Advanced Econometrics - Lecture 5

Univariate Time Series Models

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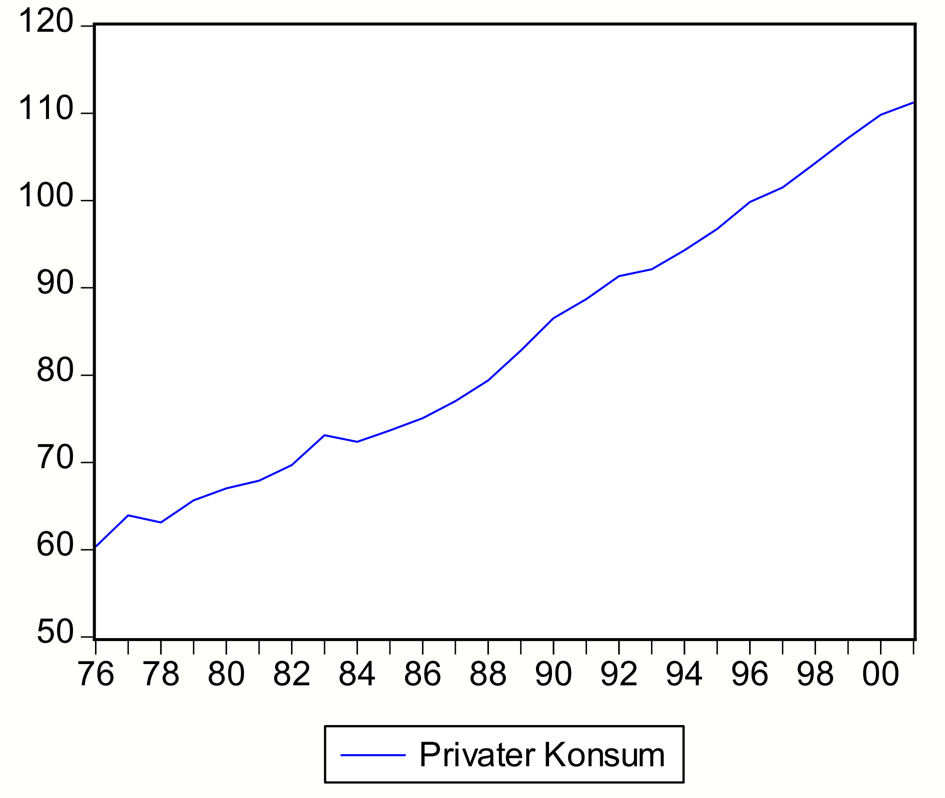
- Time Series
- Stochastic Processes
- Stationary Processes
- The ARMA Process
- Deterministic and Stochastic Trends
- Models with Trend
- Unit Root Tests
- Estimation of ARMA Models
- ARCH and GARCH Models

