W. GREEN, strana 711, příklad 6

Máme specifikován následující model

(1) $$\mathbf{y}_1 = \beta_1 \mathbf{y}_2 + \gamma_{11}\mathbf{x}_1 + \varepsilon_1$$

(2) $$\mathbf{y}_2 = \beta_2 \mathbf{y}_1 + \gamma_{22}\mathbf{x}_2 + +\gamma_{32}\mathbf{x}_3 + \varepsilon_2 .$$

Všechny proměnné jsou měřeny jako odchylky od svých průměrů. Vzorek 25 pozorování poskytl následující matici součtů čtverců a křížových součinů (proměnných) **:**

(3) $$\begin{matrix} & & (& y_1 & y_2 & x_1 & x_2 & x_3 &) \end{matrix}$$
$$\begin{pmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 & 6 & 4 & 3 & 5 \\ 6 & 10 & 3 & 6 & 7 \\ 4 & 3 & 5 & 2 & 3 \\ 3 & 6 & 2 & 10 & 8 \\ 5 & 7 & 3 & 8 & 15 \end{pmatrix}$$

Úlohy k řešení :

A) Odhadněte obě rovnice metodou OLS.

B) Odhadněte parametry obou rovnic metodou 2SLS. Odhadněte také asymptotickou kovarianční matici 2SLS-odhadové funkce.

C) Získejte LIML odhady parametrů první rovnice.

D) Odhadněte obě rovnice pomocí 3SLS.

E) Odhadněte matici koeficientů redukovaného tvaru pomocí OLS a nepřímo s použitím strukturních odhadů z části B).

A1) Odhad parametrů 1. rovnice metodou OLS

(1) $$\mathbf{y}_1 = \beta_1 \mathbf{y}_2 + \gamma_{11}\mathbf{x}_1 + \varepsilon_1$$

Výpočetní vzorec pro odhad parametrů metodou OLS pro danou situaci:

(4) $$\begin{pmatrix} {}_{OLS}\hat{\beta}_1 \\ {}_{OLS}\hat{\gamma}_{11} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_1'\mathbf{Y}_1 & \mathbf{Y}_1'\mathbf{X}_1 \\ \mathbf{X}_1'\mathbf{Y}_1 & \mathbf{X}_1'\mathbf{X}_1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathbf{Y}_1'\mathbf{y}_1 \\ \mathbf{X}_1'\mathbf{y}_1 \end{pmatrix}$$, kde $\mathbf{Y}_1 = \mathbf{y}_1$, $\mathbf{X}_1 = \mathbf{x}_1$, $\mathbf{Y}_1 = \mathbf{y}_2$.

konkrétně $\mathbf{Y}_1'\mathbf{Y}_1 = \mathbf{y}_2.\mathbf{y}_2 = 10$ $\quad \mathbf{X}_1'\mathbf{y}_1 = \mathbf{x}_1.\mathbf{y}_1 = 4$

$\mathbf{Y}_1'\mathbf{X}_1 = \mathbf{y}_2.\mathbf{x}_1 = 3$ $\quad \mathbf{X}_1'\mathbf{Y}_1 = \mathbf{Y}_1'\mathbf{X}_1 = 3$

$\mathbf{X}_1'\mathbf{X}_1 = \mathbf{x}_1.\mathbf{x}_1 = 5$ $\quad \mathbf{Y}_1'\mathbf{y}_1 = \mathbf{y}_2.\mathbf{y}_1 = 6$ $\quad \mathbf{X}_1'\mathbf{y}_1 = \mathbf{x}_1.\mathbf{y}_1 = 4$

Po dosazení do výpočetního vzorce (4) dostaneme:

$$\begin{pmatrix} {}_{OLS}\hat{\beta}_1 \\ {}_{OLS}\hat{\gamma}_1 \end{pmatrix} = \begin{pmatrix} 10 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \frac{1}{41}\begin{pmatrix} 5 & -3 \\ -3 & 10 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{30-12}{41} \\ \frac{-18+40}{41} \end{pmatrix} = \begin{pmatrix} \frac{18}{41} \\ \frac{22}{41} \end{pmatrix} = \begin{pmatrix} 0{,}439 \\ 0{,}536 \end{pmatrix}$$

A2) Odhad parametrů 2. rovnice metodou OLS

(2) $$\mathbf{y}_2 = \beta_2 \mathbf{y}_1 + \gamma_{21}\mathbf{x}_2 + \gamma_{22}\mathbf{x}_3 + \varepsilon_2$$

Výpočetní vzorec pro odhad parametrů metodou OLS pro danou situaci:

(5) $$\begin{pmatrix} {}_{OLS}\hat{\beta}_2 \\ {}_{OLS}\hat{\gamma}_{21} \\ {}_{OLS}\hat{\gamma}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_2'\mathbf{Y}_2 & \mathbf{Y}_2'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{Y}_2 & \mathbf{X}_2'\mathbf{X}_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathbf{Y}_2'\mathbf{y}_2 \\ \mathbf{X}_2'\mathbf{y}_2 \end{pmatrix}$$, kde $\mathbf{Y}_2 = \mathbf{y}_1$, $\mathbf{X}_2 = (\mathbf{x}_2, \mathbf{x}_3)$, $\mathbf{Y}_2 = \mathbf{y}_2$.

$$\mathbf{X}_2'\mathbf{Y}_2 = \begin{pmatrix} \mathbf{x}_2\mathbf{y}_1 \\ \mathbf{x}_3\mathbf{y}_1 \end{pmatrix} \qquad \mathbf{X}_2'\mathbf{y}_2 = \begin{pmatrix} \mathbf{x}_2\mathbf{y}_2 \\ \mathbf{x}_3\mathbf{y}_2 \end{pmatrix} \qquad \mathbf{X}_2'\mathbf{X}_2 = \begin{pmatrix} \mathbf{x}_2\mathbf{x}_2 & \mathbf{x}_2\mathbf{x}_3 \\ \mathbf{x}_2\mathbf{x}_3 & \mathbf{x}_3\mathbf{x}_3 \end{pmatrix}$$

$$\mathbf{Y}_2'\mathbf{Y}_2 = \mathbf{y}_1.\mathbf{y}_1 \qquad \mathbf{Y}_2'\mathbf{X}_2 = (\mathbf{y}_1\mathbf{x}_2, \mathbf{y}_1\mathbf{x}_3) \qquad \mathbf{Y}_2'\mathbf{y}_2 = \mathbf{y}_1.\mathbf{y}_2$$

