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OPTIMAL POLICIES FOR ECONOMIC STABILIZATION

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Short-term economic stabilization policy is approached as a problem in optimal control. The optimal control problem is defined as a dual discrete-time tracking problem (nominal state and nominal policy trajectories are tracked) for a linear time-invariant system with a quadratic cost functional. This problem is solved analytically, and the solution is applied to a ten-equation quarterly econometric model. Optimal stabilization policies are calculated for cost functionals designed to force single variables to follow nominal paths, to impose trade-offs between the movements of different variables, and to emphasize the use of one or another policy variable. The experimental results demonstrate that this approach is valuable both as a tool for policy planning and as a method of analyzing the dynamic properties of econometric models.

1. INTRODUCTION

SOME FIFTEEN YEARS have passed since Phillips [**15**] first showed that the application of certain types of stabilization policies to multiplier-accelerator macroeconomic models could result in undesired oscillations or instabilities. It has become clear from this and other analyses of macroeconomic policy [**1, 3, 5, 16**] that, because of the dynamic structure of the economy, well-intentioned policies may have unexpected and counterintuitive results.

In recent years a number of economists have demonstrated the potential application of the mathematical techniques of optimal control theory to economic policy formulation for stabilization [**6, 20, 22**] as well as long-run growth and development [**7, 8, 12, 13, 21**]. While much of this work has been successful in showing how optimal control could be applied to policy problems, there has been little attempt made to actually apply it to a realistic policy problem, particularly in the area of short-run stabilization. A goal of this paper is to show that if one is willing to work with a linear or linearized economic model and quadratic cost criteria, optimal control theory can provide a viable tool for both analyzing and understanding the dynamic properties of the model, and for formulating stabilization policies based on the model.

In this paper economic stabilization will be approached as a dual tracking problem in optimal control. The problem that is defined and solved involves tracking nominal state and nominal policy trajectories, subject to a quadratic cost function and the constraint of a linear system. This is actually quite general and will enable us to penalize for variations in, as well as the levels of, the state variables and control variables. Moreover, this lets us structure the problem as one without absolute limitations on the sets of allowable controls and allowable states; any restrictions that are to be imposed on the motion of control or state variables are expressed by assigning higher costs to their deviations. We will also put no restriction on the endpoint, i.e., on the final value of the state vector.

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The analytical solution to this optimal control problem will be seen to be computationally compatible even with very large economic models. It is preferable, however, to demonstrate the application of the results with a model that is reasonably small and whose dynamics are easily understood. A small linear quarterly model of the United States economy was constructed for this purpose [**17**], respecified in state-variable form,[2] and used to obtain numerical solutions for optimal stabilization policies.

2. THE OPTIMAL CONTROL PROBLEM AND ITS SOLUTION

The aim of our policy plan will be to make x_i, the vector of state variables, track as closely as possible a nominal state vector $\hat{x}_i$ but subject to u_i, the vector of control variables, tracking as closely as possible a nominal control vector $\hat{u}_i$. In other words, we would like variables such as gross national product, investment, and unemployment to follow closely nominal or "ideal" time paths throughout the planning period. The nominal time paths for gross national product and investment, for example, would probably grow at some steady rate, while that for unemployment might drop and then remain low for the remainder of the planning period. The control variables, of course, are used to make gross national product, investment, and unemployment move in the desired direction, but we are not free to manipulate the control variables in any way whatsoever—they in turn must also stay close to a set of nominal or "ideal" time paths. For example, we are not free to increase government spending or the money supply by 100 per cent in one year and decrease them by 200 per cent in the next year, etc. Manipulating policy variables has very real costs associated with it, and these costs must be embodied in the cost functional.

2.1 *The Problem*

The system of interest has the state-variable form

$$x_{i+1} - x_i = Ax_i + Bu_i + Cz_i \tag{1}$$

with known initial condition

$$x_0 = \xi. \tag{2}$$

Here x_i is the n-dimensional state vector[3] at time i, u_i the r-dimensional control vector at time i, and z_i an s-dimensional vector representing, at time i, s exogenous variables which are known for all i but cannot be controlled by the policy planner.[4] The matrices A, B, and C are $n \times n$, $n \times r$, and $n \times s$.

[2] Readers unfamiliar with the state-variable specification of a model should consult a standard text on dynamic system theory. A good introduction is provided by Athans and Falb [**2**] or Zadeh and Desoer [**23**].

[3] The number of state variables will generally be larger than the number of endogenous variables since the structural form of the model will usually contain difference equations of order greater than one.

[4] Exogenous variables might include exports, foreign investments, as well as variables sometimes thought endogenous but for which behavioral equations have not been written.

Let $\hat{x}_i$ and $\hat{u}_i$ be the nominal state and control vectors that we would like to track. At time i we would like x_i to be close to $\hat{x}_i$ and u_i to be close to $\hat{u}_i$. We assume that $\hat{x}_i$ and $\hat{u}_i$ have been specified for the entire planning period, $i = 0, 1, \ldots, N$. Note that the $\{\hat{x}_i\}$ need *not* be the result of substituting the $\{\hat{u}_i\}$ into (1).

The cost functional is given by

$$J = \frac{1}{2} \sum_{i=0}^{N} \{(x_i - \hat{x}_i)'Q(x_i - \hat{x}_i) + (u_i - \hat{u}_i)'R(u_i - \hat{u}_i)\} \tag{3}$$

where Q is an $n \times n$ positive semi-definite matrix, and R is an $r \times r$ positive definite matrix.

The optimal control problem is to find a control sequence $\{u_i^*, i = 0, 1, \ldots, N - 1\}$ such that

$$x_0^* = \xi, \tag{4}$$

$$x_{i+1}^* - x_i^* = Ax_i^* + Bu_i^* + Cz_i, \tag{5}$$

and the cost functional (3) is minimized.

It is important to keep in mind the meaning of the cost functional (3). Both Q and R will normally be diagonal matrices. The elements of Q give the relative costs for deviating from the nominal path of each state variable—for example, the cost of deviating from nominal GNP relative to the cost for deviating from nominal unemployment. Some of the elements of Q may be zero. The elements of R give the relative costs for deviating from the nominal paths of the control variables. For example, we would expect it to be more costly to manipulate the tax rate than to manipulate the money supply. All of the diagonal elements of R must be non-zero. This is both meaningful in terms of the economic problem and necessary for a mathematical solution. Finally, the comparative magnitudes of Q and R give the costs of controlling the economy relative to the costs of having the economy deviate from its "ideal" path—that is, the relative costs of means versus ends.

2.2 *The Necessary Conditions*

We begin by expressing the necessary conditions set forth by the Minimum Principle of Pontryagin [**4**, **9**, **10**, **11**, **14**, **18**, **19**].[5] The Hamiltonian is written as

$$\begin{aligned} H(x_i, p_{i+1}, u_i) = {} & \tfrac{1}{2}(x_i - \hat{x}_i)'Q(x_i - \hat{x}_i) + \tfrac{1}{2}(u_i - \hat{u}_i)'R(u_i - \hat{u}_i) \\ & + p_{i+1}'(Ax_i + Bu_i + Cz_i), \end{aligned} \tag{6}$$

where p_i is the vector of co-states (i.e., dynamic Lagrange multipliers). The necessary conditions provide the equations describing the optimal trajectories for x_i^*, p_i^*, and u_i^*

$$x_{i+1}^* - x_i^* = \left.\frac{\partial H}{\partial p_{i+1}}\right|_* = Ax_i^* + Bu_i^* + Cz_i, \tag{7}$$

$$p_{i+1}^* - p_i^* = -\left.\frac{\partial H}{\partial x_i}\right|_* = -Q(x_i^* - \hat{x}_i) - A'p_{i+1}^*. \tag{8}$$

[5] Note that our quadratic cost functional insures the sufficiency of these conditions as well.

And these are subject to the split boundary conditions

(9) $x_0^* = \xi,$

(10) $p_N^* = Q(x_N^* - \hat{x}_N),$

where equation (10) is a result of the transversality condition. Finally, the minimization of the Hamiltonian is written

(11) $\dfrac{\partial H}{\partial u_i} = 0,$

yielding

(12) $u_i^* = -R^{-1}B'p_{i+1}^* + \hat{u}_i.$

2.3 *Solving the Optimal Control Problem*

We could, if we wanted to, substitute equation (12) back into equation (7). We would then have a set of $2n$ first-order difference equations for the x_i^*'s and p_i^*'s, together with the $2n$ split boundary conditions provided by (9) and (10). These could then be solved on a computer and the resulting values for p_i^* could be substituted back into (12) to give us the optimal control u_i^*. This, however, is a tedious procedure involving a perhaps inordinate amount of computation, and we therefore follow a different tack.

We make the assumption that p_i^* is of the form

(13) $p_i^* = K_i x_i^* + g_i,$

and we will later see this to indeed be the case. Substituting this into equation (12),

(14) $u_i^* = -R^{-1}B'(K_{i+1}x_{i+1}^* + g_{i+1}) + \hat{u}_i,$

and substututing (13) and (14) into (7) and (8),

(15a) $x_{i+1}^* - x_i^* = Ax_i^* - BR^{-1}B'K_{i+1}x_{i+1}^* - BR^{-1}B'g_{i+1} + B\hat{u}_i + Cz_i,$

(15b) $p_{i+1}^* - p_i^* = -Q(x_i^* - \hat{x}_i) - A'(K_{i+1}x_{i+1}^* + g_{i+1}).$

Rearranging terms and substituting (13) into the left-hand side of (15b),

(16a) $(I + BR^{-1}B'K_{i+1})x_{i+1}^* = (I + A)x_i^* - BR^{-1}B'g_{i+1} + B\hat{u}_i + Cz_i,$

(16b) $(I + A')K_{i+1}x_{i+1}^* + Qx_i^* - K_i x_i^* = -(I + A')g_{i+1} + g_i + Q\hat{x}_i.$

Now define

(17) $E = I + BR^{-1}B'K_{i+1}.$

We will assume for now that E is non-singular and E^{-1} exists. This will indeed be the case since, as we will see, K_i must be positive semi-definite for all i. Then (16a) can be written as

(18) $x_{i+1}^* = E^{-1}(I + A)x_i^* - E^{-1}BR^{-1}B'g_{i+1} + E^{-1}B\hat{u}_i + E^{-1}Cz_i.$

Now substituting (18) into (16b),

$$(19)\qquad (I + A')K_{i+1}\{E^{-1}(I + A)x_i^* - E^{-1}BR^{-1}B'g_{i+1} + E^{-1}B\hat{u}_i + E^{-1}Cz_i\} + Qx_i^* - K_ix_i^* = -(I + A')g_{i+1} + g_i + Q\hat{x}_i.$$

Rearranging,

$$(20)\qquad \{Q + (I + A)'K_{i+1}E^{-1}(I + A)\}x_i^* - (I + A)'K_{i+1}E^{-1}BR^{-1}B'g_{i+1} + (I + A)'g_{i+1} + (I + A)'K_{i+1}E^{-1}(B\hat{u}_i + Cz_i) - Q\hat{x}_i = K_ix_i^* + g_i.$$

Now this must hold for any initial ξ, and since K_i does not depend on ξ, this must hold for all x_i^*. Hence we can equate coefficients, yielding

$$(21)\qquad K_i = Q + (I + A)'K_{i+1}E^{-1}(I + A)$$

and

$$(22)\qquad g_i = -(I + A)'K_{i+1}E^{-1}BR^{-1}B'g_{i+1} + (I + A)'g_{i+1} + (I + A)'K_{i+1}E^{-1}(B\hat{u}_i + Cz_i) - Q\hat{x}_i.$$

From (10) and (13) we have

$$(23)\qquad p_N^* = Q(x_N^* - \hat{x}_N) = K_Nx_N^* + g_N.$$

This must hold for any x_N^*, since x_N^* is unspecified. Then

$$(24)\qquad K_N = Q$$

and

$$(25)\qquad g_N = p_N^* - K_Nx_N^* = -Q\hat{x}_N.$$

Equations (24) and (25) provide the boundary conditions for equations (21) and (22) respectively.