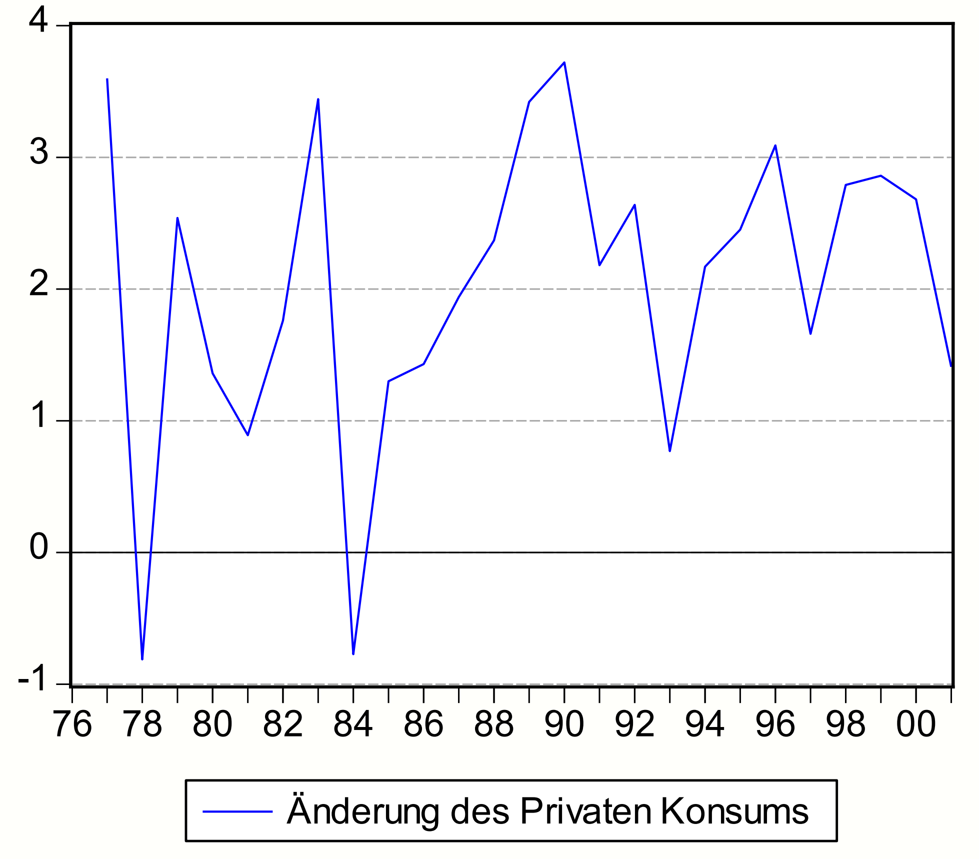
Private Consumption

Private consumption in Austria (in Bn EUR), prices at basis 1995



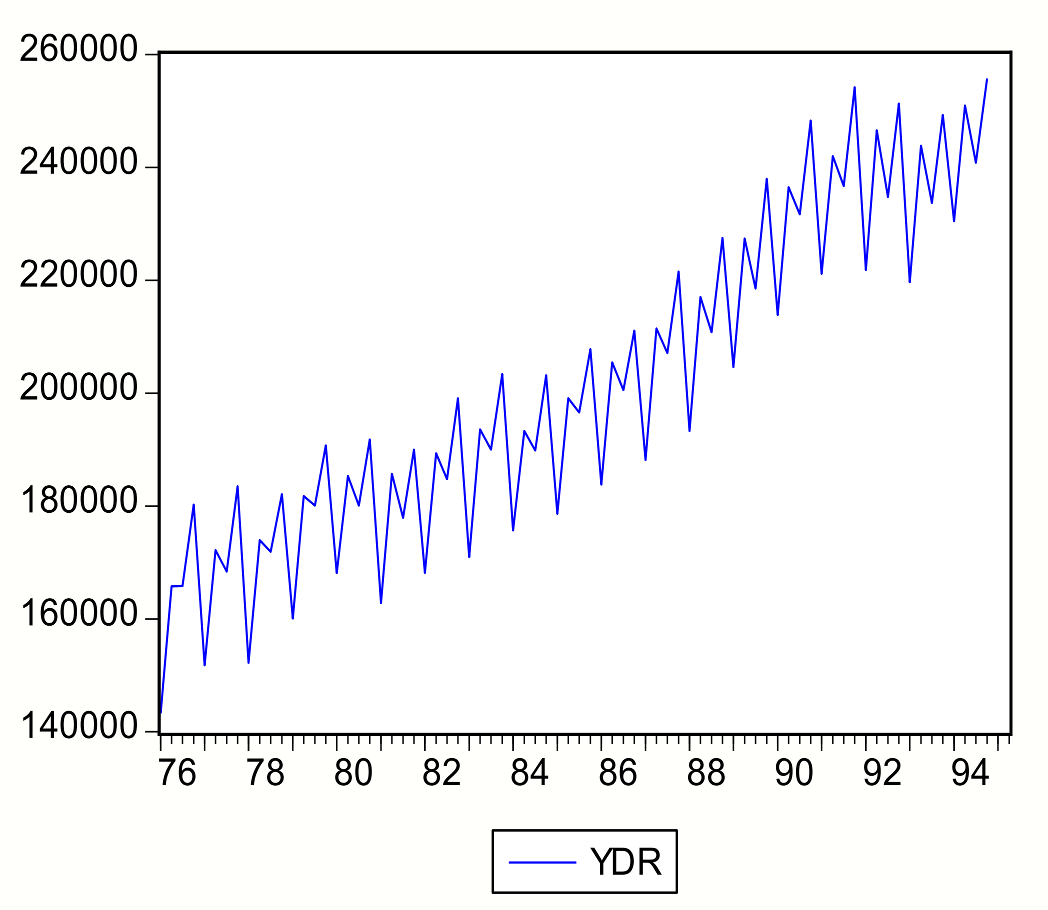
Private Consumption, cont'd

Growth of private consumption in Austria (in Bn EUR), prices at basis 1995



Disposable Income

Disposable income
in Austria (in Mio EUR)



Time Series

Is a time-ordered sequence of observations of a random variable

Examples:

- Annual values of private consumption
- Changes in expenditure on private consumption
- Quarterly values of personal disposable income
- Monthly values of imports

Notation:

- Random variable Y
- Sequence of observations $Y_1, Y_2, \ldots , Y_T$
- Deviations from the mean: $y_t = Y_t - E\{Y_t\} = Y_t - \mu$

Components of a Time Series

Components or characteristics of a time series are

- Trend
- Seasonality
- Irregular fluctuations

Time series model: represents the characteristics as well as possible

Purpose of modeling

- Describing the time series
- Forecasting the future

Example: $Y_t = \beta t + \Sigma_i \gamma_i D_{it} + \varepsilon_t$

with $D_{it} = 1$ if t corresponds to i-th quarter, $D_{it} = 0$ otherwise

for describing the development of the disposable income

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Stochastic Process

Time series: realization of a stochastic process

Stochastic process is a sequence of random variables Y_t, e.g.,

$\{Y_t, t = 1, ..., n\}$

$\{Y_t, t = -\infty, ..., \infty\}$

Joint distribution of the $Y_1, ... , Y_n$:

$p(y_1, ..., y_n)$

Of special interest

- Evolution of the expectation $\mu_t = E\{Y_t\}$ over time
- Dependence structure over time

Example: Extrapolation of a time series as a tool for forecasting

AR(1)-Process

States the dependence structure between consecutive observations as

$$Y_t = \delta + \theta Y_{t-1} + \varepsilon_t, \quad |\theta| < 1$$

with ε_t: white noise, i.e., serially uncorrelated, mean zero, $V\{\varepsilon_t\} = \sigma^2$

"Autoregressive process of order 1"

From $Y_t = \delta + \theta Y_{t-1} + \varepsilon_t = \delta + \theta\delta + \theta^2\delta + \ldots + \varepsilon_t + \theta\varepsilon_{t-1} + \theta^2\varepsilon_{t-2} + \ldots$ follows

$$E\{Y_t\} = \mu = \delta(1-\theta)^{-1}$$

In deviations from μ, $y_t = Y_t - \mu$, the model is

$$y_t = \theta y_{t-1} + \varepsilon_t$$

Autocovariances $\gamma_k = \text{Cov}\{Y_t, Y_{t-k}\}$

- k=0: $\gamma_0 = V\{Y_t\} = \theta^2 V\{Y_{t-1}\} + V\{\varepsilon_t\} = \ldots = \Sigma_i \theta^{2i} \sigma^2 = \sigma^2(1-\theta^2)^{-1}$
- k=1: $\gamma_1 = \text{Cov}\{Y_t, Y_{t-1}\} = E\{(\theta y_{t-1}+\varepsilon_t) y_{t-1}\} = \theta V\{y_{t-1}\} = \theta\sigma^2(1-\theta^2)^{-1}$
- In general: $\gamma_k = \text{Cov}\{Y_t, Y_{t-k}\} = \theta^k \sigma^2 (1-\theta^2)^{-1}$
- Depends of k, not of t!

MA(1)-Process

States the dependence structure between consecutive observations as

$$Y_t = \mu + \varepsilon_t + \alpha\varepsilon_{t-1}$$

with ε_t: white noise, $V\{\varepsilon_t\} = \sigma^2$

Moving average process of order 1

$E\{Y_t\} = \mu$

Autocovariances $\gamma_k = Cov\{Y_t, Y_{t-k}\}$

- k=0: $\gamma_0 = V\{Y_t\} = \sigma^2(1+\alpha^2)$
- k=1: $\gamma_1 = Cov\{Y_t, Y_{t-1}\} = \alpha\sigma^2$
- $\gamma_k = 0$ for $k = 2, 3, \ldots$
- Depends of *k*, not of *t*!

AR-Representation of MA-Process

The AR(1) can be represented as MA-process of infinite order

$$y_t = \theta y_{t-1} + \varepsilon_t = \Sigma^{\infty}_{i=0}\, \theta^i\, \varepsilon_{t-i}$$

given that $|\theta| < 1$

Similarly, the AR representation of the MA(1) process

$$y_t = \alpha y_{t-1} - \alpha^2 y_{t-2} + \ldots\ \varepsilon_t = \Sigma^{\infty}_{i=0}\, (-1)^i\, \alpha^{i+1} y_{t-i-1} + \varepsilon_t$$

given that $|\alpha| < 1$

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Stationary Processes

Refers to the joint distribution of Y_t's, in particular to second moments

A process is called strictly stationary if its stochastic properties are unaffected by a change of the time origin

- The joint probability distribution at any set of times is not affected by an arbitrary shift along the time axis

Covariance function:

$\gamma_{t,k} = \text{Cov}\{Y_t, Y_{t+k}\}$, $k = 0, 1, \ldots$

Properties:

$\gamma_{t,k} = \gamma_{t,-k}$

Weak stationary process:

$E\{Y_t\} = \mu$ for all t

$\text{Cov}\{Y_t, Y_{t+k}\} = \gamma_k$, $k = 0, 1, \ldots$ for all t and all k

Also called covariance stationary process

AC and PAC Function

Autocorrelation function (AC function, ACF) is independent of the scale of Y

for a stationary process:

$$\rho_k = \mathrm{Corr}\{Y_t, Y_{t-k}\} = \gamma_k/\gamma_0,\ k = 0,\ 1,\ldots$$