Po dosazení do výpočetního vzorce (5) dostaneme:

$$\begin{pmatrix} {}_{OLS}\hat{\beta}_2 \\ {}_{OLS}\hat{\gamma}_{21} \\ {}_{OLS}\hat{\gamma}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1\mathbf{y}_1 & \mathbf{y}_1\mathbf{x}_2 & \mathbf{y}_1\mathbf{x}_3 \\ \mathbf{y}_1\mathbf{x}_2 & \mathbf{x}_2\mathbf{x}_2 & \mathbf{x}_2\mathbf{x}_3 \\ \mathbf{y}_1\mathbf{x}_3 & \mathbf{x}_2\mathbf{x}_3 & \mathbf{x}_3\mathbf{x}_3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathbf{y}_1\mathbf{y}_2 \\ \mathbf{x}_2\mathbf{y}_2 \\ \mathbf{x}_3\mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} 20 & 3 & 5 \\ 3 & 10 & 8 \\ 5 & 8 & 15 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 6 \\ 6 \\ 7 \end{pmatrix} =$$

$$= \frac{1}{1575}\begin{pmatrix} 86 & -5 & -26 \\ -5 & 275 & -145 \\ -26 & -145 & 191 \end{pmatrix}\begin{pmatrix} 6 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 0{,}193 \\ 0{,}384 \\ 0{,}197 \end{pmatrix},$$

protože $$\det \mathbf{X}'\mathbf{X} = 20 \cdot 10 \cdot 15 + 3 \cdot 8 \cdot 5 + 5 \cdot 3 \cdot 8 - 20 \cdot 8 \cdot 8 - 3 \cdot 3 \cdot 15 - 5 \cdot 5 \cdot 10 =$$
$$= 3000 + 120 + 120 - 1280 - 135 - 250 = 3240 - 1665 = 1575$$

B1) Odhad parametrů 1.rovnice metodou 2SLS

(1) $$\mathbf{y}_1 = \beta_1 \mathbf{y}_2 + \gamma_{11}\mathbf{x}_1 + \varepsilon_1$$

Výpočetní vzorec pro odhad parametrů metodou 2SLS pro danou situaci:

(6) $$\begin{pmatrix} {}_{2SLS}\hat{\beta}_1 \\ {}_{2SLS}\hat{\gamma}_{11} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_1'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}_1 & \mathbf{Y}_1'\mathbf{X}_1 \\ \mathbf{X}_1'\mathbf{Y}_1 & \mathbf{X}_1'\mathbf{X}_1 \end{pmatrix}^{-1} . \begin{pmatrix} \mathbf{Y}_1'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}_1 \\ \mathbf{X}_1'\mathbf{y}_1 \end{pmatrix}$$, kde

$$\mathbf{Y}_1 = \mathbf{y}_2 \quad \mathbf{X}_1 = \mathbf{x}_1 \quad \mathbf{y}_1 = \mathbf{y}_1, \quad \mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

$$\mathbf{Y}_1'\mathbf{X}_1 = \mathbf{y}_2.\mathbf{x}_1 = 3 \qquad \mathbf{X}_1'\mathbf{y}_1 = \mathbf{x}_1.\mathbf{y}_1 = 4$$

$$\mathbf{X}_1'\mathbf{Y}_1 = \mathbf{x}_1.\mathbf{y}_2 = 3 \qquad \mathbf{X}_1'\mathbf{X}_1 = \mathbf{x}_1.\mathbf{x}_1 = 5$$

Po dosazení do výpočetního vzorce (6) dostaneme:

$$\mathbf{X}'\mathbf{Y}_1 = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} \mathbf{y}_2 = \begin{pmatrix} \mathbf{x}_1\mathbf{y}_2 \\ \mathbf{x}_2\mathbf{y}_2 \\ \mathbf{x}_3\mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} \qquad \mathbf{X}'\mathbf{y}_1 = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} \mathbf{y}_1 = \begin{pmatrix} \mathbf{x}_1\mathbf{y}_1 \\ \mathbf{x}_2\mathbf{y}_1 \\ \mathbf{x}_3\mathbf{y}_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2x_2 & x_2x_3 \\ x_1x_1 & x_2x_3 & x_3x_3 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}$$, takže máme

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{376} . \begin{pmatrix} 150-64 & 30-24 & 16-30 \\ 30-24 & 75-9 & 40-6 \\ 16-30 & 40-6 & 50-4 \end{pmatrix} = \frac{1}{376} . \begin{pmatrix} 86 & -6 & -14 \\ -6 & 66 & -34 \\ -14 & -34 & 46 \end{pmatrix}$$, protože

$$\det = 5.10.15 + 2.8.3 + 3.2.8 - 5.8.8 - 2.2.15 - 3.3.10 = 750 + 48 + 48 - 320 - 60 - 90 = 846 - 470 = 376$$

$$\begin{pmatrix} {}_{2SLS}\hat{\beta}_1 \\ {}_{2SLS}\hat{\gamma}_{11} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 & 6 & 7 \end{pmatrix} \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}^{-1} . \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} & 3 \\ 3 & 5 \end{pmatrix}^{-1} . \begin{pmatrix} \begin{pmatrix} 3 & 6 & 7 \end{pmatrix} \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}^{-1} . \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \\ 4 \end{pmatrix} =$$

$$\begin{pmatrix} {}_{2SLS}\hat{\beta}_1 \\ {}_{2SLS}\hat{\gamma}_{11} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 & 6 & 7 \end{pmatrix} . \frac{1}{376} \begin{pmatrix} 86 & -6 & -14 \\ -6 & 66 & -34 \\ -14 & -34 & 46 \end{pmatrix} . \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} & 3 \\ 3 & 5 \end{pmatrix}^{-1} . \begin{pmatrix} \begin{pmatrix} 3 & 6 & 7 \end{pmatrix} . \begin{pmatrix} 86 & -6 & -14 \\ -6 & 66 & -34 \\ -14 & -34 & 46 \end{pmatrix} . \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \\ 4 \end{pmatrix} =$$

$$\begin{pmatrix} {}_{2SLS}\hat{\beta}_1 \\ {}_{2SLS}\hat{\gamma}_{11} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 3 & 6 & 7 \end{pmatrix} . \begin{pmatrix} 0{,}2287 & -0{,}016 & -0{,}0372 \\ -0{,}016 & 0{,}1755 & -0{,}0904 \\ -0{,}0372 & -0{,}0904 & 0{,}1223 \end{pmatrix} . \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} & 3 \\ 3 & 5 \end{pmatrix}^{-1} . \begin{pmatrix} \begin{pmatrix} 3 & 6 & 7 \end{pmatrix} \begin{pmatrix} 0{,}2287 & -0{,}016 & -0{,}0372 \\ -0{,}016 & 0{,}1755 & -0{,}0904 \\ -0{,}0372 & -0{,}0904 & 0{,}1223 \end{pmatrix} . \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} {}_{2SLS}\hat{\beta}_1 \\ {}_{2SLS}\hat{\gamma}_{11} \end{pmatrix} = \begin{pmatrix} 4{,}6383 & 3 \\ 3 & 5 \end{pmatrix}^{-1} . \begin{pmatrix} 3{,}44681 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 4{,}6383 \end{pmatrix} . \begin{pmatrix} 3{,}44681 \\ 4 \end{pmatrix} = \begin{pmatrix} 0{,}36882 \\ 0{,}57871 \end{pmatrix}$$

B2) Odhad parametrů 2.rovnice metodou 2SLS

(2) $\mathbf{y}_2 = \beta_2 \mathbf{y}_1 + \gamma_{21}\mathbf{x}_2 + \gamma_{22}\mathbf{x}_3 + \varepsilon_2$

Výpočetní vzorec pro odhad parametrů metodou 2SLS pro danou situaci:

$$\begin{pmatrix} {}_{2SLS}\hat{\beta}_2 \\ {}_{2SLS}\hat{\gamma}_{21} \\ {}_{2SLS}\hat{\gamma}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_2'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}_2 & \mathbf{Y}_2'\mathbf{X}_2 \\ \mathbf{X}_2'\mathbf{Y}_2 & \mathbf{X}_2'\mathbf{X}_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathbf{Y}_2'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}_2 \\ \mathbf{X}_2'\mathbf{y}_2 \end{pmatrix}$$ **, kde**

