MATH 164, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Honors Project 2: Tracer Methods in Permeability

Potassium ions K^+ flow into and out of red blood cells. Physiologists are concerned with understanding this flow since it helps them understand the structure and behavior of red blood cells; hence it might help them in combating blood diseases.

One approach to studying the flow of potassium ions into and out of red blood cells is to use the fact that there is a radioactive isotope K^{42+} of potassium that that is indistinguishable from K^+ except for the fact that it is radioactive. Thus, if K^{42+} ions are introduced into the bloodstream, they can be sampled and measured at subsequent times to provide information about the permeability of red blood cells.

To be more concrete, suppose $C = C(t)$ represents the amount of potassium in corpuscles at time t, and $P = P(t)$ represents the amount of potassium in the plasma. If C and P measure the K^{42+} which is introduced into the bloodstream at time $t = 0$ then $C(0) = 0$ and $P(0) = P_0$, P_0 being a known quantity. Assuming that no potassium enters or leaves the bloodstream during the course of study amounts to assuming that $C(t) + P(t) = P_0$. (Since the half-life of K^{42+} is approximately 12.5 hr and this is significantly greater than the time it takes to introduce K^{42+} into the bloodstream and to perform requisite sampling tests, radioactive decay is not a significant source of error when it comes to the assumption that "no potassium enters or leaves the bloodstream during the course of study".) If we model the change dP/dt of P as a linear function of C (by K^{42+} ions moving from corpuscles to plasma) and P (by K^{42+} ions moving from plasma to corpuscles)

$$\frac{dP}{dt} = k_1 C - k_2 P \quad \text{where} \quad k_1, k_2 > 0$$

then we obtain a differential equation

$$\frac{dP}{dt} = k_1 C - k_2 P = k_1(P_0 - P) - k_2 P = k_1 P_0 - (k_1 + k_2)P \quad \text{where} \quad P(0) = P_0 \tag{1}$$

which can be solved for P.

Exercises:

1. a) Show that, under the above assumptions, the equilibrium level of K^{42+} in the plasma (the value of $P = P(t)$ for large t) is
$$P_\infty = \frac{k_1 P_0}{k_1 + k_2}.$$
   b) How could P_∞ be determined experimentally?
2. Show that the solution to Equation 1 is
$$P = \frac{k_1 P_0}{k_1 + k_2}\left(1 + \frac{k_2}{k_1} e^{-(k_1+k_2)t}\right).$$

It follows from Exercise 2 that the solution to our differential equation may be written

$$P = P_\infty \left(1 + \frac{k_2}{k_1} e^{-(k_1+k_2)t}\right) \quad \text{or} \quad \frac{P}{P_\infty} - 1 = \frac{k_2}{k_1} e^{-(k_1+k_2)t}.$$

By taking logarithms of both sides of this last equation, we find that

$$\ln\left(\frac{P}{P_\infty}-1\right) = -(k_1+k_2)t + \ln\left(\frac{k_2}{k_1}\right). \tag{2}$$

Exercises:

3. How would you use Equation 2 to determine k_1 and k_2 from a data set $\{(t_i, P(t_i)), i = 1, \ldots, N\}$?
4. Find P_0, P_∞, k_1 and k_2 given the following fictitious data set (in which we suppress units for simplicity):

t	0	1	2	3	4	5	6	7	8	9	10	11	12
P	1.60	1.37	1.27	1.22	1.19	1.18	1.18	1.18	1.17	1.17	1.17	1.17	1.17

5. Plot $P(t)$ vs. $C(t)$ (in a PC-coordinate plane) using k_1 and k_2 for the above data set (you may use Maple if you wish), and describe in words the evolution of $P(t)$ vs. $C(t)$ for $t > 0$.

The model we discussed in this project is often referred to as a *two compartment model* since the passage of K^{42+} ions occurs between two compartments: plasma and corpuscles. There are, of course, also three, four, ... compartment models, and these are of interest in the study of subjects such as epidemiology. A multiple compartment model in which quantities change with time is often referred to as a *dynamical system.* One goal of this project has been to illustrate that the study of dynamical systems has important applications in both science and engineering.