mathbb{R}$. This contradiction proves that ψ is almost periodic.

To prove the converse implication, we assume that ψ is an almost periodic function. We apply the well-known method of the diagonal extraction and modify the proof of [46, Theorem 6.6].

Let $\{t_n;\, n \in \mathbb{N}\}$ be a dense subset of $\mathbb{R}$ and $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be an arbitrarily given sequence. From the sequence $\{\psi(t_1 + s_n)\}_{n\in\mathbb{N}}$, using Lemma 5.4, we choose a subsequence $\{\psi(t_1 + r_n^1)\}_{n\in\mathbb{N}}$ such that, for any $\varepsilon > 0$, there exists $l_1(\varepsilon) \in \mathbb{N}$ with the property that

$$\varrho\left(\psi\left(t_1 + r_i^1\right), \psi(t_1 + r_j^1)\right) < \varepsilon, \quad i, j > l_1(\varepsilon),\, i, j \in \mathbb{N}.$$

Such a subsequence exists, because infinitely many values $\psi(t_1 + s_n)$ is in a neighbourhood of radius 2^{-i} for all $i \in \mathbb{N}$ (consider the method of the diagonal extraction). Analogously, from the sequence $\{\psi(t_2 + r_n^1)\}_{n\in\mathbb{N}}$, we get $\{\psi(t_2 + r_n^2)\}_{n\in\mathbb{N}}$ such that, for any $\varepsilon > 0$, there exists $l_2(\varepsilon) \in \mathbb{N}$ for which

$$\varrho\left(\psi(t_2 + r_i^2), \psi(t_2 + r_j^2)\right) < \varepsilon, \quad i, j > l_2(\varepsilon),\, i, j \in \mathbb{N}.$$

We proceed further in the same way. We obtain $\{r_n^k\} \subseteq \cdots \subseteq \{r_n^1\}$, $k \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrarily given, $p = p\,(\varepsilon/5)$ be from Definition 5.1, $\delta = \delta\,(\varepsilon/5)$ correspond to $\varepsilon/5$ from the definition of the uniform continuity of ψ (see Lemma 5.3) and let a finite set $\{t_1, \ldots, t_j\} \subset \{t_n;\, n \in \mathbb{N}\}$ satisfy

$$\min_{i\in\{1,\ldots,j\}} |\, t_i - t\,| < \delta, \quad t \in [0, p].$$

Obviously, there exists $l \in \mathbb{N}$ such that, for all integers $n_1, n_2 > l$, it holds

$$\varrho\left(\psi(t_i + r_{n_1}^{n_1}), \psi(t_i + r_{n_2}^{n_2})\right) < \frac{\varepsilon}{5}, \quad i \in \{1, \ldots, j\}.$$

Let $t \in \mathbb{R}$ be given, $s = s(t) \in [-t, -t+p]$ be an $(\varepsilon/5)$-translation number of ψ, and $t_i = t_i(s) \in \{t_1, \ldots, t_j\}$ be such that $|\, t + s - t_i\,| < \delta$. Finally, we have

$$\begin{aligned}
\varrho\left(\psi(t + r_{n_1}^{n_1}), \psi(t + r_{n_2}^{n_2})\right) \;&\leq\; \varrho\left(\psi(t + r_{n_1}^{n_1}), \psi(t + r_{n_1}^{n_1} + s)\right) \\
&+ \varrho\left(\psi(t + r_{n_1}^{n_1} + s), \psi(t_i + r_{n_1}^{n_1})\right) \;+\; \varrho\left(\psi(t_i + r_{n_1}^{n_1}), \psi(t_i + r_{n_2}^{n_2})\right) \\
&+ \varrho\left(\psi(t_i + r_{n_2}^{n_2}), \psi(t + r_{n_2}^{n_2} + s)\right) \;+\; \varrho\left(\psi(t + r_{n_2}^{n_2} + s), \psi(t + r_{n_2}^{n_2})\right).
\end{aligned}$$

Thus, we obtain

$$\varrho\left(\psi(t + r_{n_1}^{n_1}), \psi(t + r_{n_2}^{n_2})\right) < \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon \tag{5.1}$$

for all $t \in \mathbb{R}$, $n_1, n_2 > l$, $n_1, n_2 \in \mathbb{N}$. Evidently, (5.1) completes the proof of the theorem if we put $r_n := r_n^n$, $n \in \mathbb{N}$. $\square$

Remark 5.6. Using a combination of the methods used in the proofs of Theorems 1.3 and 5.5, it is possible to prove that, for a continuous function $f : G \to \mathcal{X}$ with G an abelian topological group and $\mathcal{X}$ a complete metric space, the definitions of almost periodicity in the sense of Bohr and Bochner are equivalent. For other generalizations, see [14]; for other equivalent definitions (e.g., the von Neumann and the Maak definition) of almost periodicity, see [117, 128]. We add that, for the first time, almost periodic functions on groups with values in Banach spaces were studied by S. Bochner and J. von Neumann in [24, 25].

In the recent years, many researchers study the concept of almost periodicity on time scales and analyse solutions of almost periodic (linear) dynamic equations. We refer at least to papers [86, 119, 120, 121, 127, 180, 185]. △

Analogously as for complex valued almost periodic functions or almost periodic sequences in Chapter 1, one can prove many properties of almost periodic functions with values in pseudometric spaces.

Theorem 5.7. *Let $\mathcal{X}_1, \mathcal{X}_2$ be pseudometric spaces and $\Phi : \mathcal{X}_1 \to \mathcal{X}_2$ be a uniformly continuous map. If $\psi : \mathbb{R} \to \mathcal{X}_1$ is almost periodic, then $\Phi \circ \psi$ is almost periodic as well.*

Proof. We can proceed similarly as in the proof of Theorem 1.6. If $\delta(\varepsilon) > 0$ is the number corresponding to arbitrary $\varepsilon > 0$ from the definition of the uniform continuity of Φ, then it is valid

$$T\left(\psi, \delta(\varepsilon)\right) \subseteq T\left(\Phi \circ \psi, \varepsilon\right)$$

which proves the theorem. □

Theorem 5.8. *The limit of a uniformly convergent sequence of almost periodic functions is almost periodic.*

Proof. It is possible to prove the theorem using the process from the proof of [46, Theorem 6.4]. □

Directly from Theorem 5.5, we obtain the following corollaries.

Corollary 5.9. *Let $\mathcal{X}$ be a Banach space. The sum of two almost periodic functions with values in $\mathcal{X}$ is an almost periodic function.*

Corollary 5.10. *If $\mathcal{X}_1, \dots, X_n$ are pseudometric spaces and $\psi_1, \dots, \psi_n$ are arbitrary almost periodic functions with values in $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively, then the function ψ, with values in $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ given by $\psi := (\psi_1, \dots, \psi_n)$, is almost periodic.*

We add that one can use Corollary 5.10 to obtain simple modifications of the below presented method of constructions of almost periodic functions. Moreover, from Corollary 5.10, we get:

Corollary 5.11. *The set*

$$T(\psi_1, \varepsilon) \cap T(\psi_2, \varepsilon) \cap \cdots \cap T(\psi_n, \varepsilon)$$

is relatively dense in $\mathbb{R}$ for arbitrary almost periodic functions $\psi_1, \psi_2, \dots, \psi_n$ and any $\varepsilon > 0$.

Remark 5.12. For the first time, Corollary 5.11 was proved for almost periodic functions with values in an arbitrary metric space in [153]. △

To conclude this subsection, we establish theorems which show how almost periodic functions can be characterized by almost periodic sequences.

Theorem 5.13. *A uniformly continuous function $\psi : \mathbb{R} \to \mathcal{X}$ is almost periodic if and only if there exists a sequence of positive numbers r_n, $n \in \mathbb{N}$, satisfying $r_n \to 0$ as $n \to \infty$, such that the sequence $\{\psi(r_n k)\}_{k\in\mathbb{Z}}$ is almost periodic for all $n \in \mathbb{N}$.*

Proof. One can prove the theorem using a corresponding extension of the proof of [46, Theorem 1.29]. □

Theorem 5.14. *Let $\mathcal{X}$ be a Banach space. A necessary and sufficient condition for a sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{X}$ to be almost periodic is the existence of an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$ for which $\psi(k) = \varphi_k$, $k \in \mathbb{Z}$.*

Proof. The sufficiency of the condition follows directly from Theorems 1.3 and 5.5. Conversely, assume that an almost periodic sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is given. We define

$$\psi(t) := \varphi_k + (t-k)(\varphi_{k+1} - \varphi_k), \quad k \le t < k+1,\ k \in \mathbb{Z}. \tag{5.2}$$

Evidently, $\psi : \mathbb{R} \to \mathcal{X}$ is continuous and $\psi(k) = \varphi_k$, $k \in \mathbb{Z}$. The almost periodicity of ψ follows from

$$T\left(\{\varphi_k\}, \frac{\varepsilon}{3}\right) \subseteq T(\psi, \varepsilon)$$

which can be proved using (5.2). □

Remark 5.15. Many known theorems show how almost periodic functions can be characterized by almost periodic sequences. Such theorems are used to study almost periodic solutions of differential equations as well. General examples of differential equations, for which a solution $x(t)$ defined for $t \in \mathbb{R}$ is almost periodic if and only if $\{x(k)\}_{k\in\mathbb{Z}}$ is an almost periodic sequence, are mentioned in [4, 133, 144]. △

5.2.2 Limit periodic functions

Now we briefly recall the concept of limit periodicity for continuous functions with ranges in $\mathcal{X}$.

Definition 5.16. A function $f : \mathbb{R} \to \mathcal{X}$ is called *limit periodic* if $f(x) = \lim_{n\to\infty} f_n(x)$ uniformly for $x \in \mathbb{R}$, where all $f_n : \mathbb{R} \to \mathcal{X}$ are periodic continuous functions.

Remark 5.17. As in the discrete case (cf. Remark 1.18), the periods of functions f_n in Definition 5.16 do not need to be the same for considered n. △

Theorem 5.18. *Any limit periodic function is almost periodic.*

Proof. It suffices to consider Theorem 5.8 and the above definitions. □

Theorem 5.19. *There exist almost periodic functions $f : \mathbb{R} \to \mathbb{C}$ (with respect to the usual metric) which are not limit periodic.*

Proof. The theorem follows from the characterization of limit periodic functions using the Fourier expansion, which can be found in [18] (see also [47, p. 129]). □

5.3 Constructions of almost periodic functions

Now we present the way one can generate almost periodic functions with given properties.

Theorem 5.20. *For arbitrary $a > 0$, any continuous function $\psi : \mathbb{R} \to \mathcal{X}$ such that*

$$\psi(t) \in \mathcal{O}_a\left(\psi(t-1)\right), \quad t \in (1,2],$$

$$\psi(t) \in \mathcal{O}_a\left(\psi(t+2)\right), \quad t \in (-2,0],$$
$$\psi(t) \in \mathcal{O}_{a/2}\left(\psi(t-4)\right), \quad t \in (2,6],$$
$$\psi(t) \in \mathcal{O}_{a/2}\left(\psi(t+8)\right), \quad t \in (-10,-2],$$
$$\psi(t) \in \mathcal{O}_{a/4}\left(\psi(t-2^4)\right), \quad t \in (2+2^2, 2+2^2+2^4],$$
$$\psi(t) \in \mathcal{O}_{a/4}\left(\psi(t+2^5)\right), \quad t \in (-2^5-2^3-2, -2^3-2],$$
$$\vdots$$
$$\psi(t) \in \mathcal{O}_{a\,2^{-n}}\left(\psi(t-2^{2n})\right), \quad t \in (2+2^2+\cdots+2^{2n-2}, 2+2^2+\cdots+2^{2n-2}+2^{2n}],$$
$$\psi(t) \in \mathcal{O}_{a\,2^{-n}}\left(\psi(t+2^{2n+1})\right), \quad t \in (-2^{2n+1}-\cdots-2^3-2, -2^{2n-1}-\cdots-2^3-2],$$
$$\vdots$$

is almost periodic.

Proof. Let $\varepsilon > 0$ be arbitrary and $k = k(\varepsilon) \in \mathbb{N}$ be such that $2^k > 8a/\varepsilon$. It suffices to prove that $l\,2^{2k}$ is an ε-translation number of ψ for any integer l.

First we define

$$\varphi(t) := \psi(t), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}].$$

We see that

$$\psi(t) \in \mathcal{O}_{\varepsilon/8}\left(\varphi(t-2^{2k})\right), \quad t \in [2+2^2+\cdots+2^{2k-2}, 2+2^2+\cdots+2^{2k}],$$

$$\psi(t) \in \mathcal{O}_{\varepsilon/8}\left(\psi(t+2^{2k+1})\right), \quad t \in [-2^{2k+1}-\cdots-2^3-2, -2^{2k-1}-\cdots-2^3-2],$$
$$\psi(t) \in \mathcal{O}_{\varepsilon/16}\left(\psi(t-2^{2k+2})\right), \quad t \in [2+2^2+\cdots+2^{2k}, 2+2^2+\cdots+2^{2k+2}],$$
$$\psi(t) \in \mathcal{O}_{\varepsilon/16}\left(\psi(t+2^{2k+3})\right), \quad t \in [-2^{2k+3}-\cdots-2^3-2, -2^{2k+1}-\cdots-2^3-2],$$
$$\vdots$$

In a pseudometric space $\mathcal{X}$, this observation implies

$$\psi(t+2^{2k}) \in \mathcal{O}_{\varepsilon/8}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$

$$\psi(t-2^{2k+1}) \in \mathcal{O}_{\varepsilon/8}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$

$$\psi(t-2^{2k}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/8}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$
$$\psi(t+2^{2k+1}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/16}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$
$$\psi(t+3\,2^{2k}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/8+\varepsilon/16}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$
$$\psi(t+2^{2k+2}) \in \mathcal{O}_{\varepsilon/16}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$
$$\psi(t+2^{2k}+2^{2k+2}) \in \mathcal{O}_{\varepsilon/8+\varepsilon/16}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$
$$\vdots$$

Since
$$\frac{\varepsilon}{8}+\frac{\varepsilon}{8}+\frac{\varepsilon}{16}+\frac{\varepsilon}{16}+\frac{\varepsilon}{32}+\frac{\varepsilon}{32}+\cdots=\frac{\varepsilon}{2},$$
we have
$$\psi\left(t+l\,2^{2k}\right) \in \mathcal{O}_{\varepsilon/2}\left(\varphi(t)\right), \quad t \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],\ l \in \mathbb{Z}. \tag{5.3}$$
We express any $t \in \mathbb{R}$ as the sum of numbers $p(t)$ and $q(t)$ for which
$$p(t) \in [-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}]$$
and $q(t) = j2^{2k}$ for some $j \in \mathbb{Z}$. Using (5.3), we obtain
$$\begin{aligned}\varrho\left(\psi(t), \psi(t+l\,2^{2k})\right) &\le \varrho\left(\psi(p(t)+q(t)), \varphi(p(t))\right)\\ &\quad + \varrho\left(\varphi(p(t)), \psi(p(t)+(j+l)\,2^{2k})\right) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned} \tag{5.4}$$
for any $t \in \mathbb{R}$ and $l \in \mathbb{Z}$ which terminates the proof. □

Remark 5.21. Note that the range of a function ψ generating by Theorem 5.20 does not need to be complete (for a general pseudometric space). △

The process mentioned in the previous theorem is easily modifiable. We illustrate this fact by the following two theorems.

Theorem 5.22. *Let $M > 0$, $x_0 \in \mathcal{X}$, and $j \in \mathbb{N}$ be given. Let $\varphi : [0, M] \to \mathcal{X}$ satisfy $\varphi(0) = \varphi(M) = x_0$. If $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$ has the property that*
$$\sum_{n=1}^{\infty} r_n < \infty, \tag{5.5}$$
then an arbitrary continuous function $\psi : \mathbb{R} \to \mathcal{X}$, $\psi|_{[0,M]} \equiv \varphi$ for which
$$\begin{aligned}\psi(t) = x_0, \quad t \in &\{iM,\, 2 \le i \le j+1\} \cup \{-i(j+1)M,\, 1 \le i \le j\}\\ &\bigcup_{n=1}^{\infty}\{((j+1)+\cdots+j(j+1)^{2n-2}+i(j+1)^{2n})M;\ 1 \le i \le j\}\\ &\bigcup_{n=1}^{\infty}\{-((j+1)+\cdots+j(j+1)^{2n-1}+i(j+1)^{2n+1})M;\ 1 \le i \le j\}\end{aligned} \tag{5.6}$$

and, at the same time, for which it is valid

$$\psi(t) \in \mathcal{O}_{r_1}\left(\psi(t-M)\right), \quad t \in (M, 2M),$$

$$\vdots$$

$$\begin{aligned} \psi(t) &\in \mathcal{O}_{r_1}\left(\psi(t-jM)\right), \quad t \in (jM, (j+1)M), \\ \psi(t) &\in \mathcal{O}_{r_2}\left(\psi(t+(j+1)M)\right), \quad t \in (-(j+1)M, 0), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} \psi(t) &\in \mathcal{O}_{r_2}\left(\psi(t+j(j+1)M)\right), \quad t \in (-j(j+1)M, -(j-1)(j+1)M), \\ \psi(t) &\in \mathcal{O}_{r_3}\big(\psi(t-(j+1)^2M)\big), \quad t \in \big((j+1)M, ((j+1)+(j+1)^2)M\big), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} &\psi(t) \in \mathcal{O}_{r_3}\big(\psi(t-j(j+1)^2M)\big), \\ &\qquad t \in \big(((j+1)+(j-1)(j+1)^2)M, ((j+1)+j(j+1)^2)M\big), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} &\psi(t) \in \mathcal{O}_{r_{2n}}\big(\psi(t+(j+1)^{2n-1}M)\big), \\ &\qquad t \in \big(-((j+1)^{2n-1}+j(j+1)^{2n-3}+\cdots+j(j+1)^3+j(j+1))M, \\ &\qquad\qquad -(j(j+1)^{2n-3}+\cdots+j(j+1)^3+j(j+1))M\big), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} &\psi(t) \in \mathcal{O}_{r_{2n}}\big(\psi(t+j(j+1)^{2n-1}M)\big), \\ &\qquad t \in \big(-(j(j+1)^{2n-1}+j(j+1)^{2n-3}+\cdots+j(j+1)^3+j(j+1))M, \\ &\qquad\qquad -((j-1)(j+1)^{2n-1}+j(j+1)^{2n-3}+\cdots+j(j+1)^3+j(j+1))M\big), \\ &\psi(t) \in \mathcal{O}_{r_{2n+1}}\big(\psi(t-(j+1)^{2n}M)\big), \\ &\qquad t \in \big(((j+1)+j(j+1)^2+\cdots+j(j+1)^{2n-2})M, \\ &\qquad\qquad ((j+1)+j(j+1)^2+\cdots+j(j+1)^{2n-2}+(j+1)^{2n})M\big), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} &\psi(t) \in \mathcal{O}_{r_{2n+1}}\big(\psi(t-j(j+1)^{2n}M)\big), \\ &\qquad t \in \big(((j+1)+j(j+1)^2+\cdots+j(j+1)^{2n-2}+(j-1)(j+1)^{2n})M, \\ &\qquad\qquad ((j+1)+j(j+1)^2+\cdots+j(j+1)^{2n-2}+j(j+1)^{2n})M\big), \end{aligned}$$

$$\vdots$$

is almost periodic.

Proof. We prove this theorem analogously as Theorem 5.20. Let ε be a positive number and let an odd integer $n(\varepsilon) \geq 2$ have the property that (see (5.5))

$$\sum_{n=n(\varepsilon)}^{\infty} r_n < \frac{\varepsilon}{2}. \tag{5.7}$$

We prove that $l(j+1)^{n(\varepsilon)-1}M$ is an ε-translation number of ψ for all $l \in \mathbb{Z}$. Let $l \in \mathbb{Z}$ and $t \in \mathbb{R}$ be arbitrary. If we put

$$s := l(j+1)^{n(\varepsilon)-1}M, \tag{5.8}$$

then it suffices to show that the inequality

$$\varrho\left(\psi(t), \psi(t+s)\right) < \varepsilon \tag{5.9}$$

holds; i.e., this inequality proves the theorem.

We can write t as the sum of numbers t_1 and t_2, where

$$\begin{aligned} t_1 &\geq -\left(j(j+1)^{n(\varepsilon)-2} + \cdots + j(j+1)^3 + j(j+1)\right) M, \\ t_1 &\leq \left(j+1 + j(j+1)^2 + \cdots + j(j+1)^{n(\varepsilon)-3}\right) M \end{aligned} \tag{5.10}$$

and

$$t_2 = i(j+1)^{n(\varepsilon)-1}M \quad \text{for some } i \in \mathbb{Z}. \tag{5.11}$$

Now we have (see (5.10) and the proof of Theorem 5.20)

$$\begin{aligned} \varrho\left(\psi(t), \psi(t+s)\right) &\leq \varrho\left(\psi(t_1+t_2), \psi(t_1)\right) + \varrho\left(\psi(t_1), \psi(t_1+t_2+s)\right) \\ &< \sum_{n=n(\varepsilon)}^{n(\varepsilon)+p-1} r_n + \sum_{n=n(\varepsilon)}^{n(\varepsilon)+q-1} r_n. \end{aligned} \tag{5.12}$$

Indeed, we can express (consider (5.8) and (5.11))

$$\begin{aligned} t_2 &= \left(i_1(j+1)^{n(\varepsilon)-1} + i_2(j+1)^{n(\varepsilon)} + \cdots + i_p(j+1)^{n(\varepsilon)+p-1}\right)(m+1), \\ t_2 + s &= \left(l_1(j+1)^{n(\varepsilon)-1} + l_2(j+1)^{n(\varepsilon)} + \cdots + l_q(j+1)^{n(\varepsilon)+q-1}\right)(m+1), \end{aligned}$$

where $i_1, \ldots, i_p, l_1, \ldots, l_q \subseteq \{-j, \ldots, 0, \ldots, j\}$ satisfy

$$i_1 \geq 0, \quad i_2 \leq 0, \quad \cdots \quad (-1)^p\, i_p \leq 0, \qquad l_1 \geq 0, \quad l_2 \leq 0, \quad \cdots \quad (-1)^q\, l_q \leq 0.$$

Evidently, (5.7) and (5.12) give (5.9). □

For $j = 1$, we get the most important case of Theorem 5.22.

