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Geometric constructions and correspondences in action

Habilitation thesis

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Abstract

This thesis is primarily concerned with several geometric notions and constructions that all have interesting history as well as useful applications. Nowadays, they are well established and put into the framework of Cartan, respectively parabolic, geometries. After setting the scene with some general context, we investigate selected problems with notable outcomes. In all the situations discussed here, the constructions are used or combined in such a way that is in some respect uncommon. Moreover, although our setting is of a rather functorial nature, we can often provide concrete down-to-earth interpretations. We consider this symbiosis as an interesting and not always automatic feature.

Contents

Introduction

Transforming a given problem into one of a different character is an influential approach, which has dozens of incarnations across the mathematics. This opens new perspectives that help both to solve the initial problem and, importantly, to investigate new interrelations. The success of respective techniques depends on the control of the transition between the two settings. The constructions discussed below exhibit very satisfactory level of control which is then demonstrated in applications that follow. Moreover, in these applications we are able to follow the transition in various modes.

It is a common feature of all notions and constructions discussed in this treatise that they can be dealt within the framework of Cartan, respectively parabolic, geometries. This field is nowadays well developed and provides a wide range of conceptual approaches and instruments.

Background

The background material is collected in chapter I. The very general principles we are going to use are introduced in section 1. The extension and correspondence space constructions belong among the most prominent notions, which are later combined in diverse ways.

The typical and omnipresent features of parabolic geometries—the main area of our interest—are concentrated in section 2. These include the canonical normalization condition, underlying Weyl structures, distinguished families of curves and basics of the BGG industry.

The previous abstract notions are concretized in section 3, where particular geometric structures are discussed. In this context, two constructions of ambient character related to projective and conformal geometries are presented.

Most of the discussed generalities as well as many of their concrete realizations are thoroughly described by Čap and Slovák in the monograph [15]. Therefore this book is used as the main reference throughout this thesis.

Our contributions

Selected applications of previous general concepts, containing our original contributions, are presented in chapter II. In section 4 we discuss the geometry of chains of parabolic contact structures. These are distinguished curves determined by the geometric structure which are, as unparametrized curves, uniquely given by a tangent direction in one point. In this respect they behave as geodesics of an affine, respectively projective, structure (in other respects, however, they behave very differently). As such they form a path geometry on the base manifold, which can also be described in terms of parabolic geometries.

The most interesting results are obtained for CR and Lagrangean contact structures. It is the main contribution of our attached article [17] that the correspondence between the initial geometric structure and the associated path geometry of chains can be described directly, i.e., without any prolongation, via a functorial extension construction provided that the initial structure is integrable. In these cases, the construction preserves the normality of parabolic geometries attached to the structures in question, hence allows an immediate use. The point is that we control fully the transition—as well as a backward reconstruction of the initial structure—also in non-flat cases. This represents an unusual usage of this kind of construction.

In section 5, an application of the construction of Fefferman type is discussed, which induces a conformal structure of split signature on the (weighted) cotangent bundle of a projective manifold. The construction preserves normality of induced parabolic geometries only in very limited situations (in dimension two or in flat case). It might therefore seem surprising how much can be said, including a full characterization of the induced structure, despite this unfriendly setting. This is indeed an unexpected case, which is also emphasized as the main achievement of the attached article [43]. In fact the construction can be seen as a composition of two sub-constructions with a natural intermediate Lagrangean contact structure. This turns to be half-integrable which implies the resulting conformal structure is somehow 'half-normal'.

It is also an interesting issue that our construction can be interpreted as a conformal analogue of the classical Patterson–Walker metric construction. This point of view is elaborated in our associated article [42]. As an icing on the cake, we can show that such conformal structures admit a global and explicit ambient metric, which is a rather rare phenomenon. The ambient metric can further be described as the true Patterson–Walker metric of the Thomas ambient connection associated to the very starting projective structure. This construction is the main subject of the next attached article [44].

In section 6 we consider the problem of describing a generating set of absolute conformal invariants of curves on conformal manifolds. There is a very classical solution for curves in Euclidean spaces, the Frenet frame construction, which is based on expressing the infinitesimal change of a natural frame associated to a given curve. There are several attempts to apply these ideas to other geometries, both in homogeneous and curved setting. The main complication lies in detecting reasonable invariant tools to treat the problem, which is why any careful study proceeds only case by case. Our approach in the last attached article [62] is based on a transition to the conformal standard tractor bundle, where a natural linear connection and a parallel bundle metric are ready to use. With these instruments, the original Frenet frame construction is easily adapted for general curves in conformal manifolds of any signature, including null curves. As a side result we get a new characterization of conformal circles, the distinguished curves of conformal structures.

Conception

The conception of this text is subordinated to its aim, which is introducing the results of the four attached articles. They are, of course, not chosen randomly, but rather for their common features which were already mentioned above. To give a reasonable exposition of the results, it is important to describe those common features in a wider context. That is why we spend a relatively large space with processing this background knowledge in an individual chapter I, which might help to follow the subsequent presentation more smoothly.

Also in chapter II, where the results are presented, we take the liberty to introduce each subject more widely than is common in journal articles. We always start with a brisk historical survey, including some comparisons, after which the key parts follow. We help the rushing readers by directing them to the key parts of these sections by using the label 'action' in the section title. Many of the topics discussed are still subjects of a ongoing research, which is usually commented in accompanying remarks.

Throughout the text many things are explained and many are not: we tend to explain relevant notions from Cartan geometry, whereas we do not bother with the notions and ideas from standard differential geometry on the level of classical textbooks, e.g., [48, 49]. Besides the notions of smooth manifolds, connections etc., we rely on Lie groups and their actions, principal bundles and their reductions, associated bundles and induced connections and the like. Some basics on Lie algebras and their representations are involved as well.

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CHAPTER I

Toolkit

Throughout this thesis we enjoy the opportunity to remain in the safe haven of Cartan geometries. This is our preferred point of view and this is also why we open the current chapter by introducing general concepts that will be repeatedly applied below. After focusing on the subclass of parabolic geometries, we present concrete geometric structures and other related notions that are further employed in chapter II. Individual sections are meant to be independent, although there are many corss-references. The standard reference covering most of the material collected here is [15]; generally, if we do not refer anywhere, details can be found in that book.

There is nothing really original in this chapter except for some remarks on Lagrangean contact structures in section 3.5, which are useful for later considerations. Although all those observations are rather direct and natural, we are not aware they have been published anywhere.

1 Cartan geometries

The concept of Cartan connections was developed by É. Cartan hand in hand with the method of moving frame. These techniques proved to constitute an efficient approach to the equivalence problem, associating a natural parallelism (connection) and essential invariants (curvatures) to a particular geometric problem. However, learning about Cartan connections from original sources may be somewhat troubled. One of main reasons is "the difficulty Cartan faced in trying to express notions for which there was no truly suitable language". This concerns primarily the conception of principal and associated bundles, the terms in which Cartan's ideas are nowadays usually presented. As a gentle introduction to the topic we refer the reader to the monograph [58] by Sharpe, probably the first comprehensive textbook on Cartan geometries, from which also the quotation above is taken.

The driving idea to what we now call Cartan geometries is indicated in the following diagram: Cartan geometry (of a particular type) generalizes the corresponding Klein geometry (on a homogeneous space of a particular Lie group) in an analogous way as the Riemannian geometry generalizes the Euclidean one.

$$\begin{tikzcd} \text{Riemannian} \arrow[r, rightsquigarrow] & \text{Cartan} \\ \text{Euclidean} \arrow[u, rightsquigarrow] \arrow[r, rightsquigarrow] & \text{Klein} \arrow[u, rightsquigarrow] \end{tikzcd}$$

In other words, Cartan geometry is regarded as a 'curved' geometry modeled over a homogeneous space of a particular Lie group. We employ both this terminology and this point of view consistently below.

In the following subsections we define Cartan connection and describe some related notions and alternative approaches, among which the tractor bundles and connections are emphasized. The next topic concerns a construction relating Cartan geometries, respectively connections, of different types. The motivation comes from the characterization of homogeneous Cartan connections, i.e., (non-flat) Cartan connections that are invariant under a group action. However, constructions of this type apply more generally and they may have many guises as is demonstrated in what follows.

1.1 Cartan connections

With the previous motivation, we may roughly interpret Cartan's original approach as follows (incorporating, however, some non-original terminology to be used below). Considering a particular geometric structure, take its *homogeneous model*, i.e., the most symmetric case, first. Let the model space be the coset space G/P of a pair of *principal* Lie group G and its *structure* subgroup $P \subset G$. In this situation, the absolute parallelism on G, the flat Cartan connection, is provided by the Maurer–Cartan form $\omega : TG \to \mathfrak{g}$, i.e., the left-invariant 1-form on G with values in the Lie algebra $\mathfrak{g}$ of G. The flatness of this connection means that the *structure equation* is satisfied, $\mathrm{d}\omega + \omega \wedge \omega = 0$. In order to emphasize the role of the Lie algebra $\mathfrak{g}$, the same is rather written as

$$\mathrm{d}\omega + [\omega, \omega] = 0. \tag{1.1}$$

Instead of the 1-form $\omega : TG \to \mathfrak{g}$ one may consider its pull-backs $\theta := s^*\omega : TU \to \mathfrak{g}$, the so-called *Cartan gauges*, with respect to local sections $s : U \to G$ of the canonical projection $p : G \to G/P$, where $U \subseteq G/P$. For a choice of basis of $\mathfrak{g}$, any gauge θ is represented by a bunch of ordinary 1-forms on U. The structure equation (1.1) then expands to a couple of equations relating the constituents of θ and their exterior differentials. For sections that are somehow adapted to the geometric structure in question, these equations simplify. In a more general setting, one starts with a local adapted coframe and other 1-forms that are consecutively adjusted in a sequence of steps (which may often seem mysterious) so that one ends as close to the previous structure equations as possible. This yields a 'natural' Cartan gauge, respectively Cartan connection, and the invalidity of the structure equation is interpreted as the contribution of its *curvature*. Also, among the pieces that obstruct the validity of the structure equation, one may distinguish the essential ones whose vanishing imply the vanishing of all others.

Reading this overview, several natural questions have to arise; in particular, it is not clear what are the 'natural' choices and restrictions to follow. In order to make the previous sketch more serious (and ready to use), one needs a number of illustrative examples (plus a bit of Cartan's insight) or a general theory. It is a good news that many seemingly unrelated cases can be put on a common foundation, which is the result of decades of later investigations. This brings especially the notion of normalization into the game, which we exemplify in section 2. Anyhow, the existence and description of a 'natural' Cartan connection associated to a particular problem is typically not an obvious statement. For the time being, we consider Cartan connection as the given data and focus on some related notions.

Basic definitions

We keep in mind the model Cartan geometry of type (G, P) consists of the principal P-bundle $p : G \to G/P$ with the Maurer–Cartan form $\omega : TG \to \mathfrak{g}$ which, among other things, defines an absolute parallelism on G and resonates with the principal (right) action of the structure group P so that it reproduces its infinitesimal generators (fundamental vector fields) and is equivariant. A general *Cartan geometry* of type (G, P) on a smooth manifold M consists of a principal P-bundle $p : \mathcal{G} \to M$ with *Cartan connection*, which is a $\mathfrak{g}$-valued 1-form $\omega : T\mathcal{G} \to \mathfrak{g}$ satisfying the previous three properties, i.e.,

- $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism, for each $u \in \mathcal{G}$,
- $\omega\left(\frac{d}{dt}\big|_0 r_{\exp tX}(u)\right) = X$, for each $u \in \mathcal{G}$ and $X \in \mathfrak{p}$,
- $(r_h)^*\omega = \mathrm{Ad}_{h^{-1}} \circ \omega$, for each $h \in P$,

where $\mathfrak{p} \subset \mathfrak{g}$ is the Lie algebra of the subgroup $P \subset G$ and $r_h : \mathcal{G} \to \mathcal{G}$, respectively $\mathrm{Ad}_h : \mathfrak{g} \to \mathfrak{g}$, denotes the principal right action, respectively the adjoint action, by an element $h \in P$. Remember that Cartan connections are *not* principal connections (which are $\mathfrak{p}$-valued 1-forms satisfying the last two conditions).[1]

A *morphism* of two Cartan geometries of the same type is a morphism of the corresponding principal bundles that preserves Cartan connection.

The *curvature* of Cartan connection ω is the $\mathfrak{g}$-valued 2-form on $\mathcal{G}$ given by

$$\kappa := \mathrm{d}\omega + [\omega, \omega]. \tag{1.2}$$

Note that κ is strictly horizontal, i.e., it vanishes under insertion of any vertical vector. Cartan curvature vanishes identically if and only if the Cartan geometry is locally isomorphic to its homogeneous model.

The composition of $\omega : T\mathcal{G} \to \mathfrak{g}$ with the quotient projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ is the so-called *solder form.* For any $u \in \mathcal{G}$, this yields a linear isomorphism $T_xM \to \mathfrak{g}/\mathfrak{p}$, where $x = p(u)$. For a different element over x, say $r_h(u) \in p^{-1}(x)$ with $h \in P$, this isomorphism is afflicted by the action of $\mathrm{Ad}_{h^{-1}}$. Hence the identification $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ provided by ω descends to the identification of TM with $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$, the associate bundle to $\mathcal{G}$ with respect to the indicated P-action.

$$\begin{tikzcd}
T\mathcal{G} \arrow[r, "\omega"] \arrow[d, "Tp"'] & \mathfrak{g} \arrow[d] \\
TM \arrow[r, dashed] & \mathfrak{g}/\mathfrak{p}
\end{tikzcd}$$

Under this identification, various geometric objects on M are reflected as P-invariant objects in $\mathfrak{g}/\mathfrak{p}$. Furthermore, the previous identification can be read so that a vector field on M is represented by a P-equivariant map $\mathcal{G} \to \mathfrak{g}/\mathfrak{p}$, the so-called *frame form.* The same logic applies to any tensor field on M, more generally, to sections of any associated bundle to $\mathcal{G}$. In particular, the frame form of the Cartan curvature κ is a P-equivariant function

$$\mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}. \tag{1.3}$$

Composing this with the quotient projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$, we obtain a P-equivariant map $\mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes (\mathfrak{g}/\mathfrak{p})$, which represents a tensor field of type $\Lambda^2 T^*M \otimes TM$, which is called the *torsion* of Cartan connection ω.

Remarks

Cartan gauge is defined alike to the flat case, i.e., $\theta := s^*\omega : TU \to \mathfrak{g}$, where $s : U \to \mathcal{G}$ is a local section of the projection $p : \mathcal{G} \to M$. Under change of section $s' = r_h \circ s$, for $h : U \to P$, the gauge changes so that

$$\theta' = \mathrm{Ad}_{h^{-1}}\, \theta + \delta h, \tag{1.4}$$

where $\delta h : TU \to \mathfrak{p}$ is the left logarithmic derivative of h, i.e., the pull-back of the Maurer–Cartan form on P. Cartan geometry can be dealt in terms of (atlases of) Cartan gauges with the equivalence relation given by (1.4) (on intersections of domains).

[1]Cartan connection determines a principal connection on $\mathcal{G}$ only in very specific cases of so-called reductive geometries, see section 3.1.

1.2 Related notions and connections

Since Cartan connection $\omega : T\mathcal{G} \to \mathfrak{g}$ is not a principal connection, it does not induce connection neither on TM nor on any associated bundle. It, however, determines a principal connection $\widehat{\omega} : T\widehat{\mathcal{G}} \to \mathfrak{g}$ on the principal G-bundle $\widehat{\mathcal{G}} := \mathcal{G} \times_P G$ via the equivariant extension of ω from the image of the canonical inclusion $\mathcal{G} \hookrightarrow \mathcal{G} \times_P G$ (mapping $u \in \mathcal{G}$ to the equivalence class represented by u and the unit element of G). The principal connection $\widehat{\omega}$ induces connections on all associated bundles to $\widehat{\mathcal{G}} \to M$. Hence the Cartan connection ω gives rise to a connection on any associated bundle to $\mathcal{G} \to M$ provided that the standard fibre is actually a G-module (and not only P-module).

Cartan space

As a typical instance we have to mention the *Cartan space*, $\mathcal{S} := \mathcal{G} \times_P (G/P) = \widehat{\mathcal{G}} \times_G (G/P)$ with the obvious G-action on G/P, which is important for several reasons. There is a canonical section of the projection $\mathcal{S} \to M$ (mapping $x \in M$ to the equivalence class represented by any $u \in \mathcal{G}_x$ and the origin in G/P), along which the vertical subbundle is naturally identified with $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$, i.e., the tangent bundle TM. This observation makes precise the intuitive idea of 'osculating' M at each point by the model space G/P, the idea that may be spotted in Cartan's work.

The induced connection on $\mathcal{S}$, respectively the parallel displacement, allows to define the *development* of a curve on M into the homogeneous space G/P. More concretely, for a curve $c : I \to M$ and its canonical lift $\hat{c} : I \to \mathcal{S}$ (given by the canonical section of the projection $\mathcal{S} \to M$), one may transport parallelly the points $\hat{c}(t)$ into a chosen fiber over $x = c(t_0)$. Locally, this draws curves in $\mathcal{S}_x \cong G/P$ passing through the origin. The previous identification depends on $u \in \mathcal{G}_x$ so that, for $r_h(u)$ with $h \in P$, the resulting curve is shifted by the left action of h^{-1} on G/P. However, fixing $u \in \mathcal{G}_x$, this construction provides a bijection between germs of curves through x in M and germs of curves through origin in G/P preserving the order of contact. This is an important tool for studying curves on M via their counterparts in G/P. In particular, the curves developing to homogeneous curves in G/P form a distinguished family of curves on M, cf. section 2.3.

Note also that sections of $\mathcal{S} \to M$ are in bijective correspondence with reductions of $\widehat{\mathcal{G}} \to M$ to the subgroup $P \subset G$ (in particular, the canonical section corresponds to the canonical reduction $\mathcal{G} \hookrightarrow \mathcal{G} \times_P G$ above). Hence, taking the principal connection on $\widehat{\mathcal{G}}$ as the prior object, Cartan connection on the reduced bundle is given by its pull-back. In this vein, the *holonomy* of Cartan connection is defined to be the holonomy of the corresponding principal connection on $\widehat{\mathcal{G}}$ and as such it is a subgroup of G.

Tractors

Another prominent family of associated bundles with natural induced connections are the *tractor bundles*, which are the vector bundles with standard fiber being a G-representation. Among these bundles there is a particularly important one, namely, the *adjoint tractor bundle* $\mathcal{A} := \mathcal{G} \times_P \mathfrak{g} = \widehat{\mathcal{G}} \times_G \mathfrak{g}$ given by the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$. Among other things, this bundle is equipped with a natural algebraic bracket $\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, given by the Lie bracket on $\mathfrak{g}$, and a bundle projection $\Pi : \mathcal{A} \to TM$, given by the quotient projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ and the identification $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$. Using the Cartan connection on $\mathcal{G}$, sections of $\mathcal{A} \to M$ are identified with vector fields on $\mathcal{G}$ that are invariant with respect to the principal P-action.

For a P-representation $\rho : P \to GL(V)$, let $\mathcal{V} := \mathcal{G} \times_P V$ be the associated bundle corresponding to ρ. The differentiation of any P-equivariant map $\mathcal{G} \to V$ in the direction of a P-invariant vector field is again P-equivariant, which yields a natural differential operator

$$\mathrm{D} : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}), \tag{1.5}$$

the so-called *fundamental derivative*.

If $\mathcal{V}$ is a tractor bundle, i.e., ρ is the restriction of a homomorphism $G \to GL(V)$ and $\mathcal{V} = \mathcal{G} \times_P V = \widehat{\mathcal{G}} \times_G V$, then we also have the *tractor connection*,

$$\nabla : \Gamma(TM) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}), \tag{1.6}$$

i.e., the linear connection induced by $\widehat{\omega}$. It follows that, for any $s \in \Gamma(\mathcal{A})$ and $t \in \Gamma(\mathcal{V})$, these two operations are related by

$$\nabla_{\Pi(s)} t = \mathrm{D}_s t + s \bullet t, \tag{1.7}$$

where the bundle map $\bullet : \mathcal{A} \times \mathcal{V} \to \mathcal{V}$ is given by the derivative of the defining representation, $\mathfrak{g} \times V \to V$, and $\Pi : \mathcal{A} \to TM$ is as above. (In the case $\mathcal{V} = \mathcal{A}$, the map $\bullet$ coincides with the bracket $\{\ ,\ \}$ above.) Notice an important feature behind the equality (1.7), namely, that the result does not depend on the section $s \in \Gamma(\mathcal{A})$ covering $\xi = \Pi(s) \in \Gamma(TM)$.

One of often used aspects is that the curvature κ of the Cartan connection ω is naturally interpreted as a section of $\Lambda^2 T^*M \otimes \mathcal{A}$, cf. (1.3). Denoting the curvature of the tractor connection by R, which is a section of $\Lambda^2 T^*M \otimes \mathrm{End}(\mathcal{V})$, then these two objects are related by

$$R(\xi, \eta)(t) = \kappa(\xi, \eta) \bullet t,$$

for any $\xi, \eta \in \Gamma(TM)$ and $t \in \Gamma(\mathcal{V})$. In particular, the holonomy of the tractor connection ∇ coincides with the image of the holonomy group of the principal connection $\widehat{\omega}$ under the defining representation $G \to GL(V)$.

$$\begin{tikzcd}
 & & \text{tractor}\quad (\mathcal{V},\nabla) \arrow[dddl] \\
 & \text{principal}\quad (\widehat{\mathcal{G}},\widehat{\omega}) \arrow[ur, rightsquigarrow] \arrow[dd] & \\
\text{Cartan}\quad (\mathcal{G},\omega) \arrow[ur, hook] \arrow[dr] & & \\
 & M &
\end{tikzcd}$$

Previous reasonings indicate how to deal with Cartan geometry in terms of induced tractorial objects. Among all conceivable tractor bundles, other popular candidates (besides the adjoint ones) are the *standard tractor bundles* corresponding to the standard representation of the principal group G. It is interesting that these entities, in the case of projective and conformal geometries, have roots in the work of T.Y. Thomas from the similar period in which Cartan developed notions leading to Cartan connections. Despite this coincidence, the full understanding of all relationships is of much later date, see [2, 11].

1.3 Extension functors and correspondence spaces

The previous extension of the Cartan connection $\omega : T\mathcal{G} \to \mathfrak{g}$ of type (G, P) to the principal connection $\widehat{\omega} : T\widehat{\mathcal{G}} \to \mathfrak{g}$ was simply driven by the inclusion $P \subset G$. This notion can be generalized as follows.

General principal extension

In the homogeneous case, i.e., when $\mathcal{G} = G$ and $\omega =$ Maurer–Cartan form, the bundle $\widehat{\mathcal{G}} = G \times_P G \cong (G/P) \times G$ is a homogeneous principal bundle and the principal connection $\widehat{\omega}$ is invariant under the action of G. More generally, any homomorphism of Lie groups $i : P \to \widehat{G}$ defines a homogeneous principal $\widehat{G}$-bundle $\widehat{\mathcal{G}} := G \times_P \widehat{G}$. For $\mathfrak{p} \subset \mathfrak{g}$ and $\hat{\mathfrak{g}}$ being the Lie algebras of respective Lie groups, it follows that G-invariant principal connections on $\widehat{\mathcal{G}}$ are in bijective correspondence with linear maps $\alpha : \mathfrak{g} \to \hat{\mathfrak{g}}$ that are compatible with i and P-equivariant, i.e., such that

- $\alpha|_{\mathfrak{p}} = i'$, the derivative of i,
- $\alpha \circ \mathrm{Ad}_h = \mathrm{Ad}_{i(h)} \circ \alpha$, for each $h \in P$.

It also follows that, for a given Cartan geometry of type (G, P), any such compatible pair (i, α) determines a principal $\widehat{G}$-bundle with a principal connection $\widehat{\omega}$ and this construction is functorial. One just substitutes $\mathcal{G}$ for G and a general Cartan connection ω for the Maurer–Cartan form. More concretely, $\widehat{\omega}$ extends equivariantly the form $\alpha \circ \omega : T\widehat{\mathcal{G}} \to \hat{\mathfrak{g}}$ from the image of the canonical inclusion $j : \mathcal{G} \hookrightarrow \mathcal{G} \times_P \widehat{G}$, i.e., it is determined by $j^*\widehat{\omega} = \alpha \circ \omega$. In this view, the construction at the beginning of section 1.2 corresponds to $\widehat{G} = G$ so that i is the inclusion $P \subset G$ and α is the identity map $\mathfrak{g} = \hat{\mathfrak{g}}$.

General Cartan extension

In a similar manner, one may characterize invariant Cartan connections and apply the corresponding functor generally. In addition to the previous setting, there is a new element entering the game in the place of the structure group of the extended principal bundle. To be more precise, let $P \subset G$ and $\widetilde{P} \subset \widetilde{G}$ be the pairs of Lie groups and their closed subgroups and let $\mathfrak{p} \subset \mathfrak{g}$ and $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$ be the corresponding Lie algebras. A Lie group homomorphism $i : P \to \widetilde{P}$ defines a homogeneous principal $\widetilde{P}$-bundle $\widetilde{\mathcal{G}} := G \times_P \widetilde{P}$ and it follows that G-invariant Cartan connections of type $(\widetilde{G}, \widetilde{P})$ on $\widetilde{\mathcal{G}}$ are in bijective correspondence with linear maps $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that

- $\alpha|_{\mathfrak{p}} = i'$, the derivative of i,
- $\alpha \circ \mathrm{Ad}_h = \mathrm{Ad}_{i(h)} \circ \alpha$, for each $h \in P$,
- the induced map $\underline{\alpha} : \mathfrak{g}/\mathfrak{p} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ is a linear isomorphism.

It also follows that, for a given Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), any such compatible pair (i, α) determines a Cartan geometry $(\mathcal{G} \times_P \widetilde{P} \to M, \widetilde{\omega})$ of type $(\widetilde{G}, \widetilde{P})$ and this construction is functorial. Again, the induced connection is determined by

$$j^*\widetilde{\omega} = \alpha \circ \omega$$

and its required $\widetilde{P}$-equivariance, where $j : \mathcal{G} \hookrightarrow \mathcal{G} \times_P \widetilde{P}$ is the canonical inclusion. The inner action of $\widetilde{P}$, respectively its adjoint action on $\tilde{\mathfrak{g}}$, defines an equivalence relation on the pairs of compatible maps (i, α) so that two equivalent pairs induce isomorphic functors.

The effect of this construction on the Cartan curvature follows easily from definitions. In the invariant setting, i.e., starting with flat ω, the curvature $\widetilde{\kappa}$ of the induced Cartan connection $\widetilde{\omega}$ is fully given just by $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ so that the frame form of $\widetilde{\kappa}$ is constant along the image of j with value the P-invariant element $\Psi_\alpha \in \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \tilde{\mathfrak{g}}$ given by

$$(X + \mathfrak{p}, Y + \mathfrak{p}) \mapsto [\alpha(X), \alpha(Y)] - \alpha([X, Y]). \tag{1.8}$$

In particular, the induced Cartan connection is flat if and only if α is a homomorphism of Lie algebras.

To describe the general case, we use the same symbol Ψ_α also for the corresponding section of $\Lambda^2 T^*M \otimes \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is the adjoint tractor bundle to the induced Cartan geometry. Similarly, since α is P-equivariant, it gives rise to a bundle map $\alpha : \mathcal{A} \to \widetilde{\mathcal{A}}$, where $\mathcal{A}$ is the adjoint tractor bundle to the initial Cartan geometry. Now, if $\kappa \in \Gamma(\Lambda^2 T^*M \otimes \mathcal{A})$ is a curvature of the initial Cartan connection, then the curvature $\widetilde{\kappa} \in \Gamma(\Lambda^2 T^*M \otimes \widetilde{\mathcal{A}})$ of the induced one is given by

$$\widetilde{\kappa} = \alpha \circ \kappa + \Psi_\alpha. \tag{1.9}$$

Correspondence space

Until now we considered various bundles and connections over a common base manifold. Factorizing a given bundle, respectively connection, via an intermediate manifold leads to the notion of correspondence spaces. For a Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), let $Q \subset P \subset G$ be another closed subgroup and let $\widetilde{M} := \mathcal{G}/Q$ be the corresponding quotient space. This is called the *correspondence space* of M induced by the inclusion $Q \subset P$.

Easily, $\widetilde{M}$ is the total space of the fiber bundle over M with the standard fiber P/Q, the initial Cartan geometry factorizes to the Cartan geometry $(\mathcal{G} \to \widetilde{M}, \omega)$ of type (G, Q) and this construction is functorial. It is also clear that the initial Cartan geometry over M is flat if and only if the induced Cartan geometry over $\widetilde{M}$ flat.

$$\begin{tikzcd}
(\mathcal{G},\omega) \arrow[dr] \arrow[dd] & \\
& \widetilde{M} \arrow[dl] \\
M &
\end{tikzcd}$$

By its horizontality, the curvature κ of the Cartan connection ω vanishes under insertion of any vertical vector of the projection $\widetilde{M} \to M$. Under the identification $T\widetilde{M} \cong \mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$, the vertical subbundle is $V \cong \mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{q})$ and we highlight the previous condition as

$$\iota_v \kappa = 0, \qquad \text{for any } v \in \Gamma(V). \tag{1.10}$$

It is both interesting and important result that this condition locally characterizes Cartan geometries obtained by the correspondence space construction.

Remarks and outlook

Combinations of the previous constructions provide an efficient approach for constructing (or better understanding of already existing) relations between Cartan geometries of different types. Below in this text we meet several concrete realizations belonging to this framework.

$$\begin{tikzcd}
(\mathcal{G},\omega) \arrow[dr, "Q"] \arrow[dd, "P"'] & \text{correspondence} \\
& \widetilde{M} \arrow[dl] \\
M &
\end{tikzcd}
\qquad + \qquad
\begin{tikzcd}
& (\widetilde{\mathcal{G}},\widetilde{\omega}) \arrow[dd, "\widetilde{P}"] & \\
(\mathcal{G},\omega) \arrow[ur, hook] \arrow[dr, "Q"'] & & \text{extension} \\
& \widetilde{M} &
\end{tikzcd}$$

The correspondence space construction is induced by the inclusion $Q \subset P$, the Cartan extension by the pair of maps $i : Q \to \widetilde{P}$ and $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ satisfying the compatibility conditions listed on the previous page (but substituting Q in the place of P). Let us highlight several cases:

- *Correspondence spaces* themselves (no further extension) appear in sections 3.2, 3.3 and other places.
- *Mutations* are the constructions that correspond to i being an equality $Q = \widetilde{P}$ and α being a (Q-equivariant) linear isomorphism, see section 3.1.
- Constructions of *Fefferman type* are induced by an embedding $\phi : G \to \widetilde{G}$ of Lie groups so that $i = \phi|_Q$ and $\alpha = \phi'$, see sections 3.5 and 5.2.
- General Cartan extension is discussed in section 4.2.

2 Parabolic geometries

Parabolic geometries are Cartan geometries of type (G, P) where G is a semisimple Lie group and $P \subset G$ is its parabolic subgroup. This terminology reflects the one for corresponding Lie algebras: Lie algebra $\mathfrak{g}$ is *semisimple* if it has no non-zero solvable ideal, equivalently, it is a direct sum of simple Lie algebras, equivalently, the Killing form is non-degenerate (the Cartan's criterion), etc. Subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is *parabolic* if it contains the Borel subalgebra (the maximal solvable subalgebra of $\mathfrak{g}$) and the ortho-complement of $\mathfrak{p}$ in $\mathfrak{g}$ with respect to the Killing form is the nilradical of $\mathfrak{p}$.

Previous explanations does not sound too convincing in the front of geometric audience. To make a better impression, one usually adds that parabolic geometries involve such important and well studied structures as projective, conformal, CR contact and many others. These relations are get closer in section 3. Here we focus on several uniform features of parabolic geometries, among which the conceptual and canonical normalization condition is the prominent one. Also, when establishing the correspondence between concrete and more abstract notions, the relations to ordinary G-structures with subordinate compatible connections can be dealt in generality. Next topics to be introduced here are distinguished curves and BGG sequences.

Basics

In fact, parabolic subalgebras are related to gradings of semisimple Lie algebras, which is a more appropriate starting position. Let

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

be a $|k|$-grading of a semisimple Lie algebra $\mathfrak{g}$, i.e., $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ (where we consider $\mathfrak{g}_l = 0$, for $|l| > k$), and let

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^0 \supset \cdots \supset \mathfrak{g}^k = \mathfrak{g}_k$$

be the corresponding filtration, where $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. This turns $\mathfrak{g}$ into a filtered Lie algebra, i.e., $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$. Let the grading be such that the subalgebra $\mathfrak{p}_+ := \mathfrak{g}^1 = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is generated by $\mathfrak{g}_1$ or, equivalently, $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. Then $\mathfrak{p} := \mathfrak{g}^0 = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is a parabolic subalgebra with nilradical $\mathfrak{p}_+$.

Any parabolic subalgebra can be obtained this way, in particular, their classification corresponds to the classification of gradings of $\mathfrak{g}$. It follows that this is equivalent to distinguishing subsets of simple roots of $\mathfrak{g}$ (which is pictured by crossing nodes of the corresponding Satake diagram). The number of distinguished elements corresponds to the dimension of the center of the reductive subalgebra $\mathfrak{g}_0$.

Passing to the group level, there is a freedom in choosing a Lie group G and its subgroup P whose Lie algebras are $\mathfrak{g}$ and $\mathfrak{p}$, respectively. Fixing G, the extreme choices for $P \subset G$ are the normalizer of $\mathfrak{p}$ in G and its connected component of the identity. Under the restricted adjoint action, the subgroup P preserves the filtration of $\mathfrak{g}$. Given a parabolic subgroup $P \subset G$, let $G_0 \subset P$ be the subgroup preserving the gradation of $\mathfrak{g}$ (the Lie algebra of G_0 is obviously $\mathfrak{g}_0$). This inclusion induces an isomorphism $G_0 \cong P/P_+$, where P_+ denotes the nilpotent normal subgroup $\exp \mathfrak{p}_+$.

Note that the Killing form on $\mathfrak{g}$ provides a natural isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ compatible with the filtration, respectively gradation. This leads to a natural isomorphism of P-modules $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$, respectively G_0-modules $\mathfrak{p}_+ \cong \mathfrak{g}_-^*$, which we often use below without explicit mentioning. Note also that for many considerations that follow, it is an important assumption that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$. This is a vacuous condition for simple $\mathfrak{g}$ and we will not emphasize it below.

2.1 Underlying data, regularity and normalization

For a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), we now describe the induced underlying data on M. We comment the conditions under which this association can be made bijective: the regularity and the normality.

Underlying data, regularity

The subgroup $P_+ \subset P$ acts freely on $\mathcal{G}$, let us denote by $\mathcal{G}_0 := \mathcal{G}/P_+$ the orbit space. The fibration $\mathcal{G} \to \mathcal{G}_0$, respectively $\mathcal{G}_0 \to M$, is a principal bundle with structure group P_+, respectively G_0. The P-invariant filtration of $\mathfrak{g}$ gives rise to filtrations of the tangent bundles $T\mathcal{G}$, $T\mathcal{G}_0$, respectively TM (which are of length $2k+1$, $k+1$, respectively k). Finally, the Cartan connection ω descends to a couple of partially defined G_0-equivariant 1-forms $(\underline{\omega}_i)$ on $\mathcal{G}_0$ with values in $\mathfrak{g}_i$ (where $i = -k, \dots, -1$).

$$\begin{tikzcd} & (\mathcal{G}, \omega) \arrow[dl] \arrow[dd] \\ (\mathcal{G}_0, \underline{\omega}_i) \arrow[dr] & \\ & M \end{tikzcd}$$

In the case of trivial filtration, $k = 1$, we get the classical picture of G-structures on M, with the structure group $G_0 \subseteq GL(\mathfrak{g}_{-1})$, where $\underline{\omega}_{-1} : T\mathcal{G}_0 \to \mathfrak{g}_{-1}$ plays the role of the solder form. In general, there is an important notion of regularity tightening the relation between the filtration of TM and the reduction to G_0. The parabolic geometry (respectively its underlying structure) is called *regular* if the filtration of TM is compatible with the Lie bracket of vector fields and the corresponding Levi bracket

$$\mathcal{L} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM), \tag{2.1}$$

i.e., the induced algebraic bracket on the associated graded bundle, coincides with the one given by the Lie bracket in the Lie algebra $\mathfrak{g}_-$ via the identification $\mathrm{gr}(TM) \cong \mathcal{G}_0 \times_{G_0} \mathfrak{g}_-$. Hence the notion of *symbol algebra* etc.

The regularity condition may be equivalently formulated in terms of the curvature κ of the Cartan connection ω as follows. The filtration of $\mathfrak{g}$ gives rise also to a filtration of the adjoint tractor bundle $\mathcal{A}$, which is compatible with the algebraic bracket $\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and the corresponding objects on TM via the projection $\Pi : \mathcal{A} \to TM$. This leads to the notion of homogeneity degree of sections of $\Lambda^2 T^*M \otimes \mathcal{A}$ (and similar bundles). It follows that the parabolic geometry is regular if and only if its curvature κ is of positive homogeneity. Clearly, $|1|$-graded or torsion-free parabolic geometries are automatically regular.

Normalization

It is an important achievement that, up to two exceptional cases[2], the correspondence between (isomorphic classes of) regular parabolic geometries and induced underlying data can be made bijective imposing a normalization condition. The whole story involves an abstraction of the underlying data (the notion of infinitesimal flag structures), the construction of a bigger P-bundle with a Cartan connection of type (G, P) (via a choice of compatible principal G_0-connection and an application of a convenient extension functor discussed above) and a possible normalization of the resulting Cartan connection (to be discussed below). This process should be seen as a filtration adapted generalization of the standard prolongation procedure for G-structures. In the case of $|1|$-graded parabolic geometries, these can indeed be related so that $\mathfrak{g}_1 \cong \mathfrak{g}^*_{-1}$ coincides with the first prolongation of $\mathfrak{g}_0$ and its second prolongation vanishes.

Now we focus on the natural normalization condition. First of all, one has to find a reasonable description of the fact that two Cartan connections induce the same underlying data and, in such case, a description of the difference of their curvatures. This is done in terms of the difference map of the two connections (which is seen due to its horizontality as a 1-form $\Phi : TM \to \mathcal{A}$), and

[2]These are the projective and projective contact structures, in which cases a class of connections on the base manifold has to be added as a part of underlying data.

the bundle map $\mathrm{gr}(T^*M \otimes \mathcal{A}) \to \mathrm{gr}(\Lambda^2 T^*M \otimes \mathcal{A})$ induced by the standard differential of the chain complex

$$\cdots \longrightarrow \mathfrak{g}_-^* \otimes \mathfrak{g} \xrightarrow{\partial} \Lambda^2 \mathfrak{g}_-^* \otimes \mathfrak{g} \longrightarrow \ldots \tag{2.2}$$

computing the Lie algebra cohomology of $\mathfrak{g}_-$ with coefficients in $\mathfrak{g}$ (which is indeed G_0-equivariant).

A normalization condition may be given by a subbundle of $\Lambda^2 T^*M \otimes \mathcal{A}$, respectively by a P-invariant subspace of $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, whose associated graded is complementary to $\mathrm{im}\,\partial$. For any parabolic geometry, there is a canonical choice given by the kernel of a natural adjoint map, the *Kostant codifferential*, which is the codifferential of the complex

$$\cdots \longleftarrow \mathfrak{p}_+ \otimes \mathfrak{g} \xleftarrow{\partial^*} \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g} \longleftarrow \ldots . \tag{2.3}$$

This map is indeed P-equivariant etc.; the corresponding bundle map is denoted by the same symbol. The Cartan connection, respectively the parabolic geometry, is *normal* if its curvature is a section of $\ker \partial^* \subset \Lambda^2 T^*M \otimes \mathcal{A}$.

For a given Cartan connection, its potential normalization proceeds inductively according to the homogeneity degree. A rough sketch of the process (with concrete outcomes in the case of conformal structures) is described in section 5.2.

Harmonic curvature

For normal parabolic geometries, the composition of the curvature function κ with the quotient projection to the cohomology space $\ker \partial^* \to \ker \partial^* / \mathrm{im}\,\partial^*$ yields the so-called *harmonic curvature*, denoted by κ_H. In fact, P_+ acts trivially on this space, which means the corresponding associated bundle allows an interpretation in underlying terms. The adjointness of ∂ and ∂^* gives rise to the G_0-equivariant self-adjoint endomorphism

$$\Box := \partial \circ \partial^* + \partial^* \circ \partial,$$

the *Kostant Laplacian*. This determines a G_0-invariant Hodge decomposition of the chain complex, in particular, the kernel of this operator is isomorphic to the cohomology space in each degree.

For regular and normal parabolic geometries, it follows that the lowest non-zero homogeneous component of κ has values in $\ker \Box$, i.e., it coincides with the corresponding homogeneous component of κ_H. In particular, κ vanishes identically if and only if κ_H does. In fact, the full curvature can be recovered from the harmonic one via the BGG splitting operator, see section 2.4. Finally, note that $\ker \Box$ is algorithmically computable as a $\mathfrak{g}_0$-representation and often consists only of few components, see [59].

Previous results allow to reduce many treatments involving curvature just to its harmonic counterpart. This is also behind number of conclusions discussed below.

2.2 Weyl structures

Weyl structures provide a useful tool in transition between the two descriptions of parabolic geometries (Cartan geometries vs. underlying data). In particular, they induce a canonical class of compatible affine connections on the base manifold. All notions that follow have well known preimages in conformal geometry, from which also a lot of terminology is taken.

Reductions, splittings, connections

Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type G/P and let $\mathcal{G}_0 = \mathcal{G}/P_+ \to M$ be the underlying principal G_0-bundle as above. A *Weyl structure* is a global G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the P_+-projection $\mathcal{G} \to \mathcal{G}_0$, i.e., a reduction of the P-principal bundle $\mathcal{G} \to M$ to the subgroup $G_0 \subset P$. The pull-back $\sigma^*\omega : T\mathcal{G}_0 \to \mathfrak{g}$ of the Cartan connection decomposes into G_0-equivariant pieces in accord with the grading of $\mathfrak{g}$. The negative component $\sigma^*\omega_- : T\mathcal{G}_0 \to \mathfrak{g}_-$ provides an identification of the filtered tangent bundle TM with the associated graded $\mathrm{gr}(TM)$, hence a splitting of the filtration of TM. The zeroth component $\sigma^*\omega_0 : T\mathcal{G}_0 \to \mathfrak{g}_0$ defines a principal G_0-connection, hence also an induced connection on any associated bundle; they are jointly called the *Weyl connections*. The positive component $\sigma^*\omega_+ : T\mathcal{G}_0 \to \mathfrak{p}_+$ can be interpreted as a 1-form $TM \to T^*M$, respectively a section of $T^*M \otimes T^*M$, and is called the *Rho* or *Schouten tensor*.

$$\begin{tikzcd}
 & (\mathcal{G},\omega) \arrow[dl] \arrow[d] \\
(\mathcal{G}_0,\sigma^*\omega_j) \arrow[ur, dashed, bend left, "\sigma"] \arrow[r] & M
\end{tikzcd}$$

The splitting of the filtration of the tangent, respectively cotangent, bundle caused by a choice of Weyl structure applies similarly to any associated bundle corresponding to a P-representation V that is completely reducible as a G_0-representation. There is a P-invariant filtration of V with the property that the action of P_+ shifts each subspace to the next smaller one. This yields a filtration of $\mathcal{V} = \mathcal{G} \times_P V$,

$$\mathcal{V} \supset \mathcal{V}^0 \supset \mathcal{V}^1 \supset \dots, \tag{2.4}$$

and a choice of a Weyl structure provides an identification of $\mathcal{V}$ with its associated graded $\mathrm{gr}(\mathcal{V}) \cong \mathcal{G}_0 \times_{G_0} V$. The filtration of $\mathcal{V}$ may be described directly via a natural bundle map $\bullet : T^*M \times \mathcal{V} \to \mathcal{V}$ induced by the derived action of $\mathfrak{p}_+$ on V and the identification $T^*M \cong \mathcal{G} \times_P \mathfrak{p}_+$.

In the case when $\mathcal{V}$ is a tractor bundle, this map is the restriction of the characteristic map $\bullet : \mathcal{A} \times \mathcal{V} \to \mathcal{V}$ (as in (1.7)) under the natural inclusion $T^*M \subset \mathcal{A}$ (given by $\mathfrak{p}_+ \subset \mathfrak{g}$), which is independent of a choice of Weyl structure. On the other hand, since $\mathfrak{g}_- \subset \mathfrak{g}$ as a G_0-module, a choice of Weyl structure yields an inclusion of bundles $TM \subset \mathcal{A}$, respectively their associated graded. In this context we can express the tractor connection ∇, according to (1.6), as follows. For a Weyl structure σ, let D and P be the corresponding Weyl connection on $\mathcal{V}$ and the Rho tensor, respectively. A comparison of the fundamental derivative D with these objects yields that

$$\nabla_\xi t = D_\xi t + \mathsf{P}(\xi) \bullet t + \xi \bullet t, \tag{2.5}$$

for each $\xi \in \Gamma(TM)$ and $t \in \Gamma(\mathcal{V})$. This can further be expanded with respect to the induced splitting of $\mathcal{A}$ and $\mathcal{V}$, which yields the concrete formulas that appear below.

Although the constituents of (2.5) depend on a choice of Weyl structure, the resulting tractor connection does not. In the just indicated approach this follows for free, while a direct check of this fact can be a rather tedious exercise. This would involve the knowledge of the transformation rules of respective objects according to a change of Weyl structure.

Changes of Weyl structures

Two Weyl structures σ and $\hat\sigma$ differ by a G_0-equivariant map $\Upsilon : \mathcal{G}_0 \to \mathfrak{p}_+$ so that

$$\hat\sigma(u) = r_{\exp \Upsilon(u)}(\sigma(u)), \tag{2.6}$$

for each $u \in \mathcal{G}_0$. Such Υ corresponds to a 1-form on M, more precisely, to a section of the associated graded bundle $\mathrm{gr}(T^*M)$. Weyl structures thus form an affine bundle over M modeled on this vector bundle. The *bundle of Weyl structures* can also be described as $\mathcal{W} := \mathcal{G} \times_P P/G_0 \to M$; indeed, sections of this bundle are in bijective correspondence with Weyl structures $\sigma : \mathcal{G}_0 \to \mathcal{G}$. The Cartan connection on $\mathcal{G}$ induces a surprisingly rich geometric structure on the total space of $\mathcal{W}$, which is discovered and employed only recently, see [45, 14].

The effect of a change of Weyl structure on various associated quantities can be described in terms of corresponding 1-forms. Below we meet explicit formulas only for $|1|$-graded geometries, in which cases the uniform background is as follows. For Weyl structures σ and $\hat\sigma$ differing by $\Upsilon \in \Gamma(T^*M)$ as in (2.6), let $(v_0, v_1, v_2, \dots)$ and $(\hat v_0, \hat v_1, \hat v_2, \dots)$ be the corresponding splittings of an element of a natural bundle $\mathcal{V} = \mathcal{G} \times_P V$, labeled so that the zeroth gradation component is the projecting part, i.e., a quotient of the whole $\mathcal{V}$, cf. (2.4). Then, expanding the action of $\exp\Upsilon$, it easily follows that

$$\hat{v}_0 = v_0, \quad \hat{v}_1 = v_1 - \Upsilon \bullet v_0, \quad \hat{v}_2 = v_2 - \Upsilon \bullet v_1 + \tfrac{1}{2}\Upsilon \bullet \Upsilon \bullet v_0, \quad \dots . \tag{2.7}$$

Further, the corresponding Weyl connections D and $\hat{D}$ on $\mathcal{V}$ are related as

$$\hat{D}_\xi v = D_\xi v - \{\Upsilon, \xi\} \bullet v \tag{2.8}$$

and the corresponding Rho tensors P and $\hat{\mathsf{P}}$ differ as

$$\hat{\mathsf{P}}(\xi) = \mathsf{P}(\xi) + D_\xi \Upsilon + \tfrac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\}, \tag{2.9}$$

where $\xi \in \Gamma(TM)$ and $v \in \Gamma(\mathcal{V})$.

Scales

Scales are sections of scale bundles, which are oriented line bundles convenient to a given parabolic geometry. The point is that sections of a scale bundle determine special Weyl structures, called *exact*, whereas general Weyl structures correspond to principal connections on them. Note that the terminology reflects the fact that the 1-form Υ measuring the difference between two exact Weyl structures is exact.

As associated bundles to $\mathcal{G}_0 \to M$, scale bundles are given by a representation $G_0 \to \mathbb{R}_+$. As such they can be classified by certain elements from the center of $\mathfrak{g}_0$ (describing the derived representation). Any scale bundle is identified with the quotient bundle $\mathcal{G}_0/G_0' \to M$, where G_0' is the kernel of the defining representation. In particular, sections of this bundle correspond to reductions of the principal G_0-bundle $\mathcal{G}_0 \to M$ to the structure group G_0'. Thus, exact Weyl connections have reduced holonomies, corresponding to the parallelity of the geometric quantity related to a choice of scale.

For $|1|$-graded geometries, the center of $\mathfrak{g}_0$ is 1-dimensional (consisting of multiples of the grading element), hence G_0' is just the semisimple part of the reductive group G_0.

Curvature invariants

Likewise to the decomposition of the pullback of the Cartan connection with respect to a Weyl structure, one may decompose the pullback of its curvature. These data allow an interpretation of the Cartan, respectively harmonic, curvature in underlying terms as well as an alternative treatment of parabolic geometries themselves.

Bellow we need some details only for $|1|$-graded geometries, in which case the general discussion simplifies considerably as follows. For a normal Cartan connection ω and a Weyl structure σ, let D be the corresponding Weyl connection on TM and let R be its curvature tensor. Let κ be the curvature of ω and let $\sigma^*\kappa =: T + W + Y$ be the decomposition of its pullback to $\mathcal{G}_0$ according to the grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. The individual components, interpreted as 2-forms on M with values in TM, $\mathrm{End}_0(TM)$ and T^*M, are the *torsion*, *Weyl curvature* and *Cotton–York tensor* of D, respectively. It follows the last two tensors may be expressed in previous terms as

$$W = R + \partial\mathsf{P}, \quad Y = d^D\mathsf{P},$$

where ∂ is the bundle map induced by the differential (2.2) and d^D is the covariant exterior derivative determined by D. Note that the first non-zero tensor in the sequence (T, W, Y) is an invariant of the structure, i.e., is independent of the choice of Weyl structure, as it corresponds to the lowest non-zero homogeneous component of the harmonic curvature κ_H.

2.3 Distinguished curves

As for any Cartan geometry $(\mathcal{G} \to M, \omega)$, the notion of development of curves described in section 1.2 allows to distinguish (families of) curves on M using their counterparts in the corresponding homogeneous space G/P. The prominent curves there are the exponential curves, respectively orbits of 1-parameter subgroups of G. Within the framework of parabolic geometries we usually restrict to such curves whose generators lie in $\mathfrak{g}_- \subset \mathfrak{g}$. The corresponding curves on M are called the *distinguished curves*.[3]

Besides the development, there are other ways to distinguished curves in parabolic geometry. For instance, for a curve $c : I \to M$, the following properties are equivalent:

- The curve c admits a development into a curve $L(t) = \exp(tX)/P$ in G/P, where $X \in \mathfrak{g}_-$.
- The curve c admits a lift $\hat{c} : I \to \mathcal{G}$ such that $\omega(\hat{c}'(t)) \in \mathrm{sym}_L$, for all t, where $\mathrm{sym}_L \subset \mathfrak{g}$ is the symmetry algebra of L.
- The curve c admits a lift $\hat{c} : I \to \mathcal{G}$ such that $\omega(\hat{c}'(t)) = X$, for all t.
- The curve c admits a lift $\hat{c} : I \to \mathcal{A}$ such that $\mathrm{D}_{\hat{c}} c' = 0$, where D is the fundamental derivative.
- There is a Weyl structure such that $D_{c'} c' = 0$ and $\mathsf{P}(c') = 0$, where D and P is the corresponding Weyl connection and Rho tensor, respectively.

In the second condition, the symmetry algebra $\mathrm{sym}_L \subset \mathfrak{g}$ consists of those infinitesimal automorphisms of G/P that are everywhere tangent to L. The third condition can be rephrased so that the curve is the projection to M of the flow of a constant vector field in $\mathcal{G}$. In the fourth condition, the lift to the adjoint tractor bundle $\mathcal{A}$ is the same as a P-invariant vector field in $\mathcal{G}$ over the curve, hence a lift to $\mathcal{G}$ whose tangent field is extended as required. The relation to the last condition follows from relations between D, P and D as is behind (2.5).

Remarks

It is worth emphasizing that, for different distinguished curves with the same trajectory, the freedom in possible reparametrizations is not arbitrary. It turns out that the only eventualities are either projective of affine ones.

For a finer discussion of possible types of distinguished curves and their properties, we further restrict the set of possible generators in $\mathfrak{g}_- \subset \mathfrak{g}$. It follows that these types can be classified according to G_0-invariant subsets $A \subset \mathfrak{g}_-$. Different subsets contained in a common P-hull correspond to different types of distinguished curves sharing the same tangent vectors (this may happen only for more graded parabolic geometries).

Among other approaches to distinguished curves, let us mention the one from [45] employing a natural connection on the bundle of Weyl structures. Below in section 6 we also formulate other characterizations of distinguished curves in conformal setting (where they are named *conformal circles*) using tractor calculus. Some of those ideas are easily, but not uniformly, transformable to other parabolic geometries, cf. section 6.4.

2.4 BGG sequences

BGG sequences form a really powerful instrument of parabolic geometry. Many important invariant operators, equations, respectively problems, can be interpreted in this framework. The general theory was introduced by Čap, Slovák and Souček in [16], later simplified by Calderbank and Diemer in [9]. The sequences are intrinsically constructed for any tractor bundle of given parabolic geometry. The key player is the so-called splitting operator, let us introduce the basics.

[3]Even in this generality, distinguished curves can be found under various nicknames in the literature: *canonical curves*, *parabolic geodesics*, *generalized geodesics*, *Cartan circles* and the like.

For a given tractor bundle $\mathcal{V} = \mathcal{G} \times_P V$, the starting ingredient is the exterior covariant derivative given by the normal tractor connection ∇,

$$0 \longrightarrow \Gamma(\mathcal{V}) \xrightarrow{\mathrm{d}^\nabla} \Gamma(T^*M \otimes \mathcal{V}) \xrightarrow{\mathrm{d}^\nabla} \Gamma(\Lambda^2 T^*M \otimes \mathcal{V}) \xrightarrow{\mathrm{d}^\nabla} \dots$$

In the opposite direction it operates ∂^*, the bundle map given by the Kostant codifferential as in (2.3), where only the coefficients are now taken in the $\mathfrak{g}$-module V rather than in $\mathfrak{g}$. This defines the cohomology bundles $\mathcal{H}_i := \ker \partial^* / \operatorname{im} \partial^*$ and projections $\Pi_i : \ker \partial^* \to \mathcal{H}_i$, both labeled by the corresponding form degree. The crucial fact is the existence of a unique differential operator $L_i : \Gamma(\mathcal{H}_i) \to \Gamma(\ker \partial^*)$, for any i, splitting the projection Π_i and satisfying

$$\partial^*(\mathrm{d}^\nabla(L_i(\tau))) = 0,$$

for all $\tau \in \Gamma(\mathcal{H}_i)$. These are the *splitting operators*. The displayed property allows the composition $\Theta_i = \Pi_{i+1} \circ \mathrm{d}^\nabla \circ L_i$, which yields the sequence of *BGG operators*,

$$0 \longrightarrow \Gamma(\mathcal{H}_0) \xrightarrow{\Theta_0} \Gamma(\mathcal{H}_1) \xrightarrow{\Theta_1} \Gamma(\mathcal{H}_2) \xrightarrow{\Theta_2} \dots$$

Note that his sequence forms a complex if and only if the parabolic geometry is flat.

Some applications and remarks

One of many use of BGG techniques is the reconstruction of the curvature $\kappa \in \Gamma(\Lambda^2 T^*M \otimes \mathcal{V})$ of normal Cartan connection from its harmonic counterpart $\kappa_H = \Pi_2(\kappa)$. As a consequence of the differential Bianchi identity, it follows that $\kappa = L_2(\kappa_H)$.

Another noteworthy application concerns the characterization of correspondence spaces discussed in section 1.3, which allows a much stronger formulation. With the notation as before, it follows that the condition

$$\iota_v \kappa_H = 0, \qquad \text{for any } v \in \Gamma(V), \tag{2.10}$$

implies the one in (1.10), hence a local characterization of Cartan connection obtained by the correspondence space construction. This was shown by Čap in [10].

Below we deal with first splitting, respectively first BGG, operators. Note that the first cohomology bundle $\mathcal{H}_0$ is simply the quotient $\mathcal{V}/\mathcal{V}^0$, where $\mathcal{V}^0$ is the largest filtration component in $\mathcal{V}$, cf. (2.4). In particular, the first splitting operator maps sections of the projecting part of $\mathrm{gr}(\mathcal{V})$ to sections of $\mathcal{V}$.

3 Concrete geometries

In this section we discuss mostly those structures and constructions that appear also in chapter II. It is a preparatory section, but it was not meant to be just a collection of conventions and notations. References to the previous text should indicate how to see this material within the given context. Besides concrete incarnations of the general constructions above, we are going to introduce two ambient constructions for projective and conformal structures. Except the structures in section 3.1, all others belong among parabolic geometries.

3.1 Reductive geometries, mutations etc.

Riemannian and affine geometries are not parabolic geometries, but they are particular examples of so-called reductive geometries, which form a subclass of Cartan geometries. It is worth reminding some general features of these geometries.

Reductive geometries

Reductive geometry is a Cartan geometry of type (G, H) such that the corresponding Lie algebra admits an H-invariant decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ (with respect to the adjoint action).[4] This is a very strong condition with many restrictive implications.

The Cartan connection $\omega : T\mathcal{G} \to \mathfrak{g}$ of a reductive Cartan geometry splits according to the decomposition of $\mathfrak{g}$ into well defined H-equivariant pieces $\theta : T\mathcal{G} \to \mathfrak{n}$ and $\gamma : T\mathcal{G} \to \mathfrak{h}$. The solder form θ provides an identification $TM \cong \mathcal{G} \times_H \mathfrak{n}$. If, in addition, the homomorphism $H \to GL(\mathfrak{n})$ induced by the adjoint action is (infinitesimally) injective, then we get an identification of $\mathcal{G}$ with a (covering of) a subbundle of the first order frame bundle of M, i.e., an ordinary G-structure with structure group H. The form γ is the principal connection, which induces structure adapted connections on all bundles associated to $\mathcal{G}$. These induced connections coincide with the restriction of the fundamental derivative (1.5), where the restriction (of the first argument from $\mathcal{A}$ to TM) indeed makes sense only in reductive cases (when TM is a subbundle of $\mathcal{A}$).

The reductivity influences also the behavior of distinguished curves. Considering the curves corresponding to $\mathfrak{n} \subset \mathfrak{g}$, it easily follows that all of them seem like ordinary geodesics: as unparametrized curves, they are uniquely given by a tangent direction in one point and each of them admits an affine class of distinguished parametrizations.

In the discussion above, we met reductive geometries in section 2.2: any Weyl structure σ of a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) induces a reductive Cartan geometry $(\mathcal{G}_0 \to M, \sigma^*(\omega_- \oplus \omega_0))$ of type (P^{op}, G_0), where $P^{op} \subset G$ is the opposite parabolic subalgebra to P. In particular, $P^{op} \cap P = G_0$ and the corresponding Lie algebra is $\mathfrak{p}^{op} = \mathfrak{g}_- \oplus \mathfrak{g}_0$, which is a G_0-invariant decomposition. Two particular examples, which might be seen in this scheme, follow.

Affine structures

An *affine structure* on a smooth manifold M is given by an affine connection, i.e., a linear connection on TM, which brings the notion of parallel displacement of vectors along curves. The model is the affine space $\mathbb{R}^m$ with the absolute parallelism given by the translations. This is the homogeneous space of the group of affine motions, the semidirect product of the abelian group of translations and the general linear group, $Aff(m, \mathbb{R}) = \mathbb{R}^m \ltimes GL(m, \mathbb{R})$. Under the identification $\mathbb{R}^m \ni X \mapsto \left(\begin{smallmatrix} 1 \\ X \end{smallmatrix}\right) \in \mathbb{R}^{m+1}$, the action of this group corresponds to the standard action of

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ Y & A \end{pmatrix} : Y \in \mathbb{R}^m,\ A \in GL(m, \mathbb{R}) \right\} \subset GL(m+1, \mathbb{R}). \tag{3.1}$$

The stabilizer of the origin in $\mathbb{R}^m$, which corresponds to the first vector from the standard basis of $\mathbb{R}^{m+1}$, is $H = GL(n, \mathbb{R})$, which is given by $Y = 0$ in the previous description.

The natural decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, with $\mathfrak{n} = \mathbb{R}^m$ and $\mathfrak{h} = \mathfrak{gl}(m, \mathbb{R})$ in accord with the block decomposition above, is H-invariant. Thus the Cartan geometry is reductive, but the corresponding G-structure is the full frame bundle (no reduction) and the only underlying object is just the affine connection D. Since $\mathfrak{n} \subset \mathfrak{g}$ is abelian, the torsion and the curvature of D corresponds to

$$T = \mathrm{d}\theta + [\gamma, \theta] \quad \text{and} \quad R = \mathrm{d}\gamma + [\gamma, \gamma],$$

respectively, where $\omega = \theta \oplus \gamma$ is the decomposition of the Cartan connection as above.

Riemannian structures

A *Riemannian structure* on a manifold M is given by a Riemannian metric $g \in \Gamma(S^2 T^* M)$, which brings the notion of distance and parallel displacement provided by the Levi-Civita connection, the unique metric connection with vanishing torsion. The model is the Euclidean space, i.e., the affine space $\mathbb{R}^m$ with the standard inner product. This is the homogeneous space of the group of rigid motions, the semidirect product of translations and the orthogonal group preserving the

[4] Warning: this notion has nothing to do with the reductivity of the Lie algebra $\mathfrak{g}$ in algebraic sense.

inner product, $Euc(m) = \mathbb{R}^m \ltimes O(m)$, where $H = O(m)$ is that stabilizer of the origin. According to (3.1), the group $Euc(m) \subset Aff(m,\mathbb{R})$ is described by restricting the values of A to $O(m)$.

The current Cartan geometry is reductive, the corresponding G-structure is the reduction to the orthonormal frame bundle and the Levi-Civita connection corresponds to the unique 'normal' Cartan connection: the normalization condition in this case is trivial (vanishing torsion) and the fact that the first prolongation of $\mathfrak{h} = \mathfrak{o}(m)$ is also trivial pins down the connection uniquely.

A transition from positive definite metrics to indefinite ones is reflected by a transition from orthogonal structure groups $O(m)$ to pseudo-orthogonal ones $O(p,q)$, where $p+q=m$, with the model space being the pseudo-Euclidean affine space of signature (p,q).

Mutations

As we know from the very definition of Cartan geometry of type (G,H), the crucial role is played by the structure group H and the Lie algebra $\mathfrak{g}$ of the principal group G, whereas the group itself enters mainly the model description, respectively extensions. In fact, in the definition of the Cartan connection just the H-module structure of $\mathfrak{g}$ is addressed, its Lie algebraic structure enters the curvature.

In this sense, Riemannian structures may be viewed as Cartan geometries modeled over any space form and not just the Euclidean space $\mathbb{R}^m \cong Euc(m)/O(m)$. Indeed, the round sphere, respectively the hyperbolic space, is the homogeneous space $S^m \cong O(m+1)/O(m)$, respectively $H^m \cong O(m,1)/O(m)$, and $\mathfrak{euc}(m,\mathbb{R})$, $\mathfrak{o}(m+1)$ and $\mathfrak{o}(m,1)$ are mutually isomorphic as $O(m)$-modules, but not as Lie algebras. With respect to their standard block realizations, the isomorphisms $\mathfrak{o}(m+1) \leftarrow \mathfrak{euc}(m,\mathbb{R}) \to \mathfrak{o}(m+1,1)$ are given as

$$\begin{pmatrix} 0 & -kY^t \\ kY & A \end{pmatrix} \hookleftarrow \begin{pmatrix} 0 & 0 \\ Y & A \end{pmatrix} \mapsto \begin{pmatrix} 0 & lY^t \\ lY & A \end{pmatrix},$$

where $k,l \in \mathbb{R}$ are arbitrary constants.

These are concrete examples of model *mutations* leading to a functorial construction described in section 1.3. The curvature of a mutated Cartan geometry is controlled by the (constant) contribution (1.8) so that (1.9) holds.

3.2 Projective geometry and Thomas cone

Projective geometries are one of the two exceptional types of parabolic geometries, for which the underlying data described in section 2.1 are vacuous and the structure is rather given by a class of affine connections.

Projective structures

A *projective structure* on a manifold M is given by a class of torsion-free projectively equivalent affine connections, i.e., connections having the same unparametrized geodesics. The model is the real projective space $\mathbb{RP}^m$, the projectivization of $\mathbb{R}^{m+1}$, with projective lines as the distinguished family of curves. This is the homogeneous space of the group of collineations, $Proj(m,\mathbb{R})$, which coincides with the projectivized general linear group $G = PGL(m+1,\mathbb{R})$, the quotient of $GL(m+1,\mathbb{R})$ by its center. The stabilizer of the origin in $\mathbb{RP}^m$, respectively the first vector from the standard basis of $\mathbb{R}^{m+1}$, is the subgroup $P \subset G$ represented by block upper-triangular matrices according to the block decomposition indicated in (3.1).

The corresponding Lie algebras are simple $\mathfrak{g} = \mathfrak{sl}(m+1,\mathbb{R})$ and parabolic $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with the corresponding grading indicated in the block form as

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix},$$

with blocks of sizes 1 and m. In particular, $\mathfrak{g}$ is $|1|$-graded and we identify $\mathfrak{g}_{-1} \cong \mathbb{R}^m$, $\mathfrak{g}_0 \cong \mathfrak{gl}(m,\mathbb{R})$ and $\mathfrak{g}_1 \cong \mathbb{R}^{m*}$. The semisimple part of $\mathfrak{g}_0$ is $\mathfrak{g}_0' \cong \mathfrak{sl}(m,\mathbb{R})$.

The underlying structure of the corresponding Cartan geometry consists just of family of Weyl connections (there is no reduction of the principal frame bundle since the Levi factor G_0 coincides with $GL(m,\mathbb{R})$). The general formula (2.8) for the change of Weyl connection according to a 1-form Υ expands in this case to

$$\hat{D}_\xi \eta = D_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi,$$

for $\xi, \eta \in \Gamma(TM)$, which indeed recovers the well known description of projectively equivalent connections. Moreover all these connections have the same torsion, which vanishes in the normal case. The first (and only) non-trivial harmonic curvature component is the Cotton–York tensor, for $m = 2$, respectively the Weyl tensor, for $m > 2$. These are the obstructions to the flatness of the projective structure.

Thomas cone

Another choice of principal Lie group to the Lie algebra $\mathfrak{g} = \mathfrak{sl}(m{+}1, \mathbb{R})$ clearly is $G = SL(m{+}1, \mathbb{R})$. This group acts transitively not only on lines, but also on rays in $\mathbb{R}^{m+1}$. Hence both the projective space $\mathbb{RP}^m$ and the sphere S^m, its 2-fold cover, can be considered as the homogeneous spaces of $SL(m{+}1, \mathbb{R})$ (only the action on $\mathbb{RP}^m$ is not effective). In the latter case, the family of distinguished curves is formed by great circles. The corresponding Cartan geometry modeled on S^n defines the *oriented* projective structures, i.e., projective structures on oriented manifolds.

Great circles on S^m are geodesics of the standard round metric, whose Levi-Civita connection defines the projective class of connections. Among these connections, there is a subclass of *special* ones, each of which preserves a volume form (these are the exact Weyl connections in the terminology of section 2.2). Volume forms on S^m correspond to sections of the canonical $\mathbb{R}_+$-bundle $q : \mathbb{R}^{m+1} \setminus \{0\} \to S^m$. It is an inspiring observation that any special connection on S^m can be described as a 'pull-back' of the flat ambient connection on $\mathbb{R}^{m+1}$ with respect to the corresponding section of q. This is a (simplified) model interpretation of the cone, respectively ambient, description of projective structures, which allows a straight generalization. Before doing so, let us denote $P \subset G$ the stabilizer of a ray in $\mathbb{R}^{m+1}$ and $Q \subset P$ the stabilizer of its generating vector. Thus, $S^m \cong G/P$, $\mathbb{R}^{m+1} \setminus \{0\} \cong G/Q$ and $P/Q \cong \mathbb{R}_+$.

Let $(M, [D])$ be an oriented projective structure, let $(\mathcal{G} \to M, \omega)$ be the corresponding normal Cartan geometry of type (G, P) and let $\widetilde{M} := \mathcal{G}/Q$ be the correspondence space given by the inclusion $Q \subset P$. As usually, the tangent bundle is identified as $T\widetilde{M} \cong \mathcal{G} \times_Q (\mathfrak{g}/\mathfrak{q})$, where $\mathfrak{q}$ is the Lie algebra of Q. As unusually, the Q-module $\mathfrak{g}/\mathfrak{q}$ is isomorphic to the restriction of the standard representation of $G = SL(m+1, \mathbb{R})$ on $\mathbb{R}^{m+1}$. Hence $T\widetilde{M}$ is a tractor bundle over $\widetilde{M}$ and as such it carries an induced linear connection $\widetilde{\nabla}$, which is called the *Thomas* or *ambient connection*. This connection has many characteristic properties: easily, it is torsion-free and preserves the canonical volume form (given by the standard G-invariant volume on $\mathbb{R}^{m+1}$). It is also compatible with the principal $\mathbb{R}_+$-action on $\widetilde{M}$ and any section of the projection $q : \widetilde{M} \to M$ determines a special connection from the projective class $[D]$. Finally, it is Ricci flat, which is a consequence of the normality of ω.

$$\begin{tikzcd}
(\mathcal{G}, \omega) \arrow[dd] \arrow[dr] & & \\
& (\widetilde{M}, \widetilde{\nabla}) \arrow[dl] & \text{Thomas} \\
(M, [D]) & \text{projective} &
\end{tikzcd}$$

It follows that the initial projective structure on M is fully encoded in the Thomas ambient connection on $\widetilde{M}$. Concrete coordinate expressions and direct relations of corresponding curvature quantities can be found in Thomas' original work, cf. [64, chapter III]. Let us notice that this is the place where the so-called *Thomas projective parameters*, the projectively invariant bunch of functions with suspicious transformation laws, appear. For later reference, they are given by

$$\Pi_A{}^C{}_B := \Gamma_A{}^C{}_B - \frac{2}{m+1}\,\delta^C_{(A}\Gamma_{B)}{}^D{}_D\,, \tag{3.2}$$

where $\Gamma_A{}^C{}_B$ are the Christoffel symbols of any connection from the projective class.

Relation to standard tractors

Since $\mathfrak{g}/\mathfrak{q}$ is identified with the standard representation of the principal group G, there is a close relation between the tangent bundle $T\widetilde{M} \to \widetilde{M}$ with its ambient connection $\widetilde{\nabla}$ and the standard tractor bundle $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{m+1} \to M$ with its tractor connection ∇. It follows that the latter data can be obtained by factorizing the former ones with respect to the $\mathbb{R}_+$-action (which naturally extends to $T\widetilde{M}$). In particular, sections of the standard tractor bundle $\mathcal{T}$ are in bijective correspondence with vector fields on $\widetilde{M}$ which are of homogeneity -1 with respect to the $\mathbb{R}_+$-action and the normality of ∇ is equivalent to the Ricci flatness of $\widetilde{\nabla}$.

The standard tractor bundle is filtered, $\mathcal{T} \supset \mathcal{T}^0$, so that $\mathcal{T}^0$ is the tautological line bundle, which is conventionally labeled as $\mathcal{E}(-1)$, the density bundle of projective weight -1. Under the previous identification, $\mathcal{T}^0$ corresponds to the vertical subbundle of the cone projection $\widetilde{M} \to M$. With respect to a choice of Weyl structure, the standard tractor bundle splits as

$$\mathcal{T} = TM(-1) \oplus \mathcal{E}(-1),$$

where $TM(-1) = TM \otimes \mathcal{E}(-1)$ corresponds to the projecting part $\mathcal{T}/\mathcal{T}^0$.

There are many variants of cone constructions, but we do not know any direct analogy to the previous one among $|1|$-graded parabolic geometries. Very similar approach, however, applies to projective contact structures, see [38]. Note that the ambient construction for conformal structures (discussed in sections 3.4 and 5.3) is significantly more complicated, though the model observations are of similar nature.

3.3 Flag varieties and correspondence spaces

Projective, Grassmannian and Lagrangean contact manifolds are particular examples of parabolic geometries whose homogeneous models are flag varieties, i.e., algebraic varieties of linear subspaces of particular type in a real vector space. In fact, all parabolic homogeneous spaces G/P can be realized (after a possible complexification) this way, which is why they are sometimes called generalized flag varieties. Previously mentioned examples correspond to the most ordinary flags, with simple Lie algebras $\mathfrak{g} = \mathfrak{sl}(-, \mathbb{R})$ in the background. In this context we just add some details.