Now re-examine expression (17) for E. Apply the matrix identity

$$(26)\qquad (I_n + ST')^{-1} = I_n - S(I_r + T'S)^{-1}T',$$

where S and T are $n \times r$ matrices, with $r \leqslant n$. Let $S = B$ and $T' = R^{-1}B'K_{i+1}$, yielding

$$(27)\qquad \begin{aligned} E^{-1} &= I - B(I_r + R^{-1}B'K_{i+1}B)^{-1}R^{-1}B'K_{i+1}, \\ &= I - B(R(I_r + R^{-1}B'K_{i+1}B))^{-1}B'K_{i+1}, \\ &= I - B(R + B'K_{i+1}B)^{-1}B'K_{i+1}. \end{aligned}$$

Note that $(R + B'K_{i+1}B)$ in equation (27) is an $r \times r$ matrix. Hence the only matrices to be inverted now are of size $r \times r$, and r, the dimension of the control vector, is generally much smaller than n, the dimension of the state vector.

Substituting (27) back into (21) and (22) gives us

$$K_i = Q + (I + A)'(K_{i+1} - K_{i+1}B(R + B'K_{i+1}B)^{-1}B'K_{i+1})(I + A) \tag{28}$$

and

$$\begin{aligned} g_i = &-(I + A)'(K_{i+1} - K_{i+1}B(R + B'K_{i+1}B)^{-1}B'K_{i+1}) \\ &\times BR^{-1}B'g_{i+1} + (I + A)'g_{i+1} + (I + A)'(K_{i+1} - K_{i+1}B \\ &\times (R + B'K_{i+1}B)^{-1}B'K_{i+1})(B\hat{u}_i + Cz_i) - Q\hat{x}_i. \end{aligned} \tag{29}$$

Equation (28) is the Riccati equation, and (29) is the tracking equation for our optimal control problem. Together with the boundary conditions (24) and (25) they can be solved for K_i and g_i, $i = 1 \ldots, N$. The optimal control is then determined by equation (14). Now, substituting equation (18) into (14) gives us

$$\begin{aligned} u_i^* = &- R^{-1}B'K_{i+1}\{E^{-1}(I + A)x_i^* - E^{-1}BR^{-1}B'g_{i+1} + E^{-1}B\hat{u}_i \\ &+ E^{-1}Cz_i\} - R^{-1}B'g_{i+1} + \hat{u}_i. \end{aligned} \tag{30}$$

And substituting (27) for E^{-1},

$$\begin{aligned} u_i^* = &- R^{-1}B'K_{i+1}(I - B(R + B'K_{i+1}B)^{-1}B'K_{i+1})(I + A)x_i^* \\ &+ R^{-1}B'K_{i+1}(I - B(R + B'K_{i+1}B)^{-1}B'K_{i+1})BR^{-1}B'g_{i+1} \\ &- R^{-1}B'g_{i+1} - R^{-1}B'K_{i+1}(I - B(R + B'K_{i+1}B)^{-1}B'K_{i+1}) \\ &\times (B\hat{u}_i + Cz_i) + \hat{u}_i, \end{aligned} \tag{31}$$

$$\begin{aligned} u_i^* = &- R^{-1}(I - B'K_{i+1}B(R + B'K_{i+1}B)^{-1})B'K_{i+1}(I + A)x_i^* \\ &+ R^{-1}(I - B'K_{i+1}B(R + B'K_{i+1}B)^{-1})B'K_{i+1}BR^{-1}B'g_{i+1} \\ &- R^{-1}B'g_{i+1} - R^{-1}(I - B'K_{i+1}B(R + B'K_{i+1}B)^{-1})B'K_{i+1} \\ &\times (B\hat{u}_i + Cz_i) + \hat{u}_i. \end{aligned} \tag{32}$$

Now we make use of the identity

$$I - X(Y + X)^{-1} = Y(Y + X)^{-1}; \tag{33}$$

then

$$\begin{aligned} u_i^* = &- (R + B'K_{i+1}B)^{-1}B'K_{i+1}(I + A)x_i^* + (R + B'K_{i+1}B)^{-1} \\ &\times B'K_{i+1}BR^{-1}B'g_{i+1} - R^{-1}B'g_{i+1} - (R + B'K_{i+1}B)^{-1} \\ &\times B'K_{i+1}(B\hat{u}_i + Cz_i) + \hat{u}_i. \end{aligned} \tag{34}$$

Equation (34) determines the optimal control u_i^* in terms of the present optimal state x_i^* and the solutions to the Riccati equation (28) and the tracking equation (29).

It is easy to see now that the Riccati matrices K_i must all be positive semi-definite. Since Q was specified to be positive semi-definite, K_N must be positive semi-definite (from equation (24)). Now examine equation (28). Since R is positive definite, the expression

$$K_N - K_N B(R + B'K_N B)^{-1} B'K_N$$

must be positive semi-definite. Hence K_{N-1} is the sum of two positive semi-definite terms, and is itself positive semi-definite. And similarly, $K_{N-2}, K_{N-3}, \ldots, K_1$ are all positive semi-definite. Now since all of the K_i are positive semi-definite, we can be sure that E is always non-singular and has an inverse.

2.4 *Summary of the Solution*

Once the system (the matrices A, B, and C), the exogenous variables z_i, the nominal state and control trajectories $\hat{x}_i$ and $\hat{u}_i$, and the cost functional (the matrices Q and R) have been specified, the optimal control is found as follows:

(i) Solve the Riccati equation (28) with boundary condition (24) backwards in time to get values for K_i, $i = 1 \ldots, N$. Store the resulting N $n \times n$ matrices.

(ii) Solve the tracking equation (29) with boundary condition (25) backwards in time to get values for g_i, $i = 1, \ldots, N$. Store the resulting N n-vectors.

(iii) Compute the optimal control u_0^* from equation (34) using $x_0^* = \xi$. Then compute x_1^* from equation (7), the system equation. Now x_1^* can be used in equation (34) to compute u_1^*, which can be used in equation (7) to compute x_2^*, etc. Continue this process until all of the u_i^*, $i = 0, 1, \ldots, N - 1$, and all of the x_i^*, $i = 1, \ldots, N$, have been computed.

(iv) The optimal cost J^* (if it is desired) can be computed from equation (3).

The solution to the problem may seem somewhat formidable, but it really is not. All of the above steps involve iterative solutions (and only N iterations) that require little more than multiplying and adding matrices (albeit large matrices—n might be on the order of several hundred for a large econometric model). Remember that the largest matrix that might be inverted is of dimension r, and r would normally be less than 10 and for many problems on the order of 3 or 4. On the whole, very little computer time should be required to go through steps (i) to (iv) above.

3. OPTIMAL STABILIZATION POLICIES USING A SMALL ECONOMETRIC MODEL

3.1 *The Model*

In order to demonstrate properly the application of the above results, a small, quarterly, linear model of the post-Korean War United States economy was required. The model had to include several basic macro-economic variables—consumption, investment, gross national product, an interest rate, a price level, wages, and unemployment—as well as some basic "policy" variables—the money supply, government spending, and some sort of crude tax mechanism.

To maintain linearity, a certain amount of compromise was required, and occasionally statistical fit had to be sacrificed for structural form. Nonetheless, the final form of the model correlates reasonably well with the historical time series, especially considering its size and simplicity.

The model consists of nine behavioral equations together with a tax relation and an income identity. Fiscal policy is provided for through exogenous government expenditures G and a surtax T_0, and monetary policy is realized in the money supply M (currency plus demand deposits).

GNP and its components, consumption C, total investment I, and government expenditures G, are all in real terms. Total investment is disaggregated and separate equations were estimated to explain fixed non-residential investment INR, residential investment IR, and investment in inventories IIN. GNP, then, can be written as

$$GNP = C + INR + IR + IIN + G. \tag{35}$$

Disposable income, YD, is defined by

$$YD = GNP - T, \tag{36}$$

where T is total tax flow net of transfers and is in turn given by

$$T = t \cdot GNP + T_0. \tag{37}$$

Here t is the average tax rate (estimated to be .15) and T_0 is the "surtax." If GNP is fixed at some nominal level $(GNP)_0$ then we could think of T_0 in terms of a tax surcharge of S per cent, where

$$S = \frac{T_0}{t \cdot (GNP)_0}. \tag{38}$$

T_0 is used rather than the surcharge rate S so as to have an exogenous tax variable that enters into the system in a completely linear way.

All of the behavioral equations were estimated in the period 1955-I to 1967-IV, using two-stage least squares in combination with a Hildreth-Lu autoregressive correction. The estimated equations are listed below. The first set of parentheses beneath each estimated coefficient contains the estimated standard errors of the coefficients. Also shown is the R^2, the estimated standard error of the residual, (SER), the F statistic, the Durbin-Watson statistic (DW), and the value of ρ used in the autoregressive transformation. A dynamic simulation was performed on the model as a whole using historical data for the policy variables. Along with each equation the RMS simulation error is given for the left-hand side endogenous variable, as well as the mean of that variable.

Real Consumption, C:

$$C = .4152YD - .2819YD_{-1} + 8.1743W_{-1} - 2.3676\Delta P + .7596C_{-1} \tag{39}$$
$$\quad (.0502) \quad (.0586) \quad (4.83) \quad (1.06) \quad (.0734)$$
$$+ 5.2988.$$
$$(2.44)$$

$R^2 = .9991$, $SER = 1.594$, $F = 10{,}280$,
$DW = 1.95$, $\rho = -.400$,
RMS Simulation Error = 5.24, Mean = 335.7 billion dollars.

Non-residential Investment, INR:[6]

$$\Delta INR = .1569\Delta YD + .0443\Delta YD_{-3} - 1.3563\Delta RL_{-5} + .3397\Delta INR_{-1} \tag{40}$$
$$\quad (.0221) \quad (.0225) \quad (0.863) \quad (.0910)$$
$$-.0042(INR_{-1} + INR_{-2}).$$
$$(.0017)$$

$R^2 = .718$, $SER = .744$, $F = 29.3$, $DW = 1.98$,
$\rho = -.335$,
RMS Simulation Error = 2.95,
Mean = 51.3 billion dollars.

Residential Investment, IR:

$$IR = .0127YD - .550(R_{-2} + R_{-3}) + .603IR_{-1} + 6.65. \tag{41}$$
$$\quad (.0023) \quad (.0886) \quad (.0470) \quad (1.140)$$

$R^2 = .992$, $SER = .582$, $F = 184.9$, $DW = 1.64$,
$\rho = .700$,
RMS Simulation Error = 1.73, Mean = 23.0 billion dollars.

Inventory Investment, IIN:[7]

$$IIN = .0113YD + .4647\Delta_2 YD - .6002\Delta_2 C + .4219IIN_{-1} - 2.4615. \tag{42}$$
$$\quad (.0054) \quad (.0782) \quad (.1617) \quad (.0916) \quad (2.219)$$

$R^2 = .740$, $SER = 2.228$, $F = 33.5$, $DW = 2.28$,
$\rho = .400$,
RMS Simulation Error = 4.20, Mean = 4.17 billion dollars.