$\mathbf{Y}_2 = \mathbf{y}_1$ $\quad \mathbf{X}_2 = (\mathbf{x}_2, \mathbf{x}_3)$ $\quad \mathbf{y}_2 = \mathbf{y}_2$ $\quad \mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$

$\mathbf{X}_2'\mathbf{Y}_2 = (\mathbf{x}_2\mathbf{y}_1, \mathbf{x}_3\mathbf{y}_1)$ $\quad \mathbf{X}_2'\mathbf{y}_2 = (\mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_2)$

$\mathbf{Y}_2'\mathbf{X}_2 = (\mathbf{y}_1\mathbf{x}_2, \mathbf{y}_1\mathbf{x}_3)$ $\quad \mathbf{X}'\mathbf{Y}_2 = (\mathbf{x}_1\mathbf{y}_1, \mathbf{x}_2\mathbf{y}_1, \mathbf{x}_3\mathbf{y}_1)$

$\mathbf{X}_2'\mathbf{X}_2 = \begin{pmatrix} \mathbf{x}_2\mathbf{x}_2 & \mathbf{x}_2\mathbf{x}_3 \\ \mathbf{x}_2\mathbf{x}_3 & \mathbf{x}_3\mathbf{x}_3 \end{pmatrix}$ $\quad \mathbf{X}'\mathbf{y}_2 = (\mathbf{x}_1\mathbf{y}_2, \mathbf{x}_2\mathbf{y}_2, \mathbf{x}_3\mathbf{y}_2)$

$\mathbf{Y}_2'\mathbf{X} = (\mathbf{y}_1\mathbf{x}_1, \mathbf{y}_1\mathbf{x}_2, \mathbf{y}_1\mathbf{x}_3)$

$$(\mathbf{X}'\mathbf{X})^{-1} = \frac{1}{376}\begin{pmatrix} 86 & -6 & -14 \\ -6 & 66 & -34 \\ -14 & -34 & 46 \end{pmatrix}$$

$$\begin{pmatrix} \hat{\beta}_2 \\ \hat{\gamma}_{21} \\ \hat{\gamma}_{22} \end{pmatrix} = \left(\begin{array}{c|c} (4 \;\; 3 \;\; 5)\cdot(\mathbf{X}'\mathbf{X})^{-1}\begin{pmatrix}4\\3\\5\end{pmatrix} & (3 \;\; 5) \\ \hline \begin{pmatrix}3\\5\end{pmatrix} & \begin{pmatrix}10 & 8\\ 8 & 15\end{pmatrix} \end{array}\right)^{-1} \cdot \left(\begin{array}{c} (4 \;\; 3 \;\; 5)\cdot(\mathbf{X}'\mathbf{X})^{-1}\begin{pmatrix}3\\6\\7\end{pmatrix} \\ \hline \begin{pmatrix}6\\7\end{pmatrix} \end{array}\right)$$

A po vyčíslení tedy $\begin{pmatrix} {}_{2SLS}\hat{\beta}_2 \\ {}_{2SLS}\hat{\gamma}_{21} \\ {}_{2SLS}\hat{\gamma}_{22} \end{pmatrix} = \begin{pmatrix} 0{,}4844 \\ 0{,}3672 \\ 0{,}1094 \end{pmatrix}$

Kovarianční matice OLS-odhadové funkce parametrů 2. regresní rovnice

$$Cov\begin{pmatrix} {}_{OLS}\hat{\beta}_2 \\ {}_{OLS}\hat{\gamma}_{22} \\ {}_{OLS}\hat{\gamma}_{32} \end{pmatrix} = {}_{OLS}\, s2_e (X'X)^{-1}$$

Nyní opět využijeme platnost vztahu ${}_{OLS}SSE2 = y'y - y'X\,{}_{OLS}b$ neboli po dosazení

${}_{OLS}SSE2 = \sum y_{t2}^2 - {}_{OLS}\hat{\beta}_2 . \sum y_{t1} y_{t2} - {}_{OLS}\hat{\gamma}_{22} . \sum y_{t2} x_{t2} - {}_{OLS}\hat{\gamma}_{32} . \sum y_{t2} x_{t3}$ Vyčíslíme:

${}_{OLS}SSE2 = 10 - 6.0,193 - 6.0,384 - 7.0,197 = 5,159$, z čehož plyne

$${}_{OLS}s2^2{}_e = \frac{{}_{OLS}SSE2}{T-k} = \frac{5,159}{25-3} = 0,2345$$

$${}_{OLS}Cov\begin{pmatrix} \hat{\beta}_2 \\ \hat{\gamma}_{22} \\ \hat{\gamma}_{32} \end{pmatrix} = 0,2345.\begin{pmatrix} 0,054603 & -0,00317 & -0,01651 \\ -0,00317 & 0,174603 & -0,09206 \\ -0,01651 & -0,09206 & 0,12127 \end{pmatrix} = \begin{pmatrix} 0,012804 & -0,00074 & -0,00387 \\ -0,00074 & 0,040944 & -0,02159 \\ -0,00387 & -0,02159 & 0,028438 \end{pmatrix}$$

pro porovnání

$${}_{2SLS}Cov\begin{pmatrix} \hat{\beta}_2 \\ \hat{\gamma}_{22} \\ \hat{\gamma}_{32} \end{pmatrix} = 0,164984.\begin{pmatrix} 0,4934 & -0,0287 & -0,1492 \\ -0,0287 & 0,1761 & -0,0844 \\ -0,1492 & -0,0844 & 0,1614 \end{pmatrix} = \begin{pmatrix} 0,081403 & -0,00474 & -0,02462 \\ -0,00474 & 0,02905 & -0,01392 \\ -0,02462 & -0,01392 & 0,02663 \end{pmatrix}$$

B3) Asymptotická kovarianční matice 2SLS-odhadové funkce parametrů

1. regresní rovnice

má tvar

$$\sqrt{T}(\hat{\delta}_{.i}-\delta_{.i})\approx N[\,0,\sigma_{ii}.\mathrm{plim}\left(\frac{Q_i'Q_i}{T}\right)^{-1}]$$

$$Q_i[q,m_i+q_i]=(R^{-1}[q,q]X'[q,T]Y_i[T,m_i];R^{-1}[q,q]X'[q,T]X_i[T,q_i])\ ,$$

