II. 2. Integrace racionální lomené funkce

Racionální lomenou funkci je nutné rozložit na parciální zlomky. Tyto parciální zlomky se pak postupně integrují, přičemž postup pro jejich integraci je následující:

- $$\int \frac{A}{x-x_0}\,dx \left| \begin{matrix} t = x - x_0 \\ dt = dx \end{matrix} \right| = \int \frac{A}{t}\,dt = A\ln|t| + C = A\ln|x-x_0| + C;$$
- $$\int \frac{A}{(x-x_0)^n}\,dx \left| \begin{matrix} t = x - x_0 \\ dt = dx \end{matrix} \right| = \int \frac{A}{t^n}\,dt = \frac{A\cdot t^{-n+1}}{-n+1} + C = \\ = \frac{A}{(1-n)(x-x_0)^{n-1}} + C, \quad \text{kde } n \geq 2;$$
- $$\int \frac{Ax+B}{x^2+px+q}\,dx = \frac{A}{2}\int \frac{2x+p}{x^2+px+q}\,dx \left| \begin{matrix} t = x^2+px+q \\ dt = (2x+p)\,dx \end{matrix} \right| + \\ + \left(B - \frac{Ap}{2}\right)\int \frac{1}{x^2+px+q}\,dx = \\ = \frac{A}{2}\int \frac{dt}{t} + \left(B - \frac{Ap}{2}\right)\int \frac{1}{(x-x_0)^2+a^2}\,dx = \ln|t| + \\ + \left(B - \frac{Ap}{2}\right)\frac{1}{a^2}\int \frac{dx}{\left(\frac{x-x_0}{a}\right)^2+1} \left| \begin{matrix} u = \frac{x-x_0}{a} \\ du = \frac{1}{a}dx \end{matrix} \right| = \\ = \ln\left|x^2+px+q\right| + \left(B - \frac{Ap}{2}\right)\frac{1}{a^2}\int \frac{a\,du}{u^2+1} = \ln\left|x^2+px+q\right| + \\ + \left(B - \frac{Ap}{2}\right)\frac{1}{a^2}\cdot a\cdot \operatorname{arctg} u + C = \\ = \ln\left|x^2+px+q\right| + \frac{2B-Ap}{2a}\cdot \operatorname{arctg}\frac{x-x_0}{a} + C;$$
- $$\int \frac{Ax+B}{(x^2+px+q)^n}\,dx = \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^n}\,dx \left| \begin{matrix} t = x^2+px+q \\ dt = (2x+p)\,dx \end{matrix} \right| + \\ + \left(B - \frac{Ap}{2}\right)\int \frac{dx}{(x^2+px+q)^n} = \\ = \frac{A}{2}\int \frac{dt}{t^n} + \left(B - \frac{Ap}{2}\right)\int \frac{dx}{[(x-x_0)^2+a^2]^n} = \\ = \frac{A}{2}\cdot \frac{1}{(1-n)(x^2+px+q)^{n-1}} + \left(B - \frac{Ap}{2}\right)K_n(x_0,a), \quad \text{kde } n \geq 2,$$

přičemž $K_n(x_0,a) := \int \frac{dx}{[(x-x_0)^2+a^2]^n}$. K dokončení výpočtu posledního integrálu je třeba využít následující rekurentní formule

$$K_{n+1}(x_0,a) = \frac{1}{a^2}\left(\frac{2n-1}{2n}K_n(x_0,a) + \frac{1}{2n}\frac{x-x_0}{[(x-x_0)^2+a^2]^n}\right),$$
$$K_1(x_0,a) = \frac{1}{a}\operatorname{arctg}\frac{x-x_0}{a},$$

což ve speciálním případě ($x_0 = 0$ a $a = 1$) dává

$$K_{n+1}(0,1) = \frac{2n-1}{2n} K_n(0,1) + \frac{1}{2n} \frac{x}{(x^2+1)^n},$$
$$K_1(0,1) = \operatorname{arctg} x.$$

(347) Vypočtěte

$$\int \frac{x^3+1}{x(x-1)^3}\,dx.$$

<u>**Řešení:**</u>

$$\begin{aligned}\int \frac{x^3+1}{x(x-1)^3}\,dx &= -\int \frac{dx}{x} + 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} + 2\int \frac{dx}{(x-1)^3} = \\ &= -\ln|x| + 2\ln|x-1| - \frac{1}{(x-1)^2} - \frac{1}{x-1} + C.\end{aligned}$$