Short-term Interest Rate, R:

$$R = .0071YD + .0233\Delta YD - .1648\Delta M + .4791\Delta P + .3745R_{-1} \tag{43}$$
$$\quad (.0019) \quad (.0074) \quad (.0719) \quad (.193) \quad (.1175)$$
$$- 1.4734.$$
$$(.537)$$

[6] The equation for non-residential investment was estimated in differenced form ($\Delta INR = INR - INR_{-1}$). The moving average in the level of INR represents the capital stock when undifferenced.

[7] $\Delta_2 YD = YD - YD_{-2} = \Delta YD + \Delta YD_{-1}$.

$R^2 = .883, \quad SER = .336, \quad F = 68.2, \quad DW = 1.90,$
$\rho = .400,$
RMS Simulation Error = 0.586,
Mean = 3.15 per cent per annum.

Long-term Interest Rate, RL:

(44) $$RL = \underset{(.0428)}{.0598R} + \underset{(.0027)}{.0055\Delta_2 YD} + \underset{(.0704)}{.8715RL_{-1}} + \underset{(.1931)}{.3126}.$$

$R^2 = .941, \quad SER = .1358, \quad F = 292, \quad DW = 1.97,$
$\rho = .250,$
RMS Simulation Error = 0.255,
Mean = 3.94 per cent per annum.

Price Level, P:[8]

(45) $$P = \underset{(.814)}{6.281W_{-1}} + \underset{(.0046)}{.0195(YD_{-1} - YDP_{-1})} - \underset{(.0093)}{.0328IIN_{-2}} - \underset{(.0050)}{.0156YD}$$
$$+ \underset{(.0788)}{.8040P_{-1}} + \underset{(2.220)}{14.552}.$$

$R^2 = .984, \quad SER = .195, \quad F = 710.3, \quad DW = 2.47,$
$\rho = .500,$
RMS Simulation Error = 0.704, Mean = 104.7.

Unemployment Rate, UR:

(46) $$UR = -\underset{(.00009)}{.00043\Delta YD} - \underset{(.00008)}{.00032\Delta YD_{-1}} + \underset{(.00093)}{.0024W_{-1}} - \underset{(.00007)}{.00014(YD_{-1}}$$
$$- YDP_{-1}) + \underset{(.0904)}{.8047UR_{-1}} + \underset{(.0041)}{.0065}.$$

$R^2 = .953, \quad SER = .0023, \quad F = 185.5, \quad DW = 1.11,$
$\rho = .050,$
RMS Simulation Error = 0.0081, Mean = .0537.

Money Wage Rate, W:

(47) $$W = \underset{(.0031)}{.0105P_{-3}} + \underset{(.0003)}{.0011YD_{-1}} + \underset{(.0004)}{.0012\Delta YD} - \underset{(.2441)}{.8277UR_{-4}} + \underset{(.1062)}{.6269W_{-1}}$$
$$- \underset{(.2045)}{.6850}.$$

$R^2 = .9992, \quad SER = .0117, \quad F = 11{,}940, \quad DW = 1.87,$

$\rho = .0200,$
RMS Simulation Error = 0.0233,
Mean = 2.23 dollars per hour.

[8] The exogenous variable YDP represents potential disposable income. It is based on a potential GNP trend line.

Combining the GNP and tax identities as an equation for disposable income completes the model:

$$YD = (1 - t)(C + INR + IR + IIN + G) - T_0. \tag{48}$$

3.2 *State-Variable Form of the Model*

Before applying the results of the optimal control problem, the model must be respecified in state-variable form, i.e., in the form of equation (1). This means defining new state variables to replace those variables that appear in the model with lags greater than one period, and adding their definitional equations to the model.

Before this can be done, a problem relating to the time subscript in the control variables must be resolved. Note that government spending G, the tax surcharge T_0, and the change in the money supply ΔM all appear unlagged in the structural form. In a sense these are not true control variables. The variable G, for example, is the *actual* (and observed) level of government spending, which may be different from the desired level. Decisions made at some prior time as to the desired levels of policy variables result, through some process in time, in their actual levels. As it was not our objective to describe or model these processes, we will assume that the actual values of G, T_0, and ΔM are the results of, and equal to, the desired levels that were specified in the previous quarter. This means lagging by one quarter the variables G, T_0, and ΔM as they appear in equations (43) and (48).

To complete the state-variable form, eighteen new state variables and their definitional equations must be added to the model. The following variables are defined:

$$C1 = C_{-1} \qquad P1 = P_{-1}$$
$$INR1 = INR_{-1} \qquad P2 = P1_{-1} = P_{-2},$$
$$IIN1 = IIN_{-1} \qquad UR1 = UR_{-1},$$
$$R1 = R_{-1}, \qquad UR2 = UR1_{-1} = UR_{-2},$$
$$R2 = R1_{-1} = R_{-2} \qquad UR3 = UR2_{-1} = UR_{-3},$$
$$RL1 = RL_{-1}, \qquad YD1 = YD_{-1},$$
$$RL2 = RL1_{-1} = RL_{-2} \qquad YD2 = YD1_{-1} = YD_{-2},$$
$$RL3 = RL2_{-1} = RL_{-3} \qquad YD3 = YD2_{-1} = YD_{-3}$$
$$RL4 = RL3_{-1} = RL_{-4},$$
$$RL5 = RL4_{-1} = RL_{-5}$$

The model is now in the form

$$x_t = A_0 x_t + A_1 x_{t-1} + B_1 u_{t-1} + C_1 z_{t-1}. \tag{49}$$

There are a total of twenty-eight state variables, x (ten endogenous variables and the eighteen variables defined above), three control variables, u, and two exogenous and uncontrollable variables, z (YDP and the constant 1, which corresponds to the constant term in each equation of the structural form). By recognizing that

$$(I + A) = (I - A_0)^{-1}A_1,$$
$$B = (I - A_0)^{-1}B_1,$$
$$C = (I - A_0)^{-1}C_1,$$

equation (49) can be expressed in the form of equation (1).

3.3 *Calculating Optimal Stabilization Policies*

As can be seen from equations (28), (29), and (34), obtaining computational optimal control solutions requires little more than basic matrix manipulations—additions, subtractions, multiplications, and small inversions. For this reason the optimal control solution program contains basically a combination of matrix subroutines. The program was run on time-sharing on the IBM 360/67 at the Massachusetts Institute of Technology's Computation Center.

Several experiments were performed in which the optimal control solution was used to formulate stabilization policies for the model described above. In each experiment a different cost functional was chosen to specify the relative penalties for deviations of each variable from its nominal (i.e., "ideal") path. The objective of the experiments was to gain insight into the trade-offs inherent in policy formulation in the context of the dynamic structure of the model.

All of the optimal policy experiments were run for twenty time periods, beginning with the first quarter of 1957 and ending with the first quarter of 1962. A simulation of the model was performed over this time period (i.e., beginning in 1957-I) using the historical values of the policy variables. These simulation results are used to view the results of the experiments in the perspective of the actual control and behavior of the economy.

In the experiments that follow, the nominal trajectories for the state and control variables were entered and then taken as fixed. The elements of Q and R were changed for each experiment.

The initial conditions of the state variables are shown below. The subscript "0" refers to the first quarter of 1957. All of these values are historical.

$$\begin{array}{llll}
C_0 = 286.7, & INR_0 = 47.7, & IR_0 = 20.7, & IIN_0 = 2.10, \\
R_0 = 3.1, & RL_0 = 3.27, & P_0 = 96.4, & UR_0 = .040, \\
W_0 = 1.82, & YD_0 = 386.6, & C1_0 = 284.8, & INR1_0 = 47.5, \\
IIN1_0 = 4.5, & R1_0 = 3.03, & R2_0 = 2.58, & RL1_0 = 3.3, \\
RL2_0 = 3.13, & RL3_0 = 2.99, & RL4_0 = 2.89, & RL5_0 = 2.89, \\
P1_0 = 95.4, & P2_0 = 94.6, & UR1_0 = .042, & UR2_0 = .042, \\
UR3_0 = .043. & YD1_0 = 385.8, & YD2_0 = 381.4, & YD3_0 = 382.1
\end{array}$$

The nominal trajectories for the first ten state variables (i.e., the unlagged endogenous variables) are shown in Table I. The index $i = 1$ refers to 1957-II, $i = 2$

TABLE I

Nominal Trajectories for the First Ten State Variables

i	$\hat{C}_i$	$I\hat{N}R_i$	$I\hat{R}_i$	$\hat{I}IN_i$	$\hat{R}_i$	$\hat{R}L_i$	$\hat{P}_i$	$\hat{U}R_i$	$\hat{W}_i$	$\hat{Y}D_i$
1	289.6	48.4	21.0	2.12	3.1	3.28	96.9	.02	1.85	390.5
2	292.5	49.1	21.3	2.14	3.1	3.29	97.4	.02	1.88	394.4
3	295.4	49.8	21.6	2.16	3.1	3.3	97.9	.02	1.91	398.3
4	298.3	50.5	21.9	2.18	3.1	3.3	98.4	.02	1.94	402.3
5	301.3	51.2	22.3	2.20	3.1	3.3	98.9	.02	1.97	406.3
6	304.3	52.0	22.6	2.22	3.1	3.3	99.4	.02	2.00	410.4
7	307.3	52.8	22.9	2.24	3.1	3.3	99.9	.02	2.03	414.5
8	310.4	53.6	23.2	2.26	3.1	3.3	100.4	.02	2.06	418.6
9	313.5	54.4	23.5	2.28	3.1	3.3	100.9	.02	2.09	422.8
10	316.6	55.2	23.9	2.30	3.1	3.3	101.4	.02	2.12	427.0
11	319.8	56.0	24.3	2.32	3.1	3.3	101.9	.02	2.15	431.3
12	323.0	56.8	24.7	2.34	3.1	3.3	102.4	.02	2.18	435.6
13	326.2	57.6	25.1	2.36	3.1	3.3	102.9	.02	2.21	440.0
14	329.5	58.4	25.5	2.38	3.1	3.3	103.4	.02	2.24	444.4
15	332.8	59.3	25.9	2.40	3.1	3.3	103.9	.02	2.27	448.8
16	336.1	60.2	26.3	2.42	3.1	3.3	104.4	.02	2.30	453.3
17	339.5	61.1	26.7	2.44	3.1	3.3	104.9	.02	2.34	457.8
18	342.9	62.0	27.1	2.46	3.1	3.3	105.4	.02	2.38	462.4
19	346.3	62.9	27.5	2.48	3.1	3.3	105.9	.02	2.42	467.0
20	349.7	63.8	27.9	2.50	3.1	3.3	106.4	.02	2.46	471.7

refers to 1957-III, etc. Nominal real consumption was taken to grow at a steady 4 per cent annual growth rate from its initial value. Nominal non-residential investment and nominal residential investment were both taken to grow at a 6 per cent annual rate from their initial values. Nominal inventory investment grows at a 4 per cent annual rate. The nominal short-term interest rate is taken to be constant at its initial value of 3.1 per cent, and the nominal long-term interest rate is taken to be a constant 3.3 per cent. The nominal price level has an annual growth rate of 2 per cent. The nominal unemployment rate is taken to be a constant .02. Finally, nominal real disposable income grows at an annual rate of 4 per cent.

The nominal trajectories for the policy variables are shown in Table II. The index $i = 0$ refers to 1957-I, $i = 1$ refers to 1957-II, etc. The nominal tax surcharge is taken always to be 0. Nominal government spending grows at an annual rate of 4 per cent from its actual value in the fourth quarter of 1956. The nominal quarterly change in the money supply is a constant 1.4. This represents a 4 per cent annual rate of growth for an actual money supply that averaged about 140 billion dollars during 1957 to 1962.