Výpočetní vzorec pro odhad parametrů metodou 2SLS pro danou situaci:

$${}_{2SLS}\mathrm{Cov}\begin{pmatrix}\hat{\beta}_1\\ \hat{\gamma}_{11}\end{pmatrix}={}_{2SLS}\,s_e{}^2.(Q_1'Q_1)^{-1}\ \text{, kde}$$

$$(Q_1'Q_1)^{-1}=\begin{pmatrix}Y_1'X(X'X)^{-1}X'Y_1 & Y_1'X_1\\ X_1'Y_1 & X_1'X_1\end{pmatrix}^{-1}=$$

$$=\begin{pmatrix}(3\quad 6\quad 7).\begin{pmatrix}0{,}2287 & -0{,}016 & -0{,}0372\\ -0{,}016 & 0{,}1755 & -0{,}0904\\ -0{,}0372 & -0{,}0904 & 0{,}1223\end{pmatrix}.\begin{pmatrix}3\\6\\7\end{pmatrix} & 3\\ 3 & 5\end{pmatrix}^{-1}=\begin{pmatrix}4{,}6383 & 3\\ 3 & 5\end{pmatrix}^{-1}$$

$$=\frac{1}{14{,}1915}\begin{pmatrix}5 & -3\\ -3 & 4{,}6383\end{pmatrix}=\begin{pmatrix}0{,}3523245 & -0{,}21139\\ -0{,}21139 & 0{,}326836\end{pmatrix}$$

a

$${}_{2SLS}s_e{}^2=\frac{{}_{2SLS}SSE}{T}$$

$$SSE=e'e=(y-Xb)'.(y-Xb)=y'y-y'Xb-b'X'y+b'X'Xb=$$
$$y'y-y'X(X'X)^{-1}X'y-y'X(X'X)^{-1}XX'y+y'X(X'X)^{-1}X'X(X'X)^{-1}X'y=$$
$$y'y-y'X(X'X)^{-1}X'y-y'X(X'X)^{-1}XX'y+y'X(X'X)^{-1}X'y=y'y-y'X(X'X)^{-1}X'y$$
$$y'y-y'Xb=y'\left[I-X(X'X)^{-1}X'\right]y=y'My$$

V našem případě tedy využijeme platnost vztahu

$$SSE1=y'y-y'Xb \quad \text{neboli} \quad SSE1=\sum y_t{}^2-\hat{a}.\sum y_t-\hat{b}.\sum y_t x_t$$

$${}_{2SLS}SSE1=\sum y_{t1}{}^2-{}_{2SLS}\,\hat{\beta}_1.\sum y_{t1}y_{t2}-{}_{2SLS}\,\hat{\gamma}_{11}.\sum y_{t1}x_{t1}$$

$${}_{2SLS}SSE1=20-0{,}36882*6-0{,}57871*4=15{,}47224$$

$${}_{2SLS}s1^2{}_e=\frac{15{,}47224}{25}=0{,}61889$$

$${}_{2SLS}\mathrm{Cov}\begin{pmatrix}\hat{\beta}_1\\ \hat{\gamma}_{11}\end{pmatrix}=0{,}61889.\begin{pmatrix}0{,}3523245 & -0{,}21139\\ -0{,}21139 & 0{,}326836\end{pmatrix}=\begin{pmatrix}0{,}218049 & -0{,}13084\\ -0{,}13084 & 0{,}202275\end{pmatrix}$$

B4) Asymptotická kovarianční matice 2SLS-odhadové funkce parametrů

2. regresní rovnice

$$ {}_{2SLS}\mathrm{Cov}\begin{pmatrix}\hat\beta_2\\ \hat\gamma_{22}\\ \hat\gamma_{32}\end{pmatrix} = {}_{2SLS}\, s_e^{\,2}.(Q_2'Q_2)^{-1} $$, kde

$$ (Q_2'Q_2)^{-1} = \begin{pmatrix} Y_2'X(X'X)^{-1}X'Y_2 & Y_2'X_2 \\ X_2'Y_2 & X_2'X_2 \end{pmatrix}^{-1} = $$

$$ = \begin{pmatrix} (4\ \ 3\ \ 5)(X'X)^{-1}\begin{pmatrix}4\\3\\5\end{pmatrix} & (3\ \ 5) \\ \begin{pmatrix}3\\5\end{pmatrix} & \begin{pmatrix}10 & 8\\ 8 & 15\end{pmatrix} \end{pmatrix}^{-1} = \begin{pmatrix} (4\ \ 3\ \ 5)\begin{pmatrix}0{,}2287 & -0{,}0160 & -0{,}0372\\ -0{,}016 & 0{,}1755 & -0{,}0904\\ -0{,}0372 & -0{,}0904 & 0{,}1223\end{pmatrix}\begin{pmatrix}4\\3\\5\end{pmatrix} & (3\ \ 5) \\ \begin{pmatrix}3\\5\end{pmatrix} & \begin{pmatrix}10 & 8\\ 8 & 15\end{pmatrix} \end{pmatrix}^{-1} $$

$$ = \begin{pmatrix} 3{,}71277 & 3 & 5\\ 3 & 10 & 8\\ 5 & 8 & 15\end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix} 150-64 & -(45-40) & 24-50\\ -(45-40) & 3.71277*15-25 & -(3{,}71277*8-15)\\ 24-50 & -(3{,}71277*8-15) & 37{,}1277-9\end{pmatrix} = ..= $$

$$ \begin{pmatrix} 0{,}4934 & -0{,}0287 & -0{,}1492\\ -0{,}0287 & 0{,}1761 & -0{,}0844\\ -0{,}1492 & -0{,}0844 & 0{,}1614\end{pmatrix} $$

$$ {}_{2SLS}SSE2 = \sum y_{t2}^{\,2} - {}_{2SLS}\hat\beta_2.\sum y_{t1}y_{t2} - {}_{2SLS}\hat\gamma_{22}.\sum y_{t2}x_{t2} - {}_{2SLS}\hat\gamma_{32}.\sum y_{t2}x_{t3} $$

$$ {}_{2SLS}SSE2 = 10 - 6.0{,}4844 - 6.0{,}3672 - 7.0{,}1094 = 4{,}1246 $$

$$ {}_{2SLS}s2^{2}_{e} = \frac{4{,}1246}{25} = 0{,}164984 $$

$$ {}_{2SLS}\mathrm{Cov}\begin{pmatrix}\hat\beta_2\\ \hat\gamma_{22}\\ \hat\gamma_{32}\end{pmatrix} = 0{,}164984.\begin{pmatrix} 0{,}4934 & -0{,}0287 & -0{,}1492\\ -0{,}0287 & 0{,}1761 & -0{,}0844\\ -0{,}1492 & -0{,}0844 & 0{,}1614\end{pmatrix} $$

Pro doplnění:

0,054603	0,2345	0,012804		0,4934	0,164984	0,081403
-0,00317	0,2345	-0,00074		-0,0287	0,164984	-0,00474
-0,01651	0,2345	-0,00387		-0,1492	0,164984	-0,02462
0,174603	0,2345	0,040944		0,1761	0,164984	0,029054
-0,09206	0,2345	-0,02159		-0,0844	0,164984	-0,01392
0,12127	0,2345	0,028438		0,1614	0,164984	0,026628

E) Odhad matice parametrů redukovaného tvaru

Odhad (OLS) matice parametrů redukovaného tvaru: $\hat{\Pi} = (\mathbf{X'X})\mathbf{X'Y}$

$$\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} 20 & 6 & 4 & 3 & 5 \\ 6 & 10 & 3 & 6 & 7 \\ 4 & 3 & 5 & 2 & 3 \\ 3 & 6 & 2 & 10 & 8 \\ 5 & 7 & 3 & 8 & 15 \end{pmatrix}$$

$$\mathbf{X'X} = \begin{pmatrix} \mathbf{x}_1\mathbf{x}_1 & \mathbf{x}_1\mathbf{x}_2 & \mathbf{x}_1\mathbf{x}_3 \\ \mathbf{x}_1\mathbf{x}_2 & \mathbf{x}_2\mathbf{x}_2 & \mathbf{x}_2\mathbf{x}_3 \\ \mathbf{x}_1\mathbf{x}_1 & \mathbf{x}_2\mathbf{x}_3 & \mathbf{x}_3\mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix} \qquad \mathbf{X'Y} = \begin{pmatrix} 4 & 3 \\ 3 & 6 \\ 5 & 7 \end{pmatrix}$$

