Project 2.1c: Tangential Approximation

Objective

In this project we illustrate one reason the problem of finding the equation of the tangent line to the graph of a function at a point is important.

Narrative

One of the important practical problems faced in the compilation of tables and in the programming of computational devices (such as hand-calculators and computers) is how to find the value of functions such as $\sqrt{x}$, $\sin x$, e^x, and $\ln x$ for specific choices of x. Tables and computational devices are so ubiquitous that it might not be (immediately) apparent that this is an issue, but *someone* has to program the devices we use, and evaluating functions such as these *is* an issue for anyone involved in writing low-level computer code.

Knowing the equation of the tangent line to the graph of a function $y = f(x)$ at a given point is important since the tangent line is the graph of a linear function $y = mx + b$, and linear functions are easy to evaluate. For example, later we will see that if angles are measured in radians then the tangent line to the graph of $y = f(x) = \sin x$ at $x = 0$ is the graph of $y = x$. Thus for "small" angles x, $\sin x \approx x$. So if we're interested in computing $\sin 3°$, for example, we might convert $3°$ to radians and then use this value to approximate $\sin 3°$.

Task

a) Type the command lines below into Maple in the order in which they are listed. These command lines produce graphs of $y = \sin x$ and $y = x$ at two different scales. Note that the natural unit for angle measurement (in the computation of trig functions such as sin) in Maple is radians. Also observe how close the graphs of $y = \sin x$ and $y = x$ are to one another!

```
> # Project 2.1c:  Tangential Approximation
> # Task a
> restart:
> plot({x,sin(x)},x=-Pi/2..Pi/2);
> plot({x,sin(x)},x=-Pi/6..Pi/6);
```

b) Continue by typing the following command lines into Maple. They produce numerical information about the values of the functions $f(x) = \sin x$ and $f(x) = x$ for small values of x (in degrees and in radians) and store them in a matrix (a rectangular array of numbers). (Remember that $1° = \pi/180$ radians.)

```
> # Task b
> OneDeg := evalf(Pi/180);
> M := matrix(32,4,(Row,Col) -> 0):
> M[1,1] := "x (in degrees)":  M[1,2] := "x (in radians)":
    M[1,3] := "sin(x)":  M[1,4] := "error":
> for i from 0 to 30 do
    M[i+2,1] := 2*i:  M[i+2,2] := evalf(2*i*OneDeg):
    M[i+2,3] := evalf(sin(2*i*OneDeg)):
    M[i+2,4] := evalf(M[i+2,2]-M[i+2,3]):  od:
> eval(M);
```

At this time make a hard-copy of your typed input and Maple's responses. Then, ...

c) By hand, use the data in the chart you produced to find an approximation to $\sin 3°$ to 4 decimal places of accuracy. Roughly how much error is there in this approximation? (Make sure it is clear how you draw your conclusions.)

d) Later we will see that the tangent line to the graph of $y = \sqrt{x}$ at $x = 1$ is the graph of $y = \frac{1}{2}x + \frac{1}{2}$. Using this fact, approximate $\sqrt{1.1}$. Based on the fact that Maple says $\sqrt{1.1} = 1.048808848$, roughly how much error is there in your approximation?

Comments

1. We will return to this topic later in MATH 163 in our discussion of differentials.

2. Evaluating functions such as $\sqrt{x}$, $\sin x$, e^x, and $\ln x$ is *only one* reason finding the equation of the tangent line to the graph of a function at a point is important. There are *many* others which we will discuss later in MATH 163.

3. In MATH 164 we will discuss how small "small" is, and how to handle "large" angles when evaluating functions such as sin.