Advanced Econometrics - Lecture 6

Multivariate Time Series Models

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- Econometric Models
- Multiplicators and ADL Models
- Cointegration
- VAR Models
- Cointegration and VAR Models
- Vector Error-Correction Model
- Estimation of VEC Models

The Lüdeke Model for Germany

Consumption function

$$C_t = \alpha_1 + \alpha_2 Y_t + \alpha_3 C_{t-1} + \varepsilon_{1t}$$

Investment function

$$I_t = \beta_1 + \beta_2 Y_t + \beta_3 P_{t-1} + \varepsilon_{2t}$$

Import function

$$M_t = \gamma_1 + \gamma_2 Y_t + \gamma_3 M_{t-1} + \varepsilon_{3t}$$

Identity relation

$$Y_t = C_t + I_t - M_{t-1} + G_t$$

with *C*: private consumption, *Y*: GDP, *I*: investments, *P*: profits, *M*: imports, *G*: governmental spendings

endogenous: *C*, *Y*, *I*, *M*
exogenous: *G*, P_{-1}

Econometric Models

Basis is the multiple linear regression model

Model extensions

- Dynamic models
- Systems of regression relations

Dynamic Models: Examples

Demand model: describes the quantity Q demanded of a product as a function of its price P and the income Y of households

Demand is determined by

- Current price and current income (static model):

  $$Q_t = \beta_1 + \beta_2 P_t + \beta_3 Y_t + \varepsilon_t$$

- Current price and income of the previous period (dynamic model):

  $$Q_t = \beta_1 + \beta_2 P_t + \beta_3 Y_{t-1} + \varepsilon_t$$

- Current price and demand of the previous period (dynamic autoregressive model):

  $$Q_t = \beta_1 + \beta_2 P_t + \beta_3 Q_{t-1} + \varepsilon_t$$

The Dynamic of Processes

Static processes: immediate reaction to changes in regressors, the adjustment of the dependent variables to the realizations of the independent variables will be completed within the current period, the process seems to be always in equilibrium

Static models are inappropriate

- Some processes are determined by the past, e.g., energy consumption depends on past investments into energy-consuming systems and equipment
- Actors in economic processes may respond delayed, e.g., time for decision-making and procurement processes exceeds the observation period
- Expectations: e.g., consumption depends not only on current income but also on the income expectations; modeling the expectation may be based on past development

Elements of Dynamic Models

- Lag structures, distributed lags: they describe the delayed effect of one or more regressors on the dependent variable; e.g., the distributed lag of order *s*, the DL(*s*) model, is

$$Y_t = \delta + \Sigma^{s}_{i=0}\, \varphi_i X_{t-i} + \varepsilon_t$$

- Geometric or Koyck lag structure: infinite lag structure with

$$\varphi_i = \lambda_0\, \lambda^i$$

- ADL models: autoregressive model with lag structure, e.g., the ADL(1,1) model

$$Y_t = \delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1} + \varepsilon_t$$

- Error correction model:

$$\Delta Y_t = -(1-\theta)(Y_{t-1} - \alpha - \beta X_{t-1}) + \varphi_0\, \Delta X_t + \varepsilon_t$$

  which is obtained from the ADL(1,1) model with $\alpha = \delta/(1-\theta)$ and $\beta = (\varphi_0+\varphi_1)/(1-\theta)$

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Example: Consumption Function

Data for Austria (1976:1 – 1995:2), logarithmic differences:

$$\hat{C} = 0.009 + 0.621Y$$

with $t(Y) = 2.288$, $R^2 = 0.335$

DL(2) model, same data:

$$\hat{C} = 0.006 + 0.504Y - 0.026Y_{-1} + 0.274Y_{-2}$$

with $t(Y) = 3.79$, $t(Y_{-1}) = -0.18$, $t(Y_{-2}) = 2.11$, $R^2 = 0.370$

Effect of income on consumption:

- Short term effect, i.e. effect in the current period:
  $\Delta C = 0.504$, given a change in income $\Delta Y = 1$
- Overall effect, i.e., cumulative current and future effects
  $\Delta C = 0.504 - 0.026 + 0.274 = 0.752$, given a change in income $\Delta Y = 1$

Multiplicators

Describe the effect of a change in explanatory variable X by $\Delta X = 1$ on current and future values of the dependent variable Y

- DL(s) model: $Y_t = \delta + \varphi_0 X_t + \varphi_1 X_{t-1} + \ldots + \varphi_s X_{t-s} + \varepsilon_t$
  - short run or impact multiplier: effect of the change in the same period ($\Delta Y = \varphi_0$)
  - long run or equilibrium multiplier: the effect of $\Delta X = 1$, cumulated over all future ($\Delta Y = \varphi_0 + \ldots + \varphi_s$)
- ADL(1,1) model: $Y_t = \delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1} + \varepsilon_t$
  - impact multiplier: $\Delta Y = \varphi_0$; after one period: $\Delta Y = \theta\varphi_0 + \varphi_1$; after two periods: $\Delta Y = \theta(\theta\varphi_0 + \varphi_1)$; etc.
  - equilibrium multiplier:

$$\varphi_0 + (\theta\varphi_0 + \varphi_1) + \theta(\theta\varphi_0 + \varphi_1) + \ldots = (\varphi_0 + \varphi_1)/(1 - \theta)$$

Multiplicators, cont'd

Describe the effect of a change in explanatory variable *X* on current and future values of *Y*

- ADL(1,1) model written as error correction model

$$\Delta Y_t = -(1-\theta)(Y_{t-1} - \alpha - \beta X_{t-1}) + \varphi_0 \Delta X_t + \varepsilon_t$$

effects on ΔY

  - due to changes ΔX
  - due to equilibrium error, i.e., $Y_{t-1} - \alpha - \beta X_{t-1}$, a negative adjustment $-(1-\theta)(Y_{t-1} - \alpha - \beta X_{t-1})$

The ADL(p,q) Model

ADL(p,q): generalizes the ADL(1,1) model:

$$\theta(L)Y_t = \delta + \Phi(L)X_t + \varepsilon_t$$

with lag polynomials

$$\theta(L) = 1 - \theta_1 L - \ldots - \theta_p L^p \text{ , } \Phi(L) = \varphi_0 + \varphi_1 L + \ldots + \varphi_q L^q$$

Given invertibility of $\theta(L)$, i.e., $\theta_1 + \ldots + \theta_p < 1$,

$$Y_t = \theta(1)^{-1}\delta + \theta(L)^{-1}\Phi(L)X_t + \theta(L)^{-1}\varepsilon_t$$

The coefficients of $\theta(L)^{-1}\Phi(L)$ describe the dynamic effects of X on current and future values of Y

- equilibrium multiplier

$$\theta(1)^{-1}\phi(1) = \frac{\varphi_0 + \ldots + \varphi_q}{1 - \theta_1 - \ldots - \theta_p}$$

ADL(0,q): coincides with the DL(q) model; $\theta(L) = 1$

Partial Adjustment Model

Describes the process of adapting to a desired or planned value

Example: Stock level K and revenues S

- The desired (optimal) stock level K^* depends of the revenues S

  $K^*_t = \alpha + \beta S_t + \eta_t$
- Actual stock level K_t deviates from K^*_t
- (Partial) adjustment according to

  $K_t - K_{t-1} = (1 - \theta)(K^*_t - K_{t-1})$ with $0 < \theta < 1$
- Substitution for K^*_t gives the AR form of the model

  $K_t = K_{t-1} + (1 - \theta)\alpha + (1 - \theta)\beta S_t - (1 - \theta)K_{t-1} + (1 - \theta)\eta_t$

