

Honors Project 9: The Origin of Some Interesting Curves

Curves in the plane and in space arise not only in the study of geometric problems, but also in the study of analytic problems. We hinted at this in our study of differential equations in MATH 164; there we studied curves arising from differential equations of the form $dy/dx = F(x, y)$ in the xy -coordinate plane, or $dx/dt = F(t, x)$ in the tx -coordinate plane. In this project we move further in this direction.

One of the most important second order differential equations arises both in the study of mechanics and in the study of electricity. In mechanics, the equation

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = f(t) \quad (1)$$

arises in the study of a damped spring-mass system: x represents the distance that a spring is stretched out of equilibrium, k is the spring constant of the spring, m is the mass of an object attached to the spring, c is the damping constant, and f is a driving force. In electricity, the equation

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = E(t)$$

arises in the study of circuits: Q represents charge, R represents resistance, C represents capacitance, L represents impedance, and E represents impressed voltage. For the sake of concreteness, we focus on equation (1) in this project.

In Honors Project 4 of MATH 164, we studied the solutions to equation (1) in the case $f(t) = 0$, although we did not derive them. In this project, we study the character of the solutions to equation (1), again in the case $f(t) = 0$, *without* assuming we know the solutions explicitly. We do this both to gain deeper insight into the character of the solutions and to illustrate some ideas that are useful in studying other differential equations whose solutions are not easy to find.

If we let $v = \dot{x} = dx/dt$, then we may write

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = 0$$

as a coupled system of differential equations

$$\dot{x} = v, \quad \dot{v} = -\frac{c}{m}v - \frac{k}{m}x.$$

Thus there is a curve

$$\alpha(t) = (x(t), v(t))$$

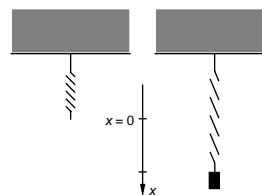
in the xv -plane whose tangent vector

$$\dot{\alpha}(t) = (\dot{x}(t), \dot{v}(t))$$

is known at each point $P(x, v)$ (even though neither x nor v is known explicitly in terms of t), and our objective is to learn what we can about this curve. The xv -plane is known as the *phase plane* of the original system, and any point in the phase plane is called a *state* of the original system (since it specifies the position x and the velocity v of the object).

If $c = 0$ then our system becomes

$$\dot{x} = v, \quad \dot{v} = -\frac{k}{m}x \quad (2)$$



from which it follows that

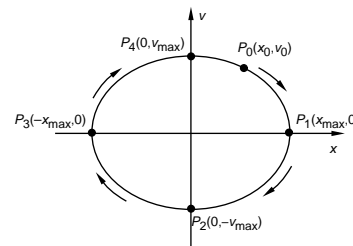
$$\frac{dv}{dx} = \frac{dv/dt}{dx/dt} = -\frac{k}{m} \frac{x}{v} \quad \text{or} \quad mv \, dv = -kx \, dx.$$

By integrating this equation we find that

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = C$$

for some constant C . Equation (2) is known as the *energy equation*, and its graph is an ellipse in the phase plane. What this curve illustrates is this (see the figure to the right below):

If, at time $t = 0$, the object starts at a displacement of x_0 from equilibrium with an initial velocity v_0 — that is, at the state $P_0 = P_0(x_0, v_0)$ — then the object moves further from equilibrium as it slows down, until it gets to the state $P_1(x_{\max}, 0)$ at which its displacement is maximal and its velocity is 0. (Do you see how the facts that the object has to be moving further from equilibrium and that it has to be slowing down are embodied in the figure to the right?) The object then moves back towards equilibrium as its speed increases negatively; at the state $P_2(0, -v_{\max})$ the object is back at equilibrium moving at its greatest negative speed. It does



not stop at its equilibrium, but continues to move past equilibrium; as it does, it nears the state $P_3(-x_{\max}, 0)$ at which its displacement is again maximal (but now at a different extreme) and its velocity is again 0. From P_3 , the object begins to move forward again with positive velocity, back towards equilibrium. At the state $P_4(0, v_{\max})$, the object is back at equilibrium moving at its maximum speed. Since the object is moving at its maximum speed, it continues to move beyond equilibrium, until it reaches the state P_0 at which it started. At this point, the pattern in the motion of the object repeats itself.

If $c = 0$ then it is easy to analyze the motion of the object since it is easy to solve equation (2). If $c \neq 0$, however, this is not the case. To facilitate further study, type the following command lines into Maple in the order in which they are listed. The affect of these command lines is to produce a direction field for the coupled system (2), as well as some integral curves of this system.

```
> # Some Interesting Curves
> restart: with(DEtools):
> x0 := x(t): v0 := v(t):
> xdot := diff(x(t),t): vdot := diff(v(t),t):
> # Assuming m=1, ...
> c := 0.0; k := 1.0;
> eq1 := xdot = v0: eq2 := vdot = -c*v0-k*x0:
> dfieldplot([eq1,eq2],[x0,v0],t=-2..2,x=-2..2,v=-2..2,arrows=large,color=yellow);
> phaseportrait([eq1,eq2],[x(t),v(t)],t=0..15,
  [[x(0)=1,v(0)=0.2],[x(0)=1.0,v(0)=0.5],[x(0)=1.0,v(0)=0.8]],
  stepsize=0.1,arrows=large,color=yellow,linecolor=[red,green,blue]);
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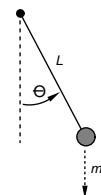
Tasks

1. Replace c by some small positive constant (such as 0.1) in the above code, and consider the graphics that you get.
 - (a) What gross behavior do you observe in the resulting system, and why does it occur? (*Hint*: You might wish to use the term *dissipation* in your response.)
 - (b) For arbitrary values of x_0 and v_0 , describe the motion of the object (as we did above in the case when $c = 0$).

The motion of a pendulum (see the figure to the right) can be described by the equation

$$L \frac{d^2\theta}{dt^2} = -g \sin \theta \quad (3)$$

where L is the length of the arm, θ is the angle from the vertical to the arm, m is the mass of the bob, and g is the force due to gravity.



2. When θ is small:

- (a) What form does equation (3) take? (*Hint*: What is the first-order Mclaurin series expansion for $\sin \theta$?)
- (b) Rewrite this form as a coupled system of differential equations.
- (c) Draw the direction field of this system and graph some integral curves of the system.
- (d) For arbitrary values of θ_0 and $(d\theta/dt)_0$, describe the motion of the pendulum.

3. Returning to equation (3):

- (a) Rewrite equation (3) as a coupled system of differential equations.
- (b) Draw the direction field of this system and identify and graph some of the integral curves of the system that illustrate the different types of behavior that are possible. (*Hint*: Some very interesting behavior occurs for large values of θ and $d\theta/dt$.)
- (c) For arbitrary values of θ_0 and $(d\theta/dt)_0$, describe the motion of the pendulum along each of the integral curves you drew in part (b).

4. So far, we have not considered damping.

- (a) How might equation (3) be modified to accomodate damping?
- (b) Draw the direction field of this system and identify and graph some of the integral curves of the system that illustrate the different types of behavior that are possible.
- (c) For arbitrary values of θ_0 and $(d\theta/dt)_0$, describe the motion of the pendulum along each of the integral curves you drew in part (b).