3. LINEAR TIME-INVARIANT SYSTEMS
(Linear transfer function models)

$$u_t \longrightarrow \boxed{\quad LS \quad} \longrightarrow X_t$$

For simplicity : • dimension $= 1$ (otherwise $\underline{u}_t, \underline{X}_t$ are vectors)
• $t \in T$ where $T = \mathbb{R}$ or $T = \mathbb{Z}$

$u, X : \left.\begin{array}{l} T \to \mathbb{R} \text{ (or } \mathbb{C}) \ldots \text{deterministic} \\ T \to \text{random var.} \ldots \text{random} \end{array}\right\}$ functions, $u(t) := u_t$; $X(t) := X_t$
$\underbrace{}$ over scalar field $\mathbb{R}$ or $\mathbb{C}$ $(=: \mathbb{F})$

$\left.\begin{array}{l} u \in \mathcal{F}_1 \ldots \text{space of inputs} \\ X \in \mathcal{F}_2 \ldots \text{space of outputs} \end{array}\right\}$ normed linear spaces (NL-spaces)
shift-invariant : $y(\cdot) \in \mathcal{F}_i \Rightarrow y(\cdot - a) \in \mathcal{F}_i \;\; \forall a \in T$

$\|\cdot\|_i$ … norm on $\mathcal{F}_i$ and $\xrightarrow{\mathcal{F}_i}$ convergence in norm : $y_n \xrightarrow{\mathcal{F}_i} y \equiv \|y - y_n\|_i \to 0$ as $n \to \infty$.
$\|\cdot\|$ … norm on $u(T)$ or $X(T)$, usually abs. value. (or in another sense)

3.1. Definition

A mapping $LS : \mathcal{F}_1 \to \mathcal{F}_2$ is called linear (time-invariant, shift-invariant) system (LTI-system) if the following holds:

! (1) Linearity: $LS(u_1 + u_2) = LS(u_1) + LS(u_2) \quad \forall u_1, u_2 \in \mathcal{F}_1$
$LS(c \cdot u) = c \cdot LS(u) \quad \forall u \in \mathcal{F}_1, c \in \mathbb{F}$

! (2) Shift-invariance (time-invariance):
$$LS(u) = X \Rightarrow LS(u(\cdot - a)) = X(\cdot - a)$$
$$\forall u \in \mathcal{F}_1 \text{ and } a \in T.$$

! (3) Continuity: $u_n \xrightarrow{\mathcal{F}_1} u \Rightarrow LS(u_n) \xrightarrow{\mathcal{F}_2} LS(u)$

Moreover LS is called

! (4) Causal if $\forall t \in T : u_1(\tau) = u_2(\tau) \;\forall \tau \le t \Rightarrow LS(u_1)(t) = LS(u_2)(t)$
($X(t) = LS(u)(t)$ depends only on input values $u(\tau), \tau \le t$)

! (5) Stable if $\|u(t)\| \le C_1 < \infty \Rightarrow \exists C_2 < \infty : \|LS(u)(t)\| \le C_2 \;\forall t$
(bounded inputs produce bounded outputs).

$\left.\begin{array}{l} T = \mathbb{R} \\ T = \mathbb{Z} \end{array}\right\} \Rightarrow LS$ is called $\left\{\begin{array}{l} \text{continuous-time linear system} \\ \text{discrete-time} \quad - \text{"} - \end{array}\right.$

3.2. Example of continuous-time LS ($T = \mathbb{R}$)

(3.1) $$\boxed{x(t) \overset{a.e.}{=} \int_{-\infty}^{\infty} u(t - \tau) h(\tau) \, d\tau} \equiv x = LS_h(u) \text{ ; } h \ldots \text{fixed}$$