(348) Vypočtěte

$$\int \frac{x^3 + x}{(x^2 - 1)(x^2 - 2)}\, dx.$$

Řešení:

$$\int \frac{x^3 + x}{(x^2 - 1)(x^2 - 2)}\, dx = -\int \frac{dx}{x - 1} - \int \frac{dx}{x + 1} + \int \frac{\frac{3}{2}}{x - \sqrt{2}}\, dx + \int \frac{\frac{3}{2}}{x + \sqrt{2}}\, dx =$$

$$= -\ln|x - 1| - \ln|x + 1| + \frac{3}{2} \ln\left|x - \sqrt{2}\right| + \frac{3}{2} \ln\left|x + \sqrt{2}\right| + C.$$

(349) Vypočtěte
$$\int \frac{x^6+2x-1}{x^5-x^2}\,dx.$$

Řešení:

$$\int \frac{x^6+2x-1}{x^5-x^2}\,dx = \int x\,dx - 2\int \frac{dx}{x} + \frac{2}{3}\int \frac{dx}{x-1} + \frac{2}{3}\int \frac{2x+1}{x^2+x+1} + \int \frac{dx}{x^2} =$$
$$= \frac{x^2}{2} - 2\ln|x| + \frac{2}{3}\ln|x-1| + \frac{2}{3}\ln\left|x^2+x+1\right| - \frac{1}{x} + C.$$

(350) Vypočtěte

$$\int \frac{3x+7}{x^2-4x+15}\,dx.$$

Řešení:

$$\begin{aligned}
\int \frac{3x+7}{x^2-4x+15}\,dx &= \frac{3}{2}\int \frac{2x-4}{x^2-4x+15}\,dx + 13\int \frac{dx}{x^2-4x+15} = \\
&= \frac{3}{2}\ln\left|x^2-4x+15\right| + 13\int \frac{dx}{(x-2)^2+11} = \\
&= \frac{3}{2}\ln\left|x^2-4x+15\right| + \frac{13}{11}\int \frac{dx}{\left(\frac{x-2}{\sqrt{11}}\right)^2+1} \left| \begin{array}{c} t = \frac{x-2}{\sqrt{11}} \\ dt = \frac{1}{\sqrt{11}}dx \end{array} \right| = \\
&= \frac{3}{2}\ln\left|x^2-4x+15\right| + \frac{13\sqrt{11}}{11}\int \frac{dt}{t^2+1} = \\
&= \frac{3}{2}\ln\left|x^2-4x+15\right| + \frac{13\sqrt{11}}{11}\operatorname{arctg} t + C = \\
&= \frac{3}{2}\ln\left|x^2-4x+15\right| + \frac{13}{\sqrt{11}}\operatorname{arctg}\frac{x-2}{\sqrt{11}} + C.
\end{aligned}$$

(351) Vypočtěte

$$\int \frac{x^3+2x+x-1}{x^2-x+1}\,dx.$$

Řešení:

$$\begin{aligned}
\int \frac{x^3+2x+x-1}{x^2-x+1}\,dx &= \int (x+3)\,dx + \int \frac{3x-4}{x^2-x+1}\,dx = \\
&= \frac{x^2}{2} + 3x + \frac{3}{2}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{5}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} = \\
&= \frac{x^2}{2} + 3x + \frac{3}{2}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{10}{3}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} \left| \begin{array}{l} t = \frac{2x-1}{\sqrt{3}} \\ dt = \frac{2}{\sqrt{3}}dx \end{array} \right| = \\
&= \frac{x^2}{2} + 3x + \frac{3}{2}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{5\sqrt{3}}{3}\int \frac{dt}{t^2+1} = \\
&= \frac{x^2}{2} + 3x + \frac{3}{2}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{5\sqrt{3}}{3}\,\mathrm{arctg}\, t + C = \\
&= \frac{x^2}{2} + 3x + \frac{3}{2}\ln\left|x^2-x+1\right| - \frac{5\sqrt{3}}{3}\,\mathrm{arctg}\,\frac{2x-1}{\sqrt{3}} + C.
\end{aligned}$$