Properties:

- $|\rho_k| \leq 1$
- $\rho_k = \rho_{-k}$
- $\rho_0 = 1$

Correlogram: graphical presentation of the AC function

Partial autocorrelation function (PAC function, PACF):

$$\theta_{kk} = \mathrm{Corr}\{Y_t,\ Y_{t-k} | Y_{t-1},\ldots,Y_{t-k+1}\},\ k = 0,\ 1,\ \ldots$$

θ_{kk} is obtained from $Y_t = \theta_{k0} + \theta_{k1} Y_{t-1} + \ldots + \theta_{kk} Y_{t-k}$

Partial correlogram: graphical representation of the PAC function

AC and PAC Function, cont'd

Examples for the AC and PAC functions

- White noise

  $\rho_0 = \theta_{00} = 1$

  $\rho_k = \theta_{kk} = 0$, if $k \neq 0$

- AR(1) process, $Y_t = \delta + \theta Y_{t-1} + \varepsilon_t$

  $\rho_k = \theta^k$, $k = 0, \quad 1, \ldots$

  $\theta_{00} = 1$, $\theta_{11} = \theta$, $\theta_{kk} = 0$ for $k > 1$

- MA(1) process, $Y_t = \mu + \varepsilon_t + \alpha\varepsilon_{t-1}$

  $\rho_0 = 1$, $\rho_1 = -\alpha/(1 + \alpha^2)$, $\rho_k = 0$ for $k > 1$

  PAC function: damped exponential if $\theta > 0$, otherwise alternating and damped exponential

AC and PAC Function: Estimates

Estimating of AC and PAC function

Estimator for ρ_k:

$$r_k = \frac{\sum_t y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_t y_t - \bar{y})^2}$$

Estimator for θ_{kk}: coefficient of Y_{t-k} in the regression of Y_t on Y_{t-1}, …, Y_{t-k}

AR(1) Processes, Verbeek, p.274

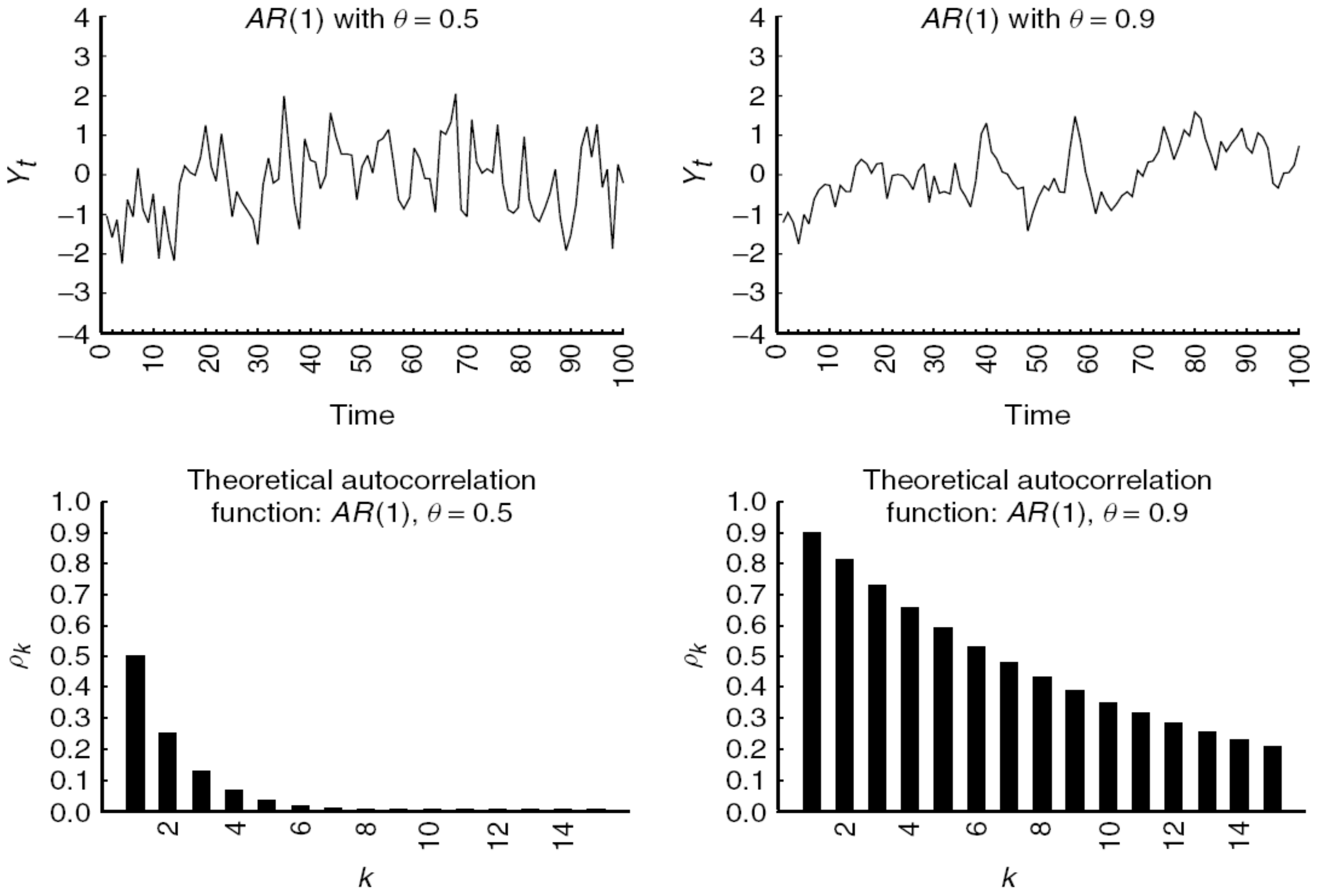

Figure 8.1 First-order autoregressive processes: data series and autocorrelation functions