  $= \delta + \theta K_{t-1} + \varphi_0 S_t + \varepsilon_t$

  which is a ADL(1,0) model

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The Drunk and her Dog

M. P. Murray, A drunk and her dog: An illustration of cointegration and error correction. *The American Statistician*, **48** (1997), 37-39

drunk: $x_t - x_{t-1} = u_t$

dog: $y_t - y_{t-1} = w_t$

Cointegration:

$$x_t - x_{t-1} = u_t + c(y_{t-1} - x_{t-1})$$

$$y_t - y_{t-1} = w_t + d(x_{t-1} - y_{t-1})$$

A Macro-Economic Example

$I(1)$ variables

- M: money stock
- P: price level
- Y: output
- R: nominal interest rate

Equilibrium relation (in logarithms)

$$m - p = \gamma_0 + \gamma_1 y + \gamma_2 r, \quad \gamma_1 > 0; \gamma_2 < 0$$

Theory implies that

$$x'\beta = (m, p, y, r)\,(1, -1, -\gamma_1, -\gamma_2)' = m - p - \gamma_1 y - \gamma_2 r \sim I(0)$$

Deviations from the equilibrium are corrected

Notation

Non-stationary variables X, Y:

$X_t \sim I(1)$, $Y_t \sim I(1)$

exists β such that

$Z_t = Y_t - \beta X_t \sim I(0)$

- X_t and Y_t are cointegrated, have a common trend
- β is the cointegration parameter
- (1, - β)’ is the cointegration vector

Cointegration implies a long-run equilibrium

Long-run Equilibrium

Equilibrium defined by

$$Y_t = \alpha + \beta X_t$$

Equilibrium error: $z_t = Y_t - \beta X_t - \alpha = Z_t - \alpha$

- $z_t \sim I(0)$: equilibrium error stationary, fluctuating around zero
- Y_t, βX_t not integrated:
  - $z_t \sim I(1)$, non-stationary process
  - $Y_t = \alpha + \beta X_t$ does not describe an equilibrium

Cointegration vector implies a long-run equilibrium relation

Identification of Cointegration

Information about cointegration:

- Economic theory
- Visual inspection of data
- Statistical test

Testing for Cointegration

Non-stationary variables $X_t \sim I(1)$, $Y_t \sim I(1)$

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

- X_t and Y_t are cointegrated: $\varepsilon_t \sim I(0)$
- X_t and Y_t are not cointegrated: $\varepsilon_t \sim I(1)$

Tests for cointegration:

- If β is known, unit root test based on differences $Y_t - \beta X_t$
- Unit root test based on residuals e_t

  $$\Delta e_t = \gamma_0 + \gamma_0 e_{t-1} + u_t$$

  Critical values, Verbeek, Tab. 9.2
- Cointegrating regression Durbin-Watson (CRDW) test: DW statistic from OLS-fitting $Y_t = \alpha + \beta X_t + \varepsilon_t$

  Critical values, Verbeek, Tab. 9.3

OLS Estimation

To be estimated:

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

cointegrated non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$

$\varepsilon_t \sim I(0)$

OLS estimator b for β

- Non-standard distribution
- Super consistent:
  - $T(b - \beta)$ converges to zero
  - In case of consistency: $\sqrt{T}(b - \beta)$ converges to zero
- Robust
- Non-normal; e.g., *t*-test misleading
- Small samples: bias

OLS Estimation, cont'd

To be estimated:

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$

If $\varepsilon_t \sim I(1)$, i.e., Y_t and X_t not cointegrated: spurious regression

OLS estimator *b* for β

- Non-standard distribution
- High values of R^2, *t*-statistic
- Highly autocorrelated residuals
- Low value of DW statistic

Error-correction Model

Granger's Representation Theorem (Engle & Granger, 1987): If a set of variables is cointegrated then an error-correction relation of the variables exists

non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$ with cointegrating vector (1, -β)': error-correction representation

$$\theta(L)\Delta Y_t = \delta + \Phi(L)\Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \alpha(L)\varepsilon_t$$

with lag polynomials $\theta(L)$ – with $\theta_0=1$ –, $\Phi(L)$, and $\alpha(L)$

