## 3.9. Orthogonal Complements and Projection Theorem

By a subspace of a Hilbert space $H$, we mean a vector subspace of $H$. A subspace of a Hilbert space is an inner product space. If we additionally assume that $S$ is a closed subspace of $H$, then $S$ is a Hilbert space itself, because a closed subspace of a complete normed space is complete.

***Definition 3.9.1 (Orthogonal Complement).*** Let $S$ be a non-empty subset of a Hilbert space $H$. An element $x \in H$ is said to be orthogonal to $S$, denoted by $x \perp S$, if $\langle x, y \rangle = 0$ for every $y \in S$. The set of all elements of $H$ orthogonal to $S$, denoted by $S^\perp$, is called the *orthogonal complement* of $S$. In symbols:

$$S^\perp = \{x \in H, x \perp S\}.$$

The orthogonal complement of $S^\perp$ is denoted by $S^{\perp\perp} = (S^\perp)^\perp$.

If $x \perp y$ for every $y \in H$, then $x = 0$. Thus, $H^\perp = \{0\}$. Similarly, $\{0\}^\perp = H$. Two subsets $A$ and $B$ of a Hilbert space are said to be *orthogonal* if $x \perp y$ for every $x \in A$ and $y \in B$. This is denoted by $A \perp B$. Note that, if $A \perp B$, then $A \cap B = \{0\}$ or $\varnothing$.

**Theorem 3.9.1.** *For any subset $S$ of a Hilbert space $H$, the set $S^\perp$ is a closed subspace of $H$.*

***Proof.*** If $\alpha, \beta \in \mathbb{C}$ and $x, y \in S^\perp$, then

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle = 0$$

for every $z \in S$. Thus, $S^-$ is a vector subspace of $H$. We next prove that $S^\perp$ is closed.

Let $(x_n) \in S^\perp$ and $x_n \to x$ for some $x \in H$. From the continuity of the inner product, we have

$$\langle x, y \rangle = \left\langle \lim_{n\to\infty} x_n, y \right\rangle = \lim_{n\to\infty} \langle x_n, y \rangle = 0$$

for every $y \in S$. This shows that $x \in S^\perp$, and thus $S^\perp$ is closed. □

The preceding theorem says that $S^\perp$ is a Hilbert space for any subset $S$ of $H$. Note that $S$ does not have to be a vector space. Since $S \perp S^\perp$, we have $S \cap S^\perp = \{0\}$ or $S \cap S^\perp = \varnothing$.

***Definition 3.9.2 (Convex Sets).*** A set $U$ in a vector space is called *convex* if for any $x, y \in U$ and $\alpha \in (0, 1)$ we have $\alpha x + (1 - \alpha) y \in U$.

Note that a vector subspace is a convex set.

The following theorem, concerning the minimization of the norm, is of fundamental importance in approximation theory.

**Theorem 3.9.2 (The Closest Point Property).** *Let $S$ be a closed convex subset of a Hilbert space $H$. For every point $x \in H$ there exists a unique point $y \in S$ such that*

$$\|x - y\| = \inf_{z \in S} \|x - z\|. \tag{3.9.1}$$

***Proof.*** Let $(y_n)$ be a sequence in $S$ such that

$$\lim_{n\to\infty} \|x - y_n\| = \inf_{z \in S} \|x - z\|.$$

Denote $d = \inf_{z \in S} \|x - z\|$. Since $\frac{1}{2}(y_m + y_n) \in S$, we have

$$\|x - \tfrac{1}{2}(y_m + y_n)\| \geq d \quad \text{for all } m, n \in \mathbb{N}.$$

Moreover, by the parallelogram law (3.3.6), we obtain

$$\begin{aligned}
\|y_m - y_n\|^2 &= 4\|x - \tfrac{1}{2}(y_m + y_n)\|^2 + \|y_m - y_n\|^2 - 4\|x - \tfrac{1}{2}(y_m + y_n)\|^2 \\
&= \|(x - y_m) + (x - y_n)\|^2 + \|(x - y_m) - (x - y_n)\|^2 \\
&\quad - 4\|x - \tfrac{1}{2}(y_m + y_n)\|^2 \\
&= 2(\|x - y_m\|^2 + \|x - y_n\|^2) - 4\|x - \tfrac{1}{2}(y_m + y_n)\|^2.
\end{aligned}$$