Corollary 5.23. *Let $M > 0$ and $x_0 \in \mathcal{X}$ be given and let $\varphi : [0, M] \to \mathcal{X}$ be such that*

$$\varphi(0) = \varphi(M) = x_0.$$

If $\{\varepsilon_i\}_{i \in \mathbb{N}} \subset \mathbb{R}_0^+$ satisfies

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty, \tag{5.13}$$

then any continuous function $\psi : \mathbb{R} \to \mathcal{X}$, $\psi|_{[0,M]} \equiv \varphi$ *for which*

$$\begin{aligned}\psi(t) = x_0, \quad t \in \{2M, -2M\} &\cup \{(2 + 2^2 + \cdots + 2^{2(i-1)} + 2^{2i})M;\, i \in \mathbb{N}\} \\ &\cup \{-(2 + 2^3 + \cdots + 2^{2i-1} + 2^{2i+1})M;\, i \in \mathbb{N}\}\end{aligned} \tag{5.14}$$

and, at the same time, for which it is valid

$$\psi(t) \in \mathcal{O}_{\varepsilon_1}(\psi(t-M)), \quad t \in (M, 2M),$$

$$\psi(t) \in \mathcal{O}_{\varepsilon_2}(\psi(t+2M)), \quad t \in (-2M, 0),$$
$$\psi(t) \in \mathcal{O}_{\varepsilon_3}\left(\psi(t-2^2M)\right), \quad t \in (2M, (2+2^2)M),$$
$$\psi(t) \in \mathcal{O}_{\varepsilon_4}\left(\psi(t+2^3M)\right), \quad t \in (-(2^3+2)M, -2M),$$
$$\psi(t) \in \mathcal{O}_{\varepsilon_5}\left(\psi(t-2^4M)\right), \quad t \in ((2+2^2)M, (2+2^2+2^4)M),$$
$$\vdots$$
$$\psi(t) \in \mathcal{O}_{\varepsilon_{2i}}\left(\psi(t+2^{2i-1}M)\right), \quad t \in (-(2^{2i-1}+\cdots+2)M, -(2^{2i-3}+\cdots+2)M),$$
$$\psi(t) \in \mathcal{O}_{\varepsilon_{2i+1}}\left(\psi(t-2^{2i}M)\right), \quad t \in ((2+2^2+\cdots+2^{2i-2})M, (2+2^2+\cdots+2^{2i})M),$$
$$\vdots$$

is almost periodic.

Theorem 5.24. *Let* $\varphi : (-r, r] \to \mathcal{X}$, $\{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^+$, *and* $\{j_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ *be arbitrary such that*

$$\sum_{n=1}^{\infty} r_n j_n < \infty \tag{5.15}$$

holds. Let a function $\psi : \mathbb{R} \to \mathcal{X}$ *satisfy* $\psi|_{(-r,r]} \equiv \varphi$ *and*

$$\psi(t) \in \mathcal{O}_{r_1}(\varphi(t-2r)), \quad t \in (r, r+2r],$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{r_1}(\varphi(t-2r)), \quad t \in (r+(j_1-1)2r, r+j_1 2r],$$
$$\psi(t) \in \mathcal{O}_{r_1}(\varphi(t+2r)), \quad t \in (-2r-r, -r],$$
$$\vdots$$
$$\psi(t) \in \mathcal{O}_{r_1}(\varphi(t+2r)), \quad t \in (-j_1 2r - r, -(j_1-1)2r - r],$$
$$\vdots$$
$$\psi(t) \in \mathcal{O}_{r_n}(\varphi(t-p_n)), \quad t \in (p_1+\cdots+p_{n-1}, p_1+\cdots+p_{n-1}+p_n],$$
$$\vdots$$
$$\psi(t) \in \mathcal{O}_{r_n}(\varphi(t-p_n)), \quad t \in (p_1+\cdots+p_{n-1}+(j_n-1)p_n, p_1+\cdots+p_{n-1}+j_n p_n],$$

$$\psi(t) \in \mathcal{O}_{r_n}\left(\varphi(t+p_n)\right), \quad t \in (-p_n - p_{n-1} - \dots - p_1, -p_{n-1} - \dots - p_1],$$

$$\vdots$$

$$\psi(t) \in \mathcal{O}_{r_n}\left(\varphi(t+p_n)\right), \quad t \in (-j_n p_n - p_{n-1} - \dots - p_1, -(j_n - 1)p_n - p_{n-1} - \dots - p_1],$$

$$\vdots$$

where

$$p_1 := r + j_1 2r, \qquad p_2 := 2(r + j_1 2r),$$

$$p_3 := (2j_2 + 1)p_2, \quad \dots \quad p_n := (2j_{n-1} + 1)p_{n-1}, \quad \dots$$

If ψ is continuous on $\mathbb{R}$, then it is almost periodic.

Proof. It is not difficult to prove Theorem 5.24 analogously as Theorems 5.20 and 5.22. For given $\varepsilon > 0$, let an integer $n(\varepsilon) \geq 2$ satisfy

$$\sum_{n=n(\varepsilon)}^{\infty} r_n\, j_n < \frac{\varepsilon}{4}.$$

One can prove the inclusion

$$\{l\, p_{n(\varepsilon)};\ l \in \mathbb{Z}\} \subseteq T(\psi, \varepsilon) \tag{5.16}$$

which guarantees the almost periodicity of ψ. □

Remark 5.25. From the proofs of Theorems 5.20, 5.22, 5.24 (see (5.4), (5.8) and (5.9), (5.16)), we get an important property of the set of all ε-translation numbers of the resulting function ψ. For any $\varepsilon > 0$, there exists non-zero $c \in \mathbb{R}$ for which

$$\{l\, c;\ l \in \mathbb{Z}\} \subseteq T(\psi, \varepsilon).$$

Hence, applying the method from the above theorems, one cannot construct almost periodic functions without this property. △

Chapter 6

Solutions of almost periodic differential systems

In this chapter, we analyse (non-)almost periodic solutions of almost periodic homogeneous linear differential systems. Sometimes this field is called the Favard theory which is based on the Favard contributions in [67] (see also [35, Theorem 1.2], [47, Chapter 5], [72, Theorem 6.3] or [145, Theorem 1]; for homogeneous case, see [44, 66]). In this context, sufficient conditions for the existence of almost periodic solutions are mentioned in [42, 54, 97] (for generalizations, see [48, 49, 69, 90, 95, 98, 105, 116, 118, 135, 159, 161]; for other extensions and supplements of the Favard theory, e.g., see [2, 34, 36, 37, 52, 53, 55, 96, 122, 166, 167]). Certain sufficient conditions, under which homogeneous systems that have non-trivial bounded solutions have also non-trivial almost periodic solutions, are given in [146].

It is a corollary of the Favard (and the Floquet) theory that any bounded solution of an almost periodic linear differential system is almost periodic if the matrix valued function, which determines the system, is periodic (see [72, Corollary 6.5]; for a generalization in the homogeneous case, see [89]). This result is no longer valid for systems with almost periodic coefficients. There exist systems for which all solutions are bounded, but none of them is almost periodic (see [102, 103, 145, 155]). Homogeneous systems have the zero solution which is almost periodic, but do not need to have other almost periodic solutions. Note that the existence of a homogeneous system, which has bounded solutions (separated from zero) and, at the same time, all systems from some neighbourhood of it do not have non-trivial almost periodic solutions, is proved in [171].

In this chapter, we consider the set of all almost periodic skew-Hermitian differential systems with the uniform topology of matrix functions on the real axis. In [170], there is proved that the systems, whose all solutions are almost periodic, form a dense subset of the set of all considered systems. We add that special cases of this result are proved in [113, 114]. Using the method for constructing almost periodic functions from Section 5.3, we prove that, in any neighbourhood of a skew-Hermitian system, there exists a system which does not possess an almost periodic solution other than the trivial one (not only with a fundamental matrix which is not almost periodic as in [172]). Then, we prove the corresponding result in the real case, i.e., for the skew-symmetric differential systems.

We use recurrent methods for constructing almost periodic functions. For non-almost periodic solutions of homogeneous linear differential equations, we refer to [140] (and [141]), where a method of constructions of minimal cocycles, which one gets as solutions of recurrent homogeneous linear differential systems, is mentioned. Special constructions of almost periodic homogeneous linear differential systems with given properties can be found in [112, 123, 124, 125] as well. A method to construct fundamental matrices for almost periodic homogeneous linear systems is introduced in [150].

6.1 Preliminaries

Let $m \in \mathbb{N}$ be arbitrarily given. In this chapter, we use the following notations: $\mathcal{I}m(\varphi)$ for the range of a function φ, $\mathcal{M}at(\mathbb{C}, m)$ for the set of all $m \times m$ matrices with complex elements, $\mathrm{Mat}\,(\mathbb{R}, m)$ for the set of all $m \times m$ matrices with real elements, $U(m) \subset \mathcal{M}at(\mathbb{C}, m)$ for the group of all unitary matrices of dimension m, $SO(m) \subset \mathrm{Mat}\,(\mathbb{R}, m)$ for the group of all orthogonal matrices with determinant 1, $so(m) \subset \mathrm{Mat}\,(\mathbb{R}, m)$ for the set of all skew-symmetric (i.e., antisymmetric) matrices, A^* for the conjugate transpose of $A \in \mathcal{M}at(\mathbb{C}, m)$, I for the identity matrix, O for the zero matrix, and symbol i for the imaginary unit. We remark that the Lie algebra associated to the Lie group $SO(m)$ consists of the skew-symmetric $m \times m$ matrices (i.e., this Lie algebra is $so(m)$ and it is sometimes called the special orthogonal Lie algebra).

6.2 Skew-Hermitian systems without almost periodic solutions

We consider systems of m homogeneous linear differential equations of the form

$$x'(t) = A(t) \cdot x(t), \quad t \in \mathbb{R}, \tag{6.1}$$

where A is an almost periodic function with $\mathcal{I}m(A) \subset \mathcal{M}at(\mathbb{C}, m)$ and with the property that $A(t) + A^*(t) = O$ for any $t \in \mathbb{R}$, i.e., $A : \mathbb{R} \to \mathcal{M}at(\mathbb{C}, m)$ is an almost periodic function of skew-Hermitian matrices. Let $\mathcal{S}$ be the set of all systems (6.1). We identify the function A with the system (6.1) which is determined by A. Especially, we write $A \in \mathcal{S}$ and $O \in \mathcal{S}$ denotes the system (6.1) given by $A(t) = O$, $t \in \mathbb{R}$.

In the vector space $\mathbb{C}^m$, we consider the absolute norm $||\cdot||_1$ (one can also consider the Euclidean norm or the maximum norm). Let $||\cdot||$ be the corresponding matrix norm. Considering that any almost periodic function is bounded (see Lemma 5.4), the distance between two systems $A, B \in \mathcal{S}$ is defined by the norm of the matrix valued functions A, B, uniformly on $\mathbb{R}$; i.e., we introduce the metric

$$\sigma\,(A, B) := \sup_{t \in \mathbb{R}} ||\, A(t) - B(t)\,||\,, \quad A, B \in \mathcal{S}. \tag{6.2}$$

For $\varepsilon > 0$, symbol $\mathcal{O}^{\sigma}_{\varepsilon}(A)$ stands for the ε-neighbourhood of A in $\mathcal{S}$.

Now we recall the notion of the frequency module and its rational hull which can be introduced for all almost periodic function with values in a Banach space. The frequency module $\mathcal{F}$ of an almost periodic function $A : \mathbb{R} \to \mathcal{M}at(\mathbb{C}, m)$ is the $\mathbb{Z}$-module of the real numbers, generated by the numbers λ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{e}^{2\pi \mathrm{i}\lambda t}\, A(t)\,\mathrm{d}t \neq O.$$

The rational hull of $\mathcal{F}$ is the set

$$\{\lambda/l;\ \lambda \in \mathcal{F},\ l \in \mathbb{Z}\}.$$

For the frequency modules of almost periodic linear differential systems and their solutions, we refer to [72, Chapters 4 and 6], [145].

In [170], there is proved that, in any neighbourhood of a system (6.1) with frequency module $\mathcal{F}$, there exists a system with a frequency module contained in the rational hull of $\mathcal{F}$ possessing all almost periodic solutions with frequencies belonging to the rational hull of $\mathcal{F}$ as well. From [174, Theorem 1] it follows that there exists a system (6.1) which cannot be approximated by the so-called reducible systems with frequency module $\mathcal{F}$ (there exists an open set of irreducible systems with a fixed frequency module; see [173] in the real case); i.e., a neighbourhood of a system (6.1) with frequency module $\mathcal{F}$ may not contain a system with almost periodic solutions and frequency module $\mathcal{F}$. In this case, see also [62] and [175] for reducible constant systems and systems reducing to diagonal forms by a Lyapunov transformation with frequency module $\mathcal{F}$, respectively.

In addition, for all $k \in \mathbb{N}$, it is proved in [172] that the systems with k-dimensional frequency basis of A, having solutions which are not almost periodic, form a subset of the second category of the space of all considered systems with k-dimensional frequency basis of A. Thus, it is known (see also [170, Corollary 1]) that the systems with k-dimensional frequency basis of A and with an almost periodic fundamental matrix form a dense set of the first category in the space of all systems (6.1) with k-dimensional frequency basis.

In this context, we formulate and prove the following result that the systems having no non-trivial almost periodic solution form a dense subset of $\mathcal{S}$.

Theorem 6.1. *For any $A \in \mathcal{S}$ and $\varepsilon > 0$, there exists $B \in \mathcal{O}^{\sigma}_{\varepsilon}(A)$ which does not have an almost periodic solution other than the trivial one.*

Proof. Let $A, C \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrary. Since the sum of skew-Hermitian matrices is also skew-Hermitian and since the sum of two almost periodic functions is almost periodic (see Corollary 5.9), we have that $A + C \in \mathcal{S}$. Let $X_A(t)$, $t \in \mathbb{R}$, and $X_C(t)$, $t \in \mathbb{R}$, be the principal (i.e., $X_A(0) = X_C(0) = I$) fundamental matrix of $A \in \mathcal{S}$ and $C \in \mathcal{S}$, respectively. If matrices $C(t)$, $X_A(t)$ commute for all $t \in \mathbb{R}$, then the matrix valued function $X_A(t)\,X_C(t)$, $t \in \mathbb{R}$, is the principal fundamental matrix of $A+C \in \mathcal{S}$. Indeed, from $X'_A(t) = A(t)\,X_A(t)$, $X'_C(t) = C(t)\,X_C(t)$, $t \in \mathbb{R}$, we obtain

$$\begin{aligned}(X_A(t) \cdot X_C(t))' &= A(t) \cdot X_A(t) \cdot X_C(t) + X_A(t) \cdot C(t) \cdot X_C(t)\\ &= A(t) \cdot X_A(t) \cdot X_C(t) + C(t) \cdot X_A(t) \cdot X_C(t) = (A + C)(t) \cdot X_A(t) \cdot X_C(t), \quad t \in \mathbb{R}.\end{aligned}$$

This fact implies that it suffices to find $C \in \mathcal{O}^{\sigma}_{\varepsilon}(O)$ for which all matrices $C(t)$, $t \in \mathbb{R}$, have the form $\mathrm{diag}\,(\mathrm{i}a, \dots, \mathrm{i}a)$, $a \in \mathbb{R}$, and for which the vector valued function $X_A(t)\, X_C(t)\, u$, $t \in \mathbb{R}$, is not almost periodic for any vector $u \in \mathbb{C}^m$, $||\, u \,||_1 = 1$.

We construct such an almost periodic function C using Theorem 5.20 for $a = \varepsilon/4$. First of all, we put

$$C(t) \equiv O, \quad t \in [0,1].$$

Then, in the first step of our construction, we define C on $(1,2]$ arbitrarily so that it is constant on $[1+1/4, 1+3/4]$ and $||\, C(t) \,|| < \varepsilon/4$ for t from this interval, $C(2) := C(1) = O$, and it is linear between values O, $C(3/2)$ on $[1, 1+1/4]$ and $[1+3/4, 2]$.

In the second step, we define continuous C satisfying $||\, C(t) - C(t+2) \,|| < \varepsilon/4$ for $t \in [-2, 0)$ arbitrarily so that it is constant on

$$[-2+1/16, -2+1-1/16], \quad [-2+1+1/4+1/16, -2+1+3/4-1/16];$$

at the same time, we put

$$C(-2) := C(0) = O, \quad C(-1+1/4) := C(1+1/4) = C(3/2),$$

$$C(-1) := C(1) = O, \quad C(-1/4) := C(2-1/4) = C(3/2),$$

and

$$C(t) \equiv C(3/2)/2, \quad t \in [-1+1/16, -1+1/4-1/16] \cup [-1/4+1/16, -1/16],$$

and we define C so that it is linear on

$$[-2, -2+1/16], \quad [-1-1/16, -1], \quad [-1, -1+1/16],$$

$$[-1+1/4-1/16, -1+1/4], \quad [-1+1/4, -1+1/4+1/16],$$

$$[-1/4-1/16, -1/4], \quad [-1/4, -1/4+1/16], \quad [-1/16, 0].$$

Analogously, in the third step, we get C on $(2,6]$ for which we can choose constant values on

$$[4-2+1/16+8^{-1}/16, 4-2+1-1/16-8^{-1}/16],$$

$$[4-2+1+1/4+1/16+8^{-1}/16, 4-2+1+3/4-1/16-8^{-1}/16],$$

$$[4-1+1/16+8^{-1}/16, 4-1+1/4-1/16-8^{-1}/16],$$

$$[4-1/4+1/16+8^{-1}/16, 4-1/16-8^{-1}/16],$$

$$[4+8^{-1}/16, 4+1-8^{-1}/16], \quad [4+1+1/4+8^{-1}/16, 4+1+3/4-8^{-1}/16]$$

arbitrarily so that $||\, C(t) - C(t-4) \,|| < \varepsilon/8$, $t \in (2,6]$; at the same time, we put

$$C(4-2+1/16) := C(-2+1/16) = C(-3/2),$$

$$C(4-2+1-1/16) := C(-1-1/16) = C(-3/2),$$

$$C(4-1) := C(-1) = O,$$

$$C(4-1+1/16) := C(-1+1/16) = C(3/2)/2,$$

$$
\begin{aligned}
C(4-1+1/4-1/16) &:= C(-1+1/4-1/16) = C(3/2)/2,\\
C(4-1+1/4) &:= C(-1+1/4) = C(3/2),\\
C(4-2+1+1/4+1/16) &:= C(-2+1+1/4+1/16) = C(-1/2),\\
C(4-2+1+3/4-1/16) &:= C(-2+1+3/4-1/16) = C(-1/2),\\
C(4-1/4) &:= C(-1/4) = C(3/2),\\
C(4-1/4+1/16) &:= C(-1/4+1/16) = C(3/2)/2,\\
C(4-1/16) &:= C(-1/16) = C(3/2)/2,\\
C(4) &:= C(0), \quad C(4+1) := C(1),\\
C(4+1+1/4) &:= C(1+1/4) = C(3/2),\\
C(4+1+3/4) &:= C(1+3/4) = C(3/2),\\
C(4+2) &:= C(2) = C(0) = O,
\end{aligned}
$$

$$
\begin{aligned}
C(t) \equiv C(-3/2)/2, \quad & t \in [4-2+8^{-1}/16, 4-2+1/16-8^{-1}/16]\\
& \cup [4-1-1/16+8^{-1}/16, 4-1-8^{-1}/16],\\
C(t) \equiv C(3/2)/4, \quad & t \in [4-1+8^{-1}/16, 4-1+1/16-8^{-1}/16]\\
& \cup [4-1/16+8^{-1}/16, 4-8^{-1}/16],\\
C(t) \equiv 3\,C(3/2)/4, \quad & t \in [4-1+1/4-1/16+8^{-1}/16, 4-1+1/4-8^{-1}/16]\\
& \cup [4-1/4+8^{-1}/16, 4-1/4+1/16-8^{-1}/16],\\
C(t) \equiv (C(3/2)+C(-1/2))/2, \quad & t \in [4-1+1/4+8^{-1}/16, 4-1+1/4+1/16-8^{-1}/16]\\
& \cup [4-1/4-1/16+8^{-1}/16, 4-1/4-8^{-1}/16],
\end{aligned}
$$

$$
\begin{aligned}
C(t) &\equiv (8\,C(4+1) + 1\,C(4+1+1/4))/9,\\
& \quad t \in [4+1+8^{-1}/16, 4+1+8^{-1}/16\cdot 3],\\
C(t) &\equiv (7\,C(4+1) + 2\,C(4+1+1/4))/9,\\
& \quad t \in [4+1+8^{-1}/16\cdot 5, 4+1+8^{-1}/16\cdot 7],\\
& \vdots\\
C(t) &\equiv (1\,C(4+1) + 8\,C(4+1+1/4))/9,\\
& \quad t \in [4+1+8^{-1}/16\cdot 29, 4+1+8^{-1}/16\cdot 31],\\
C(t) &\equiv (8\,C(4+1+3/4) + 1\,C(4+2))/9,\\
& \quad t \in [4+1+3/4+8^{-1}/16, 4+1+3/4+8^{-1}/16\cdot 3],\\
C(t) &\equiv (7\,C(4+1+3/4) + 2\,C(4+2))/9,\\
& \quad t \in [4+1+3/4+8^{-1}/16\cdot 5, 4+1+3/4+8^{-1}/16\cdot 7],\\
& \vdots\\
C(t) &\equiv (1\,C(4+1+3/4) + 8\,C(4+2))/9,\\
& \quad t \in [4+1+3/4+8^{-1}/16\cdot 29, 4+1+3/4+8^{-1}/16\cdot 31],
\end{aligned}
$$

and we define continuous C so that it is linear on the rest of subintervals.

