MATH 261, Spring 2001 Due Date:____________ Name(s): ________________________

Project 17.7a: Surface Area and Surface Integrals

Objective

The objective of this project is to use illustrate the computation of surface area and surface integrals.

Narrative

If you have not already done so, read Sections 17.6 and 17.7 of the text.

If $f : R^3 \to R$ is continuous and $\mathbf{r} = \mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, $(u, v) \in D \subset R^2$, parametrizes the surface S in R^3 then the surface integral

$$\int\int_S f\ dS = \int\int_D f(\mathbf{r}(u, v))\ |\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)|\ dA.$$

In particular, if $\mathbf{F}$ is any vector field in R^3 and

$$\mathbf{n} = \frac{\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)}{|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)|}$$

is a unit normal vector to S, then the surface integral

$$\begin{aligned} \int\int_S \mathbf{F} \cdot d\mathbf{S} &= \int\int_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v))\ dA_{uv} \\ &= \int\int_D \mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v))}{|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)|} |\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)| dA_{uv} \\ &= \int\int_S \mathbf{F} \cdot \mathbf{n}\ dS \end{aligned}$$

is the flux of $\mathbf{F}$ through S. (And this integral is of the form $\int\int_S f\ dS$ if we let $f = \mathbf{F} \cdot \mathbf{n}$.)

Task

1. a) Type the command lines below into Maple in the order in which they are listed. In doing so, you will compute the surface area $\int\int_{S^2}\ dS = \int\int_{S^2}\ 1\ dS = \int\int_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\ dA$ of the unit sphere S^2. (See Example 10, p. 1123 in the text.)

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> # Project 17.7a:  Surface Area and Surface Integrals
> restart; with(linalg):
> r := vector([sin(phi)*cos(theta),sin(phi)*sin(theta),cos(phi)]);
> r_phi := map(diff,r,phi); r_theta := map(diff,r,theta);
> cprod := crossprod(r_phi,r_theta);
> # The surface area
> integrand := sqrt(simplify(cprod[1]^2+cprod[2]^2+cprod[3]^2));
> Int(Int(integrand,phi=0..Pi),theta=0..2*Pi) =
      evalf(int(int(integrand,phi=0..Pi),theta=0..2*Pi));
```

b) Continue by typing the command lines below into Maple. In doing so, you will compute the surface integral $\int\int_{S^2} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S^2} \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F}(x, y, z) = \langle z, y, x\rangle$, over the unit sphere S^2. (See Example 5, p. 1135 in the text.)

```
> # A surface integral
> F := vector([z,y,x]);
> plot3d([r[1],r[2],r[3]],phi=0..Pi/2,theta=0..Pi/2,axes=normal,
      orientation=[30,60],scaling=constrained);
> integrand := subs({x=r[1],y=r[2],z=r[3]},dotprod(F,cprod));
> Int(Int(integrand,phi=0..Pi),theta=0..2*Pi) =
      evalf(int(int(simplify(integrand),phi=0..Pi),theta=0..2*Pi));
```

Do *not* adjust the options used to create the graphic for part (b).

At this time make a hard-copy of your typed input and Maple's responses. Then, ...

2. On the graphic you produced in part (b) of Task 1, draw the vectors $\mathbf{F}(x,y) = \langle z, y, x\rangle$ at the following points:

$$(x,y,z) = \begin{array}{l}(1,0,0),(0,1,0),(0,0,1),\\ (0,1/\sqrt{2},1/\sqrt{2}),(1/\sqrt{2},0,1/\sqrt{2}),(1/\sqrt{2},1/\sqrt{2},0),\\ (1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})\end{array}$$

Comments

Surface integrals are also often written in the form

$$\int\int_S P\ dy\ dz + Q\ dx\ dz + R\ dx\ dy.$$

To relate this form to the one described in the Narrative, observe that if $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field on R^3 and $\mathbf{n} = \langle n_x, n_y, n_z\rangle$ is the unit normal to S then

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int_S \mathbf{F}\cdot\mathbf{n}\ dS = \int\int_S \langle P,Q,R\rangle\cdot\langle n_x,n_y,n_z\rangle\ dS = \int\int_S (Pn_x + Qn_y + Rn_z)\ dS.$$

Now let us consider $\int\int_S Rn_z\ dS$. If, over the projection D_{xy} of S into the xy-coordinate plane, S is the graph of $z = f(x,y)$ then

$$\mathbf{n} = \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+f_x^2+f_y^2}},$$

so $n_z = 1/\sqrt{1+f_x^2+f_y^2}$, and $dS = \sqrt{1+f_x^2+f_y^2}\ dx\ dy$. Thus

$$\int\int_S Rn_z\ dS = \int\int_{D_{xy}} R\frac{1}{\sqrt{1+f_x^2+f_y^2}}\sqrt{1+f_x^2+f_y^2}\ dx\ dy = \int\int_{D_{xy}} R\ dx\ dy.$$

Similarly

$$\int\int_S Pn_x\ dS = \int\int_{D_{yz}} P\ dy\ dz \quad\text{and}\quad \int\int_S Qn_y\ dS = \int\int_{D_{xz}} Q\ dx\ dz.$$

Thus we may write

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int_S P\ dy\ dz + Q\ dx\ dz + R\ dx\ dy.$$