Since

$$2\left(\|x-y_m\|^2+\|x-y_n\|^2\right) \to 4d^2, \quad \text{as } m,n\to\infty,$$

and

$$\|x-\tfrac{1}{2}(y_m+y_n)\|^2 \geq d^2,$$

we have $\|y_m-y_n\|^2 \to 0$, as $m,n\to\infty$. Thus, $(y_n)$ is a Cauchy sequence. Since $H$ is complete and $S$ is closed, the limit $\lim_{n\to\infty} y_n = y$ exists and $y\in S$. From the continuity of the norm we obtain

$$\|x-y\| = \left\|x-\lim_{n\to\infty} y_n\right\| = \lim_{n\to\infty}\|x-y_n\| = d.$$

We have proved that there exists a point in $S$ satisfying (3.9.1). It remains to prove the uniqueness. Suppose there is another point $y_1$ in $S$ satisfying (3.9.1). Then, since $\frac{1}{2}(y+y_1)\in S$, we have

$$\|y-y_1\|^2 = 4d^2 - 4\left\|x-\frac{y+y_1}{2}\right\|^2 \leq 0.$$

This can only happen if $y=y_1$. □

Theorem 3.9.2 gives an existence and uniqueness result which is crucial for optimization problems. However, it does not tell us how to find that optimal point. The characterization of the optimal point in the case of a real Hilbert space, stated in the following theorem, is often useful in such problems.

**Theorem 3.9.3.** *Let $S$ be a closed convex subset of a real Hilbert space $H$, $y\in S$, and let $x\in H$. Then the following conditions are equivalent:*

(a) $\|x-y\| = \inf_{z\in S}\|x-z\|$,

(b) $\langle x-y, z-y\rangle \leq 0$ for all $z\in S$.

***Proof.*** Let $z\in S$. Since $S$ is convex, $\lambda z+(1-\lambda)y\in S$ for every $\lambda\in(0,1)$. Then, by (a), we have

$$\|x-y\| \leq \|x-\lambda z-(1-\lambda)y\| = \|(x-y)-\lambda(z-y)\|.$$

Hence, as $H$ is a real Hilbert space, we get

$$\|x-y\|^2 \leq \|x-y\|^2 - 2\lambda\langle x-y, z-y\rangle + \lambda^2\|z-y\|^2,$$

and consequently,

$$\langle x-y, z-y\rangle \leq \frac{\lambda}{2}\|z-y\|^2.$$

Thus, (b) follows by letting $\lambda\to 0$.

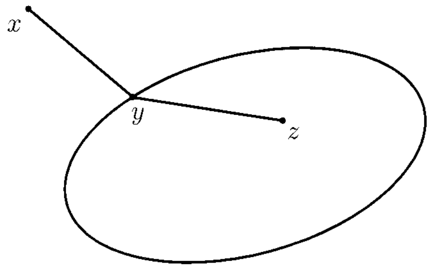


FIGURE 3.3.

Conversely, if $x \in H$ and $y \in S$ satisfy (b), then for every $z \in S$, we have

$$\|x - y\|^2 - \|x - z\|^2 = 2\langle x - y, z - y\rangle - \|z - y\|^2 \leq 0.$$

Thus, $x$ and $y$ satisfy (a). □

If $H = \mathbb{R}^2$ and $S$ is a closed convex subset of $\mathbb{R}^2$, then condition (b) has a clear geometrical meaning: The angle between the line through $x$ and $y$ and the line through $z$ and $y$ is always obtuse (see Fig. 3.3).

**Theorem 3.9.4 (Orthogonal Projection).** *If $S$ is a closed subspace of a Hilbert space $H$, then every element $x \in H$ has a unique decomposition in the form $x = y + z$ where $y \in S$ and $z \in S^\perp$.*

***Proof.*** If $x \in S$, then the obvious decomposition is $x = x + 0$. Suppose now that $x \notin S$. Let $y$ be the unique point of $S$ satisfying $\|x - y\| = \inf_{w \in S} \|x - w\|$, as in Theorem 3.9.2. We will show that $x = y + (x - y)$ is the desired decomposition.

If $w \in S$ and $\lambda \in \mathbb{C}$, then $y + \lambda w \in S$ and

$$\|x - y\|^2 \leq \|x - y - \lambda w\|^2 = \|x - y\|^2 - 2\Re\lambda\langle w, x - y\rangle + |\lambda|^2\|w\|^2.$$

Hence,

$$-2\Re\lambda\langle w, x - y\rangle + |\lambda|^2\|w\|^2 \geq 0.$$

If $\lambda > 0$, then dividing by $\lambda$ and letting $\lambda \to 0$ gives

$$\Re\langle w, x - y\rangle \leq 0. \tag{3.9.2}$$

Similarly, replacing $\lambda$ by $-i\lambda$ ($\lambda > 0$), dividing by $\lambda$, and letting $\lambda \to 0$ yields

$$\Im\langle w, x - y\rangle \leq 0. \tag{3.9.3}$$

Since $y \in S$ implies $-y \in S$, inequalities (3.9.2) and (3.9.3) hold also with $-w$ instead of $w$. Therefore $\langle w, x - y\rangle = 0$ for every $w \in S$, which means $x - y \in S^\perp$.