Vypočteme inverzi k $\mathbf{X'X}$:

$$(\mathbf{X'X})^{-1} = \frac{1}{376} \cdot \begin{pmatrix} 150-64 & 30-24 & 16-30 \\ 30-24 & 75-9 & 40-6 \\ 16-30 & 40-6 & 50-4 \end{pmatrix} = \frac{1}{376} \cdot \begin{pmatrix} 86 & -6 & -14 \\ -6 & 66 & -34 \\ -14 & -34 & 46 \end{pmatrix}$$

$$\hat{\Pi} = (\mathbf{X'X})^{-1}\mathbf{X'Y} = \frac{1}{376} \cdot \begin{pmatrix} 86 & -6 & -14 \\ -6 & 66 & -34 \\ -14 & -34 & 46 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 \\ 3 & 6 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 0{,}68085 & 0{,}32979 \\ 0{,}01064 & 0{,}37234 \\ 0{,}19149 & 0{,}20213 \end{pmatrix}$$

0,68085	0,32979
0,01064	0,37234
0,19149	0,20213

D) Simultánní odhad parametrů 1. a 2. rovnice metodou 3SLS

Příslušný estimátor má tvar ${}_{3SLS}\beta = (X'\hat{\Sigma}^{-1}X)^{-1}(X'\hat{\Sigma}^{-1}y)$, resp.rozvedeno do detailní podoby

$$\begin{pmatrix} \hat{\sigma}^{11}Y_1'X(X'X)^{-1}X'Y_1 & \hat{\sigma}^{11}Y_1'X_1 & \hat{o}^{12}Y_1'X(X'X)^{-1}X'Y_2 & \hat{o}^{12}Y_1'X_2 \\ \hat{\sigma}^{11}X_1'Y_1 & \hat{\sigma}^{11}X_1'X_1 & \hat{o}^{12}X_1'Y_2 & \hat{o}^{12}X_1'X_2 \\ \hat{o}^{21}Y_2'X(X'X)^{-1}X'Y_1 & \hat{o}^{21}Y_2'X_1 & \hat{o}^{22}Y_2'X(X'X)^{-1}X'Y_2 & \hat{o}^{22}Y_2'X_2 \\ \hat{o}^{21}X_2'Y_1 & \hat{o}^{21}X_2'X_1 & \hat{o}^{22}X_2'Y_2 & \hat{o}^{22}X_2'X_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \hat{o}^{11}Y_1'X(X'X)^{-1}X'y_{.1} + \hat{o}^{12}Y_1'X(X'X)^{-1}X'y_{.2} \\ \hat{o}^{11}X_1'y_{.1} + \hat{o}^{12}X_1'y_{.2} \\ \hat{o}^{21}Y_2'X(X'X)^{-1}X'y_1 + \hat{o}^{22}Y_2'X(X'X)^{-1}X'y_2 \\ \hat{o}^{21}X_2'y_1 + \hat{o}^{22}X_2'y_2 \end{pmatrix}$$

, kde

$$X'X = \begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2x_2 & x_2x_3 \\ x_1x_1 & x_2x_3 & x_3x_3 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 10 & 8 \\ 3 & 8 & 15 \end{pmatrix}$$

$$(X'X)^{-1} = \frac{1}{376} \cdot \begin{pmatrix} 86 & -6 & -14 \\ -6 & 66 & -34 \\ -14 & -34 & 46 \end{pmatrix} = \begin{pmatrix} 0{,}2287 & -0{,}016 & -0{,}0372 \\ -0{,}016 & 0{,}1755 & -0{,}0904 \\ -0{,}0372 & -0{,}0904 & 0{,}1223 \end{pmatrix}$$

$$\begin{pmatrix} \hat{o}^{11}Y_1'X(X'X)^{-1}X'Y_1 & \hat{\sigma}^{11}Y_1'X_1 & \hat{o}^{12}Y_1'X(X'X)^{-1}X'Y_2 & \hat{o}^{12}Y_1'X_2 \\ \hat{\sigma}^{11}X_1'Y_1 & \hat{\sigma}^{11}X_1'X_1 & \hat{o}^{12}X_1'Y_2 & \hat{o}^{12}X_1'X_2 \\ \hat{o}^{21}Y_2'X(X'X)^{-1}X'Y_1 & \hat{o}^{21}Y_2'X_1 & \hat{o}^{22}Y_2'X(X'X)^{-1}X'Y_2 & \hat{o}^{22}Y_2'X_2 \\ \hat{o}^{21}X_2'Y_1 & \hat{o}^{21}X_2'X_1 & \hat{o}^{22}X_2'Y_2 & \hat{o}^{22}X_2'X_2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \hat{o}^{11}Y_1'X(X'X)^{-1}X'y_{.1} + \hat{o}^{12}Y_1'X(X'X)^{-1}X'y_{.2} \\ \hat{o}^{11}X_1'y_{.1} + \hat{o}^{12}X_1'y_{.2} \\ \hat{o}^{21}Y_2'X(X'X)^{-1}X'y_1 + \hat{o}^{22}Y_2'X(X'X)^{-1}X'y_2 \\ \hat{o}^{21}X_2'y_1 + \hat{o}^{22}X_2'y_2 \end{pmatrix}$$

$$\begin{pmatrix} \beta_1 \\ \gamma_{11} \\ \beta_2 \\ \gamma_{22} \\ \gamma_{32} \end{pmatrix} =$$

$$\begin{pmatrix} \hat{\sigma}^{11} y_2' X(X'X)^{-1}X' y_2 & \hat{\sigma}^{11} y_2 x_1 & \hat{\sigma}^{12} y_2 X(X'X)^{-1}X' y_1 & \hat{\sigma}^{12}\begin{pmatrix} y_2 x_2 & y_2 x_3 \end{pmatrix} \\ \hat{\sigma}^{11} x_1 y_2 & \hat{\sigma}^{11} x_1 x_1 & \hat{\sigma}^{12} x_1 y_1 & \hat{\sigma}^{12}\begin{pmatrix} x_1 x_2 & x_1 x_3 \end{pmatrix} \\ \hat{\sigma}^{21} y_1 X(X'X)^{-1}X' y_2 & \hat{\sigma}^{21} y_1 x_1 & \hat{\sigma}^{22} y_1 X(X'X)^{-1}X' y_1 & \hat{\sigma}^{22}\begin{pmatrix} y_1 x_2 & y_1 x_3 \end{pmatrix} \\ \hat{\sigma}^{21}\begin{pmatrix} x_2 y_2 \\ x_3 y_2 \end{pmatrix} & \hat{\sigma}^{21}\begin{pmatrix} x_2 x_1 \\ x_3 x_1 \end{pmatrix} & \hat{\sigma}^{22}\begin{pmatrix} x_2 y_1 \\ x_3 y_1 \end{pmatrix} & \hat{\sigma}^{22}\begin{pmatrix} x_2 x_2 & x_2 x_3 \\ x_2 x_3 & x_3 x_3 \end{pmatrix} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \hat{\sigma}^{11} y_2 X(X'X)^{-1}X' y_{.1} + \hat{\sigma}^{12} y_2 X(X'X)^{-1}X' y_{.2} \\ \hat{\sigma}^{11} x_1 y_{.1} + \hat{\sigma}^{12} x_1 y_{.2} \\ \hat{\sigma}^{21} y_1 X(X'X)^{-1}X' y_1 + \hat{\sigma}^{22} y_1 X(X'X)^{-1}X' y_2 \\ \hat{\sigma}^{21}\begin{pmatrix} x_2 y_1 \\ x_3 y_1 \end{pmatrix} + \hat{\sigma}^{22}\begin{pmatrix} x_2 y_2 \\ x_3 y_2 \end{pmatrix} \end{pmatrix}$$