Values for the exogenous variables are shown in Table III. Potential disposable income is essentially a 3.5 per cent trend line. The second exogenous variable is the constant 1, which is paired to the constant terms of each equation in the structural form of the model.

3.4 *Results*

When interpreting the results it is important to keep in mind the fact that the cost functional accumulates penalties only over a finite time period—namely, twenty quarters. As a result optimal paths for some variables may behave strangely during the last few quarters of the planning period. Changes in the money supply,

TABLE II

NOMINAL TRAJECTORIES FOR THE CONTROL VARIABLES

i	$\hat{T}_{0i}$	$\hat{G}_i$	$\Delta\hat{M}_i$
0	0	97.0	1.4
1	0	98.0	1.4
2	0	99.0	1.4
3	0	100.0	1.4
4	0	101.0	1.4
5	0	102.0	1.4
6	0	103.0	1.4
7	0	104.0	1.4
8	0	105.0	1.4
9	0	106.0	1.4
10	0	107.1	1.4
11	0	108.2	1.4
12	0	109.3	1.4
13	0	110.4	1.4
14	0	111.5	1.4
15	0	112.6	1.4
16	0	113.7	1.4
17	0	114.8	1.4
18	0	115.9	1.4
19	0	117.1	1.4
20	0	117.3	1.4

TABLE III

TRAJECTORIES FOR THE EXOGENOUS VARIABLES

i	Constant	YDP_i
0	1	392.0
1	1	395.3
2	1	398.7
3	1	402.2
4	1	405.7
5	1	409.3
6	1	412.9
7	1	416.5
8	1	420.2
9	1	423.9
10	1	427.4
11	1	431.1
12	1	434.8
13	1	438.5
14	1	442.3
15	1	446.1
16	1	449.9
17	1	453.8
18	1	457.7
19	1	461.6
20	1	465.5

for example, affect the short-term interest rate immediately, but at least two quarters must elapse before there is any impact on residential investment and hence GNP. Therefore, if the cost functional does not penalize directly for interest-rate deviations from the nominal, the optimal quarterly change in the money supply will always be equal to the nominal value (\$1.4 billion) during the last two quarters of the planning period. This kind of behavior can occur in other variables as well.

The solution to this problem is to extend the planning period beyond the time horizon of actual interest. If, for example, one is interested in formulating an optimal stabilization policy for the next three years, he should extend the planning period to four or five years, obtain a numerical solution, and then ignore the results for the last year or two.

It should also be pointed out that in all experiments the cost functional contained zero weights for those endogenous variables that were considered "intermediate" in terms of policy goals. For example, the coefficients in the Q-matrix corresponding to the short- and long-term interest rates were always set to zero, although residential and non-residential investment may have non-zero coefficients. The reason for this is that we were not concerned with interest rates themselves, but only with their effects on other variables.

In the first experiment, consumption, non-residential investment, residential investment, the price level, and the unemployment rate were all weighted equally[9] in the cost functional, while zero weights were placed on inventory investment, the short- and long-term interest rates, the wage rate, and disposable income. The policy variables G (government spending) and ΔM (quarter change in the money supply) will both have weights equal to that of consumption, the price level, etc. Only the tax surcharge (T_0) was weighted differently; it was penalized twice as heavily for deviations from the nominal as the other variables.

We need only consider the first ten coefficients that appear along the diagonal of the Q matrix. The last eighteen elements of this diagonal correspond only to lagged endogenous variables (i.e., to the added state variables), and can thus be set to zero. All three elements of the diagonal of the R matrix, however, must be non-zero and positive, as required by the derivations of Section 2.

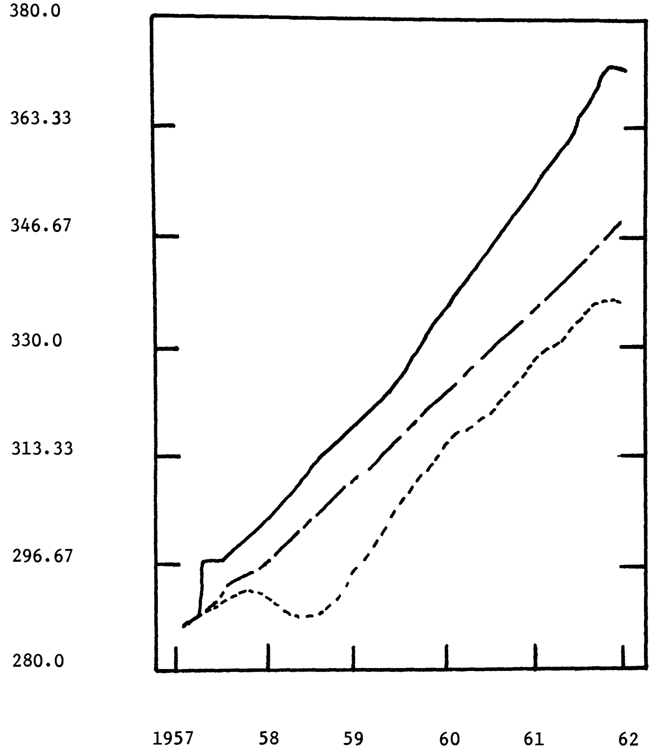
The cost functional for this first run, defined by the first ten diagonal elements of Q and the three diagonal elements of R, is given by:

	C	INR	IR	IIN	R	RL	P	UR	W	YD
Q:	1	6	15	0	0	0	6	4×10^6	0	0

	T_0	G	ΔM
R:	6	3	300.

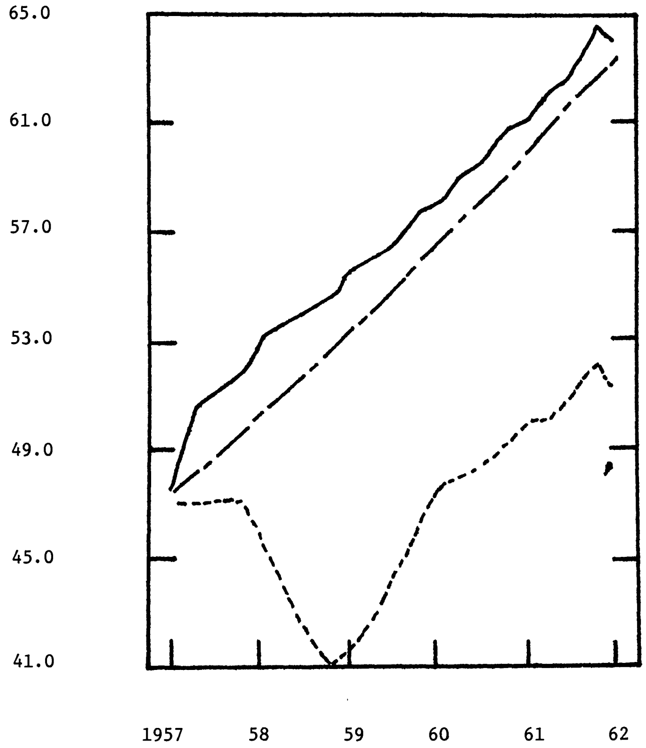
The optimal policy results for the first run are shown graphically in Figures 1–11.

[9] Note that we are penalizing for per cent deviations from the nominal in each case, and since the variables are scaled differently, their assigned weights in the cost function will have different scales.



Simulation results using historical values for policy variables
Nominal path
Optimal path

FIGURE 1.—Consumption, run 1.



Simulation results using historical values for policy variables
Nominal path
Optimal path

FIGURE 2.—Nonresidential investment, run 1.

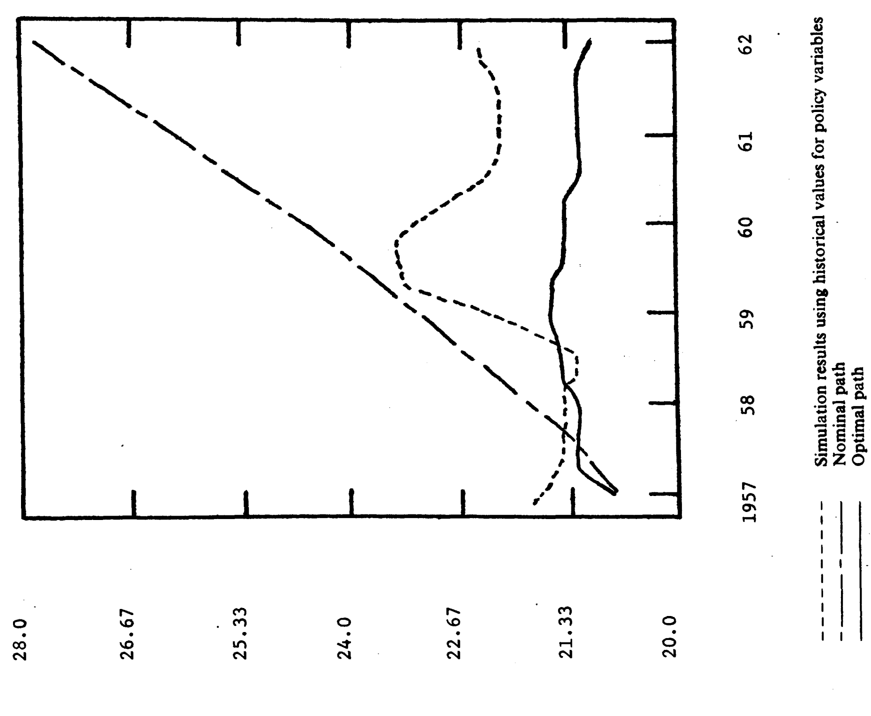


FIGURE 3.—Residential investment, run 1.

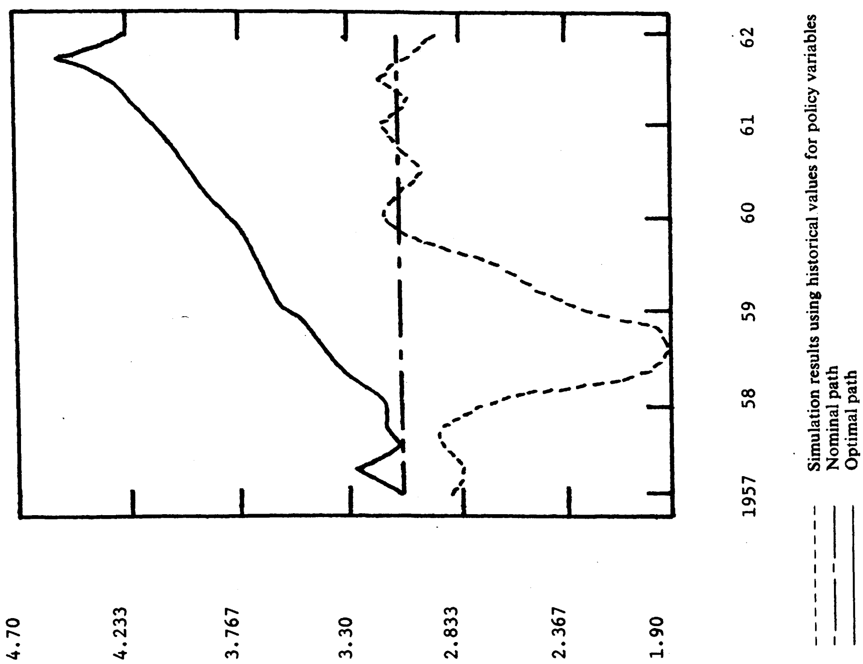


FIGURE 4.—Short-term interest rate, run 1.

