MATH 261, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Extra Project 14.4: Motion Around a Circular Path

Objective

In this project we discuss motion around a circular path. This project is *not* a Maple project: it only involves computations you can make by hand.

Narrative

If an object moves around a circular path of radius R then its position can be described by the parametric equations in time t:

$$x = x(t) = R\cos\theta(t), \quad y = y(t) = R\sin\theta(t)$$

where $\theta(t)$ is the *angular displacement* of the object at time t, or by the vector equation

$$\mathbf{r} = \mathbf{r}(t) = \langle R\cos\theta(t), R\sin\theta(t)\rangle = R\langle\cos\theta(t), \sin\theta(t)\rangle. \tag{1}$$

Differentiating (1) with respect to t, we find that

$$\mathbf{r}' = R\langle -\theta'\sin\theta, \theta'\cos\theta\rangle = R\theta'\langle -\sin\theta, \cos\theta\rangle. \tag{2}$$

Thus:

1. $\mathbf{r}'$ is perpendicular to $\mathbf{r}$ since

   $$\mathbf{r}'\cdot\mathbf{r} = (R\theta'\langle -\sin\theta, \cos\theta\rangle)\cdot(R\langle\cos\theta, \sin\theta\rangle) = R^2\theta'(-\sin\theta\cos\theta + \sin\theta\cos\theta) = 0,$$

   and the *unit tangent vector*

   $$\mathbf{T} = \mathbf{T}(t) = \frac{\mathbf{r}'}{||\mathbf{r}'||} = \langle -\sin\theta, \cos\theta\rangle$$

   is perpendicular to $\mathbf{r}(t)$ for all t.

2. the *linear velocity* $v = v(t)$ of the object

   $$v = ||\mathbf{r}'|| = ||R\theta'\langle -\sin\theta, \cos\theta\rangle|| = R|\theta'|.$$

   So if we denote the (absolute value of the) *angular velocity* θ' of the object by ω, then

   $$v = R\omega. \tag{3}$$

3. The derivative

   $$\mathbf{T}' = \langle -\theta'\cos\theta, -\theta'\sin\theta\rangle = \theta'\langle -\cos\theta, -\sin\theta\rangle$$

   of $\mathbf{T}$ with respect to t is perpendicular to $\mathbf{T}$ since

   $$\mathbf{T}'\cdot\mathbf{T} = (\theta'\langle -\cos\theta, -\sin\theta\rangle)\cdot\langle -\sin\theta, \cos\theta\rangle = \theta'(\sin\theta\cos\theta - \sin\theta\cos\theta) = 0.$$

   Indeed, the *unit normal vector*

   $$\mathbf{N} = \frac{\mathbf{T}'}{||\mathbf{T}'||} = \langle -\cos\theta, -\sin\theta\rangle = -\frac{\mathbf{r}}{R}$$

   for each t.

Differentiating (2) with respect to t, we find that

$$\mathbf{r}'' = R(\theta')^2\langle -\cos\theta, -\sin\theta\rangle + R\theta''\langle -\sin\theta, \cos\theta\rangle = R\theta''\mathbf{T} + R(\theta')^2\mathbf{N} \tag{4}$$

(or $\mathbf{r}'' = R\omega'\mathbf{T} + R\omega^2\mathbf{N}$). Using the facts that (3) implies that the linear acceleration

$$a = v' = R\omega' = R\theta''$$

and that

$$R(\theta')^2 = R\left(\frac{v}{R}\right)^2 = \frac{v^2}{R},$$

we may write (4) as

$$\mathbf{a} = a\mathbf{T} + \frac{v^2}{R}\mathbf{N}. \tag{5}$$

Thus:

1. the magnitude of the tangential component of the acceleration vector $\mathbf{a}$ is a,
2. the magnitude of the normal component of the acceleration vector $\mathbf{a}$ — the *centripetal acceleration* — is v^2/R, and
    (a) as R decreases, the centripetal acceleration increases, and
    (b) as the linear velocity v increases, the centripetal acceleration increases (by the square of v).

Task

Assuming $R = 2$ and $\theta = \theta(t) = t^3$, compute:
1. $\mathbf{r}$, **2**. $\mathbf{r}'$, **3**. $\mathbf{T}$, **4**. v, **5**. ω,
6. $\mathbf{T}'$, **7**. $\mathbf{N}$, **8**. $\mathbf{a}$, **9**. a, **10**. v^2/R.

Finally,

11. sketch the path of the projectile, and
12. sketch and label the vectors $\mathbf{T}$ and $\mathbf{N}$ when $t = \sqrt[3]{3\pi/4}$.