MATH 164, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Extra Project 10.5c: Another Population Model, Equilibria and Stability

Objective

To investigate a population model that is a refinement of the logistic model, equilibria and stability.

Narrative

If you have not already done so, do Project 10.5a on the logistic equation.

While the logistic equation models the growth of a population whose size is limited by the carrying capacity of its environment, it does not take into account the fact that some populations require a minimal number of individuals to survive. (Wolves, for example, survive in packs, and if a population of wolves drops below the size of a pack then it will become extinct.) If $P = P(t)$ denotes the size of a population at time t, L is the carrying capacity of the environment supporting this population, and $m \in [0, L]$ is the minimal number of individuals the population requires to survive, then the growth of this population can be modeled by the equation

$$\frac{dP}{dt} = -kP(m-P)(L-P) \tag{1}$$

where k is a positive constant.

Task

1. Using Maple, draw (in one graphic):

   a) the direction field associated to the equation $P' = -0.5P(1-P)(5-P)$ for $t \in [0,4]$ and $P \in [0,8]$, and

   b) the six solutions corresponding to the initial conditions $P(0) = 0.25$, $P(0) = 0.75$, $P(0) = 1.25$, $P(0) = 4.75$, $P(0) = 5.25$, and $P(0) = 6.0$.

   At this point, make a hard-copy of your typed input and Maple's responses. Then, ...

2. On the graph you produced for Task 1, label the coordinate axes, draw and label by hand the lines whose equations are $P = 1$ and $P = 5$ (in this case, $m = 1$ and $L = 5$), and label the curves corresponding to the six initial conditions. (Label the curve corresponding to $P(0) = 1.5$ by "$P(0) = 1.5$", for example.)

3. In one or two sentences, justify — on the basis of population dynamics — the fact that if the initial level $P(0)$ of a population whose growth satisfies equation (1), is:

   a) between 0 and m then P *decreases* to 0 as t increases,

   b) between m and L then P *increases* to L as t increases, and

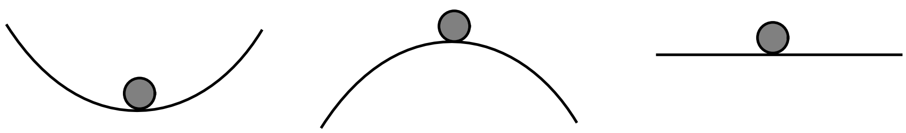
   c) greater than L then P *decreases* to L as t increases.

Comments

An *equilibrium* for a population is a level of P at which the population does not grow. Since, for the population modeled by (1), $dP/dt = 0$ when $P = 0, m$, and L, this population has three equilibrium levels: if P ever attains any one of the values 0, m, or L, it will neither increase nor decrease. As the computations you have performed above illustrate, however, the behavior of P at these equilibria is different. The equiliibria

0 and L are *stable* because if $P = 0$ or $P = L$, and P is then changed by some small amount, P will return to 0 or L, respectively. The equilibrium m is *unstable*, however: if $P = m$ and P is decreased slightly, P will eventually approach 0 (or move away from m), while if $P = m$ and P is increased slightly, P will eventually approach L (again moving away from m).

One way to think of this is in the context of a ball (see the figure below): If a ball is at rest at the bottom of a valley and it is moved a small amount one way or another, it will return to its initial position; in this case we say the ball is at a stable equilibrium. If a ball is at rest on the top of a hill and it is moved a small amount one way or another, it moves away from its initial position; in this case we say the ball is at an unstable equilibrium. A third possibility (which does not arise in considering equation (1)) is that a ball is at rest on a flat surface: in this case, the ball is moved a small amount one way or another it remains where it was moved; in this case we say the ball is at a neutral equilibrium.



Stable equilibrium   Unstable equilibrium   Neutrally stable equilibrium

From the point of view of MATH 163, what distinguishes the above situations is whether the graph of the function f which describes the "landscape" has a second derivative f'' which is positive (the case of a stable equilibrium), negative (the case of an unstable equilibrium), or sero (the case of a neutral equilibrium). From the point of view of population dynamics, the analytic condition for a stable equilibrium is $d^2P/dt^2 > 0$, for an unstable equilibrium is $d^2P/dt^2 < 0$, and for a neutral equilibrium is $d^2P/dt^2 = 0$. Can you see why 0 and L are stable equilibria of (1), and m is an unstable equilibrium of (1)? Can you think of an example that illustrates neutral stablity?

The ideas discussed in this project, and extensions of them to multiple interacting populations and stratified populations, are considered further in courses dealing with population dynamics.