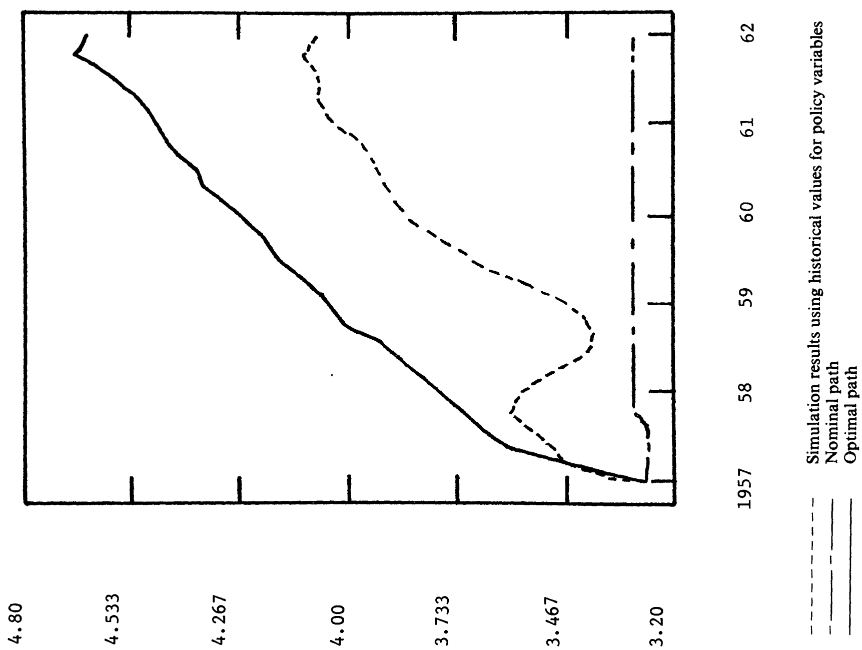


FIGURE 5.—Long-term interest rate, run 1.

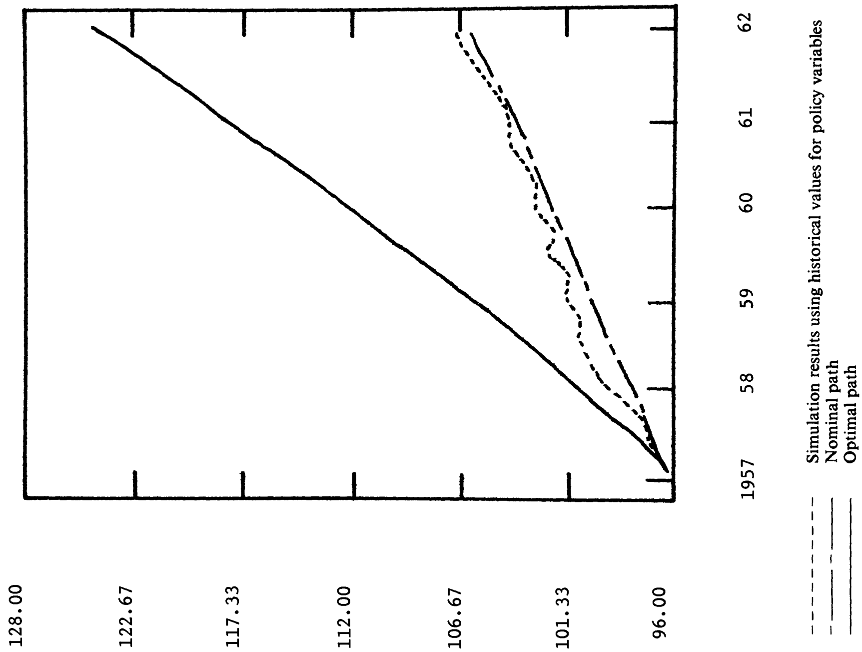


FIGURE 6.—Price level, run 1.

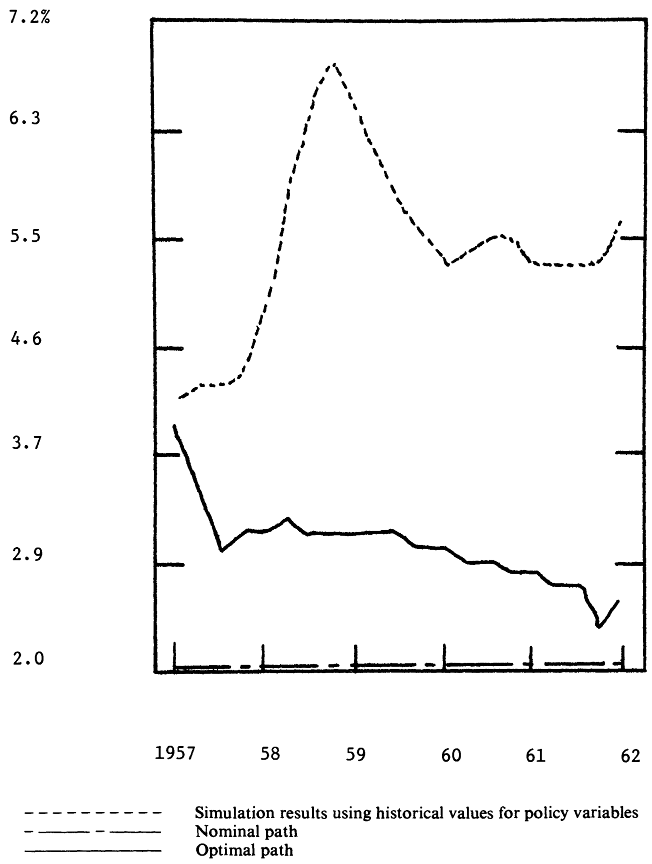


FIGURE 7.—Unemployment rate, run 1.

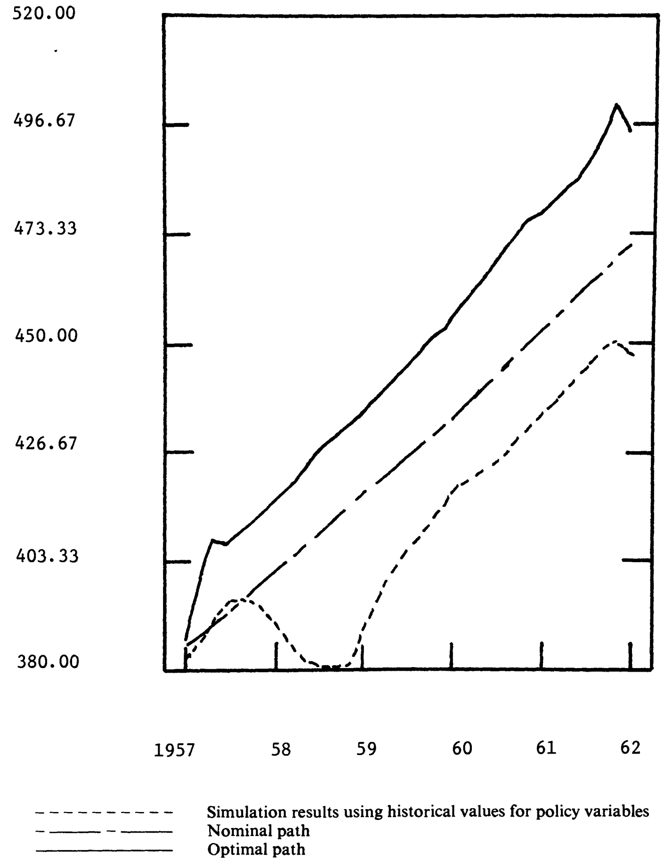


FIGURE 8.—Disposable income, run 1.

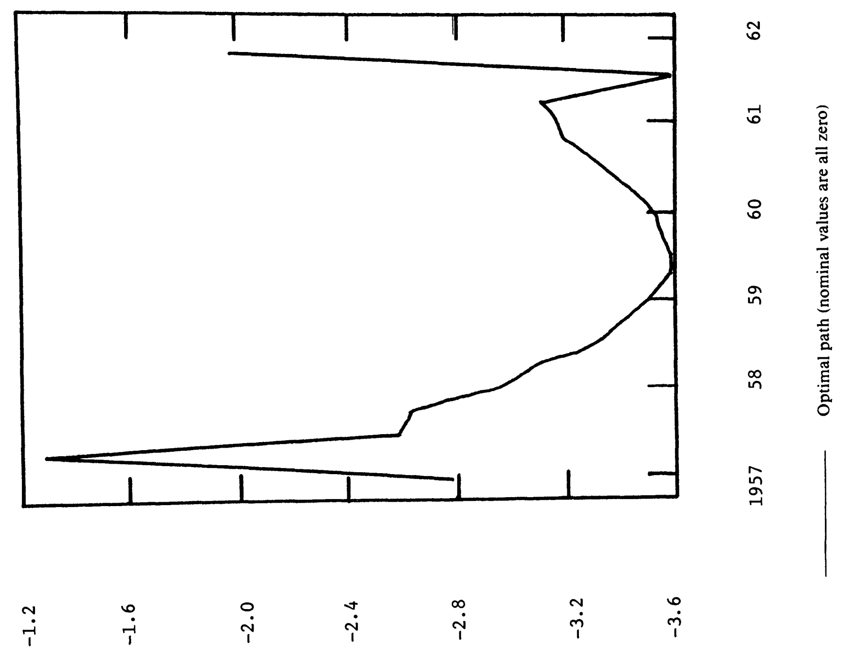


FIGURE 9.—Tax surcharge, run 1.

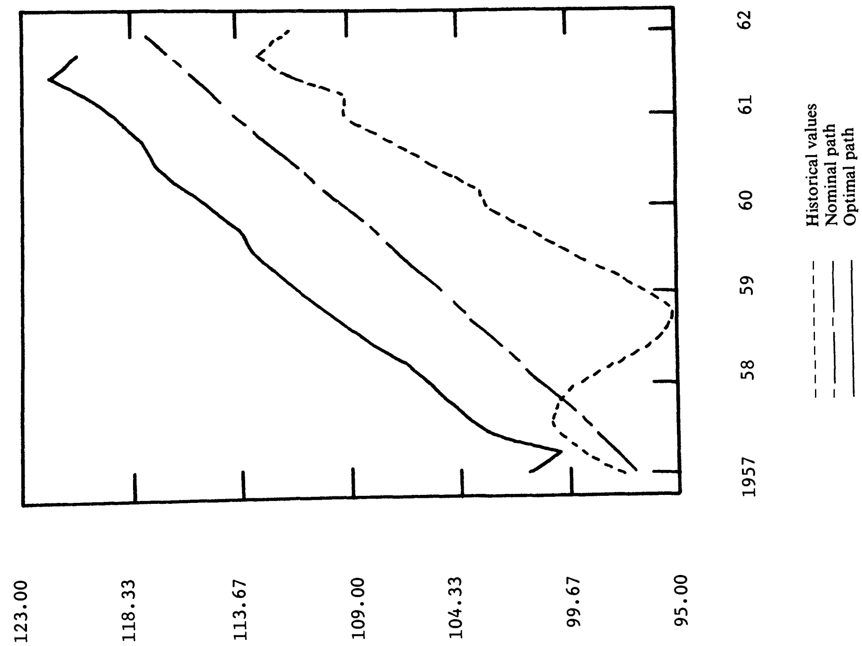


FIGURE 10.—Government spending, run 1.

