MATH 164, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

Honors Project 9: Escape Velocity

If a ball is thrown straight up into the air, it will travel upward, stop, and then fall back to earth. If a bullet is shot straight up into the air, it will travel upward, stop (at an altitude higher than the ball, unless the person who threw the ball was a real zealot), and then fall back to earth. In this project we discuss the problem of determining the velocity with which an object needs to be propelled straight up into the air so that the object goes into orbit; that is, so that it never falls back to earth. This is the escape velocity for the earth. (We add the phrase "for the earth" since the escape velocity, as we will see, depends on the mass of the body — whether it be a planet, or moon, or ... — from which the object is propelled.)

We make two simplifying assumptions in our derivation: First, that there is no air resistance. Second, that the object has constant mass throughout its flight. This second assumption is significant since it means our analysis does *not* apply to a rocket, for example, since the mass of the rocket changes as the rocket burns fuel.

Narrative

Newton's Second Law of Motion states that if an object with mass m travels along a straight-line path parametrized by $x = x(t)$ then the force which it will exert on a particle at a point along its path is given by $F = ma$ where $a = d^2x/dt^2$. Since a may also be written

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

we may rewrite Newton's Second Law

$$F = mv\frac{dv}{dx}. \tag{1}$$

Now the Universal Law of Gravitation states that the direction of the force of gravity is along a line joining the centers of mass of the earth and the object, and that the magnitude of this force is

$$-\frac{GM_Em}{(R_E+x)^2}$$

where G is the universal gravitational constant, M_E is the mass of the earth, R_E is the radius of the earth, and x is the distance of the object above the surface of the earth. The fact that the weight w (in pounds) of an object near the surface of the earth is given by $w = mg$ where m is the mass (in slugs) of the object and $g = 32$ ft/sec^2, is based on an approximation. For an object of mass m a distance x above the surface of the earth,

$$w = \frac{GM_Em}{(R_E+x)^2} \approx \frac{GM_Em}{R_E^2} = \frac{GM_E}{R_E^2}m = mg$$

where $g = GM_E/R_E^2$ and R_E is the radius of the earth. This approximation is justified by the fact that — in certain applications — x is generally much smaller than $R_E \approx 3963$ mi, so $R_E + h \approx R_E$. In any case,

$$g = \frac{GM_E}{R_E^2}$$

implies that $gR_E^2 = GM_E$, so we can write the magnitude of gravitational force

$$-\frac{GM_Em}{(R_E+x)^2} = -\frac{gR_E^2m}{(R_E+x)^2}. \tag{2}$$

Since the force given by (1) must be balanced by the gravitational force given by (2) for an object being propelled vertically into the air, it must follow that

$$-\frac{gR_E^2 m}{(R_E + x)^2} = mv\frac{dv}{dx} \quad \text{or} \quad -\frac{gR_E^2 m}{(R_E + x)^2}\ dx = mv\ dv.$$

If we integrate this equation from ground level (at which $x = 0$ and v_0 is initial velocity) to the point at which the object is no longer acted upon by the gravitational force of the earth (at which $x = \infty$ and $v = 0$)

$$\int_0^\infty -\frac{gR_E^2 m}{x^2}\ dx = \int_{v_0}^0 mv\ dv, \tag{3}$$

we obtain an equation which we can solve for the escape velocity v_0: Indeed, upon integrating, we find that

$$-gR_E = -\frac{1}{2}v_0^2, \tag{4}$$

so

$$v_0 = \sqrt{2gR_E}.$$

Tasks

1. Verify equation (4).
2. Assuming that $g = 9.8$ m/sec^2, $R_E \approx 6.38 \times 10^6$ meters, and $G = 6.67 \times 10^{-11}\mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2$:

   a) What is the mass M_E of the earth in kg? In tons?

   b) What is the escape velocity v_0 from the earth in meters per second? In miles per hour?

3. Repeat the above computations to find the escape velocity for the moon. (You will have to use references to find quantities such as the radius of the moon.)