Exercise Session
Advanced Macroeconomics II
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Using the Uhlig Toolbox[1] – Handout

This handout derives the systems of linear equations for the examples No. 0 and 1 in the Uhlig toolbox. These examples are contained in the files *exampl0.m* and *exampl1.m.* Before you start using Uhlig toolbox type *readme* in the Matlab command window.[2]

1 Example 0 – Neoclassical Stochastic Growth Model

The model we want to solve is

$$\max_{\{C_t,K_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{C_t^{1-\eta}-1}{1-\eta}\right)\right],$$

s.t.

$$\begin{aligned} C_t+K_t &= Z_tK_{t-1}^{\rho}+(1-\delta)K_{t-1} \\ \log(Z_t) &= (1-\psi)\log(\bar{Z})+\psi\log(Z_{t-1})+\varepsilon_t, \\ \varepsilon_t &\sim \text{i.i.d. } \mathrm{N}(0,\sigma_\varepsilon^2) \end{aligned}$$

First order conditions generate the following Euler Equation

$$C_t^{-\eta}=\beta E\left[C_{t+1}^{-\eta}(\rho Z_{t+1}K_t^{\rho-1}+(1-\delta))\right]$$

As we want to solve the model for five variables (C,K,Y,R,Z) we need

[1] Available at http://www2.wiwi.hu-berlin.de/institute/wpol

[2] Don't forget to put the folder with the toolbox on Matlab's search path either by the command *addpath* – then put it on the top of the search path by the argument -begin – or by copying the toolbox in your working directory.

five equations[3], which are

$$R_t = \rho Z_t K_{t-1}^{\rho-1} + (1-\delta) \quad (1.1)$$
$$C_t = Y_t + (1-\delta)K_{t-1} - K_t \quad (1.2)$$
$$Y_t = Z_t K_{t-1}^{\rho} \quad (1.3)$$
$$1 = \beta E\left[\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \quad (1.4)$$
$$\log Z_t = \psi \log \bar{Z} + (1-\psi)\log Z_{t-1} + \varepsilon \quad (1.5)$$

Eliminating the time subscripts we can solve for steady state[4]

$$\bar{K} = \left(\frac{\rho \bar{Z}}{\bar{R} - 1 + \delta}\right)^{\frac{1}{1-\rho}} \quad (1.6)$$
$$\bar{Y} = \bar{Z}\bar{K}^{\rho} \quad (1.7)$$
$$\bar{C} = \bar{Y} - \delta \bar{K} \quad (1.8)$$
$$\bar{R} = \rho \bar{Z}\bar{K}^{\rho-1} + (1-\delta) \quad (1.9)$$
$$1 = \beta \bar{R} \quad (1.10)$$

To use the Uhlig toolbox we have to linearize the system of equations (1.1)-(1.5) first. Define the lowercase variables as $c_t \equiv \log C_t - \log \bar{C}$, similarly for all the other variables. By construction we have $C_t = \bar{C}e^{c_t}$ and using the first order Taylor approximation[5] it follows $\bar{C}e^{c_t} \approx \bar{C}(1 + c_t)$.

Using this substitution in equation (1.1) gives

$$\bar{R} + \bar{R}r_t \approx \rho \bar{Z}\bar{K}^{\rho-1} + (1-\delta) + \rho \bar{Z}\bar{K}^{\rho-1}(z_t + (\rho-1)k_{t-1})$$

using (1.9)

$$\bar{R}r_t \approx (\bar{R} - (1-\delta))(z_t + (\rho-1)k_{t-1})$$

which after substituting (1.10) becomes

$$r_t \approx (1 - \beta(1-\delta))z_t - (1-\beta(1-\delta))k_{t-1} \quad (1.11)$$

[3]We could alternatively use smaller number of equations and compute the other variables later substituting the solution. However, the algorithm can handle more equations easily and there is thus no reason to reduce the system. It is,in fact, convenient to solve for all the variables simultaneously.

[4]$\bar{Z}$ is an arbitrary parameter.

[5]Recall that $e^x \approx e^0 + e^0(x-0)$.