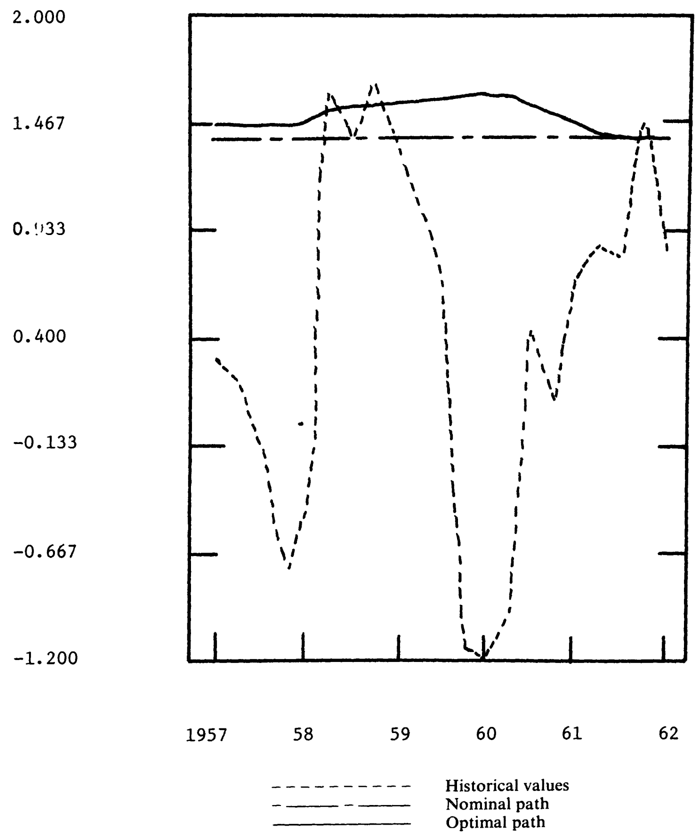


FIGURE 11.—Quarterly change in the money supply, run 1.

Observe that consumption, non-residential investment, and disposable income all run significantly higher than their nominal paths, while the unemployment rate drops to about 3 per cent (its nominal path runs at a constant 2 per cent). The optimal way to force the unemployment rate down and keep it there is to have a real *GNP* that rises faster than 4 per cent per year, high consumption and high investment (at least non-residential investment). The resulting effect on the price level is as expected. Prices go far above the nominal path with an inflation rate of about 5 per cent. Wages are also pushed up and rise at an annual rate of around 8 per cent to 12 per cent (the nominal growth rate for the wage rate is 6 per cent per year).

A rapidly rising *GNP* forces up the demand for money, and as a result interest rates rise, with the short-term rate reaching 4.5 per cent by the end of the planning period. Any increase in residential investment that could have come about as a result of the rising *GNP* is offset by the increase in interest rates, and so residential investment stagnates at a level of about 21 billion dollars. Non-residential investment, however, is much more sensitive to the GNP than it is to interest rates, and rises above its nominal path.

The optimal policy called for the predominant use of fiscal instruments. The tax surcharge ranged from 1.2 billion dollars (-1.6 per cent) to 3.6 billion dollars (-4.6 per cent) and then back to about 3 billion dollars (-3.6 per cent). Government spending was on the average around 6 billion dollars higher than its nominal path. Monetary policy was expansionary but only mildly so. The optimal quarterly change in the money supply was in the range of 1.5 billion to 1.6 billion dollars (corresponding to a 4.5 per cent to 5.5 per cent annual growth rate versus a 4 per cent nominal growth rate). The reason for this is that the response of the model to fiscal policy is much greater than its response to monetary policy, and since their cost weightings were the same, we would expect monetary policy to be used much less extensively in the optimal solution.

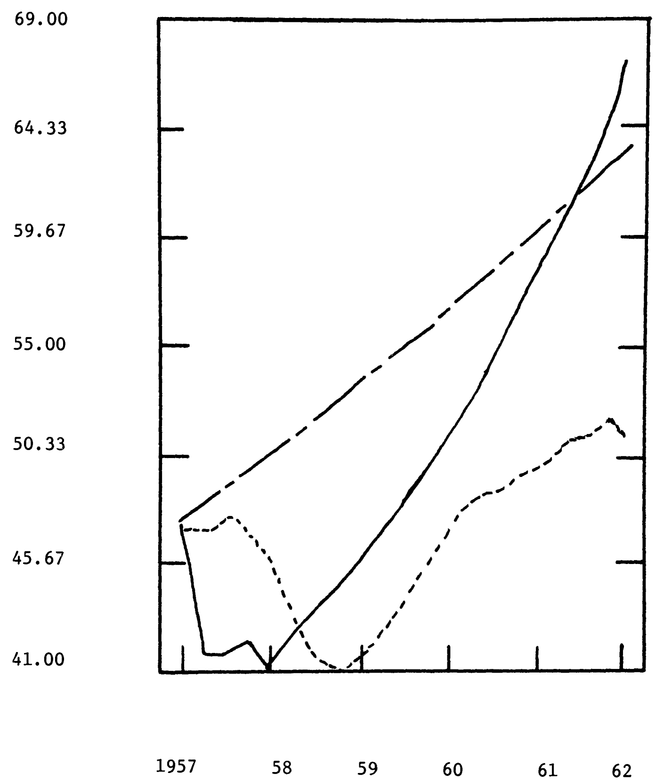
In the second experiment the cost functional is changed so as to de-emphasize the use of the tax surcharge, emphasize the use of monetary policy, and correct for the poor performance of residential investment and the price level. The cost coefficient for T_0 is increased by a factor of 10, the coefficient for ΔM is decreased by a factor of 20, the coefficient for IR is increased by a factor of 10, and the coefficient for P is increased by a factor of 20. The cost functional is now determined by:

	C	INR	IR	IIN	R	RL	P	UR	W	YD
Q:	1	6	150	0	0	0	120	4×10^6	0	0

	T_0	G	ΔM
R:	60	3	15

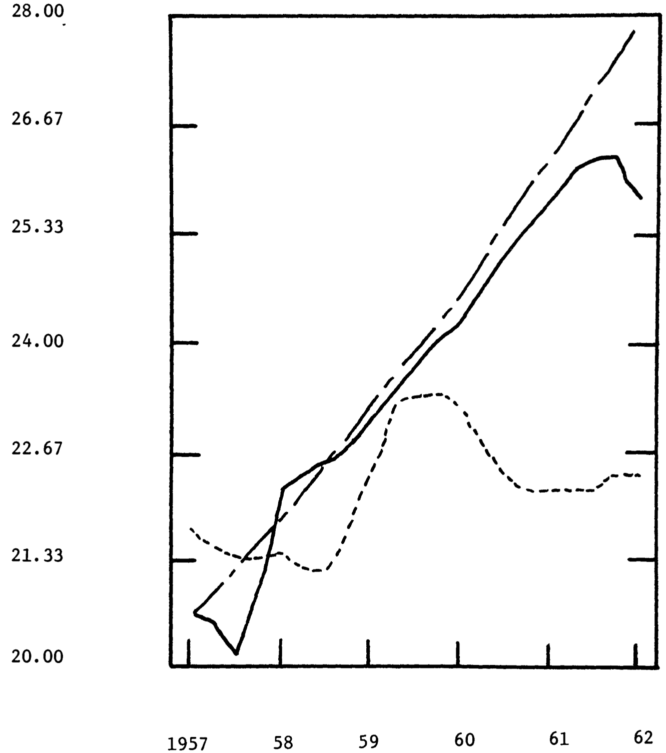
The optimal policy results for this second run are shown in Figures 12 to 19. The solution results in a level of residential investment that is indeed close to the nominal level. The optimal price level remained close to the nominal level for the first one or two years of the planning period, but the rate of inflation then picked up to about 3 per cent per year and during the second half of the planning period rose steadily to about 4.5 per cent per year.

The high level of residential investment was reached by lowering short-term interest rates to about 2 per cent for most of the planning period. This was done partly by increases in the money supply ranging from 1.5 billion dollars per quarter at the beginning of the planning period to over 8 billion dollars per quarter during the fourth year of the plan. But monetary policy operates with long lags, and to force interest rates down quickly the GNP was initially reduced so as to lower the demand for money. The optimal level of government spending was only 86 billion dollars at the beginning of the planning period (11 billion below the nominal level). The unemployment rate jumped to about 7 per cent during the first year, but then increasing disposable income resulting from both an increasing level of government spending and an expanding money supply forced it down to below 3 per cent by the end of the plan. The unemployment rate is much more



Simulation results using historical values for policy variables
Nominal path
Optimal path

FIGURE 12.—Nonresidential investment, run 2.



Simulation results using historical values for policy variables
Nominal path
Optimal path

FIGURE 13.—Residential investment, run 2.

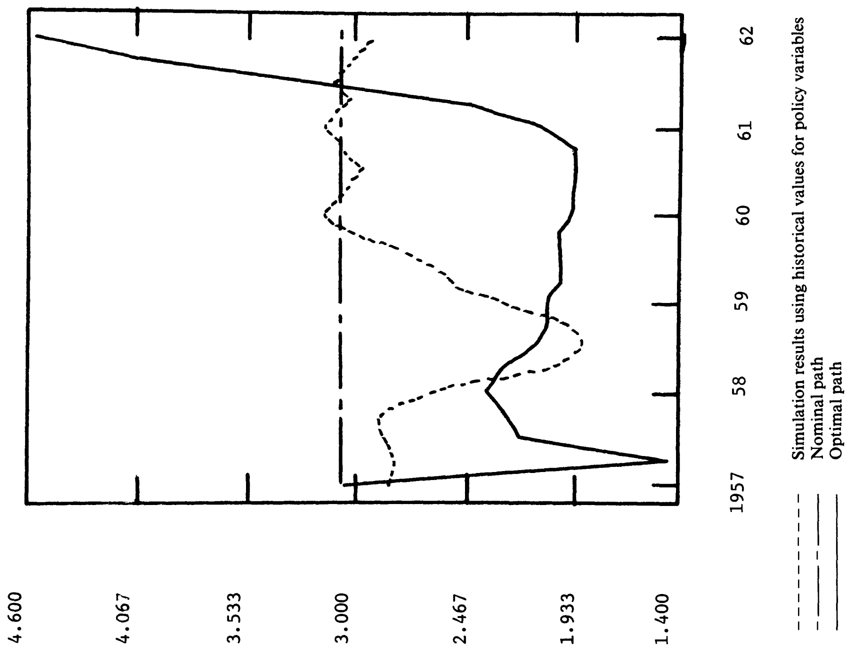


FIGURE 14.—Short-term interest rate, run 2.

