MATH 164, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Honors Project 3: Improper Integrals and Black Holes

An integral of the form

$$\int_0^{r_E} \frac{1}{r^2}\, dr$$

is an improper integral. A black hole is an object associated with a star which has collapsed on itself. In this project we investigate the connection between improper integrals and black holes.

The weight w of an object near the surface of the earth is often taken, by definition, to be $w = mg$ where m is the mass of the object and g is a constant. Weight is a result of the force of gravity. (In English units, w is measured in pounds, m in slugs, and $g = 32$ ft/sec^2.) The work W done in lifting an object of constant weight (or mass) a distance h above the surface of the earth is thus *approximately*

$$W = \text{weight} \; * \text{distance} = wh = mgh. \tag{1}$$

(If the weight (or mass) of the object varied with distance h above the surface of the earth[1]

$$w = w(h) = m(h)gh$$

then

$$W = \int_0^h w\; dh = \int_0^h m(h)g\; dh = g\int_0^h m(h)\; dh.$$

In the special case that m does *not* depend on h,

$$W = g\int_0^h m\; dh = mgh,$$

in agreement with (1).)

The above analysis assumes the object is near the surface of the earth. It is based on an approximation to the Universal Law of Gravitation, which states that the attraction between two objects of masses m_1 and m_2 which are r units apart is

$$F = \frac{Gm_1m_2}{r^2} \tag{2}$$

where G is a universal gravitational constant. For an object of constant mass m which is a distance h above the surface of the earth,

$$F = \frac{Gm_Em}{(r_E+h)^2} \approx \frac{Gm_Em}{r_E^2} = \frac{Gm_E}{r_E^2}m = mg$$

where $g = Gm_E/r_E^2$, m_E is the mass of the earth, and r_E is the radius of the earth. (This approximation is justified by the fact that h is generally much smaller than $r_E \approx 3963$ mi, so $r_E + h \approx r_E$.)

Exercise
1. *Avoiding* approximations, find the work W done in lifting an object of constant weight (or mass) a distance h away from the surface of the earth.

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[1] This would happen, for example, if the object being lifted was a bucket full of water with a hole through which water leaked.

The assumption you made in doing Exercise 1 was that the radius r_E of the earth is fixed, and the object is lifted a distance h away from the surface of the earth. Let us now change our point of view: instead of *lifting the object*, consider what would happen if we *shrank the earth while maintaining its mass.* Let us assume, for example, that we level all mountains and valleys on the earth, and eliminate all hollow spaces (such as caves) within the earth.

Exercise
2. If the radius of the shrunken earth is r_S, find the work done in lifting an object of constant weight (or mass) m from the surface of the shrunken earth to the radius r_E of the "old" earth.

Let us now assume that we shrink the earth further and further, all the while maintaining its mass. (Such a process would be possible if, for example, we were to alter the *molecular* structure of matter, making component materials denser and denser.)

Exercise
3. What is the limit of the work done in lifting an object of constant weight (or mass) m from the surface of a shrunken earth to the radius r_E of the "old" earth as the radius r_S of the shrunken earth approaches 0.
4. Interpret your answer to Exercise 3 in terms of physics.

A black hole is an object associated with a star which has (for whatever reason) collapsed on itself. If, in the process of collapsing, the star maintains its mass then the resulting black hole exerts tremendous forces on objects close to it: forces so large that not even light can escape its gravitational pull. From the point of view of mathematics, this is because an associated improper integral

$$\int_0^{r_E} \frac{1}{r^2}\, dr = \lim_{r_S \to 0} \int_{r_S}^{r_E} \frac{1}{r^2}\, dr$$

is infinite.