(352) Vypočtěte

$$\int \frac{x}{x^4 - x^3 - x + 1}\,dx.$$

<u>**Řešení:**</u>

$$\begin{aligned}
\int \frac{x}{x^4 - x^3 - x + 1}\,dx &= \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{1}{3}\int \frac{dx}{x^2+x+1} \left| \begin{array}{c} t = x-1 \\ dt = dx \end{array} \right| = \\
&= \frac{1}{3}\int \frac{dt}{t^2} - \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} = \\
&= -\frac{1}{3(x+1)} - \frac{1}{3}\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C = \\
&= -\frac{1}{3(x+1)} - \frac{2\sqrt{3}}{9}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C.
\end{aligned}$$

(353) Vypočtěte

$$\int \frac{x}{(x^2+2x+2)(x^2+2x-3)}\,dx.$$

Řešení:

$$\int \frac{x}{(x^2+2x+2)(x^2+2x-3)}\,dx = \frac{1}{20}\int \frac{dx}{x-1} + \frac{3}{20}\int \frac{dx}{x+3} - \frac{1}{5}\int \frac{x}{x^2+2x+2}\,dx =$$

$$= \frac{1}{20}\ln|x-1| + \frac{3}{20}\ln|x+3| - \frac{1}{5}\left(\frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - \int \frac{dx}{x^2+2x+2}\right) =$$

$$= \frac{1}{20}\ln|x-1| + \frac{3}{20}\ln|x+3| - \frac{1}{10}\ln\left|x^2+2x+2\right| + \frac{1}{5}\int \frac{dx}{(x+1)^2+1}\ \left|\begin{matrix} t = x+1 \\ dt = dx \end{matrix}\right| =$$

$$= \frac{1}{20}\ln|x-1| + \frac{3}{20}\ln|x+3| - \frac{1}{10}\ln\left|x^2+2x+2\right| + \frac{1}{5}\int \frac{dt}{t^2+1} =$$

$$= \frac{1}{20}\ln|x-1| + \frac{3}{20}\ln|x+3| - \frac{1}{10}\ln\left|x^2+2x+2\right| + \frac{1}{5}\operatorname{arctg}(x+1) + C.$$

(354) Vypočtěte

$$\int \frac{dx}{x^5 - x^4 + x^3 - x^2 + x - 1}.$$

Řešení:

$$\int \frac{dx}{x^5 - x^4 + x^3 - x^2 + x - 1} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{6}\int \frac{2x+1}{x^2+x+1} - \frac{1}{2}\int \frac{dx}{x^2-x+1} =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| - \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| - \frac{2}{3}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1} \left| \begin{array}{c} t = \frac{2x-1}{\sqrt{3}} \\ dt = \frac{2}{\sqrt{3}}dx \end{array} \right| =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| - \frac{\sqrt{3}}{3}\int \frac{dx}{t^2+1} =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| - \frac{\sqrt{3}}{3}\operatorname{arctg} t =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} + C.$$

(355) Vypočtěte

$$\int \frac{x^2+3x+2}{x^2+x+2}\,dx.$$

Řešení:

$$\begin{aligned}
\int \frac{x^2+3x+2}{x^2+x+2}\,dx &= \int 1\,dx + \int \frac{2x}{x^2+x+2}\,dx = \\
&= x + \int \frac{2x+1}{x^2+x+2}\,dx - \int \frac{dx}{x^2+x+2} = \\
&= x + \ln\left|x^2+x+2\right| - \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}} = \\
&= x + \ln\left|x^2+x+2\right| - \frac{4}{7}\int \frac{dx}{\left(\frac{2x+1}{\sqrt{7}}\right)^2+1} \left|\begin{array}{c} t = \frac{2x+1}{\sqrt{7}} \\ dt = \frac{2}{\sqrt{7}}\,dx \end{array}\right| = \\
&= x + \ln\left|x^2+x+2\right| - \frac{2\sqrt{7}}{7}\int \frac{dx}{t^2+1} = \\
&= x + \ln\left|x^2+x+2\right| - \frac{2\sqrt{7}}{7}\operatorname{arctg} t + C = \\
&= x + \ln\left|x^2+x+2\right| - \frac{2}{\sqrt{7}}\operatorname{arctg}\frac{2x+1}{\sqrt{7}} + C.
\end{aligned}$$