MA(1) Processes, Verbeek, p.275

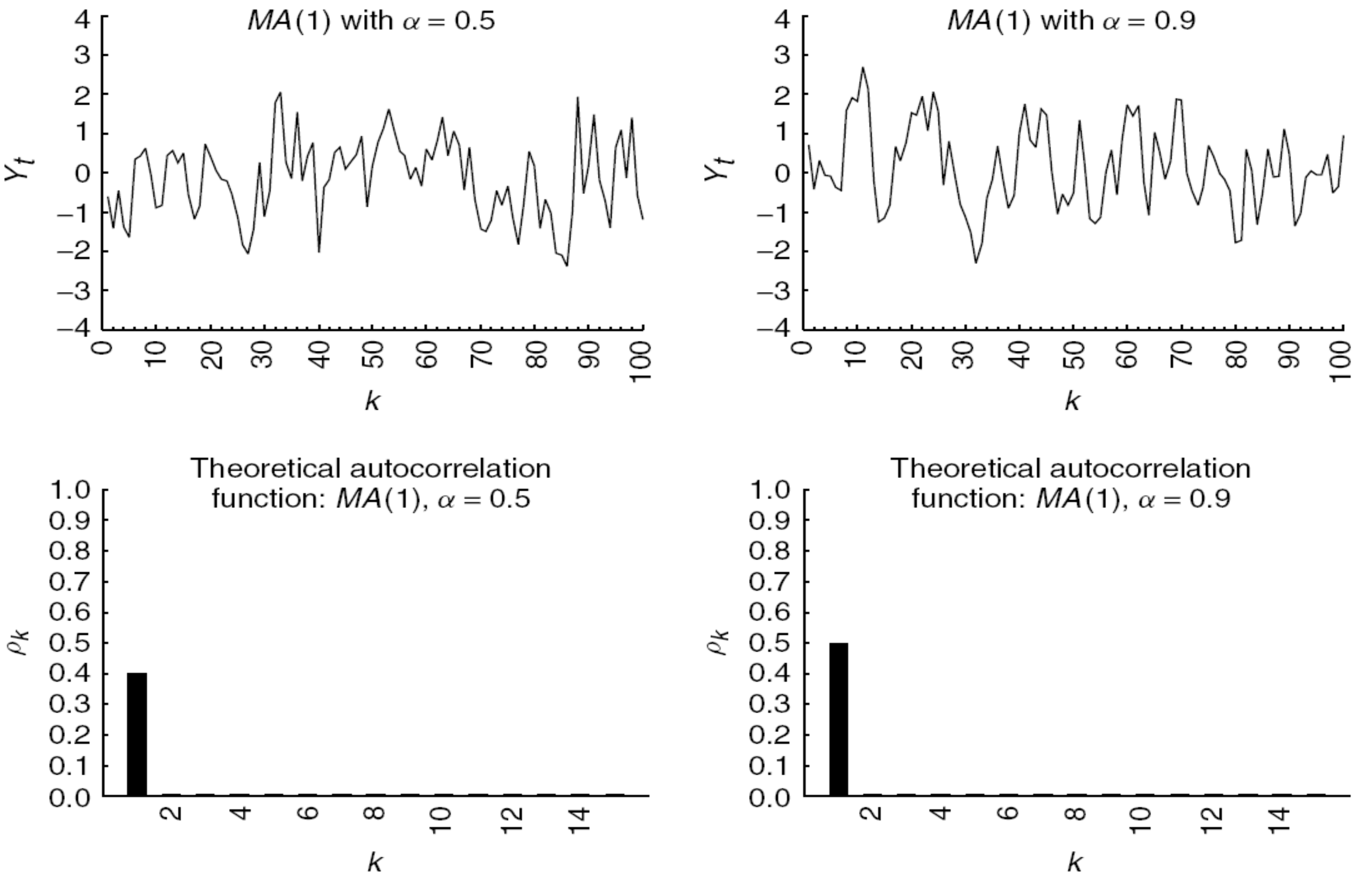

Figure 8.2 First-order moving average processes: data series and autocorrelation functions

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The ARMA(p,q) Process

Generalization of the AR and MA processes: ARMA(p,q) process

$$y_t = \theta_1 y_{t-1} + \ldots + \theta_p y_{t-p} + \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \ldots + \alpha_q \varepsilon_{t-q}$$

with ε_t: white noise

Lag (or shift) operator L ($Ly_t = y_{t-1}$, $L^0 y_t = Iy_t = y_t$, $L^p y_t = y_{t-p}$)

ARMA(p,q) process on operator notation

$$\theta(L) y_t = \alpha(L) \varepsilon_t$$

with operator polynomials $\theta(L)$ and $\alpha(L)$

$$\theta(L) = I - \theta_1 L - \ldots - \theta_p L^p, \ \alpha(L) = I + \alpha_1 L + \ldots + \alpha_q L^q$$

Lag Operator

Lag (or shift) operator L

- $Ly_t = y_{t-1}$, $L^0 y_t = Iy_t = y_t$, $L^p y_t = y_{t-p}$
- Algebra of polynomials in L like algebra of variables

Examples:

- $(I - \phi_1 L)(I - \phi_2 L) = I – (\phi_1 + \phi_2)L + \phi_1\phi_2 L^2$
- $(I - \theta L)^{-1} = \Sigma^{\infty}_{i=0}\theta^i L^i$
- MA(∞) representation of the AR(1) process

  $y_t = (I - \theta L)^{-1}\varepsilon_t$

  the infinite sum needs (e.g., finite variance) $|\theta| < 1$
- MA(∞) representation of the ARMA(p,q) process

  $y_t = [\theta (L)]^{-1}\alpha(L)\varepsilon_t$

  similarly the AR(∞) representations; invertibility condition: restrictions on parameters

Invertibility of Lag Polynomials

Invertibility condition for $I - \theta L$: $|\theta| < 1$

Invertibility condition for $I - \theta_1 L - \theta_2 L^2$:

- $\theta(L) = I - \theta_1 L - \theta_2 L^2 = (I - \phi_1 L)(I - \phi_2 L)$ with $\phi_1 + \phi_2 = \theta_1$ and $-\phi_1\phi_2 = \theta_2$
- Invertibility conditions: both $(I - \phi_1 L)$ and $(I - \phi_2 L)$ invertible; $|\phi_1| < 1$, $|\phi_2| < 1$
- Characteristic equation: $\theta(z) = (1- \phi_1 z)\ (1- \phi_2 z) = 0$
- Characteristic roots: solutions z_1, z_2 from $(1- \phi_1 z)\ (1- \phi_2 z) = 0$
- Invertibility conditions: $|z_1| > 1$, $|z_2| > 1$

Can be generalized to lag polynomials of higher order

Unit root: a characteristic root of value 1

- Polynomial $\theta(z)$ evaluated at $z = 1$: $\theta(1) = 0$, if $\Sigma_i \theta_i = 1$
- Simple check, no need to solve characteristic equation

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Types of Trend

Trend: The expected value of a process Y_t increases or decreases with time

- Deterministic trend: a function $f(t)$ of the time, describing the evolution of $E\{Y_t\}$ over time

  $Y_t = f(t) + \varepsilon_t$, ε_t: white noise

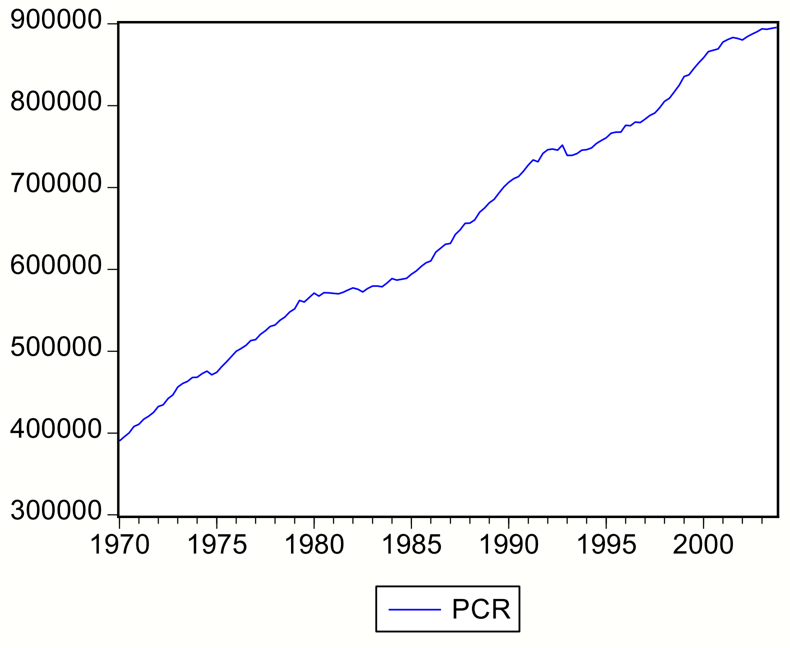
  Example: $Y_t = \alpha + \beta t + \varepsilon_t$ describes a linear trend of Y; an increasing trend corresponds to $\beta > 0$
- Stochastic trend: $Y_t = \delta + Y_{t-1} + \varepsilon_t$ or

