Honors Project 12: Interacting Species

Objective

In this project we expand on the discussion of the Lotka-Volterra equations in MATH 164.

Narrative

The system of differential equations

$$\frac{1}{x}\frac{dx}{dt} = Ax + By + C, \quad \frac{1}{y}\frac{dy}{dt} = Dx + Ey + F \tag{1}$$

is often used to model the populations of two interacting species. If $B = 0$ and $D = 0$ then (1) is said to be *decoupled*, and it represents a system in which two species exist independently of one another. (In fact, the growth of both populations is logistic.) If $B \neq 0$ or $D \neq 0$ then (1) is *coupled*, and it represents a system in which the two species interact. The fact that $dx/dt = xf(x, y)$ and $dy/dt = yg(x, y)$ for some functions f and g in (1) is known as the *parenting* assumption: if $x = 0$ or $y = 0$ then $dx/dt = 0$ or $dy/dt = 0$, so the population does not grow. (Another way to put this is that the population grows only by parenting.)

Solving a decoupled system of differential equations is relatively straightforward: it simply involves solving a pair of independent differential equations. Solving a coupled system, however, is not straightforward: the analysis can get very complicated. Still, a great deal can be said about coupled systems by studying the geometry associated with them. The insight that leads to this information is that (1) defines a vector field $\mathbf{F}$ that associates to each point $P(x, y)$ in the plane a vector

$$\mathbf{F}(x, y) = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle = \langle Ax + By + C, Dx + Ey + F \rangle \tag{2}$$

which is tangent to any solution of (1) that passes through P.

There are two major differences between what we did in MATH 164 and what we are doing here: First, in MATH 164 we considered slope fields, and a slope field is a collection of short *undirected line segments*; here we are considering vector fields, and a vector field (such as $\mathbf{F}$) is a collection of *vectors*. Second, in MATH 164 we drew the slope field for a differential equation $dx/dt = f(x)$ in the xt-coordinate plane (independent variable against dependent variable); here we draw vector fields, and the vector field (2) associated to (1) is drawn in the xy-coordinate plane (dependent variable against dependent variable).

For simplicity, we assume $A = 0$ and $E = 0$ in (1). In doing this we ignore logistic affects. (You may choose to investigate the consequences of including logistic affects, or perhaps even threshold affects on your own.) Thus, the system we will study is

$$\frac{1}{x}\frac{dx}{dt} = By + C, \quad \frac{1}{y}\frac{dy}{dt} = Dx + F. \tag{3}$$

There are three qualitatively different cases:

1. If $B > 0$ and $D > 0$ (and $C < 0$ and $F < 0$) then (3) represents a symbiotic relationship between two species. (The greater y, the greater the growth in x, and the greater x, the greater the growth in y.)

2. If $B < 0$ and $D > 0$ (and $C > 0$ and $F < 0$) then (3) represents a predator-prey relationship between two species. (The greater y, the lesser the growth in x, and the greater x, the greater the growth in y.)

3. If $B < 0$ and $D < 0$ (and $C > 0$ and $F > 0$) then (3) represents a competitive relationship between two species. (The greater y, the lesser the growth in x, and the greater x, the lesser the growth in y.)

Tasks

1. a) Type the following command lines into Maple in the order in which they are listed. The effect of these command lines is to plot the direction field for the system

$$\frac{1}{x}\frac{dx}{dt} = -y + 2, \quad \frac{1}{y}\frac{dy}{dt} = x - 3$$

for $(x, y) \in [0, 6] \times [0, 4]$, the lines whose equations are $x = 3$ and $y = 2$, and a solution to our system.

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> # Honors Project 12:  Interacting Species
> restart:  with(plots):  with(DEtools):
> x0 := x(t):  y0 := y(t):
> xdot := diff(x(t),t):  ydot := diff(y(t),t):
> eq1 := xdot = x0*(-y0+2); eq2 := ydot = y0*(x0-3);
> plot0 := dfieldplot([eq1,eq2],[x0,y0],t=-2..2,x=0..6,y=0..4,arrows=large,color=yellow):
> plot1 := plot({[[0,2],[6,2]],[[3,0],[3,4]]},color=blue):
> display({plot0,plot1});
> thesolthrough := proc(a,b,t0,t1)
    local soln,myplot:
    soln := dsolve({eq1,eq2,x(0)=a,y(0)=b},{x0,y0},type=numeric):
    myplot := odeplot(soln,[x0,y0],t0..t1,numpoints=25):
    RETURN(myplot):
    end:
> plot2 := thesolthrough(1.75,1.75,0,3):
> display({plot0,plot1,plot2});
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Some notes before going on:

 (a) We assume $x \geq 0$ and $y \geq 0$ since we are assuming x and y represent physical quantities (populations) which are always non-negative.

 (b) The lines $y = 2$ and $x = 3$ are the *horizontal and vertical isoclines* of the system: they are the lines along which $dx/dt = 0$ and $dy/dt = 0$, or the lines along which the x- and y-components of $\mathbf{F}(x, y)$ are zero, and hence the lines along which $\mathbf{F}(x, y)$ is vertical and horizontal. The horizontal and vertical isoclines divide the plane into four regions, each of which can be characterized by the signs of dx/dt and dy/dt, and hence the direction of $\mathbf{F}(x, y)$.[1]

 (c) The point of intersection of the horizontal and vertical isoclines is the *equilibrium* of the system: it is the point at which $dx/dt = 0$ and $dy/dt = 0$.

 (d) In the above code we use a procedure to draw a solution to our system. You might eventually wish to use this procedure to draw more solutions, but for now, ...

b) Make a hard copy of your typed input and Maple's responses.