Začneme-li postupně naplňovat tento výraz číselnými hodnotami, dostaneme:

Po dosazení za jednotlivé jednoduché momenty:

$$\begin{pmatrix} \hat{\sigma}^{11} y_2' X(X'X)^{-1}X' y_2 & \hat{\sigma}^{11} .3 & \hat{\sigma}^{12} y_2 X(X'X)^{-1}X' y_1 & \hat{\sigma}^{12}\begin{pmatrix} 6 & 7 \end{pmatrix} \\ \hat{\sigma}^{11} 3 & \hat{\sigma}^{11} 5 & \hat{\sigma}^{12} 4 & \hat{\sigma}^{12}\begin{pmatrix} 2 & 3 \end{pmatrix} \\ \hat{\sigma}^{21} y_1 X(X'X)^{-1}X' y_2 & \hat{\sigma}^{21} y_1 x_1 & \hat{\sigma}^{22} y_1 X(X'X)^{-1}X' y_1 & \hat{\sigma}^{22}\begin{pmatrix} 3 & 5 \end{pmatrix} \\ \hat{\sigma}^{21}\begin{pmatrix} 6 \\ 7 \end{pmatrix} & \hat{\sigma}^{21}\begin{pmatrix} 2 \\ 3 \end{pmatrix} & \hat{\sigma}^{22}\begin{pmatrix} 3 \\ 5 \end{pmatrix} & \hat{\sigma}^{22}\begin{pmatrix} 10 & 8 \\ 8 & 15 \end{pmatrix} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \hat{\sigma}^{11} y_2 X(X'X)^{-1}X' y_{.1} + \hat{\sigma}^{12} y_2 X(X'X)^{-1}X' y_{.2} \\ \hat{\sigma}^{11} 4 + \hat{\sigma}^{12} 3 \\ \hat{\sigma}^{21} y_1 X(X'X)^{-1}X' y_1 + \hat{\sigma}^{22} y_1 X(X'X)^{-1}X' y_2 \\ \hat{\sigma}^{21}\begin{pmatrix} 3 \\ 5 \end{pmatrix} + \hat{\sigma}^{22}\begin{pmatrix} 6 \\ 7 \end{pmatrix} \end{pmatrix}$$

Po dosazení hodnot inverze momentové matice X'X

$$(X'X)^{-1} = \begin{pmatrix} 0{,}2287 & -0{,}016 & -0{,}0372 \\ -0{,}016 & 0{,}1755 & -0{,}0904 \\ -0{,}0372 & -0{,}0904 & 0{,}1223 \end{pmatrix}$$ **, budou výrazy**

$$y_2 X(X'X)^{-1}X' y_2 = \begin{pmatrix} x_1 y_2 & x_2 y_2 & x_3 y_2 \end{pmatrix} \begin{pmatrix} x_1 x_1 & x_1 x_2 & x_1 x_3 \\ x_1 x_2 & x_2 x_2 & x_2 x_3 \\ x_1 x_3 & x_2 x_3 & x_3 x_3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} x_1 y_2 \\ x_2 y_2 \\ x_3 y_2 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & 6 & 7 \end{pmatrix} \begin{pmatrix} 0,2287 & -0,016 & -0,0372 \\ -0,016 & 0,1755 & -0,0904 \\ -0,0372 & -0,0904 & 0,1223 \end{pmatrix} . \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} = 4,6383$$

$$y_2 X(X'X)^{-1} X' y_1 = \begin{pmatrix} x_1y_2 & x_2y_2 & x_3y_2 \end{pmatrix} \begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2x_2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3x_3 \end{pmatrix}^{-1} . \begin{pmatrix} x_1y_1 \\ x_2y_1 \\ x_3y_1 \end{pmatrix}$$

$$y_2' X(X'X)^{-1} X' y_1 = \begin{pmatrix} 3 & 6 & 7 \end{pmatrix} \begin{pmatrix} 0,2287 & -0,016 & -0,0372 \\ -0,016 & 0,1755 & -0,0904 \\ -0,0372 & -0,0904 & 0,1223 \end{pmatrix} . \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = 3,4468$$

$$y_1' X(X'X)^{-1} X' y_2 = \begin{pmatrix} x_1y_1 & x_2y_1 & x_3y_1 \end{pmatrix} \begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2x_2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3x_3 \end{pmatrix}^{-1} . \begin{pmatrix} x_1y_2 \\ x_2y_2 \\ x_3y_2 \end{pmatrix}$$

$$y_1' X(X'X)^{-1} X' y_2 = \begin{pmatrix} 4 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0,2287 & -0,016 & -0,0372 \\ -0,016 & 0,1755 & -0,0904 \\ -0,0372 & -0,0904 & 0,1223 \end{pmatrix} . \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} = 3,4468$$

$$y_1' X(X'X)^{-1} X' y_1 = \begin{pmatrix} x_1y_1 & x_2y_1 & x_3y_1 \end{pmatrix} \begin{pmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2x_2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3x_3 \end{pmatrix}^{-1} . \begin{pmatrix} x_1y_1 \\ x_2y_1 \\ x_3y_1 \end{pmatrix}$$

$$y_1' X(X'X)^{-1} X' y_1 = \begin{pmatrix} 4 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0,2287 & -0,016 & -0,0372 \\ -0,016 & 0,1755 & -0,0904 \\ -0,0372 & -0,0904 & 0,1223 \end{pmatrix} . \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = 3,7128$$

Po dosazení jednotlivých momentů do matice (   ). Dostaneme [1]:

---

[1] Tato matice je rozměrů [5x5] , dimenze odpovídá počtu všech odhadovaných parametrů modelu.