$\mathcal{F}_1, \mathcal{F}_2$ … suitable function spaces ($\mathcal{F}_i := L_p, 1 \le p < \infty$)

integral convolution: $x = u * h$

3.3. Discretization of (3.1) with time step Δt

$u(t) \approx u_m$ for $m\Delta t \le t < (m+1)\Delta t$ ... piecewise constant function

$$\Downarrow$$

$$x_n := x(n\Delta t) \overset{(3.1)}{=} \int_{-\infty}^{\infty} u(n\Delta t - \tau)\, h(\tau)\, d\tau = \sum_{k=-\infty}^{\infty} \int_{(k-1)\Delta t}^{k\Delta t} \underbrace{u(n\Delta t - \tau)}_{(n-k)\Delta t \le n\Delta t - \tau < (n-k+1)\Delta t} h(\tau)\, d\tau \approx$$

$$\approx \sum_{k=-\infty}^{\infty} \int_{(k-1)\Delta t}^{k\Delta t} u_{n-k}\, h(\tau)\, d\tau$$

$$= \sum_{k=-\infty}^{\infty} u_{n-k} \int_{(k-1)\Delta t}^{k\Delta t} h(\tau)\, d\tau = \sum_{k=-\infty}^{\infty} u_{n-k}\, h_k \quad \text{where } h_k := \int_{(k-1)\Delta t}^{k\Delta t} h(\tau)\, d\tau.$$

$\Downarrow$ usually with $\Delta t = 1$

3.4. Example of discrete-time LS $(T = \mathbb{Z})$

! (3.2) $$\boxed{x(t) = \sum_{k=-\infty}^{\infty} u(t-k)\, h(k)}$$ $\equiv$ $x = LS_h(u)$ where u, x, h ... sequences provided that the series converges in some sense.

! Discrete Linear Convolution (DLC): $x = u * h$

$\mathcal{F}_1, \mathcal{F}_2$ ... suitable spaces of sequences, typically:

$\mathcal{F}_i = \ell_p = \left\{ \{x(t)\}_{t\in\mathbb{Z}} \;\middle|\; \sum_{t=-\infty}^{\infty} |x(t)|^p < \infty \right\}, \; 1 \le p < \infty$

$\|x\|_i := \|x\|_p := \sqrt[p]{\sum_{t=-\infty}^{\infty} |x(t)|^p}$; $\quad x_n \xrightarrow{\mathcal{F}_i} x \;\equiv\; \|x - x_n\|_i \to 0$ as $n \to \infty$.

$p = 2$ ... Euclidean norm as an analog of vector norm for sequences.

! 3.5. Notation

$\delta := \{\delta(t)\}_{t\in\mathbb{Z}}$ where $\delta(t) = \underset{\text{kronecker symbol}}{\delta_{0,t}} = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{otherwise} \end{cases}$ ... Dirac sequence

$\{\delta(t-k)\}_{t\in\mathbb{Z}}$ ... shifted Dirac sequence: $\delta(t-k) = \begin{cases} 1 & \text{for } t = k \\ 0 & \text{otherwise} \end{cases}$

$x^- := \{x(-t)\}_{t\in\mathbb{Z}}$ for $x = \{x(t)\}_{t\in\mathbb{Z}}$ ... reversed sequence to x

3.6. Theorem

$h \in \ell_1$, $u \in \ell_p \Rightarrow$ (3.2) converges in usual sense for each $t \in \mathbb{Z}$, and $x \in \ell_p$. Moreover it holds: $\|x\|_p \le \|u\|_p \cdot \|h\|_1$ which confirms continuity (3). As the validity of (1) and (2) is evident, $LS_h : \ell^p \to \ell^p$ is a linear system for each $h \in \ell_1$.

! 3.7. Theorem

If (3.2) converges in usual sense for each $t \in \mathbb{Z}$ and $LS_h : \mathcal{F}_1 \to \mathcal{F}_2$ defines a linear system, then

(1) $h \in \ell_1$, i.e. $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$ (socalled stability condition),

$\Downarrow$

LS_h is stable.

(2) LS_h is causal $\Leftrightarrow$ $h(k) = 0$ for all $k < 0$.
In such a case (3.2) attains the form:

$$(3.2') \quad \boxed{x(t) = \sum_{k=0}^{\infty} u(t-k)\, h(k)}$$

3.8. Corollary

$h \in \ell_1$, $\mathcal{F}_1 = \mathcal{F}_2 = \ell_p \Rightarrow LS_h : \ell_p \to \ell_p$ is a stable linear system.