Real projective space $\mathbb{RP}^m$ is the space of 1-dimensional subspaces in vector space $\mathbb{R}^{m+1}$. It is the homogeneous space of $G = PGL(m+1, \mathbb{R})$, respectively $SL(m+1, \mathbb{R})$, with $P \subset G$ the stabilizer of a line. This is an example of Grassmannian, respectively flag, variety of type $(1, m+1)$. General Grassmannian of type (p, N) is the variety of p-dimensional subspaces in vector space $\mathbb{R}^N$. General flag variety of type $(p_1, p_2, \ldots, N)$ is the variety of flags of subspaces $V^1 \subset V^2 \subset \cdots \subset \mathbb{R}^N$ of dimensions $p_1 < p_2 < \cdots < N$. It is a homogeneous spaces with G as above and $P \subset G$ the stabilizer of a particular flag.

Mapping any flag $V^1 \subset V^2 \subset \cdots \subset \mathbb{R}^N$ to the subspace of a particular dimension (say V^i) yields a fibration of flag manifolds (whose fiber over V^i consists of all flags of the current type containing V^i). As all these manifolds are homogeneous spaces of the same Lie group, the previous fibrations are model examples of correspondences discussed in section 1.3. In particular, for flags of depths two, we have two such fibrations:

$$\begin{tikzcd}
 & G = PGL(N, \mathbb{R}) \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \text{Flags } (p_1, p_2, N) \arrow[dl] \arrow[dr] & \\
\text{Flags } (p_1, N) & & \text{Flags } (p_2, N)
\end{tikzcd}$$

Lagrangean contact structures

Let us consider parabolic geometries modelled on flag varieties of lines in hyperplanes, i.e., of type $(1, N-1, N)$. Counting dimensions, one observes that the base manifold is odd dimensional, say $m = 2n+1$, in which case $N = n+2$. Let $P \subset G$ be the stabilizer of the flag determined by the first and the first $n-1$ vectors from the standard basis. The corresponding grading of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$ is indicated in the following block form

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^L & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^L & \mathfrak{g}_0 & \mathfrak{g}_1^R \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^R & \mathfrak{g}_0 \end{pmatrix}$$

with blocks of sizes 1, n and 1. In particular, $\mathfrak{g}$ is $|2|$-graded with $\mathfrak{g}_{-2} \cong \mathbb{R}$, $\mathfrak{g}_{-1}^L \cong \mathbb{R}^n$, $\mathfrak{g}_{-1}^R \cong \mathbb{R}^{n*}$ etc. The decomposition $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^L \oplus \mathfrak{g}_{-1}^R$ is $\mathfrak{g}_0$-invariant, the Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non-degenerate and, with respect to this map, the subspaces $\mathfrak{g}_{-1}^L$ and $\mathfrak{g}_{-1}^R$ are isotropic (aka Lagrangean, if we interpret the bracket as a symplectic form on $\mathfrak{g}_{-1}$). In particular, we have a contact grading of $\mathfrak{g}$.

Lagrangean contact structure is the structure underlying the current parabolic geometry. On a manifold M of dimension $m = 2n+1$ it consists of a contact distribution $H \subset TM$ and a decomposition $H = L \oplus R$ into two Lagrangean subspaces with respect to the Levi form $\mathcal{L} : H \times H \to TM/H$, the only non-trivial part of the Levi bracket (2.1).

Normal parabolic geometries of this type are always regular, the harmonic curvature decomposes according to the dimension of M as follows. For $m = 3$, there are two components, both of homogeneity 4; in particular, the geometry is torsion-free. An interpretation of these object in underlying terms yields two tensors of Cotton–York type, namely,

$$C_L \in \Gamma(L^* \wedge (TM/H)^* \otimes L^*), \quad C_R \in \Gamma(R^* \wedge (TM/H)^* \otimes R^*). \tag{3.3}$$

For $m > 3$, there are two components of homogeneity 1 and one component of homogeneity 2; the corresponding underlying objects are two torsions and one curvature of Weyl type,

$$T_L \in \Gamma(L^* \wedge L^* \otimes R), \quad T_R \in \Gamma(R^* \wedge R^* \otimes L), \quad W \in \Gamma(L^* \wedge R^* \otimes \operatorname{End}(H)). \tag{3.4}$$

It follows that vanishing of T_L and T_R is equivalent to the integrability of the distribution L and R, respectively.

Projective to Lagrangean correspondence

The model correspondence of flag varieties described above reduces in the current setting to

$$\begin{tikzcd}
 & \text{Flags } (1, n+1, n+2) \arrow[dl] \arrow[dr] & \\
\text{Flags } (1, n+2) \cong \mathbb{RP}^{n+1} & & \text{Flags } (n+1, n+2) \cong \mathbb{RP}^{(n+1)*}
\end{tikzcd}$$

In this picture it is easy to identify the top manifold, which is the model Lagrangean contact manifold of dimension $m = 2n+1$, with $\mathcal{P}(T^*\mathbb{RP}^{n+1})$, the projectivized cotangent bundle of real projective space of dimension $n+1$, so that its canonical contact distribution coincides with the one spanned by the vertical subbundles of the two projections. Anyway, this is the space of contact elements of hyperplane type in $\mathbb{RP}^{n+1}$. We come back to these interpretations in sections 3.5 and 4.3.

According to the general principles from section 1.3, this correspondence translates immediately to curved cases. Let $(\mathcal{G} \to X, \omega)$ be the normal parabolic geometry associated to a projective structure on X. For the inclusion of structure groups (the flag stabilizers) as above, the correspondence space of X is identified with its projectivized cotangent bundle $M := \mathcal{P}(T^*X)$ so that the vertical subbundle of the projection $M \to X$ is one of the two Lagrangean subspaces of the contact distribution $H \subset TM$, say R.

$$
\begin{tikzcd}
 & (\mathcal{G}, \omega) \arrow[d] \arrow[dl, bend right] & \\
 & (M, L \oplus R) \arrow[dl] & \text{Lagrangean} \\
(X, [D]) & \text{projective} &
\end{tikzcd}
$$

From the description of the Lagrangean harmonic curvature components, and the fact that the vertical subbundle is identified with R, we see that:

- for $\dim X = 2$, i.e., $\dim M = 3$, the induced structure on M satisfies $C_R = 0$,
- for $\dim X > 2$, i.e., $\dim M > 3$, the induced structure on M satisfies $T_R = W = 0$.

In particular, in the latter case, the induced Lagrangean contact structure is torsion-free if and only if it is flat, which means the initial projective structure is flat. Anyway, due to the key result materialized in (2.10), the previous equalities guarantee that, locally, the Lagrangean contact structure is induced by a projective structure.

Note that a discussion concerning the second fibration of the correspondence above is quite parallel because of the very symmetry of the whole picture.

Path geometries

Let us consider parabolic geometries modelled on flag varieties of lines in planes, i.e., of type $(1, 2, N)$. Counting dimensions, one observes that the base manifold is odd dimensional, say $m = 2n+1$, in which case $N = n+2$. Note that, for $n = 1$, we just recover the Lagrangean contact case in the lowest dimension, hence we assume $n > 1$ hereafter.

Let $P \subset G$ be the stabilizer of the flag determined by the first and the first two vectors from the standard basis. The corresponding grading of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{R})$ is indicated in the following block form

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^E & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^E & \mathfrak{g}_0 & \mathfrak{g}_1^V \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^V & \mathfrak{g}_0 \end{pmatrix}$$

with blocks of sizes 1, 1 and n. In particular, $\mathfrak{g}$ is $|2|$-graded with $\mathfrak{g}_{-2} \cong \mathbb{R}^n$, $\mathfrak{g}_{-1}^E \cong \mathbb{R}$, $\mathfrak{g}_{-1}^V \cong \mathbb{R}^n$ etc. The decomposition $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^V$ is $\mathfrak{g}_0$-invariant, the Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is trivial on $\mathfrak{g}_{-1}^V \times \mathfrak{g}_{-1}^V$ and induces a linear isomorphism $\mathfrak{g}_{-1}^E \otimes \mathfrak{g}_{-1}^V \to \mathfrak{g}_{-2}$. Since the bracket is also trivial on $\mathfrak{g}_{-1}^E \times \mathfrak{g}_{-1}^E$ (because of $\dim \mathfrak{g}_{-1}^E = 1$), it is fully determined by the previous two bits.

Generalized path geometry is the structure underlying a regular Cartan geometry of the current type. On a manifold M of dimension $m = 2n+1$ it consists of two transversal subbundles $E, V \subset TM$ such that, for $H := E \oplus V$, the filtration $H \subset TM$ is compatible with the vector field bracket and the corresponding Levi bracket (2.1) vanishes on $V \times V$ and induces an isomorphism $E \otimes V \to TM/H$.

The terminology abstracts a natural presence of such structures on the projectivized tangent bundle of a manifold, Y, endowed with a system of paths, i.e., unparametrized curves, with the property that through each point in each direction there passes exactly one path from the system. The tangent directional fields of paths from the system foliate $M := \mathcal{P}(TY)$ and tangent spaces of this foliation give a line subbundle $E \subset TM$. This E is obviously contained in the tautological subbundle $H \subset TM$ and complementary to the vertical subbundle $V \subset H$ of the projection $M \to Y$, which are there by nature. The decomposition $H = E \oplus V$ satisfies the above declared properties for generalized path geometries, hence system of paths on Y determines this fancy geometric structure on M. This way, geometric properties of systems of second order ODEs, modulo point transformations, may be studied.

In particular, geodesics of an affine connection on Y, respectively its projectively equivalent class, give rise to a (generalized) path geometry, which has to be somehow special. Before its full characterization, we need to learn about the harmonic curvatures.

Generally, i.e., for any $n \geq 2$, there are three harmonic curvature components, two of which are independent of n: they are of homogeneity 2 and 3 and they correspond to torsion and curvature of Weyl type, namely,

$$T_E \in \Gamma(E^* \wedge (TM/H)^* \otimes V), \quad W \in \Gamma(V^* \wedge (TM/H)^* \otimes \mathrm{End}(V)).$$

The third component is a torsion whose type depends on n: for $n = 2$, respectively $n > 2$, it is of homogeneity 1, respectively 0,

$$T_V \in \Gamma(V^* \wedge V^* \otimes E), \quad \text{respectively} \quad T_V \in \Gamma(V^* \wedge V^* \otimes TM/H). \tag{3.5}$$

Note that, for generalized path geometries in dimension $m = 2n + 1 > 5$, the bad torsion T_V vanishes by the assumption of regularity.

Projective to path geometry correspondence

The model correspondence in the current setting is

$$\begin{tikzcd}
& \text{Flags } (1, 2, n+2) \arrow[dl] \arrow[dr] & \\
\text{Flags } (1, n+2) \cong \mathbb{RP}^{n+1} & & \text{Flags } (2, n+2)
\end{tikzcd}$$

The bottom right object is the Grassmannian of planes in $\mathbb{R}^{n+2}$, whose dimension is $2n$. The top object, which is the model of generalized path geometry of dimension $m = 2n + 1$, is indeed identified with $\mathcal{P}(T\mathbb{RP}^{n+1})$ so that its tautological subbundle coincides with the one spanned by the vertical subbundles of the two projections. The curved version of this correspondence is already outlined in the previous paragraph: unparametrized geodesics of a projective structure on Y form a system of paths, which determines E, and hence a generalized path geometry on $M := \mathcal{P}(TY)$. This is covered by the normal parabolic geometry $(\mathcal{G} \to Y, \omega)$ so that we have

$$\begin{tikzcd}
& (\mathcal{G}, \omega) \arrow[d] \arrow[dl, bend right] & \\
& (M, E \oplus V) \arrow[dl] & \text{path} \\
(Y, [D]) & \text{projective} &
\end{tikzcd}$$

From the description of harmonic curvature components, and the fact that the vertical subbundle is identified with V, we see that:

- the induced structure on M satisfies $T_V = W = 0$.

In particular, the induced generalized path geometry is torsion-free if and only if it is flat, which means the initial projective structure is flat. Due to the harmonic curvature characterization of correspondence spaces, the previous equalities guarantee that, locally, the generalized path geometry is induced by a projective structure or, in other words, the paths on the local leaf space are unparametrized geodesics of an affine connection.

For $\dim M > 5$, the condition $T_V = 0$ is satisfied by regularity, in which cases M is always identified with the projectivized tangent bundle of a local leaf space. Hence only $W = 0$ is relevant in these dimensions.

Grassmannian to path geometry correspondence

In a similar manner, one may discuss the correspondence concerning the second fibration from the picture above, which relates generalized path geometries and almost Grassmannian structures. An *almost Grassmannian structure* on manifold Z is given by an isomorphism $TZ \cong K \otimes L$ of the tangent bundle with the tensor product of auxiliary vector bundles K and L of ranks 2 and n, respectively. Starting with the normal parabolic geometry $(\mathcal{G} \to Z, \omega)$ to the almost Grassmannian structure and the inclusion of parabolics given by the model description, the correspondence space M is identified with $\mathcal{P}(K)$, the projectivization of the smaller auxiliary bundle over Z.

$$
\begin{tikzcd}
& (\mathcal{G},\omega) \arrow[d] \arrow[ddrr, bend left] & & \\
\text{path} & (M, E \oplus V) \arrow[drr] & & \\
& & \text{Grassmanian} & (Z, K \otimes L)
\end{tikzcd}
$$

Almost Grassmannian structures of the current type are specific for $n = 2$, in particular, they are torsion-free, i.e., integrable. For $n > 2$, it follows that the torsion of the almost Grassmannian structure fits the torsion (3.5) of the induced Cartan geometry. Thus, such structure gives rise to a regular geometry, i.e., to a generalized path geometry, just in the integrable case. Anyhow,

- the induced structure on M satisfies $T_E = 0$.

Conversely, this condition for generalized path geometries allows to descend to a local leaf space, interpreted as the space of paths, with an (integrable) Grassmannian structure. Such path geometries, respectively systems of ODEs, are of particular importance as they generically allow to construct enough first integrals (from the Cartan curvature) to describe the paths, respectively to solve the system, see [40].

Remarks

In the previous treatment, we observed specificities in two particular dimensions. At first, for $n = 1$, the 3-dimensional Lagrangean contact structures coincide with 3-dimensional generalized path geometries as both are given by two complementary 1-dimensional subbundles in a 2-dimensional non-integrable distribution $H \subset TM$. Let us comment some aspects of one of correspondences above. Let us consider a second order ODE,

$$y'' = Q(x, y, y'), \tag{3.6}$$

whose graphs of solutions are considered as the system of paths in the plane, Y, with local coordinates (x, y). Let $(x, y, p = y')$ be the local coordinates of the space of contact elements $M := \mathcal{P}(TY)$ (which coincides canonically with $\mathcal{P}(T^*Y)$ in this dimension). The geometry of the ODE is fully encoded in the generalized path geometry on M, whose determining subbundles E and V are given by

$$\mathrm{d}p - Q\,\mathrm{d}x = 0, \quad \text{and} \quad \mathrm{d}y - p\,\mathrm{d}x = 0,$$

respectively. The two fundamental invariants, which correspond to the harmonic curvatures from (3.3), can be expressed in terms of Q and its partial derivatives with respect to y and p. These invariants were known and used ages before their current interpretations by Lie, Tresse and others. The one, whose vanishing guarantees that (3.6) is equivalent to a geodesic equation, turns out to be Q_{pppp}. In such case, the equation has the form

$$y'' = A_0 + A_1 y' + A_2 (y')^2 + A_3 (y')^3, \tag{3.7}$$

where A_i are functions only of x and y. This equation, and the associated projective Cartan connection, appears in Catran's projective papers, see, e.g., [22, chapter V]. We come back to these issues in section 5.1.

Another specific behaviour appears for generalized path geometries in dimension five, i.e., for $n = 2$. In this case, the related Grassmannian structure on Z is equivalent to a conformal structure of split signature so that its two harmonic curvatures correspond to self-dual and anti-self-dual part of the conformal Weyl tensor. In particular, the model correspondence between 3-dimensional projective space and 4-dimensional conformal quadric (via 5-dimensional flag variety) reveals the classical Klein–Plücker correspondence. Note also that this picture (but complexified and primarily read backwards) is the starting position of the far reaching Penrose twistor program, see, e.g., the survey article [1].

3.4 Conformal geometry and ambient metric

The conformal class of the flat metric on the Euclidean space $\mathbb{R}^m$ defines a flat conformal structure. But this is not an accurate conformal model since, e.g., the inversions does not act globally on $\mathbb{R}^m$. One rather takes the standard sphere $S^m \subset \mathbb{R}^{m+1}$, the compactification of $\mathbb{R}^m$ with the stereographic projection, as the right model. This is actually fine, but to describe satisfactorily the group of (global) conformal automorphisms, one would further pass to the 'celestical' sphere in a bigger Minkowski space $\mathbb{R}^{m+1,1}$, i.e., to the space of its null rays. These two spheres can, of course, be identified, but not canonically. This leads to the notion of scales etc. The current viewpoint generalizes straightly to an arbitrary signature.

Conformal structures

Conformal structure of signature (p, q) on a manifold M of dimension $m = p+q$ is given by a class of conformally equivalent pseudo-Riemannian metrics of this signature, i.e., a class of metrics that differ just by multiples of everywhere positive functions. The conformal class of metrics is viewed as a $\mathbb{R}_+$-bundle over M, which is a subbundle on S^2T^*M.

The model structure arises as follows. Let $\mathbb{R}^{p+1,q+1}$ be a vector space with an inner product of signature $(p+1, q+1)$ and let $\mathcal{N} \subset \mathbb{R}^{p+1,q+1}$ be the cone of non-zero null vectors. The space of rays in $\mathcal{N}$, i.e., the oriented projectivization of $\mathcal{N}$, is called the *Möbius sphere* and denoted $S^{p,q}$. Hence we have the projection $q : \mathcal{N} \to S^{p,q} = \mathcal{N}/\mathbb{R}_+$. The Möbius sphere carries a natural conformal structure of signature (p, q), whose each representative metric is given by pulling back the induced metric from the image of a smooth section of the projection q. This conformal structure is obviously invariant under the action of simple Lie group $G = O(p+1, q+1)$, which acts transitively both on $\mathcal{N}$ and $S^{p,q}$. Let $Q \subset P \subset G$ be the nested subgroups stabilizing a null vector and the corresponding ray, respectively. Thus, $\mathcal{N} \cong G/Q$, $S^{p,q} \cong G/P$ and $P/Q \cong \mathbb{R}_+$. The subgroup $P \subset G$ is parabolic, the so-called *Poincaré group.*

For concrete matrix realizations we need some choices. Let the inner product on $\mathbb{R}^{p+1,q+1}$ be given in the standard basis by the Gram matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & E_{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{with} \quad E_{p,q} = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}, \tag{3.8}$$

where E_i is the rank i identity matrix, and let P be the stabilizer of the first vector form the basis. Then the corresponding grading of the Lie algebra $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$ is indicated as

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ 0 & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix},$$

with blocks of sizes 1, n and 1. In particular, $\mathfrak{g}$ is $|1|$-graded with $\mathfrak{g}_{-1} \cong \mathbb{R}^m$, $\mathfrak{g}_0 \cong \mathbb{R} \oplus \mathfrak{o}(p,q)$ and $\mathfrak{g}_1 \cong \mathbb{R}^{m*}$ (of course, the blocks that are symmetric with respect to the anti-diagonal are determined by each other).

Underlying data of the parabolic geometry of the current type are G-structures with structure group $CO(m) = \mathbb{R}_+ \times O(p,q)$, the conformal orthogonal group. The Weyl connections are the ones which preserve the conformal class of metrics, Levi-Civita connections of representative metrics are the exact ones. The general formula (2.8) describing the change of Weyl connections according to parametrizing 1-forms Υ expands in this case to

$$\hat{D}_\xi \eta = D_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi - g(\xi,\eta)\Upsilon^\sharp, \tag{3.9}$$

for $\xi, \eta \in \Gamma(TM)$, where g is any metric from the conformal class (and $\Upsilon^\sharp \in \Gamma(TM)$ is the image of $\Upsilon \in \Gamma(T^*M)$ under the corresponding isomorphism $T^*M \to TM$). Moreover all these connections have the same torsion, which vanishes in the normal case. The first non-trivial harmonic curvature component is the Cotton–York tensor, for $m = 3$, respectively the Weyl tensor, for $m > 3$ (which splits into two components, for $m = 4$). These are the obstructions to the flatness of the conformal structure.

Note that different choices of the principal Lie group to the Lie algebra $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$ lead to variations on the current structure, some of which are subjects of later interests. Our choice here is motivated by similar reasons as in section 3.2.

Note also that the full projectivization of the null cone $\mathcal{N}$ (rather than the oriented one) yields a hyperquadric in $\mathbb{RP}^{n+1}$ as the conformal homogeneous model. In order to distinguish these two cases, one speaks about the *Möbius space* rather that the Möbius sphere (which is its 2-fold cover). The group $O(p+1, q+1)$ does not act effectively on the Möbius space and it may be substituted by $PO(p+1, q+1)$, the quotient by center (which is $\mathbb{Z}_2$).

Fefferman–Graham ambient metric

The model interpretations generalize easily so that the cone of conformal metrics $\mathcal{N} \subset S^2T^*M$, which forms a $\mathbb{R}_+$-bundle over M, is the correspondence space induced by the inclusion $Q \subset P$. But, since the Q-action on $\mathfrak{g}/\mathfrak{q}$ is not a restriction of a G-action, there are no easily induced ambient objects as in section 3.2. Still, the ambient description of the model conformal structure can be mimicked also in the general setting. Following the approach of [33] by Fefferman and Graham, one extends the natural $\mathbb{R}_+$-bundle $\mathcal{N} \to M$ by one more dimension and constructs an ambient metric $\mathbf{g}$ on this bigger manifold $\mathbf{M}$ so that it determines the individual metrics on M from the conformal class in the very same way as above. For this task one actually needs to control the behaviour along the cone $\mathcal{N} \subset \mathbf{M}$. Besides the obvious requirement that $\mathbf{g}$ is homogeneous of degree 2 with respect to the $\mathbb{R}_+$-action, one imposes its Ricci flatness as a natural normalization condition. The procedure is quite subtle and typically describes the ambient metric only in terms of power series expansion.

To proceed, let $I \subset \mathbb{R}$ be an interval containing 0, let $\mathbf{M} = \mathcal{N} \times I$ be an ambient space and let t, (x^i) and ρ be local coordinates on $\mathbb{R}_+$, M and I, respectively. To describe the ambient metric, one may start with the prescription

$$\mathbf{g} = t^2 g_{ij}(x,\rho)\,\mathrm{d}x^i \odot \mathrm{d}x^j + 2\rho\,\mathrm{d}t \odot \mathrm{d}t + 2t\,\mathrm{d}t \odot \mathrm{d}\rho, \tag{3.10}$$

where $g_{ij}(x, 0)$ corresponds to a representative metric in the conformal class $[g]$ on M. Now, iteratively, one determines $g_{ij}(x,\rho)$ as a Taylor series in ρ so that the Ricci flatness condition is satisfied along $\mathcal{N}$, i.e., for $\rho = 0$. Necessarily, the first order expansion is

$$g_{ij}(x, \rho) = g_{ij}(x) + 2\mathsf{P}_{ij}(x)\rho + O(\rho^2), \tag{3.11}$$

where P is the Schouten tensor of the representative metric g on M.

It turns out that, for general conformal structures in odd dimension, there is a (essentially unique) solution of the problem. But, in even dimension, the construction is obstructed at a finite order by the so-called *Fefferman–Graham tensor*, a conformally invariant tensor (which coincides with the Bach tensor in dimension 4). There are not too many examples of conformal structures for which the ambient metric is Ricci flat globally and not just asymptotically along $\mathcal{N}$. We are going to extend this rare family by a new class of members in section 5.3.

Conformal standard tractors and relation to ambient data

The conformal standard tractor bundle $\mathcal{T}$ is the tractor bundle corresponding to the standard representation $\mathbb{R}^{p+1,q+1}$ of the principal group $G = O(p+1, q+1)$. The G-invariant inner product on $\mathbb{R}^{p+1,q+1}$ gives rise to a bundle metric that is preserved by the tractor connection ∇. The standard tractor bundle is filtered, $\mathcal{T} \supset \mathcal{T}^0 \supset \mathcal{T}^1$, so that the line bundle $\mathcal{T}^1$ is identified with $\mathcal{E}[-1]$, the density bundle of conformal weight -1, and $\mathcal{T}^0$ is its ortho-complement with respect to the bundle metric. With respect to a choice of Weyl structure, the standard tractor bundle splits as

$$\mathcal{T} = \mathcal{E}[1] \oplus TM[-1] \oplus \mathcal{E}[-1], \tag{3.12}$$

where $TM[-1] = TM \otimes \mathcal{E}[-1]$ and the projecting part is $\mathcal{E}[1] \cong \mathcal{T}/\mathcal{T}^0$. General formulas (2.5), (2.7) etc. expand in this case to well known expressions that we employ repeatedly below.

Possible relation of the standard tractor bundle $\mathcal{T}$ with its bundle metric (parallel with respect to the tractor connection) and the ambient space $\mathbf{M}$ with the ambient metric (parallel with respect to its Levi-Civita connection) is not clear at all. In particular, the previous description of the general ambient space might seem slightly artificial. However, this can be sorted out so that the similar factorization as in section 3.2 is possible provided that we restrict to the cone $\mathcal{N} \subset \mathbf{M}$. This is proved by Čap and Gover in [12], where even much weaker normalization condition on the ambient metric is considered.

$$\begin{tikzcd}
 & & T\mathbf{M} \arrow[dd] & \\
 & i^*T\mathbf{M} \arrow[ur, hook] \arrow[dl] \arrow[dd] & & \\
\mathcal{T} \arrow[dd] & & (\mathbf{M}, \mathbf{g}) & \text{ambient} \\
 & \mathcal{N} \arrow[ur, hook, "i"] \arrow[dl] & & \\
(M, [g]) & \text{conformal} & &
\end{tikzcd}$$

In particular, sections of the standard tractor bundle $\mathcal{T} \to M$ are in bijective correspondence with ambient vector fields along $\mathcal{N}$ (more precisely, sections of $i^*T\mathbf{M} \to \mathcal{N}$) which are of homogeneity -1 with respect to the $\mathbb{R}_+$-action. Under this correspondence, the line subbundle $\mathcal{T}^1 \subset \mathcal{T}$ corresponds to the vertical subbundle of the cone projection $\mathcal{N} \to M$.

3.5 CR contact geometry and Fefferman space

Prototypical CR structures are induced on real submanifolds in complex spaces. Submanifolds of the same real codimension need not be mutually equivalent under biholomorphic transformations, hence their local geometry may be studied and invariants constructed. This was shown already by Poincaré in [55] for real hypersurfaces in $\mathbb{C}^2$, which are the simplest examples of CR contact structures. Popular and the most symmetric ones are the hyperquadrics given by

$$|z_1|^2 + |z_2|^2 = 1 \quad \text{and} \quad \operatorname{Im} z_2 = |z_1|^2, \tag{3.13}$$

$(z_1, z_2) \in \mathbb{C}^2$, which are the boundaries of the unit ball and the Siegel domain, respectively. It follows that they are globally biholomorphically equivalent up to one point under a complex projective map. Thus, they may be seen as different affine realizations of the same hyperquadric in $\mathbb{CP}^2$, i.e., the complex projectivization of the null cone in $\mathbb{C}^3$ with respect to a Hermitean inner product of signature $(2,1)$. This is the proper homogeneous model of 3-dimensional CR contact structures.

Almost CR contact structures

Generalizing according to dimension and signature, let $\mathbb{C}^{p+1,q+1}$ be a complex vector space with a Hermitean inner product of signature $(p+1, q+1)$ and let $\mathcal{N} \subset \mathbb{C}^{p+1,q+1}$ be the cone of non-zero null vectors. The complex projectivization of $\mathcal{N}$ yields a real hyperquadric $C^{p,q} \subset \mathbb{CP}^{n+1}$, where $n = p + q$, which is the model CR contact structure of signature (p, q). It is a homogeneous space of the simple Lie group $G = SU(p+1, q+1)$, respectively $PSU(p+1, q+1)$, and the stabilizer $P \subset G$ of a point of $C^{p,q}$, i.e., a complex line in $\mathcal{N}$, is a parabolic subgroup. Thus, $C^{p,q} \cong G/P$.

Let the Hermitean inner product on $\mathbb{C}^{p+1,q+1}$ be represented by the same matrix as in (3.8) and let P be the stabilizer of the first vector from the standard basis. Then the corresponding grading of the Lie algebra $\mathfrak{g} = \mathfrak{su}(p+1, q+1)$ has the form

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix},$$

with blocks of sizes 1, n and 1. In particular, $\mathfrak{g}$ is $|2|$-graded with $\mathfrak{g}_{-2} \cong \mathbb{R}$, $\mathfrak{g}_{-1} \cong \mathbb{C}^{p,q}$, $\mathfrak{g}_0 \cong \mathbb{C} \oplus \mathfrak{u}(p,q)$ etc. (of course, the blocks that are symmetric with respect to the anti-diagonal are determined by each other). The Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is (a multiple) of the imaginary part of the standard Hermitean product on $\mathfrak{g}_{-1}$, in particular, it is non-degenerate and of type $(1,1)$ with respect to the standard complex structure.

The underlying geometric structure of the current parabolic geometry on a manifold M (of dimension $m = 2n+1$) consists of a contact distribution $H \subset TM$ with an almost complex structure J on H such that the Levi form $\mathcal{L} : H \times H \to TM/H$ is of type $(1,1)$ with respect to J. Thus $\mathcal{L}$ is the imaginary part of a non-degenerate Hermitean form on H whose signature, (p,q) with $p+q = n$, is the signature of the structure. In other words, the signature in encoded in the map $\mathcal{L}(-, J(-))$. Such structures are known as non-degenerate partially integrable almost CR structures of hypersurface type or, shortly, *almost CR contact structures.* The structure is integrable if the almost complex structure J is integrable, i.e., the corresponding Nijenhuis tensor, which is the map $H \times H \to H$ of type $(0,2)$, vanishes.

The notion of partial integrability refers to the compatibility of the Levi form $\mathcal{L}$ with the almost complex structure J. Both this and the stronger condition of integrability can be equivalently stated in terms of complexifications: for $H^{1,0} \oplus H^{0,1}$ being the decomposition of $H \otimes \mathbb{C} \subset TM \otimes \mathbb{C}$ into the holomorphic and anti-holomorphic part, the almost CR structure is partially integrable if and only if $[H^{0,1}, H^{0,1}] \subset H \otimes \mathbb{C}$, the structure is integrable if and only if $[H^{0,1}, H^{0,1}] \subset H^{0,1}$. In this view one also relates Lagrangean contact structures and almost CR contact structures as two real forms of a complex parabolic contact structure: indeed, $\mathfrak{sl}(n+2, \mathbb{R})$ and $\mathfrak{su}(p+1, q+1)$, with $p + q = n$, are distinct real forms of complex Lie algebra $\mathfrak{sl}(n+2, \mathbb{C})$.

This remark concerns also the interpretation of the corresponding cohomologies and harmonic curvatures: the two components in the Lagrangean case that are labeled C_L and C_R in (3.3), respectively T_L and T_R in (3.4), correspond to only one component in the almost CR case. In particular, for $n = 1$ (i.e., $\dim M = 3$), there is only one harmonic curvature of Cotton–York type, for $n > 1$, there is one torsion and one curvature of Weyl type (that all are somehow compatible with the almost complex structure J). It follows that the torsion part coincides with the Nijenhuis tensor of J up to a multiple, hence its vanishing is equivalent to the integrability of the almost CR structure.

Fefferman space

Fefferman's original construction in [32] concerns hypersurfaces in complex spaces, the boundaries of strictly pseudoconvex domains $D \subset \mathbb{C}^{n+1}$. It associates a natural Lorentzian metric on $\partial D \times S^1$ so that its conformal class is invariant under biholomorphic transformations of D, respectively CR automorphisms of ∂D. This way, the conformal geometry is used to tackle many problems of the original complex, respectively CR, nature. The construction was later generalized for abstract CR contact structures by several authors until it was realized it may be considered for almost CR contact structures in any signature and, moreover, it allows a neat Cartan geometric reinterpretation, see [13].

As usual, one may start with the homogeneous model and then consider generalizations. With the usual identification of the multiplicative group $\mathbb{C}^\times$ with $S^1 \times \mathbb{R}_+$, we may factorize the projection $\mathcal{N} \to C^{p,q} = \mathcal{N}/\mathbb{C}^\times$ above via $\mathcal{N}/\mathbb{R}_+$, which gives a S^1-fibration over the model CR quadric $C^{p,q}$.

$$\begin{tikzcd}
\mathcal{N} \arrow[dr] \arrow[dd] & \\
& \mathcal{N}/\mathbb{R}_+ = S^{*,*} \arrow[dl] \\
\mathcal{N}/\mathbb{C}^\times = C^{*,*} &
\end{tikzcd}$$

To interpret this picture, we just forget about the complex structure on $\mathbb{C}^{p+1,q+1}$ and consider $\mathcal{N}$ as the cone of non-zero null vectors in $\mathbb{R}^{2p+2,2q+2}$ with respect to the inner product given by the real part of the Hermitean inner product on $\mathbb{C}^{p+1,q+1}$. Hence, $\mathcal{N}/\mathbb{R}_+$ is the Möbius sphere $S^{2p+1,2q+1}$ with its flat (oriented) conformal structure as the Fefferman conformal structure of the model CR quadric. The inner product on $\mathbb{R}^{2p+2,2q+2}$ is by its definition G-invariant, which yields an embedding of Lie groups

$$\phi : G = SU(p+1, q+1) \hookrightarrow SO(2p+2, 2q+2) = \widetilde{G}. \tag{3.14}$$

As a homogeneous space, $S^{2p+1,2q+1} \cong \widetilde{G}/\widetilde{P}$, where $\widetilde{P} \subset \widetilde{G}$ is the stabilizer of a ray in $\mathcal{N} \subset \mathbb{R}^{2p+2,2q+2}$. Since also the group G acts transitively on $\mathcal{N}$, we may identify $S^{2p+1,2q+1} \cong G/Q$, where $Q \subset G$ is the preimage of $\widetilde{P} \subset \widetilde{G}$ with respect to ϕ; roughly we write $Q = G \cap \widetilde{P}$. Obviously, $Q \subset P$, where $P \subset G$ is the stabilizer of the complex line containing the ray stabilized by Q. But this is exactly the setting of section 1.3: the Möbius sphere is the correspondence space of the model CR quadric and the Cartan connection encoding the (flat) conformal structure on the former space can be seen as the equivariant extension of the Maurer–Cartan form on G encoding the (flat) CR structure on the latter space.

$$\begin{tikzcd}
& & (\widetilde{G}, \widetilde{\omega}) \arrow[d] & \\
(G, \omega) \arrow[urr, hook, dashed, "\phi"] \arrow[r] \arrow[dd] & & S^{*,*} \cong G/Q \cong \widetilde{G}/\widetilde{P} \arrow[dll] & \text{conformal} \\
& & & \\
C^{*,*} \cong G/P & \text{CR contact} & &
\end{tikzcd}$$

In other words, the G-invariant conformal geometry on the model Fefferman space is controlled by the pair of maps $i = \phi|_Q : Q \to \widetilde{P}$ and $\alpha = \phi' : \mathfrak{g} \to \tilde{\mathfrak{g}}$ via the corresponding extension functor. The natural—and for applications also important—question is how this functor behaves in generality, in particular, when it is compatible with the normality condition. In contrast to previous correspondence space constructions (where there were no extension), there is no general answer and it demands a careful analysis of curvature interrelations given by (1.9). The only bonus in the current situation is that there is no contribution of Ψ_α, since α is a Lie algebra homomorphism. It is shown in [13] that the current construction preserves the normality condition just for integrable CR contact structures. More precisely, for $(\mathcal{G} \to M, \omega)$ being the normal Cartan geometry describing an almost CR contact structure and $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ being the Cartan geometry with an underlying conformal structure constructed via the extension given by the pair (i, α) as above,

- the Cartan connection $\widetilde{\omega}$ is normal if and only if ω is torsion-free.

Remarks

For the description of the model Fefferman construction, it is sometimes useful to switch the point of view: instead of considering a complex vector space with a Hermitean inner product and then forgetting the complex structure, one may start with a real vector space with a pseudo-Euclidean inner product and then imposing a compatible complex structure. The compatibility means that the complex structure is skew with respect to the inner product. This allows to reconstruct the Hermitean inner product which is behind the embedding (3.14). In particular, elements from its image commute with the complex structure. Note that this embedding admits a lift to the corresponding Spin group.

The previous factorization of the complex projectivization $\mathcal{N} \to C^{p,q} = \mathcal{N}/\mathbb{C}^\times$ via $\mathcal{N}/\mathbb{R}_+$ led to the model interpretation of the Fefferman construction. Naturally, one may ask what interesting issues appear when factorization via $\mathcal{N}/S^1$. It easily follows that the fibration $\mathcal{N}/S^1 \to C^{p,q}$ is just a scale bundle that can be directly identified with the bundle of (positive) contact forms.

As we already noticed above, Lagrangean and almost CR contact structure correspond to different real forms of the contact grading of $\mathfrak{sl}(n+2, \mathbb{C})$. In fact, any complex simple Lie algebra allows a unique (up to inner automorphism) contact grading. Up to several exceptions (including mainly the compact forms), real simple Lie algebras also admit unique contact gradings; the corresponding parabolic geometries are collectively called *contact parabolic geometries*. Quick comments on these structures, in association with chains, can be found in section 4.3. A slightly extended comment on Lagrangean contact structures follows.

Lagrangean contact structures revised

Lagrangean contact structures are defined in section 3.3 in terms of decomposition of the contact distribution into Lagrangean subspaces, $H = L \oplus R$, with the model being a flag variety of specific type. Almost CR structures are defined earlier in this section in terms of an almost complex structure J on H (i.e., an endomorphism such that $J^2 = -\operatorname{id}|_H$) compatible with the Levi bracket, with the model being a real hyperquadric in complex projective space. The relationship between these two structures can be emphasized by interpreting the Lagrangean decomposition $H = L \oplus R$ as the eigenspace decomposition of an almost para-complex structure K on H (i.e., an endomorphism such that $K^2 = \operatorname{id}|_H$) that is compatible with the Levi bracket. This can indeed be precised so that, in particular, vanishing of the Nijenhuis tensor of K is equivalent to the integrability of both L and R.

Similar affinity should, of course, be perceptible also on the level of homogeneous models. This is indeed possible: starting with the CR description and imposing the 'para-' prefix everywhere, we claim that the homogeneous model of the Lagrangean contact structure of dimension $m = 2n+1$ can be naturally identified with the 'para-complex projectivization' of the cone of non-zero null vectors of a para-Hermitean inner product in a 'para-complex vector space' of real dimension $2n+4$. The quote marks indicate that one has to be careful about typical para-issues, i.e., that para-complex numbers do not form a field, hence 'para-complex vector spaces' are actually modules etc. To be on the safe side, let us formulate things in real terms.

Let V be a real vector space of dimension $2n+4$ with an inner product h of signature $(2p+2, 2q+2)$, where $p+q=n$, and $\mathcal{N} \subset V$ be the cone of non-zero null vectors of h. Let us consider K to be a para-complex structure on V that is skew with respect to h. This compatibility forces the eigenspaces V_+ and V_- of K to be isotropic. In particular, h must have split signature and V_+ and V_- are dual one another with respect to h. For $X \in V$, the para-complex hull $\langle X, K(X) \rangle$ is 2-dimensional if and only if X belongs to the open subset $V_0 := V \setminus \{V_+ \cup V_-\}$; such subspaces are called *para-complex lines*. Any para-complex line $\langle X, K(X) \rangle$ intersects V_+ and V_- in 1-dimensional subspaces spanned by $X_+ := X + K(X)$ and $X_- := X - K(X)$, respectively. Since V_- is identified with the dual space of V_+, we may interpret the latter subspace as a hyperplane in V_+. Since the identification is provided by h, the pair $\langle X_+ \rangle$ and $\langle X_- \rangle$ represents a flag in V_+ if and only if $h(X_+, X_-) = 0$. This is also equivalent to the fact that the para-complex line is null, i.e., contained in $\mathcal{N}$ or, more precisely, in $\mathcal{N}_0 := \mathcal{N} \setminus \{V_+ \cup V_-\}$. Altogether, we have

Proposition 3.1. *The space of para-complex null lines in $V = \mathbb{R}^{n+2,n+2}$ is naturally identified with the flag variety of type $(1, n+1, n+2)$.*

Thus, the 'para-complex projectivization' of $\mathcal{N}_0$ recovers the model Lagrangean flag variety together with the double fibration as displayed on page 23.

Considering the real projectivization of $\mathcal{N}$, which is the Möbius space $S^{n+1,n+1}$, it has three components in accord with the decomposition

$$\mathcal{N} = V_+ \cup \mathcal{N}_0 \cup V_-. \tag{3.15}$$

The group $\widetilde{G} = SO(n+2, n+2)$ acts transitively both on $\mathcal{N}$ and $S^{n+1,n+1}$, however, the previous decomposition corresponds to the orbit decomposition of the subgroup of elements commuting with K. Such elements are represented by linear isomorphisms of V_+ (and their duals on V_-), hence the subgroup is isomorphic to $GL(n+2, \mathbb{R})$. Restricting only to isomorphisms preserving a volume form on V_+, we have an embedding

$$\phi : G = SL(n+2, \mathbb{R}) \hookrightarrow SO(n+2, n+2) = \widetilde{G}. \tag{3.16}$$

The compatibility of K and h allows to construct a para-Hermitean inner product on V that is preserved by G and whose real part is h. In this context we may see G as a special 'para-unitary' group.

Note that map (3.16) admits a lift to the corresponding Spin group. All these observations are employed later in section 5.2 in connection with a (closest possible) analogue of the Fefferman construction.

CHAPTER II

Usage

Here we collect our concrete contributions to the generalities described in the previous chapter. The exposition is based on the attached articles, extended by motivations and commentaries for which, usually, there is not enough space in journal publications. Although the individual sections concern independent stories, one can easily spot many interrelations. The ordering of the material was driven also by this aspect.

A quick summary of the results obtained is given already in chapter . The core of each of the following sections follows the introductory historical survey. In the explanations of the results we suppress all technical details and rather stress the key points and remarkable relations.

At this place we face a slight problem concerning the notation and terminology: as we refer both to the attached articles and the previous preparatory text, we hope the reader will understand that we cannot guarantee an absolute consistency.

4 Geometry of chains

Chains are non-contact distinguished curves of parabolic contact structures. Certainly, they are best known and studied in the CR contact setting, in which case they were introduced by Cartan, and later generalized by Chern and Moser, in the course of solution to the equivalence problem. Despite their innocent definition and model realization, they form a rather intricate family of curves. This is probably why chains and their behaviour became a subject of independent study. In the introductory section we make several remarks on this account.

In the main part we focus on the conceptual description of the associated path geometry following our article [17]. This can be attained by a general extension of Cartan geometries as described in section 1.3. In the CR and Lagrangean contact case, this construction preserves the normality condition for an important subclass of integrable (torsion-free) structures, in which cases we can add further non-trivial observations. This, in particular, concerns the reconstruction of the initial structure from the path geometry of chains.

Considering a similar adventure for other parabolic contact structures, one observes some limitations: in most cases the whole circle of ideas works only in the flat case, for projective contact structures the construction reduces to a construction of Fefferman type. These situations are commented in the concluding section.

4.1 History and unification

Cartan, Chern and Moser

The notion of chains (*chaînes*) was introduced in the work of Cartan on real hypersurfaces in $\mathbb{C}^2$, [21, 20]. On hyperquadrics (3.13), they are just the intersections with complex lines, which turn to be either real circles or straight lines. Passing to the model projective hyperquadric in $\mathbb{CP}^2$, which is the homogeneous space G/P with $G = PSU(2,1)$ and $P \subset G$ the parabolic subgroup as in section 3.5, it follows that chains are the homogeneous curves with generators in the 1-dimensional subspace $\mathfrak{g}_{-2} \subset \mathfrak{g}$. Easily, all chains are mutually equivalent, any two points determine a unique chain and any chain is everywhere transverse to the contact distribution. Thus, each chain (as an unparametrized curve) is uniquely determined by a tangent direction in one point, provided it is transverse to the complex contact subspace.

Cartan's generalization to general real hypersurfaces $M \subset \mathbb{C}^2$ can be phrased in our terms as follows. For $\mathcal{G} \to M$ being the associated P-bundle with the normal Cartan connection ω, chains are the projections of integral submanifolds of a closed differential ideal on $\mathcal{G}$ determined by ω. This somehow vague description may be interpreted so that the $\mathfrak{g}_{\pm 1}$-parts of a suitable Cartan gauge along each chain vanish.

Chains are further considered on real hypersurfaces in complex spaces of general dimension by Chern and Moser in [25]. Firstly, chains play a key role in the construction of normal forms, where they (and certain framings along them) are related to an external coordinate system under biholomorphic mappings. This way, chains (and the associated framings) satisfy a system of differential equations which is invariantly given by the hypersurface. Secondly, chains are described intrinsically on a general (integrable) CR manifold of hypersurface type in terms of the associated (normal) Cartan connection generalizing the above mentioned Cartan's description.

Beside the previous views, there is another one by Fefferman via his construction discussed in section 3.5: chains are understood as projections of null geodesics of the conformal Fefferman space.

Unification and remarks

It is not immediately clear how all these concepts are related, if ever. Detailed comparisons of Cartan and Chern–Moser approach can be found in [47], many illuminating remarks are also scattered in [4]. It follows that, in the Cartan geometric interpretation, chains coincide with projections to the base manifold of flows of constant vector fields on $\mathcal{G}$ that corresponds to elements from $\mathfrak{g}_{-2} \subset \mathfrak{g}$. This fits nicely to the concept of distinguished curves introduced in section 2.3. In particular, the subalgebra of $\mathfrak{g}$ with vanishing $\mathfrak{g}_{\pm 1}$-parts is the symmetry algebra of a model chain. Also, via the notion of development of curves, the above listed properties of model chains remain valid in general. And, finally, the notion of chains can be instantly transferred to any parabolic contact geometry.

Before we fully focus on the previous standpoint, let us remark several interesting features of chains that were studied by many authors, cf. [32, 46, 50, 51]: It holds that, any two nearby points on strictly pseudoconvex CR contact manifolds may be connected by a chain. But, this statement fails in general signature. Furthermore, it may happen that chains spiral into a point. Another result, which was proved in generality by Cheng [23], is that chain preserving diffeomorphisms between CR contact manifolds are either CR automorphisms or conjugate CR automorphisms. We are going to recover this result in the next section.

4.2 The action: curved Cartan extension

The fact that chains (as unparametrized curves) are determined by a tangent direction at one point means that they form a system of paths in the sense of section 3.3, restricted, however, to the non-contact directions. Since this structure also allows a parabolic geometric description, it is natural to look for a direct relation of the two structures via some extension construction described in previous chapter. Experiments with the corresponding system of ODEs in homogeneous model give some clue on the character of such relation; see [27] for concrete outcomes in small dimensions. The most interesting discussions concern the CR and Lagrangean contact structures, which we reproduce here following our article [17] stimulated by [26]. According to the current exposition, we focus on the CR case and only comment the Lagrangean one below (although the structure of [17] is right opposite).

Let M be a smooth manifold of dimension $m = 2n+1$ endowed with an almost CR contact structure (H, J) of signature (p, q), $p+q = n$. Let $G = PSU(p+1, q+1)$ and $P \subset G$ be the pair of simple Lie group and its parabolic subgroup, let $\mathfrak{p} \subset \mathfrak{g}$ be their Lie algebras and let $(\mathcal{G} \to M, \omega)$ be the corresponding regular normal parabolic geometry as in section 3.5. Let further $\widetilde{M} \subset \mathcal{P}TM$ be the open subset of the projectivized tangent bundle of M consisting of the non-contact directions and let $E \oplus V$ be the decomposition of the tautological subbundle of $T\widetilde{M}$ determined by the path geometry of chains. Let $\widetilde{G} = PGL(2n+2, \mathbb{R})$ and $\widetilde{P} \subset \widetilde{G}$ be the pair of simple Lie group and its parabolic subgroup, let $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$ be their Lie algebras and let $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ be the corresponding regular normal parabolic geometry as in section 3.3.

The construction

Searching for a direct relation of the two Cartan geometries, one has to investigate the homogeneous model first. For $M = G/P$, the first observation is that G acts transitively on $\widetilde{M}$, the space of non-contact lines in TM. The stabilizer $Q \subset G$ of an appropriate element contains G_0 and is contained in P so that its Lie algebra is $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$. In particular, $\widetilde{M}$ is the homogeneous space G/Q and the associated path geometry as well as the regular normal parabolic geometry $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ are homogeneous with respect to G. From the generalities in section 1.3 we know that such Cartan geometry can be described by a pair of compatible extension maps $i : Q \to \widetilde{P}$ and $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$. It follows that, in the current setting, such pair is actually unique up to the natural equivalence, see [17, Theorem 3.4].

$$\begin{tikzcd}
& & (\widetilde{\mathcal{G}}, \widetilde{\omega}) \arrow[dd, "\widetilde{P}"] & \\
(\mathcal{G}, \omega) \arrow[urr, hook, dashed] \arrow[drr, "Q"] \arrow[dd, "P"'] & & & \\
& & (\widetilde{M}, E \oplus V) \arrow[dll] & \text{path} \\
(M, H, J) & \text{CR contact} & &
\end{tikzcd}$$

To prescribe the pair (i, α) in concrete, one may fix as much as possible to satisfy all the compatibility conditions and eliminate the remaining freedom so that the induced parabolic geometry is normal, i.e., that $\widetilde{\partial}^* \Psi_\alpha = 0$, for Ψ_α as in (1.8). For instance, according to the block decompositions as before, the following choice guarantees that α induces a linear isomorphism $\mathfrak{g}/\mathfrak{q} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ and $\mathfrak{g}_{-2} \subset \mathfrak{g}$ hits $\tilde{\mathfrak{g}}^E_{-1} \oplus \tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$,

$$\begin{pmatrix} \cdot & Z & \cdot \\ X & \cdot & -E_{p,q}Z^* \\ ix & -X^* E_{p,q} & \cdot \end{pmatrix} \xmapsto{\alpha} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ x & \cdot & \cdot & \cdot \\ \operatorname{Re}(X) & \operatorname{Im}(E_{p,q}Z^*) & \cdot & \cdot \\ \operatorname{Im}(X) & -\operatorname{Re}(E_{p,q}Z^*) & \cdot & \cdot \end{pmatrix},$$

where $E_{p,q}$ is as in (3.8) and * denotes the conjugate transposition. This setup can be completed and integrated to $i : Q \to \widetilde{P}$ so that the remaining compatibility and normality conditions hold, which determines the pair (i, α) uniquely. The full matrix realization is given in [17, section 5.2]. It follows that the corresponding Ψ_α is non-trivial, hence we deal with general extension functor.

Normality and applications

The first serious task concerns answering the fundamental question under which conditions the extension functor yields a regular normal parabolic geometry. In order to tackle this problem, a careful analysis of the Cartan curvatures in accord with (1.9) is inevitable. With a detailed knowledge of harmonic curvatures and the properties of the map α, it turns out the direct relation between the regular normal parabolic geometries associated to an almost CR contact structure and its path geometry of chains is available exactly for the subclass of integrable structures:

Theorem 4.1 ([17, Theorem 5.2])**.** *Let (M, H, J) be an almost CR contact structure and let $(\mathcal{G} \to M, \omega)$ be the corresponding regular normal parabolic geometry of type (G, P). Then the parabolic geometry $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ constructed using the extension functor associated to the pair (i, α) as above is regular and normal if and only if ω is torsion-free, i.e. the almost CR structure is integrable.*

In particular, this is automatically satisfied when $\dim M = 3$, i.e., $n = 1$. With such direct relation we can control some properties of chains—for integrable CR contact structures—in relatively easy way:

Firstly, the general principle that the induced Cartan curvature $\widetilde{\kappa}$ is fully given by the CR Cartan curvature κ and the constant contribution Ψ_α may be nicely specified in our setting. It follows that the two ingredients, respectively their equivariant extensions, are well separable from the whole curvature $\widetilde{\kappa}$ in terms of certain projections. Note that this is not an obvious task since this has to be done without the prior knowledge of the bundle extension. In particular, these two parts can never eliminate each other, the one corresponding to Ψ_α is never vanishing and the one corresponding to κ supports just the torsion of $\widetilde{\kappa}$. These facts and the interpretation of the vanishing of individual curvature components as in section 3.3 lead to the following results:

Theorem 4.2 ([17, Theorem 5.3])**.** *Let (M, H, J) be a CR contact structure.*

- *There is no linear connection on TM which has the chains among its geodesics.*
- *The path geometry of chains is torsion-free if and only if the CR structure is flat.*

Secondly, the previous separation can be pushed even further so that the equivariant extension of Ψ_α, respectively κ, coincides up to a non-zero multiple with the component of the harmonic curvature of homogeneity 3, respectively 2 (torsion), of the induced Cartan connection. This and the explicit knowledge of Ψ_α, which is just an algebraic map given by (1.8), allows us to interpret the corresponding harmonic curvature component via the complete symmetrization of the map

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}(\xi, J(\eta))J(\zeta), \tag{4.1}$$

for $\xi, \eta, \zeta \in \Gamma(H)$, where $\mathcal{L} : H \times H \to TM/H$ is the Levi bracket. This shows an intimate relation between the initial CR contact structure and the associated path geometry of chains on the very underlying level. Now, it is an algebraic exercise to show the following:

Theorem 4.3 ([17, Theorem 5.3])**.** *The almost complex structure J can be reconstructed up to sign from the harmonic curvature of the associated path geometry of chains.*

In the proof one has to consider the complexification of (4.1) to $H \otimes \mathbb{C}$, which then allows to reconstruct the subset $H^{1,0} \cup H^{0,1}$. This determines the almost complex structure on H up to the sign, hence the ambiguity. Notice that particularly the signature of the CR structure is incorporated in the path geometry of chains and the previous reasoning indicates how.

Finally, as an easy consequence, we recover the result of [23] on chain preserving transformations:

Corollary 4.4 ([17, Corollary 5.3])**.** *A contact diffeomorphism between two CR contact manifolds which maps chains to chains is either a CR isomorphism or a CR anti-isomorphism.*

Remarks

From the viewpoint of path geometry, chains form a complicated system of curves even in the homogeneous model since the corresponding Cartan geometry is non-flat. However, it is torsion-free and has a relatively large automorphism group. From the previous we know this group is $G = PSU(p+1, q+1)$, whose dimension is $n^2 + 4n + 3$, where $n = p + q$. A natural question in similar situations is whether this realizes the submaximal case or not. It follows from [52, section 5.3] that this is not the case, since the maximal automorphism group of a non-flat path geometry has dimension $4n^2 + 4n + 6$.

Since the path geometry of chains in the homogeneous model is torsion-free, the twistor space of chains carries a canonical Grassmannian structure of type $(2, 2n)$ (equivalently, Segre or para-quaternionic structure). Again, this is an example of a non-flat Grassmannian structure with relatively large (but not submaximal) automorphism group. Moreover, this is an example of a non-flat Grassmannian (globally) symmetric space, cf. [69].

Although chains are not geodesics of any linear connection on M, there is a variational approach to their description and study. This has been recently shown in [24] via a Finsler-like metric on M, the so-called *Kropina metric*.

4.3 Other parabolic contact structures

The just described story of chains for CR contact structures has analogies for other parabolic contact structures. The Lagrangean contact ones are, of course, the closest.

Lagrangean contact structures

In this case, the development of section 4.2 is very parallel with analogous results, which are also described in [17]: the extension functor preserves the normality condition just in torsion-free cases, chains are not geodesics of any linear connection and the path geometry of chains is torsion-free just for flat Lagrangean contact structures. In the reconstruction, the harmonic curvature component corresponding to Ψ_α allows to recreate the subset $L \cup R \subset H$, i.e., the para-complex structure on H up to sign. Hence the chain preserving diffeomorphisms are either Lagrangean automorphism or anti-automorphism, i.e., they either preserve or swap the Lagrangean subspaces from the decomposition $L \oplus R = H$.

Concerning the homogeneous model, it allows several descriptions as discussed in sections 3.3 and 3.5. Anyhow, it is the correspondence space of $\mathbb{RP}^n$ which is identified with the space of contact elements in $\mathbb{RP}^n$ of hyperplane type. Consequently, chains of the model Lagrangean contact structure should be described in the underlying projective terms. Indeed, seeing them as 1-parameter families of contact elements, it easily follows that each chain consists of a pencil of hyperplanes in $\mathbb{RP}^n$ with a common projective line whose base points also form a projective line. In particular, chains project to projective geodesics and this fact holds generally for any Lagrangean contact structure induced by projective structure.

Projective contact structures

Projective contact structure (in a narrower sense) is the underlying structure of a regular normal parabolic geometry corresponding to the contact grading of simple Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n+2, \mathbb{R})$, where the dimension of base manifold is $2n + 1$. Roughly, the structure is given by a class of 'contact projectively equivalent' affine connections with vanishing 'contact torsion', i.e., basically, by a family of 'contact geodesics'. In a broader sense, the condition of vanishing contact torsion may be suppressed, which still allows a parabolic geometric description with a generalized normalization condition. This is considered by Fox in [38], where also a subordinated projective structure on the same manifold is described. The subordinateness means that the contact geodesics of the former structure are among the geodesics of the latter one.

The approach of [38] is via the ambient connections—the Thomas one for projective structures and its direct analogue for projective contact structures. Considering the structure with vanishing contact torsion, the subordinated projective structure may be interpreted as the result of a construction of Fefferman type. In such case, geodesics in non-contact directions are exactly the chains of the projective contact structure, hence the associated path geometry is covered just by the correspondence space construction. This is indeed unusual situation compared to other parabolic contact structures. Anyhow, the chain preserving transformations turn out to be automorphisms of projective contact structures. All details can be found in our article [18].

Lie contact structures

Other parabolic contact geometries are less studied than the ones mentioned above. Among them, probably the Lie contact structures are the most prominent as they are related to the classical Lie sphere geometry. Lie contact structures correspond to contact gradings of simple Lie algebras $\mathfrak{g} = \mathfrak{so}(p+2, q+2)$, where the dimension of base manifold is $2p+2q+1$. The underlying structure allows several equivalent descriptions, among which the shortest is via a para-quaternionic structure on the contact distribution compatible with the Levi bracket (the role of the signature (p, q) is rather involved in this setting).

According to the development of section 4.2 one can follow the key steps including the reconstruction of the structure from the harmonic curvature of the associated path geometry, provided that the Lie contact geometry is torsion-free. The only defect on the neck is that, for dimensions grater or equal to 7, this restriction implies the Lie contact structure is flat and in such cases the used machinery seems a bit heavy. Anyhow, in the homogeneous case the path geometry of chains is again non-flat, torsion-free, with relatively large automorphism group, see [68] for more.

Furthermore, the model Lie contact structure can be identified on the projectivized cotangent bundle of the Möbius sphere, and this observation can be covered by a construction of Fefferman type, see [67]. In the case of positive definite signature, interpreting chains as 1-parameter families of contact elements, one can show that the base points of each chain form a conformal circle which intersect the hyperplanes from the family in constant angle. In particular, chains project to conformal circles and this fact holds for any Lie contact structure induced by conformal structure.

In low dimensions, Lie contact structures are basically equivalent to other, better studied, parabolic contact structures: the Lie algebra isomorphisms $\mathfrak{so}(4, 2) \cong \mathfrak{su}(4, 2)$ and $\mathfrak{so}(3, 3) \cong \mathfrak{sl}(4, \mathbb{R})$ indicate the equivalence to almost CR contact structures of split signature and Lagrangean contact structures, respectively, in dimension 5. Note that the latter identification (respectively, its complexification) is also behind the famous line-sphere correspondence by Lie.

Remaining cases

There is also an isomorphism $\mathfrak{so}(6, 2) \cong \mathfrak{so}^*(8)$, which is an isomorphism of real forms of the same complex Lie algebra. Parabolic contact structures corresponding to contact gradings of $\mathfrak{so}^*$ algebras are almost unknown. However, it can be shown that the underlying structure consists of a quaternionic structure on the contact distribution that is compatible with the Levi bracket. The behaviour of chains is not studied in detail for these structures, but it is expected to be parallel to the Lie contact case.

One may also list the structures corresponding to contact gradings of exotic simple Lie algebras. Similarly to previous two cases, the common feature is that their harmonic curvatures consists only of torsions, hence the same restrictions as above are expected.

5 Conformal Patterson–Walker metrics

Walker manifolds are pseudo-Riemannian manifolds admitting a parallel isotropic distribution. In the case of split signature, there is an interesting subclass of Patterson–Walker metrics, which are the Walker metrics induced naturally on the cotangent bundle of a manifold with an affine connection. If the base manifold has dimension two, the construction was modified in [30] in a projectively invariant manner. We can grasp this construction as a construction of Fefferman type and consider its generalization to any dimension.

In the following we recall some basics following our three main sources of inspiration, [54, 30, 53]. Then we compare various viewpoints, which should serve as a useful background for understanding the general Fefferman-type construction relating the associated parabolic geometries. As emphasized already in the introductory chapter, this construction is typically non-normal, but it is in a sense 'half-normal', which allows its surprisingly satisfactory treatment. This is described in section 5.2 following our article [43].

It follows that many of the abstract functorial notions have very concrete counterparts. From the latter perspective, new interrelations between the two geometric structures were researched in our next article [42]. This is not treated systematically in the current exposition, but here and there we mention concerning particular results. As a nice mixture of abstract and concrete, we are able to draw also a very favourable ambient picture. This is described in section 5.3 following our last article [44].

5.1 History and comparisons

Patterson and Walker

The concept of the *Riemann extension* appeared for the first time in the article [54] by Patterson and Walker. This is a pseudo-Riemannian metric of split signature (n,n) with a parallel isotropic distribution of rank n induced by a geometric data on an n-dimensional manifold. Already in that reference, there are considered several variants of such construction, which are further modified and generalized by many authors. What we mean by the *Patterson–Walker metric* is the Riemann extension induced by a torsion-free affine connection as follows. Let (x^A) be local coordinates on M, let ${\Gamma_A}^C{}_B = {\Gamma_B}^C{}_A$ be the Christoffel symbols of a torsion-free affine connection D on M and let (p_A) be the extended coordinates on the bigger manifold.[1] Then the associated Patterson–Walker metric is given by

$$g := 2\,\mathrm{d}x^A \odot \mathrm{d}p_A - 2\,{\Gamma_A}^C{}_B\, p_C\, \mathrm{d}x^A \odot \mathrm{d}x^B. \tag{5.1}$$

It is easy to see that the distribution spanned by ∂_{p_A} is isotropic and parallel with respect to the Levi-Civita connection of g. Besides concrete local expressions and relations, it is indicated already in [54] how to interpret the metric g in large, namely, as living on the total space of the cotangent bundle T^*M. We come back to this issue later in this section.

Right after this initiation, many related contributions appeared, including a local characterization of Patterson–Walker metrics and reflections of various specific properties. For instance, we list several typical statements:

- D is flat if and only if g is flat,
- D is Ricci flat if and only if g is Ricci flat,
- D is projectively flat if and only if g is conformally flat.

These and similar results can be found in papers by Patterson, Walker, Afifi and others, see, e.g., [66, chapter VII, §15] for a quick introduction and many references.

Dunajski and Tod

In the 2-dimensional case, the previous construction was modified by Dunajski and Tod in [30] so that single affine connection is substituted by a whole projective class. This was done using the Thomas projective parameters in the place of Christoffel symbols,

$$\bar{g} := 2\,\mathrm{d}x^A \odot \mathrm{d}p_A - 2\,{\Pi_A}^C{}_B\, p_C\, \mathrm{d}x^A \odot \mathrm{d}x^B, \tag{5.2}$$

where the symbols ${\Pi_A}^C{}_B$ are defined by (3.2).

[1] We use the notation that fits best to our later development, in particular, uppercase indices range from 1 to $n = \dim M$, while lowercase ones will range from 1 to $2n$.

It follows that the conformal structure represented by $\bar{g}$ is anti-self-dual and admits a light-like conformal Killing field, namely, $k := p_A \partial_{p_A}$. A close relation between the two structures is demonstrated in [30, Theorem 4.1] which states that the initial projective structure is metrizable if and only if the induced conformal structure contains Kähler or para-Kähler metric.

A proper interpretation of the extended manifold, an alternative description of the induced conformal class and its characterization is commented below.

Nurowski and Sparling

It is a useful observation and one of our research motivations that the previous construction can be directly related to the one by Nurowski and Sparling in [53], which is demonstrably a construction of Fefferman type. In that article, authors are concerned with the geometry of 2-order ODE,

$$y'' = Q(x, y, y'), \tag{5.3}$$

in the plane (x, y) modulo point transformations; cf. remarks around (3.6). Since, in this dimension, the corresponding path geometry is equivalent to a Lagrangean contact structure, the article is designed in full analogy with the Cartan's original treatment of 3-dimensional CR contact structures. In particular, the equivalence method yields the normal Cartan connection ω associated to the ODE together with the fundamental curvature invariants.

Moreover, this approach allows to define an analogue of the Fefferman conformal metric on a 4-dimensional quotient of the Cartan's principal bundle. A representative metric from the conformal class is then explicitly given by

$$\widetilde{g} := (\mathrm{d}p - Q\,\mathrm{d}x) \odot \mathrm{d}x - (\mathrm{d}y - p\,\mathrm{d}x) \odot \left(\tfrac{2}{3} i\,\mathrm{d}\phi + \tfrac{2}{3} Q_p\,\mathrm{d}x + \tfrac{1}{6} Q_{pp}(\mathrm{d}y - p\,\mathrm{d}x)\right), \tag{5.4}$$

where $(x, y, p = y', \phi)$ are local extended coordinates on the Fefferman space and i is a non-zero real constant; see [53, formula (31)].

It is shown in [53, Proposition 1] that the conformal structure $[\widetilde{g}]$ is half-flat (i.e., self-dual or anti-self-dual) if and only if the Cartan connection ω is half-flat (i.e., one of the two fundamental invariants vanishes). In particular, this happens for ODEs which are point-equivalent to a geodesic equation or, in other words, for Lagrangean contact structures which are induced by a projective structure. In such case, the resulting conformal structure is the result of composition of two functorial construction, namely, the correspondence space construction (from projective to Lagrangean contact) and the Fefferman-type construction (from Lagrangean contact to conformal).

Direct comparison

In order to compare the current approach with the setting of Dunajski and Tod, we just need to expand the condition that the ODE (5.3) is point-equivalent to a geodesic equation. From (3.7) we know that this is equivalent to

$$Q = A_0 + A_1 p + A_2 p^2 + A_3 p^3, \tag{5.5}$$

where A_i are functions only of x and y and $p = y'$. Computing partial derivatives Q_p and Q_{pp} and substituting into (5.4) yields

$$\begin{aligned} \widetilde{g} = \mathrm{d}p \odot \mathrm{d}x + \tfrac{2i}{3} p\,\mathrm{d}\phi \odot \mathrm{d}x - \tfrac{2i}{3}\,\mathrm{d}y \odot \mathrm{d}\phi - \\ - (A_0 + \tfrac{1}{3} A_1 p)\,\mathrm{d}x \odot \mathrm{d}x - \tfrac{2}{3}(A_1 + A_2 p)\,\mathrm{d}x \odot \mathrm{d}y - (\tfrac{1}{3} A_2 + A_3 p)\,\mathrm{d}y \odot \mathrm{d}y. \end{aligned} \tag{5.6}$$

To compare this metric with the one from (5.2), we need some relations among the coefficients in (5.5), the Christoffel symbols of a representative connection and the Thomas projective parameters. Considering the representative connection is torsion-free, it easily follows the relations are

$$A_0 = -\Gamma^2_{11}, \quad A_1 = \Gamma^1_{11} - 2\Gamma^2_{12}, \quad A_2 = 2\Gamma^1_{12} - \Gamma^2_{22}, \quad A_3 = \Gamma^1_{22},$$
$$\Pi^2_{11} = -A_0, \quad \Pi^1_{11} = -\Pi^2_{21} = -\Pi^2_{12} = \tfrac{1}{3} A_1, \quad \Pi^1_{12} = \Pi^1_{21} = -\Pi^2_{22} = \tfrac{1}{3} A_2, \quad \Pi^1_{22} = A_3,$$

cf., e.g., [6]. This gives the following expression of the metric from (5.2),

$$\begin{aligned}\bar{g} = \mathrm{d}p_1 \odot \mathrm{d}x^1 + \mathrm{d}p_2 \odot \mathrm{d}x^2 - \\ - \tfrac{1}{3}A_1 p_1 \,\mathrm{d}x^1 \odot \mathrm{d}x^1 - \tfrac{2}{3}A_2 p_1 \,\mathrm{d}x^1 \odot \mathrm{d}x^2 - A_3 p_1 \,\mathrm{d}x^2 \odot \mathrm{d}x^2 + \\ + A_0 p_2 \,\mathrm{d}x^1 \odot \mathrm{d}x^1 + \tfrac{2}{3}A_1 p_2 \,\mathrm{d}x^1 \odot \mathrm{d}x^2 + \tfrac{1}{3}A_2 p_2 \,\mathrm{d}x^2 \odot \mathrm{d}x^2.\end{aligned} \tag{5.7}$$

Now, employing a coordinate transformation

$$x = x^1, \quad y = x^2, \quad p = -\tfrac{p_1}{p_2}, \quad \phi = \tfrac{3}{2i} \ln |p_2|,$$

it is a straightforward exercise to show that

Proposition 5.1. *Metrics* (5.6) *and* (5.7) *are conformally equivalent.*

Conceptual comparison

An alternative way to realize the whole story is to interpret, and then modify, appropriately the original Patterson–Walker construction. The first part of this program is scattered in literature, the second part, including a comparison with previously used Thomas projective parameters, is novel. This approach, initiated by [29], is developed in [42]. General interpretation and further investigation of this construction as a construction of Fefferman type is the subject of next section.

In what follows we consider a smooth manifold M of general dimension n and a torsion-free affine connection D. As before, let (x^A) be local coordinates on M and let $\Gamma_A{}^C{}_B$ be the Christoffel symbols of D. But, now, let (p_A) be the dual fibre coordinates of T^*M, i.e., the ones so that $\theta = p_A \,\mathrm{d}x^A$ is the tautological 1-form, and let g be the Patterson–Walker metric on T^*M defined by (5.1). The connection D determines (and is determined by) a horizontal distribution $H \subset TT^*M$ which is complementary to the vertical distribution V of the bundle projection $\pi : T^*M \to M$. Via the tangent map of π, the bundle H is isomorphic to TM, whilst V is canonically isomorphic to T^*M. Concretely, the vertical and the horizontal distribution is spanned by ∂_{p_A} and $\partial_{x^A} + \Gamma_A{}^C{}_B \, p_C \, \partial_{p_B}$, respectively. The Patterson–Walker metric g is the unique one such that both V and H are isotropic with respect to g and the value of g with one entry from V and another entry from H is given by the natural pairing between $V \cong T^*M$ and $H \cong TM$. I.e., for any $\xi, \eta \in H$ and $\varphi, \psi \in V$,

$$g(\xi, \eta) = g(\varphi, \psi) = 0, \quad g(\xi, \varphi) = \varphi(T\pi \cdot \xi).$$

Note that, extending the previous natural pairing between V and H in anti-symmetric (rather than symmetric) way, one recovers the canonical symplectic form on T^*M,

$$\mu = \mathrm{d}\theta = \mathrm{d}p_A \wedge \mathrm{d}x^A. \tag{5.8}$$

The relation to the Patterson–Walker metric g is provided by the almost para-complex structure, K, induced by the decomposition $H \oplus V = TT^*M$ so that H and V are its $+1$ and -1 eigenspace, respectively. It follows that g is para-Hermitean with respect to K, i.e., $g(K-, K-) = -g(-,-)$, and $g(-,-) = \mu(K-,-)$. The vector field corresponding to the tautological 1-form with respect to g, respectively μ, is the Euler vector field of the fiber vector structure,

$$k := p_A \partial_{p_A}. \tag{5.9}$$

This is an important companion and one of the characteristics of the Patterson–Walker metrics as it is its homothety, i.e., a conformal Killing field with constant divergence.

For the projective-to-conformal variant of the previous construction, let us consider its obvious modification to weighted cotangent bundles $T^*M(w)$, with general w. In such cases, in order to trivialize the density bundle, we have to assume that M is orientable and an affine connections on M are special, i.e., volume preserving. Now, the effect of the change of the underlying affine connection can be directly spotted with the following outcome:

Proposition 5.2 ([42, Proposition 3.1]). *Let D and $\widehat{D}$ be projectively equivalent special torsion-free affine connections on M and let g and $\hat{g}$ be the associated Patterson–Walker metrics on $T^*M(w)$. Then g and $\hat{g}$ are conformally equivalent if and only if $w = 2$.*

Note that the definition (5.2) using Thomas projective parameters can be considered in any dimension. It is an instructive exercise to directly compare that approach with the just mentioned one, see [42, remark 3.3] for the details.

