MATH 163, Spring 2001 **Due Date:**____________ **Name(s):** ________________________

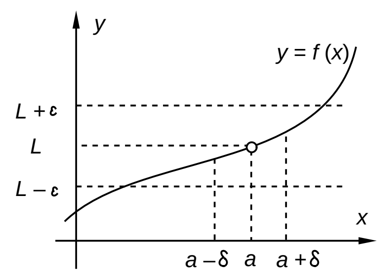
Project 2.4a: The Precise Definition of a Limit

Objective

To investigate the precise definition of limit.

Narrative

If you have not already done so, read Section 2.4 of the text.



To prove that the number L which we *guess* to be the limit of $f(x)$ at $x = a$ *really is* the limit of $f(x)$ at $x = a$, we must verify the condition in the formal definition of limit. This condition requires that for each real number $\epsilon > 0$, there is a real number $\delta > 0$ such that the values of $f(x)$ for all x in the interval $(a - \delta, a + \delta)$ — except possibly at $x = a$ itself — lie between $L - \epsilon$ and $L + \epsilon$. In this project we investigate the graphical implications of this condition.

In this project we also introduce the command `with(<package name>)` which loads the package of routines named <*package name*> into Maple. We also illustrate how to draw a line segment from the point $P(a, b)$ to the point $Q(c, d)$ using the command `<segment name> := [[a,b],[c,d]]`, and the use of plotting options such as `color=blue` and `scaling=constrained`. Finally, we illustrate "delayed plotting" and the use of the `plots` command `display`.

Task

a) Type the command lines below into Maple in the order in which they are listed. These commands are concerned with $\lim_{x\to 2}(-x^3/12 + x^2/2 + 5/3)$. Note how we terminate the `plot0 := ...` and `plot1 := ...` lines with a colon ":" rather than a semicolon ";": this suppresses the immediate display of these plot structures; later we can display them using the `plots` command `display`.

```
> # Project 2.4a:  The Precise Definition of a Limit
> restart;
> with(plots):
> f := x -> -x^3/12+x^2/2+5/3;
> a := 2.0;
> L := limit(f(x),x=a);
> xeqa := [[a,0],[a,4]];
> plot0 := plot({f(x),L,xeqa},x=-1..4,y=0..4,color=blue,scaling=constrained):
> display(plot0);
> e := 0.5;
> plot1 := plot({L-e,L+e},x=-1..4,y=0..4,color=red,scaling=constrained):
> display({plot0,plot1});
```

b) Continue by typing the command lines below into Maple in the order in which they are listed.

```
> e := 0.2;
> plot1 := plot({L-e,L+e},x=-1..4,y=0..4,color=red,scaling=constrained):
> display({plot0,plot1});
```

At this point, make a hard-copy of your typed input and Maple's responses. Then, ...

c) Label the graphs of $y = f(x)$, $y = L$, $y = L \pm e$, and $x = a$ on the second graphic you produced in part (a) by hand. Estimate by eye and state a value of `d` for which the values of `f(x)` for all `x` in the interval `a-d..a+d` — except possibly at $x = a$ — lie between `L-e` and `L+e` when `e = 0.5`. Draw the lines whose equations are `x = a+d` and `x = a-d` by hand on the second graphic you drew in part (a).

d) Label the graphs of $y = f(x)$, $y = L$, $y = L \pm e$, and $x = a$ on the graphic you produced in part (b) by hand. Estimate by eye and state a value of `d` for which the values of `f(x)` for all `x` in the interval `a-d..a+d` — except possibly at $x = a$ — lie between `L-e` and `L+e` when `e = 0.2`. Draw the lines whose equations are `x = a+d` and `x = a-d` by hand on this graphic.

Comments

1. In this project we are *not* actually proving that $L = \lim_{x \to a} f(x)$. On one hand, we are just verifying that an appropriate `d` exists for *two* given `e`'s: to verify that $L = \lim_{x \to a} f(x)$, we would have to do this *for every* `e`, not just two, three, four, or any finite number of `e`'s. On the other hand, since Maple draws the graphs of functions by "connecting-the-dots", some significant behavior could occur *between* the dots that is not revealed by Maple, so we cannot trust Maple's graphics to be completely accurate. This is one of the big reasons the $\epsilon\delta$-analysis of limits is so important.
2. You have to tell Maple you want to use a package of routines, such as `plots`, by saying `with(plots)` only once at the beginning (or right after the `restart`) of a session — not every time you use a routine in the package.
3. If the same options are to be used several times in a Maple session, they can be specified once at the beginning of the session using the `setoptions` command and omitted thereafter (saving time and reducing typing). For example, if we had included the command

   ```
   > setoptions(color=blue,scaling=constrained):
   ```

   immediately after the `with(plots)` command, then in all subsequent plots we would have automatically declared `color=blue` and `scaling=constrained`.