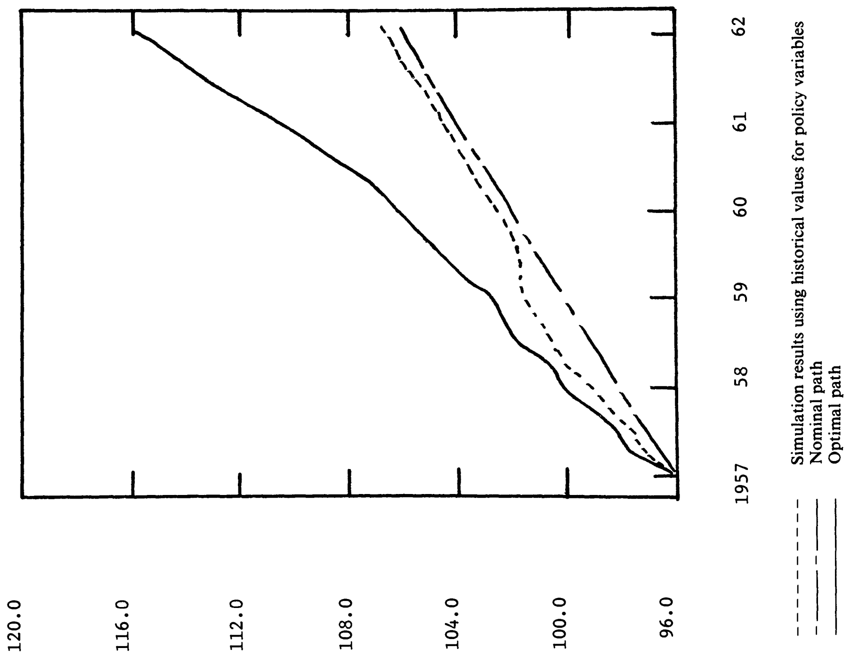
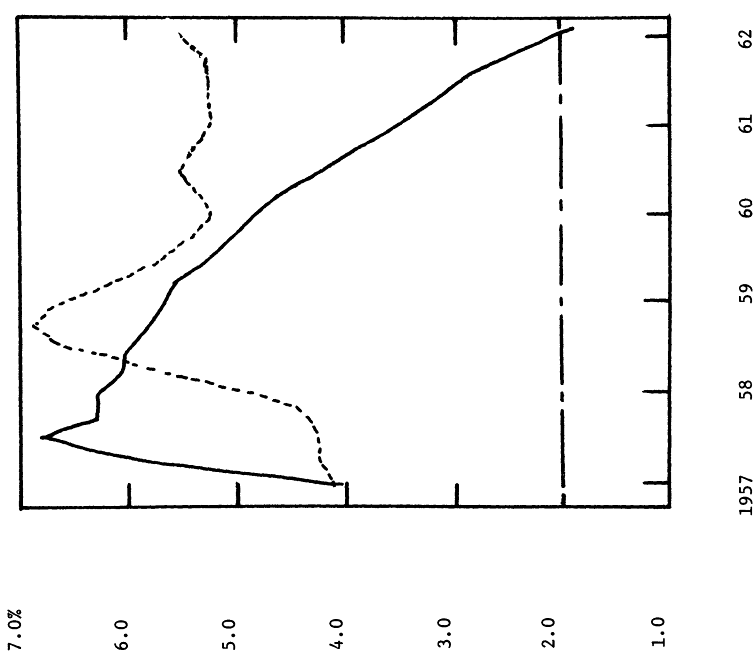
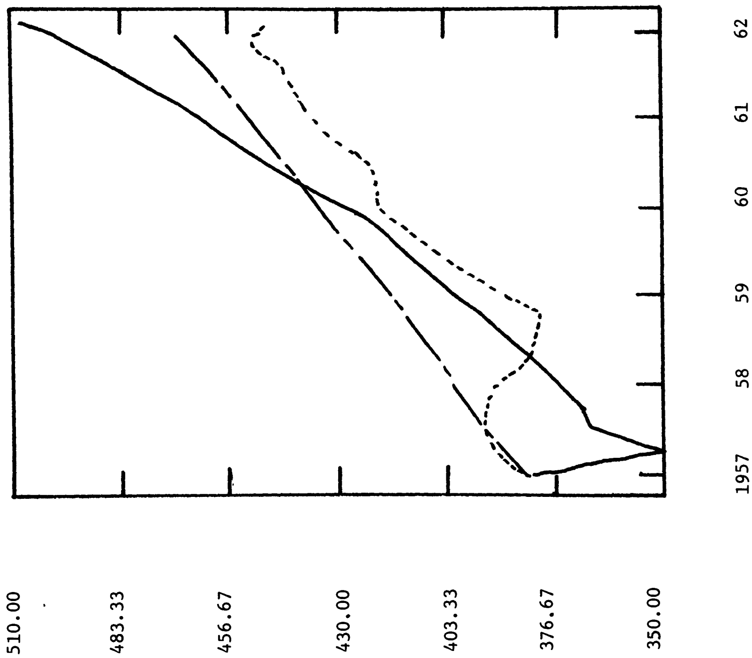


FIGURE 15.—Price level, run 2.



- - - - Simulation results using historical values for policy variables
— - — Nominal path
——— Optimal path

FIGURE 16.—Unemployment rate, run 2.



- - - - Simulation results using historical values for policy variables
— - — Nominal path
——— Optimal path

FIGURE 17.—Disposable income, run 2.

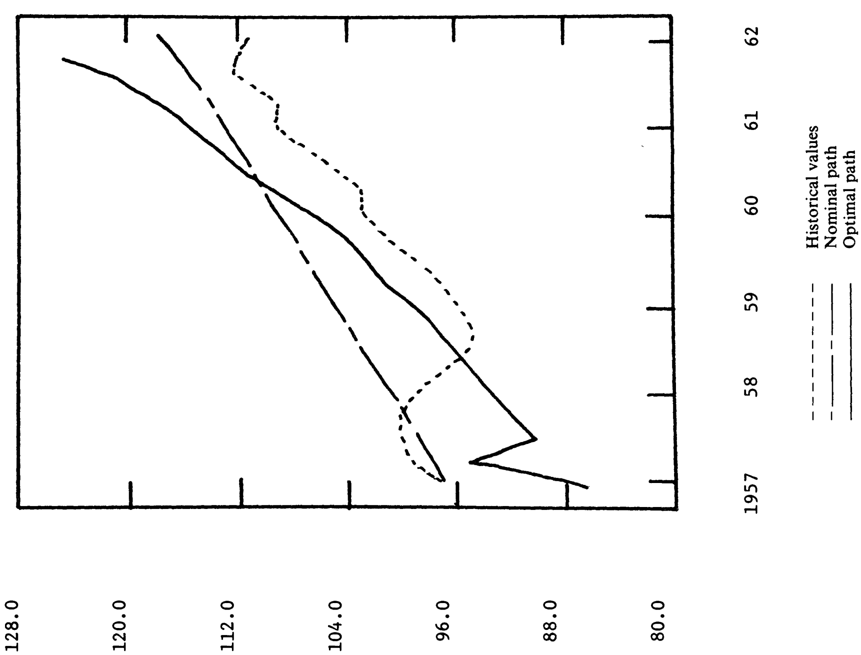


FIGURE 18.—Government spending, run 2.

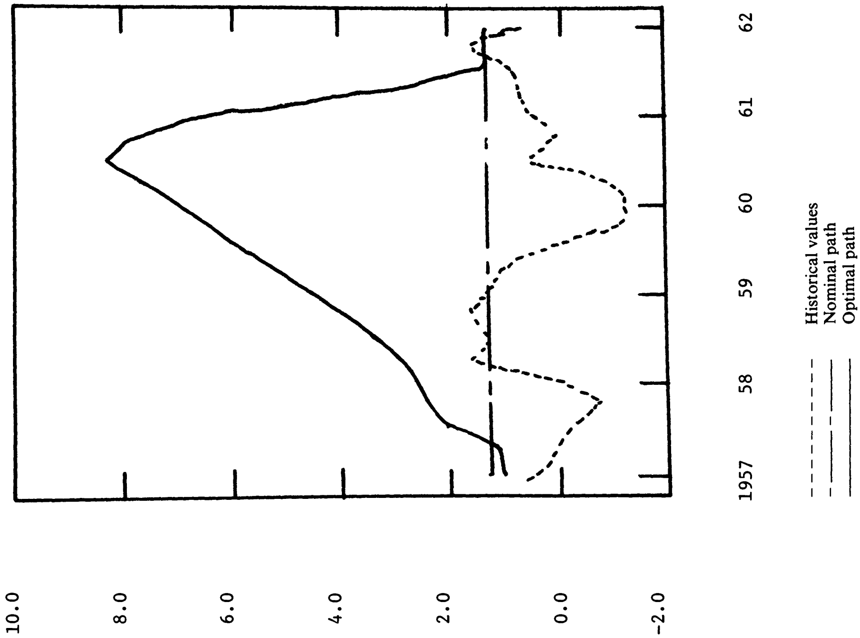


FIGURE 19.—Quarterly change in money supply, run 2.

sensitive to *changes* in disposable income than it is to the level of disposable income, and thus the low but growing level of disposable income was compatible with a low unemployment rate. For our model, this is the optimal way to combat inflation in the long run. By initially keeping disposable income low (and allowing unemployment to rise), disposable income can later be allowed to grow more rapidly, forcing unemployment back down while still maintaining a moderate rate of inflation.

Although residential investment did remain close to its nominal level during most of the plan, non-residential investment was low and only picked up in the second half of the plan. The long-term interest rate, while low, began to affect non-residential investment only after a five-quarter lag, and in the beginning of the planning period non-residential investment was responding mostly to the falling level of disposable income. Non-residential investment grows during the second half of the planning period because disposable income is rising during that time, and low levels of non-residential investment earlier in the plan resulted in a low capital stock which later stimulated non-residential investment to rise.

Finally, note that the quarterly change in the money supply dropped towards its nominal level during the last six quarters of the planning period as a result of the lags involved in monetary policy. Residential investment does not even begin to respond to changes in the short-term interest rate until after two quarters, and the mean lag of the response is about four quarters. There is a cost (although it is relatively low) to extreme monetary policy, and thus in the last year of the planning period it did not pay to have the money supply grow at an extremely rapid rate.

In the last experiment only one control variable, the money supply, was used to force one endogenous variable, disposable income, to remain close to its nominal path. The other two control variables were forced to remain on their nominal paths, and no cost was imposed on the other nine endogenous variables. The cost functional is given by:

	C	INR	IR	IIN	R	RL	P	UR	W	YD
Q:	0	0	0	0	0	0	0	0	0	1000

	T_0	G	ΔM
R:	10,000	3000	3.

The results are shown in Figures 20–25.

Disposable income did follow its nominal path almost exactly, and this was accomplished by an initial surge in the money supply (the money supply grows by 13 billion dollars—almost 10 per cent—in the first quarter of 1957 alone), and a later contraction in the money supply.

The initial surge in the money supply forced the short-term interest rate down and, by the third or fourth quarter of the planning period, residential investment up. During the second year of the plan residential investment was about 23.5 billion

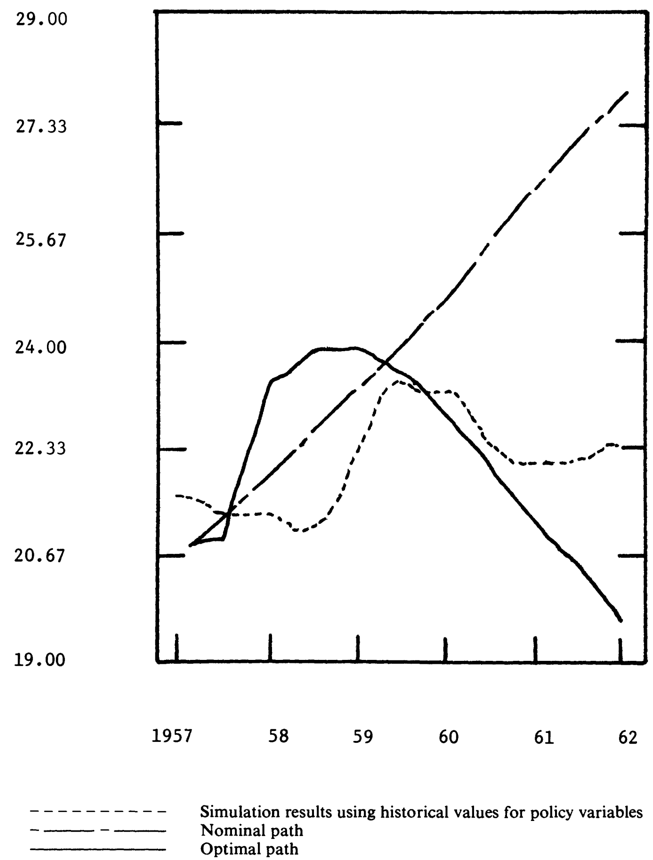


FIGURE 20.—Residential investment, run 3.

