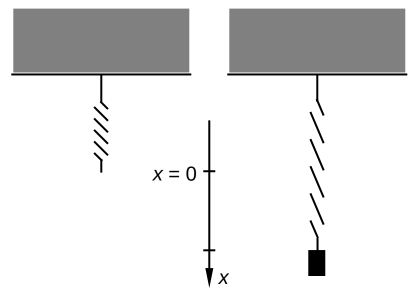
MATH 164, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Honors Project 4: Spring-Mass Damping

Introduction

In this project we investigate solutions to the spring-mass damping equation $m\ddot{x} + c\dot{x} + kx = 0$.



According to Hooke's Law, the force exerted by an ideal spring in restoring the spring to its equilibrium position is $F = -kx$, where k is a positive constant, and x is the distance the spring is stretched out of equilibrium. If an object of mass m is attached to the spring then over time t, then the force exerted on the spring on the object is $F = ma = m\ddot{x}$ where the distance $x = x(t)$ the spring is stretched out of equilibrium is a function of time. If these forces balance each other then the position of the object at time t can be described by the second order differential equation in t:

$$m\ddot{x} = -kx \quad \text{or} \quad m\ddot{x} + kx = 0. \tag{1}$$

Such an object exhibits what is known as *simple harmonic motion.*

If the motion of the object is damped by a force (such as air resistance) proportional in magnitude to the velocity of the object, then equation (1) becomes

$$m\ddot{x} = -kx - c\dot{x} \quad \text{or} \quad m\ddot{x} + c\dot{x} + kx = 0 \tag{2}$$

where c is again a positive constant. Finally, if some external force of magnitude $f = f(t)$ depending on time, acts on (or drives) the motion of the object, then

$$m\ddot{x} + c\dot{x} + kx = f(t). \tag{3}$$

Equation (3) is known as the *spring-mass damping equation.* In this project we investigate the solutions of equations (1) and (2). (In this project we do not derive the solutions of these equations: we simply *verify* and *investigate* them. The fact that these are the only solutions to (1) is proved in a course on differential equations.)

Equation (1)

Any solution to (1) may be written in the form $x = A\cos(\omega t + \phi)$ where A is the maximum displacement from equilibrium, $\omega = \sqrt{k/m}$ is the frequency, and ϕ is the phase. The values of A and ϕ depend on the initial values $x(0)$ and $\dot{x}(0)$ of the displacement x and the velocity $\dot{x}$: in general,

$$A = \sqrt{x(0)^2 + \left(\frac{\dot{x}(0)}{\omega}\right)^2} \quad \text{and} \quad \phi = \text{Tan}^{-1}\left(\frac{-\dot{x}(0)}{\omega x(0)}\right).$$

Task 1: Using Maple, verify that $x = A\cos(\omega t + \phi)$ is a solution to (1).

Task 2: Using Maple, graph $x = A\cos(\omega t + \phi)$ for $t \in [0, 4\pi]$ assuming that $m = 2.0$, $k = 3.0$, $x_0 = 1.0$, and $\dot{x}_0 = 1.25$.

Task 3: Using Maple, and assuming that $m = 2.0$, $k = 3.0$, $A = 1.5$, and $\phi = \pi/4$, find $x(0)$ and $\dot{x}(0)$.

Equation (2)

The character of solutions to equation (2) depends on the sizes of m, c, and k. In general, they are of the form:

- if $c^2 - 4mk \neq 0$ then $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where

$$r_i = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

  are the solutions to the *algebraic* equation $mX^2 + cX + k = 0$ and c_1 and c_2 are constants depending on $x(0)$ and $\dot{x}(0)$, and

- if $c^2 - 4mk = 0$ then $x = (c_1 + c_2 t)e^{-ct/2m}$ where c_1 and c_2 are constants depending on $x(0)$ and $\dot{x}(0)$.

Equation (2), Case 1: $c^2 - 4mk > 0$ ***(over damping)***

Task 4: Using Maple, graph $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ for $t \in [0, 4\pi]$ assuming that $m = 2.0$, $c = 5.0$, $k = 3.0$, $c_1 = 0.4$, and $c_2 = 1.0$.

Task 5: Using Maple, and assuming that $m = 2.0$, $c = 5.0$, $k = 3.0$, $c_1 = 0.4$, and $c_2 = 1.0$, find $x(0)$ and $\dot{x}(0)$.

Task 6: Using Maple, and assuming that $m = 2.0$, $c = 5.0$, $k = 3.0$, $x(0) = 1.0$ and $\dot{x}(0) = -1.5$, find c_1, and c_2.

Equation (2), Case 2: $c^2 - 4mk < 0$ ***(under damping)***

If $c^2 - 4mk < 0$ then the roots r_i of the equation $mX^2 + cX + k = 0$ are complex, and we may write

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{-ct/2m}(c_1 \cos \omega t + c_2 \sin \omega t)$$

where $\omega = \sqrt{4mk - c^2}/2m$.

Task 7: Using Maple, graph $x = e^{-ct/2m}(c_1 \cos \omega t + c_2 \sin \omega t)$ for $t \in [0, 4\pi]$ assuming that $m = 2.0$, $c = 2.0$, $k = 3.0$, $c_1 = 0.4$, and $c_2 = 1.0$. Using these same parameters, also plot $x = (c_1 + c_2 t)e^{-ct/2m}$, $x = \sqrt{c_1^2 + c_2^2} e^{-ct/2m}$ and $x = \sqrt{c_1^2 + c_2^2} e^{-ct/2m}$ for $t \in [0, 4\pi]$ on a single set of coordinate axes.

Task 8: Using Maple, and assuming that $m = 2.0$, $c = 2.0$, $k = 3.0$, $c_1 = 0.4$, and $c_2 = 1.0$, find $x(0)$ and $\dot{x}(0)$.

Task 9: Using Maple, and assuming that $m = 2.0$, $c = 2.0$, $k = 3.0$, $x(0) = 0.4$ and $\dot{x}(0) = 1.0$, find c_1, and c_2.

Equation (2), Case 3: $c^2 - 4mk = 0$ ***(critical damping)***

Task 10: Using Maple, graph $x = (c_1 + c_2 t)e^{-ct/2m}$ for $t \in [0, 4\pi]$ assuming that $m = 2.0$, $c = 2.0$, $k = 3.0$, $c_1 = 0.4$, and $c_2 = 1.0$.

Task 11: Using Maple, and assuming that $m = 2.0$, $c = 2.0$, $k = 3.0$, $c_1 = 0.4$, and $c_2 = 1.0$, find $x(0)$ and $\dot{x}(0)$.

Task 12: Using Maple, and assuming that $m = 2.0$, $c = 2.0$, $k = 3.0$, $x(0) = 0.4$ and $\dot{x}(0) = 0.6$, find c_1, and c_2.