3.9. Theorem

Let $LS : \mathcal{F}_1 \to \mathcal{F}_2$ be a discrete linear system, $\mathcal{F}_1$ and $\mathcal{F}_2$ spaces of number sequences such that $\delta \in \mathcal{F}_1$ and $\xrightarrow{\mathcal{F}_1}$ is either convergence in usual sense (point convergence) or convergence in norm $\|\cdot\|_p$ in case $\mathcal{F}_1 = \mathcal{F}_2 = \ell_p$.
Then there exists unique $h \in \mathcal{F}_2$: $LS = LS_h$ by (3.2) where $h = LS(\delta)$.

3.10. Corollary

$LS : \ell_p \to \ell_p$ arbitrary $\Rightarrow \underset{\text{unique}}{\exists!}\, h \in \ell_p$, $h = LS(\delta)$: $LS = LS_h$.

! 3.11. Definition

Every discrete linear system as of theorem 3.9 is uniquely determined by the sequence $h = LS(\delta)$ which is called its impulse response.

h finite ($h(k) \neq 0$ only for a finitely many $k \in \mathbb{Z}$)... LS is said to be finite impulse response linear system (FIR-LS)

h infinite ... LS is said to be an infinite impulse response linear system (IIR-LS)

$$\{\delta(t)\} \longrightarrow \boxed{\;LS\;} \longrightarrow \{h(t)\}, \quad h = \delta * h$$

! 3.12. Example

Discrete linear filter = weighted moving average = LS_h for suitable h

Geometric vizualization of (3.2):

Input: $\dots\ u(t-2)\quad u(t-1)\quad u(t)\quad u(t+1)\quad u(t+2)\ \dots$

Weights h^-: $\dots\ h(2)\quad h(1)\quad h(0)\quad h(-1)\quad h(-2)\ \dots$ → move according to t

Σ

$$x(t)=\sum_{k=-\infty}^{\infty} u(t-k)\,h(k) \overset{k\to -k}{=} \sum_{k=-\infty}^{\infty} u(t+k)\,\underbrace{h(-k)}_{h^-(k)}$$

$\Downarrow$

$x(t)$ = weighted mean from values $u(t\pm k)$ using weights $h^-(k)$

Simple moving average: $h=h^-=\{\dots 0, \underbrace{\tfrac{1}{m} \dots \underset{\substack{\uparrow\\0}}{\tfrac{1}{m}} \dots \tfrac{1}{m}}_{m\text{-times}}, 0, \dots\}$

! 3.13. Definition (z-transform)

$h \rightsquigarrow H(z) := \sum_{k=-\infty}^{\infty} h(k)\,z^k$, $z\in\mathbb{C}$ ... formal Laurent series ... z-transform of h

$$h^- \rightsquigarrow H^-(z) = \sum_{k=-\infty}^{\infty} h(-k)\,z^k \overset{k\leftrightarrow -k}{=} \sum_{k=-\infty}^{\infty} h(k)\,z^{-k} = H\left(\tfrac{1}{z}\right)$$

! 3.14. Theorem

(1) LS_h is causal $\Leftrightarrow$ $H(z)=\sum_{k=0}^{\infty} h(k)\,z^k$ is formal power series

(1') LS_h is causal FIR-LS $\Leftrightarrow$ $H(z)$ is a polynomial.

(2) LS_h stable by $h\in l_1 \Leftrightarrow H(z)$ is absolutely convergent on a ring (annulus) $\{z \mid r_1<|z|<r_2\} =: C(r_1,r_2)$ for some $0<r_1<1<r_2<\infty$ (a ring containing unit circle $\{z \mid |z|=1\} =: C(1)$).

(3) LS_h both causal and stable (by $h\in l_1$) $\Leftrightarrow$ $H(z)$ is a power series absolutely konvergent on a disc $\{z \mid |z|<r\} =: D(r)$ for some $1<r$ (a disc containing unit circle).

! (4) LS_h stable by $h\in l_1 \Rightarrow$

(i) $LS_h(z^{-t}) = H(z)\,z^{-t}$ $\forall z\in C(r_1,r_2)$ as of (2) or $z\in D(r)$ as of (3)

(ii) $LS_h(z^{t}) = H\left(\tfrac{1}{z}\right) z^{t}$ $\forall z\in C\left(\tfrac{1}{r_2},\tfrac{1}{r_1}\right)$ or $z\in\left\{z \mid |z|>\tfrac{1}{r}\right\}$

where r_1, r_2 are as of (2) or r is as of (3).