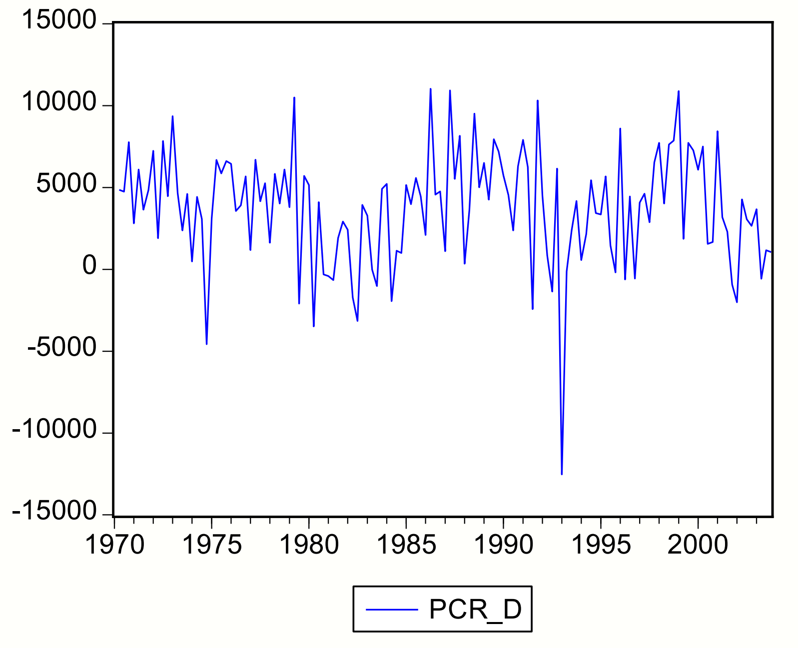
  $\Delta Y_t = Y_t - Y_{t-1} = \delta + \varepsilon_t$, ε_t: white noise

  - describes an irregular or random fluctuation of the differences ΔY_t around the expected value δ
  - AR(1) – or AR(p) – process with unit root
  - "random walk with trend"

Example: Private Consumption

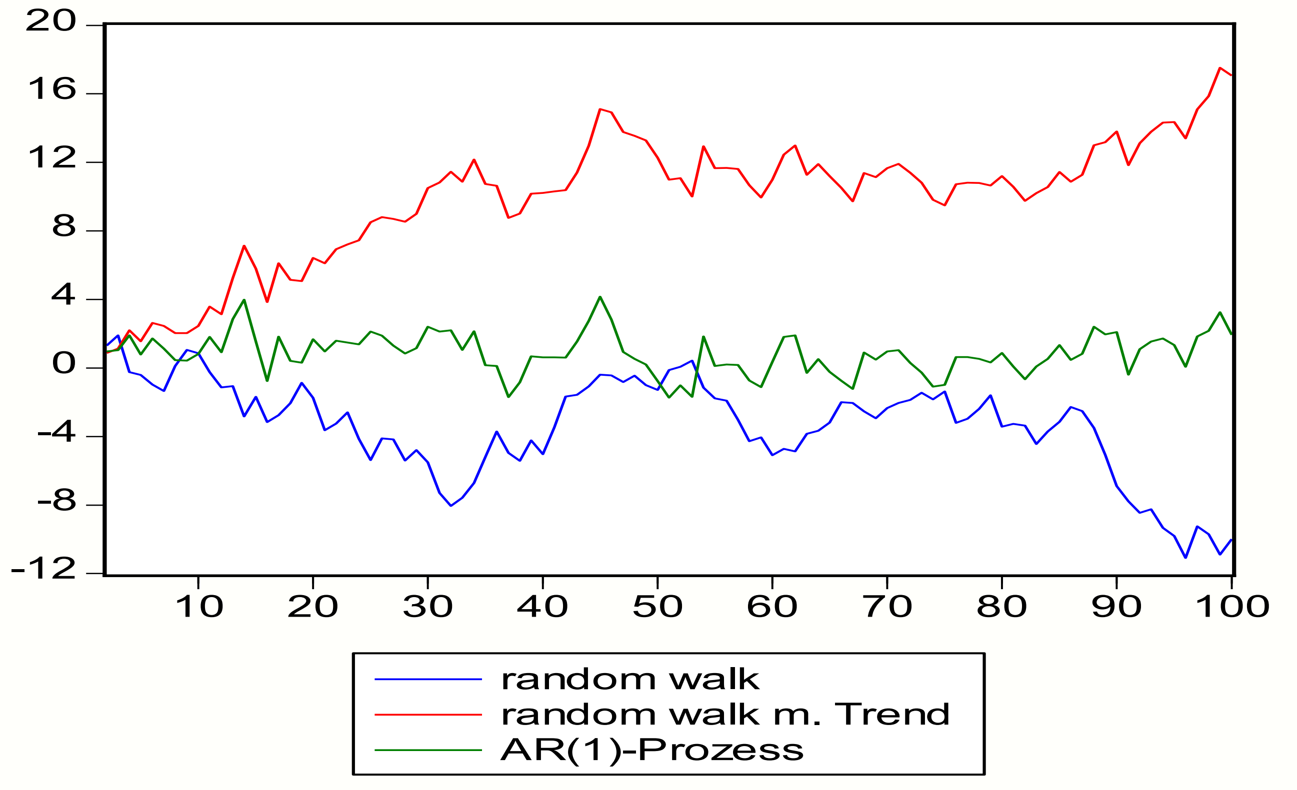
Private consumption, AWM database; level values (PCR) and first differences (PCR_D)





Trends: Random Walk and AR Process

Random walk: $Y_t = Y_{t-1} + \varepsilon_t$; random walk with trend: $Y_t = 0.1 + Y_{t-1} + \varepsilon_t$; AR(1) process: $Y_t = 0.2 + 0.7Y_{t-1} + \varepsilon_t$; ε_t simulated from $N(0,1)$



