MATH 163, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Extra Project 5.3b: The Fundamental Theorem of Calculus

Objective

To discuss an alternate approach to the Fundamental Theorem of Calculus.

Narrative

Two proofs of the Fundamental Theorem of Calculus are presented in the text. In this project we investigate the second. The individual steps are presented and you will be asked to justify them.

Tasks

1. On a separate piece of paper, justify each labeled step in the following proof that for a differentiable function f over the closed interval $[a,b]$, $\int_a^b f'(x)\ dx = f(b) - f(a)$:

$$\begin{aligned}
&\text{Step a:} & \int_a^b f'(x)\ dx &= \lim_{h\to 0} \textstyle\sum_{i=1}^N f'(w_i)\ \Delta x_i \\
&\text{Step b:} & &\approx \textstyle\sum_{i=1}^N f'(w_i)\ \Delta x_i \\
&\text{Step c:} & &\approx \textstyle\sum_{i=1}^N \dfrac{f(x_i) - f(x_{i-1})}{\Delta x} \Delta x \\
&\text{Step d:} & &= \textstyle\sum_{i=1}^N (f(x_i) - f(x_{i-1})) \\
&\text{Step e:} & &= (f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + \dots \\
& & &\qquad + (f(x_{N-1}) - f(x_{N-2})) + (f(x_N) - f(x_{N-1})) \\
&\text{Step f:} & &= f(x_N) - f(x_0) \\
&\text{Step g:} & &= f(b) - f(a)
\end{aligned}$$

(For example, for Step a, you might write, "This is the definition of the definite integral.". *Note*: Since all your responses for this part must "fit together", it would be advisable to read through this part before you begin and anticipate in early steps what you will need for later steps.)

2. On a separate piece of paper, justify each labeled step in the following proof that if f is continuous over the interval $[a,b]$ and $F(x) = \int_a^x f(t)\ dt$ for $x \in [a,b]$, then $D_x\left(\int_a^x f(t)\ dt\right) = f(x)$:

$$\begin{aligned}
&\text{Step a:} & \int_a^x F'(t)\ dt &= F(x) - F(a) \\
&\text{Step b:} & &= \int_a^x f(t)\ dt - 0 = \int_a^x f(t)\ dt \\
&\text{Step c:} & \int_a^x (F'(t) - f(t))\ dt &= 0 \\
&\text{(See below):} & F'(t) - f(t) &= 0 \\
&\text{Step d:} & F'(x) &= f(x) \\
&\text{Step e:} & D_x\left(\int_a^x f(t)\ dt\right) &= f(x)
\end{aligned}$$

(You do not have to justify the indicated step, but — for the record — it can be justified by the fact that if $\int_a^x g(t)\ dt = 0$ for all $x \in [a,b]$ then $g(x) = 0$ for all $x \in [a,b]$. To see why this is true, suppose that

$g(c) \neq 0$ for some $c \in [a,b]$; indeed, for the sake of argument, let us suppose that $g(c) > 0$. Then, since g is continuous, there is an $\epsilon > 0$ such that $g(t) > 0$ for all $t \in [c-\epsilon, c]$. Since

$$\int_a^c g(t)\ dt = \int_a^{c-\epsilon} g(t)\ dt + \int_{c-\epsilon}^c g(t)\ dt,$$

and since $\int_a^x g(t)\ dt = 0$ for all $x \in [a,b]$ this would imply that $\int_{c-\epsilon}^c g(t)\ dt = 0$; but this is impossible since g is positive on $[c-\epsilon, c]$ so $\int_{c-\epsilon}^c g(t)\ dt$ must be strictly greater than 0.)

Comments

The above proof will be useful in MATH 164 in understanding Euler's Method for solving differential equations. Also, it anticipates the concept of "a telescoping sum" which we will also discuss in MATH 164.