If we denote
$$a_1^1 := 0, \quad b_1^1 := 0, \quad c_1^1 := 1,$$
$$a_2^1 := 1, \quad b_2^1 := 1 + 1/4, \quad c_2^1 := 1 + 3/4, \quad a_3^1 := 2$$
and (compare with the situation after the second step)
$$a_1^2 := -2, \quad b_1^2 := -2, \quad c_1^2 := -2,$$
$$a_2^2 := -2, \quad b_2^2 := -2 + 1/16, \quad c_2^2 := -1 - 1/16,$$
$$a_3^2 := -1, \quad b_3^2 := -1, \quad c_3^2 := -1,$$
$$a_4^2 := -1, \quad b_4^2 := -1 + 1/16, \quad c_4^2 := -1 + 1/4 - 1/16,$$
$$a_5^2 := -1 + 1/4, \quad b_5^2 := -1 + 1/4 + 1/16, \quad c_5^2 := -1 + 3/4 - 1/16,$$
$$a_6^2 := -1 + 3/4, \quad b_6^2 := -1 + 3/4 + 1/16, \quad c_6^2 := -1/16,$$
we see that C does not need to be constant only on
$$[a_j^1 - 2, a_j^1 - 2 + 4^{-2}], \quad [b_2^1 - 2 - 4^{-2}, b_2^1 - 2], \quad [b_j^1 - 2, b_j^1 - 2 + 4^{-2}],$$
$$[c_j^1 - 2 - 4^{-2}, c_j^1 - 2], \quad [c_j^1 - 2, c_j^1 - 2 + 4^{-2}], \quad [a_{j+1}^1 - 2 - 4^{-2}, a_{j+1}^1 - 2]$$
for $j \in \{1, 2\}$, i.e., on
$$[a_j^2, b_j^2], \quad j \in \{1, \dots, 6\}, \qquad [c_j^2, a_{j+1}^2], \quad j \in \{1, \dots, 5\}, \qquad [c_6^2, 0],$$
and it has to be constant on each one of the intervals
$$[a_2^1 - 2 + 4^{-2}, b_2^1 - 2 - 4^{-2}], \quad [c_2^1 - 2 + 4^{-2}, a_3^1 - 2 - 4^{-2}],$$
$$[b_j^1 - 2 + 4^{-2}, c_j^1 - 2 - 4^{-2}], \quad j \in \{1, 2\},$$
i.e., on
$$[b_j^2, c_j^2], \quad j \in \{1, \dots, 6\}.$$
It is also seen that
$$a_1^2 = d_1^1, \quad b_1^2 = d_2^1, \quad c_1^2 = d_3^1, \quad a_2^2 = d_4^1, \quad \cdots \quad c_6^2 = d_{18}^1,$$
where $d_1^1, d_2^1, \dots, d_{18}^1$ is the non-decreasing sequence of all numbers
$$a_j^1 - 2, \quad b_j^1 - 2, \quad c_j^1 - 2,$$
$$\min\{a_j^1 - 2 + 4^{-2}, b_j^1 - 2\}, \quad \max\{a_j^1 - 2, b_j^1 - 2 - 4^{-2}\},$$
$$\min\{c_j^1 - 2, b_j^1 - 2 + 4^{-2}\}, \quad \max\{c_j^1 - 2 - 4^{-2}, b_j^1 - 2\},$$
$$\min\{c_j^1 - 2 + 4^{-2}, a_{j+1}^1 - 2\}, \quad \max\{c_j^1 - 2, a_{j+1}^1 - 2 - 4^{-2}\}$$
for $j \in \{1, 2\}$. We put $a_7^2 := 0$.

Let $d^2_1, d^2_2, \dots, d^2_{168}$ be the non-decreasing sequence of all numbers

$$b^1_1 + 4, \quad b^1_1 + 4, \quad b^1_1 + 4, \quad b^1_1 + 4, \quad b^1_1 + 4, \quad b^1_1 + 4,$$

$$b^1_1 + 4, \quad b^1_1 + 4, \quad b^1_1 + 4, \quad b^1_1 + 4, \quad b^1_1 + 4, \quad b^1_1 + 4,$$

$$c^1_1 + 4, \quad \min\{c^1_1 + 4, b^1_1 + 4 + 8^{-1}/16\}, \quad \max\{c^1_1 + 4 - 8^{-1}/16, b^1_1 + 4\},$$

$$c^1_2 + 4, \quad \min\{c^1_2 + 4, b^1_2 + 4 + 8^{-1}/16\}, \quad \max\{c^1_2 + 4 - 8^{-1}/16, b^1_2 + 4\},$$

$$a^1_1 + (4k+1)(b^1_1 - a^1_1)/32 + 4, \quad a^1_1 + (4k+3)(b^1_1 - a^1_1)/32 + 4,$$

$$a^1_1 + (4k+4)(b^1_1 - a^1_1)/32 + 4, \quad k \in \{0, 1, \dots, 7\},$$

$$c^1_1 + (4k+1)(a^1_2 - c^1_1)/32 + 4, \quad c^1_1 + (4k+3)(a^1_2 - c^1_1)/32 + 4,$$

$$c^1_1 + (4k+4)(a^1_2 - c^1_1)/32 + 4, \quad k \in \{0, 1, \dots, 7\},$$

$$a^1_2 + (4k+1)(b^1_2 - a^1_2)/32 + 4, \quad a^1_2 + (4k+3)(b^1_2 - a^1_2)/32 + 4,$$

$$a^1_2 + (4k+4)(b^1_2 - a^1_2)/32 + 4, \quad k \in \{0, 1, \dots, 7\},$$

$$c^1_2 + (4k+1)(a^1_3 - c^1_2)/32 + 4, \quad c^1_2 + (4k+3)(a^1_3 - c^1_2)/32 + 4,$$

$$c^1_2 + (4k+4)(a^1_3 - c^1_2)/32 + 4, \quad k \in \{0, 1, \dots, 7\},$$

and

$$a^2_{j+1} + 4, \quad b^2_j + 4, \quad c^2_j + 4,$$

$$\min\{a^2_j + 4 + 8^{-1}/16, b^2_j + 4\}, \quad \max\{a^2_j + 4, b^2_j + 4 - 8^{-1}/16\},$$

$$\min\{c^2_j + 4, b^2_j + 4 + 8^{-1}/16\}, \quad \max\{c^2_j + 4 - 8^{-1}/16, b^2_j + 4\},$$

$$\min\{c^2_j + 4 + 8^{-1}/16, a^2_{j+1} + 4\}, \quad \max\{c^2_j + 4, a^2_{j+1} + 4 - 8^{-1}/16\}$$

for $j \in \{1, \dots, 6\}$. We denote

$$a^3_1 := 2, \quad b^3_1 := d^2_1, \quad c^3_1 := d^2_2, \quad a^3_2 := d^2_3, \quad \cdots \quad a^3_{57} := d^2_{168}.$$

We remark that, in the sequences of d^l_j, $l \in \mathbb{N}$, values are a number of time.

In the fourth step, we define C so that

$$\left|\left| \, C(t) - C(t + 2^3) \, \right|\right| < \frac{\varepsilon}{2^3}, \quad t \in [-2^3 - 2, -2).$$

We consider the non-decreasing sequence $d^3_1, d^3_2, \dots, d^3_{21 \cdot 8^2}$ of

$$a^3_j - 2^3, \quad b^3_j - 2^3, \quad c^3_j - 2^3,$$

$$\min\{a^3_j - 2^3 + 8^{-2}/16, b^3_j - 2^3\}, \quad \max\{a^3_j - 2^3, b^3_j - 2^3 - 8^{-2}/16\},$$

$$\min\{c^3_j - 2^3, b^3_j - 2^3 + 8^{-2}/16\}, \quad \max\{c^3_j - 2^3 - 8^{-2}/16, b^3_j - 2^3\},$$

$$\min\{c^3_j - 2^3 + 8^{-2}/16, a^3_{j+1} - 2^3\}, \quad \max\{c^3_j - 2^3, a^3_{j+1} - 2^3 - 8^{-2}/16\}$$

for $j \in \{1, \dots, 7 \cdot 8\}$, 144 numbers $b^1_1 - 2^3$, and

$$c^1_1 - 2^3, \quad \min\{c^1_1 - 2^3, b^1_1 - 2^3 + 8^{-2}/16\}, \quad \max\{c^1_1 - 2^3 - 8^{-2}/16, b^1_1 - 2^3\},$$

$$
\begin{gathered}
c_2^1 - 2^3, \quad \min\{c_2^1 - 2^3, b_2^1 - 2^3 + 8^{-2}/16\}, \quad \max\{c_2^1 - 2^3 - 8^{-2}/16, b_2^1 - 2^3\}, \\
\min\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8\cdot 4) - 2^3 + 8^{-2}/16, a_1^1 + k(b_1^1 - a_1^1)/(8\cdot 4) - 2^3\}, \\
\max\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8\cdot 4) - 2^3, a_1^1 + k(b_1^1 - a_1^1)/(8\cdot 4) - 2^3 - 8^{-2}/16\}, \\
a_1^1 + k(b_1^1 - a_1^1)/(8\cdot 4) - 2^3, \quad k \in \{1, \dots, 8\cdot 4\}, \\
\min\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8\cdot 4) - 2^3 + 8^{-2}/16, c_1^1 + k(a_2^1 - c_1^1)/(8\cdot 4) - 2^3\}, \\
\max\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8\cdot 4) - 2^3, c_1^1 + k(a_2^1 - c_1^1)/(8\cdot 4) - 2^3 - 8^{-2}/16\}, \\
c_1^1 + k(a_2^1 - c_1^1)/(8\cdot 4) - 2^3, \quad k \in \{1, \dots, 8\cdot 4\}, \\
\min\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8\cdot 4) - 2^3 + 8^{-2}/16, a_2^1 + k(b_2^1 - a_2^1)/(8\cdot 4) - 2^3\}, \\
\max\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8\cdot 4) - 2^3, a_2^1 + k(b_2^1 - a_2^1)/(8\cdot 4) - 2^3 - 8^{-2}/16\}, \\
a_2^1 + k(b_2^1 - a_2^1)/(8\cdot 4) - 2^3, \quad k \in \{1, \dots, 8\cdot 4\}, \\
\min\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8\cdot 4) - 2^3 + 8^{-2}/16, c_2^1 + k(a_3^1 - c_2^1)/(8\cdot 4) - 2^3\}, \\
\max\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8\cdot 4) - 2^3, c_2^1 + k(a_3^1 - c_2^1)/(8\cdot 4) - 2^3 - 8^{-2}/16\}, \\
c_2^1 + k(a_3^1 - c_2^1)/(8\cdot 4) - 2^3, \quad k \in \{1, \dots, 8\cdot 4\}, \\
c_1^2 - 2^3, \quad \min\{c_1^2 - 2^3, b_1^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_1^2 - 2^3 - 8^{-2}/16, b_1^2 - 2^3\}, \\
c_2^2 - 2^3, \quad \min\{c_2^2 - 2^3, b_2^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_2^2 - 2^3 - 8^{-2}/16, b_2^2 - 2^3\}, \\
c_3^2 - 2^3, \quad \min\{c_3^2 - 2^3, b_3^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_3^2 - 2^3 - 8^{-2}/16, b_3^2 - 2^3\}, \\
c_4^2 - 2^3, \quad \min\{c_4^2 - 2^3, b_4^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_4^2 - 2^3 - 8^{-2}/16, b_4^2 - 2^3\}, \\
c_5^2 - 2^3, \quad \min\{c_5^2 - 2^3, b_5^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_5^2 - 2^3 - 8^{-2}/16, b_5^2 - 2^3\}, \\
c_6^2 - 2^3, \quad \min\{c_6^2 - 2^3, b_6^2 - 2^3 + 8^{-2}/16\}, \quad \max\{c_6^2 - 2^3 - 8^{-2}/16, b_6^2 - 2^3\}, \\
\min\{a_1^2 + (k-1)(b_1^2 - a_1^2)/8 - 2^3 + 8^{-2}/16, a_1^2 + k(b_1^2 - a_1^2)/8 - 2^3\}, \\
\max\{a_1^2 + (k-1)(b_1^2 - a_1^2)/8 - 2^3, a_1^2 + k(b_1^2 - a_1^2)/8 - 2^3 - 8^{-2}/16\}, \\
a_1^2 + k(b_1^2 - a_1^2)/8 - 2^3, \quad k \in \{1, \dots, 8\}, \\
\min\{c_1^2 + (k-1)(a_2^2 - c_1^2)/8 - 2^3 + 8^{-2}/16, c_1^2 + k(a_2^2 - c_1^2)/8 - 2^3\}, \\
\max\{c_1^2 + (k-1)(a_2^2 - c_1^2)/8 - 2^3, c_1^2 + k(a_2^2 - c_1^2)/8 - 2^3 - 8^{-2}/16\}, \\
c_1^2 + k(a_2^2 - c_1^2)/8 - 2^3, \quad k \in \{1, \dots, 8\}, \\
\vdots \\
\min\{a_6^2 + (k-1)(b_6^2 - a_6^2)/8 - 2^3 + 8^{-2}/16, a_6^2 + k(b_6^2 - a_6^2)/8 - 2^3\}, \\
\max\{a_6^2 + (k-1)(b_6^2 - a_6^2)/8 - 2^3, a_6^2 + k(b_6^2 - a_6^2)/8 - 2^3 - 8^{-2}/16\}, \\
a_6^2 + k(b_6^2 - a_6^2)/8 - 2^3, \quad k \in \{1, \dots, 8\}, \\
\min\{c_6^2 + (k-1)(a_7^2 - c_6^2)/8 - 2^3 + 8^{-2}/16, c_6^2 + k(a_7^2 - c_6^2)/8 - 2^3\}, \\
\max\{c_6^2 + (k-1)(a_7^2 - c_6^2)/8 - 2^3, c_6^2 + k(a_7^2 - c_6^2)/8 - 2^3 - 8^{-2}/16\}, \\
c_6^2 + k(a_7^2 - c_6^2)/8 - 2^3, \quad k \in \{1, \dots, 8\}.
\end{gathered}
$$

We put

$$a_1^4 := d_1^3, \quad b_1^4 := d_2^3, \quad c_1^4 := d_3^3, \quad \cdots \quad c_{7\cdot 8^2}^4 := d_{21\cdot 8^2}^3, \quad a_{7\cdot 8^2+1}^4 := -2.$$

We recall that C can be increasing or decreasing only on

$$[a_j^4, b_j^4], \quad [c_j^4, a_{j+1}^4], \quad j \in \{1, \dots, 7 \cdot 8^2\}.$$

We proceed further in the same way (as in the third and the fourth step). In the $2n$-th step, we define continuous C so that

$$\big|\big|\, C(t) - C(t + 2^{2n-1}) \,\big|\big| < \frac{\varepsilon}{2^{n+1}}, \quad t \in [-2^{2n-1} - \cdots - 2, -2^{2n-3} - \cdots - 2).$$

We get the non-decreasing sequence $\{d_l^{2n-1}\}$ from

$$a_j^{2n-1} - 2^{2n-1}, \quad b_j^{2n-1} - 2^{2n-1}, \quad c_j^{2n-1} - 2^{2n-1},$$

$\min\{a_j^{2n-1} - 2^{2n-1} + 8^{2-2n}/16, b_j^{2n-1} - 2^{2n-1}\}$, $\max\{a_j^{2n-1} - 2^{2n-1}, b_j^{2n-1} - 2^{2n-1} - 8^{2-2n}/16\}$,
$\min\{c_j^{2n-1} - 2^{2n-1}, b_j^{2n-1} - 2^{2n-1} + 8^{2-2n}/16\}$, $\max\{c_j^{2n-1} - 2^{2n-1} - 8^{2-2n}/16, b_j^{2n-1} - 2^{2n-1}\}$,
$\min\{c_j^{2n-1} - 2^{2n-1} + 8^{2-2n}/16, a_{j+1}^{2n-1} - 2^{2n-1}\}$, $\max\{c_j^{2n-1} - 2^{2n-1}, a_{j+1}^{2n-1} - 2^{2n-1} - 8^{2-2n}/16\}$
for $j \in \{1, \dots, 7 \cdot 8^{2n-3}\}$, from

$c_1^1 - 2^{2n-1}$, $\min\{c_1^1 - 2^{2n-1}, b_1^1 - 2^{2n-1} + 8^{2-2n}/16\}$, $\max\{c_1^1 - 2^{2n-1} - 8^{2-2n}/16, b_1^1 - 2^{2n-1}\}$,
$c_2^1 - 2^{2n-1}$, $\min\{c_2^1 - 2^{2n-1}, b_2^1 - 2^{2n-1} + 8^{2-2n}/16\}$, $\max\{c_2^1 - 2^{2n-1} - 8^{2-2n}/16, b_2^1 - 2^{2n-1}\}$,
$\min\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16, a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}\}$,
$\max\{a_1^1 + (k-1)(b_1^1 - a_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}, a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\}$,

$$a_1^1 + k(b_1^1 - a_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8 \cdot 4^{2n-3}\},$$

$\min\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16, c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}\}$,
$\max\{c_1^1 + (k-1)(a_2^1 - c_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}, c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\}$,

$$c_1^1 + k(a_2^1 - c_1^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8 \cdot 4^{2n-3}\},$$

$\min\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16, a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}\}$,
$\max\{a_2^1 + (k-1)(b_2^1 - a_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}, a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\}$,

$$a_2^1 + k(b_2^1 - a_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8 \cdot 4^{2n-3}\},$$

$\min\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1} + 8^{2-2n}/16, c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}\}$,
$\max\{c_2^1 + (k-1)(a_3^1 - c_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}, c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1} - 8^{2-2n}/16\}$,

$$c_2^1 + k(a_3^1 - c_2^1)/(8 \cdot 4^{2n-3}) - 2^{2n-1}, \quad k \in \{1, \dots, 8 \cdot 4^{2n-3}\},$$

