Extra Project 16.2b: Visualizing Regions in R^3

Objective

In this project we illustrate some techniques for visualizing regions in R^3 using Maple.

Narrative

Just as one of the basic problems in first semester calculus is to find the area of the planar region between the graphs of two functions $f = f(x)$ and $g = g(x)$, one of the big challenges in third semester calculus is to find the volume R of the solid between the graphs of two functions $f = f(x, y)$ and $g = g(x, y)$. In principle, such a volume can be represented by an iterated double integral; the difficulty is that while the integrand is generally easy to identify, the limits of integration may not be so easy to identify. In this project we present some techniques that can be applied to determining the limits of integration.

Our approach revolves around obtaining a good picture of R. In this project we illustrate, for example, how to go from a picture like the one at the top right to the one at the bottom right.

In some cases, getting a good picture might be easy; in other cases, it may not. In the later case:
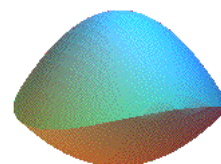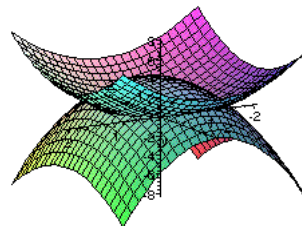
1. One fact that may be of use is that $\gamma = \{P(x, y) \mid f(x, y) = g(x, y)\}$ is the projection of the curve of intersection of the graphs of f and g into the xy-coordinate plane. In this project we assume that γ is a closed curve.

2. Another fact that we may use is that to plot only over the closed region R bounded by γ, we can treat R as a Type I or Type II region by solving for x or y (whichever is easier) and plotting from the minimum value of the independent variable to the maximum such value.

In this project we illustrate these techniques by applying them to the region bounded by the graphs of the functions $f(x, y) = x^2 + y^2$ and $g(x, y) = 4 - x^2 - 2y^2$.

The following command lines they draw an initial picture of our region in Maple.

```
> # Project 16.2b:  Visualizing Regions in R^3
> restart:  with(plots):
> setoptions3d(scaling=unconstrained,color=blue,axes=normal,orientation=[25,72]);
> f := (x,y) -> x^2+y^2;
> g := (x,y) -> 4-x^2-2*y^2;
> plot3d({f(x,y),g(x,y)},x=-2..2,y=-2..2);
```

To get a better picture, we now clip off some of the extraneous parts of the graphs of f and g. To do this, let us first draw a picture of the projection γ of the curve of intersection of the graphs of f and g into the xy-coordinate plane.

```
> implicitplot(f(x,y)=g(x,y),x=-2..2,y=-2..2);
```

This curve looks like an ellipse. To check this conjecture, we describe the curve analytically by solving the equation $f(x, y) = g(x, y)$ for y in terms of x.

```
> solve(f(x,y)=g(x,y),y);
```

It is, indeed, an ellipse. To find the x-intercepts of γ, we solve the equation $f(x,0) = g(x,0)$ for x.

```
> solve(f(x,0)=g(x,0),x);
```

Finally, we are ready to draw a better picture: todraw the graphs of f and g over the interior of γ enter the command line ...

```
> plot3d({f(x,y),g(x,y)},x=-sqrt(2)..sqrt(2),y=sqrt(12-6*x^2)/3..-sqrt(12-6*x^2)/3);
```

We can modify the appearance of this region by modifying the plotting options:

```
> plot3d({f(x,y),g(x,y)},x=-sqrt(2)..sqrt(2),y=sqrt(12-6*x^2)/3..-sqrt(12-6*x^2)/3,
         style=patchnogrid,lightmodel=light4);
```

If you want to peek inside this clam-shaped region, you simply have to restrict the values of x over which the graphs of f and g are drawn even further.

```
> plot3d({f(x,y),g(x,y)},x=-sqrt(2)..sqrt(2)/2,y=sqrt(12-6*x^2)/3..-sqrt(12-6*x^2)/3,
         style=patchnogrid,lightmodel=light4);
```

It is interesting to note that the z-values of the points of our clam-shaped region that lie above γ are *not* all the same: The points $P(\pm\sqrt{2},0)$ and $Q(0,\pm 2\sqrt{3}/3)$ all lie on γ, but

$$f(\pm\sqrt{2},0) = g(\pm\sqrt{2},0) = 2 \quad \text{while} \quad f(0,\pm 2\sqrt{3}/3) = g(0,\pm 2\sqrt{3}/3) = 4/3.$$

Tasks

1. If you have not already done so, enter all the command lines above into Maple in the order in which they were presented.

2. Draw a good graphic of the solid that lies below the graph of $z = \cos x$ and above the graph of $z = x^2 + y^2 - 1$.

3. Express the volume of the solid you drew in part (2) as an iterated double integral.