(5) $h(k)\in\mathbb{R}$ $\forall k\in\mathbb{Z} \Rightarrow H(\bar{z}) = \overline{H(z)}$ and $H\left(\tfrac{1}{\bar{z}}\right) = \overline{H\left(\tfrac{1}{z}\right)}$.

! (6) $x=u*h \Leftrightarrow X(z)=U(z)\cdot H(z)$: $U(z)$ abs. conv on G_1, $H(z)$ on $G_2 \Rightarrow X(z)$ on $G_1\cap G_2$

! 3.15. Definition

z-transform $\mathcal{H}(\frac{1}{z})$ (or alternatively $\mathcal{H}(z)$ - it is a matter of convention) of the impulse response h of a linear system LS_h is called its transfer function.

$$\{z^t\}_{t\in\mathbb{Z}} \longrightarrow \boxed{LS_h} \longrightarrow \underbrace{\mathcal{H}(\tfrac{1}{z})}_{\text{transfer coefficient}} \cdot z^t$$

3.16. Theorem (Transfer of sine/cosine waves through a stable LS_h)

Let LS_h be a stable ($h \in \ell_1$) linear system with the transfer function $\mathcal{H}(\frac{1}{z})$. Then the following holds ($c \in \mathbb{R}$ or $c \in \mathbb{C}$):

(3.3) $\boxed{LS_h(c \cdot e^{i\omega t}) = \mathcal{H}(e^{-i\omega}) \cdot c\, e^{i\omega t}}$

(3.3') $\boxed{LS_h(c \cdot e^{i(\omega t - \varphi)}) = |\mathcal{H}(e^{-i\omega})| c \cdot e^{i(\omega t - (\varphi + \psi))}}$

where $\mathcal{H}(e^{-i\omega}) = |\mathcal{H}(e^{-i\omega})| \cdot e^{-i\psi}$

$|\mathcal{H}(e^{-i\omega})|$ ... amplitude transfer coefficient

ψ ... transfer phase shift

(3.3'') $\boxed{LS_h(A \cdot \cos(\omega t - \varphi))} = \underbrace{|\mathcal{H}(e^{-i\omega})|}_{\text{amplitude transfer coeff.}} \cdot A \cdot \cos(\omega t - (\varphi + \underset{\substack{\uparrow \\ \text{transfer phase shift}}}{\psi}))$, $A \in \mathbb{R}$, $h(k) \in \mathbb{R}$

(3.3''') $\boxed{LS_h(\{c\}) = c \cdot \sum_{k=-\infty}^{\infty} h(k)}$

$\left.\begin{array}{l} \omega = 2\pi f \text{ is angle velocity } (f = \text{frequency}) \\ \varphi = \quad \text{is phase shift} \end{array}\right\}$ of the input wave function

3.17. Definition

A causal and stable LS_h, $h \in \ell_1$, is called a recursive linear system (or ARMA-LS) of order (p, q), $p > 0$, $q \geq 0$ integers, if there exist polynomials: $\Theta(z) = \Theta_0 + \Theta_1 z + \dots + \Theta_q z^q$, $\Theta_q \neq 0$ and $\Phi(z) = \Phi_0 - \Phi_1 z - \dots - \Phi_p z^p$, $\Phi_0 \neq 0$, $\Phi_p \neq 0$ such that $\mathcal{H}(z) = \dfrac{\Theta(z)}{\Phi(z)}$. [We can assume without the loss of generality, that $\Theta(z)$ and $\Phi(z)$ have no common roots and that $\Phi_0 = 1$].

3.18. Theorem

LS_h recursive of order (p,q) $\Rightarrow$ $\Phi(z)$ has all roots outside of the unit disc $\{z \mid |z| \le 1\}$
[or: z-tr. $\Phi(\frac{1}{z})$ has all roots inside of — " —]
and for each $u \in \ell_1$ is $x = LS_h(u)$ iff the following relation holds:

$$(3.4) \quad x_t = \underbrace{\phi_1 x_{t-1} + \dots + \phi_p x_{t-p}}_{\text{AR(p) component}} + \underbrace{\Theta_0 u_t + \dots + \Theta_q u_{t-q}}_{\text{MA(q) component}}$$

AR(p) component ↳ Autoregression = feedback

MA(q) component ↳ Moving Average

In such a case $x \in \ell_1$ as well.