$c_1^2 - 2^{2n-1}$, $\min\{c_1^2 - 2^{2n-1}, b_1^2 - 2^{2n-1} + 8^{2-2n}/16\}$, $\max\{c_1^2 - 2^{2n-1} - 8^{2-2n}/16, b_1^2 - 2^{2n-1}\}$,

$$\vdots$$

$$c_6^2 - 2^{2n-1},\ \min\{c_6^2 - 2^{2n-1}, b_6^2 - 2^{2n-1} + 8^{2-2n}/16\},\ \max\{c_6^2 - 2^{2n-1} - 8^{2-2n}/16, b_6^2 - 2^{2n-1}\},$$
$$\min\{a_1^2 + (k-1)(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} + 8^{2-2n}/16, a_1^2 + k(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}\},$$
$$\max\{a_1^2 + (k-1)(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, a_1^2 + k(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} - 8^{2-2n}/16\},$$
$$a_1^2 + k(b_1^2 - a_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-4}\},$$
$$\min\{c_1^2 + (k-1)(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} + 8^{2-2n}/16, c_1^2 + k(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}\},$$
$$\max\{c_1^2 + (k-1)(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, c_1^2 + k(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} - 8^{2-2n}/16\},$$
$$c_1^2 + k(a_2^2 - c_1^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-4}\},$$
$$\vdots$$
$$\min\{a_6^2 + (k-1)(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} + 8^{2-2n}/16, a_6^2 + k(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}\},$$
$$\max\{a_6^2 + (k-1)(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, a_6^2 + k(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} - 8^{2-2n}/16\},$$
$$a_6^2 + k(b_6^2 - a_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-4}\},$$
$$\min\{c_6^2 + (k-1)(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} + 8^{2-2n}/16, c_6^2 + k(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}\},$$
$$\max\{c_6^2 + (k-1)(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, c_6^2 + k(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1} - 8^{2-2n}/16\},$$
$$c_6^2 + k(a_7^2 - c_6^2)/(8\cdot 4^{2n-4}) - 2^{2n-1}, \quad k \in \{1, \dots, 8\cdot 4^{2n-4}\},$$
$$\vdots$$
$$c_1^{2n-2} - 2^{2n-1}, \quad \min\{c_1^{2n-2} - 2^{2n-1}, b_1^{2n-2} - 2^{2n-1} + 8^{2-2n}/16\},$$
$$\max\{c_1^{2n-2} - 2^{2n-1} - 8^{2-2n}/16, b_1^{2n-2} - 2^{2n-1}\},$$
$$\vdots$$
$$c_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1}, \quad \min\{c_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1}, b_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1} + 8^{2-2n}\},$$
$$\max\{c_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1} - 8^{2-2n}/16, b_{7\cdot 8^{2n-4}}^{2n-2} - 2^{2n-1}\},$$
$$\min\{a_1^{2n-2} + (k-1)(b_1^{2n-2} - a_1^{2n-2})/8 - 2^{2n-1} + 8^{2-2n}/16, a_1^{2n-2} + k(b_1^{2n-2} - a_1^{2n-2})/8 - 2^{2n-1}\},$$
$$\max\{a_1^{2n-2} + (k-1)(b_1^{2n-2} - a_1^{2n-2})/8 - 2^{2n-1}, a_1^{2n-2} + k(b_1^{2n-2} - a_1^{2n-2})/8 - 2^{2n-1} - 8^{2-2n}/16\},$$
$$a_1^{2n-2} + k(b_1^{2n-2} - a_1^{2n-2})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},$$
$$\min\{c_1^{2n-2} + (k-1)(a_2^{2n-2} - c_1^{2n-2})/8 - 2^{2n-1} + 8^{2-2n}/16, c_1^{2n-2} + k(a_2^{2n-2} - c_1^{2n-2})/8 - 2^{2n-1}\},$$
$$\max\{c_1^{2n-2} + (k-1)(a_2^{2n-2} - c_1^{2n-2})/8 - 2^{2n-1}, c_1^{2n-2} + k(a_2^{2n-2} - c_1^{2n-2})/8 - 2^{2n-1} - 8^{2-2n}/16\},$$
$$c_1^{2n-2} + k(a_2^{2n-2} - c_1^{2n-2})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},$$
$$\vdots$$
$$\min\{a_{7\cdot 8^{2n-4}}^{2n-2} + (k-1)(b_{7\cdot 8^{2n-4}}^{2n-2} - a_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1} + 8^{2-2n}/16,$$
$$a_{7\cdot 8^{2n-4}}^{2n-2} + k(b_{7\cdot 8^{2n-4}}^{2n-2} - a_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1}\},$$
$$\max\{a_{7\cdot 8^{2n-4}}^{2n-2} + (k-1)(b_{7\cdot 8^{2n-4}}^{2n-2} - a_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1},$$
$$a_{7\cdot 8^{2n-4}}^{2n-2} + k(b_{7\cdot 8^{2n-4}}^{2n-2} - a_{7\cdot 8^{2n-4}}^{2n-2})/8 - 2^{2n-1} - 8^{2-2n}/16\},$$

$$a^{2n-2}_{7\cdot 8^{2n-4}} + k(b^{2n-2}_{7\cdot 8^{2n-4}} - a^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},$$
$$\min\{c^{2n-2}_{7\cdot 8^{2n-4}} + (k-1)(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1} + 8^{2-2n}/16,$$
$$c^{2n-2}_{7\cdot 8^{2n-4}} + k(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1}\},$$
$$\max\{c^{2n-2}_{7\cdot 8^{2n-4}} + (k-1)(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1},$$
$$c^{2n-2}_{7\cdot 8^{2n-4}} + k(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1} - 8^{2-2n}/16\},$$
$$c^{2n-2}_{7\cdot 8^{2n-4}} + k(a^{2n-2}_{7\cdot 8^{2n-4}+1} - c^{2n-2}_{7\cdot 8^{2n-4}})/8 - 2^{2n-1}, \quad k \in \{1, \dots, 8\},$$

and from a number of $b^1_1 - 2^{2n-1}$ such that the total number of d^{2n-1}_l is $21 \cdot 8^{2n-2}$. We denote

$$a^{2n}_1 := d^{2n-1}_1, \quad b^{2n}_1 := d^{2n-1}_2, \quad c^{2n}_1 := d^{2n-1}_3, \quad \cdots$$
$$c^{2n-1}_{7\cdot 8^{2n-2}} := d^3_{21\cdot 8^{2n-2}}, \quad a^{2n-1}_{7\cdot 8^{2n-2}+1} := -2^{2n-3} - \cdots - 2.$$

In the $(2n+1)$-th step, we define continuous C so that

$$\left|\left| C(t) - C(t - 2^{2n}) \right|\right| < \frac{\varepsilon}{2^{n+2}}, \quad t \in (2 + \cdots + 2^{2n-2}, 2 + \cdots + 2^{2n}].$$

Now C has constant values on $[b^{2n+1}_j, c^{2n+1}_j]$, $j \in \{1, \dots, 7 \cdot 8^{2n-1}\}$, where we put

$$a^{2n+1}_1 := 2 + 2^2 + \cdots + 2^{2n-2}$$

and we obtain

$$b^{2n+1}_1, \quad c^{2n+1}_1, \quad a^{2n+1}_2, \quad \cdots \quad c^{2n+1}_{7\cdot 8^{2n-1}}, \quad a^{2n+1}_{7\cdot 8^{2n-1}+1}$$

from the non-decreasing sequence of

$$a^{2n}_{j+1} + 2^{2n}, \quad b^{2n}_j + 2^{2n}, \quad c^{2n}_j + 2^{2n},$$

$$\min\{a^{2n}_j + 2^{2n} + 8^{1-2n}/16, b^{2n}_j + 2^{2n}\}, \quad \max\{a^{2n}_j + 2^{2n}, b^{2n}_j + 2^{2n} - 8^{1-2n}/16\},$$
$$\min\{c^{2n}_j + 2^{2n}, b^{2n}_j + 2^{2n} + 8^{1-2n}/16\}, \quad \max\{c^{2n}_j + 2^{2n} - 8^{1-2n}/16, b^{2n}_j + 2^{2n}\},$$
$$\min\{c^{2n}_j + 2^{2n} + 8^{1-2n}/16, a^{2n}_{j+1} + 2^{2n}\}, \quad \max\{c^{2n}_j + 2^{2n}, a^{2n}_{j+1} + 2^{2n} - 8^{1-2n}/16\}$$

for $j \in \{1, \dots, 7 \cdot 8^{2n-2}\}$ and

$$c^1_1 + 2^{2n}, \quad \min\{c^1_1 + 2^{2n}, b^1_1 + 2^{2n} + 8^{1-2n}/16\}, \quad \max\{c^1_1 + 2^{2n} - 8^{1-2n}/16, b^1_1 + 2^{2n}\},$$

$$c^1_2 + 2^{2n}, \quad \min\{c^1_2 + 2^{2n}, b^1_2 + 2^{2n} + 8^{1-2n}/16\}, \quad \max\{c^1_2 + 2^{2n} - 8^{1-2n}/16, b^1_2 + 2^{2n}\},$$
$$\min\{a^1_1 + (k-1)(b^1_1 - a^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n} + 8^{1-2n}/16, a^1_1 + k(b^1_1 - a^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n}\},$$
$$\max\{a^1_1 + (k-1)(b^1_1 - a^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n}, a^1_1 + k(b^1_1 - a^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n} - 8^{1-2n}/16\},$$
$$a^1_1 + k(b^1_1 - a^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n}, \quad k \in \{1, \dots, 8 \cdot 4^{2n-2}\},$$
$$\min\{c^1_1 + (k-1)(a^1_2 - c^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n} + 8^{1-2n}/16, c^1_1 + k(a^1_2 - c^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n}\},$$
$$\max\{c^1_1 + (k-1)(a^1_2 - c^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n}, c^1_1 + k(a^1_2 - c^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n} - 8^{1-2n}/16\},$$
$$c^1_1 + k(a^1_2 - c^1_1)/(8 \cdot 4^{2n-2}) + 2^{2n}, \quad k \in \{1, \dots, 8 \cdot 4^{2n-2}\},$$

$$\min\{a_2^1+(k-1)(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n}+8^{1-2n}/16, a_2^1+k(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n}\},$$
$$\max\{a_2^1+(k-1)(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n}, a_2^1+k(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n}-8^{1-2n}/16\},$$
$$a_2^1+k(b_2^1-a_2^1)/(8\cdot 4^{2n-2})+2^{2n}, \quad k\in\{1,\dots,8\cdot 4^{2n-2}\},$$
$$\min\{c_2^1+(k-1)(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n}+8^{1-2n}/16, c_2^1+k(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n}\},$$
$$\max\{c_2^1+(k-1)(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n}, c_2^1+k(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n}-8^{1-2n}/16\},$$
$$c_2^1+k(a_3^1-c_2^1)/(8\cdot 4^{2n-2})+2^{2n}, \quad k\in\{1,\dots,8\cdot 4^{2n-2}\},$$
$$c_1^2+2^{2n}, \quad \min\{c_1^2+2^{2n}, b_1^2+2^{2n}+8^{1-2n}/16\}, \quad \max\{c_1^2+2^{2n}-8^{1-2n}/16, b_1^2+2^{2n}\},$$
$$\vdots$$
$$c_6^2+2^{2n}, \quad \min\{c_6^2+2^{2n}, b_6^2+2^{2n}+8^{1-2n}/16\}, \quad \max\{c_6^2+2^{2n}-8^{1-2n}/16, b_6^2+2^{2n}\},$$
$$\min\{a_1^2+(k-1)(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16, a_1^2+k(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n}\},$$
$$\max\{a_1^2+(k-1)(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n}, a_1^2+k(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},$$
$$a_1^2+k(b_1^2-a_1^2)/(8\cdot 4^{2n-3})+2^{2n}, \quad k\in\{1,\dots,8\cdot 4^{2n-3}\},$$
$$\min\{c_1^2+(k-1)(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16, c_1^2+k(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n}\},$$
$$\max\{c_1^2+(k-1)(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n}, c_1^2+k(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},$$
$$c_1^2+k(a_2^2-c_1^2)/(8\cdot 4^{2n-3})+2^{2n}, \quad k\in\{1,\dots,8\cdot 4^{2n-3}\},$$
$$\vdots$$
$$\min\{a_6^2+(k-1)(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16, a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}\},$$
$$\max\{a_6^2+(k-1)(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}, a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},$$
$$a_6^2+k(b_6^2-a_6^2)/(8\cdot 4^{2n-3})+2^{2n}, \quad k\in\{1,\dots,8\cdot 4^{2n-3}\},$$
$$\min\{c_6^2+(k-1)(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}+8^{1-2n}/16, c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}\},$$
$$\max\{c_6^2+(k-1)(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}, c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}-8^{1-2n}/16\},$$
$$c_6^2+k(a_7^2-c_6^2)/(8\cdot 4^{2n-3})+2^{2n}, \quad k\in\{1,\dots,8\cdot 4^{2n-3}\},$$
$$\vdots$$
$$c_1^{2n-1}+2^{2n}, \quad \min\{c_1^{2n-1}+2^{2n}, b_1^{2n-1}+2^{2n}+8^{1-2n}/16\},$$
$$\max\{c_1^{2n-1}+2^{2n}-8^{1-2n}/16, b_1^{2n-1}+2^{2n}\},$$
$$\vdots$$
$$c_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}, \quad \min\{c_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}, b_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}+8^{1-2n}/16\},$$
$$\max\{c_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}-8^{1-2n}/16, b_{7\cdot 8^{2n-3}}^{2n-1}+2^{2n}\},$$
$$\min\{a_1^{2n-1}+(k-1)(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}+8^{1-2n}/16, a_1^{2n-1}+k(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}\},$$
$$\max\{a_1^{2n-1}+(k-1)(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}, a_1^{2n-1}+k(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}-8^{1-2n}/16\},$$
$$a_1^{2n-1}+k(b_1^{2n-1}-a_1^{2n-1})/8+2^{2n}, \quad k\in\{1,\dots,8\},$$

$$\min\{c_1^{2n-1}+(k-1)(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}+8^{1-2n}/16, c_1^{2n-1}+k(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}\},$$
$$\max\{c_1^{2n-1}+(k-1)(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}, c_1^{2n-1}+k(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}-8^{1-2n}/16\},$$
$$c_1^{2n-1}+k(a_2^{2n-1}-c_1^{2n-1})/8+2^{2n}, \quad k\in\{1,\dots,8\},$$
$$\vdots$$
$$\min\{a_{7\cdot 8^{2n-3}}^{2n-1}+(k-1)(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}+8^{1-2n}/16,$$
$$a_{7\cdot 8^{2n-3}}^{2n-1}+k(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}\},$$
$$\max\{a_{7\cdot 8^{2n-3}}^{2n-1}+(k-1)(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n},$$
$$a_{7\cdot 8^{2n-3}}^{2n-1}+k(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}-8^{1-2n}/16\},$$
$$a_{7\cdot 8^{2n-3}}^{2n-1}+k(b_{7\cdot 8^{2n-3}}^{2n-1}-a_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}, \quad k\in\{1,\dots,8\},$$
$$\min\{c_{7\cdot 8^{2n-3}}^{2n-1}+(k-1)(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}+8^{1-2n}/16,$$
$$c_{7\cdot 8^{2n-3}}^{2n-1}+k(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}\},$$
$$\max\{c_{7\cdot 8^{2n-3}}^{2n-1}+(k-1)(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n},$$
$$c_{7\cdot 8^{2n-3}}^{2n-1}+k(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}-8^{1-2n}/16\},$$
$$c_{7\cdot 8^{2n-3}}^{2n-1}+k(a_{7\cdot 8^{2n-3}+1}^{2n-1}-c_{7\cdot 8^{2n-3}}^{2n-1})/8+2^{2n}, \quad k\in\{1,\dots,8\},$$

and the corresponding number of $b_1^1+2^{2n}$.

Using this construction, we get a continuous function C on $\mathbb{R}$. From Theorem 5.20 it follows that C is almost periodic. Since

$$||\,C(t)\,||=0, \quad t\in[0,1], \qquad ||\,C(t)\,||<\frac{\varepsilon}{4}, \quad t\in(1,2],$$

$$||\,C(t)-C(t+2)\,||<\frac{\varepsilon}{4}, \quad t\in[-2,0), \qquad ||\,C(t)-C(t-4)\,||<\frac{\varepsilon}{8}, \quad t\in(2,6],$$

$$\vdots$$

$$\left|\left|\,C(t)-C(t+2^{2n-1})\,\right|\right|<\frac{\varepsilon}{2^{n+1}}, \quad t\in[-2^{2n-1}-\dots-2^3-2,-2^{2n-3}-\dots-2^3-2),$$

$$\left|\left|\,C(t)-C(t-2^{2n})\,\right|\right|<\frac{\varepsilon}{2^{n+2}}, \quad t\in(2+2^2+\dots+2^{2n-2},2+2^2+\dots+2^{2n}],$$

we see that

$$||\,C(t)\,||<\sum_{j=1}^{\infty}\frac{2\varepsilon}{2^{j+1}}=\varepsilon, \quad t\in\mathbb{R}.$$

We denote

$$I_n:=[2+2^2+\dots+2^{2n-2},2+2^2+\dots+2^{2n}].$$

We will prove that we can choose constant values of $C(t)$, $t\in I_n$, on subintervals with the total length denoted by r_{2n+1} which is grater than 2^{2n-1} for all $n\in\mathbb{N}$. We can choose values of C on

$$[4-2+1/16+8^{-1}/16, 4-2+1-1/16-8^{-1}/16]\subset[2,6],$$

$$[4-2+1+1/4+1/16+8^{-1}/16, 4-2+1+3/4-1/16-8^{-1}/16] \subset [2,6],$$
$$[4+8^{-1}/16, 4+1-8^{-1}/16],\ [4+1+1/4+8^{-1}/16, 4+1+3/4-8^{-1}/16] \subset [2,6].$$

Hence,

$$r_3 \geq \frac{55}{64} + \frac{23}{64} + \frac{63}{64} + \frac{31}{64} = \frac{43}{16}, \tag{6.3}$$

i.e., the statement is valid for $n = 1$. We use the induction principle with respect to n. Assume that the statement is true for $1, 2, \dots, n-1$ and prove it for n. Without loss of generality (consider the below given process), we can also assume that the estimation $r_{2j} > 2^{2(j-1)}$ is valid for $j \in \{1, \dots, n\}$ (note that $r_2 = 5/4 > 2^0$) if we use the analogous notation.

In view of the construction, we see that we can choose C on any interval

$$[s + 2^{2n} + 8^{1-2n}/16, t + 2^{2n} - 8^{1-2n}/16]$$

if we can choose C on $[s, t]$, where $s = b_j^l < c_j^l = t$, $l < 2n+1$. Especially, we can choose function C on

$$[2^{2n} + 8^{1-2n}/16, 1 + 2^{2n} - 8^{1-2n}/16],$$
$$[1 + 1/4 + 2^{2n} + 8^{1-2n}/16, 1 + 3/4 + 2^{2n} - 8^{1-2n}/16],$$
$$[-2 + 1/16 + 2^{2n} + 8^{1-2n}/16, -2 + 1 - 1/16 + 2^{2n} - 8^{1-2n}/16],$$
$$[-2 + 1 + 1/4 + 1/16 + 2^{2n} + 8^{1-2n}/16, -2 + 1 + 3/4 - 1/16 + 2^{2n} - 8^{1-2n}/16]$$

and on less than $7 \cdot 8^{2n-1} - 4$ subintervals of I_n. Expressing

$$\begin{aligned} I_n = [0 + 2^{2n}, 1 + 2^{2n}] &\cup [1 + 2^{2n}, 2 + 2^{2n}] \cup [-2 + 2^{2n}, 0 + 2^{2n}] \cup \cdots \\ &\cup [2 + 2^2 + \cdots + 2^{2n-4} + 2^{2n}, 2 + 2^2 + \cdots + 2^{2n-2} + 2^{2n}] \\ &\cup [-2^{2n-1} - \cdots - 2^3 - 2 + 2^{2n}, -2^{2n-3} - \cdots - 2^3 - 2 + 2^{2n}] \end{aligned}$$

and using the induction hypothesis, the construction, and (6.3), we obtain that we can choose C on intervals of the lengths grater than or equal to

$$1 - 2 \cdot 8^{1-2n}/16, \quad 1/2 - 2 \cdot 8^{1-2n}/16,$$
$$1 - 1/8 - 2 \cdot 8^{1-2n}/16, \quad 1/2 - 1/8 - 2 \cdot 8^{1-2n}/16,$$
$$43/16 + 2^2 + 2^3 + \cdots + 2^{2n-3} + 2^{2n-2} - 2 \cdot 8^{1-2n}/16 \cdot \left(7 \cdot 8^{2n-1} - 4\right).$$

Summing, we get

$$r_{2n+1} \geq 1 + \frac{1}{2} + \frac{7}{8} + \frac{3}{8} + \frac{11}{16} + 2^{2n-1} - 2 - \frac{7}{8} > 2^{2n-1}, \tag{6.4}$$

which is the above statement. Analogously, we can prove

$$r_{2n} > 2^{2n-2}, \quad n \in \mathbb{N}. \tag{6.5}$$

Now we describe the principal fundamental matrix X_C on I_n for arbitrary $n \in \mathbb{N}$. Since C is constant and has the form diag $(\mathrm{i}a, \mathrm{i}a, \dots, \mathrm{i}a)$ for some $a \in \mathbb{R}$ on each interval $[b_j^{2n+1}, c_j^{2n+1}]$, $j \in \{1, \dots, 6 \cdot 4^{2n-1}\}$, from

$$X_C(t_2) - X_C(t_1) = \int\limits_{t_1}^{t_2} C(\tau) \cdot X_C(\tau)\, \mathrm{d}\tau, \quad t_1, t_2 \in \mathbb{R},$$

we obtain

$$\begin{aligned} &\left|\left| X_C(t) - X_C^{2n+1}(t) \right|\right| \\ &\quad \le \sum_{j=1}^{k} \left(\int_{a_j^{2n+1}}^{b_j^{2n+1}} ||\, C(\tau) \cdot X_C(\tau) \,||\, \mathrm{d}\tau + \int_{c_j^{2n+1}}^{a_{j+1}^{2n+1}} ||\, C(\tau) \cdot X_C(\tau) \,||\, \mathrm{d}\tau \right) \end{aligned} \tag{6.6}$$

if $t \le a_{k+1}^{2n+1}$, $t \in I_n$, where

$$\begin{aligned} X_C^{2n+1}(t) &:= X_C(2 + 2^2 + \dots + 2^{2n-2}), \quad t \in [2 + 2^2 + \dots + 2^{2n-2}, b_1^{2n+1}], \\ X_C^{2n+1}(t) &:= \exp\left(C(b_1^{2n+1})(t - b_1^{2n+1})\right) \cdot X_C^{2n+1}(b_1^{2n+1}), \quad t \in (b_1^{2n+1}, c_1^{2n+1}], \\ X_C^{2n+1}(t) &:= {X_C}^{2n+1}(c_1^{2n+1}), \quad t \in (c_1^{2n+1}, b_2^{2n+1}], \\ &\vdots \\ X_C^{2n+1}(t) &:= \exp\left(C(b_{7 \cdot 8^{2n-1}}^{2n+1})(t - b_{7 \cdot 8^{2n-1}}^{2n+1})\right) \cdot X_C^{2n+1}(b_{7 \cdot 8^{2n-1}}^{2n+1}), \quad t \in (b_{7 \cdot 8^{2n-1}}^{2n+1}, c_{7 \cdot 8^{2n-1}}^{2n+1}], \\ X_C^{2n+1}(t) &:= X_C^{2n+1}(c_{7 \cdot 8^{2n-1}}^{2n+1}), \quad t \in (c_{7 \cdot 8^{2n-1}}^{2n+1}, 2 + 2^2 + \dots + 2^{2n}]. \end{aligned}$$

