MATH 164, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Extra Project 12.10b: The Second Derivative Test Again

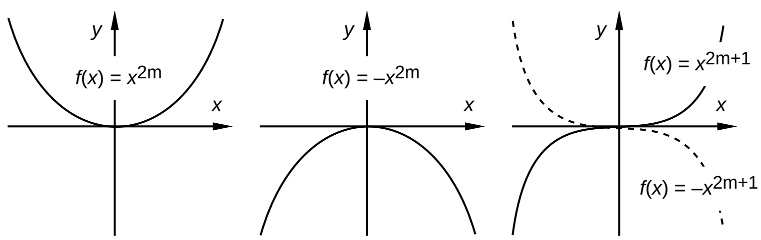
Objective

One of the major limitations of the Second Derivative Test for functions $f : R \to R$ of one real variable is that if $f'(0) = 0$ (so $x = 0$ is a critical value for f) and $f''(0) = 0$ then the Second Derivative Test provides no information about whether $f(0)$ is a relative maximum, a relative minimum, or neither. In this project we illustrate how we can get more information using concepts related to Mclaurin series.

Narrative

Our analysis is based on Mclaurin series, and the fact that if:

1. $f(x) = x^{2n}$, n an integer, then f is positive for all values of x other than $x = 0$, so $f(0) = 0$ is a relative minimum,
2. $f(x) = -x^{2n}$, n an integer, then f is negative for all values of x other than $x = 0$, so $f(0) = 0$ is a relative maximum, and
3. $f(x) = \pm x^{2n+1}$, n an integer, then the graph of f is positive for some values of x near $x = 0$ and negative for others, so $f(0)$ is a neither a relative minimum nor a relative maximum.



The Mclaurin series expansion for a real-valued function $f : R \to R$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots.$$

Thus if $x = 0$ is a critical value for f, then

$$f(x) = f(0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots.$$

If we also know that $f''(0) = 0$, then

$$f(x) = f(0) + \frac{f'''(0)}{3!}x^3 + \ldots.$$

Indeed, suppose that

$$f(x) = f(0) + \frac{f^{(N)}(0)}{N!}x^N + \frac{f^{(N+1)}(0)}{(N+1)!}x^{N+1} + \ldots$$

where N is the first integer for which $f^{(N)}(0) \neq 0$. Since x^{N+1} is considerably smaller than x^N for all values of x sufficiently close to 0, it follows that

$$f(x) \approx g(x) = f(0) + \frac{f^{(N)}(0)}{N!}x^N$$

for all values of x sufficiently close to 0. Thus we can classify the behavior of f near 0 by classifying the behavior of g near 0.

Example: If $f(x) = x^8 - x^3$ then $f'(0) = 0$ and $f''(0) = 0$, so the Second Derivative Test provides no information. However, if we write $f(x) = -x^3 + x^8$ and remember that x^8 is considerably smaller than x^3 for all values of x sufficiently close to 0, then we may approximate $f(x) = -x^3 + x^8$ by $g(x) = -x^3$. Since g has neither a relative maximum nor a relative minimum at $x = 0$, f has neither a relative maximum nor a relative minimum at $x = 0$.

Example: If $f(x) = x^2(e^{x^2} - 1)$ then $f'(0) = 0$ and $f''(0) = 0$, so the Second Derivative Test again provides no information. However, since the Mclaurin series expansion for e^x is

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots,$$

it follows that

$$e^{x^2} = 1 + \frac{1}{1!}x^2 + \frac{1}{2!}x^4 + \dots$$

so

$$f(x) = x^2(e^{x^2} - 1) = x^2\left(\left(1 + \frac{1}{1!}x^2 + \frac{1}{2!}x^4 + \dots\right) - 1\right) = \frac{1}{1!}x^4 + \frac{1}{2!}x^6 + \dots.$$

Since x^6 and all higher powers of x are considerably smaller than x^4 for all values of x sufficiently close to 0, we may approximate $f(x)$ by $g(x) = x^4$. Since g has a relative minimum $x = 0$, f has a relative minimum at $x = 0$.

Task

For each of the following functions, $f'(0) = 0$ and $f''(0) = 0$, so the Second Derivative Test provides no information. Use an analysis of the Mclaurin series of f to determine whether $f(0)$ is a relative maximum, a relative minimum, or neither.

1. $f(x) = 2x^6 - 6x^4$
2. $f(x) = 3x^4 - 4x^3 + 6$
3. $f(x) = 3x^5 - 5x^3$
4. $f(x) = 8x^2 - 2x^4$
5. $f(x) = 1 - \cos x^2$
6. $f(x) = x \ \mathrm{Tan}^{-1} x^2$
7. $f(x) = x - \ln(1 + x^2)$
8. $f(x) = e^{x^3} - 1$

Comments

1. The technique presented above can easily be extended to values c of x for which $f'(c) = 0$ and $f''(c) = 0$, other than $c = 0$. The only changes that must be made are: 1) shifting attention from the Mclaurin series expansion for f to the Taylor series expansion for f about $x = c$, and 2) shifting attention from values of x that are close to 0 to values of x that are close to c.

2. If, for a given function f, you know that $f'(0) = 0$ and $f''(0) = 0$ but you do not know the Mclaurin series expansion for f, then you can avoid the intermediate step of finding the Mclaurin series expansion for f by computing higher and higher order derivatives of f at 0, until you obtain one — the *first* one — at which $f^{(N)}(0) \neq 0$. Depending on whether N is odd or even, and if N is even whether $f^{(N)}(0) > 0$ or $f^{(N)}(0) < 0$, you can draw a conclusion about whether $f(0)$ is a relative maximum, a relative minimum, or neither. (Maple is well suited for this!) This test is a generalization of the Second Derivative Test, and although we could state it as a higher-order version of the Second Derivative Test, doing so is not necessarily useful (since the conditions are a little difficult to memorize). Instead, it is perhaps wiser to simply remember the origins of this test in the context of Mclaurin series.