MATH 261, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Honors Project 17: Lissajoux Curves

Part I

Narrative

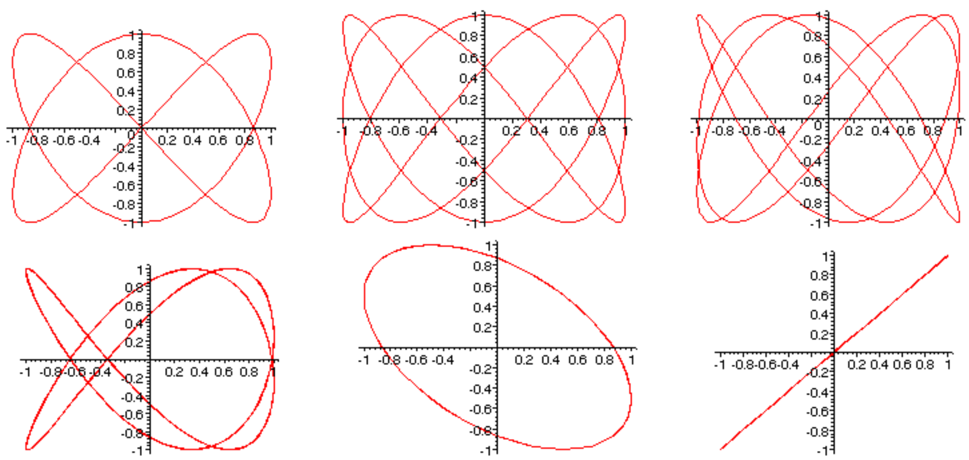
A Lissajoux curve is a parametrized curve of the form

$$x = f(m\theta + \alpha), \quad y = g(n\theta + \beta)$$

where m and n are integers, and f and g are either sine or cosine functions. For example

$$\alpha(t) = (\sin mt, \sin nt), \quad \beta(t) = (\sin mt, \cos nt), \quad \gamma(t) = (\cos mt, \cos nt)$$

are Lissajoux curves. Some Lissajoux curves are illustrated below. Lissajoux curves can be generated on an oscilloscope by plotting a known signal (or sine wave) in one coordinate direction, against an unknown signal (or sin wave) in the other. Experience with Lissajoux curves, together with a little trial-and-error modification of the known signal can help the user determine the amplitude, frequency, and phase of the unknown signal.



Task

1. Draw the graph of the Lissajoux curve $x = \sin m\theta, y = \sin n\theta$ for various combinations of relatively prime integers m and n. (Two integers are *relatively prime* if they have no common divisor other than 1.)

2. What conclusion(s) can you draw from part (1)? (*Hint*: Count the number of times the curves you drew achieve their maximum x-values, the number of times they achieve their minimum x-values, the number of times they achieve their maximum y-values, and the number of times they achieve their minimum y-values.)

3. Repeat parts (1) and (2) above using values for m and n that are not relatively prime.

4. Repeat parts (1), (2), and (3) incorporating a phase shift in one component; that is, for curves of the form $x = \sin m\theta, y = \sin(n\theta + \beta)$ for various combinations of integers m and n and come constant β.

5. What can you say about the six Lissajoux curves illustrated above if in them (reading across the first row and then across the second), $m = 2, 3, 3, 4, 4, 4$?

Part II

Narrative

Lissajoux curves can be created by projecting space curves into coordinate planes: The right circular cylinder whose cartesian equation is $x^2 + y^2 = 1$ can be parametrized by

$$x(\theta, r) = \cos m\theta, \quad y(\theta, r) = \sin m\theta, \quad z(\theta, r) = r,$$

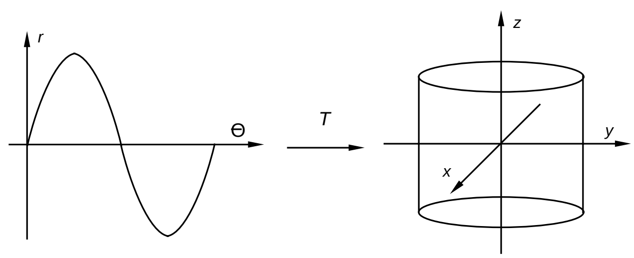
and we may think of

$$T(\theta, r) = (x(\theta, r), y(\theta, r), z(\theta, r))$$

as a transformation from the (θ, r)-coordinate plane to Cartesian (x, y, z)-space (see the figure below). The image of the curve in the θr-plane whose equation is $r = \sin n\theta$ has parametric equations

$$x(\theta) = \cos m\theta, \quad y(\theta) = \sin m\theta, \quad z(\theta) = \sin n\theta,$$

and the projection of this curve into the yz-coordinate plane has the parametric equations $y = \sin m\theta, z = \sin n\theta$.



Task

1. Illustrate the above Narrative by drawing, for some fixed values of m and n, three copies of the space curve $\alpha(\theta) = (\cos m\theta, \sin m\theta, \sin n\theta)$, one view in perspective, one view from above ("down the z-axis"), and one view from "down the x-axis". (One of these views should reveal a Lissajoux curve.)

2. Discuss the geometry associated with a nonzero phase angle ϕ in r: that is, the image under T of $r = \sin(n\theta + \phi)$.

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