Random Walk with Trends

The random walk with trend $Y_t = \delta + Y_{t-1} + \varepsilon_t$ can be written as

$$Y_t = Y_0 + \delta t + \Sigma_{i \le t}\, \varepsilon_i$$

δ: trend parameter

Components of the process

- Deterministic growth path $Y_0 + \delta t$
- Cumulative errors $\Sigma_{i \le t}\, \varepsilon_i$

Properties:

- Expectation $Y_0 + \delta t$ is not a fixed value!
- $V\{Y_t\} = \sigma^2 t$ becomes arbitrarily large!
- $Corr\{Y_t, Y_{t-k}\} = \sqrt{(1-k/t)}$
- Non-stationary

Random Walk with Trends, cont'd

From Corr{Y_t, Y_{t-k}} = $\sqrt{(1-k/t)}$ follows

- For fixed k, Y_t and Y_{t-k} are the stronger correlated, the larger t
- With increasing k, correlation tends to zero, but the slower the larger t (long memory property)

Comparison of random walk with the AR(1) process $Y_t = \delta + \theta Y_{t-1} + \varepsilon_t$

- AR(1) process: ε_{t-i} has the lesser weight, the larger i
- AR(1) process similar to random walk when θ is close to one

Non-Stationarity: Consequences

AR(1) process $Y_t = \theta Y_{t-1} + \varepsilon_t$

OLS Estimator for θ:

$$\theta = \frac{\sum_t y_t y_{t-1}}{\sum_t y_t^2}$$

For |θ| < 1: the estimator is

- Consistent
- Asymptotically normally distributed

For θ = 1 (unit root)

- θ is underestimated
- Estimator not normally distributed
- Spurious regression problem

Spurious Regression

Random walk without trend: $Y_t = Y_{t-1} + \varepsilon_t$, ε_t: white noise

- Y_t is a non-stationary process, stochastic trend?
- $V\{Y_t\}$: a multiple of t

Specified model: $Y_t = \alpha + \beta t + \varepsilon_t$

- Deterministic trend
- Constant variance
- Misspecified model!

Consequences for OLS estimator for β

- *t*- and *F*-statistics: wrong critical limits, rejection probability too large
- R^2 is about 0.45 although Y_t random walk without trend
- Granger & Newbold, 1974

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How to Model Trends?

Specification of

- Deterministic trend, e.g., $Y_t = \alpha + \beta t + \varepsilon_t$: risk of wrong decisions
- Stochastic trend: analysis of differences ΔY_t if a random walk, i.e., a unit root, is suspected

Consequences of spurious regression are more serious

Consequences of modeling differences:

- Autocorrelated errors
- Consistent estimators
- Asymptotically normally distributed estimators
- HAC correction of standard errors

Elimination of a Trend

In order to cope with non-stationarity

- Trend stationary process: the process can be transformed in a stationary process by subtracting the deterministic trend
- Difference stationary process, or integrated process: stationary process can be derived by differencing

Integrated process: stochastic process Y is called

- integrated of order one if the first differences yield a stationary process: $Y \sim I(1)$
- integrated of order d, if the d-fold differences yield a stationary process: $Y \sim I(d)$

Trend-Elimination: Examples

Random walk $Y_t = \delta + Y_{t-1} + \varepsilon_t$ with white noise ε_t

$$\Delta Y_t = Y_t - Y_{t-1} = \delta + \varepsilon_t$$

- Y_t is a stationary process
- A random walk is a difference-stationary or *I*(1) process

Linear trend $Y_t = \alpha + \beta t + \varepsilon_t$

- Subtracting the trend component $\alpha + \beta t$ provides a stationary process
- Y_t is a trend-stationary process

Integrated Stochastic Processes

Random walk $Y_t = \delta + Y_{t-1} + \varepsilon_t$ with white noise ε_t is a difference-stationary or I(1) process

Many economic time series show stochastic trends

From the AWM Database

	Variable	*d*
YER	GDP, real	1
PCR	Consumption, real	1-2
PYR	Household's Disposable Income, real	1-2
PCD	Consumption Deflator	2

ARIMA(p,d,q) process: d-th differences follow an ARIMA(p,q) process

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Unit Root Test

AR(1) process $Y_t = \delta + \theta Y_{t-1} + \varepsilon_t$ with white noise ε_t

OLS Estimator for θ:

$$\theta = \frac{\sum_t y_t y_{t-1}}{\sum_t y_t^2}$$

Distribution of *DF*

$$DF = \frac{\theta - \vartheta}{se(\theta)}$$

- If $|\theta| < 1$: approximately $t(T-1)$
- If $\theta = 1$: critical values of Dickey & Fuller

DF test for testing H_0: $\theta = 1$ against H_1: $\theta < 1$

- $\theta = 1$: characteristic polynomial has unit root

Dickey-Fuller Critical Values

Monte Carlo estimates of critical values for

DF_0: Dickey-Fuller test without intercept

DF: Dickey-Fuller test with intercept

DF_τ: Dickey-Fuller test with time trend

T		p = 0.01	p = 0.05	p = 0.10
25	DF_0	-2.66	-1.95	-1.60
	DF	-3.75	-3.00	-2.63
	DF_τ	-4.38	-3.60	-3.24
100	DF_0	-2.60	-1.95	-1.61
	DF	-3.51	-2.89	-2.58
	DF_τ	-4.04	-3.45	-3.15
N(0,1)		-2.33	-1.65	-1.28

Unit Root Test: The Practice

AR(1) process $Y_t = \delta + \theta Y_{t-1} + \varepsilon_t$ with white noise ε_t

can be written with $\pi = \theta - 1$ as

$$\Delta Y_t = \delta + \pi Y_{t-1} + \varepsilon_t$$

DF tests H_0: $\pi = 0$ against H_1: $\pi < 0$

DF test statistic

Distribution of *DF*

$$DF = \frac{\theta -}{se(\theta)} = \frac{\pi}{se(\theta)}$$

Two steps:

1. Regression of ΔY_t on Y_{t-1}: OLS-estimator for $\pi = \theta - 1$
2. Test of H_0: $\pi = 0$ against H_1: $\pi < 0$ based on *DF*; critical values of Dickey & Fuller

Unit Root Test: Extensions

DF test for model with intercept: $\Delta Y_t = \delta + \pi Y_{t-1} + \varepsilon_t$

DF test for model without intercept: $\Delta Y_t = \pi Y_{t-1} + \varepsilon_t$

DF test for model with intercept and trend: $\Delta Y_t = \delta + \gamma t + \pi Y_{t-1} + \varepsilon_t$

DF tests in all cases H_0: $\pi = 0$ against H_1: $\pi < 0$

Test statistic in all cases

$$DF = \frac{\theta -}{se(\theta)}$$

Critical values depend on cases

ADF Test

Extended model according to an AR(*p*) process:

$$\Delta Y_t = \delta + \pi Y_{t-1} + \beta_1 \Delta y_{t-1} + \dots + \beta_p \Delta y_{t-p+1} + \varepsilon_t$$

