MATH 163, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

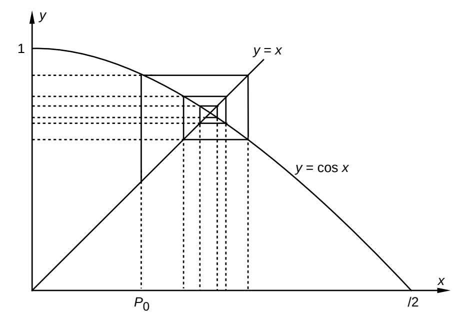
Honors Project 1b: Iteration and Chaos

The Iteration Method and Newton's Method are examples of first-order iterative processes. In general, a first-order iterative process involves starting with a quantity x_0, applying some function f to x_0 to arrive at a quantity $x_1 = f(x_0)$, applying f to x_1 to arrive at $x_2 = f(x_1)$, and so forth. The numbers $x_0, x_1, x_2, \ldots$ form an *infinite sequence*, and we will have much more to say about infinite sequences in MATH 164. The following command lines, for example, produce the first 20 terms in the infinite sequence

$$x_0 = 0.5, \quad x_{n+1} = f(x_n) = \cos x_n, \quad n = 0, 1, 2, \ldots.$$

(These are the same numbers you would get if you were to enter 0.5 into a scientific hand calculator, and repeatedly press the "cos" key 20 times!)

```
> restart:
> f := x -> cos(x);
> x0 := 0.5;
> for n from 1 to 20 do f(%); od;
```



One way to visualize a first-order iterative process is with *cob-web diagrams* as follows (see the figure to the right):

1. The vertical line through $P_0(x_0, 0)$ meets the graph of $y = \cos x$ at the point $P_1(x_0, x_1)$ whose y-coordinate is $x_1 = f(x_0)$; label P_1 now.
2. The horizontal line through $P_1(x_0, x_1)$ meets the line whose equation is $y = x$ at the point $P_2(x_1, x_1)$; label P_2 now.
3. The vertical line through $P_2(x_1, x_1)$ meets the graph of $y = \cos x$ at the point $P_3(x_1, x_2)$ whose y-coordinate is $x_2 = f(x_1)$; label P_3 now.
4. The horizontal line through $P_3(x_1, x_2)$ meets the line whose equation is $y = x$ at the point $P_4(x_2, x_2)$; label P_4 now.

We continue to repeat the steps described above over, and over again. Can you see the geometric pattern? Starting from P_0, we go vertically to curve, horizontally to line, vertically to curve, horizontally to line, vertically to curve, ... The "limit" of the points on the graph of $y = \cos x$ *and* on the line whose equation is $y = x$, is the point of intersection of the graphs of $y = x$ and $y = \cos x$; hence, we might label it $P_\infty(x_\infty, x_\infty)$. (Cob-web diagrams get their name from their cob-web-like appearance.)

The following code creates a Maple graphic of the above construction.

```
> restart:  with(plots):
> f := x -> cos(x);
> x0 := 0.5;
> a := [x0,x0,x0,f(x0)];
> S := [seq(op(map((f@@j),a)),j=0..20)]:
> A := plot({x,f(x)},x=0..1,thickness=2):
> B := pointplot(S,color=blue,connect=true):
> display({A,B});
```

One of the reasons people study cob-web diagrams is in the hope that they might provide insight into the behavior of iterative processes: while a great deal is known about iterative processes, a great deal is also *unknown* about them. To illustrate some of the the subtle behavior of iterative processes, consider the first-order iterative process

$$x_0 = a, \quad x_{n+1} = f(x_n) = kx_n(1 - x_n), \quad n = 0, 1, 2, \ldots . \tag{$*$}$$

If $k = 4$ then for some values of $a \in [0, 1]$, x_n converges, for some it repeats, and for others it wanders aimlessly over the interval $[0, 1]$. (This type of behavior is known as *chaos*. The study of chaos is important since chaotic behavior arises not only in many areas of science and engineering, but also in areas as diverse as medicine — for example, in the study of dynamic blood diseases — and economics.) If, on the other hand, $k = 3.839$ then for any value of $a \in [0, 1]$, x_n eventually settles down to cyclically repeating — or *orbit* — the three numbers 0.149888, 0.489172, and 0.959299. (This type of behavior is known as *periodicity*.)

Problems

1. Confirm the assertions made above about the behavior of (*) both numerically and geometrically.
2. What can be said about the behavior of (*) if $f(x) = x^2 - 1$: What happens if $a = 0$ or 1? What happens if a takes on any other value?
3. What can be said about the behavior of (*) if $f(x) = x^3$?