E.g., $\Delta Y_t = \delta + \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \varepsilon_t$

Error-correction model: describes

- the short-run behavior
- consistent with the long-run equilibrium

Converse statement: if $Y_t \sim I(1)$, $X_t \sim I(1)$ have an error-correction representation, they are cointegrated

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The VAR(p) Model

VAR(p) model: generalization of the AR(p) model for k-vectors Y_t

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t$$

with k-vectors Y_t, δ, and ε_t and kxk-matrices $\Theta_1, \ldots, \Theta_p$

Shorter:

$$\Theta(L) Y_t = \delta + \varepsilon_t$$

with $\Theta(L) = I - \Theta_1 L - \ldots - \Theta_p L^p$

Error terms ε_t have covariance matrix Σ; allows for contemporary correlation

In deviations $y_t = Y_t - \mu$, with $\mu = E\{Y_t\} = (I - \Theta_1 - \ldots - \Theta_p)^{-1}\delta = \Theta(1)^{-1}\delta$

$$\Theta(L) y_t = \varepsilon_t$$

MA representation:

$$Y_t = \mu + \Theta(1)^{-1}\varepsilon_t = \mu + \varepsilon_t + A_1\varepsilon_{t-1} + A_2\varepsilon_{t-2} + \ldots$$

Example: Income and Consumption

Model:

$$Y_t = \delta_1 + \theta_{11} Y_{t-1} + \theta_{12} C_{t-1} + \varepsilon_{1t}$$

$$C_t = \delta_2 + \theta_{21} C_{t-1} + \theta_{22} Y_{t-1} + \varepsilon_{2t}$$

With $Z = (Y, C)'$, 2-vectors δ and ε, and (2x2)-matrix Θ, the VAR(1) model is

$$Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$$

Represents each component of Z as a linear combination of lagged variables

Income and Consumption, cont'd

AWM data base, 1970:1-2003:4: *PCR* (real private consumption), *PYR* (real disposable income of households); respective annual growth rates: *C*, *Y*

Fitting $Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$ with $Z = (Y, C)'$ gives

		δ	Y_{-1}	C_{-1}	adj.R^2	AIC
Y	θ_{ij}	0.001	0.825	0.082	0.80	-6.94
	$t(\theta_{ij})$	0.91	12.09	1.07		
C	Θ_{ij}	0.003	0.061	0.826	0.79	-7.16
	$t(\theta_{ij})$	2.36	0.97	11.69		

with AIC = -14.45; VAR(2) model: AIC = -14.43

Compare: consumption function

$\hat{C} = 0.011 + 0.718Y$ with $t(Y) = 15.55$, adj.$R^2 = 0.65$, and AIC = -6.61

VAR Models: Advantages

Besides the generality of the specification:

- Meets assumptions of OLS estimation
- Requires no distinction between endogenous and exogenous variables (which is always arbitrary)
- Allows for non-stationarity and cointegration

The number of parameters to be estimated grows rapidly with p and k

Number of components of Θ for some values of p and k

p	1	2	3
k=2	4	8	12
k=4	16	32	56

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VAR(*p*) Model, Stationarity and Cointegration

VAR(*p*) process

- is stationary, if all roots z_i of the characteristic polynomial $\Theta(L)$,

  $\det \Theta(z) = 0$,

  fulfill $|z_i| > 1$; det Θ: determinant of $\Theta(.)$

  For kxk-matrices $\Theta_1, \ldots, \Theta_p$: kp roots
- is non-stationary, if

  $z_i = 1$

  is a (multiple) root of the characteristic polynomial $\Theta(L)$

For non-stationary variables with the same order of integration, cointegrating relationships may exist