The second equation is linearized as follows

$$\bar{C} + \bar{C}c_t \approx \bar{Z}\bar{K}^{\rho} + \bar{Z}\bar{K}^{\rho}(z_t + \rho k_{t-1}) + (1-\delta)\bar{K} + (1-\delta)\bar{K}k_{t-1} - \bar{K}k_t - \bar{K}$$

using (1.8)

$$c_t \approx \frac{\bar{Y}}{\bar{C}} z_t + \left(\rho \frac{\bar{Y}}{\bar{C}} + (1-\delta)\frac{\bar{K}}{\bar{C}}\right) k_{t-1} - \frac{\bar{K}}{\bar{C}} k_t$$

further, using the fact that $\frac{\bar{K}}{\bar{C}}\left[\rho\frac{\bar{Y}}{\bar{C}} + (1-\delta)\right] = \frac{\bar{K}}{\bar{C}}\bar{R}$ we get

$$c_t \approx \left(1 + \frac{\delta\bar{K}}{\bar{C}}\right) z_t + \frac{\bar{K}}{\bar{C}\beta} k_{t-1} - \frac{\bar{K}}{\bar{C}} k_t \tag{1.12}$$

The third equation becomes

$$\bar{Y} + \bar{Y}y_t \approx \bar{Z}\bar{K}^{\rho} + \bar{Z}\bar{K}^{\rho}(z_t + \rho k_{t-1})$$

which reduces to

$$y_t \approx z_t + \rho k_{t-1} \tag{1.13}$$

The Euler Equation yields

$$\begin{aligned} 1 &\approx \beta E\left[(\frac{\bar{C}e^{c_t}}{\bar{C}e^{c_{t+1}}})^{\eta} \bar{R}(1 + r_{t+1})\right] \\ 1 &\approx E\left[\beta\bar{R}(1 + \eta(c_t - c_{t+1}) + r_{t+1} + \eta(c_t - c_{t+1})r_{t+1})\right] \end{aligned}$$

since the last term in the brackets is negligible and using again (1.10) we get

$$0 \approx E\left[\eta(c_t - c_{t+1}) + r_{t+1}\right] \tag{1.14}$$

The last equation can be transformed as

$$\begin{aligned} \log(\bar{Z}e^{z_t}) &= (1-\psi)\log\bar{Z} + \psi\log(\bar{Z}e^{z_{t-1}}) + \varepsilon_t \\ \log\bar{Z} + z_t &= \log\bar{Z} + \psi z_{t-1} + \varepsilon_t \end{aligned}$$

$$z_t = \psi z_{t-1} + \varepsilon_t \tag{1.15}$$

The last step before running the toolbox is to rewrite the system of linear equations (1.11)-(1.15) in the form

$$\begin{aligned} 0 &= \mathbf{Ax(t)} + \mathbf{Bx(t-1)} + \mathbf{Cy(t)} + \mathbf{Dz(t)} \\ 0 &= E_t\left[\mathbf{Fx(t+1)} + \mathbf{Gx(t)} + \mathbf{Hx(t-1)} + \mathbf{Jy(t+1)} + \mathbf{Ky(t)} + \mathbf{Lz(t+1)} + \mathbf{Mz(t)}\right] \\ \mathbf{z(t+1)} &= \mathbf{Nz(t)} + \varepsilon(t+1) \end{aligned}$$

where x is the vector of endogenous state variables, y is the vector of other endogenous variables, and z is the vector of exogenous state variables. Obviously, $\mathbf{x} = k$, $\mathbf{y} = [c, r, y]$, and $\mathbf{z} = z$. After feeding the toolbox with the required matrices we are done.

2 Example 1 – Hansen's RBC Model

The model we want to solve reads

$$\max_{\{C_t,K_t,N_t,I_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty}\beta^t\left(\log C_t - AN_t\right)\right],$$

s.t.

$$\begin{aligned}
C_t + I_t &= Y_t \\
Y_t &= Z_t K_{t-1}^{\rho} N_t^{1-\rho} \\
K_t &= I_t + (1-\delta)K_{t-1} \\
\log(Z_t) &= (1-\psi)\log(\bar{Z}) + \psi\log(Z_{t-1}) + \varepsilon_t, \\
\varepsilon_t &\sim \text{i.i.d. } \mathrm{N}(0,\sigma_{\varepsilon}^2)
\end{aligned}$$