Example: AR(2) process $Y_t = \delta + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t$ can be written as

$$\Delta Y_t = \delta + (\theta_1 + \theta_2 - 1) Y_{t-1} - \theta_2 \Delta Y_{t-1} + \varepsilon_t$$

the characteristic equation $(1 - \phi_1 L)(1 - \phi_2 L) = 0$ has roots $\theta_1 = \phi_1 + \phi_2$ and $\theta_2 = -\phi_1 \phi_2$

a unit root implies $\phi_1 = \theta_1 + \theta_2 = 1$:

Augmented DF (ADF) test

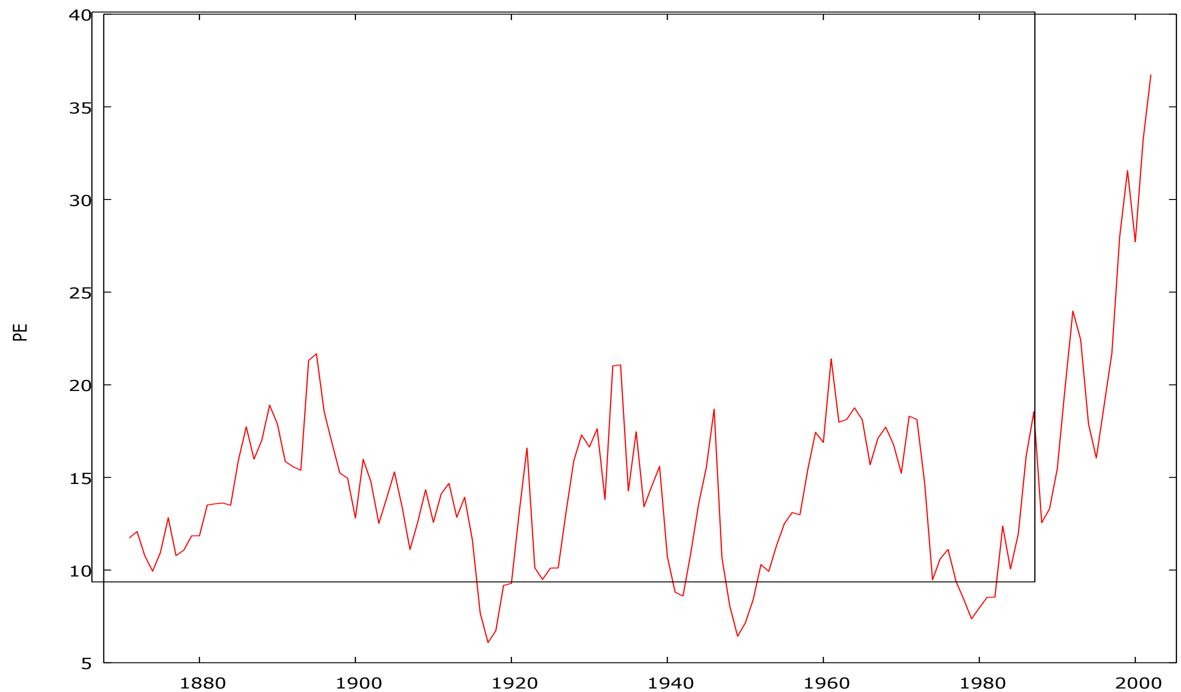
- Test of H_0: $\pi = 0$ against H_1: $\pi < 0$
- Needs its own critical values
- Extensions similar to the DF-test
- Phillips-Perron test: alternative method; uses HAC-corrected standard errors

Example: Price/Earnings Ratio

Data set PE: annual time series data on price index and the composite earnings index of the S&P500, 1871-2002

Price/earnings ratio

- Mean 14.6
- Min 6.1
- Max 36.7
- Std 5.1



Price/Earnings Ratio, cont'd

Extended model according to an AR(2) process gives:

$$\Delta Y_t = 0.366 - 0.136 Y_{t-1} + 0.152 \Delta y_{t-1} - 0.093 \Delta y_{t-2}$$

with *t*-statistics -2.487 (Y_{t-1}), 1.667 (Δy_{t-1}) and -1.007 (Δy_{t-2}) and *p*-values 0.014, 0.098 and 0.316

p-value of the DF statistic 0.121;

1% critical value: -3.48

5% critical value: -2.88

10% critical value: -2.58

Non-stationarity cannot be rejected for the log PE ratio

Unit root test for first differences: DF statistic -7.31, *p*-value 0.000 (1% critical value: -3.48)

log PE ratio is *I*(1)

However: for sample 1871-1990: DF statistic -3.52, *p*-value 0.009

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ARMA Models: Application

Application of the ARMA(p,q) model in data analysis: Three steps

1. Model specification, i.e., choice of p, q (and d if an ARIMA model is specified)
2. Parameter estimation
3. Diagnostic checking

Estimation of ARMA Models

The estimation methods are

- OLS estimation
- ML estimation

AR models: the explanatory variables are

- Lagged Y_t
- Uncorrelated with ε_t
- OLS estimation

MA Models: OLS Estimation

MA models:

- Minimization of sum of squared deviations is not straightforward
- E.g., for an MA(1) model, $S(\mu,\alpha) = \Sigma_t[Y_t - \mu - \alpha\Sigma_{j=0}(- \alpha)^j(Y_{t-j-1} - \mu)]^2$
  - $S(\mu,\alpha)$ is a nonlinear function of parameters
  - needs Y_{t-j-1} for j=0,1,…, i.e., historical Y_s, $s < 0$
- Approximate solution from minimization of

  $S^*(\mu,\alpha) = \Sigma_t[Y_t - \mu - \alpha\Sigma_{j=0}^{t-2}(- \alpha)^j(Y_{t-j-1} - \mu)]^2$
- Nonlinear minimization, grid search

ARMA models combine AR part with MA part

ML Estimation

Needs an assumption on the distribution of ε_t; usual normality

Log likelihood function, conditional on initial value

$$\log L(\alpha,\theta,\mu,\sigma^2) = -(T-1)\log(2\pi\sigma^2)/2 - (1/2)\, \Sigma_t\, \varepsilon_t^2/\sigma^2$$

ε_t are functions of the parameters

- AR(1): $\varepsilon_t = y_t - \theta_1 y_{t-1}$
- MA(1): $\varepsilon_t = \Sigma_{j=0}^{t-1}(-\alpha)^j y_{t-j}$

Initial values: y_1 for AR, $\varepsilon_0 = 0$ for MA

Extension for exact ML estimator

Again, estimation for AR models easier

ARMA models combine AR part with MA part

Model Specification

Based on the form of

- Autocorrelation function (ACF)
- Partial Autocorrelation function (PACF)

Structure of AC and PAC functions typical for AR and MA processes

Example:

- MA(1) process: $\rho_0 = 1$, $\rho_1 = \alpha/(1-\alpha^2)$; $\rho_i = 0$, $i = 2, 3, \ldots$
- AR(1) process: $\rho_k = \theta^k$, $k = 0, 1, \ldots$

