MATH 163, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Honors Project 1a: The Iteration Method

Objective

In addition to the Bisection Method and Newton's Method, an approximate solution to the equation $f(x) = 0$ can be found with the *Iteration Method*. The Iteration Method often arises in applications such as economics in the form of "cobweb cycles". In this project we discuss the Iteration Method.

Narrative

To find an approximate solution to the equation $f(x) = 0$ by the Iteration Method, look for a value of x for which $g(x) = x$ for an appropriate function $g(x)$; an x for which $g(x) = x$ is called a *fixed point* of g. For example, to find an approximate solution to the equation $x^2 - 2 = 0$ we might we choose an arbitrary value for $x_0 \in [1, 1.99]$ and repeatedly compute x_n using the formula

$$x_n = g(x_{n-1}), \quad n = 1, 2, 3, \ldots$$

where $g(x) = -\frac{1}{2}(x^2 - 2) + x$. The reason the iteration method works is that

$$g(x) = -\frac{1}{2}(x^2 - 2) + x = x \quad \text{if and only if} \quad x^2 - 2 = 0.$$

The reason we didn't simply let $g(x) = x^2 - 2 + x$ is that with this choice of g, the Iteration Method does not converge.

One strength of the Iteration Method is that after the appropriate function g and initial value x_0 are identified, it is fairly easy to implement. One weakness of it is that given f you have to come up with g, and this is not always easy.

Task

a) Type the command lines below into Maple in the order in which they are listed. These command lines illustrate how the iteration method as described in the Narrative *converges*. (Note again how little effort it takes to implement this method!)

```
> # Honors Project 1a:  The Iteration Method
> restart:
> # Task a
> g := x -> -(x^2-2)/2+x;
> g(1.5);
> for i from 1 to 20 do g(%) od;
```

b) Continue by typing the following command lines into Maple. These command lines illustrate how the iteration method as described in the Narrative *diverges*.

```
> # Task b
> g := x -> x^2-2+x;
> g(1.5);
> for i from 1 to 20 do g(%) od;
```

At this time make a hard-copy of your typed input and Maple's responses. Then, ...

The condition that determines whether the iteration method converges is $|g'(x)| < 1$ for all x in some closed interval I containing the root of the equation $g(x) = 0$.

c) Show by hand that if $g(x) = -\frac{1}{2}(x^2 - 2) + x$ then $|g'(x)| = |-x + 1| < 1$ for all $x \in [1, 1.99]$.

d) Show by hand that if $g(x) = (x^2 - 2) + x$ then $|g'(x)| = |2x + 1| \not< 1$ for any $x \in [1, 1.99]$.

Comments

1. As indicated above, one of the strengths of the iteration method is that after "set-up", it is very easy to program. Since "set-up" includes finding a function g satisfying the convergence condition, and since it is not always "obvious" how to do this, getting through the "set-up" can be a weakness of the iteration method.
2. The iteration method can be extended to multiple variables.
3. The use of the iteration method in applications is one of the factors that motivates the study of "fixed point theory" in mathematics.