(*) $$\begin{pmatrix} \hat\sigma^{11}4{,}6383 & \hat\sigma^{11}.3 & \hat\sigma^{12}3{,}4468 & \hat\sigma^{12}(6\ \ 7) \\ \hat\sigma^{11}3 & \hat\sigma^{11}5 & \hat\sigma^{12}4 & \hat\sigma^{12}(2\ \ 3) \\ \hat\sigma^{21}3{,}4468 & \hat\sigma^{21}4 & \hat\sigma^{22}3{,}7128 & \hat\sigma^{22}(3\ \ 5) \\ \hat\sigma^{21}\begin{pmatrix}6\\7\end{pmatrix} & \hat\sigma^{21}\begin{pmatrix}2\\3\end{pmatrix} & \hat\sigma^{22}\begin{pmatrix}3\\5\end{pmatrix} & \hat\sigma^{22}\begin{pmatrix}10 & 8\\8 & 15\end{pmatrix} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \hat\sigma^{11}3{,}4468+\hat\sigma^{12}4{,}6383 \\ \hat\sigma^{11}4+\hat\sigma^{12}3 \\ \hat\sigma^{21}3{,}7128+\hat\sigma^{22}3{,}4468 \\ \hat\sigma^{21}\begin{pmatrix}3\\5\end{pmatrix}+\hat\sigma^{22}\begin{pmatrix}6\\7\end{pmatrix} \end{pmatrix}$$

Závěrečným krokem pak bude dosazení prvků 2SLS- kovarianční matice reziduí do (*) a následné vyčíslení této matice :

$${}_{2SLS}\mathrm{Cov}\begin{pmatrix}e_1\\e_2\end{pmatrix} = \begin{pmatrix}\hat\sigma_{11} & \hat\sigma_{12}\\ \hat\sigma_{12} & \hat\sigma_{22}\end{pmatrix} = \begin{pmatrix}s_{11} & s_{12}\\ s_{12} & s_{22}\end{pmatrix} \text{.kde } s_{ij} = \frac{{}_{2SLS}e_i \cdot {}_{2SLS}e_j}{T}$$

$${}_{2SLS}s2^2{}_e = \frac{4{,}1246}{25} = 0{,}164984 \qquad {}_{2SLS}s1^2{}_e = \frac{15{,}47224}{25} = 0{,}61889$$

$${}_{2SLS}\mathrm{Cov}\begin{pmatrix}e_1\\e_2\end{pmatrix} = \begin{pmatrix}\hat\sigma_{11} & \hat\sigma_{12}\\ \hat\sigma_{12} & \hat\sigma_{22}\end{pmatrix} = \begin{pmatrix}0{,}61889 & s_{12}\\ s_{12} & 0{,}164984\end{pmatrix}$$

$$\begin{pmatrix} \hat\sigma^{11}.4{,}6383 & \hat\sigma^{11}.3 & \hat\sigma^{12}3{,}4468 & \hat\sigma^{12}(6\ \ 7) \\ \hat\sigma^{11}3 & \hat\sigma^{11}5 & \hat\sigma^{12}4 & \hat\sigma^{12}(2\ \ 3) \\ \hat\sigma^{21}3{,}4468 & \hat\sigma^{21}4 & \hat\sigma^{22}3{,}7128 & \hat\sigma^{22}(3\ \ 5) \\ \hat\sigma^{21}\begin{pmatrix}6\\7\end{pmatrix} & \hat\sigma^{21}\begin{pmatrix}2\\3\end{pmatrix} & \hat\sigma^{22}\begin{pmatrix}3\\5\end{pmatrix} & \hat\sigma^{22}\begin{pmatrix}10 & 8\\8 & 15\end{pmatrix} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \hat\sigma^{11}3{,}4468+\hat\sigma^{12}4{,}6383 \\ \hat\sigma^{11}4+\hat\sigma^{12}3 \\ \hat\sigma^{21}3{,}7128+\hat\sigma^{22}3{,}4468 \\ \hat\sigma^{21}\begin{pmatrix}3\\5\end{pmatrix}+\hat\sigma^{22}\begin{pmatrix}6\\7\end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix}\hat\sigma^{11} & \hat\sigma^{12}\\ \hat\sigma^{12} & \hat\sigma^{22}\end{pmatrix} = \begin{pmatrix}\hat\sigma_{11} & \hat\sigma_{12}\\ \hat\sigma_{12} & \hat\sigma_{22}\end{pmatrix}^{-1} = \begin{pmatrix}0{,}61889 & s_{12}\\ s_{12} & 0{,}164984\end{pmatrix}^{-1} =$$

$${}_{2SLS}\mathrm{Cov}\begin{pmatrix}\hat\beta_2\\ \hat\gamma_{22}\\ \hat\gamma_{32}\end{pmatrix} = 0{,}164984.\begin{pmatrix}0{,}4934 & -0{,}0287 & -0{,}1492\\ -0{,}0287 & 0{,}1761 & -0{,}0844\\ -0{,}1492 & -0{,}0844 & 0{,}1614\end{pmatrix} = .\begin{pmatrix}0{,}081403 & -0{,}00474 & -0{,}02462\\ -0{,}00474 & 0{,}02905 & -0{,}01392\\ -0{,}02462 & -0{,}01392 & 0{,}02663\end{pmatrix}$$

$$SSE12 = {}_{2SLS}e_1'\,{}_{2SLS}e_2 = (y_1 - X_{1\,2SLS}b_1)'.(y_2 - X_{2\,2SLS}b_2) = y_1'y_2 - y_1'X_2b_2 - b_1'X_1'y_2 + b_1'X_1'X$$

$$y_1'y_2 - y_1'X_2(X_2'X_2)^{-1}X_2'y_2 - y_1'X_1(X_1'X_1)^{-1}X_1'y_2 + y_1'X_1(X_1'X_1)^{-1}X_1'X_2(X_2'X_2)^{-1}X_2'y_2 =$$

$$y_1'y_2 - (y_1x_2, y_1x_3)\begin{pmatrix} x_2x_2 & x_2x_3 \\ x_2x_3 & x_3x_3 \end{pmatrix}^{-1}\begin{pmatrix} x_2y_2 \\ x_3y_2 \end{pmatrix} - (y_1x_1)(x_1x_1)^{-1}(x_1y_2)$$

$$+ (y_1x_1)'(x_1x_1)^{-1}(x_1x_2, x_1x_3)\begin{pmatrix} x_2x_2 & x_2x_3 \\ x_2x_3 & x_3x_3 \end{pmatrix}^{-1}\begin{pmatrix} x_2y_2 \\ x_3y_2 \end{pmatrix} =$$

$$SSE12 = {}_{2SLS}e_1'\,{}_{2SLS}e_2 = (y_1 - X_{1\,2SLS}b_1)'.(y_2 - X_{2\,2SLS}b_2) =$$

$$y_1'y_2 - y_1'X_2b_2 - b_1'X_1'y_2 + b_1'X_1'X_2b_2 =$$

$$y_1'y_2 - y_1(y_1 \quad x_2 \quad x_3)\begin{pmatrix} \beta_{22} \\ \gamma_{22} \\ \gamma_{32} \end{pmatrix} - (\beta_{11} \quad \gamma_{11})\begin{pmatrix} y_2 \\ x_1 \end{pmatrix} y_2 + (\beta_{11} \quad \gamma_{11})\begin{pmatrix} y_2 \\ x_1 \end{pmatrix}(y_1 \quad x_2 \quad x_3)\begin{pmatrix} \beta_{22} \\ \gamma_{22} \\ \gamma_{32} \end{pmatrix}$$

