New Definition of the Mean Value, the Contribution to the Determination of aggregate interest rate

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Abstract: This short note aimes to offer the conceptual approach to the definition of the mean value, which enable us to aggregated measured values in mathematical and economical models without quantitative misconduct, or state the relevant proxy values or comparation (funds, banks, ...)
Our definition of the man value involves the term objective function, which characterize the nature of object, which is taken to average.
There are also recalled two conceptions of definition of the mean: Azzel's one [1] and axiomtic one [2] booth buld on the assumption, tht mean is increasing function of all of its arguments. This paper also showes natural example of the mean which is not increasing function of all of its arguments.
Subject matter the is demonstrated on the case of the yields of pension funds.

Key words: Mean value, interest rate, aggregate interest rate.
MSC 2000 Classification: 26E60 Means 43A07 Means on groups, semigroups, etc.; amenable groups
JEL Classification: E400 - Money and Interest Rates: General (includes measurement and data) E430 - Determination of Interest Rates; Term Structure of Interest Rates E470 - Money and Interest Rates: Forecasting and Simulation O400 - Economic Growth and Aggregate Productivity:

1. General observation:

Funds are found of adducing of average of rate of return per some time periods. They mean the arithmetical mean. But there are only very few cases, in which the substitution of the values by their arithmetical mean does not produced a nonsense.

Let us suppose that the number of time periods is N, and rate of profit in i-th period is ξ_i. let us suppose, that mean value of rates is equal to K, hence:

$$\frac{1}{N}\sum_{i=1}^{N}\xi_i = K \tag{1}$$

overall rate of profit by all of the N time periodes is

$$\zeta = \prod_{i=1}^{N}(1+\xi_i) - 1 \tag{2}$$

If $N \geq 2$ equation (1) does not determine ξ_i in unique way. On the set of all ξ_i fulfilling (1) take the function ζ different values: its maximum is $(1+K)^N - 1$, and it is not bounded below at all, hence: *The knowledge of arithmetical mean admit to bounded the rate of the whole return above but not below. With the same arithmetical mean the rate of the whole return can be arbitrarily different.*

1.1. Proof: If $N = 2$ then if ξ_1 a ξ_2 are rates of return in two time periods po sobě jdoucí období ??? and ζ rate of return per two period and If K is the arithmetical mean of ξ_1 a ξ_2 we have:

$$\begin{gathered}(1+\xi_1)\cdot(1+\xi_2) = 1+\zeta \\ \frac{1}{2}\,\xi_1 + \frac{1}{2}\,\xi_2 = K\end{gathered} \tag{3}$$

hence

$$\begin{gathered}\xi_2 = -\xi_1 + 2\,K \\ \zeta = 2\,K - {\xi_1}^2 + 2\,\xi_1\,K\end{gathered} \tag{4}$$

The last equation gives attitude toward actual rate of return per two periods and rate of return per one of the period in the case of constant average rate of return K. The dependence is analytical so that we can find the maximum as the zero point of the derivative:

$$\frac{\partial}{\partial \xi_1} \zeta = -2\,\xi_1 + 2\,K = 0 \Longleftrightarrow \xi_1 = K \tag{5}$$

and

$$\frac{\partial^2}{\partial {\xi_1}^2} \zeta = -2 \tag{6}$$

so that if average rate of return is constant equal to K then:

- rate of return ζ have maximum $\zeta = (K+1)^2 - 1$, if $\xi_1 = \xi_2 = K$, i. e. if all of the rates are equal.
- If one of the rate is equal to zero $\xi_1 = 0$, then the second is equal to $\xi_2 = 2\,K$ a and the whole rate of return is $\zeta = 2\,K$.
- If one of the rate $\xi_1 = -1$ i. e. we lost all of the capital, it is enough, to second rate of return be $\xi_2 = 1 + 2\,K$ and the mean value will stay K, while whole rate of return will be $\zeta = -1$. It means that the investor lost all of the invested resources, but arithmetical mean of rates of the return is still K, this can show the marketing potential of arithmetical mean.
- further if $\lim_{\xi_1 \to \infty}$ then $\xi_2 = -\infty$, $\lim_{\xi_1 \to \infty} \zeta = -\infty$,
  if $\lim_{\xi_1 \to (-\infty)}$ then $\xi_2 = \infty$, $\lim_{\xi_1 \to (-\infty)} \zeta = -\infty$

A similar situation occurs also in the case when the number of time periods is more. If we know arithmetical mean K of the rates of return the rate of return per N time periods is

$$\left(1 + N\,K - \left(\sum_{i=1}^{N-1} \xi_i\right)\right) \left(\prod_{i=1}^{N-1} (1+\xi_i)\right) = 1 + \zeta \tag{7}$$

and the function ζ, with arguments made of $N-1$ rates of returns ξ_i has again maximum

$$\zeta = (1 + K/N)^N - 1 \tag{8}$$

in the point

$$(\xi_i)_{i=1}^N = (K)_{i=1}^n \tag{9}$$

and ξ is not bounded above.