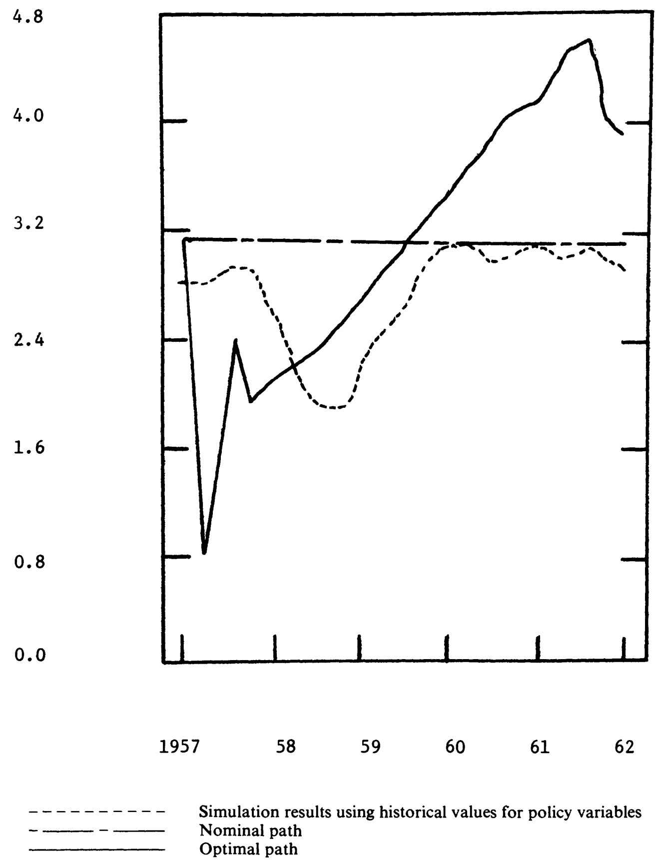


FIGURE 21.—Short-term interest rate, run 3.

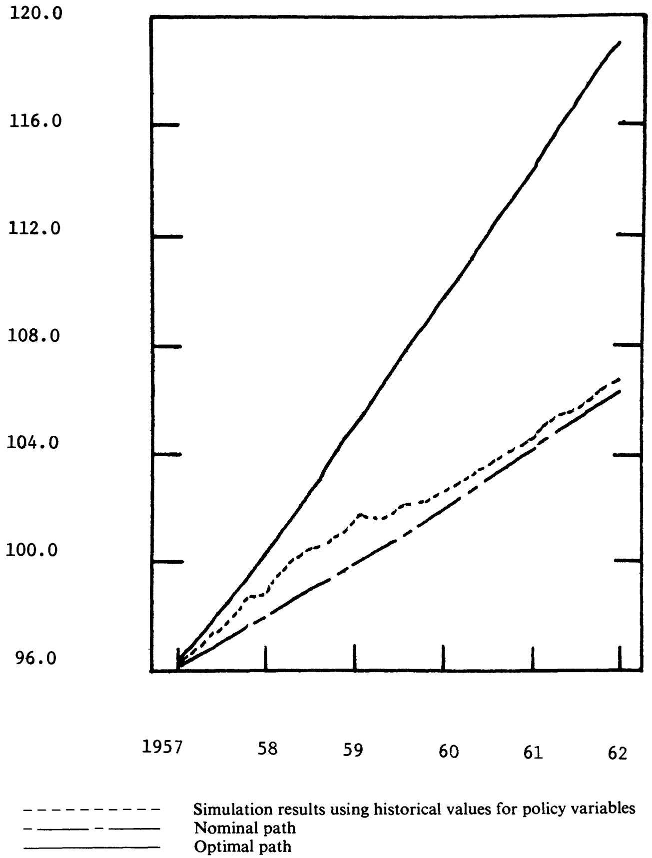


FIGURE 22.—Price level, run 3.

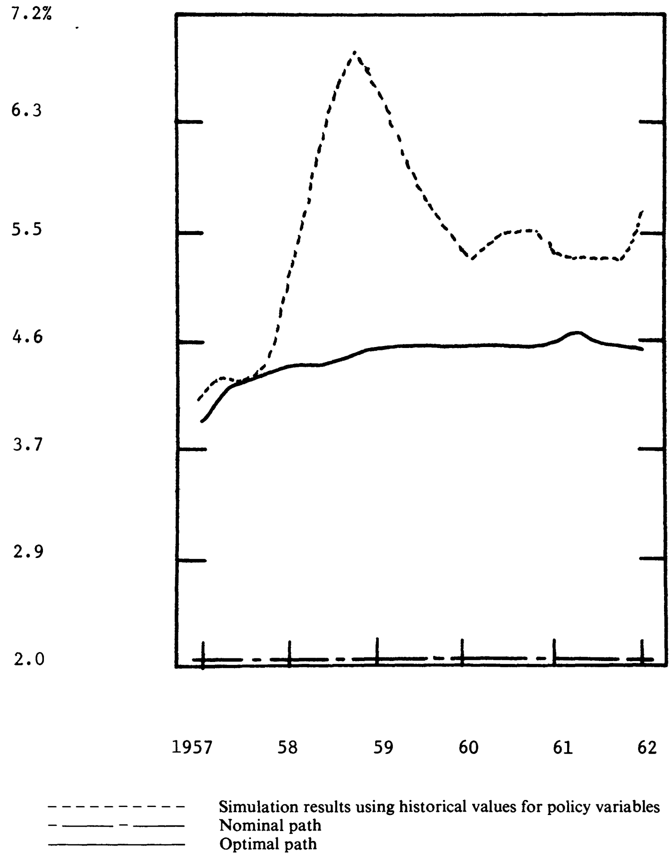


FIGURE 23.—Unemployment rate, run 3.

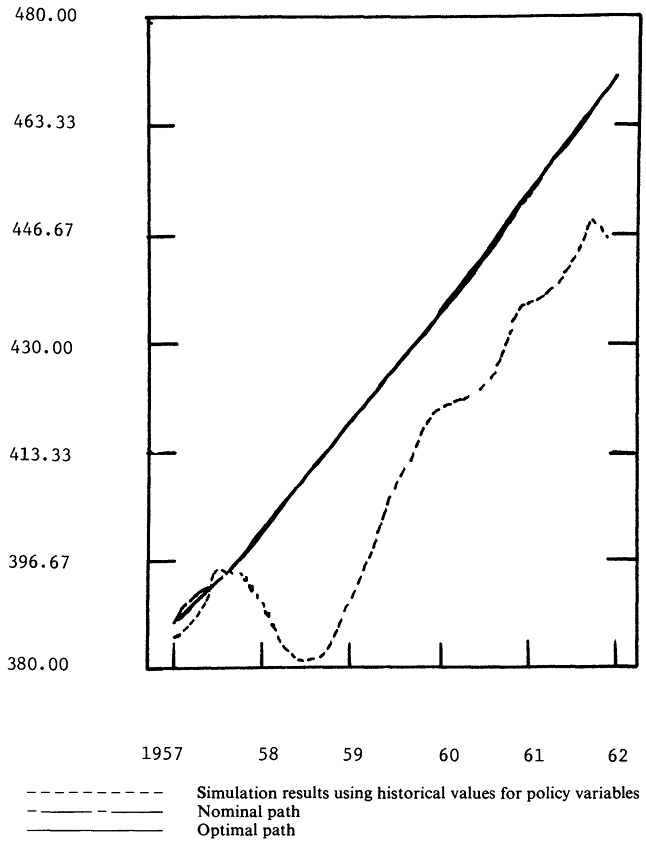


FIGURE 24.—Disposable income, run 3.

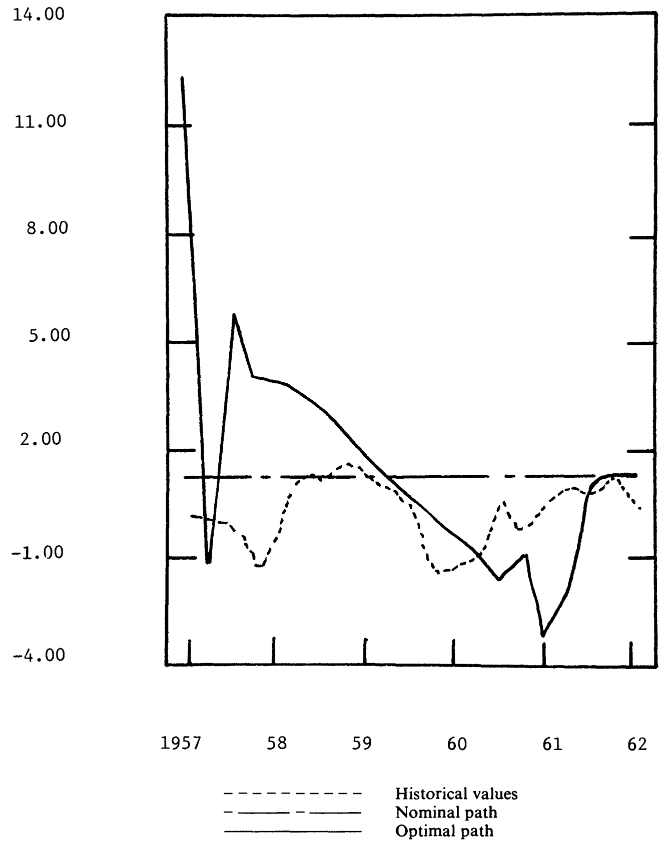


FIGURE 25.—Quarterly change in money supply, run 3.

dollars—about 1 billion dollars higher than the nominal level. This helped to keep disposable income up during the first two or three years.

Because of the time lags built into the structure of the monetary and investment sectors of the model, monetary policy was applied in short, strong bursts. The result was extreme monetary expansion at the beginning of the plan (to get the economy moving), and then monetary contraction at the end of the plan (to put the brakes on).

4. CONCLUDING REMARKS

Despite the size and simplicity of the econometric model the experimental results do provide at least some crude lessons for stabilization policy. We found, for example, that when lags are involved, as they are in monetary policy, the policy instrument must be applied in strong bursts as was the case in the third experiment. This, in fact, is a distinguishing factor between monetary and fiscal policy, and can often determine the proper mix and timing of the two. Increasing the performance of residential investment in the second experiment required an initial contraction in government spending combined with increasing monetary expansion. Thus monetary and fiscal policy may in some instances not be substitutes for each other, but must be used in combination.

The upward pressure on wages and prices is partly the result of the feedback built into the model's wage-price sector, and this is what makes it difficult to keep the inflation rate down. This, depending on one's view of the world, may be a weakness in the model. The result, in any case, is that the unemployment-inflation trade-off is dynamic in nature. Because the unemployment rate is dependent largely on changes in total demand (and output), a recession followed by rapid growth was found to be the optimal means of balancing unemployment and inflation. If, however, one is willing to live with a 5 per cent rate of inflation, then low unemployment and a healthy growth in output can be had without resorting to extreme fiscal or monetary policy.

There are other inadequacies in the model. The equation for inventory investment leaves something to be desired, and this variable performs poorly when the model is simulated. Also, the model was estimated by combining two-stage least squares with a Hildreth-Lu procedure, resulting in inconsistent estimates, since no other method[10] for dealing with the serial correlation problem was computationally available.

Certainly the value of optimal control as an aid in determining policy depends to a large extent on the richness of the model that one has to work with. We have at least tried to show that this approach can provide a viable tool both for policy planning and for analyzing and better understanding a model's dynamic behavior.

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[10] E.g., see R. Fair, "The Estimation of Simultaneous Equation Models with Lagged Endogenous Variables and First Order Serially Correlated Errors," *Econometrica*, 38 (1970), 507–516.

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