(356) Vypočtěte

$$\int \frac{dx}{(x^2-6x+8)(x^2+2x+2)}.$$

Řešení:

$$\begin{aligned}
\int \frac{dx}{(x^2-6x+8)(x^2+2x+2)} &= \frac{1}{52}\int \frac{dx}{x-4} - \frac{1}{20}\int \frac{dx}{x-2} + \frac{1}{130}\int \frac{4x+11}{x^2+2x+2}\,dx = \\
&= \frac{1}{52}\ln|x-4| - \frac{1}{20}\ln|x-2| + \frac{1}{130}\left(2\int \frac{2x+2}{x^2+2x+2}\,dx + 7\int \frac{dx}{x^2+2x+2}\right) = \\
&= \frac{1}{52}\ln|x-4| - \frac{1}{20}\ln|x-2| + \frac{2}{130}\ln\left|x^2+2x+2\right| + \frac{7}{130}\int \frac{dx}{(x+1)^2+1} \left|\begin{array}{c} t = x+1 \\ dt = dx \end{array}\right| = \\
&= \frac{1}{52}\ln|x-4| - \frac{1}{20}\ln|x-2| + \frac{2}{130}\ln\left|x^2+2x+2\right| + \frac{7}{130}\int \frac{dt}{t^2+1} = \\
&= \frac{1}{52}\ln|x-4| - \frac{1}{20}\ln|x-2| + \frac{2}{130}\ln\left|x^2+2x+2\right| + \frac{7}{130}\operatorname{arctg} t = \\
&= \frac{1}{52}\ln|x-4| - \frac{1}{20}\ln|x-2| + \frac{2}{130}\ln\left|x^2+2x+2\right| + \frac{7}{130}\operatorname{arctg}(x+1) + C.
\end{aligned}$$

(357) Vypočtěte

$$\int \frac{x^8}{x^8-1}\,dx.$$

Řešení:

$$\begin{aligned}
\int \frac{x^8}{x^8-1}\,dx &= \int 1\,dx + \frac{1}{8}\int \frac{dx}{x-1} - \frac{1}{8}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x^2+1} + \\
&\quad + \frac{1}{8}\int \frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1}dx - \frac{1}{8}\int \frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1}dx = \\
&= x + \frac{1}{8}\ln|x-1| - \frac{1}{8}\ln|x+1| - \frac{1}{4}\operatorname{arctg} x + \\
&\quad + \frac{1}{8}\left(\frac{\sqrt{2}}{2}\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx - \int \frac{dx}{x^2-\sqrt{2}x+1}\right) - \\
&\quad - \frac{1}{8}\left(\frac{\sqrt{2}}{2}\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx + \int \frac{dx}{x^2+\sqrt{2}x+1}\right) = \\
&= x + \frac{1}{8}\ln|x-1| - \frac{1}{8}\ln|x+1| - \frac{1}{4}\operatorname{arctg} x + \\
&\quad + \frac{\sqrt{2}}{16}\ln\left|x^2-\sqrt{2}x+1\right| - \frac{1}{8}\int \frac{dx}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} - \\
&\quad - \frac{\sqrt{2}}{16}\ln\left|x^2+\sqrt{2}x+1\right| - \frac{1}{8}\int \frac{dx}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} = \\
&= x + \frac{1}{8}\ln|x-1| - \frac{1}{8}\ln|x+1| - \frac{1}{4}\operatorname{arctg} x + \\
&\quad + \frac{\sqrt{2}}{16}\ln\left|x^2-\sqrt{2}x+1\right| - \frac{1}{4}\int \frac{dx}{\left(\sqrt{2}x-1\right)^2+1} \left| \begin{array}{c} t=\sqrt{2}x-1 \\ dt=\sqrt{2}\,dx \end{array} \right| - \\
&\quad - \frac{\sqrt{2}}{16}\ln\left|x^2+\sqrt{2}x+1\right| - \frac{1}{4}\int \frac{dx}{\left(\sqrt{2}x+1\right)^2+1} \left| \begin{array}{c} w=\sqrt{2}x+1 \\ dw=\sqrt{2}\,dx \end{array} \right| = \\
&= x + \frac{1}{8}\ln|x-1| - \frac{1}{8}\ln|x+1| - \frac{1}{4}\operatorname{arctg} x + \\
&\quad + \frac{\sqrt{2}}{16}\ln\left|x^2-\sqrt{2}x+1\right| - \frac{\sqrt{2}}{8}\int \frac{dt}{t^2+1} - \\
&\quad - \frac{\sqrt{2}}{16}\ln\left|x^2+\sqrt{2}x+1\right| - \frac{\sqrt{2}}{8}\int \frac{dw}{w^2+1} =
\end{aligned}$$