It is seen that X_C is bounded (see also the below given, where it is shown that $X_C(t) \in U(m)$ for all t) as almost periodic C. Any interval

$$[2 + \dots + 2^{2n-2} + l - 1, 2 + \dots + 2^{2n-2} + l], \quad l \in \{1, \dots, 2^{2n}\},\ n \in \mathbb{N},$$

contains at most 4^{2n+1} subintervals, where C can be linear. Indeed, it suffices to consider the construction. We repeat that the length of each one of the considered subintervals is $8^{1-2n}/16$ which implies that the total length of them on

$$J_n^l := [2 + 2^2 + \dots + 2^{2n-2}, 2 + 2^2 + \dots + 2^{2n-2} + 2^{2n-l}], \quad l \in \{1, \dots, n\},$$

is less than 2^{1-l}. Thus (consider also (6.6)), there exists $K \in \mathbb{R}$ such that

$$\left|\left| X_C(t) - X_C^{2n+1}(t) \right|\right| \le \frac{K}{2^l}, \quad t \in J_n^l,\ l \in \{1, \dots, n\},\ n \in \mathbb{N}. \tag{6.7}$$

From the form diag $(\mathrm{i}a(t), \dots, \mathrm{i}a(t))$ of all matrices $C(t)$, we see that

$$||\, C(t) \,|| = |\, a(t) \,|, \quad t \in \mathbb{R}.$$

For simplicity, let $a(t) \ge 0$, $t \in \mathbb{R}$. Let $a_j^n \in \mathbb{R}$, $j \in \{1, \dots, n\}$, be arbitrarily chosen. Considering the construction and combining (6.4) and (6.5), we get that we can choose constant values of

$$C(t), \quad t \in [2 + \dots + 2^{2n-2} + (l-1)\, 2^n, 2 + \dots + 2^{2n-2} + l\, 2^n],$$

on subintervals with the total length grater than 2^{n-2} for each $l \in \{1, \dots, 2^n\}$ and all sufficiently large $n \in \mathbb{N}$. Since we choose C only so that

$$\left|\left| \, C(t) - C(t - 2^{2n}) \, \right|\right| < \frac{\varepsilon}{2^{n+2}}, \quad t \in I_n,$$

we see that we can obtain

$$X_C^{2n+1}(t_j^n) = \operatorname{diag}\, \left(\exp\left(\mathrm{i}a_j^n\right), \dots, \exp\left(\mathrm{i}a_j^n\right)\right)$$

for arbitrary t_j^n such that

$$\begin{array}{ll} t_1^n \geq 2 + 2^2 + 2^4 + \cdots + 2^{2n-2} + 3^n - 3^0, & t_2^n \geq t_1^n + 3^n - 3^1, \\ \cdots \quad t_n^n \geq t_{n-1}^n + 3^n - 3^{n-1}, & 2 + 2^2 + 2^4 + \cdots + 2^{2n} \geq t_n^n, \end{array} \tag{6.8}$$

because we have

$$4^n > n(3^n - 3^0) > 3^n - 3^0 > \cdots > 3^n - 3^{n-1} > 2^{2n-k+1}$$

for sufficiently large $n \in \mathbb{N}$ and some $k = k(n) \in \{1, \dots, n\}$ satisfying

$$2^{2n-k-2} \cdot \varepsilon \cdot 2^{-n-2} > 2\pi.$$

We recall that we need to prove the existence of such C, given by the above construction, for which the vector valued function $X_A(t)\, X_C(t)\, u$, $t \in \mathbb{R}$, is not almost periodic for any $u \in \mathbb{C}^m$, $||\, u\,||_1 = 1$. Since

$$(X_A(t) \cdot X_A^*(t))' = A(t) \cdot X_A(t) \cdot X_A^*(t) - X_A(t) \cdot X_A^*(t) \cdot A(t), \quad t \in \mathbb{R},$$

and since the constant function given by I is a solution of $X' = A\,X - X\,A$, $X(0) = I$, we have $X_A(t) \in U(m)$ for all t. Thus, $X_C(t) \in U(m)$, $t \in \mathbb{R}$, as well. We add that $X_A(t)\, X_A^*(t) = I$, $t \in \mathbb{R}$, implies $A^*(t) + A(t) = O$, $t \in \mathbb{R}$.

Let $c \in \mathbb{C}$, $|\, c\,| = 1$, and $N \in U(m)$ be arbitrarily given. Obviously, for any $M \in U(m)$, we can choose a number $a(M, c) \in [0, 2\pi)$ so that all eigenvalues of matrix

$$P := M \cdot \operatorname{diag}\, \left(\exp\left(\mathrm{i}a(M,c)\right), \dots, \exp\left(\mathrm{i}a(M,c)\right)\right)$$

are not in the neighbourhood of c with a given radius which depends only on dimension m. Indeed, if M has eigenvalues $\lambda_1, \dots, \lambda_m$, then the eigenvalues of P are

$$\lambda_1 \exp\left(\mathrm{i}a(M,c)\right), \dots, \lambda_m \exp\left(\mathrm{i}a(M,c)\right).$$

Considering $P\,u - N\,u$ and expressing vectors $u \in \mathbb{C}^m$, $||\, u\,||_1 = 1$, as linear combinations of the eigenvectors of P, we see that $P\,u$ cannot be in a neighbourhood of $N\,u$ for some $c \in \mathbb{C}$, $|\, c\,| = 1$. Thus (the considered multiplication of matrices and vectors is uniformly continuous), there exist $\vartheta > 0$ and $\xi > 0$ such that, for any matrices $M, N \in U(m)$, one can find $a(M, N) \in (0, 2\pi)$ satisfying

$$||\, M \cdot \operatorname{diag}\, \left(\exp\left(\mathrm{i}\tilde{a}\right), \dots, \exp\left(\mathrm{i}\tilde{a}\right)\right) \cdot u \, - N \cdot u\, ||_1 > \vartheta \tag{6.9}$$

for $u \in \mathbb{C}^m$, $\| u \|_1 = 1$, $\tilde{a} \in (a(M,N) - \xi, a(M,N) + \xi)$.

We can construct C so that we obtain

$$X_C^{2n+1}(t_j^n) = \operatorname{diag}\left(\exp\left(\mathrm{i}a_j^n\right), \dots, \exp\left(\mathrm{i}a_j^n\right)\right)$$

for arbitrarily given $a_j^n \in [0, 2\pi)$ and any t_j^n satisfying (6.8) if $n \in \mathbb{N}$ is sufficiently large and $j \in \{1, \dots, n\}$. Especially, for sufficiently large $n \in \mathbb{N}$ and for

$$\begin{aligned} &t_1^n := 2 + 2^2 + 2^4 + \dots + 2^{2n-2} + 3^n - 3^0, \\ &t_2^n := t_1^n + 3^n - 3^1, \quad \cdots \quad t_n^n := t_{n-1}^n + 3^n - 3^{n-1}, \end{aligned} \tag{6.10}$$

we can choose all $X_C^{2n+1}(t_j^n)$ in the form without any conditions. Hence, we obtain diagonal matrices $X_C^{2n+1}(t_j^n)$, $j \in \{1, \dots, n\}$, determined by numbers

$$\exp\left(\mathrm{i}a\left(X_A(t_j^n), X_A(t_j^n - 3^n + 3^{j-1}) \cdot X_C(t_j^n - 3^n + 3^{j-1})\right)\right)$$

on their diagonals.

It is seen from (6.10) that each

$$t_j^n \in [2 + 2^2 + \dots + 2^{2n-2}, 2 + 2^2 + \dots + 2^{2n-2} + n3^n].$$

Thus (see (6.7)), for any $\eta > 0$, we have

$$\left\| X_C(t_j^n) - X_C^{2n+1}(t_j^n) \right\| < \eta \tag{6.11}$$

for sufficiently large $n = n(\eta) \in \mathbb{N}$ and $j \in \{1, \dots, n\}$. From (6.9) and (6.11) it follows that

$$\left\| X_A(t_j^n) \cdot X_C(t_j^n) \cdot u - X_A(t_j^n - 3^n + 3^{j-1}) \cdot X_C(t_j^n - 3^n + 3^{j-1}) \cdot u \right\|_1 > \vartheta \tag{6.12}$$

for any $u \in \mathbb{C}^m$, $\| u \|_1 = 1$, sufficiently large $n \in \mathbb{N}$, and $j \in \{1, \dots, n\}$.

By contradiction, suppose that there exists $u \in \mathbb{C}^m$, $\| u \|_1 = 1$, with the property that $X_A(t)\, X_C(t)\, u$, $t \in \mathbb{R}$, is almost periodic. Applying Theorem 5.5 for

$$\psi(t) = X_A(t) \cdot X_C(t) \cdot u, \quad t \in \mathbb{R}, \qquad s_n = 3^n, \quad n \in \mathbb{N}, \qquad \varepsilon = \vartheta,$$

we obtain

$$\| X_A(t + 3^{n_1}) \cdot X_C(t + 3^{n_1}) \cdot u \; - X_A(t + 3^{n_2}) \cdot X_C(t + 3^{n_2}) \cdot u \|_1 < \vartheta, \quad t \in \mathbb{R}, \tag{6.13}$$

for all n_1, n_2 from an infinite subset of $\mathbb{N}$. If we rewrite (6.13) into

$$\| X_A(t) \cdot X_C(t) \cdot u \; - X_A(t + 3^{n_2} - 3^{n_1}) \cdot X_C(t + 3^{n_2} - 3^{n_1}) \cdot u \|_1 < \vartheta, \quad t \in \mathbb{R},$$

then it is easy to see that (6.12) is not valid for infinitely many $n \in \mathbb{N}$. This contradiction proves the theorem. □

6.3 Skew-symmetric systems without almost periodic solutions

Let $m \neq 1$. Let us consider systems of m homogeneous linear differential equations of the form
$$x'(t) = A(t) \cdot x(t), \quad t \in \mathbb{R}, \tag{6.14}$$
where $A : \mathbb{R} \to so(m)$ is an almost periodic function. Let $\mathcal{S}$ denote the set of all systems (6.14). We can identify the function A with the system (6.14) which is determined by A. Especially, we write $A \in \mathcal{S}$. Let $X_S = X_S(t)$ denote the principal fundamental matrix of $S \in \mathcal{S}$ satisfying $X_S(0) = I$.

In the vector space $\mathbb{R}^m$, we use the Euclidean norm $\|\cdot\|_2$ (one can also replace it by the absolute norm or the maximum norm). Let $\|\cdot\|$ be the corresponding matrix norm in $\mathrm{Mat}(\mathbb{R}, m)$ and let ϱ be the metric given by $\|\cdot\|$. Using the boundedness of every almost periodic function, the distance between two systems $A, B \in \mathcal{S}$ is defined uniformly on $\mathbb{R}$ by the norm of the matrix valued functions A, B; i.e., we introduce the metric σ by (6.2). For $\varepsilon > 0$, symbol $\mathcal{O}^{\sigma}_{\varepsilon}(A)$ denotes the ε-neighbourhood of a system A in $\mathcal{S}$ and $\mathcal{O}^{\varrho}_{\varepsilon}(M)$ the ε-neighbourhood of a matrix M in a given subset of $\mathrm{Mat}(\mathbb{R}, m)$.

The importance of skew-symmetric systems may be illustrated by the Cameron-Johnson theorem which states that any almost periodic homogeneous linear differential system can be reduced by a Lyapunov transformation to a skew-symmetric system if all solutions of the given system and all of its limit equations are bounded (see [40]). Further, it is known (see [170]) that the skew-symmetric systems, all of whose solutions are almost periodic, form a dense subset in the space of all skew-symmetric systems (special cases are considered in [113, 114] and the corresponding result about unitary difference systems is mentioned in [176]). This fact also motivates the study of skew-symmetric systems without almost periodic solutions. Concerning basic results about skew-symmetric systems and their fundamental matrices, we refer to [33, 71, 137].

Now we repeat the basic motivation in an explicit form.

Theorem 6.2. *For any $A \in \mathcal{S}$ and $\varepsilon > 0$, there exists $B \in \mathcal{O}^{\sigma}_{\varepsilon}(A)$ whose all solutions are almost periodic.*

Proof. See [170, Theorem 1, Remark 3]. □

To prove the announced result, we need the following lemmas.

Lemma 6.3. *There exist $\xi > 0$ and a neighbourhood $\tilde{\mathcal{O}}(O)$ of the zero matrix in $so(m)$ for which the exponential map is a bijection between $\tilde{\mathcal{O}}(O)$ and $\mathcal{O}^{\varrho}_{\xi}(I) \cap SO(m)$ such that the maps*
$$A \mapsto \exp(A), \quad A \in \tilde{\mathcal{O}}(O); \qquad A \mapsto \ln(A), \quad A \in \mathcal{O}^{\varrho}_{\xi}(I) \cap SO(m), \tag{6.15}$$
are continuous in the Lipschitz sense.

Proof. It is well-known that the exponential map is a bijection between $\tilde{\mathcal{O}}(O)$ and $\mathcal{O}^{\varrho}_{\xi}(I) \cap SO(m)$ for a sufficiently small $\xi > 0$ and the corresponding neighbourhood $\tilde{\mathcal{O}}(O) \subset so(m)$.

The fact that the maps in (6.15) are continuous in the Lipschitz sense follows from the inequality

$$\| \exp (X + Y) - \exp (X) \| \leq \| Y \| \cdot \exp (\| X \|) \cdot \exp (\| Y \|), \quad X, Y \in so(m),$$

and, e.g., from the Richter theorem (see, e.g., [92, Theorem 11.1])

$$\ln (X) = \int_0^1 (X - I) \left[t (X - I) + I \right]^{-1} \mathrm{d}t, \quad X \in \mathcal{O}_\xi^\varrho (I) \cap SO(m).$$

$\square$

Remark 6.4. Any non-singular matrix has infinitely many logarithms. Here symbol $\ln (A)$ denotes the principal logarithm, which is the unique logarithm whose spectrum lies in the strip $\{z \in \mathbb{C};\ \mathrm{Im}\, z \in [-\pi, \pi)\}$. $\triangle$

Lemma 6.5. *There exists $p(\vartheta) \in \mathbb{N}$ for all $\vartheta > 0$ with the property that, for any sequence*

$$\{P_0, P_1, \dots, P_n, \dots, P_{2n}\} \subset SO(m), \quad n \geq p(\vartheta),$$

one can find matrices $Q_2, Q_4, \dots, Q_{2n} \in SO(m)$ for which

$$Q_{2i} \in \mathcal{O}_\vartheta^\varrho (P_{2i}), \quad i \in \{1, \dots, n\}, \qquad P_1 \cdot Q_2 \cdot P_3 \cdot Q_4 \cdots P_{2n-1} \cdot Q_{2n} = P_0. \tag{6.16}$$

Proof. First we recall that the group $SO(m)$ is transformable (see Example 2.3). This fact implies the existence of $q(\delta) \in \mathbb{N}$ for all $\delta > 0$ such that, for any sequence

$$\{P_0, P_1, \dots, P_q, \dots, P_n\} \subset SO(m),$$

there exist $T_1, \dots, T_q, \dots, T_n \in SO(m)$ satisfying

$$T_i \in \mathcal{O}_\delta^\varrho (P_i), \quad i \in \{1, \dots, n\}, \qquad T_1 \cdot T_2 \cdots T_n = P_0.$$

We replace matrices $P_1, \dots, P_{n-1}, P_n$ by $P_1 \cdot P_2, \dots, P_{2n-3} \cdot P_{2n-2}, P_{2n-1} \cdot P_{2n}$ and, using the transformability of $SO(m)$, we obtain matrices T_i, $i \in \{1, \dots, n\}$. We put

$$R_1 := (P_1 \cdot P_2)^{-1} \cdot T_1, \dots, R_n := (P_{2n-1} \cdot P_{2n})^{-1} \cdot T_n.$$

Since the multiplication of matrices is continuous in the Lipschitz sense on $SO(m)$ as the map $O \mapsto O^T$, there exists $L > 0$ such that

$$R_i \in \mathcal{O}_{\delta L}^\varrho (I), \quad i \in \{1, \dots, n\},$$

and, consequently, there exists $K > 0$ for which

$$P_2 \cdot R_1 \in \mathcal{O}_{\delta K}^\varrho (P_2), \dots, P_{2n} \cdot R_n \in \mathcal{O}_{\delta K}^\varrho (P_{2n}).$$

We see

$$T_1 = P_1 \cdot P_2 \cdot R_1, \dots, T_n = P_{2n-1} \cdot P_{2n} \cdot R_n,$$

i.e., we have (6.16) for $Q_2 := P_2 \cdot R_1, \dots, Q_{2n} := P_{2n} \cdot R_n$ and $p(\vartheta) := q(\vartheta / K)$. $\square$

We also need a simple method for constructing almost periodic functions with prescribed values. This method is a modification of the theorems presented in Section 5.3.

Lemma 6.6. *If the sequence of non-negative numbers $a(i)$ for $i \in \mathbb{N}$ has the property that*

$$\sum_{i=1}^{\infty} a(i) < \infty,$$

then any continuous function $\psi : \mathbb{R} \to so(m)$ for which

$$\psi(t) = \psi\left(t-1\right), \quad t \in (1,2],$$

$$\begin{gathered}
\psi(t) = \psi\left(t+2\right), \quad t \in (-2,0], \\
\psi(t) \in \mathcal{O}^{\varrho}_{a(1)}\left(\psi\left(t-4\right)\right), \quad t \in (2,6], \\
\psi(t) = \psi\left(t+8\right), \quad t \in (-10,-2]\,, \\
\psi(t) \in \mathcal{O}^{\varrho}_{a(2)}\left(\psi\left(t-2^4\right)\right), \quad t \in (2+2^2, 2+2^2+2^4], \\
\psi(t) = \psi\left(t+2^5\right), \quad t \in (-2^5-2^3-2, -2^3-2], \\
\vdots \\
\psi(t) \in \mathcal{O}^{\varrho}_{a(n)}\left(\psi\left(t-2^{2n}\right)\right), \quad t \in (2+2^2+\cdots+2^{2n-2}, 2+2^2+\cdots+2^{2n-2}+2^{2n}], \\
\psi(t) = \psi\left(t+2^{2n+1}\right), \quad t \in (-2^{2n+1}-\cdots-2^3-2, -2^{2n-1}-\cdots-2^3-2], \\
\vdots
\end{gathered}$$

is almost periodic.

Proof. Let $\varepsilon > 0$ be arbitrarily given and let $k = k(\varepsilon) \in \mathbb{N}$ satisfy

$$\sum_{i=k}^{\infty} a(i) < \frac{\varepsilon}{2}. \tag{6.17}$$

From

$$\begin{gathered}
\psi(t) \in \mathcal{O}^{\varrho}_{a(k)}\left(\psi\left(t-2^{2k}\right)\right), \quad t \in (2+2^2+\cdots+2^{k-2}, 2+2^2+\cdots+2^{k}], \\
\psi(t) = \psi\left(t+2^{2k+1}\right), \quad t \in (-2^{2k+1}-\cdots-2^3-2, -2^{2k-1}-\cdots-2^3-2], \\
\psi(t) \in \mathcal{O}^{\varrho}_{a(k+1)}\left(\psi\left(t-2^{2k+2}\right)\right), \quad t \in (2+2^2+\cdots+2^{2k}, 2+2^2+\cdots+2^{2k+2}], \\
\vdots
\end{gathered}$$

it follows

$$\psi\left(t+2^{2k}\right) \in \mathcal{O}^{\varrho}_{a(k)}(\psi(t)), \quad t \in (-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$

$$\psi\left(t-2^{2k}\right) \in \mathcal{O}^{\varrho}_{a(k)}(\psi(t)), \quad t \in (-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$

$$\psi\left(t-2^{2k+1}\right) \in \mathcal{O}^{\varrho}_{a(k)}(\psi(t)), \quad t \in (-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$
$$\psi\left(t+2^{2k+1}\right) \in \mathcal{O}^{\varrho}_{a(k)+a(k+1)}(\psi(t)), \quad t \in (-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$
$$\psi\left(t+3\cdot 2^{2k}\right) \in \mathcal{O}^{\varrho}_{a(k)+a(k+1)}(\psi(t)), \quad t \in (-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$
$$\psi\left(t+2^{2k+2}\right) \in \mathcal{O}^{\varrho}_{a(k+1)}(\psi(t)), \quad t \in (-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}],$$
$$\psi\left(t+2^{2k}+2^{2k+2}\right) \in \mathcal{O}^{\varrho}_{a(k)+a(k+1)}(\psi(t)), \quad t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],$$
$$\vdots$$

Thus (see (6.17)), it holds

$$\psi\left(t+l\cdot 2^{2k}\right) \in \mathcal{O}^{\varrho}_{\varepsilon/2}(\psi(t)), \quad t \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right],\ l \in \mathbb{Z}.$$

If we express any $t \in \mathbb{R}$ as $t = t_1 + t_2$, where

$$t_1 \in \left(-2^{2k-1}-\cdots-2^3-2, 2+2^2+\cdots+2^{2k-2}\right], \quad t_2 = j\cdot 2^{2k} \text{ for } j \in \mathbb{Z},$$

then we have

$$\begin{aligned}\varrho\left(\psi(t), \psi\left(t+l\cdot 2^{2k}\right)\right) &\leq \varrho\left(\psi\left(t_1+t_2\right), \psi\left(t_1\right)\right)\\ &+ \varrho\left(\psi\left(t_1\right), \psi\left(t_1+(j+l)\,2^{2k}\right)\right) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon, \quad t \in \mathbb{R},\ l \in \mathbb{Z}.\end{aligned}$$

This inequality implies that we can choose $l(\varepsilon) := 2^{2k(\varepsilon)}+1$ for any $\varepsilon > 0$ (see Definition 5.1); i.e., the resulting function ψ is almost periodic. □

Now we can prove the result that the systems having no non-zero almost periodic solution form an everywhere dense subset of $\mathcal{S}$.