Proof

I. $h \in \ell_1$ by definition 3.17, $u \in \ell_1$ by assumption $\Rightarrow$ $x \in \ell_1$ by 3.6 where $p=1$.

II. LS_h recursive $\Rightarrow$ LS_h causal and stable with $H(z) = \frac{\Theta(z)}{\Phi(z)} \overset{3.14(3)}{\Rightarrow}$
$H(z)$ is a power series absolutely convergent for $|z| \le 1$ $\Rightarrow$ $\Phi(z) \neq 0$ for all $z: |z| \le 1$ (otherwise such root would be pole of $H(z)$ which contradicts absolute convergence) $\Rightarrow$ all roots of $\Phi(z)$ lie outside of the unit disc.

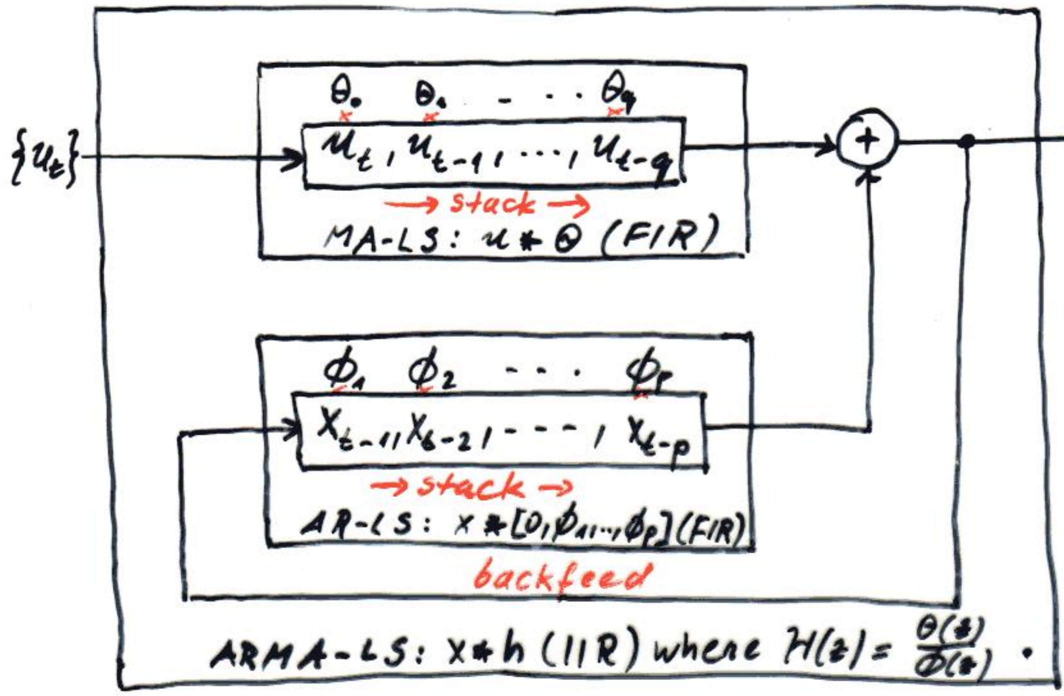
III. Let $u \in \ell_1$ be arbitrary. As $h \in \ell_1$, both $\sum_{t=-\infty}^{\infty} |u(t)| < \infty$ and $\sum_{k=0}^{\infty} |h(k)| < \infty$ are absolutely convergent $\overset{3.14(2)}{\Rightarrow}$ $U(z)$ is abs. conv. on $C(r_1, r_2)$ and $H(z)$ on $D(r)$ where we may assume without the loss of generality that $0 < r_1 < 1 < r_2 = r$. It is well-known from the calculus that their Cauchy product $X(z) := U(z) \cdot H(z)$ is a series (see 3.14(6))
$X(z) = \sum_{t=-\infty}^{\infty} x(t) z^t$ absolutely convergent at least on $C(r_1, r_2)$
where $x(t) = \sum_{k=0}^{\infty} u(t-k) h(k)$, i.e. $x = u * h$ (convolution):

$$\left(\sum_{\ell=-\infty}^{\infty} u(\ell) z^\ell\right) \cdot \left(\sum_{k=0}^{\infty} h(k) z^k\right) \overset{(*)}{=} \sum_{\ell=-\infty}^{\infty} \sum_{k=0}^{\infty} u(\ell) \cdot h(k) z^{\ell+k} \overset{t:\ell+k}{=} \sum_{t=-\infty}^{\infty} \underbrace{\left(\sum_{k=0}^{\infty} u(t-k) h(k)\right)}_{x(t)} z^t$$