$$\begin{aligned}
&= x + \frac{1}{8}\ln|x-1| - \frac{1}{8}\ln|x+1| - \frac{1}{4}\operatorname{arctg} x + \\
&\qquad + \frac{\sqrt{2}}{16}\ln\left|x^2 - \sqrt{2}x + 1\right| - \frac{\sqrt{2}}{8}\operatorname{arctg} t - \\
&\qquad - \frac{\sqrt{2}}{16}\ln\left|x^2 + \sqrt{2}x + 1\right| - \frac{\sqrt{2}}{8}\operatorname{arctg} w + C = \\
&= x + \frac{1}{8}\ln|x-1| - \frac{1}{8}\ln|x+1| - \frac{1}{4}\operatorname{arctg} x + \\
&\qquad + \frac{\sqrt{2}}{16}\ln\left|x^2 - \sqrt{2}x + 1\right| - \frac{\sqrt{2}}{8}\operatorname{arctg}\left(\sqrt{2}x - 1\right) - \\
&\qquad - \frac{\sqrt{2}}{16}\ln\left|x^2 + \sqrt{2}x + 1\right| - \frac{\sqrt{2}}{8}\operatorname{arctg}\left(\sqrt{2}x + 1\right) + C.
\end{aligned}$$

(358) Vypočtěte

$$\int \frac{x-4}{5x^2+6x+3}\,dx.$$

Řešení:

$$\begin{aligned}
\int \frac{x-4}{5x^2+6x+3}\,dx &= \frac{1}{10}\int \frac{10x+6}{5x^2+6x+3}\,dx - \frac{23}{5}\int \frac{dx}{5x^2+6x+3} = \\
&= \frac{1}{10}\ln\left|5x^2+6x+3\right| - \frac{23}{25}\int \frac{dx}{x^2+\frac{6}{5}x+\frac{3}{5}} = \\
&= \frac{1}{10}\ln\left|5x^2+6x+3\right| - \frac{23}{25}\int \frac{dx}{\left(x+\frac{3}{5}\right)^2+\frac{6}{25}} = \\
&= \frac{1}{10}\ln\left|5x^2+6x+3\right| - \frac{23}{6}\int \frac{dx}{\left(\frac{5x+3}{\sqrt{6}}\right)^2+1} \left|\begin{array}{c} t=\frac{5x+3}{\sqrt{6}} \\ dt=\frac{5}{\sqrt{6}}dx \end{array}\right| = \\
&= \frac{1}{10}\ln\left|5x^2+6x+3\right| - \frac{23\sqrt{6}}{30}\int \frac{dx}{t^2+1} = \\
&= \frac{1}{10}\ln\left|5x^2+6x+3\right| - \frac{23\sqrt{6}}{30}\operatorname{arctg} t + C = \\
&= \frac{1}{10}\ln\left|5x^2+6x+3\right| - \frac{23\sqrt{6}}{30}\operatorname{arctg}\frac{5x+3}{\sqrt{6}} + C.
\end{aligned}$$

(359) Vypočtěte

$$\int \frac{2x+1}{(x^2+4x+13)^2}\,dx.$$

Řešení:

$$\int \frac{2x+1}{(x^2+4x+13)^2}\,dx = \int \frac{2x+4}{(x^2+4x+13)^2}\,dx - 3\int \frac{dx}{(x^2+4x+13)^2} \left| \begin{array}{l} t = x^2+4x+13 \\ dt = (12x+4)\,dx \end{array} \right| =$$