Altogether, the construction we are going to investigate associates to an oriented projective structure on M a conformal structure on $T^*M(2)$, which we call the *conformal extension* or *conformal Patterson–Walker metric.* The induced conformal structure is also oriented and, moreover, it can be lifted to a conformal spin structure. We employ this addition as a convenient interpretative instrument.

5.2 The action: non-normal Fefferman-type construction

Before we launch the Fefferman-type construction covering the previous observations, we have to consult the model situation. Important preparatory observations were done in section 3.5, in which setting we continue here. In order to be in concordance with our main reference [43], we only decrease n (which controls dimensions of all the spaces around) by 1. We also consider the spin group $\widetilde{G} = Spin(n+1, n+1)$, instead of $SO(n+1, n+1)$, as the principal one. The embedding

$$\phi : G = SL(n+1, \mathbb{R}) \hookrightarrow Spin(n+1, n+1), \tag{5.10}$$

which lifts the one from (3.16), is described in [43, section 3.2].

The construction

The group $\widetilde{G}$ acts transitively on null rays in $V = \mathbb{R}^{n+1,n+1}$, whereas G has three orbits in accord with the decomposition (3.15). Let $\widetilde{P} \subset \widetilde{G}$ be the stabilizer of a ray from $\mathcal{N}_0$, i.e., a null ray not contained in $V_+ \cup V_-$, and let $Q = G \cap \widetilde{P}$. Obviously $Q \subset P' \subset P$, where $P' \subset G$ is the stabilizer of the para-complex line containing the ray stabilized by Q and $P \subset G$ is the stabilizer of the ray given by the projection to $V_+ \cong \mathbb{R}^{n+1}$. Hence the homogeneous space G/Q coincides with the open orbit in $\widetilde{G}/\widetilde{P} \cong S^{n,n}$, the Möbius sphere, G/P' coincides with the oriented 'para-complex projectivization' of $\mathcal{N}_0$, the model oriented Lagrangean flag variety, and G/P coincides with the oriented projectivization of V_+, the projective n-sphere.

$$\begin{tikzcd}
 & & (\widetilde{G},\widetilde{\omega}) \arrow[dd] & \\
 & (G,\omega) \arrow[ur, hook, dashed, "\phi"] \arrow[dr] \arrow[dd] \arrow[dddl, bend right] & & \\
 & & \mathcal{P}_+(\mathcal{N}_0) \cong G/Q \subsetneq \widetilde{G}/\widetilde{P} \cong S^{n,n} \arrow[dl] & \text{conformal} \\
 & \mathrm{Flags}_+(1,n,n+1) \cong G/P' \arrow[dl] & \text{Lagrangean} & \\
\mathcal{P}_+(V_+) \cong G/P & \text{projective} & &
\end{tikzcd}$$

The central part of this picture reminds the one from section 3.5, the left part is the standard flag correspondence. In the following, we are primarily interested in the extension from projective to conformal structures. We might therefore ignore the intermediate Lagrangean step, however, we will see it is worth remembering it is naturally there.

For the general construction, let M be an oriented manifold of dimension $n \geq 2$ endowed with a projective structure and let $(\mathcal{G} \to M, \omega)$ be the corresponding normal parabolic geometry of type (G, P). Let $\widetilde{P} \subset \widetilde{G}$ and $Q = G \cap \widetilde{P} \subset P$ be as above and let $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$ and $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ be the corresponding Lie algebras. Let us consider the extension functor of Fefferman type given by the embedding (5.10) and let $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega}^{\mathrm{ind}})$ be the induced parabolic geometry with underlying conformal spin structure on the Fefferman space $\widetilde{M} = \mathcal{G}/Q$. Let us interpret and further investigate the induced conformal structure:

First interpretations

The Fefferman space, as the quotient bundle $\widetilde{M} = \mathcal{G}/Q$ is identified with the associated bundle $\mathcal{G} \times_P (P/Q)$. Focusing on the typical fibre P/Q, one easily shows that this is isomorphic as a P-module to the set of non-zero elements of $(\mathfrak{g}/\mathfrak{p})^*(2)$, the typical fibre of $T^*M(2)$ seeing as an associated bundle to $\mathcal{G}$. Thus, the Fefferman space $\widetilde{M}$ is naturally identified with the weighted cotangent bundle of projective weight 2, without the zero section, see [43, Proposition 3.1]. Similar argument suffices to show that among the metrics from the induced conformal class there are genuine Patterson–Walker metrics, see [43, Proposition 6.2]. (The corresponding scales, i.e., the ones which correspond to projective scales on M, are called reduced.) Hence we know the current construction indeed covers the conformal Patterson–Walker extension described by Proposition 5.2.

As we indicated in previous commentaries, the induced conformal structure has number of special properties. Immediately from the algebraic setup we can easily observe several distinguished objects, which play a role in later characterizations, see [43, Proposition 3.2]. Namely, they are pure tractor spinors, denoted as $\mathbf{s}_E$ and $\mathbf{s}_F$, and a compatible involution of the standard tractor bundle, $\mathbf{K}$, which are all parallel with respect to the induced (not necessarily normal) tractor connection. The underlying objects are pure spinors, η and χ, and a null vector field, k, generating the 1-dimensional intersection of the distributions $\ker \eta$ and $\ker \chi$. Of course, one of these distributions forms the vertical subbundle of the projection $\widetilde{M} \to M$ and the notation is chosen so that $V = \ker \chi$. Note also that the sum of these two distributions coincides with the ortho-complement of k in $T\widetilde{M}$.

Normality

As usual, the critical problem is to detect in which cases the Fefferman-type construction preserves the normality condition. This can be discussed either directly or by analysing the two natural steps which are behind the current construction (cf. the model diagram above): (1) the correspondence space construction from projective to Lagrangean contact structure and (2) the Fefferman-type construction from Lagrangean contact to conformal structure.

From section 3.3 we know that the first construction always preserves the normality. Concerning the second construction, we know from section 3.5 that the classical Fefferman construction preserves the normality only in the torsion-free (integrable) case. The current Lagrangean setting can be dealt in full analogy to the CR case with the very same outcome: here the torsion-freeness means that both Lagrangean subbundles of the contact distribution are integrable. Combining these results, we conclude with the following.

Proposition 5.3 ([43, Proposition 3.8]). *Let $(\mathcal{G} \to M, \omega)$ be a normal projective parabolic geometry and let $\big(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega}^{\mathrm{ind}}\big)$ be the conformal parabolic geometry obtained by the Fefferman-type construction.*

- *If $\dim M = 2$ then $\widetilde{\omega}^{\mathrm{ind}}$ is normal.*
- *If $\dim M > 2$ then $\widetilde{\omega}^{\mathrm{ind}}$ is normal if and only if ω is flat.*

Moreover, independently of the dimension of M, $\widetilde{\omega}^{\mathrm{ind}}$ is flat if and only if ω is flat.

Hence, the normal conformal Cartan connection in the 2-dimensional case is directly obtained by the current Fefferman-type construction, which recovers the Nurowski–Sparling construction; see [43, section 3.7] for a quick summary with a full characterization. In general dimension, this is not the case. However, from section 3.3 we know that one of the two torsions of the intermediate Lagrangean contact geometry is trivial and it is just the contribution of the second one which spoils the normality of the resulting conformal parabolic geometry. Thus, even though this geometry is generally non-normal, we interpret it as 'half-normal' and this is what allows further, and eventually satisfactory, manipulation.

Normalization and characterization

To start with the normalization, one has to quantify the non-normality of the induced Cartan connection, i.e., to specify the values of $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}$. Of course, the discussion includes a lot of technicalities which obstructs formulating results in a too relaxed way. The key observation is in [43, Lemma 4.3], which is enough to follow the normalization process in a reasonable detail allowing some interpretations. The process typically (for |1|-graded geometries) has two steps,

$$\widetilde{\omega}^0 := \widetilde{\omega}^{\mathrm{ind}}, \quad \widetilde{\omega}^1 := \widetilde{\omega}^0 + \Psi^1, \quad \widetilde{\omega}^2 := \widetilde{\omega}^1 + \Psi^2 = \widetilde{\omega}^{\mathrm{nor}}, \tag{5.11}$$

where the correction 1-forms Ψ^1 and Ψ^2 have values in well described submodules of $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ and $\tilde{\mathfrak{p}}_+ = \tilde{\mathfrak{g}}_1$ and are controlled by $\widetilde{\partial}^*\widetilde{\kappa}^0$ and $\widetilde{\partial}^*\widetilde{\kappa}^1$, respectively. An important information about the curvature of the normalized Cartan connection is stated in [43, Proposition 4.6], involving also a first version of the integrability condition that controls values of $\widetilde{\kappa}^{\mathrm{nor}}$ under insertions of vectors that are vertical with respect to the projection $\widetilde{M} \to M$.

Previous results can be phrased so that the tractor spinor $\mathbf{s}_F$ is parallel with respect to the normal tractor connection. In particular, the underlying spinor field χ is pure, the tractor spinor $\mathbf{s}_F$ is its BGG splitting, $\mathbf{s}_F = L_0(\chi)$, and the holonomy of the induced conformal structure is reduced. The other allied tractorial objects, $\mathbf{s}_E$ and $\mathbf{K}$, need not be parallel but it turns out they are also BGG splittings of their underlying objects, $\mathbf{s}_E = L_0(\eta)$ and $\mathbf{K} = L_0(k)$, and, moreover, k is a conformal Killing field. All these informations are condensed in [43, Proposition 4.8].

Interpreting the integrability condition in terms of the conformal Weyl tensor, $\widetilde{W}$, we have at this stage the first half of the characterization theorem:

Theorem 5.4 ([43, Theorem 4.14]). *A split-signature (n, n) conformal spin structure on a manifold $\widetilde{M}$ is (locally) induced by an n-dimensional projective structure via the Fefferman-type construction if and only if the following properties are satisfied:*

- *$\widetilde{M}$ admits a nowhere vanishing light-like conformal Killing field k such that the corresponding tractor endomorphism $\mathbf{K} = L_0(k)$ is an involution, i.e., $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$.*
- *$\widetilde{M}$ admits a pure twistor spinor χ with $k \in \Gamma(\ker\chi)$ such that the corresponding parallel tractor spinor $\mathbf{s}_F = L_0(\chi)$ is pure.*
- *$\mathbf{K}$ acts by minus the identity on $\ker \mathbf{s}_F$.*
- *The following integrability condition holds:*

$$v^a w^c \widetilde{W}_{abcd} = 0, \qquad \text{for all } v, w \in \Gamma(\ker\chi).$$

For the converse direction one may use an auxiliary Cartan connection, given by

$$\widetilde{\omega}' := \widetilde{\omega}^{\mathrm{nor}} - \tfrac{1}{2}\iota_k \widetilde{\kappa}^{\mathrm{nor}}, \tag{5.12}$$

and an adapted version of the correspondence space descending theorem. This manoeuvre is described in [43, section 4.3]. However, a detailed analysis of the Cartan curvatures brings an interesting refinement, which simplifies the previous discussion and allows us to descend directly. It follows that the Cartan connection (5.12) coincides with the induced one and that the difference between the induced and the normal Cartan connection is, according to the notation from (5.11),

$$\Psi^1 = -\tfrac{1}{2}\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}} = \tfrac{1}{2}\iota_k \widetilde{\kappa}^{\mathrm{nor}}, \quad \Psi^2 = 0,$$

see [43, Theorem 5.7]. In particular, the above indicated normalization process completes already in the first step. Moreover, the difference term can be written explicitly using the conformal Weyl and Cotton–York tensor. These results rely on the use of reduced scales and their intrinsic characterization in [43, section 5.1].

Alternative characterization

In [42] we obtained (by different means) a slightly different version of the characterization theorem 5.4. This is formulated fully in terms of underlying objects, which is also why we speak about the conformal Patterson–Walker metric rather than Fefferman-type construction:

Theorem 5.5 ([42, Theorem 1], [43, Theorem 6.3])**.** *A split-signature (n, n) conformal spin structure on a manifold $\widetilde{M}$ is (locally) induced by an n-dimensional projective structure as a conformal Patterson–Walker metric if and only if the following properties are satisfied:*

- *$\widetilde{M}$ admits a nowhere vanishing light-like conformal Killing field k.*
- *$\widetilde{M}$ admits a pure twistor spinor χ such that $\ker\chi$ is integrable and $k \in \Gamma(\ker\chi)$.*
- *The Lie derivative of χ with respect to the conformal Killing field k is*

$$\mathcal{L}_k\chi = -\tfrac{1}{2}(n+1)\chi. \tag{5.13}$$

- *The following integrability condition holds:*

$$v^r w^s \widetilde{W}_{arbs} = 0, \qquad \text{for all } v^r, w^s \in \Gamma(\ker\chi).$$

It is easy to see that this characterization follows from the one in Theorem 5.4. For the converse direction, one has to consider the BGG splittings $\mathbf{K} = L_0(k)$ and $\mathbf{s}_F = L_0(\chi)$, translate their relationship and, in particular, show that $\mathbf{K}$ is an involution of the standard tractor bundle. All this can be found in [43, section 6.2].

In both characterizations we collect just minimal data allowing a reconstruction of the underlying projective structure. From the previous exposition we know there are more typical features of such conformal structures. For instance, the subtle condition in (5.13) implies, among other things, that k is a homothety (and not just a conformal Killing field) of a representative metric for which χ is parallel. This also plays the role of Euler vector field of the fiber vector structure of the (local) identification $\widetilde{M} \cong T^*M(2)$, cf. (5.9). Moreover, there is another accompanying object, namely, the pure spinor η, which pairs non-trivially with χ and whose tractorial counterpart was denoted as $\mathbf{s}_E$. This spinor field satisfies the twistor spinor equation only in the flat case.

Remarks

So far we concentrated on various interpretations and characterizations of conformal extensions of projective structures. Concerning further relations of the two structures, there is a great space to investigate, although a lot is already known, cf. section 5.1. In the article [42] we were able to fully describe, in underlying projective terms, conformal Killing fields of conformal extensions and Einstein metrics contained in the conformal class. This analysis naturally includes also the original Patterson–Walker construction. For an illustration and later use, let us mention several facts:

Each conformal, respectively genuine, Killing field of a Patterson–Walker metric can be uniquely written as a sum of several individuals, one of which is a lift of a projective, respectively affine, infinitesimal symmetry. It follows that the Killing field on T^*M corresponding to an infinitesimal affine symmetry on M is just its Hamiltonian lift with respect to the canonical symplectic structure (5.8).

5.3 Further action: ambient constructions

Beside the properties and relations discussed so far, there are other striking features of conformal Patterson–Walker metrics. Namely, they admit a global Fefferman–Graham ambient metric which, moreover, has a very easy form as it is also the Patterson–Walker metric of the Thomas connection, the ambient linear connection associated to the underlying projective structure. In particular, the Fefferman–Graham obstruction tensor (which is the Bach tensor in dimension four) vanishes. Following [44], we exhibit some details.

Ambient relations

The ambient description for projective and conformal structures is introduced in section 3.2 and 3.4, respectively. In the following diagram we indicate the individual associations: vertically goes the Thomas cone, respectively the Fefferman–Graham ambient, construction, horizontally goes the Patterson–Walker construction, respectively its conformal version.

$$\begin{tikzcd}
\text{affine} & (\widehat{M}, \widehat{\nabla}) \arrow[r, rightsquigarrow] & (\mathbf{M}, \mathbf{g}) & \text{Riemannian} \\
\text{projective} & (M, [D]) \arrow[r, rightsquigarrow] \arrow[u, rightsquigarrow] & (\widetilde{M}, [g]) \arrow[u, rightsquigarrow] & \text{conformal}
\end{tikzcd}$$

It is an easy observation that these constructions nicely commute in the homogeneous setting. In particular, according to the description in section 3.5, the conformal ambient space $\mathbf{M}$ coincides with $V = \mathbb{R}^{n+1,n+1} = V_+ \times V_-$, which is naturally identified with T^*V_+ for current needs. In the general setting, it is not a priori clear whether these relations still hold. One can, however, trace the individual constructions indicated in the diagram and compare the results in the upper right corner. The key facts—and strong motivations for such an endeavour—are that the normalization of both Thomas and Fefferman–Graham construction is controlled by the Ricci flatness of the respective ambient objects and this property is preserved by the Patterson–Walker construction. Elaborating these ideas in detail, we conclude with

Theorem 5.6 ([44, Theorem 2])**.** *Given a projective structure $[D]$ on M, the geometric constructions indicated in the previous diagram commute. In particular, the induced conformal structure $[g]$ admits a globally Ricci flat Fefferman–Graham ambient metric $\mathbf{g}$ which is itself a Patterson–Walker metric.*

In the proof we just compose the concrete expressions of the Thomas ambient connection and the Patterson–Walker extension and, in order to accomplish the ansatz of (3.10), one innocent coordinate transformation. This in turn yields the following explicit description of the ambient metric:

Theorem 5.7 ([44, Theorem 1])**.** *Let D be a torsion-free affine connection on M which preserves a volume form. Denote local coordinates on M by (x^A) and the induced canonical fibre coordinates on T^*M by (p_A). Let ${\Gamma_A}^C{}_B$ and Ric_{AB} denote the Christoffel symbols and the Ricci curvature of D, respectively. Let*

$$g = 2\, \mathrm{d}x^A \odot \mathrm{d}p_A - 2\, {\Gamma_A}^C{}_B\, p_C\, \mathrm{d}x^A \odot \mathrm{d}x^B$$

be the Patterson–Walker metric induced on T^*M by D. Then

$$\begin{aligned}\mathbf{g} = {} & 2\,\rho\, \mathrm{d}t \odot \mathrm{d}t + 2\,t\, \mathrm{d}t \odot \mathrm{d}\rho + \\ & + t^2 \left(2\, \mathrm{d}x^A \odot \mathrm{d}p_A - 2\, {\Gamma_A}^C{}_B\, p_C\, \mathrm{d}x^A \odot \mathrm{d}x^B + \frac{2\rho}{n-1} \mathrm{Ric}_{AB}\, \mathrm{d}x^A \odot \mathrm{d}x^B \right)\end{aligned}$$

is a globally Ricci flat Fefferman–Graham ambient metric for the conformal class $[g]$.

Comparing with (3.11) we see that we have the simplest conceivable description: the current ambient metric is linear in ρ, i.e., following the iterative procedure to obtain this local expression from a representative metric, it actually stops after the first step.

Consequences

As an immediate consequence of the existence of ambient metric, the Fefferman–Graham obstruction tensor of the conformal structure whose representative metric is a Patterson–Walker metric vanishes. This is the Bach tensor in dimension four. In general dimension, Bach tensor is not conformally invariant and it has nothing to do with the obstructions tensor. However, it follows that for Patterson–Walker metrics the Bach tensor has to vanish as well.

Beside these restrictions, there is another one that is easy to read off the current description. Namely, the Q-curvature of a Patterson–Walker metric vanishes as well. Although it is generally very difficult to compute the Q-curvature for a given metric, things simplify significantly in the current situation. This is shown in [44, Theorem 3], whose proof is based on the method from [34] and a quick analysis of the ambient Laplace operator.

Remarks

Note that the other ambient objects, namely, the cone $\mathcal{N} \subset \mathbf{M}$ and the Euler vector field $\mathbf{E}$ of the $\mathbb{R}_+$-action, are described in previous local coordinates by $\rho = 0$ and $\mathbf{E} = t\,\partial_t$, respectively. However, by the very construction, there is a more conceptual description. Firstly, as for any Patterson–Walker metric, there is a homothety $\mathbf{k}$ of $\mathbf{g}$. Secondly, the Euler vector field of the $\mathbb{R}_+$-action on the Thomas cone $\widehat{M}$ is an infinitesimal affine symmetry of $\widehat{\nabla}$, therefore its Hamiltonian lift is a Killing field $\mathbf{H}$ of $\mathbf{g}$. It easily follows, that Euler field $\mathbf{E}$ is the sum of $\mathbf{H}$ and (an appropriate) multiple of $\mathbf{k}$. Having this, it further follows that $\mathbf{g}(\mathbf{E}, \mathbf{E})$ is the defining function for $\mathcal{N} \subset \mathbf{M}$.

6 Conformal theory of curves

There is hardly more classical subject in differential geometry than the local geometry of curves in Euclidean spaces. Nowadays, many attempts to pass the study to other geometries, both homogeneous and curved, are known. In this section we present several variations of the typical construct, the Frenet–Serret frame, in the conformal geometry.

In the introductory overview we comment on the very original construction and several branches of its possible generalizations: from Euclidean to (curved) Riemannian, from Euclidean to (homogeneous) Möbius and from Riemannian to general conformal setting. In this part we follow two of three main sources of inspiration, namely, [63, 36].

$$\begin{tikzcd}
\text{Riemannian} \arrow[r, squiggly] & \text{conformal} \\
\text{Euclidean} \arrow[r, squiggly] \arrow[u, squiggly] & \text{Möbius} \arrow[u, squiggly]
\end{tikzcd}$$

The third inspiration comes from [2], where the tractorial techniques were used for the description of distinguished curves (conformal circles) and distinguished parametrizations of general curves. We can push these ideas forward so that a tractor analogue of the Frenet frame is constructed, which yields a generating set of absolute conformal invariants of a general curve. This is described in section 6.2 following our article [62], which is based on [60]. This procedure is presented as a direct passage from (homogeneous) Möbius to general conformal setting, which encompasses other approaches and, after possible expansion of tractorial formulas, allows their comfortable comparisons.

On the way, we also meet a natural family of relative conformal invariants affording further applications and generalizations. As expected, these techniques can be adapted to other parabolic geometries, which is indicated in concluding remarks.

6.1 History and initiation

Frenet, Serret and others

For a general curve in Euclidean space, there are several ways to quantify its curving. The idea of Frenet and Serret in 3-dimensional case, later generalized by Jordan to any dimension, is based on the association of a natural orthonormal frame along the curve and the expression of its rate of change with respect to the natural parametrization, i.e., the one by the arc-length. The resulting system is known as Frenet–Serret (or just Frenet, for short) equations, whose coefficients form a distinguished set of (absolute) metric invariants of the curve.

For the arc-length parametrization of a generic curve, $c : I \to \mathbb{R}^n$, the natural starting object is the unit tangent vector $e_1 = c'$; the whole Frenet frame is built from its higher derivatives, c'', c''', etc., by the Gram–Schmidt process. Denoting the Frenet frame by $(e_1, \dots, e_n)$, the Frenet equations read as

$$\begin{aligned} e_1' &= \kappa_1 e_2, \\ e_i' &= -\kappa_{i-1} e_{i-1} + \kappa_i e_{i+1}, \qquad \text{for } i = 2, \dots, n-1, \\ e_n' &= -\kappa_{n-1} e_{n-1}, \end{aligned} \tag{6.1}$$

where the functions $\kappa_1, \dots, \kappa_{n-1}$ are the curvatures of the curve. The obvious symmetries of the system reflect the fact that the frame is orthonormal. For later use, let us remark that the expressions of initial derived vectors with respect to the Frenet frame looks like

$$c' = e_1, \quad c'' = \kappa_1 e_2, \quad c''' = -\kappa_1^2 e_1 + \kappa_1' e_2 + \kappa_1 \kappa_2 e_3, \quad \text{etc.} \tag{6.2}$$

Previous concepts were generalized to Riemannian manifolds at latest in the article [5] by Blaschke. Basically, one has to substitute the Riemannian metric for the inner product and the corresponding (Levi-Civita) covariant derivative for the ordinary flat one. Beside the immediate expressions of curvatures involving the frame constituents and their derivatives, $\kappa_i = e_i' \cdot e_{i+1}$, there are expressions in terms of initial derived vectors and volumes of corresponding parallelepipeds. The latter formulas can be adapted easily with respect to an arbitrary parametrization of the curve, cf. [41, 61]. This is an easy but important remark because it shows how to express absolute invariants of curves in terms of relative ones, i.e., those which change by a functional multiple relative to a concrete parametrization of the curve. Here the natural family of relative invariants is formed by volumes of the parallelepipeds given by first k derived vectors along the curve, where $k = 2, \dots, n$.

With the help of relative invariants one may characterize special curves for which the full Frenet frame is not available. For instance, Euclidean straight lines, respectively Riemannian geodesics, are the curves for which the simplest relative invariant (corresponding to $k = 2$ in the previous description) vanishes.

Cartan

The Frenet frame is a particular instance of a much more general concept, the Cartan's moving frame. From this perspective, the Frenet frame along a curve on M is an adapted lift to the bundle of orthonormal frames $\mathcal{G} \to M$ (whose total space coincides with the group $G = Euc(n)$ of rigid motions in the flat case). If $\hat{c} : I \to \mathcal{G}$ is such a lift of the curve $c : I \to M$ and ω is the Cartan connection on $\mathcal{G}$ with values in the Lie algebra $\mathfrak{g} = \mathbb{R}^n \ltimes \mathfrak{o}(n)$ then, following the conventions as in section 3.1, the Frenet equations (6.1) are written as

$$\hat{c}^* \omega = \left(\begin{array}{c|ccccc} & & & & & \\ \hline 1 & & -\kappa_1 & & & \\ & \kappa_1 & & & & \\ & & \ddots & & \ddots & \\ & & & & & -\kappa_{n-1} \\ & & & & \kappa_{n-1} & \end{array} \right).$$

The first column in this decomposition just says that $c' = e_1$, i.e., the tangent vector of the curve is the first vector from the frame.

The point is that Cartan's method allows to determine the lift by a successive reduction of the structure group yielding the invariants with increasing order.

Schiemangk and Sulanke

In this manner, submanifolds of Möbius space S^n were studied in the series of articles starting with [56, 63] by Schiemangk and Sulanke, where the study is subordinated to the bigger group $G = O(n+1,1)$ of conformal transformations. In the second reference, a lift to G along a generic curve $c : I \to S^n$ is constructed, which is, by nature, a pseudo-orthonormal moving frame in the ambient Minkowski space $\mathbb{R}^{n+1,1}$. As a natural starting object (called the *normed representation* of the curve) it is taken a curve of null vectors in $\mathbb{R}^{n+1,1}$ whose tangent vectors have constant length one.

If $\mathfrak{b} : I \to \mathbb{R}^{n+1,1}$ denotes such a lift, $\mathfrak{b}'$, $\mathfrak{b}''$ etc. its derivatives and $\cdot$ the inner product in the ambient vector space, then the assumptions yield the following couple of relations

$$\begin{aligned} \mathfrak{b}\cdot\mathfrak{b} = 0, \qquad \mathfrak{b}\cdot\mathfrak{b}' &= 0, \qquad \mathfrak{b}\cdot\mathfrak{b}'' = -1, \\ \mathfrak{b}'\cdot\mathfrak{b}' &= 1, \qquad \mathfrak{b}'\cdot\mathfrak{b}'' = 0, \end{aligned} \tag{6.3}$$

with the first non-trivial pairing being $\mathfrak{b}''\cdot\mathfrak{b}''$. From this and the next derived vector, a conformally invariant 1-form along the curve is extracted, namely,

$$\mathrm{d}s = \sqrt[4]{\mathfrak{b}'''\cdot\mathfrak{b}''' - (\mathfrak{b}''\cdot\mathfrak{b}'')^2}\,\mathrm{d}t, \tag{6.4}$$

where t is the initial parameter of c. Integration of this 1-form gives a natural parametrization of the curve (up to additive constant), the so-called *conformal arc-length*.

At this stage it is easy to complete the full frame and to express its change with respect to the natural parametrization. Hence a conformal analogue of Frenet equations and $n-1$ distinguished conformal invariants follow.

It has to be noted that there are many particular results predating the just described ones, which would deserve careful comparisons. The list of related contributions is really huge and we intentionally ignore most of them here. Satisfactory surveys can be found, e.g., in the introductions of the articles [8, 7]. As an instance, we remark that the form (6.4) can be expressed in terms of Euclidean curvatures so that

$$\mathrm{d}s = \sqrt[4]{\kappa_1'^2 + \kappa_1^2\kappa_2^2}\,\mathrm{d}t,$$

which is a formula that appears repeatedly in older literature, cf. [63, Proposition 1.3]. (Here, t is the Euclidean arc-length parameter, κ_1, κ_2 are given by (6.1) and also (6.2) is used.)

Fialkow

To our knowledge, the first publications where the conformal invariants of curves are studied in full generality (in positive definite signature) are [35, 36] by Fialkow, based on tensorial techniques. In order to build the conformal Frenet frame and hence the distinguished conformal curvatures of a curve in analogy to the general Riemannian setting, one has to describe a couple of conformally invariant objects first:

- a natural starting object (in the place of the unit tangent vector),
- a natural derivative along the curve (in the place of the restricted Levi-Civita connection),
- a natural distinguished parametrization of the curve (in the place of the arc-length parameter).

In [36, sections 3–4], these objects are explicitly introduced and named the *first principal normal*, *conformal derivative* and *conformal arc-length*, respectively. Then, for generic curves, the full frame may be constructed and the conformal curvatures extracted. Notice that there is an exceptional curvature, completing the system of generating invariants, which does not follow from the Frenet process and is constructed independently in [36, section 5]. Opening the article, one quickly realizes that many formulas seem a bit complicated, if not mysterious. We are going to show how to recover them using tractor calculus, from which—by the way—it should be clear they cannot be easier.

Before we continue, let us remark the many observed subtleties follow from a sizable freedom in the change of covariant derivative with respect to a choice of scale, respectively Weyl structure. It follows from (3.9) that, for any curve, a metric in the conformal class may be chosen so that the curve is affinely parametrized geodesic. This, in particular, reveals the problem with conformally invariant notion of osculating subspaces etc.

Bailey, Eastwood and Gover

Here we introduce the tractor approach to conformal geometry of curves following the article [2] by Bailey, Eastwood and Gover (and adapting to our notation). For the standard tractor bundle, $\mathcal{T} \to M$, associated to a conformal Riemannian structure on M as in 3.12, there is the natural insertion $\mathcal{E}[-1] \cong \mathcal{T}^1 \hookrightarrow \mathcal{T}$ which we denote by $\boldsymbol{X}$. Considering a regular curve $c : I \to M$, a natural conformal density along the curve is the length of its tangent vector, $u := \|c'\| \in \mathcal{E}[1]$. Hence we have a lift to the standard tractor bundle, $\boldsymbol{T} := u^{-1}\boldsymbol{X}$, which is the natural starting object in the current setting.

$$
\begin{tikzcd}
 & \mathcal{T} \arrow[d] \\
I \arrow[ur, dashed, "\boldsymbol{T}"] \arrow[r, "c"'] & M
\end{tikzcd}
$$

Denoting by $\boldsymbol{U}$, $\boldsymbol{U}'$ etc. the derivatives of $\boldsymbol{T}$ with respect to the standard tractor connection along the curve and by $\cdot$ the standard tractor metric, the following relations are easily satisfied

$$
\begin{aligned}
\boldsymbol{T}\cdot\boldsymbol{T} = 0, \qquad \boldsymbol{T}\cdot\boldsymbol{U} &= 0, \qquad \boldsymbol{T}\cdot\boldsymbol{U}' = -1, \\
\boldsymbol{U}\cdot\boldsymbol{U} &= 1, \qquad \boldsymbol{U}\cdot\boldsymbol{U}' = 0,
\end{aligned}
\tag{6.5}
$$

with the first non-trivial pairing being $\boldsymbol{U}'\cdot\boldsymbol{U}'$.

It is shown in [2, Proposition 2.11] that this quantity changes under reparametrizations of the curve so that

$$
\widetilde{\boldsymbol{U}}'\cdot\widetilde{\boldsymbol{U}}' = g'^{-2}\left(\boldsymbol{U}'\cdot\boldsymbol{U}' - 2\mathcal{S}(g)\right), \tag{6.6}
$$

where the objects decorated with tildes are related to the reparametrization $\tilde{t} = g(t)$ and $\mathcal{S}$ denotes the *Schwarzian derivative*,

$$
\mathcal{S}(g) := \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 .
$$

Consequently, vanishing of $\boldsymbol{U}'\cdot\boldsymbol{U}'$ determines a preferred family of parametrizations of the curve with freedom given by the projective group of the line. In this derivation, there is no need to rummage the tractor entries (basically, the chain rule and the Leibniz rule suffice). The same result was obtained in [3, Proposition 4.1] by a direct manipulation with the expanded equation, which is seen as direct proof of one of Cartan's observations in [19]. Any parameter from this preferred family is called *projective*.

Next, it is shown in [2, Proposition 2.12] how to characterize conformal circles in terms of tractors. *Conformal circles* are distinguished curves of conformal structures in the sense of section 2.3. The name reflects the fact that on the Möbius sphere they are ordinary circles. They can be described by a system of ODEs of 3-order. Expanding the following tractorial equations and comparing with the determining system of ODEs, it turns out that the curve is a projectively parametrized conformal circle if and only if it obeys

$$
\boldsymbol{U}'\cdot\boldsymbol{U}' = 0 \quad \text{and} \quad \boldsymbol{U}'' = 0. \tag{6.7}
$$

We are going to continue this game in the next section.

Remarks

At latest after the comparison of starting relations (6.3) and (6.5), it should be clear that the two approaches have the same footing. Indeed, the curved version of the former construction means lifting to the Fefferman–Graham ambient space, $\mathbf{M}$, and from section 3.4 we know how such things are related to standard tractors:

Let the lift to the ambient space be a curve contained in the null cone, $\mathcal{N} \subset \mathbf{M}$, whose tangent vector field has constant length one. The lift can be interpreted as a choice of scale along the curve and its tangent vector field can be uniquely extended to a vector field of homogeneous degree -1 in the preimage of the curve with respect to the projection $\mathcal{N} \to M$. After factorization by the $\mathbb{R}_+$-action, this yields a section of the standard tractor bundle along the curve coinciding with $\boldsymbol{T}$ above.

6.2 The action: tractor Frenet construction

Before constructing the full Frenet frame using tractors, we prefer to start with the notion of relative invariants and an alternative characterization of conformal circles. The starting assumptions are as yet, in particular, we consider conformal structures of Riemannian signature, in which case the standard tractor metric has signature $(n+1, 1)$, where $n = \dim M$. Remarks on general indefinite signature follow.

Relative conformal invariants and conformal geodesics

A *relative conformal invariant of weight k* of a curve is a conformally invariant function, I, along the curve that transforms under any reparametrization, $\tilde{t} = g(t)$, as $\widetilde{I} = g'^{-k} I$. In particular, vanishing, respectively non-vanishing, of any relative invariant is independent of reparametrizations. Conformal invariants of weight 0 are the *absolute* ones.

The easiest relative invariant one can find by hand is the one corresponding to (6.4). Indeed, it is an exercise to show (by continuing the reasoning behind (6.6)) that the function

$$\boldsymbol{U}'' \cdot \boldsymbol{U}'' - (\boldsymbol{U}' \cdot \boldsymbol{U}')^2 \tag{6.8}$$

is a relative conformal invariant of weight 4. However, this is also (up to the sign) the first non-trivial determinant from the sequence of Gram matrices corresponding to $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \boldsymbol{U}'', \ldots$ Let us denote by Δ_i the determinant determined by the first i tractors from this sequence (in particular, (6.8) corresponds to $-\Delta_4$). It easily follows that, for $i = 4, \ldots, n+2$, the Gram determinant Δ_i is a relative conformal invariant of weight $i(i-3)$, see [62, Lemma 2.2]. Moreover, all these function are non-positive.

Hence we have a bunch of relative invariants to play with. On the one hand, in generic cases, integrating an appropriate root of any of these functions along the curve yields its natural conformally invariant parametrization. Of course, we take the simplest one, $-\Delta_4$, and call the corresponding parameter *conformal arc-length* as before. In particular, in this parametrization, $-\Delta_4$ equals identically to 1.

On the other hand, vanishing of some of these invariants (which implies vanishing of all higher ones) dictates some degeneracy, respectively specificity, of the curve. Focusing on the simplest one, it is not surprising that we recover the conformal circles:

Proposition 6.1 ([62, Proposition 3.3])**.** *The following conditions are equivalent:*

- *The curve is a conformal circle (with and arbitrary parametrization).*
- *The invariant* $\boldsymbol{U}'' \cdot \boldsymbol{U}'' - (\boldsymbol{U}' \cdot \boldsymbol{U}')^2$ *vanishes identically.*
- *The rank 3 subbundle* $\langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle \subset \mathcal{T}$ *is parallel along the curve.*

This way we extend the characterization from (6.7). Another equivalent conditions are commented in section 6.4.

Tractor Frenet frame and absolute conformal invariants

Having enough relative conformal invariants of a generic curve, one may construct the absolute ones in various ways. Constructions of Frenet type provide a natural and minimal (in the generating sense) set of such invariants. In our situation we may proceed as follows.

Consider a generic curve with the conformal arc-length parametrization and the sequence of derived tractors as before. The restriction of the tractor metric to the subbundle $\langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle$ is non-degenerate and has signature $(2,1)$. Thus its orthogonal subbundle in $\mathcal{T}$ is complementary and has positive definite signature. The initial tractor frame $(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \boldsymbol{U}'', \dots)$ can be transformed into a natural pseudo-orthonormal one $(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{U}_2; \boldsymbol{U}_3, \dots)$ in two steps: the first three tractors are taken to be

$$\boldsymbol{U}_0 := \boldsymbol{T}, \quad \boldsymbol{U}_1 := \boldsymbol{U}, \quad \boldsymbol{U}_2 := -\boldsymbol{U}' - \tfrac{1}{2}(\boldsymbol{U}' \cdot \boldsymbol{U}')\,\boldsymbol{T}, \tag{6.9}$$

the rest is completed by the standard Gram–Schmidt process.

Now the usual techniques yield the tractor version of conformal Frenet equations which are gathered in [62, formula (34)]. In particular, there are $n-1$ distinguished curvatures, $K_1, \dots, K_{n-1}$, among which the first one is slightly exceptional due to the character of the construction. This is summarized in [62, Proposition 4.2] together with immediate expressions of the curvatures in terms of Frenet tractors and their derivatives. At this stage, it is also very easy to say something about the exceptional curvature K_1 (recall the projective family of parametrizations was determined by vanishing of $\boldsymbol{U}' \cdot \boldsymbol{U}'$):

Proposition 6.2 ([62, Proposition 4.3])**.** *The conformal arc-length parameter belongs to the projective family of parameters if and only if $K_1 = 0$.*

From the conceptual point of view, we are basically done. For concrete applications and comparisons, it is useful to express the curvatures K_i with respect to an arbitrary parametrization of the curve. There are several strategies to do so, in [62, section 4.2] we use the relative conformal invariants Δ_j. Of course, the discussion for K_1 is different from the one for remaining curvatures (where we just exploit well-known ideas behind the Gram–Schmidt process). Altogether, we conclude with

Theorem 6.3 ([62, Theorem 4.6 and 4.7])**.** *With respect to an arbitrary parametrization of the curve, the conformal curvatures can be expressed as*

$$K_i = \begin{cases} -\frac{1}{2}(-\Delta_4)^{-\frac{5}{2}} \left(\Delta_4{}^2\, \boldsymbol{U}' \cdot \boldsymbol{U}' - \frac{1}{2}\Delta_4 \Delta_4'' + \frac{9}{16}\Delta_4'^2\right), & \textit{for } i = 1, \\ -(-\Delta_4)^{-\frac{1}{4}} \left(\Delta_{i+1}\Delta_{i+3}\right)^{\frac{1}{2}} \Delta_{i+2}{}^{-1}, & \textit{for } i = 2, \dots, n-1. \end{cases}$$

Remarks on indefinite signature

On a conformal manifold of indefinite signature, a rough local division of curves reflects types of tangent vectors, which may be space-, time- or light-like. The curves of first two types can be lifted to the standard tractor bundle as before (with the obvious sign adjustment in the time-like case), in the third case we have to consider higher derivatives. In any case, though the basic ideas and instruments described above remain, some subtleties appear.

Let us consider space-, respectively time-like, curves here (light-like curves are dealt individually in the next section). The first difference to the previous discussion concerns our favourite relative invariants: because now the standard tractor bundle allows isotropic subspaces of higher dimension, vanishing of Δ_i does not imply the determining tractors are linearly dependent. In particular, in the characterization of conformal circles (which are well defined for any space- or time-like direction) in Proposition 6.1, the second condition is weaker than the others. Also, these invariants may have various signs which should be taken into account when defining the conformal arc-length. In the construction of the tractor Frenet frame, both the subbundle $\langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle \subset \mathcal{T}$ and its orthogonal complement may have sundry signatures. Hence the construction has to be adapted with regard to a finer classification of types of curves, which can be done, e.g., via the sequence of signatures of subbundles generated by the derived tractors of increasing order. This is why the discussion branches quickly and it is fully manageable only in concrete situations, cf. [62, Remark 4.4(3)].

6.3 Further action: r-null curves

For the light-like curves, aka null curves, one has to find a substitute for the density $u = \|c'\| \in \mathcal{E}[1]$ that was formerly used for the lift to $\mathcal{T}$ and that vanishes now. This leads us to the notion of 'order of nullity', respectively *r-null curves*, which are the curves $c : I \to M$ whose first r derived vectors $c', \ldots, c^{(r)}$ are linearly independent and null, whereas $c^{(r+1)}$ is not null. Although the individual higher derivatives depend on the choice of scale according to (3.9), the notion of r-null curves is well defined. On conformal manifolds of signature (p, q), the highest possible r equals to the smaller of the numbers p and q.

For an r-null curve, the norm of $c^{(r+1)}$ provides an appropriate substitute for $u \in \mathcal{E}[1]$, hence the lift to $\mathcal{T}$. In this vein, space- and time-like curves can be seen as 0-null curves and with this perspective one should read the following outcomes. Indeed, most of previous concepts generalize swimmingly up to some shift in orders related to the value of r.

Projective parametrizations and relative invariants

Concerning the analogue of the starting relations (6.5), the pattern is the same only shifted so that the first non-trivial pairing is $\boldsymbol{U}^{(r+1)} \cdot \boldsymbol{U}^{(r+1)}$. Analysis of the change of this quantity with regard to a reparametrization of the curve reveals a similar behaviour as in (6.6). Hence vanishing of this function determines a preferred parametrizations of the curve with the projective freedom:

Proposition 6.4 ([62, Proposition 5.1])**.** *For any admissible r, any r-null curve carries a preferred family of projective parameters.*

Concerning the relative conformal invariants Δ_i, the first non-trivial one is Δ_{2r+4}. Both their total amount and particular weights depend on r so that

Lemma 6.5 ([62, Lemma 5.2])**.** *Along any r-null curve, the Gram determinant Δ_i, for $i = 2r + 4, \ldots, n + 2$, is a relative conformal invariant of weight $i(i - 2r - 3)$.*

As before, the simplest of these invariants can be used to define a natural parametrization of the curve, the so-called *pseudo-arc-length*, provided it is nowhere vanishing. But a simple counts show that all of them may vanish automatically even for generic r-null curves (this happens for maximal r in the case of split signature). This is a good reason to search for another source of relative conformal invariants.

The typical situation with vanishing Δ_{2r+4} occurs when the tractors $\boldsymbol{T}, \boldsymbol{U}, \ldots, \boldsymbol{U}^{(2r+2)}$ are linearly dependent. This condition can be phrased as a tractor linear ODE, whose coefficients are functions expressible in terms of pairings of the initial tractors. To this equation one may associate the so-called *Wilczynski invariants*, which form a set of $2r + 1$ essential invariants denoted as $\Theta_3, \ldots, \Theta_{2r+3}$. As a matter of fact, each Θ_i is a relative conformal invariant of weight i. Expressing the coefficients of our equation, it follows that the subleading one vanishes, in which case it is easier to gain concrete formulas for Θ_i (at least for small i) and they indeed can be found in ancient literature, cf. [57, 65]. Altogether, for an r-null curve, the first three Wilczynski invariants are

$$\Theta_3 = 0, \quad \Theta_4 = -\epsilon\beta - \frac{r(r+3)}{10}\epsilon\alpha'' + \frac{(r+3)(2r+5)(5r+4)}{10(r+1)(r+2)(2r+3)}\alpha^2, \quad \Theta_5 = 0,$$

where $\epsilon := \boldsymbol{U}^{(r)} \cdot \boldsymbol{U}^{(r)} = \pm 1$, $\alpha := \boldsymbol{U}^{(r+1)} \cdot \boldsymbol{U}^{(r+1)}$ and $\beta := \boldsymbol{U}^{(r+2)} \cdot \boldsymbol{U}^{(r+2)}$, see [62, Proposition 5.5].

Note that, for $r = 0$ and $\epsilon = 1$ (which holds in positive definite signature), the invariant Θ_4 recovers Δ_4, i.e., the one in (6.8) up to the sign. For general r, the considered tractor equation is equivalent to the trivial equation $\boldsymbol{U}^{(2r+2)} = 0$ if and only if all its Wilczynski invariants vanish. In our setting, this is equivalent to vanishing of $\boldsymbol{U}^{(j)} \cdot \boldsymbol{U}^{(j)}$, for any $j = r+1, \ldots, 2r+2$. This innocent remark serves as a preparation to an appropriate generalization of the notion of conformal circles.

Conformal null helices

For any space- or time-like curve with an arbitrary parametrization, $c : I \to M$, a metric in the conformal class may be chosen so that $c'' = 0$. For null curves, this condition holds only if the curve is *null geodesic*, i.e. the null curve whose acceleration vector c'' (and consequently all higher order vectors) is proportional to the tangent one c'. These curves, although very important, cannot be lifted to the standard tractor bundle in the sense considered here. That is why they are excluded from our considerations.

For an r-null curve, it follows that the vectors $c', \dots, c^{(2r+1)}$ are linearly independent for any choice of scale. Therefore we cannot achieve any of the conditions $c' = 0, \dots, c^{(2r+1)} = 0$ by a conformal change of metric. The simplest conceivable statement is

Lemma 6.6 ([62, Lemma 5.7])**.** *For an r-null curve with an arbitrary parametrization, a metric in the conformal class may be chosen so that $c^{(r+1)} \cdot c^{(r+1)} = \pm 1$ and $c^{(2r+2)} = 0$.*

Now we are ready to identify the closest relatives of conformal circles among r-null curves. The simplest conceivable generalization of Proposition 6.1 (and surrounding notions) turns out to be

Theorem 6.7 ([62, Theorem 5.8])**.** *For an r-null curve, the following conditions are equivalent:*

- *The curve with a projective parametrization (i.e., satisfying $\boldsymbol{U}^{(r+1)} \cdot \boldsymbol{U}^{(r+1)} = 0$) obeys $\boldsymbol{U}^{(2r+2)} = 0$.*
- *The curve with and arbitrary parametrization obeys that the subbundle $\langle \boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}^{(2r+1)} \rangle \subset \mathcal{T}$ is parallel along the curve and all Wilczynski invariants vanish $\Theta_3 = \dots = \Theta_{2r+3} = 0$.*
- *There is a metric in the conformal class such that $c^{(r+1)} \cdot c^{(r+1)} = \pm 1$, $c^{(2r+2)} = 0$ and $\mathsf{P}(c', -) = 0$.*

The parallelity assumption in the second condition implies $\Delta_{2r+4} = 0$, but the opposite implication does not hold in general (which should be corrected in the original formulation of the statement in [62]). In the flat case, the former condition means that the conformal r-null helix is contained in the conformal image of a $(2r+1)$-dimensional Euclidean subspace. With the third condition, it easily follows that these curves are just the conformal images of so-called *null Cartan helices*, cf. [28]. This is why we call the curves satisfying (any of) previous three conditions the *conformal r-null helices*. Note that these curves are not distinguished curves in the sense of section 2.3, where they are controlled by (subsets of) $\mathfrak{g}_- \subset \mathfrak{g}$. However, weakening this restriction, the currently distinguished curves may be considered in a similar way.

Tractor Frenet frame

Constructing the tractor Frenet frame, we face the very similar problems as described at the end of section 6.2. In contrast to the case $r = 0$, there is also a freedom in accomplishing the first step, i.e., in an analogy of the transformation (6.9). Of course, any additional choice influences the corresponding tractor Frenet equations and hence the resulting system of absolute invariants. This is why the construction cannot be declared as canonical.

Anyhow, we count $n - r - 1$ conformal curvatures of an r-null curve, some of which depend on the described choices. But the first of them is not influenced by any choice and its interpretation is analogous to the one in Proposition 6.2.

Instead of a toothless general reasoning, it is more illuminating to analyse some concrete situations. For the case of Lorentzian signature, this is done in [62, section 5.4], where we refer for details and inspirations for further investigations.

6.4 Comparisons and remarks

As promised in the introduction, expanding our tractorial formulas, one generates concrete (and usually ugly) tensorial expressions. These allow to compare the previously studied notions with our current approach. In particular, we can detect the Fialkow's objects listed in section 6.1 among our tractors and their individual slots. These, and other comparisons to existing literature, are discussed in [62, section 4.3]. Also the exceptional curvatures that appear both in Fialkow's and our approach can be related satisfactorily, see [62, remark 4.8].

Instead of standard tractor bundle, one may consider other ones. As a natural and prominent candidate, there is the adjoint tractor bundle. In the conformal case, $\mathcal{A} \cong \mathcal{T} \wedge \mathcal{T}$ and the natural lift of curve $c : I \to M$ to $\boldsymbol{A} : I \to \mathcal{A}$ is provided by $\boldsymbol{A} := \boldsymbol{T} \wedge \boldsymbol{U}$ (according to the notation above). With the corresponding tractor connection and parallel bundle metric, one may mimic the previous development at least in detecting distinguished parametrizations, constructing families of relative invariants and characterizing conformal circles, respectively other distinguished curves.

Since the adjoint tractor bundle carries a parallel bundle metric for any parabolic geometry (induced by the Killing form on the background Lie algebra), we consider this object as an appropriate candidate to experiment with. It also carries additional structures, whose role to the study of curves is not yet investigated. Only several experiments have been done in the context of CR and Lagrangean contact geometries.

Recently, several studies of curves in conformal and other geometries appear that use tractorial (and related) techniques, see [31, 37, 39].

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Attachments

Here we attach the articles whose main contributions are described in the previous chapter. They are [17], [43] together with [44] and [62], which correspond to sections 4, 5 and 6, respectively. For reader's convenience we collect the tables of contents of individual articles (according to their own page numberings):

On the geometry of chains

A Projective-to-Conformal Fefferman-type Construction

Fefferman–Graham ambient metrics of Patterson–Walker metrics

Conformal theory of curves with tractors

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ON THE GEOMETRY OF CHAINS

ANDREAS ČAP & VOJTĚCH ŽÁDNÍK

Abstract

The chains studied in this paper generalize Chern–Moser chains for CR structures. They form a distinguished family of one dimensional submanifolds in manifolds endowed with a parabolic contact structure. Both the parabolic contact structure and the system of chains can be equivalently encoded as Cartan geometries (of different types). The aim of this paper is to study the relation between these two Cartan geometries for Lagrangean contact structures and partially integrable almost CR structures.

We develop a general method for extending Cartan geometries which generalizes the Cartan geometry interpretation of Fefferman's construction of a conformal structure associated to a CR structure. For the two structures in question, we show that the Cartan geometry associated to the family of chains can be obtained in that way if and only if the original parabolic contact structure is torsion free. In particular, the procedure works exactly on the subclass of (integrable) CR structures.

This tight relation between the two Cartan geometries leads to an explicit description of the Cartan curvature associated to the family of chains. On the one hand, this shows that the homogeneous models for the two parabolic contact structures give rise to examples of non–flat path geometries with large automorphism groups. On the other hand, we show that one may (almost) reconstruct the underlying torsion free parabolic contact structure from the Cartan curvature associated to the chains. In particular, this leads to a very conceptual proof of the fact that chain preserving contact diffeomorphisms are either isomorphisms or anti–isomorphisms of parabolic contact structures.

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1. Introduction

Parabolic contact structures are a class of geometric structures having an underlying contact structure. They admit a canonical normal Cartan connection corresponding to a contact grading of a simple Lie algebra. The best known examples of such structures are non–degenerate partially integrable almost CR structures of hypersurface type. The construction of the canonical Cartan connection is due to Chern and Moser ([**11**]) for the subclass of CR structures, and to Tanaka ([**20**]) in general.

In the approach of Chern and Moser, a central role is played by a canonical class of unparametrized curves called *chains*. For each point x and each direction ξ at x, which is transverse to the contact distribution, there is a unique chain through x in direction ξ. In addition, each chain comes with a projective class of distinguished parametrizations. The notion of chains easily generalizes to arbitrary parabolic contact structures, and the chains are easy to describe in terms of the Cartan connection.

A *path geometry* on a smooth manifold M is given by a smooth family of unparametrized curves on M such that for each $x \in M$ and each direction ξ at x there is a unique curve through x in direction ξ. The best way to encode this structure is to pass to the projectivized tangent bundle $\mathcal{P}TM$, the space of all lines in TM. Then a path geometry is given by a line subbundle in the tangent bundle of $\mathcal{P}TM$ with certain properties, see [**12**] and [**14**] for a modern presentation. It turns out that these structures are equivalent to regular normal Cartan geometries of a certain type, which fall under the general concept of parabolic geometries, see Section 4.7 of [**4**].

In the description as a Cartan geometry, path geometries immediately generalize to open subsets of the projectivized tangent bundle. In particular, given a manifold M endowed with a parabolic contact structure, the chains give rise to a path geometry on the open subset $\mathcal{P}_0TM \subset \mathcal{P}TM$ formed by all lines transversal to the contact subbundle. The general question addressed in this paper is how to describe the resulting Cartan geometry on $\mathcal{P}_0TM$ in terms of the original Cartan geometry on M. We study this in detail in the case of Lagrangean contact structures and, in the end, briefly indicate how to deal with partially integrable almost CR structures, which can be viewed as a different real form of the same complex geometric structure. Q in the structure group P The first observation is that $\mathcal{P}_0TM$ can be obtained as a quotient of the Cartan bundle $\mathcal{G} \to M$ obtained from the parabolic contact structure. More precisely, there is a subgroup Q in the structure group P such that $\mathcal{P}_0TM \cong \mathcal{G}/Q$. In particular, $\mathcal{G}$ is a principal Q–bundle over $\mathcal{P}_0TM$ and the canonical Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ associated to the parabolic contact structure can be also viewed as a Cartan connection on $\mathcal{G} \to \mathcal{P}_0 TM$. The question then is whether the canonical Cartan geometry $(\tilde{\mathcal{G}} \to \mathcal{P}_0 TM, \tilde{\omega})$ determined by the path geometry of chains can be constructed directly from $(\mathcal{G} \to \mathcal{P}_0 TM, \omega)$.

To attack this problem, we study a class of extension functors mapping Cartan geometries of some type (G, Q) to Cartan geometries of another type $(\tilde{G}, \tilde{P})$. These functors have the property that there is a homomorphism between the two Cartan bundles, which relates the two Cartan connections. We show that in order to obtain such a functor, one needs a homomorphism $i : Q \to \tilde{P}$ (which we assume to be infinitesimally injective) and a linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ which satisfy certain compatibility conditions. There is a simple notion of equivalence for such pairs and equivalent pairs lead to naturally isomorphic extension functors.

There is a particular simple source of pairs (i, α) leading to extension functors as above. Namely, one may start from an infinitesimally injective homomorphism $\varphi : G \to \tilde{G}$, define i to be its restriction to Q and α to be the induced homomorphism φ' of Lie algebras. In a special case, this leads to the Cartan geometry interpretation of Fefferman's construction of a canonical conformal structure on a circle bundle over a CR manifold.

One can completely describe the effect of the extension functor associated to a pair (i, α) on the curvature of the Cartan geometries. Apart from the curvature of the original geometry, also the deviation from α being a homomorphism of Lie algebras enters into the curvature of the extended Cartan geometry.

An important feature of the special choice for $(\tilde{G}, \tilde{P})$ that we are concerned with, is a uniqueness result for such extension functors. We show (see Theorem 3.4) that if the extension functor associated to a pair (i, α) maps locally flat geometries of type (G, Q) to regular normal geometries of type $(\tilde{G}, \tilde{P})$, then the pair (i, α) is already determined uniquely up to equivalence. For the two parabolic contact structures studied in this paper, we show that there exist appropriate pairs (i, α) in 3.5 and 5.2.

In both cases, the resulting extension functor does *not* produce the canonical Cartan geometry associated to the path geometry of chains in general. We show that the canonical Cartan connection is obtained if and only if the original parabolic contact geometry is torsion free. For a Lagrangean contact structure this means that the two Lagrangean subbundles are integrable, while it is the usual integrability condition for CR structures. This ties in nicely with the Fefferman construction, where one obtains a conformal structure for arbitrary partially integrable almost CR structures, but the normal Cartan connection is obtained by equivariant extension if and only if the structure is integrable (and hence CR), see [**5**] and [**6**].

Finally, we discuss applications of our construction, which are based on an analysis of the curvature of the canonical Cartan connection associated to the path geometry of chains. We show that chains never are geodesics of a linear connection, and they give rise to a torsion free path geometry if and only if the original parabolic contact structure is locally flat. Then we show that the underlying parabolic contact structure can be almost reconstructed from the harmonic curvature of the path geometry of chains. In particular, this leads to a very conceptual proof of the fact that a contact diffeomorphism which maps chains to chains must (essentially) preserve the original torsion free parabolic contact structure.

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2. Parabolic contact structures, chains, and path geometries

In this section, we will discuss the concepts of chains and the associated path geometry for a parabolic contact structure, focusing on the example of Lagrangean contact structures. We only briefly indicate the changes needed to deal with general parabolic contact structures.

2.1. Lagrangean contact structures. The starting point to define a parabolic contact structure is a simple Lie algebra $\mathfrak{g}$ endowed with a *contact grading*, i.e. a vector space decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $\mathfrak{g}_{-2}$ has real dimension one, and the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non–degenerate. It is known that such a grading is unique up to an inner automorphism and it exists for each non–compact non–complex real simple Lie algebra except $\mathfrak{sl}(n, \mathbb{H})$, $\mathfrak{so}(n, 1)$, $\mathfrak{sp}(p, q)$, one real form of E_6 and one of E_7, see Section 4.2 of [**23**].

Here we will mainly be concerned with the contact grading of $\mathfrak{g} = \mathfrak{sl}(n + 2, \mathbb{R})$, corresponding to the following block decomposition with blocks of size 1, n, and 1:

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^L & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^L & \mathfrak{g}_0 & \mathfrak{g}_1^R \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^R & \mathfrak{g}_0 \end{pmatrix}.$$

We have indicated the splittings $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^L \oplus \mathfrak{g}_{-1}^R$ respectively $\mathfrak{g}_1 = \mathfrak{g}_1^L \oplus \mathfrak{g}_1^R$, which are immediately seen to be $\mathfrak{g}_0$–invariant. Further, the subspaces $\mathfrak{g}_{-1}^L$ and $\mathfrak{g}_{-1}^R$ of $\mathfrak{g}_{-1}$ are isotropic for $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$.

Put $G := PGL(n+2,\mathbb{R})$, the quotient of $GL(n+2,\mathbb{R})$ by its center. We will view G as the quotient of the group of matrices whose determinant has modulus one by the two–element subgroup generated by $\pm\,\mathrm{id}$ and work with representative matrices. The group G always has Lie algebra $\mathfrak{g}$. For odd n, one can identify G with $SL(n+2,\mathbb{R})$. For even n, G has two connected components, and the component containing the identity can be identified with $PSL(n+2,\mathbb{R})$.

By $G_0 \subset P \subset G$ we denote the subgroups formed by matrices which are block diagonal respectively block upper triangular with block sizes 1, n, and 1. Then the Lie algebras of G_0 and P are $\mathfrak{g}_0$ respectively $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. For $g \in G_0$, the map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ preserves the grading while for $g \in P$ one obtains $\mathrm{Ad}(g)(\mathfrak{g}_i) \in \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_2$ for $i = -1, \dots, 2$. This can be used as an alternative characterization of the two subgroups. The reason for the choice of the specific group G with Lie algebra $\mathfrak{g}$ is that the adjoint action identifies G_0 with the group of all automorphisms of the graded Lie algebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ which in addition preserve the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}$.

Let M be a smooth manifold of dimension $2n+1$ and let $H \subset TM$ be a subbundle of corank one. The Lie bracket of vector fields induces a tensorial map $\mathcal{L} : \Lambda^2 H \to TM/H$, and H is called a *contact structure* on M if this map is non–degenerate. A *Lagrangean contact structure* on M is a contact structure $H \subset TM$ together with a fixed decomposition $H = L \oplus R$ such that each of the subbundles is isotropic with respect to $\mathcal{L}$. This forces the two bundles to be of rank n, and $\mathcal{L}$ induces isomorphisms $R \cong L^* \otimes (TM/H)$ and $L \cong R^* \otimes (TM/H)$.

In view of the description of G_0 above, the following result is a special case of general prolongation procedures [**21**, **17**, **7**], see [**19**] and Section 4.1 of [**4**] for more information on this specific case.

Theorem 2.1. *Let $H = L \oplus R$ be a Lagrangean contact structure on a manifold M of dimension $2n+1$. Then there exists a principal P–bundle $p : \mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$ such that $L = Tp(\omega^{-1}(\mathfrak{g}^L_{-1} \oplus \mathfrak{p}))$ and $R = Tp(\omega^{-1}(\mathfrak{g}^R_{-1} \oplus \mathfrak{p}))$. The pair $(\mathcal{G},\omega)$ is uniquely determined up to isomorphism provided that one in addition requires the curvature of ω to satisfy a normalization condition discussed in* 3.6.

Similarly, for any contact grading of a simple Lie algebra $\mathfrak{g}$ and a choice of a Lie group G with Lie algebra $\mathfrak{g}$, one defines a subgroup $P \subset G$ with Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. One then obtains an equivalence of categories between regular normal parabolic geometries of type (G,P) and underlying geometric structures, which in particular include a contact structure.

The second case of such structures we will be concerned with in this paper, is partially integrable almost CR structures of hypersurface type, see Section 5.

2.2. Chains. Let $(p : \mathcal{G} \to M, \omega)$ be the canonical Cartan geometry determined by a parabolic contact structure. Then one obtains an isomorphism $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ such that $H \subset TM$ corresponds to $(\mathfrak{g}_{-1} \oplus \mathfrak{p})/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$. Of course, we may identify $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ as a vector space with $\mathfrak{g}/\mathfrak{p}$ and use this to carry over the natural P–action to $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. Let $Q \subset P$ be the stabilizer of the line $\mathfrak{g}_{-2}$ under this action. By definition, this is a closed subgroup of P. Let us denote by $G_0 \subset P$ the closed subgroup consisting of all elements whose adjoint action respects the grading of $\mathfrak{g}$. Then G_0 has Lie algebra $\mathfrak{g}_0$ and by Proposition 2.10 of [**7**], any element $g \in P$ can be uniquely written in the form $g_0 \exp(Z_1) \exp(Z_2)$ for $g_0 \in G_0$, $Z_1 \in \mathfrak{g}_1$, and $Z_2 \in \mathfrak{g}_2$.

Lemma 2.2. *(1) An element $g = g_0 \exp(Z_1) \exp(Z_2) \in P$ lies in the subgroup $Q \subset P$ if and only if $Z_1 = 0$. In particular, $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and for $g \in Q$ we have $\mathrm{Ad}(g)(\mathfrak{g}_{-2}) \subset \mathfrak{g}_{-2} \oplus \mathfrak{q}$.*
(2) Let $(p : \mathcal{G} \to M, \omega)$ be the canonical Cartan geometry determined by a parabolic contact structure. Let $x \in M$ be a point and $\xi \in T_xM \setminus H_x$ a tangent vector transverse to the contact subbundle.

Then there is a point $u \in p^{-1}(x) \subset \mathcal{G}$ and a unique lift $\tilde{\xi} \in T_u\mathcal{G}$ of ξ such that $\omega(u)(\tilde{\xi}) \in \mathfrak{g}_{-2}$. The point u is unique up to the principal right action of an element $g \in Q \subset P$.

Proof. (1) We first observe that for a nonzero element $X \in \mathfrak{g}_{-2}$, the map $Z \mapsto [Z, X]$ is a bijection $\mathfrak{g}_1 \to \mathfrak{g}_{-1}$. This is easy to verify directly for the examples discussed in 2.1 and 5.1. For general contact gradings it follows from the fact that $[\mathfrak{g}_{-2}, \mathfrak{g}_2]$ consists of all multiples of the grading element, see Section 4.2 of [**23**].

By definition, $g \in Q$ if and only if $\mathrm{Ad}(g)(\mathfrak{g}_{-2}) \subset \mathfrak{g}_{-2} \oplus \mathfrak{p}$. Now from the expression $g^{-1} = \exp(-Z_2) \exp(-Z_1) g_0^{-1}$ one immediately concludes that $\mathrm{Ad}(g^{-1})(X)$ is congruent to $-[Z_1, X] \in \mathfrak{g}_{-1}$ modulo $\mathfrak{g}_{-2} \oplus \mathfrak{p}$. Hence we see that $g \in Q$ if and only if $Z_1 = 0$, and the rest of (1) evidently follows.
(2) Choose any point $v \in p^{-1}(x)$. Since the vertical bundle of $\mathcal{G} \to M$ equals $\omega^{-1}(\mathfrak{p})$, there is a unique lift $\eta \in T_v\mathcal{G}$ of ξ such that $\omega(v)(\eta) \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. The assumption that ξ is transverse to H_x means that $\omega(v)(\eta) \notin \mathfrak{g}_{-1}$. For an element $g \in P$ we can consider $v \cdot g$ and $T_v r^g \cdot \eta \in T_{v \cdot g}\mathcal{G}$, where $v \cdot g = r^g(v)$ denotes the principal right action of g on v. Evidently, $T_v r^g \cdot \eta$ is again a lift of ξ and equivariancy of ω implies that $\omega(v \cdot g)(T_v r^g \cdot \eta) = \mathrm{Ad}(g^{-1})(\omega(v)(\eta))$.

Writing $\omega(v)(\eta) = X_{-2} + X_{-1}$ we have $X_{-2} \neq 0$, so from above we see that there is an element $Z \in \mathfrak{g}_1$ such that $[Z, X_{-2}] = X_{-1}$. Putting $g = \exp(Z) \in P$ we conclude that $\omega(v \cdot g)(T_v r^g \cdot \eta) \in \mathfrak{g}_{-2} \oplus \mathfrak{p}$. Hence putting $u = v \cdot g$ and subtracting an appropriate vertical vector from $T_v r^g \cdot \eta$, we have found a couple $(u, \tilde{\xi})$ as required.

Any other choice of a preimage of x has the form $u\cdot g$ for some $g\in P$. Any lift of ξ in $T_{u\cdot g}\mathcal{G}$ is of the form $T_ur^g\cdot\tilde{\xi}+\zeta$ for some vertical vector ζ. Clearly, there is a choice for ζ such that $\omega(T_ur^g\cdot\tilde{\xi}+\zeta)\in\mathfrak{g}_{-2}$ if and only if $\omega(T_ur^g\cdot\tilde{\xi})\in\mathfrak{g}_{-2}\oplus\mathfrak{p}$ and equivariancy of ω implies that this is equivalent to $g\in Q$. q.e.d.

This lemma immediately leads us to chains. Fix a nonzero element $X\in\mathfrak{g}_{-2}$. For a point $x\in M$ and a line ℓ in T_xM which is transverse to H_x, we can find a point $u\in\mathcal{G}$ such that $T_up\cdot\omega_u^{-1}(X)\in\ell$. Denoting by $\tilde{X}$ the "constant vector field" $\omega^{-1}(X)$ we can consider the flow of $\tilde{X}$ through u and project it onto M to obtain a (locally defined) smooth curve through x whose tangent space at x is ℓ. In Section 4 of [**9**] it has been shown that, as an unparametrized curve, this is uniquely determined by x and ℓ, and it comes with a distinguished projective family of parametrizations.

The lemma also leads us to a nice description of the space of all transverse directions. For a point $u\in\mathcal{G}$, we obtain a line in $T_{p(u)}M$ which is transverse to $H_{p(u)}$, namely $T_p(\omega_u^{-1}(\mathfrak{g}_{-2}))$. This defines a smooth map $\mathcal{G}\to\mathcal{P}TM$, where $\mathcal{P}TM$ denotes the projectivized tangent bundle of M. Since P acts freely on $\mathcal{G}$ so does Q and hence $\mathcal{G}/Q$ is a smooth manifold. By the lemma, we obtain a diffeomorphism from $\mathcal{G}/Q$ to the open subset $\mathcal{P}_0TM\subset\mathcal{P}TM$ formed by all lines which are transverse to the contact distribution H.

2.3. Path geometries. Classically, path geometries are associated to certain families of unparametrized curves in a smooth manifold. Suppose that in a manifold Z we have a smooth family of curves such that through each point of Z there is exactly one curve in each direction. Let $\mathcal{P}TZ$ be the projectivized tangent bundle of Z, i.e. the space of all lines through the origin in tangent spaces of Z. Given a line ℓ in T_xZ, we can choose the unique curve in the family which goes through x in direction ℓ. Choosing a local regular parametrization $c:I\to Z$ of this curve we obtain a lift $\tilde{c}:I\to\mathcal{P}TZ$ by defining $\tilde{c}(t)$ to be the line in $T_{c(t)}Z$ generated by $c'(t)$. Choosing a different regular parametrization, we just obtain a reparametrization of $\tilde{c}$, so the submanifold $\tilde{c}(I)\subset\mathcal{P}TZ$ is independent of all choices. These curves foliate $\mathcal{P}TZ$, and their tangent spaces give rise to a line subbundle $E\subset T\mathcal{P}TZ$.

This subbundle has a special property. Similarly to the tautological line bundle on a projective space, a projectivized tangent bundle carries a tautological subbundle $\Xi\subset T\mathcal{P}TZ$ of rank $\dim(Z)$. By definition, given a line $\ell\subset T_zZ$, a tangent vector $\xi\in T_\ell\mathcal{P}TZ$ lies in Ξ_ℓ if and only if its image under the tangent map of the projection $\mathcal{P}TZ\to Z$ lies in the line ℓ. By construction, the line subbundle E associated to a family of curves as above always is contained in Ξ and is transverse to the vertical subbundle V of $\mathcal{P}TZ\to Z$. Hence we see that $\Xi=E\oplus V$.

Conversely, having given a decomposition $\Xi = E \oplus V$ of the tautological bundle, we can project the leaves of the foliation of $\mathcal{P}TZ$ defined by E to the manifold Z to obtain a smooth family of curves in Z with exactly one curve through each point in each direction. Hence one may use the decomposition $\Xi = E \oplus V$ as an alternative definition of such a family of curves, and this decomposition is usually referred to as a *path geometry* on Z. It is easy to verify that the Lie bracket of vector fields induces an isomorphism $E \otimes V \to T\mathcal{P}TZ/\Xi$.

It turns out that path geometries also admit an equivalent description as regular normal parabolic geometries. Putting $m := \dim(Z) - 1$ we consider the Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{sl}(m+2, \mathbb{R})$ with the $|2|$–grading obtained by a block decomposition

$$\begin{pmatrix} \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1^E & \tilde{\mathfrak{g}}_2 \\ \tilde{\mathfrak{g}}_{-1}^E & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1^V \\ \tilde{\mathfrak{g}}_{-2} & \tilde{\mathfrak{g}}_{-1}^V & \tilde{\mathfrak{g}}_0 \end{pmatrix},$$

as in 2.1, but this time with blocks of size 1, 1, and m. Hence $\tilde{\mathfrak{g}}_{\pm1}^E$ has dimension 1 while $\tilde{\mathfrak{g}}_{\pm1}^V$ and $\tilde{\mathfrak{g}}_{\pm2}$ are all m–dimensional. Put $\tilde{G} := PGL(m+2, \mathbb{R})$ and let $\tilde{G}_0 \subset \tilde{P} \subset \tilde{G}$ be the subgroups formed by matrices which are block diagonal respectively block upper triangular with block sizes 1, 1, and m. Then $\tilde{G}_0$ and $\tilde{P}$ have Lie algebras $\tilde{\mathfrak{g}}_0$ respectively $\tilde{\mathfrak{p}} := \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$, where $\tilde{\mathfrak{g}}_1 = \tilde{\mathfrak{g}}_1^E \oplus \tilde{\mathfrak{g}}_1^V$.

The adjoint action identifies $\tilde{G}_0$ with the group of automorphisms of the graded Lie algebra $\tilde{\mathfrak{g}}_{-2} \oplus \tilde{\mathfrak{g}}_{-1}$ which in addition preserve the decomposition $\tilde{\mathfrak{g}}_{-1} = \tilde{\mathfrak{g}}_{-1}^E \oplus \tilde{\mathfrak{g}}_{-1}^V$. Hence the following result is a special case of the general prolongation procedures [**21**, **17**, **7**], see Section 4.7 of [**4**] for this specific case.