Theorem 6.7. *Let $A \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrary. There exists $B \in \mathcal{O}^{\sigma}_{\varepsilon}(A)$ which does not have an almost periodic solution other than the trivial one.*

Proof. Using Lemma 5.3, the almost periodicity of A implies that there exist $\delta \in (0, 1/3)$ and an almost periodic matrix valued function $\tilde{A} : \mathbb{R} \to so(m)$ satisfying $\tilde{A} \in \mathcal{O}^{\sigma}_{\varepsilon/2}(A)$ and $\tilde{A}|_{[k,k+\delta]} \equiv const.$ for any $k \in \mathbb{Z}$. Indeed, it suffices to define $\tilde{A}$ by

$$\tilde{A}(t) := A\left(k+\frac{\delta}{2}\right), \quad t \in [k, k+\delta],\ k \in \mathbb{Z},$$

$$\tilde{A}(t) := A(k-\delta) + \frac{t-(k-\delta)}{\delta}\left[A\left(k+\frac{\delta}{2}\right) - A(k-\delta)\right], \quad t \in [k-\delta, k),\ k \in \mathbb{Z},$$

$$\tilde{A}(t) := A\left(k+\frac{\delta}{2}\right) + \frac{t-(k+\delta)}{\delta}\left[A(k+2\delta) - A\left(k+\frac{\delta}{2}\right)\right], \quad t \in (k+\delta, k+2\delta],\ k \in \mathbb{Z},$$

$$\tilde{A}(t) := A(t), \quad t \notin \bigcup_{k\in\mathbb{Z}} [k-\delta, k+2\delta],$$

where $\delta > 0$ is sufficiently small. Thus, we assume without loss of generality that $A \in \mathcal{S}$ is constant on all interval $[k, k+\delta]$, $k \in \mathbb{Z}$.

Every almost periodic function is bounded. Hence, there exists $\eta \in (0,1)$ with the property that

$$\| X_S(t+s) - X_S(t) \| < \xi \tag{6.18}$$

for any $t \in \mathbb{R}$, $s \in [0,\eta]$, and $S \in \mathcal{O}^{\sigma}_{\varepsilon}(A)$, where $\xi > 0$ is taken from Lemma 6.3. We can also assume that $\delta < \eta$. Further (see again Lemma 6.3), there exists $M \in \mathbb{N}$ satisfying

$$\| A - B \| < \vartheta \quad \text{if} \quad A, B \in \tilde{\mathcal{O}}(O), \exp(A) \in \mathcal{O}^{\varrho}_{\vartheta M}(\exp(B)) \subseteq \mathcal{O}^{\varrho}_{\xi}(I) \cap SO(m). \tag{6.19}$$

We choose an increasing sequence of numbers $n(i) \in \mathbb{N} \setminus \{1\}$ for $i \in \mathbb{N}$ arbitrarily so that

$$2^{n(i)-1} \geq p\left(\frac{\varepsilon}{2^i M} \cdot \frac{\delta}{2}\right), \quad i \in \mathbb{N}, \tag{6.20}$$

where $p(\vartheta)$ is taken from Lemma 6.5.

Since the sum of skew-symmetric matrices is a skew-symmetric matrix and since the sum of two almost periodic functions is almost periodic as well (see Corollary 5.9), we have $A_1 + A_2 \in \mathcal{S}$ for any $A_1, A_2 \in \mathcal{S}$. Thus, it suffices to find $C \in \mathcal{S} \cap \mathcal{O}^{\sigma}_{\varepsilon}(O)$ for which the system $A + C$ does not have any non-zero almost periodic solution. We construct such a system C (as continuous function) applying Lemma 6.6 for

$$a(n(i)) := \frac{\varepsilon}{2^i}, \quad i \in \mathbb{N}; \qquad a(j) := 0, \quad j \notin \{n(i);\, i \in \mathbb{N}\}.$$

Let us denote

$$a_i := 2 + 2^2 + \cdots + 2^{2n(i)-2}, \quad b_i := 2 + 2^2 + \cdots + 2^{2n(i)-2} + 2^{2n(i)},$$

$$d_i^1 := \left(\frac{1}{4} - \frac{1}{2^{2n(i)}}\right)\delta, \quad d_i^2 := \left(\frac{3}{4} + \frac{1}{2^{2n(i)}}\right)\delta, \quad i \in \mathbb{N}.$$

In the first step of the construction, we put

$$C(t) := O, \quad t \in \left(-2^{2n(1)-1} - \cdots - 2^3 - 2, 2 + 2^2 + \cdots + 2^{2n(1)-2}\right],$$

$$C(t) := O, \quad t \in (a_1, b_1] \setminus \bigcup_{j \in \mathbb{N}} \left(j + d_1^1, j + d_1^2\right],$$

$$C(t) := C_1^{j-a_1+1}, \quad t \in \left(j + d_2^1, j + d_2^2\right] \subset (a_1, b_1],$$

for arbitrary matrices

$$C_1^{j-a_1+1} \in \mathcal{O}^{\varrho}_{\varepsilon/2}(O) \cap so(m), \quad j \in \{a_1, \ldots, b_1 - 1\},$$

and we define C so that it is linear on the intervals

$$\left(j + d_1^1, j + d_2^1\right], \quad \left(j + d_2^2, j + d_1^2\right], \quad j \in \{a_1, \ldots, b_1 - 1\}.$$

In the second step, we put

$$C(t) := C\left(t + 2^{2n(1)+1}\right), \quad t \in \left(-2^{2n(1)+1} - \cdots - 2^3 - 2, -2^{2n(1)-1} - \cdots - 2^3 - 2\right],$$

$$C(t) := C\left(t - 2^{2n(1)+2}\right), \quad t \in \left(2 + 2^2 + \dots + 2^{2n(1)}, 2 + 2^2 + \dots + 2^{2n(1)+2}\right],$$

$$\vdots$$

$$C(t) := C\left(t + 2^{2n(2)-1}\right), \quad t \in \left(-2^{2n(2)-1} - \dots - 2^3 - 2, -2^{2n(2)-3} - \dots - 2^3 - 2\right],$$

$$C(t) := C\left(t - 2^{2n(2)}\right), \quad t \in (a_2, b_2] \setminus \bigcup_{j \in \mathbb{N}} \left(j + d_2^1, j + d_2^2\right],$$

and we define C as linear on the intervals

$$\left(j + d_2^1, j + d_3^1\right], \quad \left(j + d_3^2, j + d_2^2\right], \quad j \in \{a_2, \dots, b_2 - 1\}.$$

At the same time, we define

$$C(t) := C_2^{j-a_2+1} \in so(m), \quad t \in \left(j + d_3^1, j + d_3^2\right], \ j \in \{a_2, \dots, b_2 - 1\},$$

arbitrarily so that

$$\left\| C(t) - C\left(t - 2^{2n(2)}\right) \right\| < \frac{\varepsilon}{4}, \quad t \in (a_2, b_2].$$

We proceed further in the same way. In the i-th step, we put

$$C(t) := C\left(t + 2^{2n(i-1)+1}\right), \quad t \in \left(-2^{2n(i-1)+1} - \dots - 2^3 - 2, -2^{2n(i-1)-1} - \dots - 2^3 - 2\right],$$

$$C(t) := C\left(t - 2^{2n(i-1)+2}\right), \quad t \in \left(2 + 2^2 + \dots + 2^{2n(i-1)}, 2 + 2^2 + \dots + 2^{2n(i-1)+2}\right],$$

$$\vdots$$

$$C(t) := C\left(t + 2^{2n(i)-1}\right), \quad t \in \left(-2^{2n(i)-1} - \dots - 2^3 - 2, -2^{2n(i)-3} - \dots - 2^3 - 2\right],$$

$$C(t) := C\left(t - 2^{2n(i)}\right), \quad t \in (a_i, b_i] \setminus \bigcup_{j \in \mathbb{N}} \left(j + d_i^1, j + d_i^2\right], \tag{6.21}$$

and we define C as a linear function on the intervals

$$\left(j + d_i^1, j + d_{i+1}^1\right], \quad \left(j + d_{i+1}^2, j + d_i^2\right], \quad j \in \{a_i, \dots, b_i - 1\},$$

and

$$C(t) := C_i^{j-a_i+1} \in so(m), \quad t \in \left(j + d_{i+1}^1, j + d_{i+1}^2\right], \ j \in \{a_i, \dots, b_i - 1\},$$

arbitrarily so that

$$\left\| C(t) - C\left(t - 2^{2n(i)}\right) \right\| < \frac{\varepsilon}{2^i}, \quad t \in (a_i, b_i].$$

For

$$\zeta := \max\left\{ \left\| C_1^j \right\|; \ j \in \{1, \dots, 2^{2n(1)}\} \right\} < \frac{\varepsilon}{2},$$

we have

$$\| C(t) \| \le \zeta, \quad t \in \left(-2^{2n(1)-1} - \dots - 2^3 - 2, 2 + 2^2 + \dots + 2^{2n(1)}\right],$$

$$\| C(t) \| < \zeta + \frac{\varepsilon}{4}, \quad t \in \left(-2^{2n(2)-1} - \dots - 2^3 - 2, 2 + 2^2 + \dots + 2^{2n(2)}\right],$$

$$\vdots$$

$$\| C(t) \| < \zeta + \frac{\varepsilon}{4} + \dots + \frac{\varepsilon}{2^i}, \quad t \in \left(-2^{2n(i)-1} - \dots - 2^3 - 2, 2 + 2^2 + \dots + 2^{2n(i)}\right],$$

$$\vdots$$

i.e., there exists $\tilde{\varepsilon} \in (0, \varepsilon)$ with the property that $\| C(t) \| < \tilde{\varepsilon}$, $t \in \mathbb{R}$. Thus, we obtain an almost periodic (continuous) function $C \in \mathcal{S} \cap \mathcal{O}^\sigma_\varepsilon(O)$.

We denote

$$I_i := [a_i, b_i] = \left[2 + 2^2 + \dots + 2^{2n(i)-2}, 2 + 2^2 + \dots + 2^{2n(i)-2} + 2^{2n(i)}\right].$$

In the construction, we can choose constant values $C_i^1, \dots, C_i^{2^{2n(i)}}$ on $2^{2n(i)}$ subintervals of I_i, where the length of each one of these intervals is

$$d^2_{i+1} - d^1_{i+1} \in \left(\frac{\delta}{2}, \delta\right). \tag{6.22}$$

Each value C_i^j can be chosen arbitrarily from the $(\varepsilon/2^i)$-neighbourhood of a skew-symmetric matrix, which is given by the previous steps of the construction. Further (see (6.21)), the function C is determined on intervals

$$\left(a_i, a_i + d_i^1\right], \quad \left(a_i + d_i^2, a_i + 1 + d_i^1\right], \quad \dots \quad \left(b_i - 2 + d_i^2, b_i - 1 + d_i^1\right], \quad \left(b_i - 1 + d_i^2, b_i\right]$$

by $C(t) = C\left(t - 2^{2n(i)}\right)$.

We repeat that C is linear on the remaining subintervals of I_i. These intervals are denoted by $J_i^1, \dots, J_i^{2^{2n(i)+1}}$, where

$$\begin{aligned} J_i^{2j-1} &:= \left(a_i + j - 1 + d_i^1, a_i + j - 1 + d^1_{i+1}\right], \quad j \in \{1, \dots, 2^{2n(i)}\}, \\ J_i^{2j} &:= \left(a_i + j - 1 + d^2_{i+1}, a_i + j - 1 + d_i^2\right], \quad j \in \{1, \dots, 2^{2n(i)}\}. \end{aligned} \tag{6.23}$$

Especially, we see that the length of each J_i^j is less than $\delta/2^{2n(i)}$ and that

$$J_i^1, \dots, J_i^{2j} \subset (a_i, a_i + j), \quad J_i^{2j+1}, \dots, J_i^{2^{2n(i)+1}} \subset (a_i + j, b_i), \quad j \in \{1, \dots, 2^{2n(i)} - 1\},$$

i.e., the total length l_i^k of all subintervals $J_i^j \subset [a_i, a_i + k]$ is

$$l_i^k < \frac{2k\delta}{2^{2n(i)}}, \quad k \in \{1, \dots, 2^{2n(i)}\}. \tag{6.24}$$

Let us consider $S = A + C \in \mathcal{O}^\sigma_\varepsilon(A)$. To describe the principal fundamental matrix X_S, we define

$$\tilde{X}^i_S(t) := X_S(t), \quad t \in \left[a_i, a_i + d_i^1\right],$$

$$\tilde{X}^i_S(t) := \tilde{X}^i_S\left(a_i + d_i^1\right), \quad t \in \left(a_i + d_i^1, a_i + d^1_{i+1}\right],$$

$$\tilde{X}^i_S(t) := \exp\left(\left(A + C_i^1\right)\left(t - a_i - d^1_{i+1}\right)\right) \cdot \tilde{X}^i_S\left(a_i + d^1_{i+1}\right), \quad t \in \left(a_i + d^1_{i+1}, a_i + d^2_{i+1}\right],$$

$$\tilde{X}^i_S(t) := \tilde{X}^i_S\left(a_i + d^2_{i+1}\right), \quad t \in \left(a_i + d^2_{i+1}, a_i + d_i^2\right],$$

$$\tilde{X}_S^i(t) := X_S\,(t)\cdot\left(X_S\left(a_i+d_i^2\right)\right)^{-1}\cdot\tilde{X}_S^i\left(a_i+d_i^2\right),\quad t\in\left(a_i+d_i^2,a_i+1+d_i^1\right],$$

$$\vdots$$

$$\tilde{X}_S^i(t) := X_S\,(t)\cdot\left(X_S\left(b_i-2+d_i^2\right)\right)^{-1}\cdot\tilde{X}_S^i\left(b_i-2+d_i^2\right),\qquad t\in\left(b_i-2+d_i^2,b_i-1+d_i^1\right],$$

$$\tilde{X}_S^i(t) := \tilde{X}_S^i\left(b_i-1+d_i^1\right),\quad t\in\left(b_i-1+d_i^1,b_i-1+d_{i+1}^1\right],$$

$$\tilde{X}_S^i(t) := \exp\left(\left(A+C_i^{2^{2n(i)}}\right)\left(t-b_i+1-d_{i+1}^1\right)\right)\cdot\tilde{X}_S^i\left(b_i-1+d_{i+1}^1\right),\qquad t\in\left(b_i-1+d_{i+1}^1,b_i-1+d_{i+1}^2\right],$$

$$\tilde{X}_S^i(t) := \tilde{X}_S^i\left(b_i-1+d_{i+1}^2\right),\quad t\in\left(b_i-1+d_{i+1}^2,b_i-1+d_i^2\right],$$

$$\tilde{X}_S^i(t) := X_S\,(t)\cdot\left(X_S\left(b_i-1+d_i^2\right)\right)^{-1}\cdot\ \tilde{X}_S^i\left(b_i-1+d_i^2\right),\quad t\in\left(b_i-1+d_i^2,b_i\right].$$

Since

$$X_S\,(t_2)-X_S\,(t_1)=\int\limits_{t_1}^{t_2}S(s)\,X_S\,(s)\;\mathrm{d}s,\quad t_1,t_2\in\mathbb{R},$$

it is valid that (see also (6.23))

$$\begin{aligned}\left\|\,X_S\,(t)-\tilde{X}_S^i\,(t)\,\right\| &\le \sum_{j=1}^{k}\int\limits_{a_i+j-1+d_i^1}^{a_i+j-1+d_{i+1}^1}\|\,S(s)\,X_S\,(s)\,\|\;\mathrm{d}s\\ &\quad+\sum_{j=1}^{k}\int\limits_{a_i+j-1+d_{i+1}^2}^{a_i+j-1+d_i^2}\|\,S(s)\,X_S\,(s)\,\|\;\mathrm{d}s\end{aligned}\tag{6.25}$$

if $t\le a_i+k$, $k\in\{1,\dots,2^{2n(i)}\}$. Considering $S\in\mathcal{O}_\varepsilon^\sigma(A)$ and $X_S\,(t)\,,\tilde{X}_S^i\,(t)\in SO(m)$, $t\in\mathbb{R}$, from (6.24) and (6.25) it follows that there exists $N\in\mathbb{N}$ satisfying

$$\left\|\,X_S\,(t)-\tilde{X}_S^i\,(t)\,\right\|<\frac{Nk}{2^{2n(i)-1}}\tag{6.26}$$

for $t\in[a_i,a_i+k]$, $k\in\{1,\dots,2^{2n(i)}\}$.

Let $n_0\in\mathbb{N}$ be such that

$$N\cdot\frac{4n(i)\,2^{n(i)}}{2^{2n(i)-1}}<\frac{1}{3},\quad i\ge n_0\,(i\in\mathbb{N}).\tag{6.27}$$

We put $X_1 := -I$, $X_2 = -I$, when m is even, and

$$X_1 := \begin{pmatrix}1&0&0&\cdots&0\\0&-1&0&\cdots&0\\0&0&-1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&-1\end{pmatrix}\in SO(m),\quad X_2 := \begin{pmatrix}-1&\cdots&0&0&0\\\vdots&\ddots&\vdots&\vdots&\vdots\\0&\cdots&-1&0&0\\0&\cdots&0&-1&0\\0&\cdots&0&0&1\end{pmatrix}\in SO(m)$$

for odd m. If we express

$$\tilde{X}_S^i\left(a_i + d_{i+1}^2\right) = \exp\left(\left(A + C_i^1\right)\left(d_{i+1}^2 - d_{i+1}^1\right)\right) \cdot \tilde{X}_S^i\left(a_i + d_{i+1}^1\right),$$

$$\tilde{X}_S^i\left(a_i + 1 + d_{i+1}^1\right) = X_S\left(a_i + 1 + d_i^1\right) \cdot \left(X_S\left(a_i + d_i^2\right)\right)^{-1} \cdot \tilde{X}_S^i\left(a_i + d_{i+1}^2\right),$$

$$\vdots$$

$$\tilde{X}_S^i\left(b_i - 1 + d_{i+1}^1\right) = X_S\left(b_i - 1 + d_i^1\right) \cdot \left(X_S\left(b_i - 2 + d_i^2\right)\right)^{-1} \cdot \tilde{X}_S^i\left(b_i - 2 + d_{i+1}^2\right),$$

$$\tilde{X}_S^i\left(b_i - 1 + d_{i+1}^2\right) = \exp\left(\left(A + C_i^{2^{2n(i)}}\right)\left(d_{i+1}^2 - d_{i+1}^1\right)\right) \cdot \tilde{X}_S^i\left(b_i - 1 + d_{i+1}^1\right),$$

$$\tilde{X}_S^i\left(b_i\right) = X_S\left(b_i\right) \cdot \left(X_S\left(b_i - 1 + d_i^2\right)\right)^{-1} \cdot \tilde{X}_S^i\left(b_i - 1 + d_{i+1}^2\right),$$

then it is seen that we can use Lemma 6.5 to choose values C_i^j on the subintervals

$$\left(a_i + j - 1 + d_{i+1}^1, a_i + j - 1 + d_{i+1}^2\right], \quad j \in \{1, \dots, 2^{2n(i)}\},$$

so that we obtain

$$\tilde{X}_S^i\left(a_i + 2^{n(i)}\right) = I, \quad \tilde{X}_S^i\left(a_i + 2^{n(i)} + \left(2^{n(i)} - 1\right)\right) = X_1,$$

$$\tilde{X}_S^i\left(a_i + 3 \cdot 2^{n(i)}\right) = I, \quad \tilde{X}_S^i\left(a_i + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 1\right)\right) = X_2,$$

$$\tilde{X}_S^i\left(a_i + 4 \cdot 2^{n(i)} + 2^{n(i)}\right) = I, \quad \tilde{X}_S^i\left(a_i + 4 \cdot 2^{n(i)} + 2^{n(i)} + \left(2^{n(i)} - 2^1\right)\right) = X_1,$$

$$\tilde{X}_S^i\left(a_i + 4 \cdot 2^{n(i)} + 3 \cdot 2^{n(i)}\right) = I, \quad \tilde{X}_S^i\left(a_i + 4 \cdot 2^{n(i)} + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 2^1\right)\right) = X_2,$$

$$\vdots$$

$$\tilde{X}_S^i\left(a_i + 4\left(n(i) - 1\right)2^{n(i)} + 2^{n(i)}\right) = I,$$

$$\tilde{X}_S^i\left(a_i + 4\left(n(i) - 1\right)2^{n(i)} + 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right)\right) = X_1,$$

$$\tilde{X}_S^i\left(a_i + 4\left(n(i) - 1\right)2^{n(i)} + 3 \cdot 2^{n(i)}\right) = I,$$

$$\tilde{X}_S^i\left(a_i + 4\left(n(i) - 1\right)2^{n(i)} + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right)\right) = X_2.$$

Indeed, it suffices to consider the form of matrices

$$\exp\left(\left(A + C_i^j\right)\left(d_{i+1}^2 - d_{i+1}^1\right)\right)$$

for which (see (6.18), (6.22))

$$\left\| \exp\left(\left(A + C_i^j\right)\left(d_{i+1}^2 - d_{i+1}^1\right)\right) - I \right\| < \xi,$$

inequality (6.20) with $M \in \mathbb{N}$ satisfying (6.19) and with

$$d_{i+1}^2 - d_{i+1}^1 > \frac{\delta}{2}, \quad 2^{n(i)} - 1 > 2^{n(i)} - 2^1 > \cdots > 2^{n(i)} - 2^{n(i)-1} = 2^{n(i)-1},$$

and the fact that we can choose all matrix C_i^j from the $(\varepsilon/2^i)$-neighbourhood of a given skew-symmetric matrix arbitrarily. Note that

$$a_i + 4\left(n(i) - 1\right)2^{n(i)} + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right) < a_i + 4n(i) \cdot 2^{n(i)} \tag{6.28}$$

and $a_i + 4n(i)2^{n(i)} < b_i$ for sufficiently large $i \in \mathbb{N}$, i.e., we can construct the resulting function C with the above mentioned properties on I_i for all $i \geq n_0$ (see also (6.27)).