$$= \int \frac{dt}{t^2} - 3\int \frac{dx}{\left[(x+2)^2+9\right]^2} =$$

$$= -\frac{1}{x^2+4x+13} - 3\int \frac{dx}{9^2\left[\left(\frac{x+2}{3}\right)^2+1\right]^2} \left| \begin{array}{l} w = \frac{x+2}{3} \\ dw = \frac{1}{3}dx \end{array} \right| =$$

$$= -\frac{1}{x^2+4x+13} - 3\frac{3}{81}\int \frac{dw}{(w^2+1)^2} =$$

$$= -\frac{1}{x^2+4x+13} - \frac{1}{9}\left(\frac{1}{2}\operatorname{arctg} w + \frac{1}{2}\frac{w}{w^2+1}\right) + C =$$

$$= -\frac{1}{x^2+4x+13} - \frac{1}{18}\operatorname{arctg}\frac{x+2}{3} - \frac{1}{18}\frac{\frac{x+2}{3}}{\left(\frac{x+2}{3}\right)^2+1} + C =$$

$$= -\frac{1}{18}\operatorname{arctg}\frac{x+2}{3} - \frac{1}{6}\frac{x+8}{x^2+4x+13} + C.$$

(360) Vypočtěte

$$\int \frac{2x^4+2x^2-5x+1}{x\left(x^2-x+1\right)^2}\,dx.$$

<u>**Řešení:**</u>

$$\int \frac{2x^4+2x^2-5x+1}{x\left(x^2-x+1\right)^2}\,dx = \int \frac{dx}{x} + \int \frac{x+3}{x^2-x+1}\,dx + \int \frac{x-6}{\left(x^2-x+1\right)^2}\,dx =$$

$$= \ln|x| + \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{7}{2}\int \frac{dx}{x^2-x+1} +$$

$$+ \frac{1}{2}\int \frac{2x-1}{(x^2-x+1)^2}\,dx - \frac{11}{2}\int \frac{dx}{(x^2-x+1)^2} =$$

$$= \ln|x| + \frac{1}{2}\ln\left|x^2-x+1\right|\,dx + \frac{7}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} +$$

$$+ \frac{1}{2}\int \frac{2x-1}{(x^2-x+1)^2}\,dx \left| \begin{array}{c} t = x^2-x+1 \\ dt = (2x-1)dx \end{array} \right| - \frac{11}{2}\int \frac{dx}{\left[\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right]^2} =$$

$$= \ln|x| + \frac{1}{2}\ln\left|x^2-x+1\right|\,dx + \frac{14}{3}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} \left| \begin{array}{c} w = \frac{2x-1}{\sqrt{3}} \\ dw = \frac{2}{\sqrt{3}}\,dx \end{array} \right| +$$

$$+ \frac{1}{2}\int \frac{dt}{t^2} - \frac{22}{3}\left(\frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{1}{2}\frac{x-\frac{1}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right) =$$

$$= \ln|x| + \frac{1}{2}\ln\left|x^2-x+1\right|\,dx + \frac{7\sqrt{3}}{3}\int \frac{dw}{w^2+1} +$$

$$- \frac{1}{2t} - \frac{44}{9}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} - \frac{11}{3}\frac{x-\frac{1}{2}}{x^2-x+1} =$$

$$= \ln|x| + \frac{1}{2}\ln\left|x^2-x+1\right|\,dx + \frac{7\sqrt{3}}{3}\operatorname{arctg} w +$$

$$- \frac{1}{2(x^2-x+1)} - \frac{44}{9}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} \left| \begin{array}{c} u = \frac{2x-1}{\sqrt{3}} \\ du = \frac{2}{\sqrt{3}}\,dx \end{array} \right| - \frac{11}{3}\frac{x-\frac{1}{2}}{x^2-x+1} =$$

$$= \ln|x| + \frac{1}{2}\ln\left|x^2-x+1\right|\,dx + \frac{7\sqrt{3}}{3}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} +$$

$$- \frac{1}{2(x^2-x+1)} - \frac{22\sqrt{3}}{9}\int \frac{du}{u^2+1} - \frac{11}{3}\frac{x-\frac{1}{2}}{x^2-x+1} =$$

$$= \ln|x| + \frac{1}{2}\ln\left|x^2 - x + 1\right| \, dx + \frac{7\sqrt{3}}{3}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} +$$