Theorem 2.3. *Let $\tilde{Z}$ be a smooth manifold of dimension $2m+1$ endowed with transversal subbundles E and V in $T\tilde{Z}$ of rank 1 and m, respectively, and put $\Xi := E \oplus V \subset T\tilde{Z}$. Suppose that the Lie bracket of two sections of V is a section of Ξ and that the tensorial map $E \otimes V \to T\tilde{Z}/\Xi$ induced by the Lie bracket of vector fields is an isomorphism.*

Then there exists a principal bundle $\tilde{p} : \tilde{\mathcal{G}} \to \tilde{Z}$ with structure group $\tilde{P}$ endowed with a Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ such that $E = T\tilde{p}(\tilde{\omega}^{-1}(\tilde{\mathfrak{g}}_{-1}^E \oplus \tilde{\mathfrak{p}}))$ and $V = T\tilde{p}(\tilde{\omega}^{-1}(\tilde{\mathfrak{g}}_{-1}^V \oplus \tilde{\mathfrak{p}}))$. The pair $(\tilde{\mathcal{G}}, \tilde{\omega})$ is uniquely determined up to isomorphism provided that $\tilde{\omega}$ is required to satisfy a normalization condition discussed in 3.6.

In particular, a family of paths on Z as before gives rise to a Cartan geometry on $\mathcal{P}TZ$. This immediately generalizes to the case of an open subset of $\mathcal{P}TZ$, i.e. the case where paths are only given through each point in an open set of directions.

It turns out that for $m \neq 2$, the assumptions of the theorem already imply that the subbundle $V \subset T\tilde{Z}$ is involutive. Then $\tilde{Z}$ is automatically locally diffeomorphic to a projectivized tangent bundle in such a way that V is mapped to the vertical subbundle and Ξ to the tautological subbundle. Hence for $m \neq 2$, the geometries discussed in the theorem are locally isomorphic to path geometries.

2.4. The path geometry of chains. From 2.2 we see that for a manifold M endowed with a parabolic contact structure the chains give rise to a path geometry on the open subset $\tilde{M} := \mathcal{P}_0 TM$ of the projectivized tangent bundle of M. We can easily describe the corresponding configuration of bundles explicitly. Denoting by $(p : \mathcal{G} \to M, \omega)$ the Cartan geometry induced by the parabolic contact structure, we know from 2.2 that $\tilde{M} = \mathcal{G}/Q$, where $Q \subset P$ denotes the stabilizer of the line in $\mathfrak{g}/\mathfrak{p}$ corresponding to $\mathfrak{g}_{-2} \subset \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. In particular, $\mathcal{G}$ is a Q–principal bundle over $\tilde{M}$ and ω is a Cartan connection on $\mathcal{G} \to \tilde{M}$. This implies that $T\tilde{M} = \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$, and the tangent map to the projection $\pi : \tilde{M} \to M$ corresponds to the obvious projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$. In particular, the vertical bundle $V = \ker(T\pi)$ corresponds to $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$. From the construction of the isomorphism $\mathcal{G}/Q \to \tilde{M}$ in 2.2, it is evident that the tautological bundle Ξ corresponds to $(\mathfrak{g}_{-2} \oplus \mathfrak{p})/\mathfrak{q}$. By part (1) of Lemma 2.2, the subspace $(\mathfrak{g}_{-2} \oplus \mathfrak{q})/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ is Q–invariant, thus it gives rise to a line subbundle E in Ξ, which is complementary to V. By construction, this exactly describes the path geometry determined by the chains.

If $\dim(M) = 2n + 1$, then the dimension of $\tilde{M}$ is $4n + 1$. Put $\tilde{G} := PGL(2n + 2, \mathbb{R})$ and let $\tilde{P} \subset \tilde{G}$ be the subgroup described in 2.3. Then by Theorem 2.3 the path geometry on $\tilde{M}$ gives rise to a canonical principal bundle $\tilde{\mathcal{G}} \to \tilde{M}$ with structure group $\tilde{P}$ endowed with a canonical normal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$. The main question now is whether there is a direct relation between the Cartan geometries $(\mathcal{G} \to \tilde{M}, \omega)$ and $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$.

The only reasonable way to relate these two Cartan geometries is to consider a morphism $j : \mathcal{G} \to \tilde{\mathcal{G}}$ of principal bundles and compare the pull–back $j^*\tilde{\omega}$ to ω. This means that j is equivariant, so we first have to choose a group homomorphism $i : Q \to \tilde{P}$ and require that $j(u \cdot g) = j(u) \cdot i(g)$ for all $g \in Q$. Having chosen i and j, we have $j^*\tilde{\omega} \in \Omega^1(\mathcal{G}, \tilde{\mathfrak{g}})$ and the only way to directly relate this to $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is to have $j^*\tilde{\omega} = \alpha \circ \omega$ for some linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$. If we have such a relation, then we can immediately recover $\tilde{\mathcal{G}}$ from $\mathcal{G}$. Consider the map $\Phi : \mathcal{G} \times \tilde{P} \to \tilde{\mathcal{G}}$ defined by $\Phi(u, \tilde{g}) := j(u) \cdot \tilde{g}$. Equivariancy of j immediately implies that $\Phi(u \cdot g, \tilde{g}) = \Phi(u, i(g)\tilde{g})$, so Φ descends to a bundle map $\mathcal{G} \times_Q \tilde{P} \to \tilde{\mathcal{G}}$, where the left action of Q on $\tilde{P}$ is defined via i. This is immediately seen to be an isomorphism of principal bundles, so $\tilde{\mathcal{G}}$ is obtained from $\mathcal{G}$ by an extension of structure group. Under this isomorphism, the given morphism $j : \mathcal{G} \to \tilde{\mathcal{G}}$ corresponds to the natural inclusion $\mathcal{G} \to \mathcal{G} \times_Q \tilde{P}$ induced by $u \mapsto (u, e)$.

3. Induced Cartan connections

In this section, we study the problem of extending Cartan connections. We derive the basic results in the setting of general Cartan geometries, and then specialize to the case of parabolic contact structures and, in particular, Lagrangean contact structures. Some of the developments in 3.1 and 3.3 below are closely related to [**15**, **22**].

3.1. Extension functors for Cartan geometries. Motivated by the last observations in 2.4, let us consider the following problem. Suppose we have given Lie groups G and $\tilde{G}$ with Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, closed subgroups $Q \subset G$ and $\tilde{P} \subset \tilde{G}$, and a homomorphism $i : Q \to \tilde{P}$ which is infinitesimally injective, i.e. such that $i' : \mathfrak{q} \to \tilde{\mathfrak{p}}$ is injective. Given a Cartan geometry $(p : \mathcal{G} \to N, \omega)$ of type (G, Q), we put $\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P}$ and denote by $j : \mathcal{G} \to \tilde{\mathcal{G}}$ the canonical map. Since i is infinitesimally injective, this is an immersion, i.e. $T_u j$ is injective for all $u \in \mathcal{G}$. We want to understand under which conditions on a linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ there is a Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ such that $j^*\tilde{\omega} = \alpha \circ \omega$, and if so, whether $\tilde{\omega}$ is uniquely determined.

Proposition 3.1. *There is a Cartan connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ such that $j^*\tilde{\omega} = \alpha \circ \omega$ if and only if the pair (i, α) satisfies the following conditions:*

1. *$\alpha \circ \mathrm{Ad}(g) = \mathrm{Ad}(i(g)) \circ \alpha$ for all $g \in Q$.*
2. *On the subspace $\mathfrak{q} \subset \mathfrak{g}$, the map α restricts to the derivative i' of $i : Q \to \tilde{P}$.*
3. *The map $\underline{\alpha} : \mathfrak{g}/\mathfrak{q} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ induced by α is a linear isomorphism.*

If these conditions are satisfied, then $\tilde{\omega}$ is uniquely determined.

Proof. Let us first assume that there is a Cartan connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ such that $j^*\tilde{\omega} = \alpha \circ \omega$. For $u \in \mathcal{G}$, the tangent space $T_{j(u)}\tilde{\mathcal{G}}$ is spanned by $T_u j(T_u \mathcal{G})$ and the vertical subspace $V_{j(u)}\tilde{\mathcal{G}}$. The behavior of $\tilde{\omega}$ on the first subspace is determined by the fact that $j^*\tilde{\omega} = \alpha \circ \omega$, while on the second subspace $\tilde{\omega}$ has to reproduce the generators of fundamental vector fields. Hence the restriction of $\tilde{\omega}$ to $j(\mathcal{G})$ is determined by the fact that $j^*\tilde{\omega} = \alpha \circ \omega$. By definition of $\tilde{\mathcal{G}}$, any point $\tilde{u} \in \tilde{\mathcal{G}}$ can be written as $j(u) \cdot \tilde{g}$ for some $u \in \mathcal{G}$ and some $\tilde{g} \in \tilde{P}$, so uniqueness of $\tilde{\omega}$ follows from equivariancy.

Still assuming that $\tilde{\omega}$ exists, condition (1) follows from equivariancy of j, ω, and $\tilde{\omega}$. Equivariancy of j also implies that for $A \in \mathfrak{q}$ and the corresponding fundamental vector field ζ_A we get $Tj \circ \zeta_A = \zeta_{i'(A)}$. Thus condition (2) follows from the fact that both ω and $\tilde{\omega}$ reproduce the generators of fundamental vector fields. Let $p : \mathcal{G} \to N$ and $\tilde{p} : \tilde{\mathcal{G}} \to N$ be the bundle projections, so $\tilde{p} \circ j = p$. For $\xi \in T_u\mathcal{G}$ we have $\alpha(\omega(\xi)) = \tilde{\omega}(T_u j \cdot \xi)$, so if this lies in $\tilde{\mathfrak{p}}$ then $T_u j \cdot \xi$ is vertical. But then ξ is vertical and hence $\omega(\xi) \in \mathfrak{q}$. Therefore, the map $\underline{\alpha}$ is injective, and since both $\mathcal{G}$ and $\tilde{\mathcal{G}}$ admit a Cartan connection, we must have $\dim(\mathfrak{g}/\mathfrak{q}) = \dim(N) = \dim(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})$, so (3) follows.

Conversely, suppose that (1)–(3) are satisfied for (i, α) and ω is given. For $\tilde{u} \in \tilde{\mathcal{G}}$ and $\tilde{\xi} \in T_{\tilde{u}}\tilde{\mathcal{G}}$ we can find elements $u \in \mathcal{G}$, $\xi \in T_u\mathcal{G}$, $A \in \tilde{\mathfrak{p}}$, and $\tilde{g} \in \tilde{P}$ such that $\tilde{u} = j(u) \cdot \tilde{g}$ and $\tilde{\xi} = Tr^{\tilde{g}} \cdot (Tj \cdot \xi + \zeta_A)$. Then we define $\tilde{\omega}(\tilde{\xi}) := \mathrm{Ad}(\tilde{g})^{-1}(\alpha(\omega(\xi)) + A)$. Using properties (1) and (2) one verifies that this is independent of all choices. By (3), it defines a linear isomorphism $T_{\tilde{u}}\tilde{\mathcal{G}} \to \tilde{\mathfrak{g}}$, and the remaining properties of a Cartan connection are easily verified directly. q.e.d.

Any pair (i, α) which satisfies the properties (1)–(3) of the proposition gives rise to an extension functor from Cartan geometries of type (G, Q) to Cartan geometries of type $(\tilde{G}, \tilde{P})$: Starting from a geometry $(p : \mathcal{G} \to N, \omega)$ of type (G, Q), one puts $\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P}$ (with Q acting on $\tilde{P}$ via i) and defines $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ to be the unique Cartan connection on $\tilde{\mathcal{G}}$ such that $j^*\tilde{\omega} = \alpha \circ \omega$, where $j : \mathcal{G} \to \tilde{\mathcal{G}}$ is the canonical map. For a morphism $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ between geometries of type (G, Q), we can consider the principal bundle map $\Phi : \tilde{\mathcal{G}}_1 \to \tilde{\mathcal{G}}_2$ induced by $\varphi \times \mathrm{id}_{\tilde{P}}$. By construction, this satisfies $\Phi \circ j_1 = j_2 \circ \varphi$ and we obtain

$$j_1^*\Phi^*\tilde{\omega}_2 = \varphi^* j_2^*\tilde{\omega}_2 = \varphi^*(\alpha \circ \omega_2) = \alpha \circ \varphi^*\omega_2 = \alpha \circ \omega_1.$$

But $\tilde{\omega}_1$ is the unique Cartan connection whose pull–back along j_1 coincides with $\alpha \circ \omega_1$, which implies that $\Phi^*\tilde{\omega}_2 = \tilde{\omega}_1$, and hence Φ is a morphism of Cartan geometries of type $(\tilde{G}, \tilde{P})$.

There is a simple notion of equivalence for pairs (i, α). We call (i, α) and $(\hat{i}, \hat{\alpha})$ equivalent and write $(i, \alpha) \sim (\hat{i}, \hat{\alpha})$ if and only if there is an element $\tilde{g} \in \tilde{P}$ such that $\hat{i}(g) = \tilde{g}^{-1} i(g) \tilde{g}$ and $\hat{\alpha} = \mathrm{Ad}(\tilde{g}^{-1}) \circ \alpha$. Notice that if (i, α) satisfies conditions (1)–(3) of the proposition, then so does any equivalent pair. In order to distinguish between different extension functors, for a geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, Q) we will often denote the geometry of type $(\tilde{G}, \tilde{P})$ obtained using (i, α) by $(\mathcal{G} \times_i \tilde{P}, \tilde{\omega}_\alpha)$.

Lemma 3.1. *Let (i, α) and $(\hat{i}, \hat{\alpha})$ be equivalent pairs satisfying conditions (1)–(3) of the proposition. Then the resulting extension functors for Cartan geometries are naturally isomorphic.*

Proof. By assumption, there is an element $\tilde{g} \in \tilde{P}$ such that $\hat{i}(g) = \tilde{g}^{-1} i(g) \tilde{g}$ and $\hat{\alpha} = \mathrm{Ad}(\tilde{g}^{-1}) \circ \alpha$. Let $j : \mathcal{G} \to \mathcal{G} \times_i \tilde{P}$ and $\hat{j} : \mathcal{G} \to \mathcal{G} \times_{\hat{i}} \tilde{P}$ be the natural inclusions, and consider the map $r^{\tilde{g}} \circ j : \mathcal{G} \to \mathcal{G} \times_i \tilde{P}$. Evidently, we have $j(u \cdot g) \cdot \tilde{g} = j(u) \cdot \tilde{g} \cdot \hat{i}(g)$. Hence, by the last observation in 2.4, we obtain an isomorphism $\Psi : \mathcal{G} \times_{\hat{i}} \tilde{P} \to \mathcal{G} \times_i \tilde{P}$ such that $\Psi \circ \hat{j} = r^{\tilde{g}} \circ j$. Now we compute

$$\hat{j}^*\Psi^*\tilde{\omega}_\alpha = j^*(r^{\tilde{g}})^*\tilde{\omega}_\alpha = \mathrm{Ad}(\tilde{g}^{-1}) \circ j^*\tilde{\omega}_\alpha = \hat{\alpha} \circ \omega.$$

By uniqueness, $\Psi^*\tilde{\omega}_\alpha = \tilde{\omega}_{\hat{\alpha}}$, so Ψ is a morphism of Cartan geometries. It is clear from the construction that this defines a natural transformation between the two extension functors and an inverse can be constructed in the same way using $\tilde{g}^{-1}$ rather than $\tilde{g}$. q.e.d.

3.2. The relation to the Fefferman construction. There is a simple source of pairs (i, α) which satisfy conditions (1)–(3) of Proposition 3.1. Suppose that $\varphi : G \to \tilde{G}$ is an infinitesimally injective homomorphism of Lie groups such that $\varphi(Q) \subset \tilde{P}$. Then $i := \varphi|_Q : Q \to \tilde{P}$ is an infinitesimally injective homomorphism and $\alpha := \varphi' : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is a Lie algebra homomorphism. Then condition (2) of Proposition 3.1 is satisfied by construction, while condition (1) easily follows from differentiating the equation $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g)^{-1}$. Hence the only nontrivial condition is (3). Note that if (i, α) is obtained from φ in this way, than any pair equivalent to (i, α) is obtained in the same way from the map $g \mapsto \tilde{g}\varphi(g)\tilde{g}^{-1}$ for some $\tilde{g} \in \tilde{G}$. The main feature of such pairs is that α is a homomorphism of Lie algebras.

In this setting, one may actually go one step further. Suppose we have fixed an infinitesimally injective $\varphi : G \to \tilde{G}$ and a closed subgroup $\tilde{P} \subset \tilde{G}$. Then we put $Q := \varphi^{-1}(\tilde{P}) \subset G$ to obtain a pair $(i := \varphi|_Q, \alpha := \varphi')$ and hence, if (3) holds, an extension functor from Cartan geometries of type (G, Q) to geometries of type $(\tilde{G}, \tilde{P})$. For a closed subgroup $P \subset G$ with $Q \subset P$, one gets a functor from geometries of type (G, P) to geometries of type (G, Q) as described in 2.2. Given a geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P), one defines $\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P (P/Q)$ and $(\mathcal{G} \to \tilde{M}, \omega)$ is a geometry of type (G, Q). Combining with the above, one gets a functor from geometries of type (G, P) to geometries of type $(\tilde{G}, \tilde{P})$.

The most important example of this is the Cartan geometry interpretation of Fefferman's construction of a Lorentzian conformal structure on the total space of a certain circle bundle over a CR manifold, see [**13**]. In this case $G = SU(n+1, 1)$, $\tilde{G} = SO(2n+2, 2)$, and φ is the evident inclusion. Putting $\tilde{P}$ the stabilizer of a real null line $\ell \subset \mathbb{R}^{2n+4}$ in $\tilde{G}$, the group $Q = G \cap \tilde{P}$ is the stabilizer of ℓ in G. Evidently, this is contained in the stabilizer $P \subset G$ of the complex null line spanned by ℓ, and $P/Q \cong \mathbb{R}P^1 \cong S^1$. Hence the above procedure defines a functor, which to a parabolic geometry of type (G, P) on M associates a parabolic geometry of type $(\tilde{G}, \tilde{P})$ on the total space $\tilde{M}$ of a circle bundle over M. More details about this can be found in [**3**, **5**, **6**]. Recently, analogs of the Fefferman construction have been developed for certain generic distributions in low dimensions, see [**18**, **2**].

3.3. The effect on curvature. We next discuss the effect of extension functors of the type discussed in 3.1 on the curvature of Cartan geometries. This will show specific features of the special case discussed in 3.2.

For a Cartan connection ω on a principal P–bundle $\mathcal{G} \to M$ with values in $\mathfrak{g}$, one initially defines the curvature $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ by $K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$. This measures the amount to which the Maurer–Cartan equation fails to hold. The defining properties of a Cartan connection immediately imply that K is horizontal and P–equivariant. In particular, $K(\xi, \eta) = 0$ for all η provided that ξ is vertical or, equivalently, that $\omega(\xi) \in \mathfrak{p}$.

Using the trivialization of $T\mathcal{G}$ provided by ω, one can pass to the curvature function $\kappa : \mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$, which is characterized by

$$\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) := K(u)(\omega^{-1}(X), \omega^{-1}(Y)).$$

This is well defined by horizontality of K, and equivariancy of K easily implies that κ is equivariant for the natural P–action on the space $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g})$, which is induced from the adjoint action on all copies of $\mathfrak{g}$.

Using the setting of 3.1, suppose that $(i : Q \to \tilde{P}, \alpha : \mathfrak{g} \to \tilde{\mathfrak{g}})$ is a pair satisfying the conditions (1)–(3) of Proposition 3.1. Consider the map $\mathfrak{g} \times \mathfrak{g} \to \tilde{\mathfrak{g}}$ defined by $(X, Y) \mapsto [\alpha(X), \alpha(Y)]_{\tilde{\mathfrak{g}}} - \alpha([X, Y]_{\mathfrak{g}})$, which measures the deviation from α being a homomorphism of Lie algebras. This map is evidently skew symmetric. By condition (1), $\alpha \circ \mathrm{Ad}(g) = \mathrm{Ad}(i(g)) \circ \alpha$ for all $g \in Q$, which infinitesimally implies that $\alpha \circ \mathrm{ad}(X) = \mathrm{ad}(i'(X)) \circ \alpha$ for all $X \in \mathfrak{q}$, and by condition (2) we have $i'(X) = \alpha(X)$ in this case. Hence this map vanishes if one of the entries is from $\mathfrak{q} \subset \mathfrak{g}$, and we obtain a well defined linear map $\Lambda^2(\mathfrak{g}/\mathfrak{q}) \to \tilde{\mathfrak{g}}$. By condition (3), α induces a linear isomorphism $\underline{\alpha} : \mathfrak{g}/\mathfrak{q} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$, and we conclude that we obtain a well defined map $\Psi_\alpha : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}}$ by putting

$$\Psi_\alpha(\tilde{X} + \tilde{\mathfrak{p}}, \tilde{Y} + \tilde{\mathfrak{p}}) = [\alpha(X), \alpha(Y)] - \alpha([X, Y]),$$

where $\alpha(X) + \tilde{\mathfrak{p}} = \tilde{X} + \tilde{\mathfrak{p}}$ and $\alpha(Y) + \tilde{\mathfrak{p}} = \tilde{Y} + \tilde{\mathfrak{p}}$.

Proposition 3.3. *Let (i, α) be a pair satisfying conditions (1)–(3) of Proposition* 3.1. *Let $(p : \mathcal{G} \to N, \omega)$ be a Cartan geometry of type (G, Q), let $(\mathcal{G} \times_i \tilde{P}, \tilde{\omega}_\alpha)$ be the geometry of type $(\tilde{G}, \tilde{P})$ obtained using the extension functor associated to (i, α), and let $j : \mathcal{G} \to \mathcal{G} \times_i \tilde{P}$ be the natural map.*

Then the curvature functions κ and $\tilde{\kappa}$ of the two geometries satisfy

$$\tilde{\kappa}(j(u))(\tilde{X}, \tilde{Y}) = \alpha(\kappa(u)(\underline{\alpha}^{-1}(\tilde{X}), \underline{\alpha}^{-1}(\tilde{Y}))) + \Psi_\alpha(\tilde{X}, \tilde{Y}),$$

for any $\tilde{X}, \tilde{Y} \in \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$, and this completely determines $\tilde{\kappa}$.

In particular, if ω is flat, then $\tilde{\omega}$ is flat if and only if α is a homomorphism of Lie algebras.

Proof. By definition, $j^*\tilde{\omega}_\alpha = \alpha \circ \omega$, and hence $j^* d\tilde{\omega}_\alpha = \alpha \circ d\omega$. This immediately implies that for the curvatures K and $\tilde{K}$ and $\xi, \eta \in \mathfrak{X}(\mathcal{G})$ we get

$$\tilde{K}(j(u))(Tj \cdot \xi, Tj \cdot \eta) = \alpha(d\omega(u)(\xi, \eta)) + [\alpha(\omega(u)(\xi)), \alpha(\omega(u)(\eta))].$$

On the other hand, we get

$$\alpha(K(u)(\xi, \eta)) = \alpha(d\omega(u)(\xi, \eta)) + \alpha([\omega(u)(\xi), \omega(u)(\eta)]).$$

Now the formula for $\tilde{\kappa}(j(u))$ follows immediately from the definition of the curvature functions. Since $\tilde{\kappa}$ is $\tilde{P}$–equivariant, it is completely determined by its restriction to $j(\mathcal{G})$. The final claim follows directly, since Ψ_α vanishes if and only if α is a homomorphism of Lie algebras.

q.e.d.

3.4. Uniqueness. A crucial fact for the further development is that, passing from parabolic contact structures to the associated path geometries of chains, there is actually no freedom in the choice of the pair (i, α) up to equivalence as introduced in 3.1 above. This result certainly is valid in a more general setting but it seems to be difficult to give a nice formulation for conditions one has to assume.

Therefore we return to the setting of Section 2, i.e. G is semisimple, $P \subset G$ is obtained from a contact grading, Q is the subgroup described in 2.2, and $\tilde{G}$ and $\tilde{P}$ correspond to path geometries in the appropriate dimension as in 2.3. In this setting we can now prove:

Theorem 3.4. *Let (i, α) and $(\hat{i}, \hat{\alpha})$ be pairs satisfying conditions (1)–(3) of Proposition* 3.1. *Suppose that there is a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, Q) such that there is an isomorphism between the geometries of type $(\tilde{G}, \tilde{P})$ obtained using (i, α) and $(\hat{i}, \hat{\alpha})$, which covers the identity on M. Then (i, α) and $(\hat{i}, \hat{\alpha})$ are equivalent.*

Proof. Using the notation of the proof of Lemma 3.1, suppose that we have an isomorphism $\Psi : \mathcal{G} \times_i \tilde{P} \to \mathcal{G} \times_{\hat{i}} \tilde{P}$ of principal bundles which covers the identity on M and has the property that $\Psi^* \tilde{\omega}_{\hat{\alpha}} = \tilde{\omega}_\alpha$. Let us denote by j and $\hat{j}$ the natural inclusions of $\mathcal{G}$ into the two extended bundles. Since Ψ covers the identity on M, there must be a smooth function $\varphi : \mathcal{G} \to \tilde{P}$ such that $\Psi(j(u)) = \hat{j}(u) \cdot \varphi(u)$.

By construction we have $j(u \cdot g) = j(u) \cdot i(g)$ and $\hat{j}(u \cdot g) = \hat{j}(u) \cdot \hat{i}(g)$, and using the fact that Ψ is $\tilde{P}$–equivariant we obtain $\hat{i}(g) = \varphi(u) i(g) \varphi(u \cdot g)^{-1}$. On the other hand, differentiating the equation $\Psi(j(u)) = \hat{j}(u) \cdot \varphi(u)$, we obtain

$$(T\Psi \circ Tj) \cdot \xi = (Tr^{\varphi(u)} \circ T\hat{j}) \cdot \xi + \zeta_{\delta\varphi(u)(\xi)}(\Psi(j(u)))$$

where $\delta\varphi \in \Omega^1(\mathcal{G}, \tilde{\mathfrak{p}})$ denotes the left logarithmic derivative of $\varphi : \mathcal{G} \to \tilde{P}$. Applying $\tilde{\omega}_{\hat{\alpha}}$ to the left hand side of this equation, we simply get

$$(j^* \Psi^* \tilde{\omega}_{\hat{\alpha}})(\xi) = (j^* \tilde{\omega}_\alpha)(\xi) = \alpha(\omega(\xi)).$$

Applying $\tilde{\omega}_{\hat{\alpha}}$ to the right hand side, we obtain

$$(\hat{j}^*(r^{\varphi(u)})^*\tilde{\omega}_{\hat{\alpha}})(\xi) + \delta\varphi(u)(\xi) = \\ \mathrm{Ad}(\varphi(u)^{-1})((\hat{j}^*\tilde{\omega}_{\hat{\alpha}})(\xi)) + \delta\varphi(u)(\xi) = \\ \mathrm{Ad}(\varphi(u)^{-1})(\hat{\alpha}(\omega(\xi))) + \delta\varphi(u)(\xi),$$

and we end up with the equation

$$(*) \qquad \alpha(\omega(\xi)) = \mathrm{Ad}(\varphi(u)^{-1})(\hat{\alpha}(\omega(\xi))) + \delta\varphi(u)(\xi)$$

for all $\xi \in T\mathcal{G}$. Together with the relation between i and $\hat{i}$ derived above, this shows that it suffices to show that $\varphi(u)$ is constant to prove that $(i,\alpha) \sim (\hat{i},\hat{\alpha})$.

By construction, $\delta\varphi(u)$ has values in $\tilde{\mathfrak{p}}$, so projecting equation $(*)$ to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ implies that

$$\alpha(\omega(\xi)) + \tilde{\mathfrak{p}} = \underline{\mathrm{Ad}}(\varphi(u)^{-1})(\hat{\alpha}(\omega(\xi)) + \tilde{\mathfrak{p}}),$$

for all $\xi \in T_u\mathcal{G}$, where $\underline{\mathrm{Ad}}$ is the action of $\tilde{P}$ on $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ induced by the adjoint action. By property (3) from Proposition 3.1 this implies that $\underline{\alpha} = \underline{\mathrm{Ad}}(\varphi(u)^{-1}) \circ \underline{\hat{\alpha}}$, so we see that $\underline{\mathrm{Ad}}(\varphi(u)^{-1})$ must be independent of u. Hence we must have $\varphi(u) = \tilde{g}_1\varphi_1(u)$ for some element $\tilde{g}_1 \in \tilde{P}$ and a smooth function $\varphi_1 : \mathcal{G} \to \tilde{P}$ which has values in the kernel of $\underline{\mathrm{Ad}}$. As in 2.2, any element of $\tilde{P}$ can be uniquely written in the form $\tilde{g}_0 \exp(\tilde{Z}_1)\exp(\tilde{Z}_2)$ with $\tilde{g}_0 \in \tilde{G}_0$ and $\tilde{Z}_i \in \tilde{\mathfrak{g}}_i$, and such an element lies in the kernel of $\underline{\mathrm{Ad}}$ if and only if $\mathrm{Ad}(\tilde{g}_0)$ restricts to the identity on $\tilde{\mathfrak{g}}_-$ and $\tilde{Z}_1 = 0$. Since $\tilde{\mathfrak{p}}_+$ is dual to $\tilde{\mathfrak{g}}_-$ and $\tilde{\mathfrak{g}}_0$ injects into $L(\tilde{\mathfrak{g}}_-,\tilde{\mathfrak{g}}_-)$ the first condition implies that $\mathrm{Ad}(\tilde{g}_0) = \mathrm{id}_{\tilde{\mathfrak{g}}}$. Since $\tilde{G} = PGL(k,\mathbb{R})$ for some k, this implies that $\tilde{g}_0$ is the identity.

Hence φ_1 has values in $\exp(\tilde{\mathfrak{g}}_2)$ and therefore $\delta\varphi(u)$ has values in $\tilde{\mathfrak{g}}_2$. Projecting equation $(*)$ to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_2$, we obtain

$$\alpha(\omega(\xi)) + \tilde{\mathfrak{g}}_2 = \underline{\mathrm{Ad}}(\varphi(u)^{-1})(\hat{\alpha}(\omega(\xi)) + \tilde{\mathfrak{g}}_2),$$

where this time $\underline{\mathrm{Ad}}$ denotes the natural action on $\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_2$. But by [**23**, Lemma 3.2] an element of $\tilde{\mathfrak{g}}_2$ vanishes provided that all brackets with elements of $\tilde{\mathfrak{g}}_{-1}$ vanish, and this easily implies that $\varphi_1(u)$ is the identity and so φ is constant. q.e.d.

This result has immediate consequences on the problem of describing the path geometry of chains associated to a parabolic contact structure. If we start with the homogeneous model G/P for a parabolic contact geometry, the induced path geometry of chains is defined on the homogeneous space G/Q. To obtain this by an extension functor as described in 3.1, we need a homomorphism $i : Q \to \tilde{P}$ and a linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$, where $(\tilde{G},\tilde{P})$ gives rise to path geometries in the appropriate dimension. The pair (i,α) has to satisfy conditions (1)–(3) of Proposition 3.1 in order to give rise to an extension functor. The only additional condition is that the extended geometry $(G \times_i \tilde{P}, \tilde{\omega}_\alpha)$ obtained from $(G \to G/Q, \omega^{MC})$ is regular and normal. (Here ω^{MC} denotes the left Maurer–Cartan form.) By Theorem 2.3, a regular normal parabolic geometry of type $(\tilde{G}, \tilde{P})$ is uniquely determined by the underlying path geometry, which is encoded into $(G \to G/Q, \omega^{MC})$, see 2.4.

The theorem above then implies that (i, α) is uniquely determined up to equivalence. In view of Lemma 3.1, the extension functor obtained from (i, α) is (up to natural isomorphism) the only extension functor of the type discussed in 3.1 which produces the right result for the homogeneous model (and hence for locally flat geometries).

The final step is then to study under which conditions on a geometry of type (G, P), the extension functor associated to (i, α) produces a regular normal geometry of type $(\tilde{G}, \tilde{P})$.

3.5. Let us return to the case of Lagrangean contact structures as discussed in 2.1. By definition, we have $G = PGL(n+2, \mathbb{R})$ and $P \subset G$ is the subgroup of all matrices which are block upper triangular with blocks of sizes 1, n, and 1. From part (1) of Lemma 2.2 one immediately concludes that $Q \subset P$ is the subgroup formed by all matrices of the block form

$$\begin{pmatrix} p & 0 & s \\ 0 & R & 0 \\ 0 & 0 & q \end{pmatrix},$$

such that $|pq \det(R)| = 1$. Since the corresponding manifolds have dimension $2n+1$, the right group for the path geometry defined by the chains is $\tilde{G} = PGL(2n+2, \mathbb{R})$. The subgroup $\tilde{P} \subset \tilde{G}$ is given by the classes of those matrices which are block upper triangular with blocks of sizes 1, 1, $2n$. In the sequel, we will always further split the last block into two blocks of size n.

Consider the (well defined) smooth map $i : Q \to \tilde{P}$ and the linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ defined by

$$i \begin{pmatrix} p & 0 & s \\ 0 & R & 0 \\ 0 & 0 & q \end{pmatrix} := \begin{pmatrix} \operatorname{sgn}(\frac{q}{p})\beta & \operatorname{sgn}(\frac{q}{p})\frac{s}{p}\beta & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & q^{-1}\beta^{-1}R & 0 \\ 0 & 0 & 0 & p\beta^{-1}(R^{-1})^t \end{pmatrix},$$

$$\alpha \begin{pmatrix} a & u & d \\ x & B & v \\ z & y & c \end{pmatrix} := \begin{pmatrix} \frac{a-c}{2} & d & \frac{1}{2}u & \frac{1}{2}v^t \\ z & \frac{c-a}{2} & \frac{1}{2}y & -\frac{1}{2}x^t \\ x & v & B - \frac{a+c}{2}\,\mathrm{id} & 0 \\ y^t & -u^t & 0 & -B^t + \frac{a+c}{2}\,\mathrm{id} \end{pmatrix},$$

where $\beta := \sqrt{|p/q|}$, and id denotes the $n \times n$ identity matrix.

Proposition 3.5. *The map $i : Q \to \tilde{P}$ is an injective group homomorphism and the pair (i, α) satisfies conditions (1)–(3) of Proposition* 3.1. *Hence it gives rise to an extension functor from Cartan geometries of type (G, Q) to Cartan geometries of type $(\tilde{G}, \tilde{P})$.*

Proof. All these facts are verified by straightforward computations, some of which are a little tedious. q.e.d.

3.6. Regularity and normality. We next have to discuss the conditions on the curvature of a Cartan connection which were used in Theorems 2.1 and 2.3. If G is a semisimple group and $P \subset G$ is parabolic, then one can identify $(\mathfrak{g}/\mathfrak{p})^*$ with $\mathfrak{p}_+$, the sum of all positive grading components, via the Killing form, see [**23**, Lemma 3.1]. Hence we can view the curvature function defined in 3.3 as having values in $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. Via the gradings of $\mathfrak{p}_+$ and $\mathfrak{g}$, this space is naturally graded, and the Cartan connection ω is called *regular* if its curvature function has values in the part of positive homogeneity. Otherwise put, if $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, then $\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) \in \mathfrak{g}_{i+j+1} \oplus \cdots \oplus \mathfrak{g}_k$.

Recall that a Cartan geometry is torsion free, if and only if κ has values in $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{p}$. Since elements of $\mathfrak{p}_+$ have strictly positive homogeneity, this subspace is contained in the part of positive homogeneity, and any torsion free Cartan geometry is automatically regular. Hence regularity should be viewed as a condition which avoids particularly bad types of torsion.

On the other hand, there is a natural map $\partial^* : \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} \to \mathfrak{p}_+ \otimes \mathfrak{g}$ defined by

$$\partial^*(Z \wedge W \otimes A) := -W \otimes [Z, A] + Z \otimes [W, A] - [Z, W] \otimes A$$

for decomposable elements. This is the differential in the standard complex computing the Lie algebra homology of $\mathfrak{p}_+$ with coefficients in the module $\mathfrak{g}$. This map is evidently equivariant for the natural P–action, so in particular, $\ker(\partial^*) \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ is a P–submodule. The Cartan connection ω is called *normal* if and only if its curvature has values in this submodule.

To proceed with the program set out in the end of 3.4 we next have to analyze the map $\Psi_\alpha : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}}$ introduced in 3.3 in the special case of the pair (i, α) from 3.5. As a linear space, we may identify $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ with $\tilde{\mathfrak{g}}_- = \tilde{\mathfrak{g}}^E_{-1} \oplus \tilde{\mathfrak{g}}^V_{-1} \oplus \tilde{\mathfrak{g}}_{-2}$. Note that using brackets in $\tilde{\mathfrak{g}}$, we may identify $\tilde{\mathfrak{g}}^V_{-1}$ with $\tilde{\mathfrak{g}}^E_1 \otimes \tilde{\mathfrak{g}}_{-2}$ if necessary. We will view $\tilde{\mathfrak{g}}_{-2}$ as $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ and correspondingly write $X \in \tilde{\mathfrak{g}}_{-2}$ as (X_1, X_2). By $\langle\ ,\ \rangle$ we denote the standard inner product on $\mathbb{R}^n$.

Lemma 3.6. *Viewing Ψ_α as an element of $\Lambda^2(\tilde{\mathfrak{g}}_-)^* \otimes \tilde{\mathfrak{g}}$, it lies in the subspace $(\tilde{\mathfrak{g}}^V_{-1})^* \wedge (\tilde{\mathfrak{g}}_{-2})^* \otimes \tilde{\mathfrak{g}}_0$. Denoting by $W_0 \in \tilde{\mathfrak{g}}^E_1$ the element whose unique nonzero entry is equal to* 1*, the trilinear map $\tilde{\mathfrak{g}}_{-2} \times \tilde{\mathfrak{g}}_{-2} \times \tilde{\mathfrak{g}}_{-2} \to$*

$\tilde{\mathfrak{g}}_{-2}$ *defined by* $(X,Y,Z) \mapsto [\Psi_\alpha(X,[Y,W_0]),Z]$ *is (up to a nonzero multiple) the complete symmetrization of the map*

$$(X,Y,Z) \mapsto \langle X_1, Y_2\rangle \begin{pmatrix} Z_1 \\ -Z_2 \end{pmatrix}.$$

Proof. Let $x \in \tilde{\mathfrak{g}}^E_{-1}$ be the element whose unique nonzero entry is equal to 1. Then an arbitrary element of $\tilde{\mathfrak{g}}_-$ can be written uniquely as $X + [Y, W_0] + ax$ for $X, Y \in \tilde{\mathfrak{g}}_{-2}$ and $a \in \mathbb{R}$. From the definition of α in 3.5 we obtain

$$\alpha \begin{pmatrix} 0 & -Y_2^t & 0 \\ X_1 & 0 & Y_1 \\ a & X_2^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}Y_2^t & \frac{1}{2}Y_1^t \\ a & 0 & \frac{1}{2}X_2^t & -\frac{1}{2}X_1^t \\ X_1 & Y_1 & 0 & 0 \\ X_2 & Y_2 & 0 & 0 \end{pmatrix},$$

so this is congruent to $X + [Y, W_0] + ax$ modulo $\tilde{\mathfrak{p}}$. Using this, one can now insert into the defining formula for Ψ_α from 3.3 and compute directly that the result always has values in $\tilde{\mathfrak{g}}_0$, and indeed only in the lower right $2n \times 2n$ block. Moreover, all the entries in that block are made up from bilinear expressions involving one entry from $\tilde{\mathfrak{g}}_{-2}$ and one entry from $\tilde{\mathfrak{g}}^V_{-1}$, so we see that $\Psi_\alpha \in (\tilde{\mathfrak{g}}_{-2})^* \wedge (\tilde{\mathfrak{g}}^V_{-1})^* \otimes \tilde{\mathfrak{g}}_0$.

For $X, Y \in \tilde{\mathfrak{g}}_{-2}$, one next computes that the only nonzero block in $\Psi_\alpha(X, [Y, W_0])$ (which is a $2n \times 2n$–matrix) is explicitly given by

$$\frac{1}{2}\begin{pmatrix} X_1Y_2^t + Y_1X_2^t + (Y_2^tX_1 + X_2^tY_1)\,\mathrm{id} & X_1Y_1^t + Y_1X_1^t \\ -X_2Y_2^t - Y_2X_2^t & -Y_2X_1^t - X_2Y_1^t - (Y_2^tX_1 + X_2^tY_1)\,\mathrm{id} \end{pmatrix}.$$

To obtain $[\Psi_\alpha(X,[Y,W_0]),Z] \in \tilde{\mathfrak{g}}_{-2}$ for another element $Z \in \tilde{\mathfrak{g}}_{-2}$, we now simply have to apply this matrix to $\left(\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}\right)$. Taking into account that $\langle v, w\rangle = v^t w = w^t v$ for $v, w \in \mathbb{R}^n$ we obtain half the sum of all cyclic permutations of

$$(\langle X_1, Y_2\rangle + \langle Y_1, X_2\rangle) \begin{pmatrix} Z_1 \\ -Z_2 \end{pmatrix},$$

which is three times the symmetrization of $(X,Y,Z) \mapsto \langle X_1, Y_2\rangle \left(\begin{smallmatrix} Z_1 \\ -Z_2 \end{smallmatrix}\right)$. q.e.d.

Using this we can now complete the first part of the program outlined in the end of 3.4:

Theorem 3.6. *The extension functor associated to the pair* (i, α) *from* 3.5 *maps locally flat Cartan geometries of type* (G, Q) *to torsion free (and hence regular), normal parabolic geometries of type* $(\tilde{G}, \tilde{P})$.

Proof. Let $(p : \mathcal{G} \to N, \omega)$ be a locally flat Cartan geometry of type (G, Q). This means that ω has trivial curvature, so by Proposition 3.3, the curvature function $\tilde{\kappa}$ of the parabolic geometry $(\mathcal{G} \times_i \tilde{P}, \tilde{\omega}_\alpha)$ has the property that

$$\tilde{\kappa}(j(u)) = \Psi_\alpha : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}},$$

where $j : \mathcal{G} \to \mathcal{G} \times_i \tilde{P}$ is the natural map. By the lemma above, $\tilde{\kappa}(j(u))$ has values in $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{p}}$, and since having values in $\tilde{\mathfrak{p}}$ is a $\tilde{P}$–invariant property, torsion freeness follows.

Similarly, since $\ker(\partial^*)$ is a $\tilde{P}$–submodule in $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$, it suffices to show that $\partial^*(\Psi_\alpha) = 0$ to complete the proof of the theorem. This may be checked by a direct computation, but there is a more conceptual argument. Tracefree matrices in the lower right $2n \times 2n$ block of $\tilde{\mathfrak{g}}_0$ form a Lie subalgebra isomorphic to $\mathfrak{sl}(2n, \mathbb{R})$ which acts on each of the spaces $\Lambda^k(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$. Hence we may decompose each of them into a direct sum of irreducible representations. Since ∂^* is a $\tilde{P}$–homomorphism, it is equivariant for this action of $\mathfrak{sl}(2n, \mathbb{R})$, and hence it can be nonzero only between isomorphic irreducible components.

In the proof of the lemma we have noted that $\tilde{\mathfrak{g}}_{-2}$ is the standard representation of $\mathfrak{sl}(2n, \mathbb{R})$, so the explicit formula for Ψ_α shows that it sits in a component isomorphic to $S^3\mathbb{R}^{2n*} \otimes \mathbb{R}^{2n}$. There is a unique trace from this representation to $S^2\mathbb{R}^{2n*}$, and the kernel of this is well known to be irreducible. One immediately checks that $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ cannot contain an irreducible component isomorphic to the kernel of this trace. Hence we can finish the proof by showing that Ψ_α lies in the kernel of that trace, which is a simple direct computation. q.e.d.

This has a nice immediate application.

Corollary 3.6. *Consider the homogeneous model $G \to G/P$ of Lagrangean contact structures. Then the resulting path geometry of chains is non–flat and hence not locally isomorphic to $\tilde{G}/\tilde{P}$, but its automorphism group contains G. In particular, for each $n \geq 1$, we obtain an example of a non–flat torsion free path geometry on a manifold of dimension $2n+1$ whose automorphism group has dimension at least n^2+4n+3.*

Remark. (1) In [**14**], the author directly constructed a torsion free path geometry from the homogeneous model of three–dimensional Lagrangean contact structures. This construction was one of the motivations for this paper and one of the guidelines for the right choice of the pair (i, α). The other main guideline for this choice are the computations needed to show that Ψ_α has values in $\tilde{\mathfrak{g}}_0$.
(2) We shall see later that in the situation of the corollary, the dimension of the automorphism group actually equals the dimension of G. In particular, for $n = 1$, one obtains a non–flat path geometry on a three manifold with automorphism group of dimension 8. To our knowledge, this is the maximal possible dimension for the automorphism group of a non–flat path geometry in this dimension.

Via the interpretation of path geometries in terms of systems of second order ODEs, we obtain examples of nontrivial systems of such ODEs with large automorphism groups.

3.7. More on curvatures of regular normal geometries. We have completed half of the program outlined in the end of 3.4 at this point. Theorem 3.6 shows that the extension functor associated to the pair (i, α) defined in 3.5 produces the regular normal parabolic geometry determined by the path geometry of chains for locally flat Lagrangean contact structures. In view of Theorem 3.4 and Lemma 3.1 this pins down the pair (i, α) up to equivalence and hence the associated extension functor up to isomorphism.

Hence it only remains to clarify under which conditions on a Lagrangean contact structure this extension procedure produces a regular normal parabolic geometry. This then tells us the most general situation in which a direct relation (as discussed in 2.4 and 3.1) between the two parabolic geometries can exist. As can already be expected from the case of the Fefferman construction (see [**3**, **6**]) this is a rather subtle question. Moreover, the result cannot be obtained by algebraically comparing the two normalization conditions, but one needs more information on the curvature of regular normal and torsion free normal geometries. In particular, the proof of part (2) of the Lemma below needs quite a lot of deep machinery for parabolic geometries.

As discussed in 3.6, the curvature function of a parabolic geometry of type (G, P) has values in $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. Since both $\mathfrak{p}_+$ and $\mathfrak{g}$ are graded, there is a natural notion of homogeneity on this space. While being of some fixed homogeneity is not a P–invariant property, the fact that all nonzero homogeneous components have at least some given homogeneity is P–invariant. This is used in the definition of regularity in 3.6, which simply says that all nonzero homogeneous components are in positive homogeneity.

The map ∂^* used in the definition of normality in 3.6 actually extends to a family of maps $\partial^* : \Lambda^\ell\mathfrak{p}_+ \otimes \mathfrak{g} \to \Lambda^{\ell-1}\mathfrak{p}_+ \otimes \mathfrak{g}$. These are the differentials in the standard complex computing the Lie algebra homology $H_*(\mathfrak{p}_+, \mathfrak{g})$. By definition, the curvature function κ of a normal parabolic geometry of type (G, P) has values in $\ker(\partial^*) \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. Hence we can naturally project to the quotient to obtain a function κ_H with values in $\ker(\partial^*)/\operatorname{im}(\partial^*) = H_2(\mathfrak{p}_+, \mathfrak{g})$. Equivariancy of κ implies that κ_H can be viewed as a smooth section of the bundle $\mathcal{G} \times_P H_2(\mathfrak{p}_+, \mathfrak{g})$. This section is called the *harmonic curvature* of the normal parabolic geometry. It turns out (see [**8**]) that P_+ acts trivially on $H_*(\mathfrak{p}_+, \mathfrak{g})$, so this bundle admits a direct interpretation in terms of the underlying structure. As we shall see below, this bundle is algorithmically computable.

Now from 3.6 we know that $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ as a P–module, and since $\mathfrak{g}_- \subset \mathfrak{g}$ is a complementary subspace (and G_0–module) to $\mathfrak{p} \subset \mathfrak{g}$ we can identify $\mathfrak{p}_+$ with $(\mathfrak{g}_-)^*$ as a G_0–module. Hence we can also view the spaces $\Lambda^\ell\mathfrak{p}_+\otimes\mathfrak{g}$ as $L(\Lambda^\ell\mathfrak{g}_-,\mathfrak{g})$, which are the chain spaces in the standard complex computing the Lie algebra cohomology of $\mathfrak{g}_-$ with coefficients in $\mathfrak{g}$. The differentials $\partial : L(\Lambda^\ell\mathfrak{g}_-,\mathfrak{g})\to L(\Lambda^{\ell+1}\mathfrak{g}_-,\mathfrak{g})$ in that complex turn out to be adjoint to the maps ∂^* with respect to a certain inner product.

Hence we obtain an algebraic Hodge theory on each of the spaces $\Lambda^\ell\mathfrak{p}_+\otimes\mathfrak{g}$, with algebraic Laplacian $\square=\partial^*\circ\partial+\partial\circ\partial^*$. This construction is originally due to Kostant (see [**16**]), whence $\square$ is usually called the Kostant Laplacian. The kernel of $\square$ is a G_0–submodule called the *harmonic subspace* of $\Lambda^\ell\mathfrak{p}_+\otimes\mathfrak{g}$. Kostant's version of the Bott–Borel–Weil theorem in [**16**] gives a complete algorithmic description of the G_0–module $\ker(\square)$. By the Hodge decomposition, $\ker(\square)$ is isomorphic to the homology group of the appropriate dimension.

We will need two general facts about the curvature of regular normal respectively torsion free normal parabolic geometries in the sequel.

Lemma 3.7. *Let $(p:\mathcal{G}\to M,\omega)$ be a regular normal parabolic geometry of type (G,P) with curvature functions $\kappa:\mathcal{G}\to\Lambda^2\mathfrak{p}_+\otimes\mathfrak{g}$ and $\kappa_H:\mathcal{G}\to H_2(\mathfrak{p}_+,\mathfrak{g})$. Then we have:*
(1) The lowest nonzero homogeneous component of κ has values in the subset $\ker(\square)\subset\Lambda^2\mathfrak{p}_+\otimes\mathfrak{g}$.
(2) Suppose that $(p:\mathcal{G}\to M,\omega)$ is torsion free and that $E_0\subset\ker(\square)\subset\Lambda^2\mathfrak{p}_+\otimes\mathfrak{g}$ is a G_0–submodule such that κ_H has values in the image of E_0 under the natural isomorphism $\ker(\square)\to H_2(\mathfrak{p}_+,\mathfrak{g})$ (induced by projecting $\ker(\square)\subset\ker(\partial^*)$ to the quotient). Then κ has values in the P–submodule of $\Lambda^2\mathfrak{p}_+\otimes\mathfrak{g}$ generated by E_0.

Proof. (1) is an application of the Bianchi identity, which goes back to [**21**], see also [**7**, Corollary 4.10]. (2) is proved in [**4**, Corollary 3.2]. q.e.d.

The final bit of information we need is the explicit form of $\ker(\square)$ for the pairs $(\mathfrak{g},\mathfrak{p})$ and $(\tilde{\mathfrak{g}},\tilde{\mathfrak{p}})$ corresponding to Lagrangean contact structures on manifolds of dimension $2n+1$ respectively path geometries in dimension $4n+1$. Obtaining the explicit description of the irreducible components of these submodules is an exercise in the application of Kostant's results from [**16**] and the algorithms from the book [**1**], see also [**4**]. The results are listed in the tables below. The first column contains the homogeneity of the component and the second column contains the subspace that it is contained in. The actual component is always the highest weight part in that subspace, so in particular, it lies in the kernel of all traces one can form.

$(\mathfrak{g},\mathfrak{p})$, $n=1$

homog.	contained in
4	$\mathfrak{g}_1^R \wedge \mathfrak{g}_2 \otimes \mathfrak{g}_1^R$
4	$\mathfrak{g}_1^L \wedge \mathfrak{g}_2 \otimes \mathfrak{g}_1^L$

$(\mathfrak{g},\mathfrak{p})$, $n>1$

homog.	contained in
2	$\mathfrak{g}_1^L \wedge \mathfrak{g}_1^R \otimes \mathfrak{g}_0$
1	$\Lambda^2\mathfrak{g}_1^L \otimes \mathfrak{g}_{-1}^R$
1	$\Lambda^2\mathfrak{g}_1^R \otimes \mathfrak{g}_{-1}^L$

$(\tilde{\mathfrak{g}},\tilde{\mathfrak{p}})$, $n=1$

homog.	contained in
3	$\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$
2	$\tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_{-1}^V$
1	$\Lambda^2\tilde{\mathfrak{g}}_1^V \otimes \tilde{\mathfrak{g}}_{-1}^E$

$(\tilde{\mathfrak{g}},\tilde{\mathfrak{p}})$, $n>1$

homog.	contained in
3	$\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$
2	$\tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_{-1}^V$
0	$\Lambda^2\tilde{\mathfrak{g}}_1^V \otimes \tilde{\mathfrak{g}}_{-2}$

3.8. We are now ready to prove the main result of this article:

Theorem 3.8. *Let $(p:\mathcal{G}\to M,\omega)$ be a regular normal parabolic geometry of type (G,P) and let $(\tilde{\mathcal{G}}:=\mathcal{G}\times_Q\tilde{P}\to\mathcal{P}_0(TM),\tilde{\omega}_\alpha)$ be the parabolic geometry obtained using the extension functor associated to the pair (i,α) defined in* 3.5*. Then this geometry is regular and normal if and only if $(p:\mathcal{G}\to M,\omega)$ is torsion free.*

Proof. We first prove necessity of torsion freeness. From the tables in 3.7 we see that for $n=1$ a regular normal parabolic geometry of type (G,P) is automatically torsion free, so we only have to consider the case $n>1$. If $\tilde{\omega}_\alpha$ is regular and normal, then all nonzero homogeneous components of $\tilde{\kappa}$ are homogeneous of positive degrees. The table in 3.7 shows that then the homogeneity is at least two, and by part (1) of Lemma 3.7 the homogeneous component of degree two sits in the subspace $\tilde{\mathfrak{g}}_1^E\wedge\tilde{\mathfrak{g}}_2\otimes\tilde{\mathfrak{g}}_{-1}^V$. In particular, for any $\tilde{u}\in\tilde{\mathcal{G}}$, the restriction of $\tilde{\kappa}(\tilde{u})$ to $\Lambda^2\tilde{\mathfrak{g}}_{-2}$ is homogeneous of degree at least three, which implies that $\tilde{\kappa}(\tilde{u})$ has values in $\tilde{\mathfrak{g}}_{-1}\oplus\tilde{\mathfrak{p}}$, i.e. for the natural projection $\pi:\tilde{\mathfrak{g}}\to\tilde{\mathfrak{g}}/(\tilde{\mathfrak{g}}_{-1}\oplus\tilde{\mathfrak{p}})$ we get $\pi\circ\tilde{\kappa}(\tilde{u})=0$.

Using the notation of the proof of Lemma 3.6, consider two elements $X,Y\in\tilde{\mathfrak{g}}_{-2}$. From that proof we see that

$$(\pi\circ\tilde{\kappa}(j(u)))(X,Y)=(\pi\circ\alpha\circ\kappa(u))\left(\begin{pmatrix}0&0&0\\X_1&0&0\\0&X_2^t&0\end{pmatrix},\begin{pmatrix}0&0&0\\Y_1&0&0\\0&Y_2^t&0\end{pmatrix}\right).$$

By regularity, $\kappa(u)(\Lambda^2\mathfrak{g}_{-1})\subset\mathfrak{g}_{-1}\oplus\mathfrak{p}$. From the definition in 3.5 it is evident that α induces a linear isomorphism $\mathfrak{g}/(\mathfrak{g}_{-2}\oplus\mathfrak{p})\to\tilde{\mathfrak{g}}/(\tilde{\mathfrak{g}}_{-1}\oplus\tilde{\mathfrak{p}})$. Hence we conclude that if $\tilde{\omega}_\alpha$ is regular and normal, then $\kappa(u)(\Lambda^2\mathfrak{g}_{-1})\subset\mathfrak{p}$. From the table in 3.7 we see that this implies that the homogeneous component of degree one of κ has to vanish identically, and then further that the homogeneous component of degree two has values in $\mathfrak{p}$. Since $\Lambda^2\mathfrak{g}_{-2} = 0$, components of homogeneity at least three automatically have values in $\mathfrak{p}$, so we see that ω is torsion free.

To prove sufficiency, we first need two facts on the curvature function κ of a torsion free normal parabolic geometry of type (G, P). On the one hand, the map ∂^* as defined in 3.6 can be written as the sum $\partial_1^* + \partial_2^*$ of two P–equivariant maps, with ∂_1^* corresponding to the first two summands and ∂_2^* corresponding to the last summand in the definition. We claim that κ has values in the kernels of both operators ∂_i^*. On the other hand, one easily verifies that the subspace $\widehat{\mathfrak{p}} \subset \mathfrak{p}$ formed by all matrices of the form

$$\begin{pmatrix} 0 & u & d \\ 0 & B & v \\ 0 & 0 & 0 \end{pmatrix}$$

is a P–submodule. (Indeed, this is the preimage in $\mathfrak{p}$ of the semisimple part of the reductive algebra $\mathfrak{g}_0 = \mathfrak{p}/\mathfrak{p}_+$.) Our second claim is that $\kappa(u)(X, Y) \in \widehat{\mathfrak{p}}$ for all $u \in \mathcal{G}$ and all X, Y.

To prove both claims, it suffices to show that κ has values in the P–submodule $\Lambda_0^2\mathfrak{p}_+ \otimes \widehat{\mathfrak{p}} \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{p}$. Here $\Lambda_0^2\mathfrak{p}_+$ is the kernel of the P–homomorphism $\Lambda^2\mathfrak{p}_+ \to \mathfrak{p}_+$ defined by the Lie bracket on $\mathfrak{p}_+$, so $\Lambda_0^2\mathfrak{p}_+ \otimes \mathfrak{g} = \ker(\partial_2^*)$.

In the case $n = 1$, this is evident, since from the table in 3.7 we see that the lowest nonzero homogeneous component of $\kappa(u)$ is of degree 4, vanishes on $\Lambda^2\mathfrak{g}_{-1}$ and has values in $\mathfrak{p}_+$. For homogeneous components of higher degree, these two properties are automatically satisfied, and we conclude that $\kappa(u) \in \mathfrak{g}_1 \wedge \mathfrak{g}_2 \otimes \mathfrak{p}_+ \subset \Lambda_0^2\mathfrak{p}_+ \otimes \widehat{\mathfrak{p}}$.

In the case $n > 1$, we see from the table in 3.7 that by torsion freeness the lowest homogeneous component of $\kappa(u)$ must be of homogeneity 2. By part (1) of Lemma 3.7 it has values in $\ker(\Box) \subset \Lambda^2\mathfrak{g}_1 \otimes \mathfrak{g}_0$. Since this component of $\ker(\Box)$ is a highest weight part, it lies in the kernel of all possible traces, and hence it must be contained in the tensor product of $\Lambda^2\mathfrak{g}_1 \cap \Lambda_0^2\mathfrak{p}_+$ with the semisimple part of $\mathfrak{g}_0$. Hence $\ker(\Box)$ is contained in the P–submodule $\Lambda_0^2\mathfrak{p}_+ \otimes \widehat{\mathfrak{p}}$ so, by part (2) of Lemma 3.7, the curvature function κ has values in that submodule.

In view of Proposition 3.3 and the proof of Theorem 3.6, to prove that $\tilde{\omega}_\alpha$ is normal, it suffices to verify that the map $F(u) : \Lambda^2\tilde{\mathfrak{g}}_- \to \tilde{\mathfrak{g}}$ defined by $F(u)(X, Y) := \alpha(\kappa(u)(\underline{\alpha}^{-1}(X), \underline{\alpha}^{-1}(Y)))$ lies in the kernel of ∂^* for all $u \in \mathcal{G}$. To compute $\partial^* F(u)$, it is better to view $F(u)$ as an element of $\Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$, and we want to relate this to $\kappa(u)$, viewed as an element of $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. We have to compute the map $\varphi : \mathfrak{p}_+ \to \tilde{\mathfrak{p}}_+$, which is dual to the composition of the canonical projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$ with $\underline{\alpha}^{-1} : \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \to \mathfrak{g}/\mathfrak{q}$, since by construction $F(u) = (\Lambda^2\varphi \otimes \alpha)(\kappa(u))$. Recall that the duality between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$ (and likewise for the other algebra) is induced by the Killing form. Since the Killing form of a simple Lie algebra is uniquely determined up to a nonzero multiple by invariance, we may as well use the trace form on both sides, which leads to a nonzero multiple of φ. But then the computation is very easy, showing that

$$\varphi\begin{pmatrix} 0 & Z & \psi \\ 0 & 0 & W \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \psi & Z & W^t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, $\varphi(\mathfrak{p}_+) \subset \tilde{\mathfrak{g}}_1^E \oplus \tilde{\mathfrak{g}}_2$, which implies that $\partial_2^*(F(u)) = 0$ for all u.

On the other hand, the formula for α from 3.5 shows that $\alpha(\widehat{\mathfrak{p}}) \subset \tilde{\mathfrak{g}}_{-1}^V \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1^E \oplus \tilde{\mathfrak{g}}_2$, and the $\tilde{\mathfrak{g}}_0$–component is contained in the bottom right $2n \times 2n$ block. This shows that $\tilde{\omega}_\alpha$ is regular, and that for $Z \in \mathfrak{p}_+$ and $A \in \widehat{\mathfrak{p}}$ we have $[\varphi(Z), \alpha(A)] \in \tilde{\mathfrak{g}}_1^E \oplus \tilde{\mathfrak{g}}_2$. One immediately verifies directly that the $\tilde{\mathfrak{g}}_1^E$–component of $[\varphi(Z), \alpha(A)]$ equals the $\tilde{\mathfrak{g}}_1^E$–component of $\alpha([Z, A])$, while the $\tilde{\mathfrak{g}}_2$–component of $[\varphi(Z), \alpha(A)]$ equals twice the $\tilde{\mathfrak{g}}_2$–component of $\alpha([Z, A])$. From the definition of ∂_1^* we now conclude that $\Lambda^2\varphi \otimes \alpha$ maps $\ker(\partial_1^*)$ to $\ker(\partial_1^*)$, so we also get $\partial_1^*(F(u)) = 0$ for all u.

q.e.d.

4. Applications

For torsion free Lagrangean contact structures, Theorem 3.8 provides us with an explicit description of the parabolic geometry determined by the path geometry of chains. In particular, we obtain an explicit formula for the Cartan curvature which is the basis for the applications discussed in this section. The main result is that one can essentially reconstruct the torsion free Lagrangean contact structure from the harmonic curvature of this parabolic geometry. In particular, this implies that a contact diffeomorphism which maps chains to chains has to either preserve or swap the subbundles defining the Lagrangean contact structure. On the way, we can prove that chains can never be described by linear connections and that only locally flat Lagrangean contact structures give rise to torsion free path geometries of chains.

4.1. Decomposing the Cartan curvature. For a torsion free Lagrangean contact structure with curvature κ, the curvature $\tilde{\kappa}$ of the normal Cartan connection associated to the path geometry of chains is determined by the formula from Proposition 3.3, which holds on $j(\mathcal{G}) \subset \mathcal{G} \times_i \tilde{P}$. In this formula, there are two terms, one of which depends on κ while the other one only comes from the map α. Our main task is to extract parts of $\tilde{\kappa}$ which only depend on one of the two terms. The difficulty is that this has to be done in a geometric way without knowing the subset $j(\mathcal{G})$ in advance.

The curvature function $\tilde{\kappa}$ has values in the P–module $\Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$, and using the map φ from the proof of Theorem 3.8, the formula from Proposition 3.3 reads as $\tilde{\kappa}(j(u)) = (\Lambda^2\varphi \otimes \alpha)(\kappa(u)) + \Psi_\alpha$. Now $\tilde{\mathfrak{p}}_+$ contains the P–invariant subspace $\tilde{\mathfrak{g}}_2$. Correspondingly, we obtain P–invariant subspaces $\Lambda^2\tilde{\mathfrak{g}}_2 \subset \tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \subset \Lambda^2\tilde{\mathfrak{p}}_+$. In the proof of Theorem 3.8, we have seen that φ has values in $\tilde{\mathfrak{g}}_1^E \oplus \tilde{\mathfrak{g}}_2$, whence $\Lambda^2\varphi$ has values in $\tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2$. From Lemma 3.6 we know that $\Psi_\alpha \in \tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$, so we conclude that $\tilde{\kappa}(j(u))$ lies in this $\tilde{P}$–submodule. By equivariancy, all values of the curvature function lie in $\tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}} \subset \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$.

On the quotient $\tilde{\mathfrak{p}}_+/\tilde{\mathfrak{g}}_2$, the subgroup $\tilde{P}_+ \subset \tilde{P}$ acts trivially, so we can identify this quotient with the $\tilde{G}_0$–module $\tilde{\mathfrak{g}}_1 = \tilde{\mathfrak{g}}_1^E \oplus \tilde{\mathfrak{g}}_1^V$. Correspondingly, we get $\tilde{P}$–equivariant projections

$$\pi^E : \tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}},$$
$$\pi^V : \tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}.$$

From the description of the image of φ in the proof of Theorem 3.8 we conclude that $(\Lambda^2\varphi \otimes \alpha)(\kappa(u)) \in \ker(\pi^V)$. On the other hand, Lemma 3.6 in particular shows that $\pi^V(\Psi_\alpha) \neq 0$ and $\Psi_\alpha \in \ker(\pi^E)$.

Theorem 4.1. *Let (M, L, R) be a torsion free Lagrangean contact structure.*
(1) There is no linear connection on the tangent bundle TM which has the chains among its geodesics.
(2) The parabolic geometry associated to the path geometry of chains on $\tilde{M} = \mathcal{P}_0(TM)$ is torsion free if and only if (M, L, R) is locally flat, i.e. locally isomorphic to the homogeneous model G/P.

Proof. (1) Suppose that ∇ is a linear connection on TM whose geodesics in directions transverse to $L \oplus R$ are parametrizations of the chains. Since symmetrizing a connection does not change the geodesics, we may without loss of generality assume that ∇ is torsion free. Then we can look at the associated projective structure $[\nabla]$ on M and use the machinery of correspondence spaces from [**4**]. The fact that the geodesics of ∇ are the chains exactly means that the path geometry of chains on $\tilde{M}$ is isomorphic to an open subgeometry of the correspondence space $\mathcal{C}(M, [\nabla])$, see 4.7 of [**4**]. In particular, the Cartan curvature $\tilde{\kappa}$ is the restriction of the curvature of this correspondence space. By [**4**, Proposition 2.4] this curvature has the property that it vanishes upon insertion of any tangent vector contained in the vertical bundle of $\tilde{M} \to M$. But this contradicts the fact that $\pi^V \circ \tilde{\kappa} \neq 0$ we have observed above.
(2) By Theorem 3.6, the path geometry of chains associated to a locally flat Lagrangean contact structure is torsion free. Conversely, if the Cartan connection $\tilde{\omega}$ is torsion free, then according to part (1) of Lemma 3.7 and the tables in 3.7, the lowest nonzero homogeneous component of $\tilde{\kappa}$ must be of degree at least three, and the harmonic curvature must

have values in $\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0 \subset \ker(\pi^E)$. By part (2) of Lemma 3.7 the whole curvature $\tilde{\kappa}$ has values in $\ker(\pi^E)$. Above, we have observed that $\Psi_\alpha \in \ker(\pi^E)$ so we conclude that for each $u \in \mathcal{G}$ we get $\pi^E \circ (\Lambda^2\varphi \otimes \alpha)(\kappa(u)) = 0$.

In the proof of Theorem 3.8 we have seen that φ is a linear isomorphism $\mathfrak{p}_+ \to \tilde{\mathfrak{g}}_1^E \oplus \tilde{\mathfrak{g}}_2$, and hence $\tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2$ is contained in the image of $\Lambda^2\varphi$. Hence we conclude that $\alpha \circ \kappa(u) = 0$ and since α is injective, the result follows. q.e.d.

4.2. Harmonic curvature. We have discussed the definition of harmonic curvature already in 3.7. Let π_H be the natural projection from $\ker(\partial^*) \subset \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$ to the quotient $\ker(\partial^*)/\operatorname{im}(\partial^*)$. Since this is a $\tilde{P}$–equivariant map, the composition $\tilde{\kappa}_H = \pi_H \circ \tilde{\kappa} : \tilde{\mathcal{G}} \to \ker(\partial^*)/\operatorname{im}(\partial^*)$ defines a smooth section of the associated bundle $\tilde{\mathcal{G}} \times_{\tilde{P}} \ker(\partial^*)/\operatorname{im}(\partial^*)$, which is the main geometric invariant of the parabolic geometry associated to the path geometry of chains.

From 3.7 we also know that on the quotient $\ker(\partial^*)/\operatorname{im}(\partial^*)$ the group $\tilde{P}_+$ acts trivially, and we may identify it with the $\tilde{G}_0$–module $\ker(\Box) \subset \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$. From the table in 3.7, we see that this module contains two irreducible components in positive homogeneity, which are the highest weight components of the subrepresentations $\tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_{-1}^V$ respectively $\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$. Correspondingly, we obtain decompositions $\pi_H = \pi_H^E + \pi_H^V$ and $\tilde{\kappa}_H = \tilde{\kappa}_H^E + \tilde{\kappa}_H^V$.

Lemma 4.2. *Let π^E and π^V be the projections on $\tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$ defined in* 4.1. *Then the restriction of π_H^E (respectively π_H^V) to*

$$\ker(\partial^*) \cap (\tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}})$$

factorizes through π^E (respectively π^V).

Proof. By Kostant's version of the Bott–Borel–Weil theorem, see [**16**], the $\tilde{G}_0$–irreducible components contained in $\ker(\Box)$ occur with multiplicity one, even within $\Lambda^*\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$. To obtain π^E and π^V, we used the projection $\tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$ with kernel $\Lambda^2\tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$. By the multiplicity one result and the fact that both components of $\ker(\Box)$ are contained in $\tilde{\mathfrak{g}}_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$, there is no nonzero $\tilde{G}_0$–equivariant map $\Lambda^2\tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}} \to \ker(\partial^*)/\operatorname{im}(\partial^*)$. Hence each of the projections π_H, π_H^E and π_H^V factorizes through $\tilde{\mathfrak{g}}_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$. Looking at the resulting map for π_H^E, we see that again by multiplicity one, the subspace $\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$ must be contained in the kernel, so we conclude that π_H^E factorizes through π^E. In the same way one shows that π_H^V factorizes through π^V. q.e.d.

Proposition 4.2. *Let (M, L, R) be a torsion free Lagrangean contact structure, and let $\tilde{\kappa}_H = \tilde{\kappa}_H^E + \tilde{\kappa}_H^V$ be the harmonic curvature of the regular normal parabolic geometry determined by the path geometry of chains.*

Then the function $\tilde{\mathcal{G}} \to \tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$ corresponding to $\tilde{\kappa}_H^V$ is a nonzero multiple of the unique equivariant extension of the constant function Ψ_α (compare with Lemma 3.6) on $j(\mathcal{G})$.

Proof. We have to compute the function $\pi_H^V \circ \tilde{\kappa}$. By the lemma, π_H^V factorizes through the projection π^V introduced in 4.1, and from there we know that $\pi_V(\tilde{\kappa}(j(u))) = \pi_V(\Psi_\alpha)$. Hence we see that $(\pi_H^V \circ \tilde{\kappa})|_{j(\mathcal{G})} = \pi_H^V(\Psi_\alpha)$. Now $\Psi_\alpha \in \tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$ by Lemma 3.6, and the values even lie in the semisimple part of $\tilde{\mathfrak{g}}_0$, which may be identified with $\mathfrak{sl}(\tilde{\mathfrak{g}}_{-2})$. Evidently, $\tilde{\mathfrak{g}}_1^V \cong \tilde{\mathfrak{g}}_{-1}^E \otimes \tilde{\mathfrak{g}}_2$ as a $\tilde{G}_0$–module, so we may interpret Ψ_α as an element of $\tilde{\mathfrak{g}}_{-1}^E \otimes (\otimes^3 \tilde{\mathfrak{g}}_2) \otimes \tilde{\mathfrak{g}}_{-2}$. In Lemma 3.6 and the proof of Theorem 3.6 we have seen that in this picture Ψ_α lies in the irreducible component $\tilde{\mathfrak{g}}_{-1}^E \otimes (S^3 \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_{-2})_0$, where the subscript denotes the trace free part. Passing back to $\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$ this exactly means that Ψ_α lies in the highest weight subspace, which is the intersection with $\ker(\square)$. Now π_H^V restricts to $\tilde{G}_0$–equivariant linear isomorphism on this intersection, which implies the result. q.e.d.

Remark. Similarly to the proof above, one shows that the harmonic curvature component $\tilde{\kappa}_H^E$ is the extension of a component of $j(u) \mapsto (\Lambda^2 \varphi \otimes \alpha)(\kappa(u))$. Since we explicitly know $\Lambda^2 \varphi \otimes \alpha$, this can be used to obtain a more explicit description of the second harmonic curvature component. From part (2) of Theorem 4.1 and [**4**, 4.7] we see that vanishing of $\tilde{\kappa}_H^E$ is equivalent to local flatness of the original Lagrangean contact structure, so κ is completely encoded in $\tilde{\kappa}_H^E$.