ARMA(p,q)-Processes

Condition for	**AR(p)** $\theta(L)Y_t = \varepsilon_t$	**MA(q)** $Y_t = \alpha(L)\,\varepsilon_t$	**ARMA(p,q)** $\theta(L)Y_t=\alpha(L)\,\varepsilon_t$
Stationarity	roots z_i of $\theta(z)=0$: $\lvert z_i\rvert > 1$	always stationary	roots z_i of $\theta(z)=0$: $\lvert z_i\rvert > 1$
Invertibility	always invertible	roots z_i of $\alpha(z)=0$: $\lvert z_i\rvert > 1$	roots z_i of $\alpha(z)=0$: $\lvert z_i\rvert > 1$
AC function	damped, infinite	$\rho_k = 0$ for $k > q$	damped, infinite
PAC function	$\phi_{kk} = 0$ for $k > p$	damped, infinite	damped, infinite

Empirical AC and PAC Function

Estimation of the AC and PAC functions

AC ρ_k:

$$r_k = \frac{\sum_t (y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_t (y_t - \bar{y})^2}$$

PAC θ_{kk}: coefficient of Y_{t-k} in regression of Y_t on $Y_{t-1}, \ldots, Y_{t-k}$

MA(q) process: standard errors for r_k, $k > q$ from

$$\sqrt{T}(r_k - \rho_k) \rightarrow N(0, v_k)$$

with $v_k = 1 + 2\rho_1^2 + \ldots + 2\rho_k^2$

- test of H_0: $\rho_1 = 0$: compare $\sqrt{T} r_1$ with critical value from N(0,1), etc.

AR(p) process: test of H_0: $\rho_k = 0$ for $k > p$ based on asymptotic distribution

$$\sqrt{T}\,\theta_{\kappa k} \rightarrow N(0,1)$$

Diagnostic Checking

ARMA(*p*,*q*): Adequacy of choices *p* and *q*

Analysis of residuals from fitted model:

- Correct specification: residuals are realizations of white noise
- Portmanteau test: for a ARMA(*p*,*q*) process

$$Q_K = T(T+2)\sum_{k=1}^{K} \frac{1}{T-k} r_k^2$$

follows the Chi-squared distribution with *K*-*p*-*q* *df*

Overfitting

- Starting point: a general model
- Comparison with a model with reduced number of parameters: *AIC* or *BIC*
- *AIC*: tends to result asymptotically in overparameterized models

Advanced Econometrics - Lecture 5

- Time Series
- Stochastic Processes
- Stationary Processes
- The ARMA Process
- Deterministic and Stochastic Trends
- Models with Trend
- Unit Root Tests
- Estimation of ARMA Models
- ARCH and GARCH Models

ARCH Processes

Autoregressive Conditional Heteroskedasticity (ARCH):

- Special case of heteroskedasticity
- Error variance: autoregressive behavior
- Allows to model successive periods with high, other periods with small volatility
- Typical for asset markets

Example:

$$y_t = x_t'\theta + \varepsilon_t$$

with $\varepsilon_t = \sigma_t v_t$, $v_t \sim NID(0,1)$

- the conditional error variance, given the information I_{t-1}, is σ_t^2
- ARCH(1) process

  $$\sigma_t^2 = E\{\varepsilon_t^2 | I_{t-1}\} = \varpi + \alpha\varepsilon_{t-1}^2$$

- I_{t-1} is the information set containing all past including ε_{t-1}

The ARCH(1) Process

ARCH(1) process describes the conditional error variance, i.e., the variance conditional on information dated *t*-1 and earlier

$$\sigma_t^2 = E\{\varepsilon_t^2 | \mathcal{I}_{t-1}\} = \varpi + \alpha \varepsilon_{t-1}^2$$

- $\mathcal{I}_{t-1}$ is the information set containing all past including ε_{t-1}
- Conditions for $\sigma_t^2 \geq 0$: $\varpi \geq 0$, $\alpha \geq 0$
- A big shock at *t*-1, i.e., a large value $|\varepsilon_{t-1}|$,
  - Induces high volatility, i.e., large σ_t^2
  - makes large values $|\varepsilon_t|$ more likely at *t* (and later)
- ARCH process does not imply correlation!

The unconditional variance of ε_t is

$$\sigma^2 = E\{\varepsilon_t^2\} = \varpi + \alpha E\{\varepsilon_{t-1}^2\} = \varpi/(1 - \alpha)$$

given that $0 \leq \alpha < 1$

- The ε_t process is stationary

More ARCH Processes

Various generalizations

ARCH(*p*) process

$$\sigma_t^2 = \varpi + \alpha_1\varepsilon_{t-1}^2 + \ldots \alpha_p\varepsilon_{t-p}^2 = \varpi + \alpha(L)\varepsilon_{t-1}^2$$

with lag polynomial $\alpha(L)$ of order p-1

- Conditions for $\sigma_t^2 \geq 0$: $\varpi \geq 0$; $\alpha_i \geq 0$, $i = 1,\ldots p$
- Condition for stationarity: $\alpha(1) < 1$

GARCH(*p*,*q*) process

- „Generalized ARCH"
- Similar to the ARMA representation of levels

$$\sigma_t^2 = \varpi + \alpha_1\varepsilon_{t-1}^2 + \ldots \alpha_p\varepsilon_{t-p}^2 + \beta_1\sigma_{t-1}^2 + \ldots + \beta_q\sigma_{t-q}^2 =$$

$$= \varpi + \alpha(L)\varepsilon_{t-1}^2 + \beta(L)\sigma_{t-1}^2$$

E.g., GARCH(1,1): $\sigma_t^2 = \varpi + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$; with "surprises" $v_t = \varepsilon_{t-1}^2 - \sigma_t^2$:

$$\varepsilon_t^2 = \varpi + (\alpha + \beta)\varepsilon_{t-1}^2 + v_t - \beta v_{t-1}^2,$$ i.e. ε_t^2 follow ARMA(1,1)

Test for ARCH Processes

Null hypothesis of homoskedasticity, to be tested against the alternative ARCH(q)

1. Estimate the model of interest using OLS: residuals e_t
2. Auxiliary regression of squared residuals e_t^2 on a constant and q lagged e_t^2
3. Test statistic TR_e^2 with R_e^2 from the auxiliary regression, p-value from the chi squared distribution with q df

More ARCH Processes, cont'd

EGARCH or exponential GARCH

$$\log \sigma_t^2 = \varpi + \beta\sigma_{t-1}^2 + \gamma\varepsilon_{t-1}/\sigma_{t-1} + \alpha|\varepsilon_{t-1}|/\sigma_{t-1}$$

- Asymmetric if $\gamma \neq 0$
  - $\gamma < 0$: positive shocks („good news“) reduce volatility

Time Series Models in GRETL

Model > Time Series > ARIMA

- Estimates an ARMA model, with or without exogenous regressors

Model > Time Series > ARCH

- Estimates the specified model allowing for ARCH: (1) model estimated via OLS, (2) auxiliary regression of the squared residual on its own lagged values, (3) weighted least squares estimation

Model > Time Series > GARCH

- Estimates a GARCH model, with or without exogenous regressors

Exercise

Answer questions a. to e. of Exercise 8.2 of Verbeek

- data from the data sets “SP500” containing daily returns on Standard & Poor's 500 index from January 1981 to April 1991, computed as the change in log index