2. In turn, modify the code of Task 1 to cover the three systems:

$$\frac{dx}{dt} = y - 2 \text{ and } \frac{dy}{dt} = x - 3, \qquad \frac{dx}{dt} = -y + 2 \text{ and } \frac{dy}{dt} = -x + 3, \qquad \frac{dx}{dt} = y - 2 \text{ and } \frac{dy}{dt} = -x + 3$$

and make a hard copy of each.

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[1]Recall that if a, b, and c are not all zero, then the set of points $P(x, y)$ for which $ax + by + c > 0$ is a half-plane determined by the line whose equation is $ax + by + c = 0$.

3. a) By hand, add an arrow head to the graph of the solution produced by the code in each of the four above cases indicating the direction of increasing t, and draw several more solutions near each equilibrium adding an arrow head to each to indicate the direction of increasing t.

b) In each of the four above cases, describe in words the behavior of solutions near the equilibrium $E(2,3)$.

c) In each case, state whether the equilibrium $E(2,3)$ is stable or unstable. (An equilibrium is *stable* if, after any small change in the state of the system away from it, the system returns to it; otherwise the equilibrium is *unstable*.)

Comments

We close this project with a few comments.

First, while it may not appear that we have yet made great strides into mathematical ecology, we have.

Let $A = A(t)$ represent a harmful insect population (such as aphids) which, under natural conditions, is held somewhat in check by a benign predator insect population $L = L(t)$ (such as ladybugs). Assume that $A(t)$ and $L(t)$ satisfy the predator-prey equations:

$$\frac{1}{A}\frac{dA}{dt} = -bL + a \quad \frac{1}{L}\frac{dL}{dt} = dA - c$$

where $a > 0$, $b > 0$, $c > 0$, and $d > 0$, so that the equilibrium populations are $A_\infty = c/d$ and $L_\infty = a/b$. Now suppose that an insecticide is used *on a continual basis* to kill (per unit time) the same (positive) fraction $f < a$ of each species of insect. What happens? The equilibrium of the harmful population is *increased* and that the equilibrium of the benign population is *decreased. The use of the insecticide is thus counterproductive!*

The key to understanding this is understanding how the introduction of the insecticide affects our equations.

If the insecticide is used just once, then it would have no effect on the equilibrium populations of aphids and ladybugs. At the time t_0 the insecticide is used, the numbers of aphids and ladybugs would decrease from $A(t_0)$ and $L(t_0)$ to $fA(t_0)$ and $fL(t_0)$, respectively. This, however, has nothing to do with the equilibria: the equilibria $A_\infty = c/d$ and $L_\infty = a/b$ are determined by the constants in the equations that govern growth — they have nothing to do with the initial populations of aphids and ladybugs — so the equilibrium populations of aphids and ladybugs would be unchanged. (We'd be just jumping from one solution curve to another.)

In this problem, however, we are assuming that the insecticide is used *on a continual basis*, and the effect of this continued use is to lower both $\frac{1}{A}\frac{dA}{dt}$ and $\frac{1}{L}\frac{dL}{dt}$: the use of the insecticide changes the equations that govern growth to

$$\frac{1}{A}\frac{dA}{dt} = -bL + a - f \quad \frac{1}{L}\frac{dL}{dt} = dA - c - f.$$

The change from a to $a - f$ in the first equation reflects the fact that the growth rate of the aphids is affected by the use of the insecticide, and the lack of change in the interaction term $-bL$ reflects the fact that each ladybug that survives the insecticide will consume the same number of aphids (as if the insecticide had not been used). The $-c$ term in our original second equation reflects the fact that in the absence of aphids (the food source for the ladybugs), the number of ladybugs will dwindle. The insecticide will just make things worse for the ladybugs: the survival rate (in the absence of aphids) will slip from $-c$ down to $-c - f$ if the insecticide is used. The lack of change in the interaction term dA reflects the fact that each ladybug that survives the insecticide will consume the same number of aphids (as if the insecticide had not been used).

Thus, if the insecticide is used on a continual basis, the new equilibria are

$$A_\infty = \frac{c+f}{d} = \frac{c}{d} + \frac{f}{d} \quad \text{and} \quad L_\infty = \frac{a-f}{b} = \frac{a}{b} - \frac{f}{b}:$$

the equilibrium of the harmful aphid population has *increased* from c/d to $c/d + f/d$, and the equilibrium of the benign ladybug population has *decreased* from a/b to $a/b - f/b$.

On one hand, this example is interesting since it illustrates the value of a qualitative analysis: here we have verified a principle without knowing specific values for a, b, c, d, and f. On the other hand, this example is also interesting since it *actually happened*: In 1868, a scale insect from Australia was accidentally introduced into the United States. Since this insect threatened the United States citrus crop, steps were sought to keep it under control. To this end, the ladybird beetle, a natural predator of the scale insect in Australia, was introduced into the north American ecosystem. This measure was relatively successful in keeping the spread of the scale insect in check. But when farmers found out that DDT could be used to kill the scale insect, they decided to spray the scale insect to reduce its numbers even further. What they did not know, however, was that DDT also killed the ladybird beetle, that the fractions of each species killed were the same, and that this would lead to a larger population of scale insects. And this is exactly what happened.

A better strategy would have been to introduce more ladybugs into the environment. If the ladybugs are introduced at the correct point in the life cycle of the ecosystem, the maximum number of aphids would be reduced. (Can you see why?) On the other hand, if the ladybugs are introduced at the wrong point in the life cycle of the ecosystem, the maximum number of aphids will be increased. (Again, can you see why?) Thus even this proposal could lead to undesirable results. In summary, it is important to perform a detailed analysis of an ecosystem before any proposal is pursued.

Second, in addition to modeling interacting species, (1) could be used to model the affects of interacting drugs, another very important topic which we do not have time to pursue here.