Now we use (6.26) and (6.27) in connection with (6.28). For $k \in \{1, \dots, 4n(i)2^{n(i)}\}$, where $i \geq n_0$, we have

$$\left\| X_S(t) - \tilde{X}_S^i(t) \right\| < N \cdot \frac{4n(i)\, 2^{n(i)}}{2^{2n(i)-1}} < \frac{1}{3}, \quad t \in [a_i, a_i + k]. \tag{6.29}$$

Especially, for all $i \geq n_0$ ($i \in \mathbb{N}$), we obtain

$$\left\| X_S\left(s_j^i\right) - \tilde{X}_S^i\left(s_j^i\right) \right\| < \frac{1}{3}, \quad j \in \{1, \dots, 4n(i)\}, \tag{6.30}$$

where

$$\begin{gathered}
s_1^i := a_i + 2^{n(i)}, \quad s_2^i := a_i + 2^{n(i)} + \left(2^{n(i)} - 1\right), \\
s_3^i := a_i + 3 \cdot 2^{n(i)}, \quad s_4^i := a_i + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 1\right), \\
\vdots \\
s_{4n(i)-3}^i := a_i + 4\left(n(i) - 1\right) 2^{n(i)} + 2^{n(i)}, \\
s_{4n(i)-2}^i := a_i + 4\left(n(i) - 1\right) 2^{n(i)} + 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right), \\
s_{4n(i)-1}^i := a_i + 4\left(n(i) - 1\right) 2^{n(i)} + 3 \cdot 2^{n(i)}, \\
s_{4n(i)}^i := a_i + 4\left(n(i) - 1\right) 2^{n(i)} + 3 \cdot 2^{n(i)} + \left(2^{n(i)} - 2^{n(i)-1}\right).
\end{gathered}$$

We recall that we need to prove that any non-trivial solution of S is not almost periodic. By contradiction, suppose that the solution

$$x(t) = X_S(t) \cdot u \tag{6.31}$$

of the Cauchy problem

$$x'(t) = S(t) \cdot x(t), \quad x(0) = u,$$

where $u \in \mathbb{R}^m$, $\|u\|_2 = 1$, is almost periodic. Applying Theorem 5.5 for $\varepsilon = 1/3$ and $s_i = 2^{n(i)}$, $i \in \mathbb{N}$, we obtain

$$\left\| x\left(t + 2^{n(i(1))}\right) - x\left(t + 2^{n(i(2))}\right) \right\|_2 < \frac{1}{3}, \quad t \in \mathbb{R}, \tag{6.32}$$

for all $i(1), i(2)$ from an infinite set $N_0 \subseteq \mathbb{N}$.

It is immediately seen that

$$\max \{\| X_1 \cdot u - u \|_2, \| X_2 \cdot u - u \|_2\} \geq 1. \tag{6.33}$$

Thus, from the construction, (6.30), (6.33), and from

$$\begin{aligned}
\left\| \tilde{X}_S^i(t) \cdot u - \tilde{X}_S^i(s) \cdot u \right\|_2 &\leq \left\| \tilde{X}_S^i(t) \cdot u - X_S(t) \cdot u \right\|_2 \\
&\quad + \| X_S(t) \cdot u - X_S(s) \cdot u \|_2 + \left\| X_S(s) \cdot u - \tilde{X}_S^i(s) \cdot u \right\|_2
\end{aligned}$$

for

$$t = s^i_4, \quad s = s^i_3; \qquad t = s^i_2, \quad s = s^i_1;$$
$$\vdots$$
$$t = s^i_{4n(i)}, \quad s = s^i_{4n(i)-1}; \qquad t = s^i_{4n(i)-2}, \quad s = s^i_{4n(i)-3},$$

respectively, it follows

$$1 < \frac{1}{3} + \left\| X_S\left(s^i_{4j}\right) \cdot u - X_S\left(s^i_{4j-1}\right) \cdot u \right\|_2 + \frac{1}{3}$$

or

$$1 < \frac{1}{3} + \left\| X_S\left(s^i_{4j-2}\right) \cdot u - X_S\left(s^i_{4j-3}\right) \cdot u \right\|_2 + \frac{1}{3}$$

for $j \in \{1, \dots, n(i)\}$. Hence, we have

$$\max\left\{ \left\| X_S\left(s^i_{4j}\right) \cdot u - X_S\left(s^i_{4j-1}\right) \cdot u \right\|_2, \left\| X_S\left(s^i_{4j-2}\right) \cdot u - X_S\left(s^i_{4j-3}\right) \cdot u \right\|_2 \right\} > \frac{1}{3} \quad (6.34)$$

for all $j \in \{1, \dots, n(i)\}$ and $i \geq n_0$. Since

$$s^i_2 - s^i_1 = 2^{n(i)} - 1 = s^i_4 - s^i_3,$$
$$s^i_6 - s^i_5 = 2^{n(i)} - 2^1 = s^i_8 - s^i_7,$$
$$\vdots$$
$$s^i_{4n(i)-2} - s^i_{4n(i)-3} = 2^{n(i)} - 2^{n(i)-1} = s^i_{4n(i)} - s^i_{4n(i)-1},$$

inequality (6.34) implies (see (6.31))

$$\sup_{t \in \mathbb{R}} \left\| x(t) - x\left(t + 2^{n(i)} - 2^{j-1}\right) \right\|_2 > \frac{1}{3} \tag{6.35}$$

for all $i \geq n_0$ and $j \in \{1, \dots, n(i)\}$. Of course, we can rewrite (6.32) into

$$\sup_{t \in \mathbb{R}} \left\| x(t) - x\left(t + 2^{n(i(2))} - 2^{n(i(1))}\right) \right\|_2 \leq \frac{1}{3}, \quad i(1), i(2) \in N_0.$$

Considering (6.35), we see that (6.32) cannot be true for all $i(1), i(2)$ from an infinite set N_0. This contradiction proves the theorem. □

The presented process can be applied to prove the existence of systems from $\mathcal{S}$ with several properties. For example, we mention the following result.

Theorem 6.8. *Let $A \in \mathcal{S}$ and $\varepsilon > 0$ be arbitrarily given. There exists $B \in \mathcal{O}^\sigma_\varepsilon(A)$ with the property that*

$$\overline{\{X_B(t);\, t \in \mathbb{R}\}} = SO(m).$$

Proof. Let a sequence $\{X_k\}_{k\in\mathbb{N}} \subset SO(m)$ be dense in $SO(m)$. In the proof of Theorem 6.7, we can replace considered matrices X_1, X_2 by arbitrary matrices X_k, X_{k+1}. Thus, there is shown the existence of a system $S = A + C \in \mathcal{O}^{\sigma}_{\varepsilon}(A)$ with the property that (see (6.29))

$$\left\| X_S\left(s_j^i\right) - X_j \right\| < N \cdot \frac{4n(i)\,2^{n(i)}}{2^{2n(i)-1}}$$

for some $s_j^i \in \mathbb{R}$ and all $j \in \{1, \dots, 2n(i)\}$, $i \geq n_0$. Now it suffices to consider that

$$\lim_{i\to\infty} N \cdot \frac{4n(i)\,2^{n(i)}}{2^{2n(i)-1}} = 0.$$

□

At the end, we remark that the question of generalizations of Theorems 6.1 and 6.7 concerning other homogeneous linear differential systems, which can have only almost periodic solutions, remains open (contrary to the corresponding discrete case, see Chapters 2 and 4).

Chapter 7

Values of almost periodic and limit periodic functions

In this chapter, we prove two theorems about almost periodic and limit periodic functions having given values. These theorems correspond to Theorem 3.1, where it is shown that, for any countable and totally bounded set, there exists a limit periodic sequence whose range is this set. In the statements of the presented results, we need that the totally bounded set is the range of a uniformly continuous function φ for which the set $\{\varphi(k);\ k \in \mathbb{Z}\}$ is finite (in the almost periodic case) or the uniformly continuous function takes a value periodically (in the limit periodic case). We also construct limit periodic functions whose ranges contain arbitrarily given totally bounded sequences if one requires the connection by arcs of the space of values.

7.1 Preliminaries

We put $\mathbb{R}_0^+ := [0, \infty)$. Let $\mathcal{X} \neq \emptyset$ be an arbitrary set and let $\varrho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ be a pseudometric on $\mathcal{X}$. For given $\varepsilon > 0$ and $x \in \mathcal{X}$, the ε-neighbourhood of x is denoted by $\mathcal{O}_\varepsilon(x)$.

7.2 Functions with given values

At first, we construct an almost periodic function with given values. Concerning a continuous counterpart of Theorem 3.1 (or directly Definition 5.1 and Lemma 5.4), the given set of values has to be the totally bounded range of a continuous function. In addition, any almost periodic function is uniformly continuous (see Lemma 5.3). Considering these facts, we formulate the following theorem.

Theorem 7.1. *Let $\varphi : \mathbb{R} \to \mathcal{X}$ be a uniformly continuous function such that the set $\{\varphi(k);\ k \in \mathbb{Z}\}$ is finite and the set $\{\varphi(t);\ t \in \mathbb{R}\}$ is totally bounded. Then, there exists an* *almost periodic function* ψ *with the property that*

$$\{\psi(k);\, k \in \mathbb{Z}\} = \{\varphi(k);\, k \in \mathbb{Z}\}, \quad \{\psi(t);\, t \in \mathbb{R}\} = \{\varphi(t);\, t \in \mathbb{R}\} \tag{7.1}$$

and that, for any $l \in \mathbb{Z}$, *there exists* $q(l) \in \mathbb{N}$ *for which*

$$\psi(l+s) = \psi(l+s+jq(l)), \quad j \in \mathbb{Z},\, s \in [0,1). \tag{7.2}$$

Proof. We construct $\psi : \mathbb{R} \to \mathcal{X}$ applying Corollary 5.23 similarly as $\{\psi_k\}$ applying Corollary 1.28 in the proof of Theorem 3.1. Considering that the set $\{\varphi(k);\, k \in \mathbb{Z}\}$ is finite, let sufficiently large $M, N \in \mathbb{Z}$ have the property that $\varphi(M) = \varphi(N)$ and that, for any $l \in \mathbb{Z}$, there exists $j(l) \in \{N, N+1, \dots, M-1\}$ for which

$$\varphi(l) = \varphi(j(l)), \quad \varphi(l+1) = \varphi(j(l)+1). \tag{7.3}$$

Without loss of generality, we can assume that $N = 0$ because, if $N < 0$, then we can redefine finitely many the below given ε_i and put $\psi \equiv \varphi$ on a sufficiently large interval.

Since φ is uniformly continuous with totally bounded range (see also (7.3)), for arbitrarily small $\varepsilon > 0$, there exist $l_1(\varepsilon), \dots, l_{m(\varepsilon)}(\varepsilon) \in \mathbb{Z}$ such that, for any $l \in \mathbb{Z}$, we have

$$\varrho\left(\varphi(l+s), \varphi(l_i+s)\right) < \varepsilon, \quad s \in [0,1],$$

for at least one integer $l_i \in \{l_1(\varepsilon), \dots, l_{m(\varepsilon)}(\varepsilon)\}$. We put $\varepsilon_i := 2^{-i}$, $i \in \mathbb{N}$, i.e.,

$$l_1^i := l_1(2^{-i}), \dots, l_{m(i)}^i := l_{m(2^{-i})}\big(2^{-i}\big), \quad i \in \mathbb{N}.$$

In addition, we assume that

$$\{l_j^i;\, j \in \{1, \dots, m(i)\}, i \in \mathbb{N}\} = \mathbb{Z}. \tag{7.4}$$

First we define

$$\psi(t) := \varphi(t), \quad t \in [0, M]. \tag{7.5}$$

We choose arbitrary $n(1) \in \mathbb{N}$ for which $2^{2n(1)}M > m(1)$. There exist (see (7.3))

$$j_1^1, j_2^1, \dots, j_{m(1)}^1 \in \{0, 1, \dots, M-1\}$$

such that

$$\begin{gathered}
\varphi(l_1^1) = \psi(j_1^1), \quad \varphi(l_1^1+1) = \psi(j_1^1+1), \\
\varphi(l_2^1) = \psi(j_2^1), \quad \varphi(l_2^1+1) = \psi(j_2^1+1), \\
\vdots \\
\varphi(l_{m(1)}^1) = \psi(j_{m(1)}^1), \quad \varphi(l_{m(1)}^1+1) = \psi(j_{m(1)}^1+1).
\end{gathered}$$

We define

$$\begin{gathered}
\psi(s+M+j_1^1) := \varphi(s+l_1^1), \quad s \in [0,1], \\
\psi(t) := \psi(t-M), \quad t \in (M, 2M] \setminus [M+j_1^1, M+j_1^1+1], \\
\psi(s+2M+j_2^1) := \varphi(s+l_2^1), \quad s \in [0,1],
\end{gathered}$$

$$\psi(t) := \psi(t-2M), \quad t \in (2M, 3M] \setminus [2M + j_2^1, 2M + j_2^1 + 1],$$

$$\vdots$$

$$\psi(s + m(1)M + j_{m(1)}^1) := \varphi(s + l_{m(1)}^1), \quad s \in [0,1],$$

$$\psi(t) := \psi(t - m(1)M), \quad t \in (m(1)M, (m(1)+1)M] \setminus [m(1)M + j_{m(1)}^1, m(1)M + j_{m(1)}^1 + 1],$$

and we define ψ as periodic with period M on

$$[-(2^{2n(1)-1} + \cdots + 2^3 + 2)M, (2 + 2^2 + \cdots + 2^{2n(1)})M] \setminus (M, (m(1)+1)M).$$

It is easily to see that we construct ψ as in Corollary 5.23 for

$$\varepsilon_i := L, \quad i \in \{1, \ldots, 2n(1)+1\}, \tag{7.6}$$

if $L > 0$ is sufficiently large.

In the second step, we choose $n(2) > n(1) + m(2)$ ($n(2) \in \mathbb{N}$) and we put

$$\psi(t) := \psi(t + 2^{2n(1)+1}M), \quad t \in [-(2^{2n(1)+1} + \cdots + 2)M, \cdots, -(2^{2n(1)-1} + \cdots + 2)M),$$

$$\psi(t) := \psi(t - 2^{2n(1)+2}M), \quad t \in ((2 + \cdots + 2^{2n(1)})M, \ldots, (2 + \cdots + 2^{2n(1)+2})M],$$

$$\vdots$$

$$\psi(t) := \psi(t + 2^{2n(2)-1}M), \quad t \in [-(2^{2n(2)-1} + \cdots + 2)M, \cdots, -(2^{2n(2)-3} + \cdots + 2)M),$$

and

$$\varepsilon_i := 0, \quad i \in \{2n(1)+2, \ldots, 2n(2)\}, \qquad \varepsilon_{2n(2)+1} := 2^{-1}. \tag{7.7}$$

From $n(2) > n(1) + m(2)$ and the above construction, we see that, for each integer j, $1 \le j \le m(1)$, there exist at least $2m(2) + 2$ intervals of the form

$$[a, a+1] \subset [-(2^{2n(2)-1} + \cdots + 2)M, \ldots, (2^{2n(2)-2} + \cdots + 2)M]$$

such that $a \in \mathbb{Z}$ and

$$\psi|_{[a,a+1]} \equiv \varphi|_{[l_j^1, l_j^1+1]}, \qquad \text{i.e.,} \qquad \psi(s+a) = \varphi(s + l_j^1), \quad s \in [0,1].$$

Hence, we can define continuous

$$\psi(t) \in \mathcal{O}_{\varepsilon_{2n(2)+1}}\left(\psi(t - 2^{2n(2)}M)\right), \quad t \in ((2 + \cdots + 2^{2n(2)-2})M, \ldots, (2 + \cdots + 2^{2n(2)})M],$$

for which

$$\psi|_{[2^{2n(2)}M, 2^{2n(2)}M+1]} \equiv \psi|_{[0,1]},$$

$$\psi|_{[k,k+1]} \equiv \psi|_{[0,1]} \quad \text{for some } k,$$
$$k \in \{(2 + \cdots + 2^{2n(2)-2})M, \ldots, (2 + \cdots + 2^{2n(2)})M - 1\} \setminus \{2^{2n(2)}M\},$$

and

$$\psi|_{[l,l+1]} \equiv \varphi|_{[j(l),j(l)+1]}, \quad l \in \{(2 + \cdots + 2^{2n(2)-2})M, \ldots, (2 + \cdots + 2^{2n(2)})M - 1\},$$
$$\text{some } j(l) \in \{0, \ldots, M-1, l_1^1, \ldots, l_{m(1)}^1, l_1^2, \ldots, l_{m(2)}^2\},$$

$$\{\varphi(t);\ t \in [l_1^1, l_1^1+1] \cup \dots \cup [l_{m(1)}^1, l_{m(1)}^1+1] \cup [l_1^2, l_1^2+1] \cup \dots \cup [l_{m(2)}^2, l_{m(2)}^2+1]\} \\ \subseteq \{\psi(t);\ t \in [(2+\dots+2^{2n(2)-2})M, \dots, (2+\dots+2^{2n(2)})M]\}.$$

In the third step, we choose $n(3) > n(2) + m(3)$ ($n(3) \in \mathbb{N}$) and we construct ψ for

$$\varepsilon_i := 0, \quad i \in \{2n(2)+2, \dots, 2n(3)\}, \qquad \varepsilon_{2n(3)+1} := 2^{-2}. \tag{7.8}$$

We have continuous

$$\psi(t) \in \mathcal{O}_{\varepsilon_{2n(3)+1}}\left(\psi(t - 2^{2n(3)}M)\right), \quad t \in ((2+\dots+2^{2n(3)-2})M, \dots, (2+\dots+2^{2n(3)})M],$$

satisfying

$$\psi|_{[l,l+1]} \equiv \varphi|_{[j(l),j(l)+1]}, \quad l \in \{(2+\dots+2^{2n(3)-2})M, \dots, (2+\dots+2^{2n(3)})M - 1\}, \\ \text{at least one } j(l) \in \{0, \dots, M-1, l_1^1, l_2^1, \dots, l_{m(3)}^3\},$$

$$\{\varphi(t); t \in [l_1^1, l_1^1+1] \cup [l_2^1, l_2^1+1] \cup \dots \cup [l_{m(3)}^3, l_{m(3)}^3+1]\} \\ \subseteq \{\psi(t);\ t \in [(2+\dots+2^{2n(3)-2})M, \dots, (2+\dots+2^{2n(3)})M]\}.$$

In addition, we have

$$\psi|_{[l,l+1]} \equiv \psi|_{[0,1]}, \quad l \in \{j\,2^{2n(2)}M;\ j \in \mathbb{Z}\} \\ \cap \{-(2^{2n(3)-1} + \dots + 2)M, \dots, (2+\dots+2^{2n(3)})M - 1\},$$

$$\psi|_{\left[2^{2n(3)}M+1, 2^{2n(3)}M+2\right]} \equiv \psi|_{[1,2]}, \quad \psi|_{\left[2^{2n(3)}M-1, 2^{2n(3)}M\right]} \equiv \psi|_{[-1,0]},$$

$$\psi|_{[k,k+1]} \equiv \psi|_{[1,2]} \quad \text{for some } k, \\ k \in \{(2+\dots+2^{2n(3)-2})M, \dots, (2+\dots+2^{2n(3)})M - 1\} \setminus \{2^{2n(3)}M + 1\},$$

$$\psi|_{[k,k+1]} \equiv \psi|_{[-1,0]} \quad \text{for some } k, \\ k \in \{(2+\dots+2^{2n(3)-2})M, \dots, (2+\dots+2^{2n(3)})M - 1\} \setminus \{2^{2n(3)}M - 1\}.$$

Continuing in the same manner, in the i-th step, we choose $n(i) > n(i-1) + m(i)$ ($n(i) \in \mathbb{N}$) and we construct ψ for

$$\varepsilon_k := 0, \quad k \in \{2n(i-1)+2, \dots, 2n(i)\}, \qquad \varepsilon_{2n(i)+1} := 2^{-i+1}. \tag{7.9}$$

For simplicity, let $i - 2 < 2^{2n(2)}M$ (see also the proof of Theorem 3.1 for $j\,2^{2n(2)}$ replaced by $[j\,2^{2n(2)}M, j\,2^{2n(2)}M+1]$, $1 + j\,2^{2n(3)}$ by $[1 + j\,2^{2n(3)}M, 1 + j\,2^{2n(3)}M + 1]$, and so on). Again, for each $j(1) \in \{1, \dots, i-1\}$, $j(2) \in \{1, \dots, m(j(1))\}$, there exist at least $2m(i)+2$ integers

$$l \in \{-(2^{2n(i)-1} + \dots + 2)M, \dots, (2+\dots+2^{2n(i)-2})M - 1\} \\ \setminus (\{j\,2^{2n(2)}M;\ j \in \mathbb{Z}\} \cup \{1 + j\,2^{2n(3)}M;\ j \in \mathbb{Z}\} \cup \{-1 + j\,2^{2n(3)}M;\ j \in \mathbb{Z}\} \cup \\ \dots \cup \{i-3 + j\,2^{2n(i-1)}M;\ j \in \mathbb{Z}\} \cup \{3-i + j\,2^{2n(i-1)}M;\ j \in \mathbb{Z}\})$$

such that

$$\psi|_{[l,l+1]} \equiv \varphi|_{\left[l_{j(2)}^{j(1)}, l_{j(2)}^{j(1)}+1\right]}.$$