To prove the uniqueness note that if $x = y_1 + z_1$, $y_1 \in S$, and $z_1 \in S^\perp$, then $y - y_1 \in S$ and $z - z_1 \in S^\perp$. Since $y - y_1 = z_1 - z$, we must have $y - y_1 = z_1 - z = 0$. □

According to Theorem 3.9.4, every element of $H$ can be uniquely represented as the sum of an element of $S$ and an element of $S^\perp$. This can be stated symbolically as

$$H = S \oplus S^\perp. \tag{3.9.4}$$

We say that $H$ is the direct sum of $S$ and $S^\perp$. Equality (3.9.4) is called an *orthogonal decomposition* of $H$. Note that the union of a basis of $S$ and a basis of $S^\perp$ is a basis of $H$.

Theorem 3.9.2 allows us to define a mapping $P_S(x) = y$, where $y$ is as in (3.9.1). Mapping $P_S$ is called the *orthogonal projection* onto $S$. Such mappings will be discussed in Section 4.7.

***Example 3.9.1.*** Let $H = \mathbb{R}^2$. Figure 3.4 exhibits the geometric meaning of the orthogonal decomposition in $\mathbb{R}^2$. Here $x \in \mathbb{R}^2$, $x = y + z$, $y \in S$, and $z \in S^\perp$. Note that, if $s_0$ is a unit vector in $S$, then $y = \langle x, s_0 \rangle s_0$.

***Example 3.9.2.*** If $H = \mathbb{R}^3$, given a plane $P$, any vector $x$ can be projected onto the plane $P$. Figure 3.5 illustrates this situation.

**Theorem 3.9.5.** *If $S$ is a closed subspace of a Hilbert space $H$, then* $S^{\perp\perp} = S$.

***Proof.*** If $x \in S$, then for every $z \in S^\perp$ we have $\langle x, z \rangle = 0$, which means $x \in S^{\perp\perp}$. Thus, $S \subset S^{\perp\perp}$. To prove that $S^{\perp\perp} \subset S$ consider an $x \in S^{\perp\perp}$.

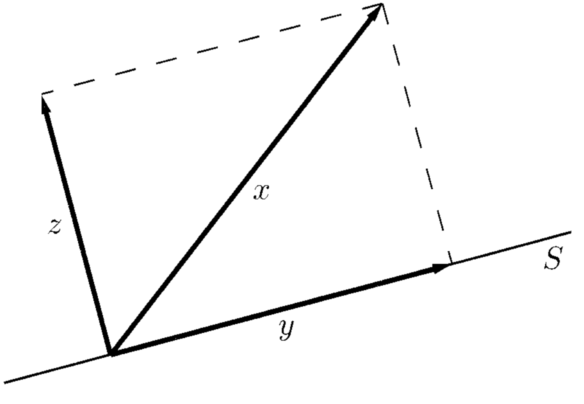


FIGURE 3.4. Orthogonal decomposition in $\mathbb{R}^2$.

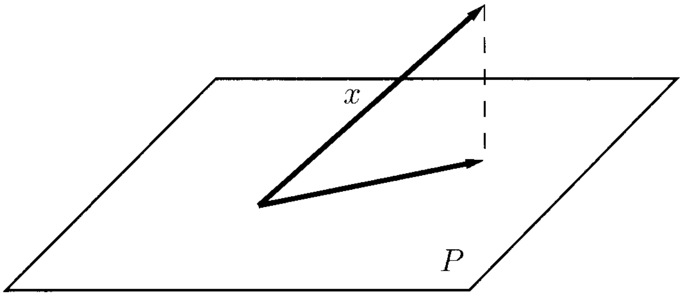


FIGURE 3.5. Orthogonal projection onto a plane.

Since $S$ is closed, $x = y + z$ for some $y \in S$ and $z \in S^{\perp}$. In view of the inclusion $S \subset S^{\perp\perp}$, we have $y \in S^{\perp\perp}$ and thus $z = x - y \in S^{\perp\perp}$, because $S^{\perp\perp}$ is a vector subspace. But $z \in S^{\perp}$, so we must have $z = 0$, which means $x = y \in S$. This shows that $S^{\perp\perp} \subset S$, completing the proof. □