4.3. Passing to the underlying manifold. The harmonic curvature component determined by the function $\tilde{\kappa}_H^V$ is a section of the bundle associated to $\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$. In the proof of Proposition 4.2 we have seen that we can replace that space by $\tilde{\mathfrak{g}}_{-1}^E \otimes (\otimes^3 \tilde{\mathfrak{g}}_2) \otimes \tilde{\mathfrak{g}}_{-2}$. The corresponding bundle is $E \otimes \otimes^3 F^* \otimes F \to \tilde{M}$, where $F := T\tilde{M}/(E \oplus V)$. Since $E \subset T\tilde{M}$ is a line bundle, we can view $\tilde{\kappa}_H^V$ as a section of $\otimes^3 F^* \otimes F$ which is determined up to a nonzero multiple.

To relate this to the underlying manifold M, recall that $\tilde{M}$ is an open subset in the projectivized tangent bundle of M. A point in $\tilde{M}$ is a line in some tangent space T_xM that is transversal to $L_x \oplus R_x$. We have noted in 2.4 that $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ and $T\tilde{M} \cong \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$, and the tangent map of the projection $\pi : \tilde{M} \to M$ corresponds to the natural projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$. Fix a point $\ell \in \pi^{-1}(x)$. Then for each $\xi \in T_xM$ there is a lift $\tilde{\xi} \in T_\ell \tilde{M}$ and we can consider the class of $\tilde{\xi}$ in $F_\ell = T_\ell \tilde{M}/(E_\ell \oplus V_\ell)$. Since V_ℓ is the vertical subbundle, this class is independent of the choice of the lift and from the explicit description of $T\pi$ we see that restricting to $L_x \oplus R_x$, we obtain a linear isomorphism $L_x \oplus R_x \cong F_\ell$.

Fixing x and ℓ we therefore see that the harmonic curvature component corresponding to $\tilde{\kappa}_H^V$ gives rise to an element of $\otimes^3 (L_x \oplus R_x)^* \otimes$

$(L_x \oplus R_x)$, which is determined up to a nonzero multiple. To write down this map explicitly, we first need the Levi bracket

$$\mathcal{L} : (L_x \oplus R_x) \times (L_x \oplus R_x) \to T_xM/(L_x \oplus R_x).$$

Since this has values in a one–dimensional space, we may view it as a real valued bilinear map determined up to a nonzero multiple. Further, we denote by $\mathbb{J}$ the almost product structure corresponding to the decomposition $L \oplus R$. This means that $\mathbb{J}$ is the endomorphism of $L \oplus R$ which is the identity on L and minus the identity on R. Using this we can now formulate:

Lemma 4.3. *The element of $\otimes^3(L_x \oplus R_x)^* \otimes (L_x \oplus R_x)$ obtained from $\tilde{\kappa}_H^V$ above is (a nonzero multiple of) the complete symmetrization of the map*

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}(\xi, \mathbb{J}(\eta))\mathbb{J}(\zeta).$$

Proof. This is a reinterpretation of the proof of Lemma 3.6. Observe that $\mathbb{J}$ corresponds to the map

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mapsto \begin{pmatrix} X_1 \\ -X_2 \end{pmatrix}$$

in the notation there. Since $\mathcal{L}$ corresponds to $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$, computing the bracket

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ X_1 & 0 & 0 \\ 0 & X_2^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ Y_1 & 0 & 0 \\ 0 & -Y_2^t & 0 \end{pmatrix} \right],$$

we see that the expression $\langle X_1, Y_2 \rangle + \langle Y_1, X_2 \rangle$ in the proof of Lemma 3.6 corresponds to $\mathcal{L}(\xi, \mathbb{J}(\eta))$. q.e.d.

4.4. Reconstructing the Lagrangean contact structure. Now we can finally show that the Cartan curvature of the path geometry of chains can be used to (almost) reconstruct the Lagrangean contact structure on M that we have started from:

Theorem 4.4. *Let (M, L, R) be a torsion free Lagrangean contact structure. Then for each $x \in M$, the subset $L_x \cup R_x \subset T_xM$ can be reconstructed from the harmonic curvature of the normal parabolic geometry associated to the path geometry of chains.*

Proof. In view of the results in 4.2 and 4.3 it suffices to show that $L_x \cup R_x$ can be recovered from the complete symmetrization S of the map

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}(\xi, \mathbb{J}(\eta))\mathbb{J}(\zeta).$$

First we see that $S(\xi, \xi, \xi) = 0$ if and only if $\mathcal{L}(\xi, \mathbb{J}(\xi)) = 0$. Note that this is always satisfied for $\xi \in L_x \cup R_x$. Fixing an element ξ with this property, we see that

$$S(\xi, \xi, \eta) = 2\mathcal{L}(\xi, \mathbb{J}(\eta))\mathbb{J}(\xi).$$

By non–degeneracy of $\mathcal{L}$, given a nonzero element ξ we can always find η such that $\mathcal{L}(\xi, \mathbb{J}(\eta)) \neq 0$. Hence we see that ξ is an eigenvector for $\mathbb{J}$ (which by definition is equivalent to $\xi \in L_x \cup R_x$) if and only if $S(\xi, \xi, \xi) = 0$ and there is an element η such that $S(\xi, \xi, \eta)$ is a nonzero multiple of ξ. q.e.d.

Corollary 4.4. *Let (M, L, R) be a torsion free Lagrangean contact structure and let $f : M \to M$ be a contact diffeomorphism which maps chains to chains. Then either f is an automorphism or an anti–automorphism of the Lagrangean contact structure. Here anti–automorphism means that $T_x f(L_x) = R_{f(x)}$ and $T_x f(R_x) = L_{f(x)}$ for all $x \in M$.*

Proof. By assumption, f induces an automorphism $\tilde{f}$ of the path geometry of chains associated to (M, L, R). This automorphism has to pull back the Cartan curvature $\tilde{\kappa}$ and also the harmonic curvature $\tilde{\kappa}_H$ to itself. From the theorem we conclude that this implies $T_x f(L_x \cup R_x) = L_{f(x)} \cup R_{f(x)}$, and this is only possible if f is an automorphism or an anti–automorphism. q.e.d.

5. Partially integrable almost CR structures

What we have done for Lagrangean contact structures so far can be easily adapted to deal with partially integrable almost CR structure. We will only briefly sketch the necessary changes in this section.

5.1. A non–degenerate partially integrable almost CR structure on a smooth manifold M is given by a contact structure $H \subset TM$ together with an almost complex structure J on H such that the Levi bracket $\mathcal{L}$ has the property that $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all ξ, η. Then $\mathcal{L}$ is the imaginary part of a non–degenerate Hermitian form and we denote the signature of this form by (p, q). Such a structure of signature (p, q) is equivalent to a regular normal parabolic geometry of type (G, P), where $G = PSU(p+1, q+1)$ and $P \subset G$ is the stabilizer of a point in $\mathbb{C}P^{n+1}$, $n = p+q$, corresponding to a null line, see [**7**, 4.15]. The group G is the quotient of $SU(p+1, q+1)$ by its center (which is isomorphic to $\mathbb{Z}_{n+2}$) and we will work with representative matrices as before.

We will use the Hermitian form of signature (p, q) on $\mathbb{C}^{n+1}$ corresponding to

$$(z_0, \dots, z_{n+1}) \mapsto z_0 \bar{z}_{n+1} + z_{n+1} \bar{z}_0 + \textstyle\sum_{j=1}^{p} |z_j|^2 - \sum_{j=p+1}^{n} |z_j|^2.$$

Then the decomposition on $\mathfrak{sl}(n+2, \mathbb{C})$ with block sizes 1, n, and 1 restricts to a contact grading on the Lie algebra $\mathfrak{g}$ of G. The explicit form for signature $(n, 0)$ can be found in [**7**, 4.15]. In general, $\mathfrak{g}$ consists of all matrices of the form

$$\begin{pmatrix} w & Z & iz \\ X & A & -\mathbb{I}Z^* \\ ix & -X^*\mathbb{I} & -\bar{w} \end{pmatrix}$$

with blocks of sizes 1, n, and 1, $w \in \mathbb{C}$, $x, z \in \mathbb{R}$, $X \in \mathbb{C}^n$, $Z \in \mathbb{C}^{n*}$, and $A \in \mathfrak{u}(p,q)$ such that $w - \bar{w} + \operatorname{tr}(A) = 0$. Here $\mathbb{I}$ is the diagonal matrix with the first p entries equal to 1 and the remaining q entries equal to -1.

It is easy to show that the subgroup $Q \subset G$ corresponds to matrices of the form

$$\begin{pmatrix} \varphi & 0 & ia\varphi \\ 0 & \Phi & 0 \\ 0 & 0 & \bar{\varphi}^{-1} \end{pmatrix},$$

with $\varphi \in \mathbb{C} \setminus \{0\}$, $a \in \mathbb{R}$ and $\Phi \in U(p,q)$ such that $(\varphi^2/|\varphi|^2)\det(\Phi) = 1$.

5.2. Next we need an analog of the pair (i, α) introduced in 3.5. As before we start with a manifold M of dimension $2n+1$, so again $\tilde{G} = PGL(2n+2, \mathbb{R})$. We will use a block decomposition into blocks of sizes 1, 1, n, and n as before. The right choice turns out to be

$$i \begin{pmatrix} \varphi & 0 & ia\varphi \\ 0 & \Phi & 0 \\ 0 & 0 & \bar{\varphi}^{-1} \end{pmatrix} := \begin{pmatrix} |\varphi| & -a|\varphi| & 0 & 0 \\ 0 & |\varphi|^{-1} & 0 & 0 \\ 0 & 0 & \Re\left(\frac{|\varphi|}{\varphi}\Phi\right) & -\Im\left(\frac{|\varphi|}{\varphi}\Phi\right) \\ 0 & 0 & \Im\left(\frac{|\varphi|}{\varphi}\Phi\right) & \Re\left(\frac{|\varphi|}{\varphi}\Phi\right) \end{pmatrix},$$

$$\alpha \begin{pmatrix} w & Z & iz \\ X & A & -\mathbb{I}Z^* \\ ix & -X^*\mathbb{I} & -\bar{w} \end{pmatrix} :=$$

$$\begin{pmatrix} \Re(w) & -z & \Re(Z) & -\Im(Z) \\ x & -\Re(w) & -\Im(X^*\mathbb{I}) & -\Re(X^*\mathbb{I}) \\ \Re(X) & \Im(\mathbb{I}Z^*) & \Re(A) & -\Im(A) + \Im(w) \\ \Im(X) & -\Re(\mathbb{I}Z^*) & \Im(A) - \Im(w) & \Re(A) \end{pmatrix},$$

where $\Re$ and $\Im$ denote real and imaginary part, respectively, and we write $\Im(w)$ for the appropriate multiple of the identity matrix.

There is an analog of Lemma 3.6 (with similar proof), the only change one has to make is that the map whose alternation has to be used is given by

$$(X, Y, Z) \mapsto (\langle X_1, \mathbb{I}Y_1\rangle + \langle X_2, \mathbb{I}Y_2\rangle) \begin{pmatrix} -Z_2 \\ Z_1 \end{pmatrix}.$$

This map has similar properties as the one from 3.6 so the analogs of Theorem 3.6 and Corollary 3.6 hold.

Concerning the structure of $\ker(\Box)$ the situation is also similar to the case of Lagrangean contact structures, since the decomposition of $\ker(\Box)$ can be determined from the complexifications of $\mathfrak{g}$ and $\mathfrak{p}$ which are the same in both cases. The only difference is that the two irreducible components for $n = 1$ respectively the two irreducible components contained in homogeneity 1 in the case $n > 1$ in the Lagrangean case correspond to only one component here. This component however has a complex structure and it consists of maps $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-2} \to \mathfrak{g}_1$ which are complex linear in the first variable respectively maps $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$, which are conjugate linear in both variables. For $n > 1$ this component is a torsion which is up to a nonzero multiple given by the Nijenhuis tensor. Vanishing of this component is equivalent to torsion freeness and to integrability of the almost CR structure, see [**7**, 4.16].

Theorem 5.2. *Let (M, H, J) be a partially integrable almost CR structure and let $(p : \mathcal{G} \to M, \omega)$ be the corresponding regular normal parabolic geometry of type (G, P). Then the parabolic geometry $(\mathcal{G} \times_Q \tilde{P} \to \mathcal{P}_0(TM), \tilde{\omega}_\alpha)$ constructed using the extension functor associated to the pair (i, α) from* 5.1 *is regular and normal if and only if ω is torsion free, i.e. the almost CR structure is integrable.*

Proof. Apart from some numerical factors which cause no problems, this is completely parallel to the proof of Theorem 3.8. q.e.d.

Hence the direct relation between the regular normal parabolic geometries associated to a partially integrable almost CR structure respectively to the associated path geometry of chains works exactly on the subclass of CR structures.

5.3. Applications. The developments of Section 4 can be applied to the CR case with only minimal changes. In analog of Lemma 4.3, one obtains $S \in \otimes^3 H_x^* \otimes H_x$, which is the complete symmetrization of

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}(\xi, J(\eta))J(\zeta),$$

where J is the almost complex structure on H.

Theorem 5.3. *Let (M, H, J) be a CR structure.*
(1) There is no linear connection on TM which has the chains among its geodesics.
(2) The path geometry of chains is torsion free if and only if the CR structure is locally flat.
(3) The almost complex structure J can be reconstructed up to sign from the harmonic curvature of the associated path geometry of chains.

Proof. The only change compared to Section 4 is that one has to extend S to the complexified bundle $H \otimes \mathbb{C}$. As in the proof of Theorem

4.4 one then reconstructs the subset $H_x^{1,0} \cup H_x^{0,1} \subset H_x \otimes \mathbb{C}$ for each $x \in M$, i.e. the union of the holomorphic and the anti–holomorphic part. This union determines J up to sign. q.e.d.

This theorem now also implies that the signature of the CR structure, which is encoded in $\mathcal{L}(-, J(-))$, can be reconstructed from the path geometry of chains. As a corollary, we obtain a completely independent proof of the analog of Corollary 4.4, which is due to [**10**] for CR structures:

Corollary 5.3. *A contact diffeomorphism between two CR manifolds which maps chains to chains is either a CR isomorphism or a CR anti–isomorphism.*

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A Projective-to-Conformal Fefferman-Type Construction

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Abstract. We study a Fefferman-type construction based on the inclusion of Lie groups $\mathrm{SL}(n+1)$ into $\mathrm{Spin}(n+1,n+1)$. The construction associates a split-signature (n,n)-conformal spin structure to a projective structure of dimension n. We prove the existence of a canonical pure twistor spinor and a light-like conformal Killing field on the constructed conformal space. We obtain a complete characterisation of the constructed conformal spaces in terms of these solutions to overdetermined equations and an integrability condition on the Weyl curvature. The Fefferman-type construction presented here can be understood as an alternative approach to study a conformal version of classical Patterson–Walker metrics as discussed in recent works by Dunajski–Tod and by the authors. The present work therefore gives a complete exposition of conformal Patterson–Walker metrics from the viewpoint of parabolic geometry.

Key words: parabolic geometry; projective structure; conformal structure; Cartan connection; Fefferman spaces; twistor spinors

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1 Introduction

In *conformal geometry* the geometric structure is given by an equivalence class of pseudo-Riemannian metrics: two metrics g and $\hat{g}$ are considered to be equivalent if they differ by a positive smooth rescaling, $\hat{g} = e^{2f} g$. In *projective geometry* the geometric structure is given by an equivalence class of torsion-free affine connections: two connections D and $\hat{D}$ are considered as equivalent if they share the same geodesics (as unparametrised curves). While conformal and projective structures both determine a corresponding class of affine connections, neither of them induces a single distinguished connection on the tangent bundle. Instead, both structures have canonically associated *Cartan connections* that govern the respective geometries and encode prolonged geometric data of the respective structures. It is therefore often useful when studying projective and conformal structures to work in the framework of *Cartan geometries*.

The present paper investigates a geometric construction that produces a conformal class of split-signature metrics on a $2n$-dimensional manifold arising naturally from a projective class of connections on an n-dimensional manifold. Split-signature conformal structures of this type have appeared in several places in the literature before. The projective-to-conformal construction studied in this paper should be understood as a generalisation of the classical *Riemann extensions* of affine spaces by E.M. Patterson and A.G. Walker [26]. One of the main authors motivations for the present study was the article [15] by M. Dunajski and P. Tod, where the Patterson–Walker construction was generalised to a projectively invariant setting in dimension $n = 2$. On the other hand, in [25] conformal structures of signature $(2, 2)$ were constructed using Cartan connections that contain the conformal structures arising from 2-dimensional projective structures as a special case. A generalisation of this Cartan-geometric approach to higher dimensions can be found in [24].

In this paper the construction is studied as an instance of a *Fefferman-type construction*, as formalised in [6, 11], based on an inclusion of the respective Cartan structure groups $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$. We show that in the general situation $n \geq 3$ the induced conformal Cartan geometry is *non-normal*. To obtain information on the conformal structure it is thus important to understand how the normal conformal Cartan connection differs from the induced one, and the main part of the paper concerns the study of this modification. We may summarise the main contributions of the paper as follows:

- A comprehensive treatment of the projective-to-conformal Fefferman-type construction including a discussion of the intermediate Lagrangean contact structure (Section 3) and a comparison with Patterson–Walker metrics (Section 6.1).
- A thorough study of the normalisation process (Section 4) and an explicit formula for the modification needed to obtain the normal conformal Cartan connection (Section 5.2).
- The characterisation of the conformal structures obtained via our Fefferman-type construction (culminating in Theorem 4.14).

Let us comment upon the characterisation in more detail. This is formulated in terms of a conformal Killing field k and a twistor spinor χ on the conformal space together with a (conformally invariant) integrability curvature condition. In Theorem 4.14 the properties of k and χ are specified in terms of corresponding conformal tractors, which nicely reflects the algebraic setup of the Fefferman-type construction in geometric terms.

An alternative equivalent characterisation theorem was obtained by the authors in [20, Theorem 1] by different means, namely, by direct computations based on spin calculus in the spirit of [28, 29]. The conformal properties are given purely in underlying terms and do not refer to tractors. In Section 6.2 (Theorem 6.3) we indicate how this alternative characterisation can be obtained in the current framework.

We remark that, to our knowledge, the present work is the first comprehensive treatment of a non-normal Fefferman-type construction and we expect that the techniques developed should have considerable scope for applications to other similar constructions. A particularly interesting case of this sort is the Fefferman construction for (non-integrable) almost CR-structures. Possible further applications concern relations between solutions of so-called *BGG-equations* and special properties of the induced conformal structures. Several such relationships were already obtained by the authors in [20]. For instance, we can give a full description of Einstein metrics contained in the resulting conformal class in terms of the initial projective structure. Moreover, in [21] we were able to show that the obstruction tensor of the induced conformal structure vanishes.

2 Projective and conformal parabolic geometries

The standard reference for the background material on Cartan and parabolic geometries presented here is [11].

2.1 Cartan and parabolic geometries

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $P \subseteq G$ a closed subgroup with Lie algebra $\mathfrak{p}$. A *Cartan geometry* $(\mathcal{G}, \omega)$ of type (G, P) over a smooth manifold M consists of a P-principal bundle $\mathcal{G} \to M$ together with a *Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. The canonical principal bundle $G \to G/P$ endowed with the Maurer–Cartan form constitutes the *homogeneous model* for Cartan geometries of type (G, P).

The *curvature* of a Cartan connection ω is the 2-form

$$K \in \Omega^2(\mathcal{G}, \mathfrak{g}), \qquad K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)], \qquad \text{for all } \xi, \eta \in \mathfrak{X}(\mathcal{G}),$$

which is equivalently encoded in the P-equivariant *curvature function*

$$\kappa\colon\ \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}, \qquad \kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) := K\big(\omega^{-1}(u)(X), \omega^{-1}(u)(Y)\big). \tag{2.1}$$

The curvature is a complete obstruction to a local equivalence with the homogeneous model. If the image of κ is contained in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$ the Cartan geometry is called *torsion-free*.

A *parabolic geometry* is a Cartan geometry of type (G, P), where G is a semi-simple Lie group and $P \subseteq G$ is a parabolic subgroup. A subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is parabolic if and only if its maximal nilpotent ideal, called nilradical $\mathfrak{p}_+$, coincides with the orthogonal complement $\mathfrak{p}^\perp$ of $\mathfrak{p} \subseteq \mathfrak{g}$ with respect to the Killing form. In particular, this yields an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of P-modules. The quotient $\mathfrak{g}_0 = \mathfrak{p}/\mathfrak{p}_+$ is called the Levi factor; it is reductive and decomposes into a semi-simple part $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$ and the center $\mathfrak{z}(\mathfrak{g}_0)$. The respective Lie groups are $G_0^{ss} \subseteq G_0 \subseteq P$ and $P_+ \subseteq P$ so that $P = G_0 \ltimes P_+$ and $P_+ = \exp(\mathfrak{p}_+)$. An identification of $\mathfrak{g}_0$ with a subalgebra in $\mathfrak{p}$ yields a grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, where $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. We set $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$. If k is the depth of the grading the parabolic geometry is called $|k|$-*graded*.

The grading of $\mathfrak{g}$ induces a grading on $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} \cong \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. A parabolic geometry is called *regular* if the curvature function κ takes values only in the components of positive homogeneity. In particular, any torsion-free or $|1|$-graded parabolic geometry is regular.

Given a $\mathfrak{g}$-module V, there is a natural $\mathfrak{p}$-equivariant map, the *Kostant co-differential*,

$$\partial^*\colon\ \Lambda^k(\mathfrak{g}/\mathfrak{p})^* \otimes V \to \Lambda^{k-1}(\mathfrak{g}/\mathfrak{p})^* \otimes V, \tag{2.2}$$

defining the Lie algebra homology of $\mathfrak{p}_+$ with values in V; see, e.g., [11, Section 3.3.1] for the explicit form. For $V = \mathfrak{g}$, this gives rise to a natural normalisation condition: parabolic geometries satisfying $\partial^*(\kappa) = 0$ are called *normal*. The *harmonic curvature* κ_H of a normal parabolic geometry is the image of κ under the projection $\ker \partial^* \to \ker \partial^* / \operatorname{im} \partial^*$. For regular and normal parabolic geometries, the entire curvature κ is completely determined just by κ_H.

A *Weyl structure* $j\colon \mathcal{G}_0 \hookrightarrow \mathcal{G}$ of a parabolic geometry $(\mathcal{G}, \omega)$ over M is a reduction of the P-principal bundle $\mathcal{G} \to M$ to the Levi subgroup $G_0 \subseteq P$. The class of all Weyl structures, which are parametrised by one-forms on M, includes a particularly important subclass of *exact Weyl structures*, which are parametrised by functions on M: For $|1|$-graded parabolic geometries, these correspond to further reductions of $\mathcal{G}_0 \to M$ just to the semi-simple part G_0^{ss} of G_0 or, equivalently, to sections of the principal $\mathbb{R}_+$-bundle $\mathcal{G}_0/G_0^{ss} \to M$. The latter bundle is called the *bundle of scales* and its sections are the *scales*.

For a Weyl structure $j\colon \mathcal{G}_0 \hookrightarrow \mathcal{G}$, the pullback $j^*\omega = j^*\omega_- + j^*\omega_0 + j^*\omega_+$ of the Cartan connection may be decomposed according to $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$. The $\mathfrak{g}_0$-part $j^*\omega_0$ is a principal connection on the G_0-bundle $\mathcal{G}_0 \to M$; it induces connections on all associated bundles, which are called (*exact*) *Weyl connections*. The $\mathfrak{p}_+$-part $j^*\omega_+$ is the so-called *Schouten tensor*.

2.2 Tractor bundles and BGG operators

Every Cartan connection ω on $\mathcal{G} \to M$ naturally extends to a principal connection $\hat{\omega}$ on the G-principal bundle $\hat{\mathcal{G}} := \mathcal{G} \times_P G \to M$, which further induces a linear connection $\nabla^{\mathcal{V}}$ on any associated vector bundle $\mathcal{V} := \mathcal{G} \times_P V = \hat{\mathcal{G}} \times_G V$ for a G-representation V. Bundles and connections arising in this way are called *tractor bundles* and *tractor connections*. The tractor connections induced by normal Cartan connections are called normal tractor connections.

In particular, for the adjoint representation we obtain the *adjoint tractor bundle* $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$. The canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ and the identification $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ yield a bundle projection $\Pi\colon \mathcal{A}M \to TM$; the inclusion $\mathfrak{p}_+ \subseteq \mathfrak{g}$ and the identification $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ yield a bundle inclusion $T^*M \hookrightarrow \mathcal{A}M$. This allows us to interpret the Cartan curvature κ from (2.1) as a 2-form Ω on M with values in $\mathcal{A}M$.

The holonomy group of the principal connection $\hat{\omega}$ is by definition the *holonomy of the Cartan connection* ω, i.e., $\mathrm{Hol}(\omega) := \mathrm{Hol}(\hat{\omega}) \subseteq G$. By the holonomy of a geometric structure we mean the holonomy of the corresponding normal Cartan connection.

In [12], and later in a simplified manner in [4], it was shown that for a tractor bundle $\mathcal{V} = \mathcal{G} \times_P V$ one can associate a sequence of differential operators, which are intrinsic to the given parabolic geometry $(\mathcal{G}, \omega)$,

$$\Gamma(\mathcal{H}_0) \overset{\Theta_0^{\mathcal{V}}}{\to} \Gamma(\mathcal{H}_1) \overset{\Theta_1^{\mathcal{V}}}{\to} \cdots \overset{\Theta_{n-1}^{\mathcal{V}}}{\to} \Gamma(\mathcal{H}_n).$$

The operators $\Theta_k^{\mathcal{V}}$ are the *BGG-operators* and they operate between the sections of subquotients $\mathcal{H}_k = \ker \partial^*/\mathrm{im}\, \partial^*$ of the bundles of $\mathcal{V}$-valued k-forms, where $\partial^*\colon \Lambda^k T^*M \otimes \mathcal{V} \to \Lambda^{k-1} T^*M \otimes \mathcal{V}$ denotes the bundle map induced by the Kostant co-differential (2.2).

The first BGG-operator $\Theta_0^{\mathcal{V}}\colon \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1)$ is constructed as follows. The bundle $\mathcal{H}_0$ is simply the quotient $\mathcal{V}/\mathcal{V}'$, where $\mathcal{V}' \subseteq \mathcal{V}$ is the subbundle corresponding to the largest P-invariant filtration component in the G-representation V. It turns out, there is a distinguished differential operator that splits the projection $\Pi_0\colon \mathcal{V} \to \mathcal{H}_0$, namely, the *splitting operator*, which is the unique map $L_0^{\mathcal{V}}\colon \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{V})$ satisfying

$$\Pi_0(L_0^{\mathcal{V}}(\sigma)) = \sigma, \qquad \partial^*(d^{\nabla^{\mathcal{V}}} L_0^{\mathcal{V}}(\sigma)) = 0, \qquad \text{for all } \sigma \in \Gamma(\mathcal{H}_0).$$

The latter condition allows to define the first BGG-operator by $\Theta_0^{\mathcal{V}} := \Pi_1 \circ d^{\nabla^{\mathcal{V}}} \circ L_0^{\mathcal{V}}$, where $\Pi_1\colon \ker \partial^* \to \Gamma(\mathcal{H}_1)$. The first BGG-operator defines an overdetermined system of differential equations on $\sigma \in \Gamma(\mathcal{H}_0)$, $\Theta_0^{\mathcal{V}}(\sigma) = 0$, which is termed the *first BGG-equation*.

2.3 Further notations and conventions

In order to distinguish various objects related to projective and conformal structures, the symbols referring to conformal data will always be endowed with tildes. To write down explicit formulae, we employ abstract index notation, cf., e.g., [27]. Furthermore, we will use different types of indices for projective and conformal manifolds. E.g., on a projective manifold M we write $\mathbb{E}_A := T^*M$, $\mathbb{E}^A := TM$, and multiple indices denote tensor products, as in $\mathbb{E}_A{}^B := T^*M \otimes TM$. Indices between squared brackets are skew, as in $\mathbb{E}_{[AB]} := \Lambda^2 T^*M$, and indices between round brackets are symmetric, as in $\mathbb{E}^{(AB)} := S^2 TM$. Analogously, on a conformal manifold $\widetilde{M}$ we write $\widetilde{\mathbb{E}}_a := T^*\widetilde{M}$, $\widetilde{\mathbb{E}}^a := T\widetilde{M}$ etc. By $\mathbb{E}(w)$ and $\widetilde{\mathbb{E}}[w]$ we denote the density bundle over M and $\widetilde{M}$, respectively. Tensor products with other natural bundles are denoted as $\mathbb{E}_A(w) := \mathbb{E}_A \otimes \mathbb{E}(w)$, $\widetilde{\mathbb{E}}_{[ab]}[w] := \widetilde{\mathbb{E}}_{[ab]} \otimes \widetilde{\mathbb{E}}[w]$, and the like.

2.4 Projective structures

Let M be a smooth manifold of dimension $n \geq 2$. A *projective structure* on M is given by a class, $\mathbf{p}$, of torsion-free projectively equivalent affine connections: two connections D and $\hat{D}$

are projectively equivalent if they have the same geodesics as unparametrised curves. This is the case if and only if there is a one-form $\Upsilon_A \in \Gamma(\mathbb{E}_A)$ such that, for all $\xi^A \in \Gamma(\mathbb{E}^A)$,

$$\hat{D}_A \xi^B = D_A \xi^B + \Upsilon_A \xi^B + \Upsilon_P \xi^P \delta_A{}^B.$$

An *oriented projective structure* $(M, \mathbf{p})$, which is a projective structure $\mathbf{p}$ on an oriented manifold M, is equivalently encoded as a normal parabolic geometry of type (G, P), where $G = \mathrm{SL}(n+1)$ and $P = \mathrm{GL}_+(n) \ltimes \mathbb{R}^{n*}$ is the stabiliser of a ray in the standard representation $\mathbb{R}^{n+1}$.

Affine connections from the projective class $\mathbf{p}$ are precisely the Weyl connections of the corresponding parabolic geometry. Exact Weyl connections are those $D \in \mathbf{p}$ which preserve a volume form — these are also known as *special* affine connections. In particular, a choice of $D \in \mathbf{p}$ reduces the structure group to $G_0 = \mathrm{GL}_+(n)$, if D is special, the structure group is further reduced to $G_0^{ss} = \mathrm{SL}(n)$.

For later purposes we now give explicit expressions of the main curvature quantities, cf., e.g., [2, 17]. For $D \in \mathbf{p}$, the Schouten tensor is determined by the Ricci curvature of D; if D is special, then the Schouten tensor is $\mathsf{P}_{AB} = \frac{1}{n-1} R_{PA}{}^P{}_B$, in particular, it is symmetric. The projective Weyl curvature and the Cotton tensor are

$$W_{AB}{}^C{}_D = R_{AB}{}^C{}_D + \mathsf{P}_{AD} \delta^C{}_B - \mathsf{P}_{BD} \delta^C{}_A, \qquad Y_{CAB} = 2 D_{[A} \mathsf{P}_{B]C}.$$

Henceforth, we use a suitable normalisation of densities so that the line bundle associated to the canonical one-dimensional representation of P has projective weight -1. Hence, comparing with the usual notation, the *density bundle of projective weight* w, denoted by $\mathbb{E}(w)$, is just the bundle of ordinary $\left(\frac{-w}{n+1}\right)$-densities. As an associated bundle to $\mathcal{G} \to M$, $\mathbb{E}(w)$ corresponds to the 1-dimensional representation of P given by

$$\mathrm{GL}_+(n) \ltimes \mathbb{R}^{n*} \to \mathbb{R}_+, \qquad (A, X) \mapsto \det(A)^w. \tag{2.3}$$

The *projective standard tractor bundle* is the tractor bundle associated to the standard representation of $G = \mathrm{SL}(n+1)$. The *projective dual standard tractor bundle* is denoted by $\mathcal{T}^*$, i.e., $\mathcal{T}^* := \mathcal{G} \times_P \mathbb{R}^{n+1*}$. With respect to a choice of $D \in \mathbf{p}$, we write

$$\mathcal{T}^* = \begin{pmatrix} \mathbb{E}_A(1) \\ \mathbb{E}(1) \end{pmatrix}, \qquad \nabla^{\mathcal{T}^*}_C \begin{pmatrix} \varphi_A \\ \sigma \end{pmatrix} = \begin{pmatrix} D_C \varphi_A + \mathsf{P}_{CA} \sigma \\ D_C \sigma - \varphi_C \end{pmatrix}.$$

2.5 Conformal spin structures and tractor formulas

Let $\widetilde{M}$ be a smooth manifold of dimension $2n \geq 4$. A *conformal structure* of signature (n, n) on $\widetilde{M}$ is given by a class, $\mathbf{c}$, of conformally equivalent pseudo-Riemannian metrics of signature (n, n): two metrics g and $\hat{g}$ are conformally equivalent if $\hat{g} = f^2 g$ for a nowhere-vanishing smooth function f on $\widetilde{M}$. It may be equivalently described as a reduction of the frame bundle of $\widetilde{M}$ to the structure group $\mathrm{CO}(n, n) = \mathbb{R}_+ \times \mathrm{SO}(n, n)$. An *oriented conformal structure* of signature (n, n) is a conformal structure of signature (n, n) together with fixed orientations both in time-like and space-like directions, equivalently, a reduction of the frame bundle to the group $\mathrm{CO}_\mathrm{o}(n, n) = \mathbb{R}_+ \times \mathrm{SO}_\mathrm{o}(n, n)$, the connected component of the identity. An equivariant lift of such a reduction with respect to the 2-fold covering $\mathrm{CSpin}(n, n) = \mathbb{R}_+ \times \mathrm{Spin}(n, n) \to \mathrm{CO}_\mathrm{o}(n, n)$ is referred to as a *conformal spin structure* $(\widetilde{M}, \mathbf{c})$ of signature (n, n).

A conformal spin structure of signature (n, n) is equivalently encoded as a normal parabolic geometry of type $(\widetilde{G}, \widetilde{P})$, where $\widetilde{G} = \mathrm{Spin}(n+1, n+1)$ and $\widetilde{P} = \mathrm{CSpin}(n, n) \ltimes \mathbb{R}^{n,n*}$ is the stabiliser of an isotropic ray in the standard representation $\mathbb{R}^{n+1,n+1}$.

A general Weyl connection is a torsion-free affine connection $\widetilde{D}$ such that $\widetilde{D}g \in \mathbf{c}$ for any $g \in \mathbf{c}$. If $\widetilde{D}g = 0$, i.e., $\widetilde{D}$ is the Levi-Civita connection of a metric $g \in \mathbf{c}$, it is an exact Weyl connection. A choice of Weyl connection reduces the structure group to $\widetilde{G}_0 = \mathrm{CSpin}(n,n)$. If the Weyl connection is exact the structure group is further reduced to $\widetilde{G}_0^{ss} = \mathrm{Spin}(n,n)$.

Now we briefly introduce the main curvature quantities of conformal structures, cf., e.g., [16]. For $g \in \mathbf{c}$, the *Schouten tensor*,

$$\widetilde{\mathsf{P}} = \widetilde{\mathsf{P}}(g) = \frac{1}{2n-2}\left(\widetilde{\mathrm{Ric}}(g) - \frac{\widetilde{\mathrm{Sc}}(g)}{2(2n-1)}g\right),$$

is a trace modification of the Ricci curvature $\widetilde{\mathrm{Ric}}(g)$ by a multiple of the scalar curvature $\widetilde{\mathrm{Sc}}(g)$; its trace is denoted $\widetilde{J} = g^{pq}\widetilde{\mathsf{P}}_{pq}$. The conformal Weyl curvature and the Cotton tensors are

$$\widetilde{W}_{ab}{}^{c}{}_{d} = \widetilde{R}_{ab}{}^{c}{}_{d} - 2\delta^{c}_{[a}\widetilde{\mathsf{P}}_{b]d} + 2g_{d[a}\widetilde{\mathsf{P}}_{b]}{}^{c}, \qquad \widetilde{Y}_{cab} = 2\widetilde{D}_{[a}\widetilde{\mathsf{P}}_{b]c}.$$

As for projective structures, we will employ a suitable parametrisation of densities so that the canonical 1-dimensional representation of $\widetilde{P}$ has conformal weight -1. Hence, the *density bundle of conformal weight* w, denoted as $\widetilde{\mathbb{E}}[w]$, is just the bundle of ordinary $\left(\frac{-w}{2n}\right)$-densities. As an associated bundle to the Cartan bundle $\widetilde{\mathcal{G}} \to \widetilde{M}$, it corresponds to the 1-dimensional representation of $\widetilde{P}$ given by

$$(\mathbb{R}_+ \times \mathrm{Spin}(n,n)) \ltimes \mathbb{R}^{2n*} \to \mathbb{R}_+, \qquad (a, A, Z) \mapsto a^{-w}. \tag{2.4}$$

In particular, the conformal structure may be seen as a section of $\widetilde{\mathbb{E}}_{(ab)}[2]$, which is called the *conformal metric* and denoted by $\boldsymbol{g}_{ab}$.

The spin bundles corresponding to the irreducible spin representations of $\mathrm{Spin}(n,n)$ are denoted by $\widetilde{\Sigma}_+$ and $\widetilde{\Sigma}_-$, and $\widetilde{\Sigma} = \widetilde{\Sigma}_+ \oplus \widetilde{\Sigma}_-$. We employ the weighted conformal gamma matrix $\gamma \in \Gamma\big(\widetilde{\mathbb{E}}_a \otimes \big(\mathrm{End}\,\widetilde{\Sigma}\big)[1]\big)$ such that $\gamma_p\gamma_q + \gamma_q\gamma_p = -2\boldsymbol{g}_{pq}$. For $\xi \in \mathfrak{X}\big(\widetilde{M}\big)$ and $\chi \in \Gamma\big(\widetilde{\Sigma}\big)$, the Clifford multiplication of ξ on χ is then written as $\xi \cdot \chi = \xi^p\gamma_p\chi$.

The *conformal standard tractor bundle* is the associated bundle $\widetilde{\mathcal{T}} := \widetilde{\mathcal{G}} \times_{\widetilde{P}} \mathbb{R}^{n+1,n+1}$ with respect to the standard representation. It carries the canonical tractor metric $\mathbf{h}$ and the conformal standard tractor connection $\widetilde{\nabla}^{\widetilde{\mathcal{T}}}$, which preserves $\mathbf{h}$. With respect to a metric $g \in \mathbf{c}$, we have

$$\widetilde{\mathcal{T}} = \begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ \widetilde{\mathbb{E}}_a[1] \\ \widetilde{\mathbb{E}}[1] \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \widetilde{\nabla}^{\widetilde{\mathcal{T}}}_c \begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} = \begin{pmatrix} \widetilde{D}_c\rho - \widetilde{\mathsf{P}}_c{}^{b}\varphi_b \\ \widetilde{D}_c\varphi_a + \sigma\widetilde{\mathsf{P}}_{ca} + \rho\boldsymbol{g}_{ca} \\ \widetilde{D}_c\sigma - \varphi_c \end{pmatrix}. \tag{2.5}$$

The BGG-splitting operator is given by

$$L_0^{\widetilde{\mathcal{T}}}\colon\ \Gamma\big(\widetilde{\mathbb{E}}[1]\big) \to \Gamma\big(\widetilde{\mathcal{T}}\big), \qquad \sigma \mapsto \begin{pmatrix} \frac{1}{2n}\big(-\widetilde{D}^p\widetilde{D}_p - \widetilde{J}\big)\sigma \\ \widetilde{D}_a\sigma \\ \sigma \end{pmatrix}. \tag{2.6}$$

The *spin tractor bundle* is the associated bundle $\widetilde{\mathcal{S}} := \widetilde{\mathcal{G}} \times_{\widetilde{P}} \Delta^{n+1,n+1}$, where $\Delta^{n+1,n+1}$ is the spin representation of $\widetilde{G} = \mathrm{Spin}(n+1,n+1)$. Since we work in even signature, it decomposes into irreducibles $\Delta^{n+1,n+1} = \Delta_+^{n+1,n+1} \oplus \Delta_-^{n+1,n+1}$; the corresponding bundles are denoted by $\widetilde{\mathcal{S}}_\pm = \widetilde{\mathcal{G}} \times_{\widetilde{P}} \Delta_\pm^{n+1,n+1}$. Under a choice of $g \in \mathbf{c}$, these decompose as $\widetilde{\mathcal{S}}_\pm = \begin{pmatrix} \widetilde{\Sigma}_\mp[-\frac{1}{2}] \\ \widetilde{\Sigma}_\pm[\frac{1}{2}] \end{pmatrix}$, where $\widetilde{\Sigma}_\pm$

are the natural spin bundles as before. For later use we record the formulas for the Clifford action of $\widetilde{\mathcal{T}}$ on $\widetilde{\mathcal{S}}$ and for the spin tractor connections on $\widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}_+ \oplus \widetilde{\mathcal{S}}_-$,

$$\begin{pmatrix} \rho \\ \varphi_a \\ \sigma \end{pmatrix} \cdot \begin{pmatrix} \tau \\ \chi \end{pmatrix} = \begin{pmatrix} -\varphi_a\gamma^a\tau + \sqrt{2}\rho\chi \\ \varphi_a\gamma^a\chi - \sqrt{2}\sigma\tau \end{pmatrix}, \qquad \widetilde{\nabla}^{\widetilde{\mathcal{S}}}_c \begin{pmatrix} \tau \\ \chi \end{pmatrix} = \begin{pmatrix} \widetilde{D}_c\tau + \frac{1}{\sqrt{2}}\widetilde{\mathsf{P}}_{cp}\gamma^p\chi \\ \widetilde{D}_c\chi + \frac{1}{\sqrt{2}}\gamma_c\tau \end{pmatrix}, \tag{2.7}$$

cf. [19]. The BGG-splitting operator of $\widetilde{\mathcal{S}}_\pm$ is

$$L_0^{\widetilde{\mathcal{S}}_\pm}\colon\ \Gamma\big(\widetilde{\Sigma}_\pm\big[\tfrac{1}{2}\big]\big) \to \Gamma\big(\widetilde{\mathcal{S}}_\pm\big), \qquad \chi \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}n}\not{D}\chi \\ \chi \end{pmatrix}, \tag{2.8}$$

where $\not{D}\colon \Gamma\big(\widetilde{\Sigma}_\pm\big) \to \Gamma\big(\widetilde{\Sigma}_\mp\big)$, $\not{D} := \gamma^p\widetilde{D}_p$, is the *Dirac* operator. The first BGG-operator associated to $\widetilde{\mathcal{S}}_\pm$ is the *twistor operator*

$$\Theta_0^{\widetilde{\mathcal{S}}}\colon\ \Gamma\big(\widetilde{\Sigma}_\pm\big[\tfrac{1}{2}\big]\big) \to \Gamma\big(\widetilde{\mathbb{E}}_a \otimes \widetilde{\Sigma}_\pm\big[\tfrac{1}{2}\big]\big), \qquad \chi \mapsto \widetilde{D}_a\chi + \tfrac{1}{2n}\gamma_a\not{D}\chi,$$

cf., e.g., [3]. Elements in the kernel of $\Theta_0^{\widetilde{\mathcal{S}}}$ are called *twistor spinors*. It is well known that $\Pi_0^{\widetilde{\mathcal{S}}}$ induces an isomorphism between $\widetilde{\nabla}^{\widetilde{\mathcal{S}}}$-parallel sections of $\widetilde{\mathcal{S}}$ with $\ker\Theta_0^{\widetilde{\mathcal{S}}}$.

The *adjoint tractor bundle* is the associated bundle $\mathcal{A}\widetilde{M} := \widetilde{\mathcal{G}} \times_{\widetilde{P}} \tilde{\mathfrak{g}}$ with respect to the adjoint representation of $\widetilde{G}$ on $\tilde{\mathfrak{g}} = \mathfrak{so}(n+1, n+1) \cong \Lambda^2\mathbb{R}^{n+1,n+1}$. The standard pairing on $\mathcal{A}\widetilde{M}$ induced by the Killing form on $\tilde{\mathfrak{g}}$ is denoted as $\langle\cdot,\cdot\rangle\colon \mathcal{A}\widetilde{M} \times \mathcal{A}\widetilde{M} \to \mathbb{R}$. Henceforth we identify $\mathcal{A}\widetilde{M}$ with $\Lambda^2\widetilde{\mathcal{T}}$. With respect to a metric $g \in \mathbf{c}$,

$$\mathcal{A}\widetilde{M} = \begin{pmatrix} \widetilde{\mathbb{E}}_a[0] \\ \widetilde{\mathbb{E}}_{[a_0a_1]}[2] \mid \widetilde{\mathbb{E}}[1] \\ \widetilde{\mathbb{E}}_a[2] \end{pmatrix}.$$

The standard representation of $\tilde{\mathfrak{g}}$ on $\mathbb{R}^{n+1,n+1}$ gives rise to the map

$$\bullet\colon\ \mathcal{A}\widetilde{M} \otimes \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}, \qquad \begin{pmatrix} \rho_a \\ \mu_{a_0a_1} \mid \varphi \\ \beta_a \end{pmatrix} \bullet \begin{pmatrix} \nu \\ \omega_b \\ \sigma \end{pmatrix} = \begin{pmatrix} \rho^r\omega_r - \varphi\nu \\ \mu_b{}^r\omega_r - \sigma\rho_b - \nu\beta_b \\ \beta^r\omega_r + \varphi\sigma \end{pmatrix}. \tag{2.9}$$

The normal tractor connection is given by

$$\widetilde{\nabla}^{\mathcal{A}\widetilde{M}}_c \begin{pmatrix} \rho_a \\ \mu_{a_0a_1} \mid \varphi \\ k_a \end{pmatrix} = \begin{pmatrix} \widetilde{D}_c\rho_a - \widetilde{\mathsf{P}}_c{}^p\mu_{pa} - \widetilde{\mathsf{P}}_{ca}\varphi \\ \begin{pmatrix} \widetilde{D}_c\mu_{a_0a_1} + 2\boldsymbol{g}_{c[a_0}\rho_{a_1]} \\ +2\widetilde{\mathsf{P}}_{c[a_0}k_{a_1]} \end{pmatrix} \mid \big(\widetilde{D}_c\varphi - \widetilde{\mathsf{P}}_c{}^pk_p + \rho_c\big) \\ \widetilde{D}_ck_a - \mu_{ca} + \boldsymbol{g}_{ca}\varphi \end{pmatrix}. \tag{2.10}$$

Written as a two-form $\widetilde{\Omega}$ with values in $\Lambda^2\widetilde{\mathcal{T}}$, the curvature of $\widetilde{\nabla}^{\widetilde{\mathcal{T}}}$ is

$$\widetilde{\Omega}_{c_0c_1} = \begin{pmatrix} -\widetilde{Y}_{ac_0c_1} \\ \widetilde{W}_{c_0c_1a_0a_1} \mid 0 \\ 0 \end{pmatrix} \in \Gamma\big(\widetilde{\mathbb{E}}_{[c_0c_1]} \otimes \mathcal{A}\widetilde{M}\big). \tag{2.11}$$

The BGG-splitting operator

$$L_0^{\mathcal{A}\widetilde{M}}\colon\ \Gamma\big(\widetilde{\mathbb{E}}^a\big) = \Gamma\big(\widetilde{\mathbb{E}}_a[2]\big) \to \Gamma\big(\mathcal{A}\widetilde{M}\big), \qquad k_a \mapsto \begin{pmatrix} \rho_a \\ \mu_{a_0a_1} \mid \varphi \\ k_a \end{pmatrix},$$

is determined by

$$\mu_{a_0a_1} = \widetilde{D}_{[a_0}k_{a_1]}, \qquad \varphi = -\frac{1}{2n}\boldsymbol{g}^{pq}\widetilde{D}_pk_q,$$
$$\rho_a = -\frac{1}{4n}\widetilde{D}^p\widetilde{D}_pk_a + \frac{1}{4n}\widetilde{D}^p\widetilde{D}_ak_p + \frac{1}{4n^2}\widetilde{D}_a\widetilde{D}^pk_p + \frac{1}{n}\widetilde{\mathsf{P}}^p{}_ak_p - \frac{1}{2n}\widetilde{J}k_a, \tag{2.12}$$

and the corresponding first BGG-operator of $\mathcal{A}\widetilde{M}$ is computed as

$$\Theta_0^{\mathcal{A}\widetilde{M}}\colon\ \Gamma\big(\widetilde{\mathbb{E}}_a[2]\big) \to \Gamma\big(\widetilde{\mathbb{E}}_{(ab)_0}[2]\big), \qquad \xi_a \mapsto \widetilde{D}_{(c}\xi_{a)_0},$$

where the subscript 0 denotes the trace-free part. Thus $\Theta_0^{\mathcal{A}\widetilde{M}}$ is the conformal Killing operator and solutions to the first BGG-equation are conformal Killing fields. In a prolonged form, the conformal Killing equation is equivalent to

$$\widetilde{\nabla}_b^{\mathcal{A}\widetilde{M}}s = \xi^a\widetilde{\Omega}_{ab}, \tag{2.13}$$

where $s = L_0^{\mathcal{A}\widetilde{M}}(\xi)$, see [7, 18].

3 The Fefferman-type construction

The construction of split-signature conformal structures from projective structures discussed in this section fits into a general scheme relating parabolic geometries of different types. Namely, it is an instance of the so-called Fefferman-type construction, whose name and general procedure is motivated by Fefferman's construction of a canonical conformal structure induced by a CR structure, see [6] and [11] for a detailed discussion.

3.1 General procedure

Suppose we have two pairs of semi-simple Lie groups and parabolic subgroups, (G,P) and $\big(\widetilde{G},\widetilde{P}\big)$, and a Lie group homomorphism $i\colon G\to\widetilde{G}$ such that the derivative $i'\colon\mathfrak{g}\to\tilde{\mathfrak{g}}$ is injective. Assume further that the G-orbit of the origin in $\widetilde{G}/\widetilde{P}$ is open and that the parabolic $P\subseteq G$ contains $Q := i^{-1}\big(\widetilde{P}\big)$, the preimage of $\widetilde{P}\subseteq\widetilde{G}$.

Given a parabolic geometry $(\mathcal{G}\to M,\omega)$ of type (G,P), one first forms the *Fefferman space*

$$\widetilde{M} := \mathcal{G}/Q = \mathcal{G}\times_P P/Q. \tag{3.1}$$

Then $\big(\mathcal{G}\to\widetilde{M},\omega\big)$ is automatically a Cartan geometry of type (G,Q). As a next step, one considers the extended bundle $\widetilde{\mathcal{G}} := \mathcal{G}\times_Q\widetilde{P}$ with respect to the homomorphism $Q\to\widetilde{P}$. This is a principal bundle over $\widetilde{M}$ with structure group $\widetilde{P}$ and $j\colon\mathcal{G}\hookrightarrow\widetilde{\mathcal{G}}$ denotes the natural inclusion. The equivariant extension of $\omega\in\Omega^1(\mathcal{G},\mathfrak{g})$ yields a unique Cartan connection $\widetilde{\omega}^{\mathrm{ind}}\in\Omega^1\big(\widetilde{\mathcal{G}},\tilde{\mathfrak{g}}\big)$ of type $\big(\widetilde{G},\widetilde{P}\big)$ such that $j^*\widetilde{\omega}^{\mathrm{ind}} = i'\circ\omega$. Altogether, one obtains a functor from parabolic geometries $(\mathcal{G}\to M,\omega)$ of type (G,P) to parabolic geometries $\big(\widetilde{\mathcal{G}}\to\widetilde{M},\widetilde{\omega}^{\mathrm{ind}}\big)$ of type $\big(\widetilde{G},\widetilde{P}\big)$.

The relation between the corresponding curvatures is as follows: The previous assumptions yield a linear isomorphism $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}\cong\mathfrak{g}/\mathfrak{q}$ and an obvious projection $\mathfrak{g}/\mathfrak{q}\to\mathfrak{g}/\mathfrak{p}$, where $\mathfrak{q}\subseteq\mathfrak{p}$ is the Lie algebra of $Q\subseteq P$. Composing these two maps one obtains a linear projection $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}\to\mathfrak{g}/\mathfrak{p}$, whose dual map is denoted as $\varphi\colon(\mathfrak{g}/\mathfrak{p})^*\to(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$. Since $i'\colon\mathfrak{g}\to\tilde{\mathfrak{g}}$ is a homomorphism of Lie algebras, the curvature function $\widetilde{\kappa}^{\mathrm{ind}}\colon\widetilde{\mathcal{G}}\to\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*\otimes\tilde{\mathfrak{g}}$ is related to $\kappa\colon\mathcal{G}\to\Lambda^2(\mathfrak{g}/\mathfrak{p})^*\otimes\mathfrak{g}$ by $\widetilde{\kappa}^{\mathrm{ind}}\circ j = (\Lambda^2\varphi\otimes i')\circ\kappa$. We note that $\widetilde{\kappa}^{\mathrm{ind}}$ is fully determined by this formula.

Since i' is an embedding, the notation is in most cases simplified such that we write $\mathfrak{g}\subseteq\tilde{\mathfrak{g}}$, $\mathfrak{q} = \mathfrak{g}\cap\tilde{\mathfrak{p}}$, etc.

3.2 Algebraic setup and the homogeneous model

Here we specify the general setup for Fefferman-type constructions from Section 3.1 according to the description of oriented projective and conformal spin structures given in Sections 2.4 and 2.5, respectively. Let $\mathbb{R}^{n+1,n+1}$ be the real vector space $\mathbb{R}^{2n+2}$ with an inner product, h, of split-signature. Let $\Delta_+^{n+1,n+1}$ and $\Delta_-^{n+1,n+1}$ be the irreducible spin representations of

$$\widetilde{G} := \mathrm{Spin}(n+1, n+1)$$

as in Section 2.5. We fix two pure spinors $s_F \in \Delta_-^{n+1,n+1}$ and $s_E \in \Delta_\pm^{n+1,n+1}$ with non-trivial pairing, which is assigned for later use to be $\langle s_E, s_F\rangle = -\frac{1}{2}$. Note that s_E lies in $\Delta_+^{n+1,n+1}$ if n is even or in $\Delta_-^{n+1,n+1}$ if n is odd.

Let us denote by $E, F \subseteq \mathbb{R}^{n+1,n+1}$ the kernels of s_E, s_F with respect to the Clifford multiplication, i.e.,

$$E := \big\{X \in \mathbb{R}^{n+1,n+1} \colon X \cdot s_E = 0\big\}, \qquad F := \big\{X \in \mathbb{R}^{n+1,n+1} \colon X \cdot s_F = 0\big\}.$$

The purity of s_E and s_F means that E and F are maximally isotropic subspaces in $\mathbb{R}^{n+1,n+1}$. The other assumptions guarantee that E and F are complementary and dual each other via the inner product h. Hence we use the decomposition

$$\mathbb{R}^{n+1,n+1} = E \oplus F \cong \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1*} \tag{3.2}$$

to identify the spinor representation $\Delta^{n+1,n+1} = \Delta_+^{n+1,n+1} \oplus \Delta_-^{n+1,n+1}$ with the exterior power algebra $\Lambda^\bullet E \cong \Lambda^\bullet \mathbb{R}^{n+1}$, whose irreducible subrepresentations are $\Delta_-^{n+1,n+1} \cong \Lambda^{\mathrm{even}}\mathbb{R}^{n+1}$ and $\Delta_+^{n+1,n+1} \cong \Lambda^{\mathrm{odd}}\mathbb{R}^{n+1}$. When n is even, respectively, odd, we can identify $\big(\Delta_-^{n+1,n+1}\big)^* \cong \Delta_+^{n+1,n+1}$, respectively $\big(\Delta_-^{n+1,n+1}\big)^* \cong \Delta_-^{n+1,n+1}$.

Now, let us consider the subgroup in $\widetilde{G}$ defined by

$$G := \{g \in \mathrm{Spin}(n+1, n+1) \colon g \cdot s_E = s_E,\ g \cdot s_F = s_F\}.$$

This subgroup preserves the decomposition (3.2) so that the restriction of the action to F is dual to the restriction to E. It further preserves the volume form on E, respectively $F \cong E^*$, which is determined by s_E and s_F according to the previous identifications. Hence $G \cong \mathrm{SL}(n+1)$ and this defines an embedding $i\colon \mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$.[1]

The G-invariant decomposition (3.2) determines a G-invariant skew-symmetric involution $K \in \mathfrak{so}(n+1, n+1)$ acting by the identity on E and minus the identity on F. The relationship among K, s_E and s_F may be expressed as

$$h(X, K(Y)) = -h(K(X), Y) = 2\langle s_E, (X \wedge Y) \cdot s_F\rangle, \tag{3.3}$$

where

$$(X \wedge Y) \cdot s_F = \frac{1}{2}(X \cdot Y \cdot s_F - Y \cdot X \cdot s_F) = X \cdot Y \cdot s_F + h(X, Y)s_F.$$

The spin action of $\tilde{\mathfrak{g}}$ is denoted by $\bullet$, and thus $A \bullet s = -\frac{1}{4}A \cdot s$, for any $A \in \tilde{\mathfrak{g}}$ and $s \in \Delta$. In particular, $K \bullet s_F = -\frac{1}{2}(n+1)s_F$ and $K \bullet s_E = \frac{1}{2}(n+1)s_E$. Here we identify $\tilde{\mathfrak{g}} = \mathfrak{so}(n+1, n+1)$ with $\Lambda^2\mathbb{R}^{n+1,n+1}$. It is convenient to split $\tilde{\mathfrak{g}}$ in terms of irreducible $\mathfrak{g}$-modules as

$$\tilde{\mathfrak{g}} = \Lambda^2(E \oplus F) = \underbrace{(E \otimes F)_0}_{\mathfrak{g}=\mathfrak{sl}(n+1)} \oplus \underbrace{(E \otimes F)_{Tr} \oplus \Lambda^2 E \oplus \Lambda^2 F}_{\mathfrak{g}^\perp}, \tag{3.4}$$

[1] Instead of the embedding $\mathrm{SL}(n+1) \hookrightarrow \mathrm{Spin}(n+1, n+1)$ we could also consider the embedding $\mathrm{SL}(n+1) \hookrightarrow \mathrm{SO}(n+1, n+1)$. The advantage of employing the embedding into the spin group is two-fold: on the one hand, it is then seen directly that the induced conformal structure has a canonical spin structure, and, on the other hand, we can then use convenient spinorial objects for its characterisation.

where $(E\otimes F)_{Tr}=\mathbb{R}K$, and K acts as $[K,\phi]=2\phi$, $[K,\psi]=-2\psi$, $[K,\lambda]=0$, for any $\phi\in\Lambda^2E$, $\psi\in\Lambda^2F$ and $\lambda\in E\otimes F$. Further, the annihilators of s_E and s_F in $\tilde{\mathfrak{g}}$ are the subalgebras $\ker s_E=\mathfrak{sl}(n+1)\oplus\Lambda^2E$ and $\ker s_F=\mathfrak{sl}(n+1)\oplus\Lambda^2F$.

The homogeneous model for conformal spin structures of signature (n,n) is the space of isotropic rays in $\mathbb{R}^{n+1,n+1}$, $\widetilde{G}/\widetilde{P}\cong S^n\times S^n$. The subgroup $G\subseteq\widetilde{G}$ does not act transitively on that space. According to the decomposition (3.2), there are three orbits: the set of rays contained in E, the set of rays contained in F, and the set of isotropic rays that are neither contained in E nor in F. Note that only the last orbit is open in $\widetilde{G}/\widetilde{P}$, which is one of the requirements from Section 3.1. Therefore, we define $\widetilde{P}\subseteq\widetilde{G}$ to be the stabiliser of a ray through a light-like vector $\tilde{v}\in\mathbb{R}^{n+1,n+1}\setminus(E\cup F)$. Denoting by $Q=i^{-1}(\widetilde{P})$ the stabiliser of the ray $\mathbb{R}_+\tilde{v}$ in G, we have the identification of G/Q with the open orbit of the origin in $\widetilde{G}/\widetilde{P}$. The subgroup Q, which is not parabolic, is contained in the parabolic subgroup $P\subseteq G$ defined as the stabiliser in G of the ray through the projection of $\tilde{v}$ to E. In particular, G/P is the standard projective sphere S^n, the homogeneous model of oriented projective structures of dimension n, and $G/Q\to G/P$ is the canonical fibration with the standard fibre P/Q, whose total space is the model Fefferman space.

Let us denote by $L=\mathbb{R}\tilde{v}$ the line spanned by the light-like vector $\tilde{v}$ and let $L^\perp$ be the orthogonal complement in $\mathbb{R}^{n+1,n+1}$ with respect to h. The tangent space of G/Q at the origin can be seen in three different ways, namely,

$$\big(L^\perp/L\big)[1]\cong\mathfrak{g}/\mathfrak{q}\cong\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}.$$

The latter isomorphism is induced by the embedding $\mathfrak{g}\subseteq\tilde{\mathfrak{g}}$, the former one by the standard action of $\mathfrak{g}\subseteq\tilde{\mathfrak{g}}$ on the vector $\tilde{v}\in\mathbb{R}^{n+1,n+1}$. Both these identifications are Q-equivariant.

There are several natural Q-invariant objects that in turn yield distinguished geometric objects on the general Fefferman space. The n-dimensional Q-invariant subspace

$$f:=\big((\bar{F}+L)/L\big)[1]\subseteq\big(L^\perp/L\big)[1],\qquad\text{where}\quad\bar{F}:=F\cap L^\perp,$$

which is isomorphic to $\mathfrak{p}/\mathfrak{q}\subseteq\mathfrak{g}/\mathfrak{q}$, the kernel of the projection $\mathfrak{g}/\mathfrak{q}\to\mathfrak{g}/\mathfrak{p}$. Another n-dimensional Q-invariant subspace is

$$e:=\big((\bar{E}+L)/L\big)[1]\subseteq\big(L^\perp/L\big)[1],\qquad\text{where}\quad\bar{E}:=E\cap L^\perp.$$

The intersection $e\cap f$ is 1-dimensional with a distinguished Q-invariant generator that corresponds to the G-invariant involution $K\in\tilde{\mathfrak{g}}$,

$$k:=K+\tilde{\mathfrak{p}}\in\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}.$$

Note that all these objects are isotropic with respect to the natural conformal class induced by the restriction of h to $L^\perp\subseteq\mathbb{R}^{n+1.n+1}$. In particular, both e and f are maximally isotropic subspaces such that

$$k\in e\cap f\subseteq k^\perp=e+f.\tag{3.5}$$

In Section 3.1 we introduced a map $\varphi\colon(\mathfrak{g}/\mathfrak{p})^*\to(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$, the dual map to the projection $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}\cong\mathfrak{g}/\mathfrak{q}\to\mathfrak{g}/\mathfrak{p}$. The kernel of this projection is just f and the image of φ is identified with its annihilator, which will be denoted by f°. Since f is a maximally isotropic subspace in $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}\cong\mathfrak{g}/\mathfrak{q}$,

$$f^\circ\cong f[-2].$$

Since $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*\cong\tilde{\mathfrak{p}}_+$, we may conclude with the help of explicit matrix realisations from Appendix A that $f^\circ=\tilde{\mathfrak{p}}_+\cap\ker s_F$. Moreover, we note that

$$\big(\tilde{\mathfrak{p}}_+\cap\ker s_F\big)_{E\otimes F}=\mathfrak{p}_+,\qquad\big(\tilde{\mathfrak{p}}\cap\ker s_F\big)_{E\otimes F}=\mathfrak{p},\tag{3.6}$$

$$\Lambda^2F\cap\tilde{\mathfrak{p}}=\Lambda^2\bar{F}\subseteq\tilde{\mathfrak{g}}_0,\qquad\big[\tilde{\mathfrak{p}}_+,\Lambda^2\bar{F}\big]=f^\circ,\qquad\big[f^\circ,\Lambda^2\bar{F}\big]=0.\tag{3.7}$$

3.3 The Fefferman space and induced structure

The pairs of Lie groups (G, P) and $(\widetilde{G}, \widetilde{P})$ from the previous subsection satisfy all the properties to launch the Fefferman-type construction.

Proposition 3.1. *The Fefferman-type construction for the pairs of Lie groups (G, P) and $(\widetilde{G}, \widetilde{P})$ yields a natural construction of conformal spin structures $(\widetilde{M}, \mathbf{c})$ of signature (n, n) from n-dimensional oriented projective structures $(M, \mathbf{p})$. The Fefferman space $\widetilde{M}$ is identified with the total space of the weighted cotangent bundle without the zero section $T^*M(2)\backslash\{0\}$.*

Proof. The first part of the statement is obvious from the general setting for Fefferman-type constructions and the Cartan-geometric description of oriented projective and conformal spin structures.

The second part is shown due to two natural identifications: On the one hand, the Fefferman space is by (3.1) equal to the total space of the associated bundle $\widetilde{M} \cong \mathcal{G} \times_P P/Q$ over M. On the other hand, the weighted cotangent bundle to M is identified with the associated bundle $T^*M(2) \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})^*(2)$ with respect to action of P induced by the adjoint action and the representation (2.3) for $w = 2$. Hence it remains to verify that the action of P on $(\mathfrak{g}/\mathfrak{p})^*(2) \setminus \{0\}$ is transitive and Q is a stabiliser of a non-zero element. But this is a purely algebraic task, which may be easily checked in a concrete matrix realisation. ■

From the algebraic setup in Section 3.2 we easily conclude number of specific features of the induced conformal structure on $\widetilde{M}$:

Proposition 3.2. *The conformal spin structure $(\widetilde{M}, \mathbf{c})$ induced from an oriented projective structure $(M, \mathbf{p})$ by the Fefferman-type construction admits the following tractorial objects that are all parallel with respect to the induced tractor connection:*

(*a*) *pure tractor spinors $\mathbf{s}_E \in \Gamma(\widetilde{\mathcal{S}}_\pm)$ and $\mathbf{s}_F \in \Gamma(\widetilde{\mathcal{S}}_-)$ with non-trivial pairing,*

(*b*) *a tractor endomorphism $\mathbf{K} \in \Gamma(\mathcal{A}\widetilde{M})$ which is an involution, i.e., $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$, and which acts by the identity, respectively minus the identity on the maximally isotropic complementary subbundles $\widetilde{\mathcal{E}} := \ker \mathbf{s}_E$, respectively $\widetilde{\mathcal{F}} := \ker \mathbf{s}_F$ of $\widetilde{\mathcal{T}}$.*

The corresponding underlying objects $\eta = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_E)$, $\chi = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_F)$ and $k = \Pi_0^{\mathcal{A}\widetilde{M}}(\mathbf{K})$ satisfy:

(*c*) *$\eta \in \Gamma(\widetilde{\Sigma}_\pm[\frac{1}{2}])$ and $\chi \in \Gamma(\widetilde{\Sigma}_-[\frac{1}{2}])$ are pure spinors, whose kernels $\widetilde{e} := \ker \eta$ and $\widetilde{f} := \ker \chi$ have 1-dimensional intersection and $\widetilde{f}$ coincides with the vertical subbundle of $\widetilde{M} \to M$,*

(*d*) *$k \in \Gamma(T\widetilde{M})$ is a nowhere-vanishing light-like vector field generating the intersection $\widetilde{e} \cap \widetilde{f}$.*

Proof. The G-invariant spinor $s_E \in \Delta_\pm$ gives rise to the tractor spinor $\mathbf{s}_E \in \Gamma(\widetilde{\mathcal{S}}_\pm = \mathcal{G} \times_Q \Delta_\pm)$ such that it corresponds to the constant (Q-equivariant) map $\mathcal{G} \to \Delta_\pm$. Hence $\mathbf{s}_E$ is automatically parallel with respect to the induced tractor connection on $\widetilde{\mathcal{S}}_\pm$. Similar reasoning for other G-invariant objects and their compatibility described above yield the first part of the statement. In particular, $\widetilde{\mathcal{E}} = \mathcal{G} \times_Q E$, $\widetilde{\mathcal{F}} = \mathcal{G} \times_Q F$ and the decomposition $\widetilde{\mathcal{T}} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{F}}$ corresponds to the decomposition (3.2).

The filtration $L \subseteq L^\perp \subseteq \mathbb{R}^{n+1,n+1}$ gives rise to the filtration of the standard tractor bundle, which can be written as

$$\begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ 0 \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ \widetilde{\mathbb{E}}_a[1] \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} \widetilde{\mathbb{E}}[-1] \\ \widetilde{\mathbb{E}}_a[1] \\ \widetilde{\mathbb{E}}[1] \end{pmatrix} = \widetilde{\mathcal{T}}.$$

In particular, the subbundles associated to $\bar{E}, \bar{F} \subseteq L^\perp$ are distinguished by the middle slot. The corresponding Q-invariant maximally isotropic subspaces $e, f \subseteq \mathfrak{g}/\mathfrak{q}$ determine the distributions $\mathcal{G} \times_Q e$ and $\mathcal{G} \times_Q f$ in $T\widetilde{M} = \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$. According to the tractor Clifford action (2.7) it follows that these are precisely the kernels of the spinors η and χ. Since these subspaces are maximally isotropic, the corresponding spinors are pure. Since $f \cong \mathfrak{p}/\mathfrak{q}$ is the kernel of the projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$, the corresponding subbundle $\widetilde{f}$ is identified with the vertical subbundle of the projection $\widetilde{M} \to M$. The intersection $e \cap f$ is 1-dimensional and it is generated by the projection of $K \in \tilde{\mathfrak{g}}$ to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Indeed, K cannot be contained in $\tilde{\mathfrak{p}}$, since K acts by the identity on E and minus the identity on F and $\tilde{\mathfrak{p}}$ is the stabiliser of a line that is neither contained in E nor in F. Altogether, the corresponding vector field k on $\widetilde{M}$ is a nowhere-vanishing generator of $\widetilde{e} \cap \widetilde{f}$, in particular, it is light-like. ■

3.4 Relating tractors, Weyl structures and scales

As a technical preliminary for further study we now relate natural objects associated to the original projective Cartan geometry $(\mathcal{G}, \omega)$ on M and the induced conformal geometry $(\widetilde{\mathcal{G}}, \widetilde{\omega}^{\mathrm{ind}})$ on the Fefferman space $\widetilde{M}$.

Since $G \subseteq \widetilde{G}$, any $\widetilde{G}$-representation V is also a G-representation, which yields compatible tractor bundles over M and $\widetilde{M}$ with compatible tractor connections: $\mathcal{V} = \mathcal{G} \times_P V \to M$ with the tractor connection ∇ induced by ω and $\widetilde{\mathcal{V}} = \widetilde{\mathcal{G}} \times_{\widetilde{P}} V = \mathcal{G} \times_Q V \to \widetilde{M}$ with the tractor connection $\widetilde{\nabla}^{\mathrm{ind}}$ induced by $\widetilde{\omega}^{\mathrm{ind}}$. Sections of $\mathcal{V}$ bijectively correspond to P-equivariant functions $\varphi\colon \mathcal{G} \to V$, while sections of $\widetilde{\mathcal{V}}$ correspond to Q-equivariant functions $\varphi\colon \mathcal{G} \to V$. Since $Q \subseteq P$, every section of $\mathcal{V}$ gives rise to a section of $\widetilde{\mathcal{V}}$, and we can view $\Gamma(\mathcal{V}) \subseteq \Gamma(\widetilde{\mathcal{V}})$. Now, Proposition 3.2 in [8] admits a straightforward generalisation to Fefferman-type constructions for which P/Q is connected and thus, in particular, to the one studied in this article:

Proposition 3.3.

(*a*) *A section $s \in \Gamma(\widetilde{\mathcal{V}})$ is contained in $\Gamma(\mathcal{V})$ (i.e., the corresponding Q-equivariant function φ is indeed P-equivariant) if and only if $\widetilde{\nabla}^{\mathrm{ind}} s$ is strictly horizontal (i.e., $v^a \widetilde{\nabla}^{\mathrm{ind}}_a s = 0$ for all $v^a \in \Gamma(\widetilde{f})$).*

(*b*) *The restriction of $\widetilde{\nabla}^{\mathrm{ind}}$ to $\Gamma(\mathcal{V}) \subseteq \Gamma(\widetilde{\mathcal{V}})$ coincides with the tractor connection ∇.*

Remark 3.4. Another instance of compatible bundles over M and $\widetilde{M}$ is provided by the density bundles $\mathbb{E}(w)$ and $\widetilde{\mathbb{E}}[w]$, which are defined via the representation of P and $\widetilde{P}$ as in (2.3) and (2.4), respectively. Restricting these representations to Q, it easily follows that the notation is indeed compatible so that we can view $\Gamma(\mathbb{E}(w)) \subseteq \Gamma(\widetilde{\mathbb{E}}[w])$.

Both projective and conformal density bundles can be described as associated bundles to the respective bundles of scales. Hence everywhere positive sections of density bundles are considered as scales. In particular, the inclusion $\Gamma(\mathbb{E}_+(1)) \subseteq \Gamma(\widetilde{\mathbb{E}}_+[1])$ may be interpreted so that any projective scale induces a conformal one. Such conformal scales will be called *reduced scales*. An intrinsic characterisation of reduced scales among all conformal ones is formulated in Proposition 5.2.

The previous remark yields that any projective exact Weyl structure on M induces a conformal exact Weyl structure on $\widetilde{M}$. This fact can be generalised as follows:

Proposition 3.5. *Any projective (exact) Weyl structure on M induces a conformal (exact) Weyl structure on the Fefferman space $\widetilde{M}$.*

Proof. A version of this result in a more general context was proved in [1, Proposition 6.1]: any Weyl structure for ω induces a Weyl structure for $\widetilde{\omega}^{\mathrm{ind}}$ if $P_+ \subseteq \widetilde{P}$ and $(G_0 \cap \widetilde{P}) \subseteq \widetilde{G}_0$. But both these conditions are satisfied as follows from the setup in Section 3.2 and explicit realisations in Appendix A. ■

Conformal Weyl structures induced by projective ones as above will be called *reduced Weyl structures*.