VAR(1) Model and k=2

VAR(1) model for 2-vector y_t: characteristic polynomial $\Theta(L)$ for

$$\Theta(z) = I - \Theta z$$

has 2 roots; three cases

a) If both roots of det $\Theta(z) = 0$ have value one, then both variables in y integrated, no cointegrating relationship between them
b) If exactly one of the two roots has value one, then the variables cointegrated
c) If none of the roots has value one, both variables are stationary, no cointegrating relationship

Example

Model for Y and Y:

$$X_t + \alpha Y_t = u_{1t},\ u_{1t} = \rho_1 u_{1,t-1} + \varepsilon_{1t} \quad \text{(A)}$$

$$X_t + \beta Y_t = u_{2t},\ u_{2t} = \rho_2 u_{2,t-1} + \varepsilon_{2t} \quad \text{(B)}$$

with $\alpha \neq \beta$, independent IID$(0,\sigma_i^2)$-processes ε_i, $i = 1,2$

Reduced forms for Y and Y: linear combinations of the u_i

VAR(1) model

$$X_t = (\rho_1 - \alpha\delta)X_{t-1} - \alpha\beta\delta Y_{t-1} + v_{1t}$$

$$Y_t = \delta X_{t-1} + (\rho_1 + \beta\delta) Y_{t-1} + v_{2t}$$

with $\delta = (\rho_1 - \rho_2)/(\alpha - \beta)$ and linear combinations v_i of the ε_i

Characteristic polynomial

$$\det \Theta(z) = 1 - z(\rho_1 - \rho_2) + z^2\rho_1\rho_2 = 0$$

has characteristic roots $z_1 = 1/\rho_1$ and $z_2 = 1/\rho_2$

Example, cont'd

1. Let $\rho_1 = 1$, $|\rho_2| <1$, i.e., $u_1 \sim I(1)$, $u_2 \sim I(0)$, then $z_1 = 1$, $z_2 > 1$; the reduced forms imply $X \sim I(1)$ and $Y \sim I(1)$, i.e., both are non-stationary, equation (B) indicates cointegration of X and Y; see case b)
2. $|\rho_1| <1$, $|\rho_2| <1$; i.e., $u_1 \sim I(0)$, $u_2 \sim I(0)$, also $X \sim I(0)$ and $Y \sim I(0)$; all are stationary; see case c)
3. Let $\rho_1 = \rho_2 = 1$, i.e., $u_1 \sim I(1)$, $u_2 \sim I(1)$, both are non-stationary, also X and Y are non-stationary, no cointegrating relation between X and Y; see case a)

VAR(*p*) Model: Cointegration

VAR(1) model for k-vector y_t: characteristic polynomial $\Theta(L)$ for

$$\Theta(z) = \mathrm{I} - \Theta z$$

has k roots

- if k-r roots of the characteristic polynomial

  $$\det \Theta(z) = 0$$

  have the value one, then r cointegrating relationships exist between the variables of y_t

Example, cont'd

Case $\rho_1 = 1$, $|\rho_2| < 1$: differences

$$\Delta X_t = -\alpha\delta X_{t-1} - \alpha\beta\delta Y_{t-1} + v_{1t}$$

$$\Delta Y_t = \delta X_{t-1} + \beta\delta Y_{t-1} + v_{2t}$$

In matrix notation with $Z = (X, Y)'$: going from $Z_t = \Theta Z_{t-1} + \delta + \varepsilon_t$ to differences gives

$$\Delta Z_t = -(I - \Theta)Z_{t-1} + \delta + \varepsilon_t = -\Theta(1)Z_{t-1} + \delta + \varepsilon_t$$

with

$$\Theta(1) = I - \Theta = \begin{pmatrix} \alpha\delta & \alpha\beta\delta \\ -\delta & -\beta\delta \end{pmatrix}$$

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Granger's Representation Theorem