$$-\frac{22\sqrt{3}}{9}\operatorname{arctg} u - \frac{1}{3}\frac{11x-4}{x^2-x+1} + C =$$

$$= \ln|x| + \frac{1}{2}\ln\left|x^2 - x + 1\right| \, dx + \frac{7\sqrt{3}}{3}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} +$$

$$-\frac{22\sqrt{3}}{9}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} - \frac{1}{3}\frac{11x-4}{x^2-x+1} + C =$$

$$= \ln|x| + \frac{1}{2}\ln\left|x^2 - x + 1\right| \, dx - \frac{\sqrt{3}}{9}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} - \frac{1}{3}\frac{11x-4}{x^2-x+1} + C.$$

(361) Vypočtěte

$$\int \frac{5x^2-12}{(x^2-6x+13)^2}\,dx.$$

<u>**Řešení:**</u>

$$\int \frac{5x^2-12}{(x^2-6x+13)^2}\,dx = 5\int \frac{dx}{x^2-6x+13} + \int \frac{30x-77}{(x^2-6x+13)^2}\,dx =$$
$$= 5\int \frac{dx}{(x-3)^2+4} + 15\int \frac{2x-6}{(x^2-6x+13)^2}dx + 13\int \frac{dx}{(x^2-6x+13)^2} =$$
$$= \frac{5}{4}\int \frac{dx}{\left(\frac{x-3}{2}\right)^2+1} \left| \begin{array}{l} t = \frac{x-3}{2} \\ dt = \frac{1}{2}\,dx \end{array} \right| + 15\int \frac{2x-6}{(x^2-6x+13)^2}dx \left| \begin{array}{l} w = x^2-6x+13 \\ dw = (2x-6)\,dx \end{array} \right| +$$
$$+ 13\int \frac{dx}{\left[(x-3)^2+4\right]^2} =$$
$$= \frac{5}{2}\int \frac{dx}{t^2+1} + 15\int \frac{dw}{w^2} + \frac{13}{4}\left(\frac{1}{4}\operatorname{arctg}\frac{x-3}{2} + \frac{1}{2}\frac{x-3}{x^2-6x+13}\right) =$$
$$= \frac{5}{2}\operatorname{arctg} t - \frac{15}{w} + \frac{13}{16}\operatorname{arctg}\frac{x-3}{2} + \frac{13}{8}\frac{x-3}{x^2-6x+13} + C =$$
$$= \frac{5}{2}\operatorname{arctg}\frac{x-3}{2} - \frac{15}{x^2-6x+13} + \frac{13}{16}\operatorname{arctg}\frac{x-3}{2} + \frac{13}{8}\frac{x-3}{x^2-6x+13} + C =$$
$$= \frac{53}{16}\operatorname{arctg}\frac{x-3}{2} + \frac{13x-159}{8(x^2-6x+13)} + C.$$

(362) Vypočtěte

$$\int \frac{5\ln x}{x(\ln^3 x + \ln^2 x - 2)}\,dx.$$

Řešení:

$$\int \frac{5\ln x}{x(\ln^3 x + \ln^2 x - 2)}\,dx = \left| \begin{array}{c} t = \ln x \\ dt = \frac{1}{x}dx \end{array} \right| = \int \frac{5t}{t^3 + t^2 - 2}\,dt =$$

$$= \int \frac{1}{t-1} + \frac{-t+2}{t^2+2t+2}\,dt = \int \left(\frac{1}{t-1} - \frac{1}{2} \cdot \frac{2t+2}{t^2+2t+2} + \frac{3}{t^2+2t+2} \right) dt =$$

$$= \ln|t-1| - \frac{1}{2}\ln(t^2+2t+2) + 3\int \frac{1}{(t+1)^2+1}\,dt \left| \begin{array}{c} s = t+1 \\ ds = dt \end{array} \right| =$$

$$= \ln|t-1| - \frac{1}{2}\ln(t^2+2t+2) + 3\int \frac{1}{s^2+1}\,ds =$$

$$= \ln|t-1| - \frac{1}{2}\ln(t^2+2t+2) + 3\operatorname{arctg}(t+1) + C =$$

$$= \ln|\ln x - 1| - \frac{1}{2}\ln(\ln^2 x + 2\ln x + 2) + 3\operatorname{arctg}(\ln x + 1) + C.$$