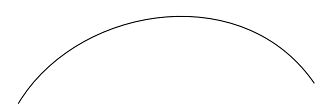
MATH 164, Spring 2001 **Due Date:____________** **Name(s): ________________________**

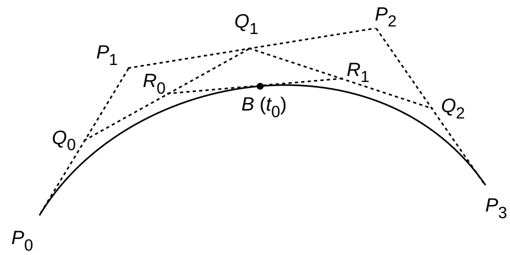
Honors Project 8a: Bezier Curves

Bezier curves (such as the one to the right) are frequently used in computer drawing programs. A curves such as this is drawn by locating numerous points along it and "connecting the dots" (the assumption being that the eye cannot distinguish between a smooth curve and a many-sided polygonal arc). In this project we describe an algorithm — the *deCastlejau algorithm* — for constructing Bezier curves.



Narrative

To construct a Bezier curve segment of degree n we begin by selecting $n+1$ points in the plane. The first and last points are the endpoints of the curve segment; the intermediate points help "guide the curve", but they usually do not lie on it. The points $n+1$ are called *control points* of the Bezier curve, and the segments joining them in order form its *control polygon.* In this project we consider the case $n = 3$, although everything we say is true if $n = 2$ or $n > 3$.



Suppose the points we have selected are $P_0(x_0, y_0), \ldots, P_3(x_3, y_3)$. The deCastlejau algorthm for finding the point on the curve that corresponds to the parameter value $t = t_0, 0 \le t_0 \le 1$, is based on linear interpolation, and, indeed, applying it repeatedly: A parametrization of the segment P_0P_1 is given by

$$x = (1-t)x_0 + tx_1, y = (1-t)y_0 + ty_1,$$

where $t \in [0, 1]$. We can abbreviate the above notation by writing $(1-t)P_0 + tP_1$, meaning we perform the indicated operation on both components. For a given parameter value t_0 we find a point Q_0 on P_0P_1 that is t_0 of the distance from P_0 to P_1:

$$Q_0 = (1-t_0)P_0 + t_0P_1.$$

(This process is known as *linear interpolation.*) Similarly we find points on P_1P_2 and P_2P_3:

$$Q_1 = (1-t_0)P_1 + t_0P_2 \quad \text{and} \quad Q_2 = (1-t_0)P_2 + t_0P_3.$$

Then we interpolate again using the same parameter value t_0 to find points R_0 and R_1 on Q_0Q_1 and Q_1Q_2:

$$R_0 = (1-t_0)Q_0 + t_0Q_1 \quad \text{and} \quad R_1 = (1-t_0)Q_1 + t_0Q_2.$$

The point $B(t_0)$ on the curve we are constructing, is

$$B(t_0) = (1-t_0)R_0 + t_0R_1.$$

Tasks

1. Choose 4 points in the plane and carry out this construction for the values $t_0 = 1/3$ and $t_0 = 2/3$.
2. Show that the expanded and simplified formula for $B(t)$ is
$$B(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3.$$
Confirm that the curve starts at P_0 when $t = 0$ and ends at P_3 when $t = 1$.
3. a) Compute the derivative $B'(t)$.
   b) Find the derivatives at $t = 0$ and $t = 1$.
   c) Find the lines tangent to the curve at the beginning and ending points.
   d) Show that the tangent line at S is the line through R_0 and R_1.
4. Carry out the deCastlejau construction for the case $n = 2$, with 3 points.
5. Find the formula for $B(t)$ in the case $n = 2$.
6. Show that if $n = 2$ then $B(1/2)$ lies midway between P_1 and M, the midpoint of segment P_0P_2. This point is called the *shoulder point* of the curve.
7. Show that for $n = 2$ the curve is an arc of a parabola from P_0 to P_2. (*Hint*: Consider eliminating the parameter t.)

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