VAR(1) model: $y_t = \Theta_1 y_{t-1} + \delta + \varepsilon_t$, in differences with $\Theta(L) = I - \Theta_1 L$

$$\Delta y_t = -\Theta(1) y_{t-1} + \delta + \varepsilon_t$$

$r\{\Theta(1)\}$: rank of $\Theta(1)$, cointegrating rank

1. If $r\{\Theta(1)\} = 0$, then $\Delta y_t = \delta + \varepsilon_t$, i.e., y is a k-dimensional random walk, each component is $I(1)$, no cointegrating relationship
2. If $r\{\Theta(1)\} = r < k$, there is a $(k - r)$-fold unit root, $(k\text{x}r)$-matrices γ and β can be found, both of rank r, with
   $$\Theta(1) = \gamma\beta'$$
   the r columns of β are the cointegrating vectors of r cointegrating relations (β in standardized form, i.e., the main diagonal elements of β are ones)
3. If $r\{\Theta(1)\} = k$, VAR(1) process is stationary, all components of y are $I(0)$

Vector Error-Correction Model

VAR(1) model $\Delta y_t = - \Theta(1)y_{t-1} + \delta + \varepsilon_t$

$$\Theta(1) = \gamma\beta'$$

with $r\{\Theta(1)\} = r < k$

- Δy_t is stationary
- $\beta' y_t$ is stationary, each of the r columns of β defines a cointegrating relation

y_t consists of k-r independent deterministic and r independent stochastic trends

Vector error-correction model, VEC model (VECM)

$$\Delta y_t = - \gamma\beta' y_{t-1} + \delta + \varepsilon_t$$

- Cointegrating rank r
- γ: Adaptation parameters

Example, cont'd

Case $\rho_1 = 1$, $|\rho_2| < 1$:

$$\Theta(1) = I - \Theta_1 = \begin{pmatrix} \alpha\delta & \alpha\beta\delta \\ -\delta & -\beta\delta \end{pmatrix} = \begin{pmatrix} \alpha \\ -1 \end{pmatrix} \begin{pmatrix} \delta & \beta\delta \end{pmatrix} = \gamma\pi'$$

Standardized form: $\pi = (1, \beta)'$

Cointegrating relation:

$$\pi' y_{t-1} = X_{t-1} + \beta Y_{t-1}$$

The VEC(1) Model

VAR(1) model $y_t = \Theta_1 y_{t-1} + \delta + \varepsilon_t$ in form of the VEC model

$$\Delta y_t = - \gamma\beta' y_{t-1} + \delta + \varepsilon_t$$

with cointegrating rank *r* describes changes Δy_t as function of

- the intercept vector δ
- of *r* equilibrium relations

Deviations from equilibrium in period *t* -1 are partly corrected in period *t* by $- \gamma\beta' y_{t-1}$

adaptation parameters (elements of matrix γ) indicate the proportion of the correction, are a measure of the correction speed

The VEC(p) Model

Extension of the VAR(1) process for the k-vector y_t to

$$\Theta(L)\, y_t = \delta + \varepsilon_t$$

with $\Theta(L) = I - \Theta_1 L - \ldots - \Theta_p L^p$

For $r\{\Theta(1)\} = r < k$

$$\Theta(L) = -\Theta(1)\, L + (1 - L)\Gamma(L) \text{ with } \Theta(1) = \gamma\beta'$$

the VAR(p) model can be written as

$$\Delta y_t = -\Gamma_1 \Delta y_{t-1} - \ldots - \Gamma_p \Delta y_{t-p} - \gamma\beta' y_{t-1} + \delta + \varepsilon_t$$

Columns of β are cointegrating vectors, define r cointegrating relations

Example: VAR(2) process $y_t = \Theta_1 y_{t-1} + \Theta_2 y_{t-2} + \delta + \varepsilon_t$ results in

$$\Delta y_t = \Gamma_1 \Delta y_{t-1} + \Pi y_{t-1} + \delta + \varepsilon_t$$

with $\Gamma_1 = (-\Theta_2)$, $\Pi = -(I - \Theta_1 - \Theta_2) = -\Theta(1)$

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Estimation of VEC Models

Assumption: k-vector $y_t \sim I(1)$

Estimation of the parameters:

1. Specification of intercepts and deterministic trends
   - in the components of y_t and
   - in the cointegrating relations
2. Choice of the cointegrating rank r, R3-method of Johansen
3. Estimation of the cointegrating relations, standardizing
4. Estimation of the VEC model

Johansen‘s R3 Methode

Reduced rank regression or R3 method, also Johansen's test: an iterative method for specifying the cointegrating rank r

The test is based on the k eigenvalues λ_i ($\lambda_1 > \lambda_2 > \ldots > \lambda_k$) of

$$Y_1'Y_1 - Y_1\Delta Y(\Delta Y'\Delta Y)^{-1}\Delta Y'Y_1,$$

with ΔY: (Txk) matrix of differences Δy_t, Y_1: (Txk) matrix of y_{t-1}

- if $r\{\Theta(1)\} = r$, the k-r smallest eigenvalues obey: $\log(1-\lambda_i) = 0$

Iterative test procedures

- Trace test
- Maximum eigenvalue test or max test

Tests

LR tests, based on the assumption of normally distributed errors

- Trace test: for $r_0 = 0, 1, \ldots$, test of H_0: $r \le r_0$ against H_1: $r > r_0$

  $\lambda_{trace}(r_0) = - T\, \Sigma^{k}_{j=r0+1} \log(1- \hat{l}_j)$

  $\hat{l}_j$: estimator of λ_j

  Stops when H_0 is rejected for the first time

  Critical values from simulations

- Max test: tests for $r_0 = 0, 1, \ldots$: H_0: $r = r_0$ against H_1: $r = r_0+1$

  $\lambda_{max}(r_0) = - T \log(1 - \hat{l}_{r0+1})$

  Stops when H_0 is rejected for the first time

  Critical values from simulations

Example: Income and Consumption

Model:

$$Y_t = \delta_1 + \theta_{11} Y_{t-1} + \theta_{12} C_{t-1} + \varepsilon_{1t}$$

$$C_t = \delta_2 + \theta_{21} C_{t-1} + \theta_{22} Y_{t-1} + \varepsilon_{2t}$$

With $Z = (Y, C)'$, 2-vectors δ and ε, and (2x2)-matrix Θ, the VAR(1) model is

$$Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$$

Represents each component of *y* as a linear combination of lagged variables

Income and Consumption, cont'd

AWM data base: *PCR* (real private consumption), *PYR* (real disposable income of households); logarithms: *C*, *Y*

1. Check whether C and Y are non-stationary:

   $C \sim I(1)$, $Y \sim I(1)$

2. Johansen test for cointegration: given that *C* and *Y* have no trends and the cointegrating relationship has an intercept:

   $r = 1$ ($p < 0.05$)

   the cointegrating relationship is

   $C = 8.55 – 1.61Y$

   with $t(Y) = 18.2$

Income and Consumption, cont'd

3. VEC(1) model (same specification as in 2.)

$$\Delta y_t = - \gamma(\beta' y_{t-1} + \delta) + \Gamma \Delta y_{t-1} + \varepsilon_t$$

		coint	Y_{-1}	C_{-1}	adj.R^2	AIC
ΔY	γ_{ij}	0.029	0.167	0.059	0.14	-7.42
	$t(\gamma_{ij})$	5.02	1.59	0.49		
ΔC	Γ_{ij}	0.047	0.226	-0.148	0.18	-7.59
	$t(\gamma_{ij})$	2.36	2.34	1.35		

The model explains growth rates of *PCR* and *PYR*; AIC = -15.41 is smaller than that of the VAR(1)-Modell (AIC = -14.45)

Exercise

Answer questions a. to f. of Exercise 9.3 of Verbeek