Thus, we can define continuous

$$\psi(t) \in \mathcal{O}_{\varepsilon_{2n(i)+1}}\left(\psi(t - 2^{2n(i)}M)\right), \quad t \in ((2 + \cdots + 2^{2n(i)-2})M, \ldots, (2 + \cdots + 2^{2n(i)})M],$$

satisfying

$$\begin{aligned} \psi|_{[l,l+1]} \equiv \varphi|_{[j(l),j(l)+1]}, \quad & l \in \{(2 + \cdots + 2^{2n(i)-2})M, \ldots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ & \text{at least one } j(l) \in \{0, \ldots, M-1, l_1^1, l_2^1, \ldots, l_{m(i)}^i\}, \end{aligned}$$

$$\begin{aligned} &\{\varphi(t); t \in [l_1^1, l_1^1 + 1] \cup [l_2^1, l_2^1 + 1] \cup \cdots \cup [l_{m(i)}^i, l_{m(i)}^i + 1]\} \\ &\qquad \subseteq \{\psi(t);\, t \in [(2 + \cdots + 2^{2n(i)-2})M, \ldots, (2 + \cdots + 2^{2n(i)})M]\}. \end{aligned} \tag{7.10}$$

In addition, we can define ψ so that

$$\begin{aligned} \psi|_{[l,l+1]} \equiv \psi|_{[0,1]}, \quad & l \in \{j\, 2^{2n(2)}M;\ j \in \mathbb{Z}\} \\ & \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \ldots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ \psi|_{[l,l+1]} \equiv \psi|_{[1,2]}, \quad & l \in \{1 + j\, 2^{2n(3)}M;\ j \in \mathbb{Z}\} \\ & \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \ldots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ \psi|_{[l,l+1]} \equiv \psi|_{[-1,0]}, \quad & l \in \{-1 + j\, 2^{2n(3)}M;\ j \in \mathbb{Z}\} \\ & \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \ldots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ & \vdots \\ \psi|_{[l,l+1]} \equiv \psi|_{[i-3,i-2]}, \quad & l \in \{i - 3 + j\, 2^{2n(i-1)}M;\ j \in \mathbb{Z}\} \\ & \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \ldots, (2 + \cdots + 2^{2n(i)})M - 1\}, \\ \psi|_{[l,l+1]} \equiv \psi|_{[3-i,4-i]}, \quad & l \in \{3 - i + j\, 2^{2n(i-1)}M;\ j \in \mathbb{Z}\} \\ & \cap \{-(2^{2n(i)-1} + \cdots + 2)M, \ldots, (2 + \cdots + 2^{2n(i)})M - 1\}, \end{aligned}$$

$$\psi|_{\left[2^{2n(i)}M+i-2,2^{2n(i)}M+i-1\right]} \equiv \psi|_{[i-2,i-1]}, \quad \psi|_{\left[2^{2n(i)}M+2-i,2^{2n(i)}M+3-i\right]} \equiv \psi|_{[2-i,3-i]},$$

$$\begin{aligned} &\psi|_{[k,k+1]} \equiv \psi|_{[i-2,i-1]} \quad \text{for some } k, \\ &\qquad k \in \{(2 + \cdots + 2^{2n(i)-2})M, \ldots, (2 + \cdots + 2^{2n(i)})M - 1\} \setminus \{2^{2n(i)}M + i - 2\}, \\ &\psi|_{[k,k+1]} \equiv \psi|_{[2-i,3-i]} \quad \text{for some } k, \\ &\qquad k \in \{(2 + \cdots + 2^{2n(i)-2})M, \ldots, (2 + \cdots + 2^{2n(i)})M - 1\} \setminus \{2^{2n(i)}M + 2 - i\}. \end{aligned}$$

Evidently, it is valid

$$\varrho\left(\varphi(i), \varphi(j)\right) \geq 2^{-K} \quad \text{or} \quad \varrho\left(\varphi(i), \varphi(j)\right) = 0$$

for all $i, j \in \mathbb{Z}$ and some $K \in \mathbb{N}$. If we begin the construction by

$$l_1^K := l_1\big(2^{-K}\big), \ldots, l_{m(K)}^K := l_{m(2^{-K})}\big(2^{-K}\big),$$

then we have to obtain

$$\psi(k) = \psi(k + M), \quad k \in \mathbb{Z}.$$

Hence, we can construct the above ψ in order that the sequence $\{\psi(k)\}_{k\in\mathbb{Z}}$ is periodic with period M which gives (5.14) and the continuity of ψ. We construct ψ using the process from Corollary 5.23 for all $i \in \mathbb{N}$ and we obtain an almost periodic function $\psi : \mathbb{R} \to \mathcal{X}$. Indeed, we have (7.5) and, summarizing (7.6), (7.7), (7.8), ..., (7.9), ..., we get (5.13) (see also (3.22)). For periodic $\{\psi(k)\}_{k\in\mathbb{Z}}$, the first identity in (7.1) follows from (7.3) and (7.5) and the second one from the construction, (7.4), and (7.10). As in the proof of Theorem 3.1, we see that, for any $l \in \mathbb{Z}$, there exists $i(l) \in \mathbb{N}$ satisfying

$$\psi|_{[k,k+1]} \equiv \psi|_{[l,l+1]}, \quad k \in \{l + j\, 2^{2n(i(l))}M;\ j \in \mathbb{Z}\}.$$

Thus, we obtain (7.2) for $q(l) = 2^{2n(i(l))}M$. □

As an example which illustrates the previous theorem, we mention the following result.

Corollary 7.2. *For any continuous function $F : [0,1] \to \mathcal{X}$, there exists an almost periodic function ψ with the property that*

$$\{\psi(t);\ t \in \mathbb{R}\} = \{F(t);\ t \in (0,1)\}.$$

Proof. It suffices to show, that there exists a uniformly continuous function $\varphi : \mathbb{R} \to \mathcal{X}$ for which $\{\varphi(k);\ k \in \mathbb{Z}\} = \{F(1/2)\}$ and $\{\varphi(t);\ t \in \mathbb{R}\} = \{F(t);\ t \in (0,1)\}$, and to apply Theorem 7.1. For example, one can put

$$\varphi(k+s) := F\left(\frac{1}{2} + s\right), \quad k \in \mathbb{N},\ s \in \left[0, \frac{k}{2k+1}\right),$$

$$\varphi(k+s) := F\left(\frac{1}{2} + \frac{k}{2k+1}\right), \quad k \in \mathbb{N},\ s \in \left[\frac{k}{2k+1}, 1 - \frac{k}{2k+1}\right),$$

$$\varphi(k+s) := F\left(\frac{1}{2} + 1 - s\right), \quad k \in \mathbb{N},\ s \in \left[1 - \frac{k}{2k+1}, 1\right);$$

$$\varphi(k+s) := F\left(\frac{1}{2} - s\right), \quad k \in \mathbb{Z} \setminus \mathbb{N},\ s \in \left[0, \frac{k}{2k-1}\right),$$

$$\varphi(k+s) := F\left(\frac{1}{2} - \frac{k}{2k-1}\right), \quad k \in \mathbb{Z} \setminus \mathbb{N},\ s \in \left[\frac{k}{2k-1}, 1 - \frac{k}{2k-1}\right),$$

$$\varphi(k+s) := F\left(\frac{1}{2} + s - 1\right), \quad k \in \mathbb{Z} \setminus \mathbb{N},\ s \in \left[1 - \frac{k}{2k-1}, 1\right).$$

□

In the limit periodic case, we obtain:

Theorem 7.3. *Let $F : \mathbb{R} \to \mathcal{X}$ be a uniformly continuous function. If the range of F is totally bounded and $F(p) = F(p + kr)$ for all $k \in \mathbb{Z}$ and some $p \in \mathbb{R}$, $r > 0$, then there exists a limit periodic function $f : \mathbb{R} \to \mathcal{X}$ such that*

$$\{f(t);\ t \in \mathbb{R}\} = \{F(t);\ t \in \mathbb{R}\} \tag{7.11}$$

and that, for every $l \in \mathbb{Z}$, one can find $q(l) \in \mathbb{N}$ for which

$$f(l+s) = f\left(l + s + jq(l)\right), \quad j \in \mathbb{Z},\ s \in [0,1]. \tag{7.12}$$

Proof. We consider the function $G : \mathbb{R} \to \mathcal{X}$ given by the formula

$$G(t) := F\,(rt + p)\,, \quad t \in \mathbb{R}.$$

We see that $G(0) = G(k)$, $k \in \mathbb{Z}$, and that

$$\{G(t);\, t \in \mathbb{R}\} = \{F(t);\, t \in \mathbb{R}\}. \tag{7.13}$$

Since G is a uniformly continuous function whose range is totally bounded, for all $\varepsilon > 0$, there exists $m = m(\varepsilon) \in \mathbb{N} \setminus \{1\}$ satisfying

$$\varrho\,(G(l + s), G(j + s)) < \varepsilon \tag{7.14}$$

for all $l \in \mathbb{Z}$, $s \in [0, 1]$, and at least one $j = j(l) \in \{-m, \dots, 0, \dots, m-1\}$. We can assume that

$$m(\varepsilon) \to \infty \quad \text{if} \quad \varepsilon \to 0^+. \tag{7.15}$$

Firstly, we put

$$X := \{G(t);\, t \in \mathbb{R}\}, \tag{7.16}$$

$$m(n) := m\left(2^{-n}\right), \quad n \in \mathbb{N},$$

and

$$f^1(s + 2lm(1)) := G(s), \quad s \in [-m(1), m(1)),\, l \in \mathbb{Z}.$$

In the second step, we define a periodic continuous function $f^2 : \mathbb{R} \to X$ with period $2^2 m(1)m(2)$ arbitrarily so that

$$f^2(s) = f^1(s), \quad s \in [-m(1), m(1)),$$

$$f^2\,(s + 2m(1)m(2)) = f^1(s), \quad s \in [-1, 1),$$

$$\left\{f^2(s);\, s \in [-2m(1)m(2) + 1, -1) \cup [1, 2m(1)m(2) - 1)\right\} = \{G(s);\, s \in [-m(2), m(2)]\}\,,$$

and

$$\varrho\left(f^2(t), f^1(t)\right) < \frac{1}{2}, \quad t \in \mathbb{R}. \tag{7.17}$$

In fact, using (7.14), one can choose this function f^2 in such a way that, for each $j \in \{-2m(1)m(2), \dots, 2m(1)m(2) - 1\}$, there exists

$$i = i(j) \in \{-m(2), \dots, m(2) - 1\}$$

having the property

$$f^2(s + j) = G(s + i), \quad s \in [0, 1].$$

Henceforth, we assume that we choose all functions f^n in this way.

In the third step, we define a periodic continuous function $f^3 : \mathbb{R} \to X$ with period $2^3 m(1)m(2)m(3)$ arbitrarily so that

$$f^3(s) = f^2(s), \quad s \in [-2m(1)m(2), 2m(1)m(2)),$$

$$f^3\,(s + 2m(1)m(2)k) = f^2(s), \quad s \in [-1, 1),\, k \in \mathbb{Z},$$

$$f^3\left(s+2^2m(1)m(2)m(3)\right)=f^2(s),\quad s\in[-2,2),$$

and

$$\varrho\left(f^3(t),f^2(t)\right)<\frac{1}{2^2},\quad t\in\mathbb{R},\tag{7.18}$$

$$\left\{f^3(s);\ s\in I_3\right\}=\left\{G(s);\ s\in[-m(3),m(3)]\right\},$$

where

$$I_3:=\left\{\left[-2^2m(1)m(2)m(3)+2,-2\right)\cup\left[2,2^2m(1)m(2)m(3)-2\right)\right\}\\ \setminus\bigcup_{k\in\mathbb{Z}}[2m(1)m(2)k-1,2m(1)m(2)k+1)\,.$$

In the general n-th step, we define a periodic continuous function $f^n:\mathbb{R}\to X$ with period $2^nm(1)m(2)\cdots m(n)$ arbitrarily so that

$$f^n(s)=f^{n-1}(s),\quad s\in\left[-2^{n-2}m(1)m(2)\cdots m(n-1),2^{n-2}m(1)m(2)\cdots m(n-1)\right),$$

$$f^n\left(s+2m(1)m(2)k\right)=f^{n-1}(s),\quad s\in[-1,1),\ k\in\mathbb{Z},$$
$$f^n\left(s+2^2m(1)m(2)m(3)k\right)=f^{n-1}(s),\quad s\in[-2,2),\ k\in\mathbb{Z},$$
$$\vdots$$
$$f^n\left(s+2^{n-2}m(1)m(2)\cdots m(n-1)k\right)=f^{n-1}(s),\quad s\in[-n+2,n-2),\ k\in\mathbb{Z},$$
$$f^n\left(s+2^{n-1}m(1)m(2)\cdots m(n)\right)=f^{n-1}(s),\quad s\in[-n+1,n-1),$$

and

$$\varrho\left(f^n(t),f^{n-1}(t)\right)<\frac{1}{2^{n-1}},\quad t\in\mathbb{R},\tag{7.19}$$

$$\left\{f^n(s);\ s\in I_n\right\}=\left\{G(s);\ s\in[-m(n),m(n)]\right\},\tag{7.20}$$

where

$$\begin{aligned}I_n:=&\left\{\left[-2^{n-1}\cdots m(n)+n-1,-n+1\right)\cup\left[n-1,2^{n-1}\cdots m(n)-n+1\right)\right\}\\ &\setminus\left\{\bigcup_{k\in\mathbb{Z}}[2m(1)m(2)k-1,2m(1)m(2)k+1)\cup\right.\\ &\quad\bigcup_{k\in\mathbb{Z}}\left[2^2m(1)m(2)m(3)k-2,2^2m(1)m(2)m(3)k+2\right)\cup\cdots\cup\\ &\quad\left.\bigcup_{k\in\mathbb{Z}}\left[2^{n-2}m(1)\cdots m(n-1)k-n+2,2^{n-2}m(1)\cdots m(n-1)k+n-2\right)\right\}.\end{aligned}$$

We can define function $f:\mathbb{R}\to X$ by

$$f(t):=\lim_{n\to\infty}f^n(t),\quad t\in\mathbb{R},\tag{7.21}$$

because

$$f^n(s)=f^{n+i}(s),\quad n,i\in\mathbb{N},\ s\in\left[-2^{n-1}m(1)\cdots m(n),2^{n-1}m(1)\cdots m(n)\right).\tag{7.22}$$

The inequality

$$\varrho\left(f(t), f^n(t)\right) \leq \varrho\left(f^{n+1}(t), f^n(t)\right) + \varrho\left(f^{n+2}(t), f^{n+1}(t)\right) + \cdots$$
$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots = \frac{1}{2^{n-1}}, \quad t \in \mathbb{R},\, n \in \mathbb{N},$$

which follows from (7.19) (consider together with (7.17), (7.18)), implies the limit periodicity of f defined in (7.21).

The resulting function f satisfies (7.11) and (7.12). Immediately, we obtain (7.11) from (7.15), (7.20) (see also (7.13), (7.16)), and (7.22). Let $n \in \mathbb{N}$ be arbitrary. Considering the construction above, we have

$$f^{n+i}\left(s + 2^n m(1) \cdots m(n+1) j\right) = f^{n+1}(s), \quad s \in [-n, n],\, j \in \mathbb{Z},\, i \in \mathbb{N}.$$

Finally, we get (7.12) for

$$q(l) = 2^{|\,l\,|+1} m(1) m(2) \cdots m(|\,l\,| + 2), \quad l \in \mathbb{Z}.$$

□

Remark 7.4. As in Section 3.2, we can formulate Theorem 7.3 in the form when $\mathcal{X}$ is a uniform space with a countable fundamental system of entourages. We refer to the references mentioned in Section 3.2. △

Remark 7.5. Here, we comment the periodic condition on F in the theorem above. Without the requirement that there exist $p \in \mathbb{R}$ and $r > 0$ such that $F(p) = F(p + kr)$, $k \in \mathbb{Z}$, Theorem 7.3 is not valid.

For example, there exists a uniformly continuous function $F : \mathbb{R} \to \mathbb{R}^2$ whose range is

$$\{F(t);\, t \in \mathbb{R}\} = \left\{ \left[x, \sin \frac{1}{x} \right];\, x \in (0, 1] \right\}. \tag{7.23}$$

Suppose that a limit periodic function $f : \mathbb{R} \to \mathbb{R}^2$ satisfies (7.11) and (7.12). There exist $j \in \mathbb{Z}$ and $a \in (0, 1]$ for which (see (7.11) and (7.23))

$$\{f(t);\, t \in [j, j+1]\} = \left\{ \left[x, \sin \frac{1}{x} \right];\, x \in [a, 1] \right\}. \tag{7.24}$$

Let us consider $q(j)$ from the statement of Theorem 7.3. Since f is uniformly continuous, there exists $b \in (0, a)$ such that

$$\{f(t);\, t \in [i, i + 1 + q(j)]\} \subseteq \left\{ \left[x, \sin \frac{1}{x} \right];\, x \in [b, 1] \right\}$$

for all $i \in \mathbb{Z}$ if

$$[1, \sin 1] \in \{f(t);\, t \in [i, i + 1 + q(j)]\}. \tag{7.25}$$

Nevertheless, (7.25) follows from (7.12) and (7.24). Thus, we have

$$\{f(t);\, t \in \mathbb{R}\} \subseteq \left\{ \left[x, \sin \frac{1}{x} \right];\, x \in [b, 1] \right\}. \tag{7.26}$$

This contradiction (given by (7.11) and (7.26)) shows that the periodic condition on F cannot be omitted. △

Immediately, from Theorem 7.3, we get the following consequence.

Corollary 7.6. *Let $F : \mathbb{R} \to \mathcal{X}$ be a uniformly continuous function. If the range of F is totally bounded and $F(p) = F(p + kr)$ for all $k \in \mathbb{Z}$ and some $p \in \mathbb{R}$, $r > 0$, then there exists an almost periodic function $f : \mathbb{R} \to \mathcal{X}$ such that (7.11) is valid and that, for every $l \in \mathbb{Z}$, one can find $q(l) \in \mathbb{N}$ for which (7.12) is valid.*

Important cases of pseudometric spaces are covered by the spaces whose elements can be connected by arcs.

Corollary 7.7. *Let $F : \mathbb{R} \to \mathcal{X}$ be a uniformly continuous function whose range $X := \{F(t);\, t \in \mathbb{R}\}$ is totally bounded. If all $x, y \in X$ can be connected in $\mathcal{X}$ by continuous curves which depend uniformly continuously on x and y, then there exists a limit periodic function $f : \mathbb{R} \to \mathcal{X}$ such that*

$$\{F(t);\, t \in \mathbb{R}\} \subseteq \{f(t);\, t \in \mathbb{R}\} \tag{7.27}$$

and that, for every $l \in \mathbb{Z}$, one can find $q(l) \in \mathbb{N}$ for which (7.12) is valid.

Proof. Let us choose $t_0 \in \mathbb{R}$ arbitrarily and consider continuous functions $g_k : [0, 1] \to \mathcal{X}$, $k \in \mathbb{Z}$, for which

$$g_k(0) = F(k), \quad g_k\left(\frac{1}{2}\right) = F(t_0), \quad g_k(1) = F(k).$$

We define $G : \mathbb{R} \to \mathcal{X}$ as

$$G(2k + s) := F(k + s), \quad G(2k + 1 + s) := g_{k+1}(s), \quad k \in \mathbb{Z},\ s \in [0, 1).$$

We can assume (consider directly the statement of the corollary) that, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varrho\left(g_k(s), g_l(s)\right) < \varepsilon, \quad s \in [0, 1],$$

if

$$\varrho\left(g_k(0), g_l(0)\right) = \varrho\left(F(k), F(l)\right) < \delta(\varepsilon)$$

for arbitrary $k, l \in \mathbb{Z}$.

Thus, there exists a uniformly continuous function $G : \mathbb{R} \to \mathcal{X}$ with a totally bounded range and with the properties

$$G\left(\frac{3}{2} + 2k\right) = F(t_0), \quad k \in \mathbb{Z},$$

$$\{F(t);\, t \in \mathbb{R}\} \subseteq \{G(t);\, t \in \mathbb{R}\}. \tag{7.28}$$

Using Theorem 7.3 for this function G, we get a limit periodic function $f : \mathbb{R} \to \mathcal{X}$ satisfying

$$\{f(t);\, t \in \mathbb{R}\} = \{G(t);\, t \in \mathbb{R}\} \tag{7.29}$$

and (7.12). Finally, (7.28) and (7.29) give (7.27). □

Especially, from the proof of Corollary 7.7, we obtain the last result.

Corollary 7.8. *Let $F : \mathbb{R} \to \mathcal{X}$ be a uniformly continuous function whose range $X := \{F(t);\, t \in \mathbb{R}\}$ is totally bounded. If there exists $x \in X$ such that x and $F(k)$ for $k \in \mathbb{Z}$ can be connected in X by continuous curves which depend uniformly continuously on $F(k)$, then one can construct a limit periodic function $f : \mathbb{R} \to \mathcal{X}$ satisfying (7.11) and (7.12).*

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All mentioned papers can be found in MathSciNet (18 of them in WoS). The status is on the date 2015/09/21.

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