Extra Project 14.3: Curvature of a Curve

Objective

In this project we try to convince ourselves that the definition of curvature for a curve is intuitively correct.

Narrative

One of the most important quantities associated to a curve $\mathbf{x} : R \to R^3$ in space beyond its tangent vector $\dot{\mathbf{x}} = d\mathbf{x}/dt$, its speed $||\dot{\mathbf{x}}||$, and its unit tangent vector $\mathbf{T} = \dot{\mathbf{x}}/||\dot{\mathbf{x}}||$, is its curvature. By definition, the curvature $\kappa = \kappa(t)$ of a curve is the magnitude in the change of $\mathbf{T}$ relative to the change $ds/dt = ||\dot{\mathbf{x}}||$ in arclength:

$$\kappa = \left|\left|\frac{d\mathbf{T}}{ds}\right|\right| = \left|\left|\frac{d\mathbf{T}/dt}{ds/dt}\right|\right| = \frac{||d\mathbf{T}/dt||}{||d\mathbf{x}/dt||} = \frac{||\dot{\mathbf{T}}||}{||\dot{\mathbf{x}}||}.$$

In this project we compute the curvature for several curves with the goal of convincing ourselves that this definition is intuitively correct.

Tasks

1. Type the following command lines into maple in the order in which they are listed. They define a procedure `kappa` which takes as input the three component functions of a curve $\mathbf{x}(t) = (x_0(t), x_1(t), x_2(t))$ in R^3 and returns as output the curvature function for $\mathbf{x}$.

```
> # Project 14.3:  Curvature of a Curve
> restart:
> kappa := proc(x0,x1,x2)
    local x0d,x1d,x2d,speed,T0,T1,T2:
    x0d(t) := diff(x0(t),t):  x1d(t) := diff(x1(t),t):  x2d(t) := diff(x2(t),t):
    speed(t) := sqrt((x0d(t))^2+(x1d(t))^2+(x2d(t))^2):
    T0(t) := x0d(t)/speed(t):  T1(t) := x1d(t)/speed(t):  T2(t) := x2d(t)/speed(t):
    RETURN(sqrt((diff(T0(t),t))^2+(diff(T1(t),t))^2+(diff(T2(t),t))^2)/speed(t)):
    end:
```

2. What would you expect the curvature of a circle with radius R should be? To test your code and check the definition of curvature, continue by typing the following command line into Maple: it computes the curvature of the circle $\mathbf{x}(t) = (R\cos t, R\sin t, 5)$ in the plane $z = 5$ whose center is $C(0,0,5)$ and whose radius is R.

```
> kappa(t -> R*cos(t), t -> R*sin(t), t -> 5);
```

3. What would you expect the curvature of the line whose equation is $\mathbf{x}(t) = (a_0 t + b_0, a_1 t + b_1, a_2 t + b_2)$ should be? Check the definition of curvature by using your Maple code as you did in part (2).

4. What would you expect the curvature of the exponential spiral whose equation is $\mathbf{x}(t) = (e^t \cos t, e^t \sin t, 0)$ should be? If you cannot guess what it should be, what properties should it have? Check the definition of curvature by using your Maple code as you did in part (2). Justify the agreement of your guess and Maple's output.

5. What would you expect the curvature of the right circular helix whose equation is $\mathbf{x}(t) = (R\cos t, R\sin t, ht)$ should be? Check the definition of curvature by using your Maple code as you did in part (2). Justify the agreement of your guess and Maple's output.

6. What would you expect the curvature of the ellipse whose equation is $\mathbf{x}(t) = (a\cos t, b\sin t, 5)$ should be? If you cannot guess what it should be, what properties should it have? Try to check the definition of curvature by using your Maple code as you did in part (2). Justify the agreement of your guess and Maple's output. (In this case, unlike the previous cases, simply computing the curvature function will likely be inadequate for immediately checking your intuition: you will likely have to perform some further graphical or numerical analysis. You may or may not wish to do this in Maple, but whatever you do, make sure you justify your answer.)