3.5 Normality

Here we show that our Fefferman-type construction does not preserve the normality in general, see Proposition 3.8. This can be shown directly as we did in a previous version of the article, see arXiv:1510.03337v2. Alternatively, we can treat the construction as the composition of two other constructions via a natural intermediate Lagrangean contact structure.

A *Lagrangean contact structure* on M' consists of a contact distribution $\mathcal{H} \subseteq TM'$ together with a decomposition $\mathcal{H} = e' \oplus f'$ into two subbundles that are maximally isotropic with respect to the Levi form $\mathcal{H} \times \mathcal{H} \to TM'/\mathcal{H}$. Such structure on a manifold M' of dimension $2n-1$ is equivalently encoded as a normal parabolic geometry of type (G, P'), where $G = \mathrm{SL}(n+1)$ and $P' \subseteq G$ is the stabiliser of a flag of type line-hyperplane in the standard representation $\mathbb{R}^{n+1}$. For $n > 2$ there are three harmonic curvatures, two of which are torsions whose vanishing is equivalent to the integrability of the respective subbundles $e', f' \subseteq \mathcal{H}$. For $n = 2$ there are two harmonic curvatures of homogeneity 4, hence the Cartan connection is torsion-free. In that case both e' and f' are 1-dimensional and thus automatically integrable.

On the one hand, P' is contained in P, where $P \subseteq G$ is the stabiliser of a ray in $\mathbb{R}^{n+1}$. For suitable choices as in Appendix A, the Lie algebra to P' consists of matrices of the form

$$\mathfrak{p}' = \begin{pmatrix} a & U^t & w \\ 0 & B & V \\ 0 & 0 & c \end{pmatrix}.$$

Given a projective Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), it turns out that the correspondence space $M' := \mathcal{G}/P'$ can be identified with the projectivised cotangent bundle $\mathcal{P}(T^*M)$. The Cartan geometry $(\mathcal{G} \to M', \omega)$ of type (G, P') is regular and thus it covers a natural Lagrangean contact structure on M'. In particular, the canonical contact distribution on $\mathcal{P}(T^*M)$ coincides with $\mathcal{H}$ and the vertical subbundle of the projection $M' \to M$ coincides with one of the two distinguished subbundles, say $f' \subseteq \mathcal{H}$. As in general, this construction preserves normality. In accord with [5], respectively [11, Section 4.4.2] we may state:

Proposition 3.6. *Let $(\mathcal{G} \to M, \omega)$ be a normal projective parabolic geometry and let $(\mathcal{G} \to M', \omega)$ be the corresponding normal Lagrangean contact parabolic geometry. The latter geometry is torsion-free if and only if $n = 2$ or it is flat, i.e., the initial projective structure is flat.*

On the other hand, P' contains Q, where $Q = G \cap \widetilde{P}$ as before. This allows us to consider the Fefferman-type construction for the pairs (G, P') and $(\widetilde{G}, \widetilde{P})$. Given a Lagrangean contact structure on M', it induces a conformal spin structure on $\widetilde{M} = \mathcal{G}/Q$. This construction is indeed very similar to the original Fefferman construction; one deals with different real forms of the same complex Lie groups in the two cases. That is why the following statement and its proof is analogous to the one for the CR case. Following [8], respectively [11, Section 4.5.2] we may state:

Proposition 3.7. *Let $(\mathcal{G} \to M', \omega)$ be the normal Lagrangean contact parabolic geometry and let $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega}^{\mathrm{ind}})$ be the conformal parabolic geometry obtained by the Fefferman-type construction. Then $\widetilde{\omega}^{\mathrm{ind}}$ is normal if and only if ω is torsion-free.*

Altogether, composing the two previous steps we obtain our projective-to-conformal Fefferman-type construction with the desired control of the normality. Note that from (3.5) and the respective matrix realisations it follows that the induced objects on $\widetilde{M} = T^*M(2) \setminus \{0\}$ from Proposition 3.2 correspond to the induced objects on $M' = \mathcal{P}(T^*M)$. In particular, the vertical subbundle of the projection $\widetilde{M} \to M'$ is spanned by k and the decomposition $k^\perp = \widetilde{e} \oplus \widetilde{f} \subseteq T\widetilde{M}$ descends to the decomposition $\mathcal{H} = e' \oplus f' \subseteq TM'$

$$\begin{tikzcd}
& & \widetilde{\mathcal{G}} \arrow[dd, "\widetilde{P}"] \\
\mathcal{G} \arrow[urr, hook] \arrow[drr, "Q"] \arrow[ddr, "P'"] \arrow[ddd, "P"'] & & \\
& & \widetilde{M} \arrow[dl] \\
& M' \arrow[dl] & \\
M. & &
\end{tikzcd}$$

Proposition 3.8. *Let $(\mathcal{G} \to M, \omega)$ be a normal projective parabolic geometry and let $\big(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega}^{\mathrm{ind}}\big)$ be the conformal parabolic geometry obtained by the Fefferman-type construction.*

(a) *If* $\dim M = 2$ *then* $\widetilde{\omega}^{\mathrm{ind}}$ *is normal.*

(b) *If* $\dim M > 2$ *then* $\widetilde{\omega}^{\mathrm{ind}}$ *is normal if and only if* ω *is flat.*

Moreover, independently of the dimension of M, $\widetilde{\omega}^{\mathrm{ind}}$ is flat if and only if ω is flat.

3.6 Remarks on torsion-free Lagrangean contact structures

At this stage it is easy to formulate a local characterisation of split-signature conformal structures arising from torsion-free Lagrangean contact structures, see Proposition 3.10. As before, the results and their proofs are very analogous to those in the CR case, therefore we just quickly indicate the reasoning and point to differences.

As in Proposition 3.2, the G-invariant algebraic objects induce the tractor fields $\mathbf{s}_E$, $\mathbf{s}_F$ and $\mathbf{K}$ on the conformal Fefferman space that are parallel with respect to the induced tractor connection and have the required compatibility properties. But, starting with a torsion-free Lagrangean contact structure, the induced connection is already normal. In particular, the corresponding underlying objects χ, η and k are pure twistor spinors and a light-like conformal Killing field, respectively.

The existence of parallel tractors $\mathbf{s}_E$, $\mathbf{s}_F$ and $\mathbf{K}$ with the algebraic properties as in Proposition 3.2 are by no means independent conditions:

Proposition 3.9. *Let $\big(\widetilde{M}, \mathbf{c}\big)$ be a conformal spin structure of split-signature (n, n). Then the following conditions are locally equivalent:*

(a) *The spin tractor bundle admits two pure parallel tractor spinors $\mathbf{s}_E \in \Gamma\big(\widetilde{\mathcal{S}}_\pm\big)$ and $\mathbf{s}_F \in \Gamma\big(\widetilde{\mathcal{S}}_-\big)$ with non-trivial pairing.*

(b) *The conformal holonomy* $\mathrm{Hol}(\mathbf{c})$ *reduces to* $\mathrm{SL}(n+1) \subseteq \mathrm{Spin}(n+1, n+1)$ *preserving a decomposition into maximally isotropic subspaces $E \oplus F = \mathbb{R}^{n+1,n+1}$.*

(c) *The adjoint tractor bundle admits a parallel involution $\mathbf{K} \in \Gamma\big(\mathcal{A}\widetilde{M}\big)$, i.e., $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$.*

The only subtle point within the proof concerns the consequences of property (c). The existence of a parallel skew-symmetric involution $\mathbf{K}$ on the standard tractor bundle immediately implies that the conformal holonomy $\mathrm{Hol}(\mathbf{c})$ is reduced to $\mathrm{GL}(n+1)$. But, analogously to the corresponding discussion for the CR case in [9] or [23], one can show that $\mathrm{Hol}(\mathbf{c})$ is actually contained in $\mathrm{SL}(n+1)$. The rest follows easily.

It turns out that conformal spin structures induced by torsion-free Lagrangean contact structures are locally characterised by any of the three equivalent conditions above. Indeed, according to results from [10], the holonomy reduction of the conformal structure to $G = \mathrm{SL}(n+1) \subseteq \mathrm{Spin}(n+1, n+1) = \widetilde{G}$ yields the so-called curved orbit decomposition of $\widetilde{M}$, which corresponds to the decomposition of the homogeneous model $\widetilde{G}/\widetilde{P}$ with respect to the action of G. Each subset from the decomposition of $\widetilde{M}$, provided it is non-empty, further carries a geometry of the same type as its counterpart in the homogeneous model. From Section 3.2 we know there is one open and two closed n-dimensional orbits. The closed n-dimensional orbits carry Cartan geometries of type (G, P), and thus inherit projective structures, the open orbit carries a Cartan geometry of type (G, Q). Note that the two closed orbits coincide with the zero sets of χ and η, the open subset is the one where both spinors, and thus k, are non-vanishing. Since k is the conformal Killing field corresponding to the parallel adjoint tractor $\mathbf{K}$, it inserts trivially into the curvature of the normal Cartan connection, cf. (2.13). Hence, according to [5], the Cartan geometry of type (G, Q) on the open orbit of $\widetilde{M}$ descends to a Cartan geometry of type (G, P') on the local leaf space M' determined by k. It follows that this Cartan geometry is torsion-free and thus determines a torsion-free Lagrangean contact structure. Altogether, following [9] we may state the following characterisation:

Proposition 3.10. *A split-signature conformal spin structure is locally induced by a torsion-free Lagrangean contact structure via the Fefferman-type construction if and only if any of the equivalent conditions from Proposition 3.9 holds and the underlying twistor spinors χ and η and the conformal Killing field k are nowhere-vanishing.*

3.7 The exceptional case: dimension $n = 2$

From Section 3.5 we know that the intermediate 3-dimensional Lagrangean contact structure on M' induced by a 2-dimensional projective structure on M is torsion-free. Hence the induced conformal Cartan geometry on $\widetilde{M}$ is normal and thus all the equivalent conditions from Proposition 3.9 are satisfied. Moreover, the fact that it comes from a projective structure implies that any vertical vector of the projection $\widetilde{M} \to M$ inserts trivially into the Cartan curvature, i.e.,

$$i_X \widetilde{\kappa}(u) = 0, \qquad \text{for all } X \in f,\ u \in \widetilde{\mathcal{G}}. \tag{3.8}$$

Analogously to the discussion before Proposition 3.10 we may conclude:

Proposition 3.11. *A conformal spin structure of signature $(2, 2)$ is locally induced by a 2-dimensional projective structure via the Fefferman-type construction if and only if any of the equivalent conditions from Proposition 3.9 holds, the underlying twistor spinors χ and η and the conformal Killing field k are nowhere-vanishing and the curvature of the normal conformal Cartan connection satisfies (3.8).*

Remark 3.12. Conformal structures induced from 2-dimensional projective structures are well-studied, see, e.g., [14, 15, 25]. Notably, the intermediate 3-dimensional Lagrangean contact structure can be equivalently viewed as a path geometry (or the geometry associated to second order ODEs modulo point transformations). Such structure is induced by a projective structure (i.e., the paths are the unparametrised geodesics of the projective class of connections) if and only if one of the two harmonic curvatures vanishes. It follows from [25] that this is equivalent to vanishing of the self-dual, respectively anti-self-dual part of the Weyl curvature of the induced conformal structure. In particular, the condition (3.8) in the previous proposition can be replaced by the condition that the conformal structure is half-flat.

4 Normalisation and characterisation

By Proposition 3.8, for $n \geq 3$, the induced conformal Cartan connection associated to a non-flat n-dimensional projective structure differs from the normal conformal Cartan connection for the induced conformal structure. In this section we will analyse the form of the difference and thus derive properties of the induced conformal structures. Furthermore, we will show that any split-signature conformal manifold having these properties is locally equivalent to the conformal structure on the Fefferman space over a projective manifold.

4.1 The normalisation process

We are going to normalise the conformal Cartan connection $\widetilde{\omega}^{\mathrm{ind}} \in \Omega^1\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$ that is induced by a normal projective Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Any other conformal Cartan connection $\widetilde{\omega}'$ differs from $\widetilde{\omega}^{\mathrm{ind}}$ by some $\Psi \in \Omega^1\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$ so that $\widetilde{\omega}' = \widetilde{\omega}^{\mathrm{ind}} + \Psi$. This Ψ must vanish on vertical fields and be $\widetilde{P}$-equivariant. The condition on $\widetilde{\omega}'$ to induce the same conformal structure on $\widetilde{M}$ as $\widetilde{\omega}^{\mathrm{ind}}$ is that Ψ has values in $\tilde{\mathfrak{p}} \subseteq \tilde{\mathfrak{g}}$. One can therefore regard Ψ as a $\widetilde{P}$-equivariant function $\Psi\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$. According to the general theory as outlined in [11, Section 3.1.13] there is a unique such Ψ such that the curvature function $\widetilde{\kappa}'$ of $\widetilde{\omega}'$ satisfies $\widetilde{\partial}^*\widetilde{\kappa}' = 0$, and then $\widetilde{\omega}'$ is the normal conformal Cartan connection $\widetilde{\omega}^{\mathrm{nor}}$.

The failure of $\widetilde{\omega}^{\mathrm{ind}}$ to be normal is given by $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$. The normalisation of $\widetilde{\omega}^{\mathrm{ind}}$ proceeds by homogeneity of $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$, which decomposes into two homogeneous components according to the decomposition $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{p}}_+$. In the first step of normalisation one looks for a Ψ^1 such that $\widetilde{\omega}^1 = \widetilde{\omega} + \Psi^1$ has $\widetilde{\partial}^*\widetilde{\kappa}^1$ taking values in the highest homogeneity, i.e., $\widetilde{\partial}^*\widetilde{\kappa}^1\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}_+$.

To write down this first normalisation we employ Weyl structures $\widetilde{\mathcal{G}}_0 \hookrightarrow \widetilde{\mathcal{G}}$. By Proposition 3.5 we can take a reduced Weyl structure, i.e., one that is induced by a reduction $\mathcal{G}_0 \hookrightarrow \mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ with respect to the structure group $Q_0 := Q \cap G_0$. This allows us to project $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}$ to $\big(\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}\big)_0\colon \mathcal{G}_0 \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$ and to employ the $\widetilde{G}_0$-equivariant Kostant Laplacian $\widetilde{\Box}\colon (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0 \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$, $\widetilde{\Box} := \widetilde{\partial} \circ \widetilde{\partial}^* + \widetilde{\partial}^* \circ \widetilde{\partial}$. For the first normalisation step we need to form a map $\Psi^1\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$ that agrees with $-\widetilde{\Box}^{-1}\big(\widetilde{\partial^*\widetilde{\kappa}}^{\mathrm{ind}}\big)_0$ in the $\tilde{\mathfrak{g}}_0$-component. If we have formed any such Ψ^1 along $\mathcal{G}_0 \hookrightarrow \widetilde{\mathcal{G}}$ we can just equivariantly extend this to all of $\widetilde{\mathcal{G}}$.

To proceed with the analysis of the normalisation we need to establish a couple of technical lemmas. As before, we denote by $f^\circ \subset \tilde{\mathfrak{p}}_+ \cong (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$ the annihilator of $f = \mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Recall that $f^\circ = \varphi(\mathfrak{p}_+) \cong f[-2]$.

Lemma 4.1. *Let V be a $\mathfrak{g}$-representation contained in a $\tilde{\mathfrak{g}}$-representation $\widetilde{V}$ and denote by $\phi \mapsto \widetilde{\phi}$ the inclusion $\Lambda^k\mathfrak{p}_+ \otimes V \hookrightarrow \Lambda^k\tilde{\mathfrak{p}}_+ \otimes \widetilde{V}$ induced by $\varphi\colon \mathfrak{p}_+ \to \tilde{\mathfrak{p}}_+$ and $V \hookrightarrow \widetilde{V}$. Then, for any $\phi \in \Lambda^k\mathfrak{p}_+ \otimes V$,*

$$\widetilde{\partial^*\phi} - \widetilde{\partial}^*\widetilde{\phi} \in \Lambda^{k-1}f^\circ \otimes \big(\Lambda^2\bar{F} \bullet V\big) \subseteq \Lambda^{k-1}\tilde{\mathfrak{p}}_+ \otimes \widetilde{V}.$$

In particular, for the adjoint representations, $\partial^*\phi = 0$ if and only if $\widetilde{\partial}^*\widetilde{\phi} \in \Lambda^{k-1}f^\circ \otimes \Lambda^2\bar{F}$.

Proof. For the sake of presentation, assume that ϕ is decomposable, i.e., of the form $\phi = Z_1 \wedge \cdots \wedge Z_k \otimes v$, where $Z_i \in \mathfrak{p}_+$ and $v \in V$. Let us denote by the same symbols also the images of these elements under the inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ and $V \hookrightarrow \widetilde{V}$, i.e., $Z_i \in \tilde{\mathfrak{p}}$ and $v \in \widetilde{V}$, respectively. Let $\widetilde{Z}_i \in f^\circ$ be the images of Z_i under the inclusion $\varphi\colon \mathfrak{p}_+ \to \tilde{\mathfrak{p}}_+$. Now, by definition of the

Kostant co-differential, the difference $\widetilde{\partial^*\phi} - \widetilde{\partial}^*\widetilde{\phi}$ evaluated on any $k-1$ elements from $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ is a linear combination of terms of the form

$$\big(Z_i - \widetilde{Z}_i\big) \bullet v. \tag{4.1}$$

However, the differences $Z_i - \widetilde{Z}_i \in \tilde{\mathfrak{p}}$ are represented by the matrices as in (A.1) in the Appendix where only the Z-entries are non-vanishing and hence contained in $\Lambda^2 F \cap \tilde{\mathfrak{p}} = \Lambda^2 \bar{F}$. Thus (4.1) belong to the image of $\bullet\colon \Lambda^2\bar{F} \times V \to \widetilde{V}$ and the first claim follows.

For the second claim we use that $\Lambda^2\bar{F} \bullet \mathfrak{g} = \big[\Lambda^2\bar{F}, \mathfrak{g}\big] \subseteq \Lambda^2 F$ and $\Lambda^2 F \cap \mathfrak{g} = 0$: since $\widetilde{\partial^*\phi}$ (evaluated on any $k-1$ elements from $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$) has values in $\mathfrak{g} \subset \tilde{\mathfrak{g}}$, vanishing of $\partial^*\phi$ is equivalent to $\widetilde{\partial}^*\widetilde{\phi}$ having values in $\Lambda^2 F$. But $\widetilde{\partial}^*\widetilde{\phi}$ has generally values in $\tilde{\mathfrak{p}}$ and $\Lambda^2 F \cap \tilde{\mathfrak{p}} = \Lambda^2\bar{F}$, hence the claim follows. ■

Lemma 4.2. *If $\psi \in \tilde{\mathfrak{p}}_+ \wedge f^\circ \otimes \Lambda^2\bar{F} \subseteq \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{p}}$ then $\widetilde{\partial}^*\psi \in f^\circ \otimes f^\circ \subseteq \tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{p}}_+$.*

Proof. ψ is a sum of terms of the form $Z_1 \wedge Z_2 \otimes A$, where $Z_1 \in \tilde{\mathfrak{p}}_+$, $Z_2 \in f^\circ$ and $A \in \Lambda^2\bar{F}$. Applying the Kostant co-differential gives

$$\widetilde{\partial}^*(Z_1 \wedge Z_2 \otimes A) = Z_1 \otimes [Z_2, A] - Z_2 \otimes [Z_1, A].$$

Now $[Z_2, A]$ belongs to $\big[f^\circ, \Lambda^2\bar{F}\big] = 0$ and $[Z_1, A]$ belongs to $\big[\tilde{\mathfrak{p}}_+, \Lambda^2\bar{F}\big] = f^\circ$, hence the claim follows. ■

The following lemma contains the crucial information which is necessary to perform our normalisation. We are going to specify the curvature function $\widetilde{\kappa}^{\mathrm{ind}}$ (later also $\widetilde{\kappa}^{\mathrm{nor}}$) by describing its values along the natural Q-reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ over $\widetilde{M}$. Recall from Section 3.2 that $\Lambda^2\bar{F}$ is a Q-invariant subspace in $\tilde{\mathfrak{g}}_0$, which can be identified with $(\Lambda^2 f)[-2]$.

Lemma 4.3. *For any $u \in \mathcal{G}$, we have*

$$\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}(u) \in f^\circ \otimes \Lambda^2\bar{F} \subseteq \tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}_0.$$

Identifying $\Lambda^2\bar{F} \cong \big(\Lambda^2 f\big)[-2]$ and $f^\circ \cong f[-2]$, we have in fact $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}(u) \in \big(f \odot \Lambda^2 f\big)[-4]$, i.e., $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}(u)$ is contained in the kernel of the alternation map

$$\mathrm{alt}\colon\ \big(f \otimes \Lambda^2 f\big)[-4] \to \big(\Lambda^3 f\big)[-4].$$

Proof. It is a general assumption that $\widetilde{\omega}^{\mathrm{ind}}$ is induced by a normal projective Cartan connection on $\mathcal{G}$, i.e., $\partial^*\kappa(u) = 0$, for any $u \in \mathcal{G}$. Hence it follows from Lemma 4.1 that $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}(u)$ belongs to $f^\circ \otimes \Lambda^2\bar{F} \cong \big(f \otimes \Lambda^2 f\big)[-4]$.

Further we need a finer discussion involving the properties of $\kappa\colon \mathcal{G} \to \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ to show that $\kappa(u)$ belongs to the kernel of the Q-equivariant map $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g} \to \big(\Lambda^3 f\big)[-4]$ given by

$$\phi \mapsto \mathrm{alt}\,\big(\widetilde{\partial}^*\widetilde{\phi}\big). \tag{4.2}$$

Note that any element $\phi \in \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{p}_+$ for which $\partial^*\phi = 0$ is mapped to zero: since $\widetilde{\phi} \in \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{p}}$ and $[\tilde{\mathfrak{p}}_+, \tilde{\mathfrak{p}}] = \tilde{\mathfrak{p}}_+$, the co-differential $\widetilde{\partial}^*\widetilde{\phi}$ has values in $f^\circ \otimes \tilde{\mathfrak{p}}_+$. But, by Lemma 4.1, it also has values in $f^\circ \otimes \Lambda^2\bar{F}$ and $\tilde{\mathfrak{p}}_+ \cap \Lambda^2\bar{F} = 0$.

Thus it suffices to consider the harmonic elements from $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}_0$, i.e., the ones corresponding to the projective Weyl tensor. For that purpose we consider the simple part of $Q_0 = Q \cap G_0$ which is isomorphic to $\mathrm{SL}(n-1)$, cf. the matrix realisation (A.2) in the Appendix where it corresponds to the A-block. Considering both $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}_0 \cong \Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n$ and $\big(\Lambda^3 f\big)[-4] \cong \Lambda^3\mathbb{R}^{n*}$ as representations of $\mathrm{SL}(n-1)$, the map (4.2) is either trivial or an isomorphism on each $\mathrm{SL}(n-1)$-irreducible component.

One can check that there is only one $\mathrm{SL}(n-1)$-irreducible component that occurs in both spaces, and it is isomorphic to $\Lambda^2\mathbb{R}^{n-1*}$. Hence it suffices to compute (4.2) on one element contained in such component. Let $X_n \in \mathfrak{g}_-$ and $Z_n \in \mathfrak{p}_+$ be the two dual basis vectors stabilised by $\mathrm{SL}(n-1)$ and consider an element

$$\phi = Z_1 \wedge Z_2 \otimes X_n \otimes Z_n - Z_1 \wedge Z_2 \otimes X_1 \otimes Z_1 + Z_n \wedge Z_2 \otimes X_n \otimes Z_1.$$

Indeed ϕ is completely trace-free, satisfies the algebraic Bianchi identity and the $\mathrm{SL}(n-1)$-orbit of ϕ is isomorphic to $\Lambda^2\mathbb{R}^{n-1*}$. Now,

$$\widetilde{\partial}^*\widetilde{\phi} = -\widetilde{Z}_1 \otimes \widetilde{Z}_n \wedge \widetilde{Z}_2 - \widetilde{Z}_n \otimes \widetilde{Z}_1 \wedge \widetilde{Z}_2,$$

which indeed lies in the kernel of the alternation map. Hence the statement follows. ∎

We can now determine the form of the normal conformal Cartan connection:

Proposition 4.4. *The normal conformal Cartan connection is of the form*

$$\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \Psi^1 + \Psi^2,$$

where $\Psi^1 = -\frac{1}{2}\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}} \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{p}}\big)$ *and* $\Psi^2 \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{p}}_+\big)$. *Furthermore, along the reduction* $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ *we have* $\Psi^1 \in \Omega^1_{\mathrm{hor}}\big(\mathcal{G}, \Lambda^2\bar{F}\big)$, $\Psi^2 \in \Omega^1_{\mathrm{hor}}(\mathcal{G}, f^\circ)$.

Remark 4.5. Since Ψ^1 and Ψ^2 are horizontal, they may equivalently be regarded as bundle-valued 1-forms on $\widetilde{M}$. Denoting by $\Lambda^2\widetilde{\bar{\mathcal{F}}}$ the associated bundle $\mathcal{G} \times_Q \Lambda^2\bar{F}$ over $\widetilde{M}$ and by $\widetilde{f}^\circ \subseteq T^*\widetilde{M}$ the annihilator of $\widetilde{f} = \ker\chi \subseteq T\widetilde{M}$, Proposition 4.4 says

$$\Psi^1 \in \Omega^1\big(\widetilde{M}, \Lambda^2\widetilde{\bar{\mathcal{F}}}\big), \qquad \Psi^2 \in \Omega^1\big(\widetilde{M}, \widetilde{f}^\circ\big), \qquad \Psi^1(v) = \Psi^2(v) = 0, \qquad \text{for all } v \in \Gamma(\ker\chi).$$

Below we also use the corresponding frame forms, i.e., the $\widetilde{P}$-equivariant functions $\phi^1\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$ and $\phi^2\colon \widetilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}_+$ such that, for any $u \in \widetilde{\mathcal{G}}$, $\Psi^1 = \phi^1(u) \circ \widetilde{\omega}^{\mathrm{ind}}$ and $\Psi^2 = \phi^2(u) \circ \widetilde{\omega}^{\mathrm{ind}}$. In these terms, the proposition means that along the reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ these maps restrict to Q-equivariant functions

$$\phi^1\colon\ \mathcal{G} \to f^\circ \otimes \Lambda^2\bar{F}, \qquad \phi^2\colon\ \mathcal{G} \to f^\circ \otimes f^\circ.$$

Further we put $\Psi = \Psi^1 + \Psi^2$ and $\phi = \phi^1 + \phi^2$.

Proof. The Kostant Laplacian $\widetilde{\Box}$ restricts to an invertible endomorphism of $((\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0) \cap \operatorname{im}\widetilde{\partial}^*$ that acts by scalar multiplication on each of the $\widetilde{G}_0$-irreducible components. Now, restricting to $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ and suppressing all arguments $u \in \mathcal{G}$, it was shown in Lemma 4.3 that $\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}$ is contained in one of the irreducible components, namely in $\big(f \odot \Lambda^2 f\big)[-4]$. On this component $\widetilde{\Box}$ acts by multiplication by 2. Thus, the modification map accomplishing the first normalisation step is

$$\phi^1\colon\ \mathcal{G} \to f^\circ \otimes \Lambda^2\bar{F}, \qquad \phi^1 := -\frac{1}{2}\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}} = -\widetilde{\Box}^{-1}\widetilde{\partial}^*\widetilde{\kappa}^{\mathrm{ind}}.$$

Now, let $\widetilde{\omega}^1 := \widetilde{\omega}^{\mathrm{ind}} + \phi^1 \circ \widetilde{\omega}^{\mathrm{ind}}$ be the modified Cartan connection. The corresponding curvature function $\widetilde{\kappa}^1$ can be expressed in terms of $\widetilde{\kappa}^{\mathrm{ind}}$, ϕ^1 and its differential $d\phi^1$ so that

$$\begin{aligned}\widetilde{\kappa}^1(X,Y) = \widetilde{\kappa}^{\mathrm{ind}}(X,Y) &+ \big[X, \phi^1(Y)\big] - \big[Y, \phi^1(X)\big] \\ &+ d\phi^1(\xi)(Y) - d\phi^1(\eta)(X) - \phi^1([X,Y]) + \big[\phi^1(X), \phi^1(Y)\big],\end{aligned} \tag{4.3}$$

where $X, Y \in \mathfrak{g}$ and $\xi = \big(\widetilde{\omega}^{\mathrm{ind}}\big)^{-1}(X)$, $\eta = \big(\widetilde{\omega}^{\mathrm{ind}}\big)^{-1}(Y)$, cf. [11, formula (3.1)]. For the last term we have $\big[\phi^1(X), \phi^1(Y)\big] = 0$ since $\phi^1(X)$ has values in $\Lambda^2\bar{F}$. The first three terms are

$$\widetilde{\kappa}^{\mathrm{ind}}(X,Y) + \big[X, \phi^1(Y)\big] - \big[Y, \phi^1(X)\big] = \widetilde{\kappa}^{\mathrm{ind}}(X,Y) + \widetilde{\partial}\phi^1(X,Y),$$

which by construction vanishes upon application of the Kostant co-differential, i.e., $\widetilde{\partial}^*\big(\widetilde{\kappa}^{\mathrm{ind}} + \widetilde{\partial}\phi^1\big) = 0$. The remaining terms in (4.3) can be combined into a map $\Lambda^2\mathfrak{g} \to \Lambda^2\bar{F}$,

$$(X,Y) \mapsto d\phi^1(\xi)(Y) - d\phi^1(\eta)(X) - \phi^1([X,Y]),$$

which vanishes upon insertion of two elements $X, Y \in \mathfrak{p}$. Therefore, applying Lemma 4.2, we conclude that $\widetilde{\partial}^*\widetilde{\kappa}^1$ has values in $f^\circ \otimes f^\circ$. Thus the second modification map is

$$\phi^2\colon\ \mathcal{G} \to f^\circ \otimes f^\circ, \qquad \phi^2 := -\widetilde{\Box}^{-1}\widetilde{\partial}^*\widetilde{\kappa}^1. \qquad \blacksquare$$

4.2 Properties

The information provided in the previous proposition allows us to determine the properties satisfied by the normal conformal Cartan curvature:

Proposition 4.6. *The normal conformal Cartan curvature $\widetilde{\kappa}^{\mathrm{nor}}$ restricts to a map*

$$\widetilde{\kappa}^{\mathrm{nor}}\colon\ \mathcal{G} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \big(\mathfrak{sl}(n+1) \oplus \Lambda^2 F\big). \tag{4.4}$$

Moreover, the following integrability condition holds:

$$i_X\widetilde{\kappa}^{\mathrm{nor}}(u) \in f^\circ \otimes \big(\Lambda^2\bar{F} \oplus f^\circ\big), \qquad \text{for all } X \in f,\ u \in \mathcal{G}. \tag{4.5}$$

Proof. Let $\widetilde{\kappa}^{\mathrm{nor}}$ be the curvature function of the normal Cartan connection $\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \phi \circ \widetilde{\omega}^{\mathrm{ind}}$, where $\phi = \phi^1 + \phi^2$. With the same conventions as in the proof of Proposition 4.4, [11, formula (3.1)] yields

$$\begin{aligned}\widetilde{\kappa}^{\mathrm{nor}}(X,Y) = {} & \widetilde{\kappa}^{\mathrm{ind}}(X,Y) + [X,\phi(Y)] - [Y,\phi(X)] \\ & + d\phi(\xi)(Y) - d\phi(\eta)(X) - \phi([X,Y]) + [\phi(X),\phi(Y)].\end{aligned}$$

Clearly, $\widetilde{\kappa}^{\mathrm{ind}}(X,Y)$ has values in $\mathfrak{sl}(n+1)$ and vanishes upon insertion of $X \in \mathfrak{p}$. A term of the form $[X,\phi(Y)]$ vanishes if $Y \in \mathfrak{p}$ and has values in $\big[\mathfrak{p}, \Lambda^2\bar{F} \oplus f^\circ\big] \subseteq \Lambda^2\bar{F} \oplus f^\circ$ for $X \in \mathfrak{p}$. A term of the form $d\phi(\xi)(Y)$ has values in $\Lambda^2\bar{F} \oplus f^\circ$ and vanishes for $Y \in \mathfrak{p}$. The term $\phi([X,Y])$ has values in $\Lambda^2\bar{F} \oplus f^\circ$ and vanishes for $X, Y \in \mathfrak{p}$. The last term $[\phi(X),\phi(Y)]$ vanishes for all $X, Y \in \mathfrak{g}$ since $\phi(X)$ has values in $\Lambda^2\bar{F} \oplus f^\circ$. Altogether, we obtain (4.4) and (4.5). $\blacksquare$

We observe here that it follows directly from (4.4) that the pairing of $\widetilde{\kappa}^{\mathrm{nor}}$ with the involution K vanishes, $\langle\widetilde{\kappa}^{\mathrm{nor}}, K\rangle = 0$.

To derive properties of induced tractorial and underlying objects on the conformal structure we will need the following preparatory lemma.

Lemma 4.7. *Let V be a $\widetilde{G}$-representation and $v \in V$ an element which is stabilised under $G \subseteq \widetilde{G}$. Let $\mathbf{v} \in \Gamma\big(\widetilde{\mathcal{V}}\big)$ be the section of the associated tractor bundle $\widetilde{\mathcal{G}} \times_{\widetilde{P}} V$ corresponding to the constant function $\mathcal{G} \to V$, $u \mapsto v$, along $\mathcal{G}$. Then the covariant derivative $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{v}$ corresponds to the Q-equivariant function*

$$\mathcal{G} \to f^\circ \otimes V, \qquad u \mapsto \phi^1(u) \bullet v + \phi^2(u) \bullet v.$$

Proof. The covariant derivative $\widetilde{\nabla}^{\text{nor}}\mathbf{v}$ corresponds to the map

$$X \in \mathfrak{g} \mapsto \left(\widetilde{\omega}^{\text{nor}}\right)^{-1}(X) \cdot v + X \bullet v. \tag{4.6}$$

The first term in (4.6) vanishes since it is the directional derivative of the constant function v. Now $\widetilde{\omega}^{\text{nor}} = \widetilde{\omega}^{\text{ind}} + \phi^1 + \phi^2$, and since $X \bullet v = 0$ the claim follows. ∎

We now show that the distinguished tractors $\mathbf{s}_E$, $\mathbf{s}_F$ and $\mathbf{K}$ on the Fefferman space are all given as BGG-splittings from their underlying objects. Moreover, several stronger properties hold:

Proposition 4.8. *Let $\mathbf{s}_E \in \Gamma(\widetilde{\mathcal{S}}_\pm)$, $\mathbf{s}_F \in \Gamma(\widetilde{\mathcal{S}}_-)$ and $\mathbf{K} \in \Gamma(\mathcal{A}\widetilde{M})$ be the tractor spinors and the adjoint tractor, respectively, and let $\eta = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_E)$, $\chi = \Pi_0^{\widetilde{\mathcal{S}}}(\mathbf{s}_F)$ and $k = \Pi_0^{\mathcal{A}\widetilde{M}}(\mathbf{K})$ be the corresponding underlying objects as in Proposition 3.2.*

(a) *The tractor spinor $\mathbf{s}_F$ is parallel, i.e., $\widetilde{\nabla}^{\text{nor}}\mathbf{s}_F = 0$. In particular, χ is a pure twistor spinor, $\mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}_-}(\chi)$ and* $\operatorname{Hol}(\mathbf{c}) \subseteq \operatorname{SL}(n+1) \ltimes \Lambda^2\left(\mathbb{R}^{n+1}\right)^* \subseteq \operatorname{Spin}(n+1, n+1)$.

(b) *The tractor spinor $\mathbf{s}_E$ is a BGG-splitting, i.e., $\widetilde{\partial}^*\left(\widetilde{\nabla}^{\text{nor}}\mathbf{s}_E\right) = 0$ and $\mathbf{s}_E = L_0^{\widetilde{\mathcal{S}}_\pm}(\eta)$.*

(c) *The adjoint tractor $\mathbf{K}$ is a BGG-splitting and k is a conformal Killing field, i.e., $\widetilde{\partial}^*\widetilde{\nabla}^{\text{nor}}\mathbf{K} = 0$, $\mathbf{K} = L_0^{\mathcal{A}\widetilde{M}}(k)$ and $\widetilde{\nabla}^{\text{nor}}\mathbf{K} = i_k\widetilde{\Omega}^{\text{nor}}$. Moreover, we have*

$$i_k\widetilde{\kappa}^{\text{nor}} = 2\phi_{\Lambda^2 F}. \tag{4.7}$$

Proof. (a) Since ϕ^1, ϕ^2 have values in $\ker s_F$ we have $\phi^1 \bullet s_F + \phi^2 \bullet s_F = 0$. Thus, according to Lemma 4.7, we have $\widetilde{\nabla}^{\text{nor}}\mathbf{s}_F = 0$ and the rest is obvious.

(b) The spinor s_E is of the form $s_E = \left(\begin{smallmatrix} * \\ \eta \end{smallmatrix}\right)$. According to Lemmas 4.3 and 4.7, ϕ^1 has values in $\left(f \odot \Lambda^2 f\right)[-4]$ and $\widetilde{\partial}^*\left(\widetilde{\nabla}^{\text{nor}}\mathbf{s}_E\right)$ corresponds to

$$\widetilde{\partial}^*\left(\phi^1 \bullet s_E\right) = \begin{pmatrix} \left(\phi^1 \bullet \eta\right)_{\widetilde{\Sigma}_\mp[-\frac{1}{2}]} \\ 0 \end{pmatrix}.$$

The projection $\left(\phi^1 \bullet \eta\right)_{\widetilde{\Sigma}_\mp[-\frac{1}{2}]}$ can be realised as the full (triple) Clifford action on $\phi^1(u) \in \left(\bigotimes^3 f\right)[-4]$, where $u \in \mathcal{G}$. Now it is easy to see that this action must vanish for a $\phi^1(u) \in \left(f \odot \Lambda^2 f\right)[-4]$: We realise $\phi^1(u)$ equivalently in $\left(S^2 f \otimes f\right)[-4]$ by symmetrisation in the first two slots, then the complete Clifford action on η vanishes because the action of the first two slots is just a (trivial) trace multiplication.

(c) According to Lemma 4.7, $\widetilde{\nabla}^{\text{nor}}\mathbf{K}$ corresponds to $\phi^1 \bullet K + \phi^2 \bullet K$. Since $K/\tilde{\mathfrak{p}} = k \in f$, the previous element lies in $\tilde{\mathfrak{p}}$. In particular, $\widetilde{\nabla}^{\text{nor}}\mathbf{K}$ has trivial projecting slot, and thus $k = \Pi_0(\mathbf{K})$ is a conformal Killing field. Since $\phi^2 \bullet K \in \tilde{\mathfrak{p}}_+$, we have that $\widetilde{\partial}^*\left(\widetilde{\nabla}^{\text{nor}}\mathbf{K}\right)$ corresponds to $\widetilde{\partial}^*\left(\phi^1 \bullet K\right)$. Now $\phi^1 \bullet K = -K \bullet \phi^1 = 2\phi^1$, since K acts by multiplication with -2 on $\Lambda^2\bar{F}$. But $\phi^1 \in \operatorname{im}\widetilde{\partial}^* \subseteq \ker\widetilde{\partial}^*$, and the expression $\widetilde{\partial}^*\left(\phi^1 \bullet K\right)$ therefore vanishes. The equality $\widetilde{\nabla}^{\text{nor}}\mathbf{K} = i_k\widetilde{\Omega}^{\text{nor}}$ is just (2.13) for the conformal Killing field k with its BGG-splitting $\mathbf{K}$. In terms of the Q-equivariant functions $\phi = \phi^1 + \phi^2$ and $\widetilde{\kappa}^{\text{nor}}$ along $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$, this can be expressed as $\phi \bullet K = i_k\widetilde{\kappa}^{\text{nor}}$, which yields (4.7). ∎

We now collect the essential information about the induced conformal structure $(\widetilde{M}, \mathbf{c})$ which we derived:

Proposition 4.9. *Let $\left(\widetilde{M}, \mathbf{c}\right)$ be the conformal spin structure induced from an oriented projective structure $(M, \mathbf{p})$ via the Fefferman-type construction. Then the following properties are satisfied:*

(*a*) $(\widetilde{M}, \mathbf{c})$ *admits a nowhere-vanishing light-like conformal Killing field* k *such that the corresponding tractor endomorphism* $\mathbf{K} = L_0^{\mathcal{A}\widetilde{M}}(k)$ *is an involution, i.e.,* $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$.

(*b*) $(\widetilde{M}, \mathbf{c})$ *admits a pure twistor spinor* $\chi \in \Gamma\big(\widetilde{\Sigma}_-\big[\frac{1}{2}\big]\big)$ *with* $k \in \Gamma(\ker \chi)$ *such that the corresponding parallel tractor spinor* $\mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}_-}(\chi)$ *is pure.*

(*c*) $\mathbf{K}$ *acts by minus the identity on* $\ker \mathbf{s}_F$.

(*d*) *The following integrability condition holds:*

$$v^a w^c \widetilde{W}_{abcd} = 0, \qquad \textit{for all } v, w \in \Gamma(\ker \chi). \tag{W}$$

The only thing left to show for Proposition 4.9 is that the integrability condition (4.5) is equivalent to the condition (W) on the Weyl tensor:

Lemma 4.10. *Let* $(\widetilde{M}, \mathbf{c})$ *be a split-signature conformal spin structure endowed with tractors* $\mathbf{s}_E$, $\mathbf{s}_F$ *and* $\mathbf{K}$ *satisfying conditions* (*a*) *and* (*b*) *from Proposition 4.9. Then condition* (4.5) *is equivalent to* (W).

Proof. The implication (4.5) $\Longrightarrow$ (W) is obvious. It remains to prove the converse implication (W) $\Longrightarrow$ (4.5).

By (W), one has that $\big(i_X\widetilde{\kappa}^{\mathrm{nor}}\big)_{\tilde{\mathfrak{g}}_0}(u) \in \big(f \otimes \Lambda^2 f\big)[-4] \subseteq f^\circ \otimes \Lambda^2 \bar{F}$ for $X \in f$, $u \in \mathcal{G}$. Since $\mathbf{s}_F$ is parallel with respect to $\widetilde{\nabla}^{\mathrm{nor}}$, we have $\widetilde{\kappa}^{\mathrm{nor}}(u) \in \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \big(\tilde{\mathfrak{p}} \cap \ker s_F\big)$. The projection of $\tilde{\mathfrak{p}} \cap \ker s_F$ to $\tilde{\mathfrak{p}}_+$ is precisely f°, hence it follows that $\big(i_X\widetilde{\kappa}^{\mathrm{nor}}\big)_{\tilde{\mathfrak{p}}_+}(u) \in (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes f^\circ$, and we obtain $i_X\widetilde{\kappa}^{\mathrm{nor}}(u) \in (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \big(\Lambda^2 \bar{F} \oplus f^\circ\big)$.

We now prove that $i_{X_1} i_{X_2} \widetilde{\kappa}^{\mathrm{nor}} = 0$ for all $X_1, X_2 \in f$. For this purpose it will be useful to work with the curvature form $\widetilde{\Omega}^{\mathrm{nor}}$, which we can represent as in (2.11). By (W) and the algebraic Bianchi identity, $\widetilde{W}_{abcd}$ vanishes upon insertion of $v, w \in \Gamma(\ker \chi)$ into any two slots, and in particular $v^a w^b \widetilde{W}_{abcd} = 0$. Thus, it remains to check that $v^a w^b \widetilde{Y}_{dab} = 0$. As in the proof of Proposition 3.2, a vector field $w \in \Gamma(\ker \chi)$ corresponds to a section $\left(\begin{smallmatrix} * \\ w^d \\ 0 \end{smallmatrix}\right) \in \Gamma\big(\widetilde{\bar{\mathcal{F}}}\big)$. According to (2.9),

$$v^a \widetilde{\Omega}_{ab}^{\mathrm{nor}} \bullet \begin{pmatrix} * \\ w^d \\ 0 \end{pmatrix} = \begin{pmatrix} -v^a \widetilde{Y}_{rab} w^r \\ v^a \widetilde{W}_{ab}{}^c{}_d w^d \\ 0 \end{pmatrix} \in \Gamma\big(\widetilde{\mathbb{E}}_b \otimes \widetilde{\mathcal{T}}\big).$$

Since $i_v \widetilde{\Omega}^{\mathrm{nor}}$ annihilates $\widetilde{\bar{\mathcal{F}}}$, it follows that $v^a w^r \widetilde{Y}_{rab} = 0$. Using $\widetilde{Y}_{rab} = -\widetilde{Y}_{bra} - \widetilde{Y}_{abr}$, we obtain also $v^a w^b \widetilde{Y}_{rab} = 0$. ■

4.3 Characterisation

We are now going to characterise the induced conformal structures. For this purpose we will introduce the following ('intermediate') Cartan connection form:

$$\widetilde{\omega}' := \widetilde{\omega}^{\mathrm{nor}} - \frac{1}{2} i_k \widetilde{\kappa}^{\mathrm{nor}}. \tag{4.8}$$

The following observation then follows immediately from Proposition 4.4 and formula (4.7):

Lemma 4.11. *The pullbacks of the Cartan connection forms* $\widetilde{\omega}' \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$, $\widetilde{\omega}^{\mathrm{nor}} \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$ *and* $\widetilde{\omega}^{\mathrm{ind}} \in \Omega^1_{\mathrm{hor}}\big(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}}\big)$ *to* $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$ *agree modulo forms with values in* $\mathfrak{p}_+ \subseteq \mathfrak{sl}(n+1) \subseteq \tilde{\mathfrak{g}}$.

For the rest of this section, we will start with a given split-signature conformal spin structure $\big(\widetilde{M}, \mathbf{c}\big)$ satisfying all the properties of Proposition 4.9. In particular, $\widetilde{M}$ is endowed with a conformal Killing field $k \in \Gamma\big(T\widetilde{M}\big)$, and we can still use formula (4.8) to define a Cartan connection $\widetilde{\omega}'$. The corresponding tractor connection will be denoted by $\widetilde{\nabla}'$ and the curvature by $\widetilde{\Omega}'$ or $\widetilde{\kappa}'$. The following proposition now shows that the so constructed Cartan connection $\widetilde{\omega}'$ is in fact an $\mathrm{SL}(n+1)$-connection.

Proposition 4.12. *Let $\big(\widetilde{M}, \mathbf{c}\big)$ be a split-signature conformal (spin) structure satisfying all the properties of Proposition 4.9. Then the sections $\mathbf{s}_F$ and $\mathbf{K}$ are parallel with respect to the tractor connection $\widetilde{\nabla}'$, i.e., $\widetilde{\nabla}'\mathbf{s}_F = 0$ and $\widetilde{\nabla}'\mathbf{K} = 0$.*

In particular, $\mathrm{Hol}(\widetilde{\omega}') \subseteq \mathrm{SL}(n+1) \subseteq \mathrm{Spin}(n+1, n+1)$ and $\widetilde{\omega}'$ pulls back to a Cartan connection of type $(\mathrm{SL}(n+1), Q)$ with respect to the Q-reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$. Along that reduction, the curvature functions $\widetilde{\kappa}'$ and $\widetilde{\kappa}^{\mathrm{nor}}$ are related according to $\kappa' = \big(\widetilde{\kappa}^{\mathrm{nor}}\big)_{\mathfrak{sl}(n+1)}$ and $\widetilde{\kappa}'$ satisfies the following integrability condition:

$$i_X\widetilde{\kappa}'(u) \in f^\circ \otimes \mathfrak{p}_+, \qquad \text{for all } X \in f,\ u \in \mathcal{G}. \tag{4.9}$$

Proof. A tractor connection induced by $\widetilde{\omega}'$ can be written as $\widetilde{\nabla}' = \widetilde{\nabla}^{\mathrm{nor}} + \Psi$ with $\Psi = -\frac{1}{2} i_k \widetilde{\Omega}^1$. That $\widetilde{\nabla}'\mathbf{s}_F = 0$ follows immediately from the fact that $\widetilde{\nabla}' - \widetilde{\nabla}^{\mathrm{nor}} = -\frac{1}{2} i_k \widetilde{\Omega}^{\mathrm{nor}}$ has values in $\Lambda^2\widetilde{\mathcal{F}}$. Since k is a conformal Killing field we have $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} = i_k\widetilde{\Omega}^{\mathrm{nor}}$. By definition

$$\widetilde{\nabla}'\mathbf{K} = \widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} - \frac{1}{2} i_k \widetilde{\Omega}^{\mathrm{nor}} \bullet \mathbf{K},$$

which vanishes, since $i_k\widetilde{\Omega}^{\mathrm{nor}}$ has values in $\Lambda^2\widetilde{\mathcal{F}}$ and therefore $\frac{1}{2} i_k\widetilde{\Omega}^{\mathrm{nor}} \bullet \mathbf{K} = i_k\widetilde{\Omega}^{\mathrm{nor}}$. As in Proposition 4.9, we write the decomposition of $\widetilde{\mathcal{T}}$ into maximally isotropic eigenspaces of $\mathbf{K}$ with eigenvalues ± 1 as $\widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{F}}$. Since $\mathbf{K}$ is $\widetilde{\nabla}'$-parallel, it follows that this decomposition is preserved by $\widetilde{\nabla}'$. Moreover, since $\widetilde{\mathcal{F}}$ is the kernel of the pure tractor spinor $\mathbf{s}_F$ it follows that $\mathrm{Hol}(\widetilde{\omega}') \subseteq \mathrm{SL}(n+1)$. In particular, $\widetilde{\omega}'$ reduces to a Cartan connection of type $(\mathrm{SL}(n+1), Q)$ on a Q-principal bundle $\mathcal{G} \subseteq \widetilde{\mathcal{G}}$.

We further compute that

$$\begin{aligned}
\widetilde{\Omega}' &= \widetilde{\Omega}^{\mathrm{nor}} - \frac{1}{2} d^{\widetilde{\nabla}^{\mathrm{nor}}} i_k \widetilde{\Omega}^{\mathrm{nor}} = \widetilde{\Omega}^{\mathrm{nor}} - \frac{1}{2} d^{\widetilde{\nabla}^{\mathrm{nor}}} \widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} \\
&= \widetilde{\Omega}^{\mathrm{nor}} - \frac{1}{2}\widetilde{\Omega}^{\mathrm{nor}} \bullet \mathbf{K} = \widetilde{\Omega}^{\mathrm{nor}} + \frac{1}{2}\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}} = \big(\widetilde{\Omega}^{\mathrm{nor}}\big)_{(\widetilde{\mathcal{E}} \otimes \widetilde{\mathcal{F}})_0},
\end{aligned}$$

where we are again using $\widetilde{\nabla}^{\mathrm{nor}}\mathbf{K} = i_k\widetilde{\Omega}^{\mathrm{nor}}$ for the conformal Killing field k and that $\widetilde{\Omega}^{\mathrm{nor}}$ has values in $\widetilde{\mathcal{E}} \otimes \widetilde{\mathcal{F}} \oplus \Lambda^2\widetilde{\mathcal{F}}$. Stated for the corresponding curvature functions, this yields $\kappa' = \big(\widetilde{\kappa}^{\mathrm{nor}}\big)_{\mathfrak{sl}(n+1)}$. Moreover, since $\widetilde{\kappa}^{\mathrm{nor}}$ has values in $\ker s_F \cap \widetilde{\mathfrak{p}}$, it follows from (3.6) that $(\widetilde{\mathfrak{p}} \cap \ker s_F)_{\mathfrak{sl}(n+1)} = \mathfrak{p}$, thus $\widetilde{\kappa}'$ has values in $\mathfrak{p}$.

We know from (4.5) that $i_X\widetilde{\kappa}^{\mathrm{nor}}$ has values in $\Lambda^2\bar{F} \oplus f^\circ$ for $X \in f$. But since $\big(\Lambda^2\bar{F}\big)_{\mathfrak{sl}(n+1)} = 0$ and $(\tilde{\mathfrak{p}}_+ \cap \ker s_F)_{\mathfrak{sl}(n+1)} = \mathfrak{p}_+$, we obtain that $(i_X\widetilde{\kappa}^{\mathrm{nor}})_{\mathfrak{sl}(n+1)}$ has values in $\mathfrak{p}_+$. Finally, $\big(i_{X_1} i_{X_2}\widetilde{\kappa}^{\mathrm{nor}}\big)_{\mathfrak{sl}(n+1)} = 0$ for $X_1, X_2 \in f$ follows immediately from (4.5), and altogether we obtain (4.9). ∎

Next, before proving the main characterisation Theorem 4.14, we will show the following proposition on factorisations of particular Cartan geometries. This proposition can be understood as an adapted variant of [5, Theorem 2.7].

Proposition 4.13. *Let $\big(\mathcal{G} \to \widetilde{M}, \omega\big)$ be a Cartan geometry of type $(\mathrm{SL}(n+1), Q)$ with curvature $\kappa\colon \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{q})^* \otimes \mathfrak{g}$ and let the following conditions be satisfied:*

$$i_{X_1} i_{X_2}\kappa(u) \in \mathfrak{p}, \qquad \text{for all } X_1, X_2 \in \mathfrak{g}/\mathfrak{q},\ u \in \mathcal{G},$$

$$i_{X_1} i_{X_2} \kappa(u) \in \mathfrak{p}_+, \qquad \textit{for all } X_1 \in \mathfrak{p}/\mathfrak{q},\ X_2 \in \mathfrak{g}/\mathfrak{q},\ u \in \mathcal{G},$$
$$i_{X_1} i_{X_2} \kappa(u) = 0, \qquad \textit{for all } X_1, X_2 \in \mathfrak{p}/\mathfrak{q},\ u \in \mathcal{G}.$$

Then $\mathcal{G}$ is locally a P-bundle over $M = \mathcal{G}/P$ and ω defines a canonical projective structure on M.

Proof. The third of the above listed conditions implies that $\mathcal{G}$ is locally a P-bundle $\mathcal{G} \to M$ by [5]. We will restrict $\mathcal{G}$ to assume this globally. We define $M = \mathcal{G}/P$ and $\mathcal{G}_0 = \mathcal{G}/P_+$.

Let $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$ be a G_0-equivariant splitting. It follows from the second of the above listed conditions that

$$\mathcal{L}_{\zeta_X} \omega = -\mathrm{ad}(X) \circ \omega \mod \mathfrak{p}_+, \qquad \text{for all } X \in \mathfrak{p}.$$

Now define $\theta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_-)$, $\gamma \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_0)$ and $\rho \in \Omega^1(\mathcal{G}_0, \mathfrak{p}_+)$ via the decomposition $\sigma^*\omega = \theta \oplus \gamma \oplus \rho$. Since σ is G_0-equivariant and the Lie derivative is compatible with pullbacks it follows that

$$\mathcal{L}_{\bar{\zeta}_X} (\theta \oplus \gamma) = -\mathrm{ad}(X) \circ (\theta \oplus \gamma), \qquad \text{for all } X \in \mathfrak{g}_0.$$

In particular, θ and γ are G_0-equivariant and define a (reductive) Cartan geometry $(\mathcal{G}_0 \to M, \theta \oplus \gamma)$ of type $(\mathbb{R}^n \rtimes \mathrm{SL}(n), \mathrm{SL}(n))$, i.e., an affine connection on M. Since by assumption Ω has values in $\mathfrak{p}$, $\theta \oplus \gamma$ is torsion-free and so is the affine connection.

Now take another splitting $\sigma' = \sigma \cdot \exp \Upsilon$, for some $\Upsilon\colon \mathcal{G} \to \mathfrak{p}_+$. Since $\mathrm{Ad}(\exp \Upsilon)$ acts by the identity on $\mathfrak{g}_- = \mathfrak{g}/\mathfrak{p}$, one has $(r^{\exp \Upsilon})^*\omega = \omega \mod \mathfrak{p}$, and thus θ is independent of the choice of splitting. Then $\sigma'^*\omega = \theta \oplus \gamma' \oplus \rho'$ and $\theta \oplus \gamma' = \mathrm{Ad}(\exp \Upsilon) \circ (\theta \oplus \gamma)$, projected to $\mathfrak{g}_- \oplus \mathfrak{g}_0$. But since $\exp \Upsilon \in P_+$, this shows that γ' is projectively equivalent to γ. We thus obtain a well-defined projective structure on M.

Since ω is P-torsion-free and P-equivariant modulo $\mathfrak{p}_+$, it can be (uniquely) modified to a normal Cartan connection $\omega^{\mathrm{nor}} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ with $\omega^{\mathrm{nor}} - \omega \in \Omega^1(\mathcal{G}, \mathfrak{p}_+)$. In particular, each splitting $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$ is in fact a Weyl structure of the projective structure on M. ∎

Theorem 4.14. *A split-signature (n, n) conformal spin structure $\mathbf{c}$ on a manifold $\widetilde{M}$ is (locally) induced by an n-dimensional projective structure via the Fefferman-type construction if and only if the following properties are satisfied:*

- *(a) $\big(\widetilde{M}, \mathbf{c}\big)$ admits a nowhere-vanishing light-like conformal Killing field k such that the corresponding tractor endomorphism $\mathbf{K} = L_0^{\mathcal{A}\widetilde{M}}(k)$ is an involution, i.e., $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$.*
- *(b) $\big(\widetilde{M}, \mathbf{c}\big)$ admits a pure twistor spinor $\chi \in \Gamma\big(\widetilde{\Sigma}_-\big[\tfrac{1}{2}\big]\big)$ with $k \in \Gamma(\ker \chi)$ such that the corresponding parallel tractor spinor $\mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}_-}(\chi)$ is pure.*
- *(c) $\mathbf{K}$ acts by minus the identity on $\ker \mathbf{s}_F$.*
- *(d) The following integrability condition holds:*

$$v^a w^c \widetilde{W}_{abcd} = 0, \qquad \textit{for all } v, w \in \Gamma(\ker \chi).$$

Proof. Starting with a projective structure $(M, \mathbf{p})$, it follows from Proposition 4.9 that the induced conformal structure $\big(\widetilde{M}, \mathbf{c}\big)$ has all the stated properties. On the other hand, let $\big(\widetilde{M}, \mathbf{c}\big)$ be a conformal structure with the stated properties. Then, by Proposition 4.12, $\widetilde{\omega}'$ restricts to a Q-equivariant Cartan connection form with values in $\mathfrak{sl}(n+1)$ on the reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$. The corresponding curvature $\widetilde{\kappa}'$ takes values in $\mathfrak{p}$ and for $X \in f$ we have that $i_X \widetilde{\kappa}'$ takes values in $\mathfrak{p}_+$. It follows from Proposition 4.13 that $\widetilde{\omega}'$ factorises to a projective structure $\mathbf{p}$ on the leaf space M.

Let us now show that the two constructions are inverse to each other. Assume first that a conformal structure $(\widetilde{M}, \mathbf{c})$ is induced by a projective structure $(M, \mathbf{p})$. Then according to Lemma 4.11 $\widetilde{\omega}'$ and $\widetilde{\omega}^{\mathrm{ind}}$ agree modulo $\mathfrak{p}_+$. This implies that the projective structure defined by $\widetilde{\omega}'$ is equal to the original projective structure. Conversely, assume now that $(M, \mathbf{p})$ is a projective structure with associated Cartan geometry $(\mathcal{G}, \omega')$ that is induced from a conformal structure $(\widetilde{M}, \mathbf{c})$ with associated Cartan geometry $(\widetilde{\mathcal{G}}, \widetilde{\omega}^{\mathrm{nor}})$. Since $\widetilde{\omega}'$ is not normal, but torsion-free, there is $\varphi \in \Omega^1_{\mathrm{hor}}(\mathcal{G}, \mathfrak{p}_+)$ such that $\omega' + \varphi$ is the normal projective Cartan connection. Since $\mathfrak{p}_+ \subseteq \widetilde{\mathfrak{p}}$ the induced conformal structure on $\widetilde{M}$ agrees with the original conformal structure. Thus, the Fefferman-type construction (with normalisation) and the described factorisation are (locally) inverse to each other. ■

For a reformulation of the characterisation theorem in terms of underlying objects, see Section 6.2.

5 Reduced scales and explicit normalisation

Although we obtained the desired characterisation in Theorem 4.14, we do not yet know the explicit relationship between the induced Cartan connection form $\widetilde{\omega}^{\mathrm{ind}}$ and the normal conformal Cartan connection form $\widetilde{\omega}^{\mathrm{nor}}$. One of the aims of the present section is to obtain a formula for this difference, which is achieved in Theorem 5.7. As a consequence, we also obtain an explicit formula for the curvature $\widetilde{\Omega}^{\mathrm{ind}}$ in terms of the normal conformal Cartan curvature $\widetilde{\Omega}^{\mathrm{nor}}$ in Corollary 5.8. In this more refined analysis, reduced scales will play an important role.

5.1 Characterisation of reduced scales

The notion of reduced Weyl structures and reduced scales is introduced in Section 3.4. Here we shall find an intrinsic characterisation (i.e., using the conformal structure only) of reduced scales and discuss their properties.

As the scale bundle on the projective manifold M we may consider the positive elements in the density bundle $\mathbb{E}(1)$, which is the projecting part of the dual standard tractor bundle $\mathcal{T}^*$, see Section 2.4. Similarly, on the Fefferman space $\widetilde{M}$ we take the positive elements in the density bundle $\widetilde{\mathbb{E}}(1)$, the projecting part of the conformal standard tractor bundle $\widetilde{\mathcal{T}}$. Hence for a projective scale $\rho \in \Gamma(\mathbb{E}_+(1))$ we have the tractor $L_0^{\mathcal{T}^*}(\rho) \in \Gamma(\mathcal{T}^*)$; similarly, for a conformal scale $\sigma \in \Gamma\big(\widetilde{\mathbb{E}}_+(1)\big)$ we have the tractor $L_0^{\widetilde{\mathcal{T}}}(\sigma) \in \Gamma\big(\widetilde{\mathcal{T}}\big)$. These will be termed *scale tractors*.

On the one hand, reduced scales correspond to the sections of $\mathbb{E}_+(1) \to M$ seen as a subset of all sections of $\widetilde{\mathbb{E}}_+(1) \to \widetilde{M}$, see Remark 3.4. On the other hand, sections of $\mathcal{T}^* \to M$ are understood as specific sections of the bundle $\widetilde{\mathcal{F}} \to \widetilde{M}$, which is a subbundle in $\widetilde{\mathcal{T}} \to \widetilde{M}$, see the generalities in Section 3.4 and the setup of our construction in Section 3.2. It follows that these two natural inclusions commute with the BGG-splitting operators.

Lemma 5.1. *The full arrows in the following diagram commute:*

$$\begin{tikzcd}
\Gamma(\mathcal{T}^*) \arrow[r, hook] \arrow[d, dashed, shift left, "\Pi_0"] & \Gamma\big(\widetilde{\mathcal{T}}\big) \arrow[d, dashed, shift right, "\widetilde{\Pi}_0"'] \\
\Gamma(\mathbb{E}_+(1)) \arrow[r, hook] \arrow[u, bend left, "L_0^{\mathcal{T}^*}"] & \Gamma\big(\widetilde{\mathbb{E}}_+[1]\big). \arrow[u, bend right, "L_0^{\widetilde{\mathcal{T}}}"']
\end{tikzcd}$$

Proof. Consider a projective density $\rho \in \Gamma(\mathbb{E}_+(1))$ on M, the corresponding tractor $L_0^{\mathcal{T}^*}(\rho) \in \Gamma(\mathcal{T}^*)$, and its extension to $\widetilde{\mathcal{F}} \subseteq \widetilde{\mathcal{T}}$, which is denoted by s'. The extension of $\rho \in \Gamma(\mathbb{E}_+(1))$ to $\widetilde{\mathbb{E}}_+[1]$ obviously coincides with the projection $\widetilde{\Pi}_0(s')$, and it is denoted by σ. We need to show that $s' = L_0^{\widetilde{\mathcal{T}}}(\sigma)$, i.e., that $\widetilde{\partial}^* \widetilde{\nabla}^{\mathrm{nor}} s' = 0$. According to Proposition 4.4, $\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \Psi^1 + \Psi^2$ with $\Psi^1 \in \Omega^1(\widetilde{M}, \Lambda^2 \widetilde{\widetilde{\mathcal{F}}})$, $\Psi^2 \in \Omega^1(\widetilde{M}, \widetilde{f}^\circ)$, hence we have

$$\widetilde{\nabla}^{\mathrm{nor}} s' = \widetilde{\nabla}^{\mathrm{ind}} s' + \Psi^1 \bullet s' + \Psi^2 \bullet s'.$$

Since $\Lambda^2 \widetilde{\widetilde{\mathcal{F}}}$ acts trivially on $\widetilde{\mathcal{F}} \subseteq \widetilde{\mathcal{T}}$, we have $\Psi^1 \bullet s' = 0$. Since $\widetilde{f}^\circ \subseteq T^*\widetilde{M}$, it follows that $\widetilde{\partial}^*(\Psi^2 \bullet s') = 0$. It thus follows that $\widetilde{\partial}^*(\widetilde{\nabla}^{\mathrm{nor}} s') = \widetilde{\partial}^*(\widetilde{\nabla}^{\mathrm{ind}} s')$. Let ϕ be the frame form of $\nabla L_0^{\mathcal{T}^*}(\rho)$. Then, according to Lemma 4.1, we have that $\widetilde{\partial}^* \widetilde{\phi} = 0$ since $\Lambda^2 F \bullet F = 0$, and in particular $\widetilde{\partial}^*(\widetilde{\nabla}^{\mathrm{nor}} s') = 0$. ∎

We can now characterise reduced scales in terms of the corresponding scale tractors:

Proposition 5.2. *Let $(\widetilde{M}, \mathbf{c})$ be a conformal spin structure associated to an oriented projective structure $(M, \mathbf{p})$ via the Fefferman-type construction. Let $\sigma \in \Gamma(\widetilde{\mathbb{E}}_+[1])$ be a conformal scale and let $s := L_0^{\widetilde{\mathcal{T}}}(\sigma) \in \Gamma(\widetilde{\mathcal{T}})$ be the corresponding scale tractor. Then the following statements are equivalent:*

- (*a*) *The scale σ is reduced.*
- (*b*) *The tractor s is a section of $\widetilde{\mathcal{F}} \subseteq \widetilde{\mathcal{T}}$.*
- (*c*) *The twistor spinor χ is parallel with respect to the Levi-Civita connection $\widetilde{D}$ of the metric corresponding to the scale σ.*

Furthermore, in a reduced scale, the Schouten tensor is strictly horizontal, i.e., it satisfies

$$v^a \widetilde{\mathsf{P}}_{ab} = 0, \qquad \text{for all } v^a \in \Gamma(\ker \chi), \tag{5.1}$$

and the scalar curvature $\widetilde{J}$ vanishes.

Proof. (a) $\Longrightarrow$ (b): This follows from definitions and Lemma 5.1.

(b) $\Longrightarrow$ (c): The condition (b) means that $s \cdot \mathbf{s}_F = 0$. According to (2.6), (2.8) and (2.7), this condition expanded in slots yields

$$s \cdot \mathbf{s}_F = L_0^{\widetilde{\mathcal{T}}}(\sigma) \cdot L_0^{\widetilde{\mathcal{S}}}(\chi) = \begin{pmatrix} -\frac{1}{2n}\widetilde{J}\sigma \\ 0 \\ \sigma \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{n\sqrt{2}} \not{D}\chi \\ \chi \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2n}\widetilde{J}\chi\sigma \\ -\frac{1}{n}\not{D}\chi\sigma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where we use the Levi-Civita connection $\widetilde{D}$ corresponding to σ. In particular, $\not{D}\chi = 0$ and, since χ is a twistor spinor, the condition (c) follows.

(c) $\Longrightarrow$ (b): The condition (c) yields $\mathbf{s}_F = \binom{0}{\chi}$ according to (2.8). The fact that $\widetilde{\nabla}_a \mathbf{s}_F = 0$ yields $\widetilde{\mathsf{P}}_{ac}\gamma^c\chi = 0$ according to (2.7). Hence (5.1) holds, which in particular means $\widetilde{J} = 0$. Summarising, we have

$$s = L_0^{\widetilde{\mathcal{T}}}(\sigma) = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix} \qquad \text{and} \qquad \mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}}(\chi) = \begin{pmatrix} 0 \\ \chi \end{pmatrix}. \tag{5.2}$$

Hence $s \cdot \mathbf{s}_F = 0$ and the condition (b) follows.

(b) and (c) $\Longrightarrow$ (a): According to the previous reasoning and (2.5), we have

$$\widetilde{\nabla}_a^{\mathrm{nor}} s = \begin{pmatrix} 0 \\ \sigma\widetilde{\mathsf{P}}_{ab} \\ 0 \end{pmatrix}.$$

Hence $\widetilde{\nabla}^{\mathrm{nor}} s$ is strictly horizontal, i.e., $v^a \widetilde{\nabla}_a^{\mathrm{nor}} s = 0$ for every $v^a \in \Gamma(\ker \chi)$. Since $\widetilde{\nabla}^{\mathrm{nor}} = \widetilde{\nabla}^{\mathrm{ind}} + \Psi$ and Ψ is horizontal, the horizontality of $\widetilde{\nabla}^{\mathrm{nor}} s$ is equivalent to the horizontality of $\widetilde{\nabla}^{\mathrm{ind}} s$. Altogether, the condition (a) follows from Proposition 3.3 and Lemma 5.1. ∎

We will need some finer discussion on the slots of the distinguished tractor

$$\mathbf{K} = \begin{pmatrix} \rho_a \\ \mu_{ab} \,|\, \varphi \\ k_a \end{pmatrix} \in \Gamma\big(\mathcal{A}\widetilde{M}\big) \tag{5.3}$$

in reduced scales. From Proposition 4.8 we know that $\mathbf{K}$ is the BGG-splitting $L_0^{\mathcal{A}\widetilde{M}}(k)$, which in particular means that $\mu_{ab} = \widetilde{D}_{[a}k_{b]}$ and $\varphi = -\frac{1}{2n}\widetilde{D}^r k_r$ according to (2.12). However, in the following statement we only exploit the algebraic properties of $\mathbf{K}$, namely, that it acts by minus and plus the identity on $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{E}}$, respectively.

Lemma 5.3. *Let us fix a reduced scale. Then the expression of* $\mathbf{K}$ *as in* (5.3) *satisfy* $\rho_a = 0$, $\varphi = -1$, ${\mu_a}^r v_r = -v_a$ *for every* $v^a \in \Gamma(\ker\chi)$ *and* ${\mu_a}^r \mu_{rb} = g_{ab}$. *Further we have* $\mu_{ab} = \langle \gamma_{[a}\gamma_{b]}\chi, \bar\eta\rangle$ *for some* $\bar\eta \in \Gamma\big(\widetilde{\Sigma}_\mp\big[-\frac{1}{2}\big]\big)$.

Proof. Firstly, we use $\mathbf{K} \bullet s = -s$ for any $s \in \Gamma\big(\widetilde{\mathcal{F}}\big)$. The scale tractor $s = L_0^{\widetilde{\mathcal{T}}}(\sigma)$ of a reduced scale σ is a section of $\widetilde{\mathcal{F}}$ and it has the form as in (5.2). Thus it follows from (2.9) that $\rho_a = 0$ and $\varphi = -1$. Next, for every $v^a \in \Gamma(\ker\chi)$, the tractor $s = \left(\begin{smallmatrix} 0 \\ v_a \\ 0 \end{smallmatrix}\right)$ is clearly a section of $\widetilde{\mathcal{F}}$, since $s \cdot \mathbf{s}_F = 0$. Thus it follows from (2.9) that ${\mu_a}^r v_r = -v_a$.

Secondly, we use $\mathbf{K} \bullet s = s$ for any $s \in \Gamma\big(\widetilde{\mathcal{E}}\big)$. Considering the tractor $s = \left(\begin{smallmatrix} 0 \\ \omega_a \\ 0 \end{smallmatrix}\right)$ with arbitrary $\omega^a \in \Gamma\big(T\widetilde{M}\big)$, the tractor $\bar s := s + \mathbf{K} \bullet s$ is a section of $\widetilde{\mathcal{E}}$, whose middle slot is $\omega_a + {\mu_a}^r\omega_r$. It follows again from (2.9) that ${\mu_a}^r \mu_{rb} = g_{ab}$.

Thirdly, we use (3.3) which shows how $\mathbf{K}$ is built from $\mathbf{s}_E$ and $\mathbf{s}_F$. Since the top slot of $\mathbf{s}_F$ vanishes, middle slots of $\mathbf{K}$ are given by a suitable tensor product of χ and the top slot of $\mathbf{s}_E$. ■

We will also need more properties of conformal curvature quantities in reduced scales.