Then the following equivalence relations clearly hold for all $z \in C(r_1, r_2)$:
$x = LS_h(u) \Leftrightarrow x = u * h \Leftrightarrow X(z) = U(z) \cdot H(z) = U(z) \cdot \frac{\Theta(z)}{\Phi(z)} \overset{①}{\Leftrightarrow}$
$X(z) \cdot \Phi(z) = U(z) \cdot \Theta(z) \overset{\text{by } (*)}{\Leftrightarrow} x * \phi = u * \theta \Leftrightarrow$
$\phi_0 x_t - \phi_1 x_{t-1} - \dots - \phi_p x_{t-p} = \theta_0 u_t + \theta_1 u_{t-1} + \dots + \theta_q u_{t-q} \Leftrightarrow (3.4)$ □

Note ① We must have $\Phi(z) \neq 0$ for $z \in C(r_1, r_2)$. Indeed, all roots z_0 of $\Phi(z)$ satisfy $|z_0| > 1$ and as $H(z) = \frac{\Theta(z)}{\Phi(z)}$ has no poles in $C(r_1, r_2)$ clearly $|z_0| \ge r_2 = r > 1$ must hold.

3.19. Flowchart of recursive (ARMA) LS



Infinitely many $h(k)$
↓ replaced
by finitely many parameters ϕ_i, θ_j.

Note: Every causal and stable IIR LS_h may be with arbitrary precision approximated by a suitable ARMA-LS.
Higher precision ⇒ higher orders p, q ⇒ more parameters ϕ_i, θ_j needed.
Analogy: any real number $h \in \mathbb{R}$ with infinitely many decimal digits may be with arbitrary precision approximated by a rational number $\frac{\theta}{\phi}$, where θ, ϕ have finite number of digits (ϕ, θ are integers).
Observe that $h = \sum_{k=-\infty}^{N} h_k 10^k$ is Laurent expansion similar to $H(z)$,
and $\theta = \sum_{i=0}^{q} \theta_i 10^i$, $\phi = 1 - \sum_{i=1}^{p} \phi_i 10^i$ finite power expansion similar to $\Theta(z), \Phi(z)$,
here playing the role of decadic representations.

3.20. Alternative notation for $x_t = LS_h(u_t)$, $h \in \ell_1$

(1) By (3.2): $\boxed{x_t = \sum_{k=-\infty \atop =0}^{\infty} h_k u_{t-k}}$ or $\boxed{x = h * u}$ ...explicit form

(2) By 3.14(6): $\boxed{X(z) = H(z) \cdot U(z)}$ or with z-transforms: $\boxed{X(\tfrac{1}{z}) = U(\tfrac{1}{z}) H(\tfrac{1}{z})}$

preferred by electrical engineers

(3) Using backward-shift operator B: $B u_t := u_{t-1}$
and forward-shift operator B^{-1}: $B^{-1} u_t = u_{t+1}$

$$x_t = \sum_{k=-\infty}^{\infty} h_k u_{t-k} = \sum_{k=-\infty \atop =0}^{\infty} h_k B^k u_t = \Big(\sum_{k=-\infty \atop =0}^{\infty} h_k B^k\Big) u_t = H(B) u_t$$

$\boxed{x_t = H(B) u_t}$ ... Preferred notation in ARMA time series modeling

In particular for ARMA-LS we obtain:

(1) $x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = \theta_0 u_t + \theta_1 u_{t-1} + \cdots + \theta_q u_{t-q}$

(2) $\Phi(z) X(z) = \Theta(z) U(z)$
(3) $\Phi(B) x_t = \Theta(B) u_t$

where $\Phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$
$\Theta(z) = \theta_0 z + \theta_1 z + \cdots + \theta_q z^q$.