The first order conditions yield

$$\begin{aligned}
A &= \frac{1}{C_t}(1-\rho)Z_t K_{t-1}^{\rho} N_t^{-\rho} \\
\frac{1}{C_t} &= \beta E\left[\frac{1}{C_{t+1}}\left(\rho Z_{t+1} K_t^{\rho-1} N_{t+1}^{1-\rho} + (1-\delta)\right)\right]
\end{aligned}$$

We have the following seven equations describing the model

$$C_t = Y_t - I_t \tag{2.1}$$

$$K_t = I_t + (1-\delta)K_{t-1} \tag{2.2}$$

$$Y_t = Z_t K_{t-1}^{\rho} N_t^{1-\rho} \tag{2.3}$$

$$A = \frac{1}{C_t}(1-\rho)Z_t K_{t-1}^{\rho} N_t^{-\rho} \tag{2.4}$$

$$R_t = \rho Z_t K_{t-1}^{\rho-1} N_t^{1-\rho} + (1-\delta) \tag{2.5}$$

$$1 = \beta E\left[\left(\frac{C_t}{C_{t+1}}\right)^{\eta} R_{t+1}\right] \tag{2.6}$$

$$\log Z_t = \psi\log\bar{Z} + (1-\psi)\log Z_{t-1} + \varepsilon \tag{2.7}$$

The steady state values are given by[6]

$$
\begin{aligned}
\bar{I} &= \delta \bar{K} && (2.8)\\
\bar{C} &= \bar{Y} - \delta \bar{K} && (2.9)\\
\bar{Y} &= \bar{Z}\bar{K}^{\rho}\bar{N}^{1-\rho} && (2.10)\\
\bar{R} &= \frac{1}{\beta} && (2.11)\\
\frac{\bar{Y}}{\bar{K}} &= \frac{\rho}{\bar{R}+\delta-1} \qquad \text{(from (2.10))} && (2.12)\\
\bar{K} &= \left(\frac{\frac{\bar{Y}}{\bar{K}}}{\bar{Z}}\right)^{\frac{1}{\rho-1}} \bar{N} \qquad \text{(from (2.3))} && (2.13)\\
A &= \frac{(1-\rho)\bar{Y}}{\bar{C}\bar{N}} \qquad \text{(from (2.4) and using (2.10))} && (2.14)
\end{aligned}
$$

Now we have to linearize the system of equations (2.1)-(2.7). By the same procedure as in the previous example the first three equations simply

$$
\begin{aligned}
\bar{C}c_t &\approx \bar{Y}y_t - \bar{I}i_t && (2.15)\\
\bar{K}k_t &\approx \bar{I}i_t + (1-\delta)\bar{K}k_{t-1} && (2.16)\\
y_t &\approx z_t + \rho k_{t-1} + (1-\rho)n_t && (2.17)
\end{aligned}
$$

Using eq. (2.17) the eq. (2.4) becomes

$$
c_t \approx y_t - n_t \qquad (2.18)
$$

The expression for interest rate (eq. (2.5)) can be approximated as

$$
\bar{R} + \bar{R}r_t \approx \rho\bar{Z}\bar{K}^{\rho 1-}\bar{N}^{1-\rho} + (1-\delta) + \rho\bar{Z}\bar{K}^{\rho 1-}\bar{N}^{1-\rho}(z_t + (\rho-1)k_{t-1} + (1-\rho)n_t)
$$

which after using eqs. (2.11) and (2.17) becomes

$$
\bar{R}r_t \approx \rho\frac{\bar{Y}}{\bar{K}}y_t - \rho\frac{\bar{Y}}{\bar{K}}k_{t-1} \qquad (2.19)
$$

The last two equations are the same as in the previous example and they read

$$
\begin{aligned}
0 &\approx E\left[\eta(c_t - c_{t+1}) + r_{t+1}\right] && (2.20)\\
z_t &= \psi z_{t-1} + \varepsilon_t && (2.21)
\end{aligned}
$$

The vectors $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ now become $\mathbf{x} = k$, $\mathbf{y} = [c, y, n, r, i]$, and $\mathbf{z} = z$.

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[6]Following Uhlig's example we take $\bar{N}$ as a parameter and solve for the steady state value of A. Usually, however, A is calibrated and $\bar{N}$ is derived. This would give $\bar{N} = \frac{1-\rho}{A}\frac{\frac{\bar{Y}}{\bar{K}}}{\frac{\bar{Y}}{\bar{K}}-\delta}$.