Lemma 5.4. *In a reduced scale,*

$$\widetilde{W}_{abcd}\mu^{cd} = 0, \qquad \text{where } \mu_{ab} = \widetilde{D}_{[a}k_{b]}, \tag{5.4}$$

$$v^c \widetilde{Y}_{abc} = 0, \qquad \text{for all } v^a \in \Gamma(\ker\chi). \tag{5.5}$$

Proof. In slots, the condition $\widetilde{\Omega}^{\text{nor}}_{ab} \bullet \mathbf{s}_F = 0$ implies that $\widetilde{W}_{abcd}\gamma^c\gamma^d\chi = 0$. Pairing both sides of the latter equality with a spinor $\bar\eta$ from Lemma 5.3 yields (5.4).

Consider two arbitrary sections v^a, w^b of $\widetilde{f} = \ker\chi$. Conditions (W) and (5.1) imply $\widetilde{R}_{abcd}v^a w^d = 0$. Now, inserting v^a and w^e into the Bianchi identity $\widetilde{D}_{[a}\widetilde{R}_{bc]de} = 0$, we obtain $v^a w^e \widetilde{D}_a \widetilde{R}_{bcde} = 0$, where we used the fact that $\widetilde{f}$ is parallel. Since we can always regard g^{ab} as a section of $\widetilde{f} \otimes \widetilde{f}^*$, this implies $0 = g^{ec} v^a \widetilde{D}_a \widetilde{R}_{bcde} = v^a \widetilde{D}_a \widetilde{\mathrm{Ric}}_{bc}$ where $\widetilde{\mathrm{Ric}}_{bc}$ is the Ricci tensor of $\widetilde{D}_a$. Since $\widetilde{J} = 0$, we have that $\widetilde{\mathsf{P}}_{ab}$ is proportional to $\widetilde{\mathrm{Ric}}_{ab}$ by a constant factor. Thus $v^a \widetilde{D}_a \widetilde{\mathsf{P}}_{bc} = 0$. From (5.1) and since $\widetilde{f}$ is parallel we also have $v^b \widetilde{D}_a \widetilde{\mathsf{P}}_{bc} = 0$. Altogether, (5.5) follows by the definition of the Cotton tensor. ■

5.2 Explicit normalisation formula

So far we discussed three Cartan connections on the Fefferman space $\widetilde{M}$: the induced one $\widetilde{\omega}^{\text{ind}}$ (Section 3.3), the corresponding normal one $\widetilde{\omega}^{\text{nor}}$ (Section 4.1) and the modified auxiliary one $\widetilde{\omega}'$ (Section 4.3). Various properties of these and derived objects are enumerated in Propositions 4.9 and 4.12. The following proposition refines the integrability conditions included there.

Proposition 5.5. *Let $(\widetilde{M}, \mathbf{c})$ be the conformal spin structure induced from an oriented projective structure $(M, \mathbf{p})$ via the Fefferman-type construction. Then, along the reduction $\mathcal{G} \hookrightarrow \widetilde{\mathcal{G}}$,*

$$i_X \widetilde{\kappa}^{\mathrm{nor}}(u) \in f^\circ \otimes \Lambda^2 \bar{F}, \qquad \textit{for all } X \in f,\ u \in \mathcal{G}, \tag{5.6}$$

$$i_X \widetilde{\kappa}'(u) = 0, \qquad \textit{for all } X \in f,\ u \in \mathcal{G}. \tag{5.7}$$

Proof. From (4.5) we already know that $i_X \widetilde{\kappa}^{\mathrm{nor}}$ has values in $f^\circ \otimes \big(\Lambda^2 \bar{F} \oplus f^\circ\big)$. We note that the top slot of sections of $\Lambda^2 \widetilde{\bar{\mathcal{F}}}$ vanishes in reduced scales, cf. (3.7). Thus the part in f° corresponds to $v^r \widetilde{Y}_{abr}$ for a $v \in \Gamma(\widetilde{f})$, which however has to vanish by (5.5). Hence (5.6) follows. The last condition (5.7) follows from $\widetilde{\kappa}' = \big(\widetilde{\kappa}^{\mathrm{nor}}\big)_{\mathfrak{sl}(n+1)}$, cf. Proposition 4.12. ∎

Since $\widetilde{\omega}'$ is an $\mathrm{SL}(n+1)$-connection on $\widetilde{\mathcal{G}} \to \widetilde{M}$, it is the extension of a Cartan connection ω', on $\mathcal{G} \to \widetilde{M}$. Now, due to (5.7), any section $v \in \Gamma(\ker \chi)$ inserts trivially into its curvature. But this is the standard condition on the connection ω' to be a Cartan connection also on the bundle $\mathcal{G} \to M$, i.e., to be a projective Cartan connection, cf. [5].

Furthermore, we will show that the descended Cartan connection is normal, i.e., $\omega' = \omega$. To do this, we first compute $\widetilde{\partial}^* \widetilde{\kappa}'$ and then use the relation between the co-differentials ∂^* on M and $\widetilde{\partial}^*$ on $\widetilde{M}$ discussed in Lemma 4.1.

Proposition 5.6. *The curvature $\widetilde{\kappa}'$ satisfies*

$$\widetilde{\partial}^* \widetilde{\kappa}'(u) = i_k \widetilde{\kappa}^{\mathrm{nor}}(u) \in f^\circ \otimes \Lambda^2 \bar{F}, \qquad \textit{for all } u \in \mathcal{G}. \tag{5.8}$$

Proof. We shall compute $\widetilde{\partial}^* \widetilde{\Omega}'$ directly. First observe that using Proposition 4.12 we have $\widetilde{\Omega}' = \big(\widetilde{\Omega}^{\mathrm{nor}}\big)_{\mathfrak{sl}(n+1)} = \widetilde{\Omega}^{\mathrm{nor}} + \frac{1}{2} \mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}$. Hence $\widetilde{\partial}^* \widetilde{\Omega}' = \frac{1}{2} \widetilde{\partial}^* \big(\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}\big)$, since $\widetilde{\partial}^* \widetilde{\Omega}^{\mathrm{nor}} = 0$. Using (5.3) and (2.11), we compute $\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}_{ab}$ as

$$\begin{pmatrix} \rho_c \\ \mu_{c_0 c_1} \mid \varphi \\ k_c \end{pmatrix} \bullet \begin{pmatrix} -\widetilde{Y}_{dab} \\ \widetilde{W}_{abd_0 d_1} \mid 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \rho^r \widetilde{W}_{abrc} - \mu_c{}^r \widetilde{Y}_{rab} + \varphi \widetilde{Y}_{cab} \\ -2 \widetilde{W}_{ab}{}^r{}_{[c_0} \mu_{c_1]r} + 2 k_{[c_0} \widetilde{Y}_{c_1]ab} \mid k^r \widetilde{Y}_{rab} \\ k^r \widetilde{W}_{abrc} \end{pmatrix}.$$

In a reduced scale, from the previous display together with Lemmas 5.3 and 5.4 we compute

$$\widetilde{\partial}^* \big(\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}_{ab}\big) = \begin{pmatrix} 0 \\ 2 k^r \widetilde{W}_{rac_0 c_1} \mid 0 \\ 0 \end{pmatrix} = 2 k^r \widetilde{\Omega}^{\mathrm{nor}}_{ra},$$

which yields (5.8). ∎

Theorem 5.7. *Let $(\mathcal{G}, \omega)$ be a projective normal Cartan geometry over M and let $\big(\widetilde{\mathcal{G}}, \widetilde{\omega}^{\mathrm{ind}}\big)$ be the conformal Cartan geometry over $\widetilde{M}$ induced via the Fefferman-type construction. Then*

(*a*) $\widetilde{\omega}^{\mathrm{ind}} = \widetilde{\omega}' = \widetilde{\omega}^{\mathrm{nor}} - \frac{1}{2} i_k \widetilde{\kappa}^{\mathrm{nor}}$,

(*b*) $\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \Psi^1$, *where* $\Psi^1 = -\frac{1}{2} \widetilde{\partial}^* \widetilde{\kappa}^{\mathrm{ind}} = \frac{1}{2} i_k \widetilde{\kappa}^{\mathrm{nor}}$.

Proof. (a) We use that $i_X \widetilde{\kappa}' = 0$ for all $X \in f$ according to (5.7). Then Proposition 5.6 together with Lemma 4.1 imply that $\partial^* \kappa' = 0$. Thus ω' is projectively normal, and therefore we obtain $\widetilde{\omega}' = \widetilde{\omega}^{\mathrm{ind}}$.

(b) The normalisation process of Proposition 4.4 provides $\Psi = \Psi^1 + \Psi^2$ such that $\widetilde{\omega}^{\mathrm{nor}} = \widetilde{\omega}^{\mathrm{ind}} + \Psi$, where Ψ^1, Ψ^2 are the first and second normalisation steps. However since $\widetilde{\omega}' = \widetilde{\omega}^{\mathrm{ind}}$, it follows from Proposition 5.6 and (4.8) that $\widetilde{\partial}^* \widetilde{\kappa}' = \widetilde{\partial}^* \widetilde{\kappa}^{\mathrm{ind}}$ is, up to a constant multiple, the difference between $\widetilde{\omega}^{\mathrm{nor}}$ and $\widetilde{\omega}^{\mathrm{ind}}$. Therefore already the first normalisation step completes the normalisation, i.e., $\Psi^2 = 0$. ∎

Using the explicit relationship provided in Theorem 5.7 we can also obtain a detailed description of the difference between the induced and the normal Cartan curvatures:

Corollary 5.8. *In a reduced scale, we have the following relation between the curvatures of the induced and the normal conformal Cartan connection:*

$$\widetilde{\Omega}^{\mathrm{ind}}_{ab} = \widetilde{\Omega}^{\mathrm{nor}}_{ab} + \frac{1}{2}\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}_{ab} = \begin{pmatrix} -\widetilde{Y}_{cab} \\ \widetilde{\widetilde{W}}_{abc_0c_1} - \widetilde{\widetilde{W}}_{ab}{}^r{}_{[c_0}\mu_{c_1]r} + k_{[c_0}\widetilde{Y}_{c_1]ab} \mid 0 \\ \frac{1}{2}k^r\widetilde{\widetilde{W}}_{abrc} \end{pmatrix}. \tag{5.9}$$

In particular, $\frac{1}{2}i_k\widetilde{\widetilde{W}}$ is the torsion of the induced Cartan connection $\widetilde{\omega}^{\mathrm{ind}}$.

Proof. We obtained the concrete expression of $\mathbf{K} \bullet \widetilde{\Omega}^{\mathrm{nor}}$ in the proof of Proposition 5.6. Now Lemmas 5.3 and 5.4, and a short computation yields (5.9). ■

6 Comparison with Patterson–Walker metrics and alternative characterisation

In this section we will show that the Fefferman-type construction studied in this article is closely related to the construction of so-called *Patterson–Walker metrics*. These are the Riemann extensions of affine connected spaces, firstly described in [26]. A conformal version of this construction was obtained by [15] for dimension $n = 2$, and was treated by the authors of the present article in general dimension in [20].

6.1 Comparison

Let M be a smooth manifold and $p\colon T^*M \to M$ its cotangent bundle. The vertical subbundle $V \subseteq T(T^*M)$ of this projection is canonically isomorphic to T^*M. An affine connection D on M determines a complementary horizontal distribution $H \subseteq T(T^*M)$ that is isomorphic to TM via the tangent map of p.

Definition 6.1. The *Riemann extension* or the *Patterson–Walker metric* associated to a torsion-free affine connection D on M is the pseudo-Riemannian metric g on T^*M fully determined by the following conditions:

(a) both V and H are isotropic with respect to g,

(b) the value of g with one entry from V and another entry from H is given by the natural pairing between $V \cong T^*M$ and $H \cong TM$.

It follows that V is parallel with respect to the Levi-Civita connection of the just constructed metric. Hence Patterson–Walker metrics are special cases of Walker metrics, i.e., metrics admitting a parallel isotropic distribution. The explicit description of the metric g in terms of the Christoffel symbols of D can be found in [20, 26].

The previous definition can be adapted to weighted cotangent bundles $T^*M(w) = T^*M \otimes \mathbb{E}(w)$, provided that M is oriented and D is special, i.e., preserving a volume form on M, which allows a trivialisation of $\mathbb{E}(w)$. It turns out that Patterson–Walker metrics induced by projectively equivalent connections are conformally equivalent if and only if $w = 2$ (interpreted as a projective weight according to the conventions from Section 2.4). Altogether, we have a natural split-signature conformal structure on $T^*M(2)$ induced by an oriented projective structure $(M, \mathbf{p})$.

From Section 3.3 we know that $\widetilde{M} = T^*M(2) \setminus \{0\}$ is the Fefferman space. Special affine connections from $\mathbf{p}$ are just the exact Weyl connections of the corresponding parabolic geometry.

The corresponding objects on $\widetilde{M}$ are the reduced Weyl connections, respectively reduced scales, which correspond to distinguished metrics in the conformal class, see Section 3.4. We are going to show that these metrics are just the Patterson–Walker metrics.

Proposition 6.2. *Let $(\widetilde{M}, \mathbf{c})$ be the conformal structure of signature (n,n) associated to an n-dimensional projective structure $(M, \mathbf{p})$ via the Fefferman-type construction. Then any metric in $\mathbf{c}$ corresponding to a reduced scale is a Patterson–Walker metric.*

Proof. Within the proof we refer to the notation and explicit matrix realisations from Appendix A. By definition, the Fefferman space is $\widetilde{M} = \mathcal{G}/Q$, which yields $T\widetilde{M} \cong \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$. Conformally invariant objects on $\widetilde{M}$, respectively objects related to a choice of reduced scale, correspond to data on $\mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ that are invariant under the action of Q, respectively $G_0^{ss} \cap Q$. Elements in $\mathfrak{g}/\mathfrak{q}$ will be represented by matrices of the form

$$\begin{pmatrix} -\frac{z}{2} & * & * \\ X & * & * \\ w & Y^t & -\frac{z}{2} \end{pmatrix},$$

where $z, w \in \mathbb{R}$ and $X, Y \in \mathbb{R}^{n-1}$. Firstly, one verifies that

$$Y^t X - zw, \tag{6.1}$$

is the only quadratic form that is invariant under $G_0^{ss} \cap Q$. Hence any reduced-scale metric in $\mathbf{c}$ corresponds to the quadratic form (6.1) in a suitable frame. Secondly, the vertical subbundle $V \subseteq T\widetilde{M}$ corresponds to the Q-invariant subspace $f = \mathfrak{p}/\mathfrak{q} \subseteq \mathfrak{g}/\mathfrak{q}$ given by $X = 0$ and $w = 0$. The horizontal distribution $H \subseteq T\widetilde{M}$ induced by a linear connection from $\mathbf{p}$ corresponds to the unique $(G_0^{ss} \cap Q)$-invariant subspace $h \subseteq \mathfrak{g}/\mathfrak{q}$ complementary to f, which is given by $Y = 0$ and $z = 0$. Obviously, both f and h are isotropic with respect to (6.1). Hence any reduced-scale metric in $\mathbf{c}$ satisfies the condition (a) from Definition 6.1. Thirdly, the canonical identification $V \cong T^*M(2)$ corresponds to an isomorphism $f \cong (\mathfrak{g}/\mathfrak{p})^*(2)$ of Q-modules. Identifying $(\mathfrak{g}/\mathfrak{p})^*(2)$ with $\mathfrak{p}_+(2)$, it turns out to be given by

$$\begin{pmatrix} -\frac{z}{2} & * & * \\ 0 & * & * \\ 0 & Y^t & -\frac{z}{2} \end{pmatrix} \mapsto \begin{pmatrix} 0 & Y^t & -z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, the inner product of any $v \in f$ and $u \in h$ coincides with the pairing of the corresponding elements $v \in \mathfrak{p}_+(2)$ and $u \in \mathfrak{g}/\mathfrak{p}$. Hence any reduced-scale metric in $\mathbf{c}$ satisfies also the condition (b) from Definition 6.1 and so it is a Patterson–Walker metric. ■

6.2 Alternative characterisation

We have characterised split-signature (n,n) conformal structures $\mathbf{c}$ on $\widetilde{M}$ induced by an n-dimensional projective structure via the Fefferman-type construction in Theorem 4.14. Now we know these structures correspond to conformal Patterson–Walker metrics discussed in [20]. There we found the following characterisation in terms of underlying objects by direct computations and spin calculus. Our aim here is to indicate how to reach the same result in the current framework.

Theorem 6.3. *A split-signature (n,n) conformal spin structure $\mathbf{c}$ on a manifold $\widetilde{M}$ is (locally) induced by an n-dimensional projective structure via the Fefferman-type construction if and only if the following properties are satisfied:*

(a) *$(\widetilde{M}, \mathbf{c})$ admits a nowhere-vanishing light-like conformal Killing field k.*

(*b*) $(\widetilde{M}, \mathbf{c})$ *admits a pure twistor spinor* χ *such that* $\widetilde{f} = \ker \chi$ *is integrable and* $k \in \Gamma(\widetilde{f})$.

(*c*) *The Lie derivative of* χ *with respect to the conformal Killing field* k *is* $\mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi$.

(*d*) *The following integrability condition holds:*

$$v^r w^s \widetilde{W}_{arbs} = 0, \qquad \text{for all } v^r, w^s \in \Gamma(\ker \chi). \tag{W}$$

We now express the conditions from Proposition 4.9 in underlying terms:

(i) For a conformal Killing field k with the splitting $\mathbf{K}$ as in (5.3), a straightforward computation shows that the condition $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$ is equivalent to

$$\begin{aligned} &k^a k_a = 0, && \rho^a \rho_a = 0, \\ &\mu^a{}_b k^b = \varphi k^a, && \mu^a{}_b \rho^b = -\varphi \rho^a, \\ &k^a \rho_a = \varphi^2 - 1, && \mu_a{}^c \mu_{cb} = g_{ab} + 2k_{(a}\rho_{b)}. \end{aligned} \tag{6.2}$$

(ii) For a twistor spinor χ, the corresponding tractor spinor $\mathbf{s}_F = \binom{\bar{\chi}}{\chi} \in \Gamma(\widetilde{\mathcal{S}}_-)$ is parallel with respect to $\widetilde{\nabla}^{\mathrm{nor}}$. In particular, purity of $\mathbf{s}_F$ can be checked at one point. If $\chi = 0$, respectively $\bar{\chi} = 0$, this tractor spinor is pure whenever $\bar{\chi}$, respectively χ, is pure. If $\chi \neq 0$ and $\bar{\chi} \neq 0$, the purity of $\binom{\bar{\chi}}{\chi}$ is equivalent to χ and $\bar{\chi}$ being pure and their kernels having $(n-1)$-dimensional intersection, cf. [13, Proposition III-1.12] or [22, 30].

(iii) Let k be a conformal Killing field which splits to $\mathbf{K}$ and χ a twistor spinor which splits to $\mathbf{s}_F$. Then the condition $\mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi$ is equivalent to $\mathbf{K} \bullet \mathbf{s}_F = -\frac{1}{2}(n+1)\mathbf{s}_F$. If the tractor spinor $\mathbf{s}_F$ is pure it has an $(n+1)$-dimensional maximally isotropic kernel $\ker \mathbf{s}_F$. Then $\mathbf{K} \bullet \mathbf{s}_F = -\frac{1}{2}(n+1)\mathbf{s}_F$ is equivalent to $\mathbf{K}$ acting by minus the identity on $\ker \mathbf{s}_F$, which therefore coincides with the eigenspace of $\mathbf{K}$ corresponding to -1.

The assumption on the pure twistor spinor χ in Theorem 6.3 guarantees the existence of a suitable compatible metric for which χ is parallel. This result is proved in [20, Proposition 4.2]. Henceforth we shall assume $\widetilde{D}\chi = 0$ where $\widetilde{D}$ is the corresponding Levi-Civita connection. In particular, we have $\mathbf{s}_F = L_0^{\widetilde{\mathcal{S}}_-}(\chi) = \binom{0}{\chi}$, which is pure and parallel by observation (ii). Expanding the latter condition according to (2.7) yields

$$v^a \widetilde{\mathsf{P}}_{ab} = 0, \qquad \text{for all } v^a \in \Gamma(\ker \chi), \tag{6.3}$$

For $\mathbf{K} = L_0^{\mathcal{A}\widetilde{M}}(k)$, we know from observation (iii) that $\mathbf{K}$ acts by $-\mathrm{id}$ on $\ker \mathbf{s}_F$. By the very same reasoning as in the first part of the proof of Lemma 5.3 it follows that

$$\rho_a = 0, \qquad \varphi = -1, \qquad \mu_a{}^r v_r = -v_a, \qquad \text{for all } v^a \in \Gamma(\ker \chi). \tag{6.4}$$

Now we are prepared to prove the theorem:

Proof of Theorem 6.3. If $(\widetilde{M}, \mathbf{c})$ is induced by a projective structure, the stated properties hold according to Proposition 4.9 and previous observations. For the converse direction, it remains to show that $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$, which is equivalently characterised by the identities (6.2). Then the properties (a)–(d) of Proposition 4.9 will be satisfied and the result will follow from the characterisation Theorem 4.14.

The expansion of the prolonged conformal Killing equation (2.13) for k gives

$$\widetilde{D}_a k_b = \mu_{ab} + g_{ab}, \tag{6.5}$$

$$\widetilde{D}_a \mu_{br} = -2\widetilde{\mathsf{P}}_{a[b}k_{r]} - W_{bras}k^s, \tag{6.6}$$

according to (2.10) and (2.11). Next, from (6.4) we especially have ${\mu_b}^r k_r = -k_b$. Applying $\widetilde{D}_a$ to both sides of this equality and using (6.5) we obtain

$$\big(\widetilde{D}_a \mu_{br}\big)k^r + {\mu_b}^r \mu_{ar} + \mu_{ba} = -(\mu_{ab} + g_{ab}).$$

From (6.6), (W) and (6.3) we have $\big(\widetilde{D}_a \mu_{br}\big)k^r = 0$, hence the previous display shows ${\mu_a}^r \mu_{rb} = g_{ab}$. This together with (6.4) implies that all identities from (6.2) are satisfied, hence $\mathbf{K}^2 = \mathrm{id}_{\widetilde{\mathcal{T}}}$. ■

A Explicit matrix realisations

Here we provide explicit realisations of the Lie algebras introduced in Section 3.2 in terms of matrices. We will consider the inner product h and the involution K on $\mathbb{R}^{n+1,n+1}$ given by the block matrices

$$h := \begin{pmatrix} 0 & I_{n+1} \\ I_{n+1} & 0 \end{pmatrix} \qquad \text{and} \qquad K := \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_{n+1} \end{pmatrix}$$

with respect to the standard basis $(e_1, \dots, e_{2n+2})$. Then $E = \langle e_1, \dots, e_{n+1}\rangle$ and $F = \langle e_{n+2}, \dots, e_{2n+2}\rangle$ and the decomposition (3.4) can be written as

$$\tilde{\mathfrak{g}} = \Lambda^2(E \oplus F) = \begin{pmatrix} E \otimes F & \Lambda^2 E \\ \Lambda^2 F & E \otimes F \end{pmatrix}.$$

For $\tilde{v} := e_1 + e_{2n+2}$, the Lie algebra $\tilde{\mathfrak{p}}$ of the parabolic subgroup $\widetilde{P} \subseteq \widetilde{G}$ is of the following form

$$\tilde{\mathfrak{p}} = \left(\begin{array}{ccc|ccc} a & U^t & w & 0 & -W^t & -b \\ X & B & V & W & C & -X \\ 0 & Y^t & c & b & X^t & 0 \\ \hline 0 & -Y^t & -d & -a & -X^t & 0 \\ Y & D & -Z & -U & -B^t & -Y \\ d & Z^t & 0 & -w & -V^t & -c \end{array}\right), \tag{A.1}$$

where $a, b, c, d, w \in \mathbb{R}$ with $a - b = d - c$, $U, V, W, X, Y, Z \in \mathbb{R}^{n-1}$, $B \in \mathfrak{gl}(n-1)$ and $C, D \in \mathfrak{so}(n-1)$. The nilradical $\tilde{\mathfrak{p}}_+ = \tilde{\mathfrak{p}}^\perp$ is then of the form

$$\tilde{\mathfrak{p}}_+ = \left(\begin{array}{ccc|ccc} a & U^t & w & 0 & -V^t & -a \\ 0 & 0 & V & V & 0 & 0 \\ 0 & 0 & a & a & 0 & 0 \\ \hline 0 & 0 & -a & -a & 0 & 0 \\ 0 & 0 & -U & -U & 0 & 0 \\ a & U^t & 0 & -w & -V^t & -a \end{array}\right).$$

A choice of Levi subalgebra $\tilde{\mathfrak{g}}_0 \subseteq \tilde{\mathfrak{p}}$ determines a grading $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_- \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{p}}_+$. We shall choose $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{p}} \cap \tilde{\mathfrak{p}}_{\mathrm{op}}$, where $\tilde{\mathfrak{p}}_{\mathrm{op}} \subseteq \tilde{\mathfrak{g}}$ is the stabiliser of the light-like vector e_{n+2}. Explicitly,

$$\tilde{\mathfrak{g}}_0 = \left(\begin{array}{ccc|ccc} a & 0 & 0 & 0 & 0 & 0 \\ X & B & V & 0 & C & -X \\ 0 & Y^t & c & 0 & X^t & 0 \\ \hline 0 & -Y^t & -a-c & -a & -X^t & 0 \\ Y & D & -Z & 0 & -B^t & -Y \\ a+c & Z^t & 0 & 0 & -V^t & -c \end{array}\right).$$

The embedding $i'\colon \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ of Lie algebras has the form $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & -A^t \end{smallmatrix}\right)$. The subgroup $Q = i^{-1}(\widetilde{P})$ is contained in P, the stabiliser in G of $v = (\tilde{v})_E = e_1$; the inclusion of corresponding Lie algebras is

$$\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}} = \begin{pmatrix} a & U^t & w \\ 0 & A & V \\ 0 & 0 & -a \end{pmatrix} \subseteq \begin{pmatrix} a & U^t & w \\ 0 & B & V \\ 0 & X^t & c \end{pmatrix} = \mathfrak{p},$$

where $\operatorname{tr}(A) = 0$ and $a + \operatorname{tr}(B) + c = 0$. The standard projective grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$,

$$\mathfrak{g}_- = \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ y & 0 & 0 \end{pmatrix}, \qquad \mathfrak{g}_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & B & V \\ 0 & X^t & c \end{pmatrix}, \qquad \mathfrak{p}_+ = \begin{pmatrix} 0 & U^t & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

is compatible with the previous conformal grading so that the reduced Lie subalgebra $\mathfrak{q}_0 := \mathfrak{q} \cap \mathfrak{g}_0$ coincides with the intersection of $\mathfrak{g}_0 \cap \tilde{\mathfrak{g}}_0$. Explicitly,

$$\mathfrak{q}_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & A & V \\ 0 & 0 & -a \end{pmatrix}, \tag{A.2}$$

where $\operatorname{tr}(A) = 0$.

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Fefferman–Graham ambient metrics of Patterson–Walker metrics

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Abstract

Given an n-dimensional manifold N with an affine connection D, we show that the associated Patterson–Walker metric g on T^*N admits a global and explicit Fefferman–Graham ambient metric. This provides a new and large class of conformal structures which are generically not conformally Einstein but for which the ambient metric exists to all orders and can be realised in a natural and explicit way. In particular, it follows that Patterson–Walker metrics have vanishing Fefferman–Graham obstruction tensors. As an application of the concrete ambient metric realisation we show in addition that Patterson–Walker metrics have vanishing Q-curvature. We further show that the relationship between the geometric constructions mentioned above is very close: the explicit Fefferman–Graham ambient metric is itself a Patterson–Walker metric.

1. *Introduction and main result*

Given a signature (p, q) conformal structure $[g]$ on an $m = p + q$ dimensional manifold M, it was shown in seminal work by Fefferman and Graham (see [**7**, **8**]) that under specific conditions the conformal structure can be encoded equivalently as a signature $(p+1, q+1)$ pseudo-Riemannian metric $(\mathbf{M}, \mathbf{g})$ with vanishing Ricci curvature. This description has been fundamental in constructing and classifying conformal invariants (see, for example, [**3**, **7**]) and for constructing and studying conformally invariant differential operators (see [**11**, **12**]).

To build the Fefferman–Graham ambient metric for given local coordinates x on M, one first considers the ray bundle of metrics in the conformal class $[g]$, written as $\mathbb{R}_+ \times \mathbb{R}^m$ with coordinates (t, x). The ambient space $\mathbf{M}$ is obtained by adding a new transversal coordinate $\rho \in \mathbb{R}$, and then an *ansatz* for the Fefferman–Graham ambient metric $\mathbf{g}$ is

$$\mathbf{g} = t^2 g_{ij}(x, \rho) dx^i \odot dx^j + 2\rho dt \odot dt + 2t dt \odot d\rho, \tag{1}$$

where $g = g_{ij}(x, 0) dx^i dx^j$ is a representative metric in the conformal class. It is directly visible from the formula that $\mathbf{g}$ is homogeneous of degree 2 with respect to the *Euler field* $t\partial_t$ on $\mathbf{M}$.

To show existence of a Fefferman–Graham ambient metric $\mathbf{g}$ for given g, the *ansatz* (1) determines an iterative procedure to determine $g_{ij}(x, \rho)$ as a Taylor series in ρ satisfying $\mathrm{Ric}(\mathbf{g}) = 0$ to infinite order at $\rho = 0$. For m odd, the existence (and a natural version of uniqueness) of $\mathbf{g}$ as an infinity-order series expansion in ρ is guaranteed for general $g_{ij}(x)$. For $m = 2n$ even, the existence of an infinity-order jet for $g_{ij}(x, \rho)$ with $\mathrm{Ric}(\mathbf{g}) = 0$ asymptotically at $\rho = 0$ is obstructed at order n, and is controlled by the vanishing of the conformally invariant Fefferman–Graham tensor $\mathcal{O}$. Moreover, existence does not in general guarantee uniqueness.

Results which provide global Fefferman–Graham ambient metrics, where $\mathbf{g}$ can then be constructed in a natural way from g and satisfies $\mathrm{Ric}(\mathbf{g}) = 0$ globally and not just asymptotically at $\rho = 0$, are rare, in both even and odd dimensions. A special instance, where global ambient

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metrics can at least be shown to exist, occurs for g real-analytic and m either being odd or m even with the obstruction tensor $\mathcal{O}$ of g vanishing. The simplest case of geometric origin for which one has global ambient metrics consists of locally conformally flat structures $(M,[g])$, where $(\mathbf{M},\mathbf{g})$ exists and is unique up to diffeomorphisms (see [8, Chapter 7]). Another well-known geometric case are conformal structures $(M,[g])$ which contain an Einstein metric g: If $\mathrm{Ric}(g) = 2\lambda(m-1)g$, then $\mathbf{g}$ on $\mathbb{R}_+ \times M \times \mathbb{R}$ can be written directly in terms of g as

$$\mathbf{g} = t^2(1+\lambda\rho)^2 g + 2\rho dt \odot dt + 2t dt \odot d\rho. \tag{2}$$

In work by Thomas Leistner and Pawel Nurowski it was shown that the so-called *pp-waves* admit global ambient metrics in the odd-dimensional case and under specific assumptions in the even-dimensional case, see [14]. Concrete and explicit ambient metrics for specific examples of families of conformal structures induced by generic 2-distributions on 5-manifolds and generic 3-distributions on 6-manifolds have been constructed in [2, 15, 16, 18]. Recent progress in obtaining classes of conformal structures for which the Fefferman–Graham ambient metric equations become linear and have explicit solutions was obtained in [1].

The present article expands the class of metrics for which canonical ambient metrics exist globally to *Patterson–Walker metrics:* Given an affine connection D on an n-manifold N with $n \geqslant 2$, which is supposed to be torsion-free and to preserve a volume form, the *Patterson–Walker metric* g is a natural split-signature (n,n) metric on the total space of the co-tangent bundle T^*N, see [13] for historical background on Patterson–Walker metrics, references and a modern treatment. Our main result is as follows.

THEOREM 1. *Let D be a torsion-free affine connection on N which preserves a volume form. Denote local coordinates on N by x^A and the induced canonical fibre coordinates on T^*N by p_A. Let $\Gamma_A{}^C{}_B$ and Ric_{AB} denote the Christoffel symbols and the Ricci curvature of D, respectively. Let*

$$g = 2\,\mathrm{d}x^A \odot \mathrm{d}p_A - 2\,\Gamma_A{}^C{}_B\, p_C\, \mathrm{d}x^A \odot \mathrm{d}x^B \tag{3}$$

be the Patterson–Walker metric induced on T^*N by D. Then

$$\begin{aligned}\mathbf{g} &= 2\rho dt \odot dt + 2t dt \odot d\rho \\ &\quad + t^2\Big(2dx^A \odot dp_A - 2p_C\Gamma_A{}^C{}_B dx^A \odot dx^B + \frac{2\rho}{n-1}\mathrm{Ric}_{AB} dx^A \odot dx^B\Big)\end{aligned} \tag{4}$$

is a globally Ricci-flat Fefferman–Graham ambient metric for the conformal class $[g]$.

For generic D, the resulting Patterson–Walker metric g is not conformally Einstein, see [13, Theorem 2]. In particular, Theorem 1 provides a large class of conformal structures which are not conformally Einstein but which admit globally Ricci-flat and explicit ambient metrics.

As an immediate consequence of the existence of the ambient metric for $(M,[g])$, the conformally invariant Fefferman–Graham obstruction tensor $\mathcal{O}$ associated to $[g]$ vanishes.

It is not difficult to check Ricci-flatness of (4) directly: Specifically, one employs [8, formula (3.17)], which is applicable to any ambient metric in the normal form (1). The computation is then based on the following key facts: The Ricci curvature of the Patterson–Walker metric g is up to a constant multiple just the pullback of the Ricci curvature of D and this tensor and its covariant derivative are totally isotropic, see [13]. We note that formula (4) for $\mathbf{g}$ says that the ambient metric is in fact linear in ρ and the iterative procedure determining the ambient metric stops after the first step.

A geometric proof of vanishing Ricci curvature is presented in the next section. This is based on a combination of the well-known Patterson–Walker and Thomas cone constructions, both of which we recall. It follows directly from the geometric construction outlined below that the Fefferman–Graham ambient metric of a Patterson–Walker metric is again a Patterson–Walker metric, namely, the one associated to the Thomas cone connection. This nice and interesting fact is elaborated in Theorem 2.

2. *Geometric construction of the ambient metric*

The association $D \rightsquigarrow g$ generalises to a natural association from projective to conformal structures. Recall that two affine connections D, D' on N are called projectively related or equivalent if they have the same geodesics as unparameterised curves, which is the case if and only if there exists a 1-form $\Upsilon \in \Omega^1(N)$ with

$$D'_X Y = D_X Y + \Upsilon(X)Y + \Upsilon(Y)X \tag{5}$$

for all $X, Y \in \mathfrak{X}(N)$. It is sufficient to restrict ourselves to *special* connections in a projective class, that is, to those that preserve some volume form. For projective structures it is useful to employ suitably scaled *projective density bundles*, defined as $\mathcal{E}(w) := (\wedge^n TN)^{-w/(n+1)}$, for arbitrary *weight* w. Then a section $s : N \to \mathcal{E}_+(1)$ corresponds to a choice of a special affine connection D in the projective equivalence class $[D]$, and any $s' = e^f s$ corresponds to D' projectively related to D via (5) with $\Upsilon = df$. We define $M := T^*N \otimes \mathcal{E}(2)$ the (projectively) weighted co-tangent bundle of N. Then, as was shown in [**13**], two projectively related affine connections D, D' on N induce two conformally related metrics g, g' on M, and we therefore have a natural association $(N, [D]) \rightsquigarrow (M, [g])$.

The cone $\mathcal{C} := \mathcal{E}_+(1)$ carries the canonical and well-known Ricci-flat *Thomas cone connection* ∇ associated to the projective class $[D]$, see [**5**, **17**]. (The specific weight 1, which is different from the one in [**5**], is more convenient for our computations.) We will need a local formula for ∇: Let $s : N \to \mathcal{C}$ be the scale corresponding to an affine connection $D \in [D]$, providing a trivialisation $\mathcal{C} \cong \mathbb{R}_+ \times N$ via $(x^0, x) \mapsto s(x)x^0$. In this trivialisation the Thomas cone connection is given by

$$\nabla_X Y = D_X Y - \frac{1}{n-1}\mathrm{Ric}(X,Y)Z, \quad \nabla_X Z = X, \tag{6}$$

where $X, Y \in \mathfrak{X}(N)$ and $Z = x^0\partial_{x^0}$ is the Euler field on $\mathcal{C}$. It is in fact easy to see directly from formula (6) that the thus defined affine connection ∇ on the Thomas cone $\mathcal{C}$ is independent of the choice of scale and it is Ricci-flat.

Employing a local coordinate patch on N which induces coordinates x^A, y_A on the co-tangent bundle T^*N and coordinates x^0, x^A, y_A, y_0 on $T^*\mathcal{C} \cong \mathbb{R}_+ \times T^*N \times \mathbb{R}$, the Patterson–Walker metric $\mathbf{g}$ associated to ∇ is

$$\begin{aligned} \mathbf{g} = 2dx^A \odot dy_A + 2dx^0 \odot dy_0 - \frac{4}{x^0} y_B dx^0 \odot dx^B \\ - 2y_C \Gamma_A{}^C{}_B dx^A \odot dx^B + 2\frac{x^0 y_0}{n-1}\mathrm{Ric}_{AB} dx^A \odot dx^B. \end{aligned} \tag{7}$$

Ricci-flatness of $\mathbf{g}$ follows directly from Ricci-flatness of ∇, see [**13**, Theorem 2]. Via the change of coordinates $t = x^0$, $\rho = y_0/x^0$, $p_A = y_A/(x^0)^2$ the metric $\mathbf{g}$ transforms to (4), which is the form of a Fefferman–Graham ambient metric (1). In particular, this shows Theorem 1.

We conclude this section by summarising the construction:

THEOREM 2. *Given a projective structure $(N,[D])$ on an n-dimensional manifold N, the geometric constructions indicated in the following diagram commute:*

$$\begin{tikzcd} (\mathcal{C},\nabla) \arrow[r, rightsquigarrow] & (\mathbf{M},\mathbf{g}) \\ (N,[D]) \arrow[u, rightsquigarrow] \arrow[r, rightsquigarrow] & (M,[g]) \arrow[u, rightsquigarrow] \end{tikzcd}$$

In particular, the induced conformal structure $[g]$ admits a globally Ricci-flat Fefferman–Graham ambient metric $\mathbf{g}$ which is itself a Patterson–Walker metric.

REMARK. As a Patterson–Walker metric, $(\mathbf{M},\mathbf{g})$ carries a naturally induced homothety $\mathbf{k}$ of degree 2, which takes the form $2p_A\partial_{p_A} + 2\rho\partial_\rho$. According to [**13**, Lemma 5.1] the infinitesimal affine symmetry Z of ∇ lifts to a Killing field, which one computes as $t\partial_t - 2p_A\partial_{p_A} - 2\rho\partial_\rho$. In particular it follows that the Euler field $t\partial_t$ of the Fefferman–Graham ambient metric $\mathbf{g}$ can be written as the sum of this Killing field and the homothety $\mathbf{k}$. The tangent bundle $T\mathbf{M}$ carries the maximally isotropic $(n+1)$-dimensional subspace spanned by $\{\partial_{p_A},\partial_\rho\}$ which is preserved by $\boldsymbol{\nabla}$. This subspace can be equivalently described by a $\boldsymbol{\nabla}$-parallel pure spinor $\mathbf{s}$ on $\mathbf{M}$. The ambient Killing field $\mathbf{k}$ and the ambient parallel pure spinor $\mathbf{s}$ correspond to a homothety k of g and a parallel pure spinor χ on M that belong to the characterising objects of the Patterson–Walker metric g, see [**13**, Theorem 1].

3. *Vanishing 'Q-curvature'*

The Q-curvature Q_g of a given metric g is a Riemannian scalar invariant with a particularly simple transformation law with respect to conformal change of metric. It has been introduced by Branson in [**4**] and has been the subject of intense research in recent years, see, for example, [**6**] for an overview. Computation of Q-curvature is notoriously difficult, see, for example, [**10**]. An explicit form of the Fefferman–Graham ambient metric $\mathbf{g}$ for a given metric g allows a computation of Q_g. Using the fact that our $\mathbf{g}$ is actually a Patterson–Walker metric, this computation is particularly simple.

THEOREM 3. *The Patterson–Walker metric g associated to a volume-preserving, torsion-free affine connection D has vanishing Q-curvature Q_g.*

Proof. We follow the computation method for Q_g from [**9**]: For this it is necessary to compute $-\boldsymbol{\Delta}^n\log(t)$, where $\boldsymbol{\Delta}$ is the ambient Laplacian on $\mathbf{M} = \mathbb{R}_+ \times T^*N \times \mathbb{R}$ associated to $\mathbf{g}$ and $t : \mathbf{M} \to \mathbb{R}_+$ is the first coordinate. Restricting $-\boldsymbol{\Delta}^n\log(t)$ to the cone $\mathbb{R}_+ \times T^*N \times \{0\}$ and evaluating at $t=1$ yields Q_g. To show that the Q-curvature vanishes for g, it is in particular sufficient to show that $\boldsymbol{\Delta}\log(t) = 0$.

However, the function $t : \mathbf{M} \to \mathbb{R}_+$ is horizontal since it is just the pullback of the coordinate function $x^0 : \mathcal{C} \to \mathbb{R}_+$ on the Thomas cone $\mathcal{C} \cong \mathbb{R}_+ \times N$ via the canonical projection $T^*\mathcal{C} \to \mathcal{C}$. It follows from the explicit formula for the Christoffel symbols of a Patterson–Walker metric that $\boldsymbol{\Delta}$ vanishes on any horizontal function, see [**13**, Section 2.1]. Thus in particular $\boldsymbol{\Delta}\log(t) = 0$, and then also $Q_g = 0$. $\square$

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Conformal theory of curves with tractors

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A R T I C L E I N F O

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We present the general theory of curves in conformal geometry using tractor calculus. This primarily involves a specification of distinguished parametrizations and relative and absolute conformal invariants of generic curves. The absolute conformal invariants are defined via a tractor analogue of the classical Frenet frame construction and then expressed in terms of relative ones. Our approach applies likewise to conformal structures of any signature; in the case of indefinite signature we focus especially on the null curves. It also provides a conceptual tool for handling distinguished families of curves (conformal circles and conformal null helices) and conserved quantities along them.

1. Introduction

The local geometry of curves is a classical subject, nowadays developed for various geometric structures using various views and methods. The study generally starts in flat spaces, the passage to general curved cases demands new ideas and attitudes. We are going to evolve an approach based on conformal tractor calculus. In this section, we present main sources of inspiration, summarize main results and introduce main tools needed later.

1.1. Main sources and methods

To our knowledge, the first general treatment of curves in conformal Riemannian manifolds is due to Fialkow [9], which is therefore used as a basic reference for comparisons. In that paper, a conformally invariant Frenet-like approach is developed. This primarily requires a natural distinguished parametrization of the curve (in the place of the arc-length parameter in the Riemannian setting), a natural starting object (in the place of the unit tangent vector) and a notion of derivative along the curve (in the place of the

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restricted Levi-Civita connection). Having all these instruments, it is easy to derive a conformal analogue of Fernet formulas with a distinguished set of conformal invariants, the conformal curvatures of the curve. For generic curves in an n-dimensional manifold, this yields $n-1$ conformal curvatures, one of which has an exceptional flavour.

The Frenet frame can be seen as an example of a more general concept of moving frame by Cartan. An application of the latter method for immersed submanifolds, especially curves, in the homogeneous Möbius space is presented by Schiemangk and Sulanke in [16] and [20], which we adduce as important sources of inspiration. That way, the curve in the homogeneous space is covered by a curve in the principal group so that the restriction of the Maurer–Cartan form to the lift reveals the generating set of invariants. The lift can be identified with, and practically is constructed as, a curve of (pseudo-)orthonormal frames in the ambient Minkowski space, whose dimension is two more higher than the dimension of the Möbius space.

Both the homogeneous principal bundle with the Maurer–Cartan form and the associated homogeneous vector bundle, whose fibre is the ambient Minkowski space with the induced linear connection, has a counterpart over general conformal manifolds: the former leads to the notion of conformal Cartan connection, the latter to the Thomas standard tractor bundle with its linear connection and parallel bundle metric. These two approaches are basically equivalent, in this paper we exploit the latter one. One of its main advantages is that everything can be set off pretty directly in terms of underlying data, while it still has very conceptual flavour. This, on the one hand, allows one to easily adapt key ideas from the homogeneous setting to the tractorial one. On the other hand, expanding any tractorial formula yields a very concrete tensorial expression that allows comparisons to results obtained by that means. We refer to the essential work [2] by Bailey, Eastwood and Gover both for the generalities on tractor calculus and for an initial step in the study of (distinguished) curves in this manner.

Besides the just cited main sources, there is a wide literature discussing curves in conformal and other geometries from various aspects. A short review of the development of the topic for conformal structures can be found in the introduction of [5]. Another references that are close to our purposes are [3], [4] and [13]. An exhibition of tensorial techniques in the study of curves in related geometries can be found in [14]. For typical subtleties in dealing with null curves in pseudo-Riemannian geometry, see e.g. [7]. For an application of the Cartan's method of moving frame to curves in a broad class of homogeneous spaces, see [6].

1.2. Aim, structure and results

We study the local differential geometry of curves in general conformal manifolds of general signature using the tractor calculus. Although the subject is indeed classic, the systematic tractorial approach is novel. This approach has not only obvious formal benefits, but also a potential for discovering new results and relations. In particular, the discussion for null curves in the case of indefinite signature (for which almost nothing is known) is very parallel to the one in positive definite case and leads to results that we, at least, would never obtain by other means.

The rough structure of the paper is as follows: In sections 2 and 4 we develop the theory for conformal Riemannian structures, in section 5 we discuss its analogies in indefinite signature. We primarily deal with generic curves, whereas special cases are briefly mentioned in accompanying remarks. However, the most special case—*conformal circles*—is dealt individually in section 3.

In section 2 we follow the setting of [2], where one already finds the tractorial determination of the preferred class of *projective* parametrizations on any curve as well as the projectively parametrized conformal circles. Continuing further with tractors of higher order, we build a natural reservoir of relative conformal invariants associated to any curve (subsection 2.3). The simplest of these leads to the notion of *conformal arc-length*, the distinguished conformally invariant parametrization of the curve (subsection 2.4). This invariant vanishes identically along the curve if and only if the curve is an arbitrarily parametrized conformal circle (Proposition 3.3). As a demonstration of the ease of use of tractors we describe some conserved quantities along projectively parametrized conformal circles on manifolds admitting an almost Einstein scale, respectively normal conformal Killing field (subsection 3.2).

In section 4 we launch the Frenet-like procedure to absolute conformal invariants of curve:

- construct a pseudo-orthonormal tractor Frenet frame along the given curve,
- differentiate with respect to the conformal arc-length and extract the tractor Frenet formulas,
- the coefficients of that system determine the generating set of invariants.

For the curve in an n-dimensional manifold, we have $n-1$ conformal curvatures, one of which has an exceptional flavour. The vanishing of the exceptional curvature has an immediate interpretation relating the above mentioned parametrizations (Proposition 4.3). By construction, all conformal curvatures are expressed via the tractors from the tractor Frenet frame, i.e. with respect to the conformal arc-length parametrization. Alternative expressions in terms of initial tractors are also possible. In particular, for all nonexceptional curvatures, we have very simple formulas using the previously defined relative conformal invariants, i.e. with respect to an arbitrary parametrization (Theorem 4.6). For a given scale, we also indicate how to express conformal curvatures in terms of the Riemannian ones (subsection 4.3). Besides the pure pleasure, this effort allows a comparison of our invariants with those in the literature, especially in [9].

In section 5 we adapt the previous scheme to conformal manifolds of indefinite signature. In that case we distinguish space-, time- and light-like curves according to the type of their tangent vectors. A full type classification of curves (via associated tractors) becomes very rich, depending on the dimension and signature. The construction of the tractor Frenet frame has to be adapted to the respective type, which makes any attempt on its universal description impossible. Avoiding the complicated branching of the discussion, we only point out the main features (Remark 4.4(3)) and focus on the light-like curves. Among these curves we identify an appropriate analogue of conformal circles, the *conformal null helices*. Notably, in their characterization another family of relative conformal invariants of Wilczynski type appears (Theorem 5.8). More details on the construction of the tractor Frenet frame and expressions of the corresponding conformal curvatures are discussed in the case of Lorentzian signature (subsection 5.4).

1.3. Notation and conventions

Most of the following conventions is taken from [2]. A *conformal structure* of signature (p,q) on a smooth manifold M of dimension $n=p+q$ is a class of pseudo-Riemannian metrics of signature (p,q) that differ by a multiple of an everywhere positive function. For all tensorial objects on M we use the standard abstract index notation. Thus, the symbol μ^a and μ_a refers to a section of the tangent and cotangent bundle, which is denoted as $\mathcal{E}^a := TM$ and $\mathcal{E}_a := T^*M$, respectively, multiple indices denote tensor products, e.g. ${\mu_a}^b$ is a section of ${\mathcal{E}_a}^b := T^*M \otimes TM$ etc. Round brackets denote symmetrization and square brackets denote skew symmetrization of enclosed indices, e.g. sections of $\mathcal{E}_{[ab]} = T^*M \wedge T^*M$ are 2-forms on M. By $\mathcal{E}[w]$ we denote the density bundle of *conformal weight* w, which is just the bundle of ordinary $(-\frac{w}{n})$-densities. Tensor products with another bundles are denoted as $\mathcal{E}^a[w] := \mathcal{E}^a \otimes \mathcal{E}[w]$ etc. In what follows, the notation as $\mu^a \in \mathcal{E}^a[w]$ always means that μ^a is a section (and not an element) of $\mathcal{E}^a[w]$, global or local according to the context.

Conformal structure on M can be described by the *conformal metric* $\mathbf{g}_{ab}$ which is a global section of $\mathcal{E}_{(ab)}[2]$. Any raising and lowering of indices is provided by the conformal metric, e.g. for $\mu^a \in \mathcal{E}^a[w]$ we have $\mu_a = \mathbf{g}_{ab}\mu^b \in \mathcal{E}_a[w+2]$. A *conformal scale* is an everywhere positive section of $\mathcal{E}[1]$. The choice of scale $\sigma \in \mathcal{E}[1]$ corresponds to the choice of metric $g_{ab} \in \mathcal{E}_{(ab)}$ from the conformal class so that $g_{ab} = \sigma^{-2}\mathbf{g}_{ab}$. The corresponding Levi-Civita connection is denoted as ∇. The Schouten tensor, which is a trace modification of the Ricci tensor, is denoted as P_{ab}. Transformations of quantities under the change of scale will be denoted by hats. In particular, for $\widehat{\sigma} = f\sigma$, $\Upsilon_a = f^{-1}\nabla_a f$ and any $\mu^a \in \mathcal{E}^a$, the Levi-Civita connection and the Schouten tensor change as

$$\widehat{\nabla}_a \mu^b = \nabla_a \mu^b + \Upsilon_a \mu^b - \mu_a \Upsilon^b + \mu^c \Upsilon_c \delta_a{}^b, \tag{1}$$

$$\widehat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \tfrac{1}{2}\Upsilon_c \Upsilon^c g_{ab}. \tag{2}$$

Let $\Gamma \subset M$ be a curve with a parametrization $t : \Gamma \to \mathbb{R}$ and the tangent vector U^a so that $U^a \nabla_a t = 1$. Regular curve is called space-like, time-like, respectively light-like (or null), in accord with the type of its tangent vector U^a, i.e. if $U_c U^c > 0$, $U_c U^c < 0$, respectively $U_c U^c = 0$. Of course, the curve may change the type of its tangent vectors. Since the whole study that follows is of very local nature, we restrict ourselves only to the (segments of) curves of fixed type. In the first two cases, the density

$$u := \sqrt{|U_c U^c|} \in \mathcal{E}[1] \tag{3}$$

is nowhere vanishing and will be employed later. Henceforth we always assume Γ is smooth. Along the curve we use notation $\frac{d}{dt} := U^c \nabla_c$ and

$$U'^{\,a} := \tfrac{d}{dt} U^a, \qquad U''^{\,a} := \tfrac{d^2}{dt^2} U^a, \qquad \dots, \qquad U^{(i)a} := \tfrac{d^i}{dt^i} U^a \tag{4}$$

for the derived vectors. By abuse of notation we often write $U^a \in \mathcal{E}^a$ which should be read as $U^a \in \mathcal{E}^a|_\Gamma$, and similarly for any other quantities defined only along the curve Γ. Note that the vector $U^{(i)a}$ has order $i+1$ (with respect to Γ). Obviously, none of vectors in (4) is conformally invariant. For instance, the acceleration vector transforms according to (1) as

$$\widehat{U}'^{a} = U'^{\,a} - U^c U_c \Upsilon^a + 2U^c \Upsilon_c U^a. \tag{5}$$

From this it follows that, for space- and time-like curves (but not for null curves), a metric in the conformal class may be chosen so that $U'^{\,a} = 0$, i.e. the curve is an affinely parametrized geodesic of the corresponding Levi-Civita connection. This indicates the problem with a conformally invariant notion of osculating subspaces. Instead, we are going to make use of tractors.

The *conformal standard tractor bundle* $\boldsymbol{\mathcal{T}}$ over the conformal manifold M of signature (p, q), where $p + q = n = \dim M$, is the tractor bundle corresponding to the standard representation $\mathbb{R}^{p+1,q+1}$ of the conformal principal group $O(p + 1, q + 1)$. Specifically, $\boldsymbol{\mathcal{T}}$ has rank $n + 2$ and for any choice of scale, it is identified with the direct sum

$$\boldsymbol{\mathcal{T}} = \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$$

whose components change under the conformal rescaling as

$$\begin{pmatrix} \widehat{\sigma} \\ \widehat{\mu}^a \\ \widehat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a \sigma \\ \rho - \Upsilon_c \mu^c - \frac{1}{2}\Upsilon_c \Upsilon^c \sigma \end{pmatrix}. \tag{6}$$

Here Υ_a is the 1-form corresponding to the change of scale as before. Note that the projecting (or primary) slot is the top one. The bundle $\boldsymbol{\mathcal{T}}$ is endowed with the *standard tractor connection* $\boldsymbol{\nabla}$, the linear connection that is given by

$$\boldsymbol{\nabla}_a \begin{pmatrix} \sigma \\ \mu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu^b + \delta_a{}^b \rho + \mathsf{P}_a{}^b \sigma \\ \nabla_a \rho - \mathsf{P}_{ac} \mu^c \end{pmatrix}.$$

It follows this definition is indeed conformally invariant. The bundle $\boldsymbol{\mathcal{T}}$ also carries the *standard tractor metric*, the bundle metric of signature $(p+1, q+1)$ that is schematically represented as

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

i.e., for any sections $\boldsymbol{U} = (\sigma, \mu^a, \rho)$ and $\boldsymbol{V} = (\tau, \nu^a, \pi)$ of $\boldsymbol{\mathcal{T}}$,

$$\boldsymbol{U} \cdot \boldsymbol{V} = \mu_a \nu^a + \sigma\pi + \rho\tau.$$

It follows the standard tractor metric is parallel with respect to the standard tractor connection.

2. Relative conformal invariants

Throughout this section we consider conformal structures of positive definite signature. The discussion for space- and time-like curves in indefinite signature shows only minor differences, the null case is more involved, see section 5. In the first two subsections we only slightly extend the setting of [2, section 2.8]. Then we introduce a natural family of relative conformal invariants and the notion of conformal arc-length parameter of curve.

2.1. Canonical lift and initial relations

In the positive definite signature, any vector is space-like. Hence, for a regular smooth curve $\Gamma \subset M$ with a fixed parametrization t, the density (3) becomes $u = \sqrt{U_c U^c}$ and it is nowhere vanishing. This provides a lift of Γ to the standard tractor bundle $\boldsymbol{\mathcal{T}}$ that is given by

$$\boldsymbol{T} := \begin{pmatrix} 0 \\ 0 \\ u^{-1} \end{pmatrix}. \tag{7}$$

Along the curve we use the notation $\frac{d}{dt} := U^c \boldsymbol{\nabla}_c$ and

$$\boldsymbol{U} := \tfrac{d}{dt}\boldsymbol{T}, \qquad \boldsymbol{U}' := \tfrac{d^2}{dt^2}\boldsymbol{T}, \qquad \dots, \qquad \boldsymbol{U}^{(i)} := \tfrac{d^{i+1}}{dt^{i+1}}\boldsymbol{T} \tag{8}$$

for the derived tractors. The notation is chosen so that the highest order term in the middle slot of $\boldsymbol{U}^{(i)}$ is a multiple of $U^{(i)a}$. Note that the tractor $\boldsymbol{U}^{(i)}$ has order $i+2$ (with respect to Γ). By construction, both the tractor lift $\boldsymbol{T}$ and all tractors in (8) are conformally invariant objects. Explicitly, the first two derived tractors are

$$\boldsymbol{U} = \begin{pmatrix} 0 \\ u^{-1} U^a \\ -u^{-3} U_c U'^{\,c} \end{pmatrix}, \tag{9}$$

$$\boldsymbol{U}' = \begin{pmatrix} -u \\ u^{-1} U'^{\,a} - 2u^{-3} U_c U'^{\,c} U^a \\ -u^{-3} U_c U''^{\,c} - u^{-3} U'_c U'^{\,c} + 3u^{-5} (U_c U'^{\,c})^2 - u^{-1} \mathsf{P}_{cd} U^c U^d \end{pmatrix}. \tag{10}$$

The tractors $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}'$ are linearly independent, and satisfy

$$\begin{aligned} \boldsymbol{T} \cdot \boldsymbol{T} = 0, \qquad \boldsymbol{T} \cdot \boldsymbol{U} &= 0, \qquad \boldsymbol{T} \cdot \boldsymbol{U}' = -1, \\ \boldsymbol{U} \cdot \boldsymbol{U} &= 1, \qquad \boldsymbol{U} \cdot \boldsymbol{U}' = 0. \end{aligned} \tag{11}$$

The first nontrivial identity is

$$\boldsymbol{U}' \cdot \boldsymbol{U}' = 3u^{-2}U_c'U'^{\,c} + 2u^{-2}U_cU''^{\,c} - 6u^{-4}(U_cU'^{\,c})^2 + 2\mathsf{P}_{cd}U^cU^d. \tag{12}$$

In order to simplify expressions, we will use the following notation

$$\alpha := \boldsymbol{U}' \cdot \boldsymbol{U}', \qquad \beta := \boldsymbol{U}'' \cdot \boldsymbol{U}'', \qquad \gamma := \boldsymbol{U}''' \cdot \boldsymbol{U}''', \qquad \text{etc.}$$

Initial relations above and their consequences may be schematically indicated by the Gram matrix which, for the sequence $(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \boldsymbol{U}'', \boldsymbol{U}''')$, has the form

$$\begin{pmatrix} 0 & 0 & -1 & 0 & \alpha \\ 0 & 1 & 0 & -\alpha & -\frac{3}{2}\alpha' \\ -1 & 0 & \alpha & \frac{1}{2}\alpha' & \frac{1}{2}\alpha'' - \beta \\ 0 & -\alpha & \frac{1}{2}\alpha' & \beta & \frac{1}{2}\beta' \\ \alpha & -\frac{3}{2}\alpha' & \frac{1}{2}\alpha'' - \beta & \frac{1}{2}\beta' & \gamma \end{pmatrix}. \tag{13}$$

Clearly, the generating rule may be written as

$$\boldsymbol{U}^{(i)} \cdot \boldsymbol{U}^{(i+j+1)} = \big(\boldsymbol{U}^{(i)} \cdot \boldsymbol{U}^{(i+j)}\big)' - \boldsymbol{U}^{(i+1)} \cdot \boldsymbol{U}^{(i+j)}.$$

Later we will need some details on the next derived tractor $\boldsymbol{U}''$. As a consequence of $\boldsymbol{T} \cdot \boldsymbol{U}'' = 0$, the projecting slot of $\boldsymbol{U}''$ must vanish. With the help of (12), the middle slot of $\boldsymbol{U}''$ may be expressed as

$$u^{-1}U''^{\,a} - 3u^{-3}U_cU'^{\,c}\,U'^{\,a} + \big(\tfrac{3}{2}u^{-3}U_c'U'^{\,c} - \tfrac{3}{2}u^{-1}(\boldsymbol{U}' \cdot \boldsymbol{U}') + 2u^{-1}U^cU^d\mathsf{P}_{cd}\big)\,U^a - uU^c\mathsf{P}_c{}^a.$$

2.2. Reparametrizations

Let $\tilde{t} = g(t)$ be a reparametrization of the curve Γ. All objects related to the new parameter $\tilde{t}$ will be denoted by tildes in accord with $\frac{d}{d\tilde{t}} = g'^{\,-1}\frac{d}{dt}$, where $g' = \frac{dg}{dt}$. In particular, $\widetilde{U}^a = g'^{\,-1}U^a$, $\widetilde{u} = g'^{\,-1}u$ and $\widetilde{\boldsymbol{T}} = g'\boldsymbol{T}$. Using just the chain rule and the Leibniz rule, one easily verifies that

$$\begin{aligned} \widetilde{\boldsymbol{U}} &= \boldsymbol{U} + g'^{\,-1}g''\boldsymbol{T}, \\ \widetilde{\boldsymbol{U}}' &= g'^{\,-1}\boldsymbol{U}' + g'^{\,-2}g''\boldsymbol{U} + (g'^{\,-2}g''' - g'^{\,-3}g''^{\,2})\boldsymbol{T}, \\ \widetilde{\boldsymbol{U}}'' &= g'^{\,-2}\boldsymbol{U}'' + 2g'^{\,-2}\mathcal{S}(g)\boldsymbol{U} + g'^{\,-2}\mathcal{S}(g)'\,\boldsymbol{T}, \end{aligned} \tag{14}$$

where $\mathcal{S}$ is the *Schwarzian derivative*,

$$\mathcal{S}(g) := \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 = (\ln g')'' - \frac{1}{2}\,(\ln g')'^{\,2}.$$

According to the starting relations (11), we have

$$\begin{aligned} \widetilde{\boldsymbol{U}}' \cdot \widetilde{\boldsymbol{U}}' &= g'^{\,-2}\left(\boldsymbol{U}' \cdot \boldsymbol{U}' - 2\mathcal{S}(g)\right), \\ \widetilde{\boldsymbol{U}}'' \cdot \widetilde{\boldsymbol{U}}'' &= g'^{\,-4}\left(\boldsymbol{U}'' \cdot \boldsymbol{U}'' - 4\mathcal{S}(g)\boldsymbol{U}' \cdot \boldsymbol{U}' + 4\mathcal{S}(g)^2\right). \end{aligned} \tag{15}$$

Hence vanishing of $\boldsymbol{U}' \cdot \boldsymbol{U}'$ determines a natural projective structure on any curve, cf. [2, Proposition 2.11]:

Proposition 2.1 *([2]). The equation $\boldsymbol{U}' \cdot \boldsymbol{U}' = 0$, regarded as a condition on the parametrization of a curve, determines a preferred family of parametrizations with freedom given by the projective group of the line.*

Any parameter from this family will be called *projective*.

From (15) it further follows that

$$\widetilde{\boldsymbol{U}}'' \cdot \widetilde{\boldsymbol{U}}'' - (\widetilde{\boldsymbol{U}}' \cdot \widetilde{\boldsymbol{U}}')^2 = g'^{\,-4} \left(\boldsymbol{U}'' \cdot \boldsymbol{U}'' - (\boldsymbol{U}' \cdot \boldsymbol{U}')^2\right).$$

Hence the function

$$\Phi := \boldsymbol{U}'' \cdot \boldsymbol{U}'' - (\boldsymbol{U}' \cdot \boldsymbol{U}')^2 = \beta - \alpha^2 \tag{16}$$

is a relative conformal invariant of the curve.

2.3. Relative conformal invariants

In general, a *relative conformal invariant of weight k* of the curve Γ is a conformally invariant function $I : \Gamma \to \mathbb{R}$ that transforms under a reparametrization $\tilde{t} = g(t)$ of the curve as

$$\widetilde{I} = g'^{\,-k} I.$$

In particular, vanishing, respectively nonvanishing, of any relative invariant is independent on reparametrizations. Conformal invariants of weight 0 are the *absolute* invariants.

Let us denote

$$\Delta_i := \det\left(\mathrm{Gram}\left(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \dots, \boldsymbol{U}^{(i-2)}\right)\right), \tag{17}$$

the determinant of the Gram matrix corresponding to the first i tractors from the derived sequence $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \dots$. From (11) it is obvious that $\Delta_1 = \Delta_2 = 0$ and $\Delta_3 = -1$ independently of the curve and its parametrization. These three determinants are thus absolute, but trivial conformal invariants. The first nontrivial invariant is Δ_4. From (13) it follows that

$$\Delta_4 = -\beta + \alpha^2 = -\Phi. \tag{18}$$

In general, we have

Lemma 2.2. *For $i = 4, \dots, n+2$, the Gram determinant Δ_i is a relative conformal invariant of weight $i(i-3)$.*

Proof. As a generalization of (14) we have

$$\widetilde{\boldsymbol{U}}^{(j)} = g'^{\,-j} \boldsymbol{U}^{(j)} \mod \langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \dots, \boldsymbol{U}^{(j-1)} \rangle,$$

for any $j = 1, \dots, n$. From this and properties of the determinant it follows that (17) changes under the reparametrization as

$$\widetilde{\Delta}_i = g'^{\,-2\cdot 2} \cdots g'^{\,-2(i-2)} \Delta_i = g'^{\,-i(i-3)} \Delta_i,$$

for any $i = 4, \dots, n+2$. □

For $i = 4, \dots, n+2$, the tractor metric restricted to $\langle \boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}^{(i-2)} \rangle$ is nondegenerate, thus vanishing of Δ_i is equivalent to the fact that the determining tractors are linearly dependent. In particular, for $i \geq 4$, vanishing of Δ_i implies vanishing of Δ_{i+1}. Note also that the weight of any Δ_i is even, thus its sign is independent of all reparametrizations.

2.4. Conformal arc-length

By (16), respectively (18), we defined a nontrivial relative conformal invariant of the lowest possible weight and order. It is actually nonnegative:

Lemma 2.3. $\Phi \geq 0$.

Proof. For a parametrization belonging to the projective family of Proposition 2.1 we have $\Phi = \boldsymbol{U}'' \cdot \boldsymbol{U}''$. Since the top slot of $\boldsymbol{U}''$ vanishes, the previous expression equals to the norm squared of the middle slot of $\boldsymbol{U}''$ and so it is nonnegative. Since the weight of Φ is even, it is a nonnegative function independently of the parametrization of the curve. □

Suppose that Γ is a curve with nowhere vanishing Φ. Then

$$ds := \sqrt[4]{\Phi(t)}\, dt$$

is a well-defined conformally invariant 1-form along the curve whose integration yields a distinguished parametrization of the curve; it is given uniquely up to an additive constant. Such parameter is called the *conformal arc-length*. Note that if s is the conformal arc-length then $\Phi(s) = 1$.

It will be clear from later alternative expressions that this is the same distinguished parameter as one finds in literature, e.g. in [9] or [5]. In the later reference, one also finds a notion of *vertex*, which is the point of curve where $\Phi = 0$. The vertices of curves are clearly invariant under conformal transformations. Generic curves are vertex-free, the opposite extreme is discussed in the next section.

Remark 2.4. For space- and time-like curves in the case of general indefinite signature, the relative conformal invariant Φ defined by (16), respectively (18), is not necessarily nonnegative as in Lemma 2.3. It may even happen that it vanishes although the corresponding tractors $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \boldsymbol{U}''$ are linearly independent. The notion of vertex in such cases would rather be defined by the linear dependence of these tractors than by vanishing of Φ, cf. Remark 3.4(2).

3. Conformal circles and conserved quantities

We continue the exposition with the assumption of positive definite signature. The only difference in indefinite signature concerns the notion of conformal circles that are defined just for the space- or time-like directions. Reasonable analogies for null directions are discussed in subsection 5.3.

3.1. Conformal circles

In this subsection we consider the curves for which Φ vanishes identically, i.e. the curves consisting only of vertices. The conformal arc-length is not defined for such curves and they are excluded from the discussion in the main part of this paper as the most degenerate cases. However, these curves form a very distinguished family of curves that coincide with the so-called *conformal circles*, the curves which are in some sense the closest conformal analogues of usual geodesics on Riemannian manifolds. In the flat case, they are just the circles, respectively straight lines, i.e. the conformal images of Euclidean geodesics.

Conformal circles are well known and studied (under various nicknames[1]) in the literature. A quick survey with an explanation of the relationship of the definition below to distinguished curves of Cartan's conformal connection can be found in [13, section 5.1]. Conformal circles on a general conformal manifold can also be related to ordinary circles in the flat model via the notion of Cartan's development of curves.

Here we adapt the notation of [1], where *conformal circles* are defined as the solutions to the system of conformally invariant third order ODE's

$$U''^{\,a} = 3u^{-2}U_cU'^{\,c}\,U'^{\,a} - \tfrac{3}{2}u^{-2}U'_cU'^{\,c}\,U^a + u^2U^c\mathsf{P}_c{}^a - 2U^cU^d\mathsf{P}_{cd}\,U^a. \tag{19}$$

Contracting, respectively skew symmetrizing (19), with U^a yields the following pair of equations equivalent to (19):

$$U_aU''^{\,a} = 3u^{-2}(U_cU'^{\,c})^2 - \tfrac{3}{2}U'_cU'^{\,c} - u^2U^cU^d\mathsf{P}_{cd}, \tag{20}$$

$$U_{[a}U''_{b]} = 3u^{-2}U_cU'^{\,c}\,U_{[a}U'_{b]} + u^2U^c\mathsf{P}_{c[b}U_{a]}. \tag{21}$$

From (5) we know that, for any curve with an arbitrary parametrization, a metric in the conformal class may be chosen so that it is an affinely parametrized geodesic of the corresponding Levi-Civita connection, i.e. $U'^{\,a} = 0$. It is then easy to see from (20) and (21) that

Proposition 3.1 *([1]). The curve is a conformal circle if and only if there is a metric in the conformal class such that the curve is an affinely parametrized geodesic and $U^c\mathsf{P}_{ca} = 0$.*

From the explicit expressions in subsection 2.1 we may read the following tractorial interpretations of the equations above: The equation (19) is equivalent to vanishing of the middle slot of the tractor $\boldsymbol{U}'' + \frac{3}{2}(\boldsymbol{U}' \cdot \boldsymbol{U}')\boldsymbol{U}$ (since the primary slot is zero, this is indeed a conformally invariant condition). This implies that the middle slots of tractors $\boldsymbol{U}''$ and $\boldsymbol{U}$ are collinear, which is just the condition (21). In particular, this condition is independent of any reparametrization, i.e. solutions to (21) are the conformal circles parametrized by an arbitrary parameter. The equation (20) is equivalent to $\boldsymbol{U}'\cdot\boldsymbol{U}' = 0$. In particular, solutions to (20) correspond to the projective parameters of the curve. Considering further the injecting slot of the tractor $\boldsymbol{U}''$, it is shown in [2, Proposition 2.12] that

Proposition 3.2 *([2]). The curve is a projectively parametrized conformal circle if and only if it obeys*

$$\boldsymbol{U}' \cdot \boldsymbol{U}' = 0 \qquad \textit{and} \qquad \boldsymbol{U}'' = 0. \tag{22}$$

We may easily generalize this statement as follows:

Proposition 3.3. *The following conditions are equivalent:*

(a) *the curve is a conformal circle (with and arbitrary parametrization),*
(b) *all its points are vertices, i.e.*

$$\Phi = \boldsymbol{U}'' \cdot \boldsymbol{U}'' - (\boldsymbol{U}' \cdot \boldsymbol{U}')^2 = 0,$$

(c) *the rank 3 subbundle $\langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle \subset \boldsymbol{\mathcal{T}}$ is parallel along the curve.*

[1] E.g. they are called conformal null curves in [9] or conformal geodesics in [10] and [13]. Note that the conformal geodesics studied in [8] or [15] form a different, more general, class of curves.

Proof. From subsection 2.1 we know that the tractors $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}'$ are linearly independent for any curve and its parametrization. From subsection 2.3 we know that $\Phi = 0$ is equivalent to the fact that the tractors $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \boldsymbol{U}''$ are linearly dependent, i.e. $\boldsymbol{U}''$ belongs to the subbundle $\langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle \subset \boldsymbol{\mathcal{T}}$ which is therefore parallel. Thus (b) and (c) are equivalent.

Among tractors $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \boldsymbol{U}''$, only $\boldsymbol{U}'$ has nonzero projecting slot. Hence the previous condition means that $\boldsymbol{U}''$ is a linear combination of $\boldsymbol{T}$ and $\boldsymbol{U}$. Since the middle slot of $\boldsymbol{T}$ vanishes, the previous is equivalent to the middle slots of $\boldsymbol{U}''$ and $\boldsymbol{U}$ being collinear, which is just the condition (21). Thus (c) and (a) are equivalent. □

Remarks 3.4.

(1) Note the two conditions in (22) are not independent: if $\boldsymbol{U}'' = 0$ then $\boldsymbol{U}' \cdot \boldsymbol{U}' = 0$.
(2) Conformal circles in the case of indefinite signature exist only for space- or time-like directions. They have all of the above stated properties except the condition (b) of Proposition 3.3, which is no more equivalent to the others, cf. Remark 2.4.