$$= y_1y_2 - (y_1y_1 \quad y_1x_2 \quad y_1x_3).\begin{pmatrix} \beta_{22} \\ \gamma_{22} \\ \gamma_{32} \end{pmatrix} - (\beta_{11} \quad \gamma_{11})\begin{pmatrix} y_2y_2 \\ x_1y_2 \end{pmatrix} + (\beta_{11} \quad \gamma_{11})\begin{pmatrix} y_2y_1 & y_2x_2 & y_2x_3 \\ x_1y_1 & x_1x_2 & x_1x_3 \end{pmatrix}\begin{pmatrix} \beta_{22} \\ \gamma_{22} \\ \gamma_{32} \end{pmatrix} =$$

$$= 6 - (20 \quad 3 \quad 5).\begin{pmatrix} {}_{2SLS}\beta_{22} \\ {}_{2SLS}\gamma_{22} \\ {}_{2SLS}\gamma_{32} \end{pmatrix} - ({}_{2SLS}\beta_{11} \quad {}_{2SLS}\gamma_{11})\begin{pmatrix} 10 \\ 3 \end{pmatrix} + ({}_{2SLS}\beta_{11} \quad {}_{2SLS}\gamma_{11})\begin{pmatrix} 6 & 6 & 7 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} {}_{2SLS}\beta_{22} \\ {}_{2SLS}\gamma_{22} \\ {}_{2SLS}\gamma_{32} \end{pmatrix} =$$

$$SSE12 = {}_{2SLS}e_1'\,{}_{2SLS}e_2 = (y_1 - X_{1\,2SLS}b_1)'.(y_2 - X_{2\,2SLS}b_2) =$$

$$= 6 - (20 \quad 3 \quad 5).\begin{pmatrix} 0{,}4844 \\ 0{,}3672 \\ 0{,}1094 \end{pmatrix} - (0{,}36882 \quad 0{,}57871)\begin{pmatrix} 10 \\ 3 \end{pmatrix} + (0{,}36882 \quad 0{,}57871)\begin{pmatrix} 6 & 6 & 7 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0{,}4844 \\ 0{,}3672 \\ 0{,}1094 \end{pmatrix} =$$

nebot' $\begin{pmatrix} {}_{2SLS}\hat{\beta}_2 \\ {}_{2SLS}\hat{\gamma}_{21} \\ {}_{2SLS}\hat{\gamma}_{22} \end{pmatrix} = \begin{pmatrix} 0{,}4844 \\ 0{,}3672 \\ 0{,}1094 \end{pmatrix}$ $\begin{pmatrix} {}_{2SLS}\hat{\beta}_1 \\ {}_{2SLS}\hat{\gamma}_{11} \end{pmatrix} = \begin{pmatrix} 0{,}36882 \\ 0{,}57871 \end{pmatrix}$

$$SSE12 = {}_{2SLS}e_1'\,{}_{2SLS}e_2 = (y_1 - X_{1\,2SLS}b_1)'.(y_2 - X_{2\,2SLS}b_2) =$$
$$= 6 - 11{,}3366 - 5{,}4243 + 3{,}9032 = -6{,}8555$$

$${}_{2SLS}s12_e = \frac{-6{,}8555}{25} = -0{,}27422$$

$$\begin{pmatrix}\hat\sigma^{11} & \hat\sigma^{12}\\ \hat\sigma^{12} & \hat\sigma^{22}\end{pmatrix} = \begin{pmatrix}\hat\sigma_{11} & \hat\sigma_{12}\\ \hat\sigma_{12} & \hat\sigma_{22}\end{pmatrix}^{-1} = \begin{pmatrix}0{,}61889 & -0{,}27422\\ -0{,}27422 & 0{,}164984\end{pmatrix}^{-1} = \begin{pmatrix}6{,}1309 & 10{,}1901\\ 10{,}1901 & 22{,}9982\end{pmatrix}$$

$$\begin{pmatrix}6{,}1309.4{,}6383 & 6{,}1309.3 & 10{,}1901.3{,}4468 & 10{,}1901.(6\quad 7)\\ 6{,}1309.3 & 6{,}1309.5 & 10{,}1901.4 & 10{,}1901(2\quad 3)\\ 10{,}1901.3{,}4468 & 10{,}1901.4 & 22{,}9982.3{,}7128 & 22{,}9982.(3\quad 5)\\ 10{,}1901\begin{pmatrix}6\\7\end{pmatrix} & 10{,}1901.\begin{pmatrix}2\\3\end{pmatrix} & 22{,}9982\begin{pmatrix}3\\5\end{pmatrix} & 22{,}9982.\begin{pmatrix}10 & 8\\ 8 & 15\end{pmatrix}\end{pmatrix}^{-1} . \begin{pmatrix}6{,}1309.3{,}4468 + 10{,}1901.4{,}6383\\ 6{,}1309.4 + 10{,}1901.3\\ 10{,}1901.3{,}7128 + 22{,}9982.3{,}4468\\ 10{,}1901\begin{pmatrix}3\\5\end{pmatrix} + 22{,}9982\begin{pmatrix}6\\7\end{pmatrix}\end{pmatrix}$$

$$\begin{pmatrix}{}_{3SLS}\hat\beta_1\\ {}_{3SLS}\hat\gamma_{11}\\ {}_{3SLS}\hat\beta_2\\ {}_{3SLS}\gamma_{22}\\ {}_{3SLS}\hat\gamma_{23}\end{pmatrix} = \begin{pmatrix}10{,}43695 & 18{,}3927 & 35{,}12324 & 61{,}1406 & 71{,}3307\\ 18{,}3927 & 30{,}6545 & 40{,}7604 & 20{,}3802 & 30{,}5703\\ 35{,}12324 & 40{,}7604 & 85{,}38772 & 68{,}9946 & 114{,}991\\ 61{,}1406 & 20{,}3802 & 68{,}9946 & 229{,}982 & 344{,}973\\ 71{,}3307 & 30{,}5703 & 114{,}991 & 344{,}973 & 211{,}9379\end{pmatrix}^{-1} . \begin{pmatrix}68{,}39673\\ 55{,}0939\\ 117{,}104\\ 168{,}5595\\ 211{,}9379\end{pmatrix} = \begin{pmatrix}0{,}3688\\ 0{,}5787\\ 0{,}4717\\ 0{,}3112\\ 0{,}1636\end{pmatrix}$$

Takže výsledný 3SLS-odhad parametrů celé pětice parametrů je (bez záruky) :

$$\begin{pmatrix}{}_{3SLS}\hat\beta_1\\ {}_{3SLS}\hat\gamma_{11}\\ {}_{3SLS}\hat\beta_2\\ {}_{3SLS}\gamma_{22}\\ {}_{3SLS}\hat\gamma_{23}\end{pmatrix} = \begin{pmatrix}0{,}3688\\ 0{,}5787\\ 0{,}4717\\ 0{,}3112\\ 0{,}1636\end{pmatrix},\ \text{zatímco}\ \begin{pmatrix}{}_{2SLS}\hat\beta_1\\ {}_{2SLS}\hat\gamma_{11}\end{pmatrix} = \begin{pmatrix}0{,}36882\\ 0{,}57871\end{pmatrix}\ \text{a}\ \begin{pmatrix}{}_{2SLS}\hat\beta_2\\ {}_{2SLS}\hat\gamma_{21}\\ {}_{2SLS}\hat\gamma_{22}\end{pmatrix} = \begin{pmatrix}0{,}4844\\ 0{,}3672\\ 0{,}1094\end{pmatrix}$$