3.2. Conserved quantities

As an application of the current approach we describe some conserved quantities (or first integrals) along conformal circles for some specific conformal structures. Here we consider conformal structures admitting almost Einstein scales or conformal Killing fields with some additional property. The description of the conserved quantities is easily given due to the tractorial characterization of these additional data, which can be found e.g. in [11].

An *almost Einstein scale* is a section $\sigma \in \mathcal{E}[1]$ satisfying the conformally invariant condition that the trace-free part of $\nabla_a \nabla_b \sigma + \mathsf{P}_{ab}\sigma \in \mathcal{E}_{ab}[1]$ vanishes. Then σ is nonvanishing on an open dense set where the metric $g_{ab} = \sigma^{-2}\mathbf{g}_{ab}$ is Einstein. The tractorial characterization of this fact is that the tractor field $L^{\boldsymbol{\mathcal{T}}}(\sigma) \in \boldsymbol{\mathcal{T}}$ is parallel, where $L^{\boldsymbol{\mathcal{T}}} : \mathcal{E}[1] \to \boldsymbol{\mathcal{T}}$ denotes the BGG splitting operator.

Proposition 3.5. *Let Γ be a conformal circle with a projective parameter t. Assume the conformal manifold M admits an almost Einstein scale $\sigma \in \mathcal{E}[1]$ and let $\boldsymbol{S} = L^{\boldsymbol{\mathcal{T}}}(\sigma) \in \boldsymbol{\mathcal{T}}$ be the corresponding parallel tractor. Then the function $\mathfrak{s} := \boldsymbol{U}' \cdot \boldsymbol{S}$ is a conserved quantity along Γ, i.e. $\frac{d}{dt}\mathfrak{s} = 0$.*

Proof. A projectively parametrized circle satisfies $\boldsymbol{U}'' = 0$, the tractor $\boldsymbol{S}$ corresponding to an almost Einstein scale satisfies $\boldsymbol{S}' = 0$. Thus $\frac{d}{dt}(\boldsymbol{U}' \cdot \boldsymbol{S}) = 0$. □

A *conformal Killing field* $k^a \in \mathcal{E}^a$ is an infinitesimal symmetry of the conformal structure. This is equivalent to the vanishing of the trace-free part of $\nabla_{(a}k_{b)} \in \mathcal{E}_{(ab)}[2]$. The tractorial characterization of this fact is that the tractor field $\boldsymbol{K} := L^{\Lambda^2\boldsymbol{\mathcal{T}}}(k^a) \in \Lambda^2\boldsymbol{\mathcal{T}}$ satisfies

$$\boldsymbol{\nabla}_a \boldsymbol{K} = k^c \boldsymbol{\Omega}_{ca}, \tag{23}$$

where $L^{\Lambda^2\boldsymbol{\mathcal{T}}} : \mathcal{E}^a \to \Lambda^2\boldsymbol{\mathcal{T}}$ denotes the BGG splitting operator and $\boldsymbol{\Omega}_{cd} \in \mathcal{E}_{[cd]} \otimes \Lambda^2\boldsymbol{\mathcal{T}}$ is the curvature of the tractor connection $\boldsymbol{\nabla}$. Further, along the given curve, we introduce the 1-form

$$\ell_a := U^b \boldsymbol{\Omega}_{ab}(\boldsymbol{U}, \boldsymbol{U}'). \tag{24}$$

Proposition 3.6. *Let Γ be a conformal circle with a projective parameter t. Assume the conformal manifold M admits a conformal Killing field $k^a \in \mathcal{E}^a$ satisfying $k^a\ell_a = 0$, and let $\boldsymbol{K} := L^{\Lambda^2\boldsymbol{\mathcal{T}}}(k^a) \in \Lambda^2\boldsymbol{\mathcal{T}}$ be the corresponding tractor field. Then the function $\mathfrak{k} := \boldsymbol{K}(\boldsymbol{U}, \boldsymbol{U}')$ is a conserved quantity along Γ, i.e. $\frac{d}{dt}\mathfrak{k} = 0$.*

Proof. A projectively parametrized circle satisfies $\boldsymbol{U}'' = 0$. The tractor $\boldsymbol{K}$ corresponding to k^a satisfies (23) and hence $(\frac{d}{dt}\boldsymbol{K})(\boldsymbol{U}, \boldsymbol{U}') = k^a \ell_a = 0$. Also, $\boldsymbol{K}$ is skew, thus $\frac{d}{dt}(\boldsymbol{K}(\boldsymbol{U}, \boldsymbol{U}')) = 0$. □

Note that the assumption $k^a \ell_a = 0$ in the Proposition is automatically satisfied for *normal* conformal Killing fields which are characterized by the vanishing of (23).

Remarks 3.7.

(1) It is somewhat tedious to expand tractorial formulas for the conserved quantities $\mathfrak{s}$ and $\mathfrak{k}$. To proceed, one needs the respective splitting operators and some manipulation. From the explicit expressions it in particular follows that the considered quantities are nontrivial. All that can be found already in [19]. Note also that both $\mathfrak{s}$ and $\mathfrak{k}$ are quantities of order 2, while the conformal circle equation is of order 3. Further related and interesting results can be found in recent article [12].

(2) Expanding (24) yields

$$\ell_a = U^b \mathsf{W}_{abcd} U^c U'^d - 2u^2 U^b \nabla_{[a} \mathsf{P}_{b]c} U^c,$$

where $\mathsf{W}_{ab}{}^c{}_d$ is the conformal Weyl tensor. Remarkably, the condition $\ell_a = 0$ plays a role in the variational approach to conformal circles, see [1, p. 218].

4. Absolute conformal invariants

For a generic curve, the relative conformal invariants Δ_i from section 2 are all nontrivial, provided that $4 \leq i \leq n+2$. One may therefore easily build a galaxy of absolute conformal invariants. In this section we construct a minimal set of such invariants, which is done in a natural if not canonical way. In order to make the construction complete, the assumption of positive definite signature is important here. In particular, the standard tractor metric has signature $(n+1, 1)$.

4.1. Tractor Frenet formulas and conformal curvatures

Let Γ be a generic curve, let s be its conformal arc-length parameter and let $\boldsymbol{T}, \boldsymbol{U} = \frac{d}{ds}\boldsymbol{T}, \boldsymbol{U}' = \frac{d}{ds}\boldsymbol{U}, \dots$ be the corresponding tractors as above. The restriction of the tractor metric to the subbundle $\langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle$ is nondegenerate and has signature $(2, 1)$. Thus its orthogonal subbundle in $\boldsymbol{\mathcal{T}}$ is complementary and has positive definite signature. We are going to transform the initial tractor frame $(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \dots)$ into a natural pseudo-orthonormal one. Firstly, under the transformation

$$\boldsymbol{U}_0 := \boldsymbol{T}, \qquad \boldsymbol{U}_1 := \boldsymbol{U}, \qquad \boldsymbol{U}_2 := -\boldsymbol{U}' - \tfrac{1}{2}(\boldsymbol{U}' \cdot \boldsymbol{U}')\,\boldsymbol{T}, \tag{25}$$

the initial relations (11) transform to

$$\begin{aligned} \boldsymbol{U}_0 \cdot \boldsymbol{U}_0 = 0, \qquad & \boldsymbol{U}_0 \cdot \boldsymbol{U}_1 = 0, \qquad && \boldsymbol{U}_0 \cdot \boldsymbol{U}_2 = 1, \\ & \boldsymbol{U}_1 \cdot \boldsymbol{U}_1 = 1, && \boldsymbol{U}_1 \cdot \boldsymbol{U}_2 = 0, \\ & && \boldsymbol{U}_2 \cdot \boldsymbol{U}_2 = 0. \end{aligned} \tag{26}$$

Secondly, by the standard orthonormalization process we may complete this triple to a tractor frame

$$(\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{U}_2; \boldsymbol{U}_3, \dots, \boldsymbol{U}_{n+1}) \tag{27}$$

such that $\boldsymbol{U}_i$ belongs to the span $\langle \boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}^{(i-1)} \rangle$, for all admissible i, and the corresponding Gram matrix is

$$\left(\begin{array}{ccc|ccc} & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right). \tag{28}$$

More concretely, for any $i = 3, \dots, n$, the frame tractors are

$$\begin{aligned} \boldsymbol{U}_i &= \boldsymbol{V}_i / \sqrt{\boldsymbol{V}_i \cdot \boldsymbol{V}_i}, \qquad \text{where} \\ \boldsymbol{V}_i &= \boldsymbol{U}^{(i-1)} - (\boldsymbol{U}^{(i-1)} \cdot \boldsymbol{U}_{i-1})\, \boldsymbol{U}_{i-1} - \cdots \\ &\quad \cdots - (\boldsymbol{U}^{(i-1)} \cdot \boldsymbol{U}_0)\, \boldsymbol{U}_2 - (\boldsymbol{U}^{(i-1)} \cdot \boldsymbol{U}_1)\, \boldsymbol{U}_1 - (\boldsymbol{U}^{(i-1)} \cdot \boldsymbol{U}_2)\, \boldsymbol{U}_0. \end{aligned} \tag{29}$$

The last tractor $\boldsymbol{U}_{n+1}$ is determined by the orthocomplement in $\boldsymbol{\mathcal{T}}$ to the codimension one subbundle $\langle \boldsymbol{T}, \dots, \boldsymbol{U}^{(n-1)} \rangle = \langle \boldsymbol{U}_0, \dots, \boldsymbol{U}_n \rangle$ up to orientation. We do not need to fix the orientation now, it is done only later. The frame (27) is called the *tractor Frenet frame* associated to the curve Γ.

Now, differentiating the constituents of the tractor Frenet frame with respect to the conformal arc-length s and expressing the result within that frame yields Frenet-like identities. As in the classical situation, it follows that the coefficients of that system determine the generating set of absolute invariants of the curve. The pseudo-orthonormality of the tractor Frenet frame implies a lot of symmetries among these coefficients.

Just from (25) we have

$$\boldsymbol{U}_0' = \boldsymbol{U}_1, \tag{30}$$

$$\boldsymbol{U}_1' = K_1 \boldsymbol{U}_0 - \boldsymbol{U}_2, \qquad \text{where} \qquad K_1 := -\tfrac{1}{2} \boldsymbol{U}' \cdot \boldsymbol{U}' \tag{31}$$

is the first nontrivial coefficient, the *first conformal curvature* of the curve. In general, for any admissible i, the derivative $\boldsymbol{U}_i'$ is a linear combination of tractors $\boldsymbol{U}_0, \boldsymbol{U}_1, \dots, \boldsymbol{U}_{i+1}$, namely,

$$\boldsymbol{U}_i' = (\boldsymbol{U}_i' \cdot \boldsymbol{U}_2)\, \boldsymbol{U}_0 + (\boldsymbol{U}_i' \cdot \boldsymbol{U}_1)\, \boldsymbol{U}_1 + (\boldsymbol{U}_i' \cdot \boldsymbol{U}_0)\, \boldsymbol{U}_2 + (\boldsymbol{U}_i' \cdot \boldsymbol{U}_3)\, \boldsymbol{U}_3 + \cdots + (\boldsymbol{U}_i' \cdot \boldsymbol{U}_{i+1})\, \boldsymbol{U}_{i+1},$$

where the coefficients satisfy $\boldsymbol{U}_i' \cdot \boldsymbol{U}_j = -\boldsymbol{U}_i \cdot \boldsymbol{U}_j'$ and they have to vanish if $i = j$ or $j \geq i + 2$. In particular, from (30) we may read $\boldsymbol{U}_0' \cdot \boldsymbol{U}_1 = 1$ and $\boldsymbol{U}_0' \cdot \boldsymbol{U}_j = 0$, for $j = 0, 2, \dots, n+1$. Similarly, from (31) we further read $\boldsymbol{U}_1' \cdot \boldsymbol{U}_2 = K_1$ and $\boldsymbol{U}_1' \cdot \boldsymbol{U}_j = 0$, for $j = 1, 3, \dots, n+1$. Substituting these facts into the previous display, for $i = 2$, we see that the only unknown coefficient is the one of $\boldsymbol{U}_3$, i.e. $\boldsymbol{U}_2' \cdot \boldsymbol{U}_3$. It however follows that this coefficient is constant so that we have

Lemma 4.1.

$$\boldsymbol{U}_2' = -K_1 \boldsymbol{U}_1 - \boldsymbol{U}_3. \tag{32}$$

Proof. From (29), according to (25) and (11), we obtain

$$\boldsymbol{V}_3 = \boldsymbol{U}'' - (\boldsymbol{U}'' \cdot \boldsymbol{U})\, \boldsymbol{U} + (\boldsymbol{U}'' \cdot \boldsymbol{U}')\, \boldsymbol{T}.$$

Using $\boldsymbol{U}'' \cdot \boldsymbol{U} = -\boldsymbol{U}' \cdot \boldsymbol{U}'$, which is a consequence of $\boldsymbol{U} \cdot \boldsymbol{U}' = 0$, we obtain

$$\boldsymbol{V}_3 \cdot \boldsymbol{V}_3 = \boldsymbol{U}'' \cdot \boldsymbol{U}'' - (\boldsymbol{U}' \cdot \boldsymbol{U}')^2 = \Phi = 1$$

and thus $\boldsymbol{U}_3 = \boldsymbol{V}_3$. From (25) and (29) we know that $\boldsymbol{U}_2'$ equals to $-\boldsymbol{U}''$ mod $\langle \boldsymbol{T}, \boldsymbol{U} \rangle$, respectively $-\boldsymbol{V}_3$ mod $\langle \boldsymbol{U}_0, \boldsymbol{U}_1 \rangle$. Hence $\boldsymbol{U}_2' \cdot \boldsymbol{U}_3 = -\boldsymbol{V}_3 \cdot \boldsymbol{U}_3 = -1$. $\square$

Following the same ideas as before, one gradually and easily obtains

$$\begin{aligned} &\boldsymbol{U}_3' = \boldsymbol{U}_0 + K_2 \boldsymbol{U}_4, \\ &\boldsymbol{U}_i' = -K_{i-2} \boldsymbol{U}_{i-1} + K_{i-1} \boldsymbol{U}_{i+1}, \qquad \text{for } i = 4, \dots, n, \\ &\boldsymbol{U}_{n+1}' = -K_{n-1} \boldsymbol{U}_n, \end{aligned} \tag{33}$$

where $K_2, \dots, K_{n-1}$ are the *higher conformal curvatures* of the curve. The bunch of equations (30), (31), (32) and (33) form the *tractor Frenet equations* associated to the curve. Schematically they may be written as

$$\begin{pmatrix} \boldsymbol{U}_0' \\ \boldsymbol{U}_1' \\ \boldsymbol{U}_2' \\ \hline \boldsymbol{U}_3' \\ \boldsymbol{U}_4' \\ \vdots \\ \boldsymbol{U}_{n+1}' \end{pmatrix} = \left(\begin{array}{ccc|ccccc} & 1 & & & & & & \\ K_1 & & -1 & & & & & \\ & -K_1 & & -1 & & & & \\ \hline 1 & & & & K_2 & & & \\ & & & -K_2 & & & & \\ & & & & \ddots & & \ddots & \\ & & & & & & & K_{n-1} \\ & & & & & & -K_{n-1} & \end{array}\right) \cdot \begin{pmatrix} \boldsymbol{U}_0 \\ \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \\ \hline \boldsymbol{U}_3 \\ \boldsymbol{U}_4 \\ \vdots \\ \boldsymbol{U}_{n+1} \end{pmatrix}. \tag{34}$$

Altogether, we summarize as follows:

Proposition 4.2. *The system of tractor Frenet equations determines a generating set of absolute conformal invariants of the curve. On a conformal Riemannian manifold of dimension n, it consists of $n-1$ conformal curvatures that are expressed, with respect to the conformal arc-length parametrization, as*

$$K_i = \begin{cases} \boldsymbol{U}_1' \cdot \boldsymbol{U}_2, & \textit{for } i = 1, \\ \boldsymbol{U}_{i+1}' \cdot \boldsymbol{U}_{i+2}, & \textit{for } i = 2, \dots, n-1. \end{cases} \tag{35}$$

In (31) we have an alternative expression of the first conformal curvature in terms of initial tractors. Thus, as a consequence of Proposition 2.1 and the ensuing definition, we obtain an interpretation of that invariant:

Proposition 4.3. *The conformal arc-length parameter belongs to the projective family of parameters if and only if $K_1 = 0$.*

Remarks 4.4.

(1) The matrix in (34) is skew with respect to the inner product corresponding to (28). A change of basis leads to a different matrix realization, e.g., just a permutation of the frame tractors leads to following rewriting of (34):

$$
\begin{pmatrix} \boldsymbol{U}_0' \\ \hline \boldsymbol{U}_1' \\ \boldsymbol{U}_3' \\ \vdots \\ \boldsymbol{U}_n' \\ \boldsymbol{U}_{n+1}' \\ \hline \boldsymbol{U}_2' \end{pmatrix}
=
\left(\begin{array}{c|ccccc|c}
 & 1 & & & & & \\ \hline
K_1 & & & & & & -1 \\
1 & & & K_2 & & & \\
 & & -K_2 & & & & \\
 & & & \ddots & & \ddots & \\
 & & & & & K_{n-1} & \\
 & & & & -K_{n-1} & & \\ \hline
 & -K_1 & -1 & & & &
\end{array}\right)
\cdot
\begin{pmatrix} \boldsymbol{U}_0 \\ \hline \boldsymbol{U}_1 \\ \boldsymbol{U}_3 \\ \vdots \\ \boldsymbol{U}_n \\ \boldsymbol{U}_{n+1} \\ \hline \boldsymbol{U}_2 \end{pmatrix}.
$$

This block decomposition reflects the standard grading of the Lie algebra $\mathfrak{so}(n+1,1)$ with the parabolic subalgebra corresponding to the block lower triangular matrices. This choice is very close to the ones in [20] or [5].

(2) For generic curves, the $(n+2)$-tuple of derived tractors is linearly independent. For specific curves, this is not the case and we cannot build the full tractor Frenet frame and all tractor Frenet formulas. One may however mimic the previous procedure as long as the tractors are independent:
If, say, the tractors $\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \dots, \boldsymbol{U}^{(i)}$ are linearly independent and $\boldsymbol{U}^{(i+1)}$ belongs to their span, then they form a parallel subbundle in $\boldsymbol{\mathcal{T}}$ of rank $i+2$ with the corresponding orthonormal tractors $\boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{U}_2, \dots, \boldsymbol{U}_{i+1}$. This yields a part of the tractor Frenet formulas above, in which the first $i-1$ conformal curvatures occur; the remaining ones may be considered to be zero. (In this vein, conformal circles may be considered as the most degenerate curves whose all conformal curvatures vanish, cf. Proposition 3.3.) In the flat case, this means that the curve itself is contained in an i-sphere, respectively i-plane, i.e. in the conformal image of an i-dimensional Euclidean subspace. See also [9, section 12] for further details.

(3) For space- and time-like curves in the case of indefinite signature, we remark the following: The same transformations as in (25) lead to the relations (26) where the type of curve is reflected just in the sign of the middle element, hence also in the signature of the subbundle $\langle \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}' \rangle$ in $\boldsymbol{\mathcal{T}}$. The orthogonal complement of this subbundle in $\boldsymbol{\mathcal{T}}$ has in general indefinite signature. Thus, the tractors $\boldsymbol{V}_i$ from (29) may have any sign, including zero. If the sign is negative, we just adjust the corresponding definition of $\boldsymbol{U}_i$ and, accordingly, one sign in the bottom-right block of (28) changes. If the tractor $\boldsymbol{V}_i$ is isotropic then we cannot satisfy both the diagonal form of the bottom-right block of (28) and the condition $\boldsymbol{U}_i \in \langle \boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}^{(i-1)} \rangle$. Of course, one can always transform given tractors to a pseudo-orthonormal tractor frame, but only with some additional choices. The rest remains basically the same, only the tractor Frenet formulas may look different and some of the resulting invariants has to be related to the respective choices. (Nevertheless, it is clear that the exceptional conformal curvature K_1 and its interpretation as in Proposition 4.3 are not influenced by these issues.) These observations lead to a finer type characterization of space- and time-like curves. Its complete branching structure is manageable only in a concrete dimension and signature.

4.2. Conformal curvatures revised

In the definition above, the orientation of the last tractor from the tractor Frenet frame was a matter of choice. In what follows we assume the orientation of $\boldsymbol{U}_{n+1}$ is chosen so that it belongs to the same half-space as $\boldsymbol{U}^{(n)}$ with respect to the hyperplane $\langle \boldsymbol{U}_0, \dots, \boldsymbol{U}_n \rangle$. Thus, according to (29), we have

$$
\boldsymbol{U}^{(i)} = \boldsymbol{V}_{i+1} \mod \langle \boldsymbol{U}_0, \boldsymbol{U}_1, \dots, \boldsymbol{U}_i \rangle, \tag{36}
$$

for any $i = 3, \dots, n$. Also, from definitions and the tractor Frenet formulas, one progressively obtains

$$\boldsymbol{T} = \boldsymbol{U}_0, \qquad \boldsymbol{U} = \boldsymbol{U}_1, \qquad \boldsymbol{U}' = -\boldsymbol{U}_2 + K_1 \boldsymbol{U}_0, \qquad \boldsymbol{U}'' = \boldsymbol{U}_3 + 2K_1 \boldsymbol{U}_1 + K_1' \boldsymbol{U}_0, \qquad \text{etc.} \tag{37}$$

In particular, it holds $\boldsymbol{U}'' = \boldsymbol{U}_3 \mod \langle \boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{U}_2 \rangle$. Similarly, it follows that $\boldsymbol{U}''' = K_2 \boldsymbol{U}_4 \mod \langle \boldsymbol{U}_0, \boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3 \rangle$ and, inductively,

$$\boldsymbol{U}^{(i)} = K_2 \cdots K_{i-1} \boldsymbol{U}_{i+1} \mod \langle \boldsymbol{U}_0, \boldsymbol{U}_1, \dots, \boldsymbol{U}_i \rangle, \tag{38}$$

for $i = 3, \dots, n$. Altogether, from (36), (38) and $\boldsymbol{U}_i = \boldsymbol{V}_i / \sqrt{\boldsymbol{V}_i \cdot \boldsymbol{V}_i}$, we obtain

$$K_2 \cdots K_{i-1} = \sqrt{\boldsymbol{V}_{i+1} \cdot \boldsymbol{V}_{i+1}},$$

which leads to the following conclusion:

Proposition 4.5. *With respect to the conformal arc-length parametrization, the higher conformal curvatures can be expressed as*

$$K_i = \sqrt{\frac{\boldsymbol{V}_{i+2} \cdot \boldsymbol{V}_{i+2}}{\boldsymbol{V}_{i+1} \cdot \boldsymbol{V}_{i+1}}}, \qquad \textit{for } i = 2, \dots, n-1. \tag{39}$$

In particular, all these curvatures are positive. The first conformal curvature K_1 does not fit to this uniform description and its sign may be arbitrary, including zero.

An expansion of (39), according to (25) and (29), provides expressions of higher conformal curvatures in terms of initial tractors. Instead of exploiting this demanding procedure, we are going to use the relative conformal invariants from subsection 2.3. From the construction of the tractor Frenet frame it follows that $\Delta_{i+1} = \Delta_i (\boldsymbol{V}_i \cdot \boldsymbol{V}_i)$, for all admissible i. Since $\Delta_3 = -1$, we see that all admissible determinants are negative. Hence we can express $\boldsymbol{V}_i \cdot \boldsymbol{V}_i$ as a quotient of determinants, whose substitution into (39) yields

$$K_i = \frac{\sqrt{\Delta_{i+1} \Delta_{i+3}}}{-\Delta_{i+2}}. \tag{40}$$

Up to this point, everything has been related to the parametrization by the conformal arc-length. Now we consider an arbitrary parametrization of the curve.

Theorem 4.6. *With respect to an arbitrary parametrization of the curve, the higher conformal curvatures can be expressed as*

$$K_i = \frac{\sqrt{\Delta_{i+1} \Delta_{i+3}}}{-\Delta_{i+2} \sqrt[4]{-\Delta_4}}, \qquad \textit{for } i = 2, \dots, n-1. \tag{41}$$

Proof. According to Lemma 2.2, the weight of the right hand side of (41) is

$$\tfrac{1}{2}(i+1)(i-2) + \tfrac{1}{2}(i+3)i - (i+2)(i-1) - 1 = 0,$$

thus it defines an absolute conformal invariant. With respect to the conformal arc-length parametrization, (41) coincides with (40), hence the statement follows. $\square$

Alternatively, the statement is deducible from (40) by the reparametrization according to subsection 2.2 and Lemma 2.2, where $g' = \Phi^{\frac{1}{4}}$. That way, we may obtain also an expression of the very first conformal invariant, K_1, which does not fit to the just given description: Its expression with respect to the conformal arc-length is given in (31), which transforms under reparametrizations according to (15). The substitution of $g' = \Phi^{\frac{1}{4}}$ and a small computation reveals that

Proposition 4.7. *With respect to an arbitrary parametrization of the curve, the first conformal curvature can be expressed as*

$$\begin{aligned} K_1 &= -\tfrac{1}{2}\Phi^{-\frac{1}{2}}\left(\boldsymbol{U}' \cdot \boldsymbol{U}' - \tfrac{1}{2}(\ln\Phi)'' + \tfrac{1}{16}(\ln\Phi)'^2\right) = \\ &= -\tfrac{1}{2}\Phi^{-\frac{5}{2}}\left(\Phi^2\,\boldsymbol{U}' \cdot \boldsymbol{U}' - \tfrac{1}{2}\Phi\Phi'' + \tfrac{9}{16}\Phi'^{\,2}\right). \end{aligned} \tag{42}$$

Remarks 4.8.

(1) Not only the conformal curvatures, but the tractor Frenet frame itself can be built with respect to an arbitrary parametrization. E.g. expressions of the first three tractors (25) transform, under the reparametrization according to (14) with $g' = \Phi^{\frac{1}{4}}$, so that

$$\begin{aligned} \boldsymbol{U}_0 &= \Phi^{\frac{1}{4}}\,\boldsymbol{T}, \\ \boldsymbol{U}_1 &= \boldsymbol{U} + \tfrac{1}{4}(\ln\Phi)'\,\boldsymbol{T}, \\ \boldsymbol{U}_2 &= -\Phi^{-\frac{1}{4}}\left(\boldsymbol{U}' + \tfrac{1}{4}(\ln\Phi)'\,\boldsymbol{U} + \left(\tfrac{1}{2}\boldsymbol{U}' \cdot \boldsymbol{U}' + \tfrac{1}{32}(\ln\Phi)'^2\right)\boldsymbol{T}\right). \end{aligned} \tag{43}$$

(2) Our exceptional invariant K_1 corresponds to a similarly exceptional invariant J_{n-1} of [9, section 5] as follows. For any relative conformal invariant Q of weight 1, an appropriate combination of Q and its derivatives leads to an appropriate transformation of the new quantity under reparametrizations of the curve. Namely, with the same conventions as yet,

$$2\widetilde{Q}\widetilde{Q}'' - 3\widetilde{Q}'^{\,2} = g'^{\,-4}\left(2QQ'' - 3Q'^{\,2} - 2\mathcal{S}(g)Q^2\right).$$

Combining with (15), it easily follows that the function

$$Q^{-2}\,\boldsymbol{U}' \cdot \boldsymbol{U}' - 2Q^{-3}Q'' + 3Q^{-4}Q'^{\,2}$$

is an absolute conformal invariant of the curve. Now, the substitution $Q = \Phi^{\frac{1}{4}}$ leads to an invariant, which differs from our K_1, respectively Fialkow's J_{n-1}, just by a constant multiple. (For the comparison with J_{n-1}, the formula (44) from the next subsection is needed.)

4.3. Conformal curvatures in terms of Riemannian ones

Expanding any of the above expressions for the conformal curvatures yields very concrete and very ugly formulas in terms of the underlying derived vectors. As a compromise between the explicitness and the ugliness we show how to rewrite the previous formulas in terms of the Riemannian curvatures of the curve. This way we will also be able to compare our invariants with their more classical counterparts.

Within this paragraph we consider a generic curve parametrized by the Riemannian arc-length parameter, according to a chosen scale. The corresponding Riemannian Frenet frame is denoted by $(e_1, \ldots, e_n)$ and the Riemannian Frenet formulas are[2]

[2] Warning: in this subsection we relax the abstract indices, i.e. we write e_i rather than e_i^a etc. The raising and lowering indices with respect to the chosen metric is denoted by $\sharp$ and $\flat$, respectively.

$$
\begin{aligned}
e_1' &= \kappa_1 e_2, \\
e_i' &= -\kappa_{i-1} e_{i-1} + \kappa_i e_{i+1}, \qquad \text{for } i = 2, \dots, n-1, \\
e_n' &= -\kappa_{n-1} e_{n-1},
\end{aligned}
$$

where primes denote the derivative with respect to the Riemannian arc-length and $\kappa_1, \dots, \kappa_{n-1}$ are the Riemannian curvatures of the curve. Accordingly, the expressions (3) and (4) read as

$$
u = 1, \qquad U = e_1, \qquad U' = \kappa_1 e_2, \qquad U'' = -\kappa_1^2 e_1 + \kappa_1' e_2 + \kappa_1 \kappa_2 e_3, \qquad \text{etc.}
$$

Following the development of subsection 2.1, the first tractors are

$$
\boldsymbol{T} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{U} = \begin{pmatrix} 0 \\ e_1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{U}' = \begin{pmatrix} -1 \\ \kappa_1 e_2 \\ -\mathsf{P}(e_1, e_1) \end{pmatrix},
$$
$$
\boldsymbol{U}'' = \begin{pmatrix} 0 \\ -\kappa_1^2 e_1 + \kappa_1' e_2 + \kappa_1 \kappa_2 e_3 - \mathsf{P}(e_1, e_1) e_1 - \mathsf{P}(e_1, -)^\sharp \\ * \end{pmatrix}.
$$

Hence we obtain

$$
\begin{aligned}
\boldsymbol{U}' \cdot \boldsymbol{U}' &= \kappa_1^2 + 2\mathsf{P}(e_1, e_1), \\
\Phi = \boldsymbol{U}'' \cdot \boldsymbol{U}'' - (\boldsymbol{U}' \cdot \boldsymbol{U}')^2 &= \kappa_1'^2 + \kappa_1^2 \kappa_2^2 + \text{terms involving } \mathsf{P}.
\end{aligned} \tag{44}
$$

Substitution of (44) into (42) leads to an expression of the first conformal curvature K_1 in terms of first two Riemannian curvatures κ_1, κ_2, some terms involving the Schouten tensor and their derivatives. The full expansion leads to a huge formula even in the flat case.

Formula enthusiasts may continue further in this spirit and express the higher derived tractors. We add some details on the Gram matrix corresponding to the first five tractors. It is displayed in (13) where, modulo terms involving P,

$$
\alpha = \kappa_1^2, \qquad \beta = \kappa_1'^2 + \kappa_1^4 + \kappa_1^2 \kappa_2^2,
$$
$$
\gamma = 9\kappa_1^2 \kappa_1'^2 + (\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2)^2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')^2 + \kappa_1^2 \kappa_2^2 \kappa_3^2.
$$

Of course, the term κ_3 is nontrivial only if $n \geq 4$. From this and (41) one could deduce an expression for the second conformal curvature K_2 in terms of κ_1, κ_2, κ_3, P and their derivatives. Although the result is expected to be messy, it can be condensed into the following neat form:

$$
K_2 = \left(\kappa_1'^2 + \kappa_1^2 \kappa_2^2\right)^{-\frac{5}{4}} \left((2\kappa_1'^2 \kappa_2 + \kappa_1^2 \kappa_2^3 + \kappa_1 \kappa_1' \kappa_2' - \kappa_1 \kappa_1'' \kappa_2)^2 + (\kappa_1'^2 + \kappa_1^2 \kappa_2^2) \kappa_1^2 \kappa_2^2 \kappa_3^2\right)^{\frac{1}{2}},
$$

modulo terms involving P.

Remark 4.9. In the flat case, for $n = 3$, the previous display agrees precisely with the invariant which is called *conformal torsion* in [5] (the other invariant there corresponds to our K_1). Note that vanishing of K_2 in such case is equivalent to $\kappa_2 = 0$, or $\kappa_2 \neq 0$ and $\frac{\kappa_2}{\kappa_1} = \left(\frac{\kappa_1'}{\kappa_1^2 \kappa_2}\right)'$. The former condition means that the curve is planar, the latter one means the curve is spherical. By virtue of Remark 4.4(2), this is an expected behaviour.

For another comparisons let us also describe first few tractors from the tractor Frenet frame constructed in subsection 4.1. According to (43), the easy tractors are

$$\boldsymbol{U}_0 = \Phi^{\frac{1}{4}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{U}_1 = \begin{pmatrix} 0 \\ e_1 \\ \frac{1}{4}(\ln\Phi)' \end{pmatrix}, \qquad \boldsymbol{U}_2 = -\Phi^{-\frac{1}{4}} \begin{pmatrix} -1 \\ \frac{1}{4}(\ln\Phi)' e_1 + \kappa_1 e_2 \\ \frac{1}{32}(\ln\Phi)'^{\,2} + \frac{1}{2}\kappa_1^2 \end{pmatrix},$$

where Φ is given in (44). The first nontrivial step concerns an expression of $\boldsymbol{U}_3$, which one calculates as

$$\boldsymbol{U}_3 = \Phi^{-\frac{1}{2}} \begin{pmatrix} 0 \\ \kappa_1' e_2 + \kappa_1\kappa_2 e_3 + \mathsf{P}(e_1, e_1)e_1 - \mathsf{P}(e_1, -)^\sharp \\ \kappa_1\kappa_1' \end{pmatrix}. \tag{45}$$

Remarks 4.10.

(1) Incidentally, the middle slot of the tractor in (45) corresponds to the starting quantity in Fialkow's approach, the so-called *first conformal normal* of the curve, cf. [9, equation (4.21)]. Using this and an invariantly defined derivative along the curve (with respect to the conformal arc-length parameter), the conformal Frenet identities and the corresponding $n-2$ conformal curvatures are deduced in [9, section 4]. The last Fialkow's conformal curvature is constructed ad hoc in [9, section 5], see the discussion in Remark 4.8(2).

(2) At this stage we can easily count the orders of individual invariants although some estimates could be done earlier. The relative conformal invariant Φ is of order 3, cf. (44). From this and (42) we see that the first conformal curvature K_1 is of order 5. For $i \geq 2$, an estimate based on (41) yields the order of ith conformal curvature K_i is at most $i+3$, but the current expressions show it is actually $i+2$.

5. Null curves in general signature

The next step is to consider possible generalizations of the previous treatment to conformal manifolds of arbitrary signature. Thus, in the following we suppose that M is a conformal manifold of dimension $n = p + q$ and signature (p, q) and $\boldsymbol{\mathcal{T}}$ is the standard tractor bundle with the tractor metric of signature $(p+1, q+1)$. Generally, we should distinguish curves according to the type of tangent vectors/tractors and also according to the type of further tractors in the tractor Frenet frame. This leads to a diversity of possible cases, cf. Remark 4.4(3). The typical situation we need to understand is when the tangent vector together with its several derivatives generate a totally isotropic subspace of the tangent space, cf. the notion of r-null curves below. One can view this as a counterpart of the Riemannian setting discussed so far. In the last subsection we discuss the Lorentzian signature in more detail.

5.1. r-null curves

For null curves the density (3) vanishes identically and cannot be used for the lift to $\boldsymbol{\mathcal{T}}$. Depending on the signature, we may consider curves that are more and more isotropic, without being degenerate. On a conformal manifold of signature (p, q), for any $r \leq \min\{p, q\}$, a curve is called the *r-null curve* if the vectors $U^a, \dots, U^{(r-1)a}$ are linearly independent and null, whereas $U^{(r)a}$ is not null. Consequently, it holds

$$U_c^{(i)} U^{(j)c} = \begin{cases} 0, & \text{for } i + j < 2r, \\ (-1)^{r-i} U_c^{(r)} U^{(r)c}, & \text{for } i + j = 2r. \end{cases} \tag{46}$$

Note that the notion of r-null curves is well defined as it does not depend on the chosen scale. Indeed, one can easily verify by induction using (1) that

$$\widehat{U}^{(i)a} = U^{(i)a} \mod \langle U^a, \dots, U^{(i-1)a} \rangle,$$

for all $1 \le i \le 2r$. In this vein, space- and time-like curves may be termed *0-null curves*. As an opposite extreme, we use the notation ∞-*null curves* for curves, whose all vectors $U^{(i)a}$ are isotropic; such curves are necessarily degenerate. The most prominent—and the most degenerate—instance of such curves are the null geodesics.

For an r-null curve, it follows from (1) that the norm squared of $U^{(r)c}$ is a nowhere vanishing density of conformal weight 2. Thus,

$$u := \sqrt{|U^{(r)}_c U^{(r)c}|} \in \mathcal{E}[1]$$

provides a lift $\boldsymbol{T} \in \boldsymbol{\mathcal{T}}$ as in (7), to which we add the derived tractors $\boldsymbol{U}, \boldsymbol{U}'$ etc. as before. Explicit expressions are very analogous to those in (9), (10) etc. up to some shift. For $i \le 2r$, we have

$$\boldsymbol{U}^{(i)} = \begin{pmatrix} 0 \\ u^{-1} U^{(i)a} + \cdots \\ * \end{pmatrix},$$

where the zero in the primary slot appears as $-u^{-1} U_c U^{(i-1)c} = 0$ and the dots in the middle slot denote terms of lower order. In particular, we have $\boldsymbol{T} \cdot \boldsymbol{T} = \cdots = \boldsymbol{U}^{(r-1)} \cdot \boldsymbol{U}^{(r-1)} = 0$ and

$$\epsilon := \boldsymbol{U}^{(r)} \cdot \boldsymbol{U}^{(r)} = \pm 1. \tag{47}$$

The first nontrivial quantity is $\boldsymbol{U}^{(r+1)} \cdot \boldsymbol{U}^{(r+1)}$. In order to simplify expressions, we will use the notation

$$\alpha := \boldsymbol{U}^{(r+1)} \cdot \boldsymbol{U}^{(r+1)}, \qquad \beta := \boldsymbol{U}^{(r+2)} \cdot \boldsymbol{U}^{(r+2)}, \qquad \gamma := \boldsymbol{U}^{(r+3)} \cdot \boldsymbol{U}^{(r+3)}, \qquad \text{etc.} \tag{48}$$

The pattern of the initial relations remains the same as in (13) up to the signs concerning ϵ and a shift in the south-east direction. After some computation, one shows that the first few antidiagonals of the Gram matrix read as

$$\boldsymbol{U}^{(i)} \cdot \boldsymbol{U}^{(j)} = \begin{cases} 0, & \text{for } i+j < 2r, \\ (-1)^{r-i}\,\epsilon, & \text{for } i+j = 2r, \\ 0, & \text{for } i+j = 2r+1, \\ (-1)^{r+1-i}\,\alpha, & \text{for } i+j = 2r+2, \\ (-1)^{r+1-i}\,\frac{2(r-i)+3}{2}\alpha', & \text{for } i+j = 2r+3, \\ (-1)^{r-i}\,\beta + (-1)^{r+1-i}\,\frac{(r-i+2)^2}{2}\alpha'', & \text{for } i+j = 2r+4, \end{cases} \tag{49}$$

where we use the convention $\boldsymbol{U}^{(-1)} := \boldsymbol{T}$.

Under the reparametrization $\tilde{t} = g(t)$ as in subsection 2.2, one easily verifies that $\widetilde{\boldsymbol{T}} = g'^{\,r+1}\boldsymbol{T}$, from which one deduces the transformations of higher order tractors. Considering $\boldsymbol{U}^{(i)} := 0$, for $i < -1$, it turns out that

$$\widetilde{\boldsymbol{U}}^{(k)} = g'^{\,r-k}\left(\boldsymbol{U}^{(k)} + A_k g'^{\,-1} g'' \boldsymbol{U}^{(k-1)} + (B_k g'^{\,-1} g''' - C_k g'^{\,-2} g''^{\,2})\boldsymbol{U}^{(k-2)} + \cdots\right), \tag{50}$$

for $k = 0, 1, \ldots$, where the coefficients A_k, B_k, C_k are given by the recurrence relations

$$\begin{aligned} A_{k+1} &= A_k + r - k, \\ B_{k+1} &= B_k + A_k, \\ C_{k+1} &= C_k - (r-k-1)A_k, \end{aligned}$$

with the initial conditions $A_0 = r+1$, $B_0 = 0$ and $C_0 = 0$. It is an elementary, but tedious, exercise to solve and tidy up this system. For $k = r+1$, it turns out that

$$\begin{aligned} A_{r+1} &= \tfrac{1}{2}(r+1)(r+2), \\ B_{r+1} &= \tfrac{1}{6}(r+1)(r+2)(2r+3), \\ C_{r+1} &= -\tfrac{1}{8}(r+1)(r+2)(r^2-r-4). \end{aligned}$$

From this and the initial relations it follows that

$$\widetilde{\alpha} = g'^{\,-2}\left(\alpha - 2\epsilon B_{r+1} g'^{\,-1} g''' + \epsilon (A_{r+1}^2 + 2C_{r+1}) g'^{\,-2} g''^{\,2}\right).$$

Since $A_{r+1}^2 + 2C_{r+1} = 3B_{r+1}$, the previous simplifies to

$$\widetilde{\alpha} = g'^{\,-2}\left(\alpha - 2\epsilon B_{r+1}\mathcal{S}(g)\right). \tag{51}$$

Thus, the equation $\alpha = 0$, regarded as a condition on the parametrization of an r-null curve, determines a preferred family of parametrizations with freedom the projective group of the line. This is a generalization of Proposition 2.1 which just corresponds to $r = 0$. Altogether we may conclude with

Proposition 5.1. *For any admissible r, any r-null curve carries a preferred family of projective parameters.*

It is well known that null geodesics carry a preferred family of affine parameters.

Further, the Gram determinants are defined and denoted just the same as in (17). Initial relations for an r-null curve lead to $\Delta_i = 0$, for $i = 1, \ldots, 2r+2$, and $\Delta_{2r+3} = -\epsilon$. The first potentially nontrivial determinant is Δ_{2r+4}. Analogously to Lemma 2.2 we have

Lemma 5.2. *Along any r-null curve, the Gram determinant Δ_i, for $i = 2r+4, \ldots, n+2$, is a relative conformal invariant of weight $i(i-2r-3)$.*

For any $i = 2r+4, \ldots, n+2$, the vanishing of Δ_i is equivalent to the fact that the determining tractors are linearly dependent. The weight of any Δ_i is even, hence its sign is not changed by reparametrizations. In contrast to Lemma 2.3 (corresponding to $r = 0$), we cannot say anything about the particular sign of Δ_{2r+4}. For an r-null curve with nowhere vanishing Δ_{2r+4}, we define the *conformal pseudo-arc-length* parameter by integration of the 1-form

$$ds := \sqrt[2r+4]{|\Delta_{2r+4}(t)|}\, dt \tag{52}$$

along the curve; it is given uniquely up to an additive constant. The invariant Δ_{2r+4} (hence also all higher ones) may vanish. In particular, this happens automatically if $2r+4 > n+2$, i.e. if $n < 2r+2$. From the introductory counts we know that r and n are related by $n \geq 2r$, therefore the critical dimensions are $n = 2r$ and $2r+1$. In such cases, none of the just considered invariants can be used to define a natural parameter of the curve. However, there is another couple of relative conformal invariants as shown in subsection 5.2 below; see also Remark 5.15 for further comments.

At this stage, we sketch how to adapt the construction of the tractor Frenet frame from subsection 4.1 for r-null curves with the pseudo-arc-length defined. The smallest nondegenerate subbundle in $\boldsymbol{\mathcal{T}}$ gradually built from the derived tractors is $\langle \boldsymbol{T}, \boldsymbol{U}, \ldots, \boldsymbol{U}^{(2r+1)} \rangle$. The restriction of the tractor metric to this subbundle has signature $(r+2, r+1)$, respectively $(r+1, r+2)$, thus its orthogonal complement in $\boldsymbol{\mathcal{T}}$ has signature $(p-r-1, q-r)$, respectively $(p-r, q-r-1)$. We may always start with the prescription

$$\boldsymbol{U}_0 := \boldsymbol{T}, \qquad \boldsymbol{U}_1 := \boldsymbol{U}, \qquad \dots, \qquad \boldsymbol{U}_{r+1} := \boldsymbol{U}^{(r)}. \tag{53}$$

Then we have to consider transformations determining $\boldsymbol{U}_i \in \langle \boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}^{(i-1)} \rangle$, for $i = r+2, \dots, 2r+2$, so that the Gram matrix corresponding to the tractors $(\boldsymbol{U}_0, \dots, \boldsymbol{U}_{2r+2})$ is antidiagonal with values ± 1 (this substitutes the upper-left block in (28)). Note that in contrast to the case $r = 0$, the definition for $\boldsymbol{U}_{r+2}, \dots, \boldsymbol{U}_{2r+2}$ is not unique. This freedom is already visible from the first tractor of this subsequence which may be given as

$$\boldsymbol{U}_{r+2} = -\epsilon \boldsymbol{U}^{(r+1)} - \tfrac{1}{2}\epsilon\alpha \boldsymbol{U}^{(r-1)} + \cdots, \tag{54}$$

where the dots stay for any linear combination of lower order tractors $\boldsymbol{U}^{(r-2)}, \dots, \boldsymbol{T}$. To accomplish the full tractor frame, i.e. to determine $(\boldsymbol{U}_{2r+3}, \dots, \boldsymbol{U}_{n+1})$, we continue by an orthonormalization process. Going along, we may meet an isotropic tractor: in such case we face the same problem as discussed in Remark 4.4(3) and some additional choices are inevitable.

Having constructed the tractor Frenet frame, we differentiate with respect to the conformal pseudo-arc-length to obtain the tractor Frenet equations with the generating set of absolute conformal invariants. Just from (53) and (54) we easily deduce first few Frenet-like identities:

$$\begin{gathered} \boldsymbol{U}_0' = \boldsymbol{U}_1, \qquad \dots, \qquad \boldsymbol{U}_r' = \boldsymbol{U}_{r+1}, \\ \boldsymbol{U}_{r+1}' = K_1 \boldsymbol{U}_r - \epsilon \boldsymbol{U}_{r+2}, \qquad \text{where} \qquad K_1 := -\tfrac{1}{2}\epsilon\alpha. \end{gathered}$$

Clearly, the first conformal curvature is not affected by the freedom in the construction of the tractor Frenet frame. From the expression of K_1 and the remark preceding Proposition 5.1 we immediately have the following generalization of Proposition 4.3.

Proposition 5.3. *The conformal pseudo-arc-length parameter belongs to the projective family of parameters of an r-null curve if and only if $K_1 = 0$.*

The freedom in the construction of $(\boldsymbol{U}_{r+2}, \dots, \boldsymbol{U}_{2r+2})$ influences the corresponding part of tractor Frenet equations and so the corresponding conformal curvatures, namely, $K_2, \dots, K_{r+1}$. The respective freedom in the construction of $(\boldsymbol{U}_{2r+3}, \dots, \boldsymbol{U}_{n+1})$ influences some of the remaining conformal curvatures $K_{r+2}, \dots, K_{n-r-1}$. Notably, they are $n-r-1$ in total.

Various expressions analogous to (35) and (39) (with respect to the conformal pseudo-arc-length parameter), or (41) and (42) (with respect to an arbitrary parameter) are deducible. We provide more details only in subsection 5.4 for the case of Lorentzian signature.

5.2. Invariants of Wilczynski type

For an r-null curve, vanishing of Δ_{2r+4} means that the tractors $\boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}^{(2r+2)}$ are linearly dependent. Since the first $2r+3$ tractors from this sequence are linearly independent and $\boldsymbol{U}^{(k)} = \boldsymbol{T}^{(k+1)}$, this condition may be interpreted as a tractor linear ODE,

$$\boldsymbol{T}^{(2r+3)} + q_{2r+2}\boldsymbol{T}^{(2r+2)} + \cdots + q_1 \boldsymbol{T}' + q_0 \boldsymbol{T} = 0, \tag{55}$$

where $q_0, \dots, q_{2r+2}$ are functions expressible in terms of pairings of the initial tractors. Following [21, ch. II, §4], we may consider the so-called Wilczynski invariants of this equation, i.e. the set of $2r+1$ essential invariants denoted as $\Theta_3, \dots, \Theta_{2r+3}$ (the indices indicate weights). Of course, we may associate Wilczynski invariants to the linear differential operator on the left hand side of (55), i.e. also to general r-null curves with nonzero Δ_{2r+4}. As a matter of fact, these invariants are relative conformal invariants of the curve.

Assuming $q_{2r+2} = 0$, i.e. the so-called semi-canonical form, an explicit form of first two of these invariants is given by the following Lemma.

Lemma 5.4 *([17]). Let $y^{(k)} + q_{k-2}y^{(k-2)} + \cdots + q_0 y = 0$ be a linear ODE of kth order in the semi-canonical form ($q_{k-1} = 0$). Then the first two Wilczynski invariants are*

$$\Theta_3 = q_{k-3} - \frac{k-2}{2} q'_{k-2},$$
$$\Theta_4 = q_{k-4} - \frac{k-3}{2} q'_{k-3} + \frac{(k-2)(k-3)}{10} q''_{k-2} - \frac{(5k+7)(k-2)(k-3)}{10k(k+1)(k-1)} q^2_{k-2}.$$

The formula for Θ_3 and Θ_4 was known already to Laguerre and Schlesinger, respectively, both predating the work of Wilczynski. Accurate references and modern treatment of this classical subject can be found in [18, section 4.1].

Now we need to relate coefficients q_i from the equation (55) to quantities α, β, γ etc. from (48). By pairing both sides of (55) with $\boldsymbol{T}$, it follows from (49) that $q_{2r+2} = 0$, i.e. the equation is in the semi-canonical form. Considering similarly the pairing of both sides of (55) with $\boldsymbol{T}' = \boldsymbol{U}$, $\boldsymbol{T}'' = \boldsymbol{U}'$ and with $\boldsymbol{T}''' = \boldsymbol{U}''$, one computes

$$q_{2r+2} = 0, \qquad q_{2r+1} = \epsilon\alpha, \qquad q_{2r} = \frac{2r+1}{2}\epsilon\alpha', \qquad q_{2r-1} = -\epsilon\beta + \frac{r^2}{2}\epsilon\alpha'' + \alpha^2. \tag{56}$$

Combining this with Lemma 5.4, we obtain the resulting form of Θ_4:

Proposition 5.5. *Let ϵ, α and β be the quantities associated to an r-null curves (with respect to an arbitrary parametrization) as in* (47) *and* (48)*. Then $\Theta_3 = \Theta_5 = 0$ and*

$$\Theta_4 = -\epsilon\beta - \frac{r(r+3)}{10}\epsilon\alpha'' + \frac{(r+3)(2r+5)(5r+4)}{10(r+1)(r+2)(2r+3)}\alpha^2$$

is a relative conformal invariant of the weight 4.

Proof. Both vanishing of Θ_3 and the form of Θ_4 follows from (56) and Lemma 5.4, where $k = 2r+3$. The particular weight of Θ_4 follows from the definitions (48) and the exponent of g' in (50).

To show $\Theta_5 = 0$, we shall for simplicity assume a projective parametrization of the curve, i.e. $\alpha = 0$. In particular, the equation is in the canonical Laguerre–Forsyth form for which the expressions of Wilczynski invariants are well known, see [21, ch. II, eqn. (48)]. Analogously to the computation leading to (56), we obtain

$$\boldsymbol{U}^{(i)} \cdot \boldsymbol{U}^{(2r+5-i)} = (-1)^{r-i}\frac{2(r-i)+5}{2}\beta' \qquad \text{and} \qquad q_{2r-2} = -\frac{2r-1}{2}\epsilon\beta'.$$

Now substituting coefficients q_{2r-1} and q_{2r-2} into the just referred formula, one easily verifies that $\Theta_5 = 0$. □

Observe that Θ_4 recovers Δ_4 from (18) for $r = 0$.

Remarks 5.6.

(1) The Laguerre–Forsyth form of the equation is preserved by transformations with freedom the projective group of the line, which corresponds precisely to the condition $\alpha = 0$. Also, it follows that the linear equation is equivalent to the trivial equation if and only if all Wilczynski invariants vanish. In our case, this is equivalent to the vanishing of α, β, etc. from (48) up to $\boldsymbol{U}^{(2r+2)} \cdot \boldsymbol{U}^{(2r+2)}$.
(2) Experiments with specific r's suggest the equation (55) could be self-adjoint, i.e. all odd Wilczynski invariants vanish. This would mean that we have just r (rather than $2r$) new and potentially nontrivial invariants $\Theta_4, \dots, \Theta_{2r+2}$.

5.3. Conformal null helices

For any space- or time-like curve with an arbitrary parametrization, a metric in the conformal class may be chosen so that $U'^{\,a} = 0$, cf. (5). For null curves, this condition holds only if the curve is *null geodesic*, i.e. the null curve whose acceleration vector $U'^{\,a}$ is proportional to U^a. Consequently, all higher order vectors are proportional to U^a. These curves, although very important, cannot be lifted to the standard tractor bundle in the sense considered above. That is why they are excluded from our considerations.

For an r-null curve, we see from (46) and the subsequent discussion that the vectors $U^a, \dots, U^{(2r)a}$ are linearly independent for any choice of scale. Therefore we cannot achieve any of the conditions $U'^{\,a} = 0$, $\dots$, $U^{(2r)a} = 0$ by a conformal change of metric. The simplest conceivable statement is

Lemma 5.7. *For an r-null curve with an arbitrary parametrization, a metric in the conformal class may be chosen so that $U^{(r)}_c U^{(r)c} = \pm 1$ and $U^{(2r+1)a} = 0$.*

Proof. This can be proved by various means. For the sake of latter references, we employ the tractors anyhow it may seem artificial. Firstly, we may always choose the metric so that $U^{(r)}_c U^{(r)c} = \pm 1$. With this assumption, the tractors associated to the curve have the form

$$\boldsymbol{T} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{U} = \begin{pmatrix} 0 \\ U^a \\ 0 \end{pmatrix}, \qquad \boldsymbol{U}' = \begin{pmatrix} 0 \\ U'^{\,a} \\ \rho_1 \end{pmatrix}, \qquad \dots,$$
$$\boldsymbol{U}^{(j)} = \begin{pmatrix} \varepsilon_j \\ U^{(j)a} \mod \langle U^a, \dots, U^{(j-2)a} \rangle \\ \rho_j \end{pmatrix},$$

for $j = 2, \dots, 2r+1$, where ρ_j are in general nonzero while all ε_j vanish except for $\varepsilon_{2r+1} = (-1)^{r-1}$.

Secondly, the choice of metric may be adjusted so that the bottom slot of $\boldsymbol{U}'$ vanishes and the expressions of previous tractors are unchanged: according to (6), the corresponding Υ_a has to satisfy $\Upsilon_c U'^{\,c} = \rho_1$ and $\Upsilon_c U^c = 0$. Hence $\boldsymbol{U}'' = \begin{pmatrix} 0 \\ U''^{\,a} \\ \rho_2 \end{pmatrix}$, i.e. there is only the leading term in the middle slot. Inductively, we may achieve a rescaling so that $\boldsymbol{U}^{(j)} = \begin{pmatrix} 0 \\ U^{(j)a} \\ 0 \end{pmatrix}$, for all $j \le 2r$. Hence $\boldsymbol{U}^{(2r+1)} = \begin{pmatrix} (-1)^{r-1} \\ U^{(2r+1)a} \\ * \end{pmatrix}$. Finally, we may consider a rescaling so that the middle slot of the last tractor vanishes: according to (6), the corresponding Υ^a equals to $(-1)^r U^{(2r+1)a}$ along the curve. Hence the statement follows. □

Now we are ready to identify the closest relatives of conformal circles among r-null curves. We know from preceding subsections that the first $2r+3$ derived tractors $\boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}^{(2r+1)}$ are linearly independent. Therefore we can never achieve any of the conditions $\boldsymbol{U}'' = 0$, $\dots$, $\boldsymbol{U}^{(2r+1)} = 0$ for r-null curves. The simplest conceivable condition appears in item (a) of the following Theorem.

Theorem 5.8. *For an r-null curve, the following conditions are equivalent:*

(a) *the curve with a projective parametrization ($\alpha = 0$) obeys $\boldsymbol{U}^{(2r+2)} = 0$,*
(b) *the curve (with and arbitrary parametrization) obeys $\Delta_{2r+4} = 0$ and $\Theta_4 = \cdots = \Theta_{2r+3} = 0$,*
(c) *there is a metric in the conformal class such that $U_c^{(r)} U^{(r)c} = \pm 1$, $U^{(2r+1)a} = 0$ and $U^c \mathsf{P}_{ca} = 0$.*

Proof. Expressing the condition $\Delta_{2r+4} = 0$ as (55), the equivalence of (a) and (b) follows from the general theory of linear differential equations and the fact that Θ_3 vanishes automatically, cf. Proposition 5.5 and Remark 5.6(1).

For the rest we assume the scale is chosen as in the proof of Lemma 5.7. In such scale, the derived tractors are

$$\boldsymbol{T} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} 0 \\ U^a \\ 0 \end{pmatrix}, \quad \ldots, \quad \boldsymbol{U}^{(2r)} = \begin{pmatrix} 0 \\ U^{(2r)a} \\ 0 \end{pmatrix}, \quad \boldsymbol{U}^{(2r+1)} = \begin{pmatrix} (-1)^{r-1} \\ 0 \\ -U^c U^{(2r)d} \mathsf{P}_{cd} \end{pmatrix}.$$

The additional assumption $U^c \mathsf{P}_{ca} = 0$ from (c) then yields $\boldsymbol{U}^{(2r+2)} = 0$ and so (a) holds.

Conversely, from (a) it in particular follows $\boldsymbol{U}^{(2r+1)} \cdot \boldsymbol{U}^{(2r+1)} = 0$, hence the bottom slot of $\boldsymbol{U}^{(2r+1)}$ has to vanish. Then $\boldsymbol{U}^{(2r+2)} = 0$ implies that $U^c \mathsf{P}_{ca} = 0$ and so (c) holds. □

Any curve satisfying (any of) the conditions (a)–(c) is called *conformal r-null helix*. The condition $\Delta_{2r+4} = 0$ is equivalent to the fact that the subbundle $\langle \boldsymbol{T}, \boldsymbol{U}, \ldots, \boldsymbol{U}^{(2r+1)} \rangle \subset \boldsymbol{\mathcal{T}}$ of rank $2r + 3$ is parallel along the curve. In the flat case, this means that the conformal r-null helix is contained in the conformal image of a $(2r + 1)$-dimensional Euclidean subspace. With condition (c), it easily follows that these curves are just the conformal images of so-called *null Cartan helices*, cf. [7]. Note that the conformal circles fit to the just given description for $r = 0$.

Remark 5.9. As in subsection 3.3, one may construct conserved quantities along conformal r-null helices on specific conformal manifolds. The two examples mentioned in Propositions 3.5 and 3.6 have obvious counterparts for general r, provided that the respective functions are replaced by

$$\mathfrak{s} := \boldsymbol{U}^{(2r+1)} \cdot \boldsymbol{S} \qquad \text{and} \qquad \mathfrak{k} := \boldsymbol{K}(\boldsymbol{U}, \boldsymbol{U}^{(2r+1)}).$$

The reasoning is the same, explicit expressions are analogous, but more complicated.

5.4. Remarks on Lorentzian signature

In this subsection we suppose an n-dimensional conformal manifold of Lorentzian signature $(n - 1, 1)$, hence the standard tractor bundle $\boldsymbol{\mathcal{T}}$ with the bundle metric of signature $(n, 2)$. Following the scheme of previous subsections, a lot of things simplify as the maximal isotropic subspace in the tangent space has dimension one.

The only admissible r-null curves in this signature correspond to $r = 1$. Therefore we may speak without a risk of confusion just about null curves, instead of 1-null curves. The assumption $U_c U^c = 0$ implies that $U_c U'^{\,c} = 0$, thus the vector $U'^{\,a}$ must be space-like. The lift to $\boldsymbol{\mathcal{T}}$ is provided by the density $u = \sqrt{U'_c U'^{\,c}}$. The smallest nondegenerate subbundle in $\boldsymbol{\mathcal{T}}$ built from the derived tractors is of rank m5 and has signature $(3, 2)$. Hence its orthogonal complement is positive definite. The first few starting relations are indicated in the following Gram matrix for the sequence $(\boldsymbol{T}, \boldsymbol{U}, \boldsymbol{U}', \boldsymbol{U}'', \boldsymbol{U}''', \boldsymbol{U}'''')$, cf. (48) and (49):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & \alpha \\ 0 & 0 & 1 & 0 & -\alpha & -\frac{3}{2}\alpha' \\ 0 & -1 & 0 & \alpha & \frac{1}{2}\alpha' & \frac{1}{2}\alpha'' - \beta \\ 1 & 0 & -\alpha & \frac{1}{2}\alpha' & \beta & \frac{1}{2}\beta' \\ 0 & \alpha & -\frac{3}{2}\alpha' & \frac{1}{2}\alpha'' - \beta & \frac{1}{2}\beta' & \gamma \end{pmatrix}.$$

The projective parametrization of the null curve corresponds to $\alpha = \boldsymbol{U}'' \cdot \boldsymbol{U}'' = 0$.

The trivial Gram determinants are $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 0$ and $\Delta_5 = 1$. The first nontrivial one is

$$\Delta_6 = \gamma + \alpha''\alpha - 2\beta\alpha - \tfrac{9}{4}\alpha'^{\,2} + \alpha^3,$$

provided that the dimension of conformal manifold is $n \geq 4$, which is assumed in the rest of this subsection (the special case $n = 3$ is dealt individually in Remark 5.15). This invariant is nonnegative:

Lemma 5.10. $\Delta_6 \geq 0$.

Proof. For a projective parametrization of the null curve, $\alpha = 0$, we have $\Delta_6 = \gamma$. According to this choice, one easily verifies that the tractor

$$\boldsymbol{V}_5 := \boldsymbol{U}'''' - \beta\,\boldsymbol{U} - \tfrac{1}{2}\beta'\,\boldsymbol{T}$$

belongs to the orthogonal complement of the subbundle $\langle \boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}''' \rangle$ in $\boldsymbol{\mathcal{T}}$. This complement has the positive definite signature, hence we conclude with $\gamma = \boldsymbol{U}'''' \cdot \boldsymbol{U}'''' = \boldsymbol{V}_5 \cdot \boldsymbol{V}_5 \geq 0$. Since the weight of Δ_6 is even, it is a nonnegative function independently of the parametrization of the curve. □

Accordingly, the 1-form (52) leading to the definition of conformal pseudo-arc-length may be substituted by

$$ds := \sqrt[6]{\Delta_6(t)}\,dt. \tag{57}$$

Clearly, vanishing of Δ_6 is equivalent to vanishing of $\boldsymbol{V}_5$, which yields exactly the equation (55), for $r = 1$. Among the corresponding Wilczynski invariants, there is only one nontrivial, namely,

$$\Theta_4 = -\tfrac{1}{25}(25\beta + 10\alpha'' - 21\alpha^2), \tag{58}$$

cf. Proposition 5.5. According to Theorem 5.8, we may conclude with

Proposition 5.11. *Conformal null helices are characterized by the pair of equations*

$$\Delta_6 = 0 \qquad \textit{and} \qquad \Theta_4 = 0.$$

To build the tractor Frenet frame, we start with

$$\boldsymbol{U}_0 := \boldsymbol{T}, \qquad \boldsymbol{U}_1 := \boldsymbol{U}, \qquad \boldsymbol{U}_2 := \boldsymbol{U}'. \tag{59}$$

Now, all admissible transformations determining $\boldsymbol{U}_3$ and $\boldsymbol{U}_4$, so that the 5-tuple $(\boldsymbol{U}_0, \dots \boldsymbol{U}_4)$ has the antidiagonal Gram matrix, are

$$\begin{aligned} \boldsymbol{U}_3 &:= -\boldsymbol{U}'' - \tfrac{1}{2}\alpha\,\boldsymbol{U} + k\,\boldsymbol{T}, \\ \boldsymbol{U}_4 &:= \boldsymbol{U}''' + \alpha\,\boldsymbol{U}' + \ell\,\boldsymbol{U} + \tfrac{1}{2}(\alpha^2 - \beta)\,\boldsymbol{T}, \end{aligned} \tag{60}$$

where $k + \ell = \frac{1}{2}\alpha'$. For the rest of this demonstration we choose

$$k = 0 \qquad \text{and} \qquad \ell = \tfrac{1}{2}\alpha'. \tag{61}$$

As mentioned above, the orthocomplement to the subbundle $\langle \boldsymbol{T}, \dots, \boldsymbol{U}''' \rangle = \langle \boldsymbol{U}_0, \dots, \boldsymbol{U}_4 \rangle$ has positive definite signature. Therefore, we easily complete the tractor Frenet frame

$$(\boldsymbol{U}_0, \dots, \boldsymbol{U}_4; \boldsymbol{U}_5, \dots, \boldsymbol{U}_{n+1})$$

in the spirit of (29) so that $\boldsymbol{U}_i \in \langle \boldsymbol{T}, \boldsymbol{U}, \dots, \boldsymbol{U}^{(i-1)} \rangle$, for all admissible i, and the corresponding Gram matrix is

$$\left(\begin{array}{ccc|ccc} & & 1 & & & \\ & \iddots & & & & \\ 1 & & & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right).$$

Now, differentiating with respect to the conformal pseudo-arc-length we approach the identities with the generating set of absolute conformal invariants. Just from (59) and (60), with respect to the choice (61), we easily deduce first four Frenet-like identities:

$$\boldsymbol{U}_0' = \boldsymbol{U}_1, \tag{62}$$

$$\boldsymbol{U}_1' = \boldsymbol{U}_2, \tag{63}$$

$$\boldsymbol{U}_2' = K_1 \boldsymbol{U}_1 - \boldsymbol{U}_3, \qquad \text{where} \qquad K_1 := -\tfrac{1}{2}\alpha, \tag{64}$$

$$\boldsymbol{U}_3' = K_2 \boldsymbol{U}_0 - K_1 \boldsymbol{U}_2 - \boldsymbol{U}_4, \qquad \text{where} \qquad K_2 := \tfrac{1}{2}(\alpha^2 - \beta). \tag{65}$$

The next step is subtle, but fully analogous to the one announced in Lemma 4.1: the only coefficient in the expression of $\boldsymbol{U}_4'$, which cannot be deduced from the previous identities, is the one of $\boldsymbol{U}_5$, i.e. $\boldsymbol{U}_4' \cdot \boldsymbol{U}_5$. It however follows that this coefficient is constant, namely,

$$\boldsymbol{U}_4' = -K_2 \boldsymbol{U}_1 + \boldsymbol{U}_5. \tag{66}$$

(The crucial point in the argument is the expression of $\boldsymbol{V}_5$ and the observation that $\boldsymbol{V}_5 \cdot \boldsymbol{V}_5 = \Delta_6$. Since we assume parametrization by the conformal pseudo-arc-length, this equals to 1, thus $\boldsymbol{U}_5 = \boldsymbol{V}_5$.) The rest is easy so that one quickly summarizes as

$$\begin{aligned} \boldsymbol{U}_5' &= -\boldsymbol{U}_0 + K_3 \boldsymbol{U}_6, \\ \boldsymbol{U}_i' &= -K_{i-3} \boldsymbol{U}_{i-1} + K_{i-2} \boldsymbol{U}_{i+1}, \qquad \text{for } i = 6, \dots, n, \\ \boldsymbol{U}_{n+1}' &= -K_{n-2} \boldsymbol{U}_n. \end{aligned} \tag{67}$$

The cluster of equations from (62) through (66) to (67) forms the tractor Frenet equations associated to the null curve determining its conformal curvatures. Schematically, the tractor Frenet equations may be written as

$$
\begin{pmatrix} \boldsymbol{U}_0' \\ \boldsymbol{U}_1' \\ \boldsymbol{U}_2' \\ \boldsymbol{U}_3' \\ \boldsymbol{U}_4' \\ \hline \boldsymbol{U}_5' \\ \boldsymbol{U}_6' \\ \vdots \\ \boldsymbol{U}_{n+1}' \end{pmatrix}
=
\left(\begin{array}{ccccc|ccccc}
 & 1 & & & & & & & & \\
 & & 1 & & & & & & & \\
 & K_1 & & -1 & & & & & & \\
K_2 & & -K_1 & & -1 & & & & & \\
 & -K_2 & & & & 1 & & & & \\ \hline
-1 & & & & & & K_3 & & & \\
 & & & & & -K_3 & & & & \\
 & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & K_{n-2} \\
 & & & & & & & & -K_{n-2} &
\end{array}\right)
\cdot
\begin{pmatrix} \boldsymbol{U}_0 \\ \boldsymbol{U}_1 \\ \boldsymbol{U}_2 \\ \boldsymbol{U}_3 \\ \boldsymbol{U}_4 \\ \hline \boldsymbol{U}_5 \\ \boldsymbol{U}_6 \\ \vdots \\ \boldsymbol{U}_{n+1} \end{pmatrix}
$$

Altogether, we summarize as follows:

Proposition 5.12. *The system of tractor Frenet equations determines a generating set of absolute conformal invariants of the null curve. On a conformal manifold of Lorentzian signature and dimension $n \geq 4$, it consists of $n-2$ conformal curvatures that are expressed, with respect to the conformal pseudo-arc-length parametrization, as*

$$
K_i = \begin{cases} \boldsymbol{U}_{i+1}' \cdot \boldsymbol{U}_{i+2}, & \text{for } i = 1, 2 \\ \boldsymbol{U}_{i+2}' \cdot \boldsymbol{U}_{i+3}, & \text{for } i = 3, \dots, n-2. \end{cases}
$$

Among these curvatures, only K_2 depends on additional choices in the construction of $\boldsymbol{U}_3$ and $\boldsymbol{U}_4$.

As in subsection 4.2, there are alternative ways of expressing the conformal curvatures. The first two, i.e. the two exceptional, conformal curvatures K_1 and K_2 are given in terms of initial tractors with respect to the conformal pseudo-arc-length parametrization in (64) and (65). One may obtain the expressions with respect to an arbitrary reparametrization by a substitution according to (51), respectively (50), where $g' = \sqrt[6]{\Delta_6}$. In the former case, one ends up with the formula very similar to the one in (42), the later case is more ugly. For the higher conformal curvatures we may proceed as follows. Firstly, an analogue of Proposition 4.5 turns out to be

Proposition 5.13. *With respect to the conformal arc-length parametrization, the higher conformal curvatures can be expressed as*

$$
K_i = \sqrt{\frac{\boldsymbol{V}_{i+3} \cdot \boldsymbol{V}_{i+3}}{\boldsymbol{V}_{i+2} \cdot \boldsymbol{V}_{i+2}}}, \qquad \text{for } i = 3, \dots, n-2. \tag{68}
$$

Secondly, it holds $\Delta_{i+1} = \Delta_i(\boldsymbol{V}_i \cdot \boldsymbol{V}_i)$, for all admissible i. Since $\Delta_5 = 1$, we in particular see that all admissible determinants are positive. Now, expressing $\boldsymbol{V}_i \cdot \boldsymbol{V}_i$ via the determinants, substituting into (68) and passing to an arbitrary parametrization, we obtain the following substitute of Theorem 4.6:

Theorem 5.14. *With respect to an arbitrary parametrization, the higher conformal curvatures can be expressed as*

$$
K_i = \frac{\sqrt{\Delta_{i+2}\Delta_{i+4}}}{\Delta_{i+3}\sqrt[6]{\Delta_6}}, \qquad \text{for } i = 3, \dots, n-2.
$$

Remark 5.15. As we already noticed, the study of null curves in dimension $n = 3$ requires an extra care. The relative invariant Δ_6 vanishes automatically in that case, therefore it cannot be used to define the distinguished parametrization of the curve. However, we still have the Wilczynski invariant Θ_4 given by (58). Thus, for a generic null curve in 3-dimensional manifold, we define an alternative *conformal pseudo-arc-length* parameter by integrating the 1-form

$$ds := \sqrt[4]{|\Theta_4(t)|}\, dt,$$

instead of (57). Accordingly, we may follow the construction of conformal invariants as above with the following conclusion: the whole tractor Frenet frame consists just from the first five tractors displayed in (59)–(60) and the corresponding tractor Frenet equations are just (62)–(66), with $\boldsymbol{U}_5 = 0$. In those equations, two conformal invariants appear, namely, K_1 and K_2. It however follows from their expressions, and the fact that $\Theta_4 = 1$ for the current conformal pseudo-arc-length parametrization, that K_2 is expressible as a function of K_1 and its derivatives, namely,

$$K_2 = -\frac{2}{5}K_1'' + \frac{8}{25}K_1^2 + \frac{1}{2}.$$

Therefore we end up with only one significant absolute conformal invariant, K_1, as expected. An interpretation of that invariant is still the same as in Propositions 5.3.

In the case that Θ_4 vanishes identically, we recover the conformal null helices discussed above, cf. Proposition 5.11.

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