- Present value of invested capital should be equal to 0 independently on the arithmetical mean of rates of return. It is necessary and sufficient if one of the rates is equal to -1.
- If one of a fund has (arithmetical) average rate of return highest than other, it does not necessary mean, that it has higher rate of return per all of the periods (that it brings higher profit). For the marketing point of view to show mean o rates of returns is better advertisement for the funds which have higher differences between rates of return.

We can see, that arithmetical mean does not gives to us any substantive information.

One can say, that we used bad mean. That usage of the geometrical mean, brings to us much more better result. If we should compute the mean using the rule:

$$\zeta = \left(\prod_{i=1}^{N} (1+\xi_i)\right)^{\left(\frac{1}{N}\right)} - 1 \tag{10}$$

expect of

$$\zeta = \frac{1}{N}\sum_{i=1}^{N} \xi_i \tag{11}$$

we should obtain the rate of return. Particularly funds with higher mean rate of return should have higher whole return.

2. Two general conceptions of mean:

The average value of two values is binary operation $\circ$. Geometrical mean as well as exponential mean, which plays role in financial mathematics too, can be watch as some generalisation of arithmetical mean for well choose function k in the form

$$x \circ y = \left(k^{(-1)}\right)\left(\frac{1}{2}\,\mathrm{k}\,(x) + \frac{1}{2}\,\mathrm{k}\,(y)\right) \tag{12}$$

General properties of this mean are described in [Azzel str. 229]. main result, which shows quite general character of this definition is:

2.2. Theorem: There exists a continuous and strictly monotonic function k which gives a value of a mean; (12) holds if, and only if, $\circ\colon I^2 \mapsto I$ is continuous and strictly increasing in booth variables, idempotent, commutative (symmetric) and medial (bisymetric) which means:

$$(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w) \tag{13}$$

In the Book P. S. Bullen, D. S. Mitrinović P. M. Vasić; Means and Their Inequalities, D. Reidel publishing Company, Dordecht, Boston., Lancaster, Tokyo, 1988, ISBN 90–277–2629–9, p. 372 is attempt to axiomatic definition of the mean:

- it is symmetric,
- homogeneous of degree 1,
- reflexive
- associative, $\mathrm{f}\,(a_1, \ldots, a_n) = \mathrm{f}\,(\mathrm{f}\,(a_1, \ldots, a_p), \ldots, \mathrm{f}\,(a_1, \ldots, a_p), a_{p+1}, \ldots, a_n)$
- increasing in each variable.

This definition is not goo enough for us, because of average interest rate of saving is not symmetrical (and the symmetry is not disrupt only by adding of weights).

The drawback of these conceptions is, that the mean is related only to the values, it is computed from but not to their meaning, which mens to this what we are going to do with the values.

(For instance: arithmetical mean has sense only for additive magnitudes it means magnitudes we are going to sum but common definition does not include this restriction.) Our conception is more simple, more general and does not have this drawback. The character of which are input of the mean is expressed by objective function (and the values are arguments of the function)

3. General definition of the mean:

3.3. Definition: Z is the mean value of $(z_i)_{i=1}^{n}$ *with respect to function* $\mathrm{F}\colon \coprod_{i\in I} X^i \to Y$ if

$$\mathrm{F}\left((z_i)_{i=1}^{n}\right) = \mathrm{F}\left((Z)_{i=1}^{n}\right) \tag{14}$$

(i. e. $F(z_1, z_2, \ldots, z_n) = F(z, z, \ldots, z)$).

If F is injection, then the mean value is determined by unique way.

If $\mathrm{Im}\,(f) = \mathrm{Im}\,(f|_\Delta)$, where Δ is diagonal: $\Delta = \{(x)_{i=1}^{n}\,|x \in X\}$, mean exists for every input values.

3.4. Example: If F is adding, we obtain aritmetical mean. If F is multiplying, we obtain geometrical mean.

Inthe case of an election, the votes are summed. Aritmetical mean of number of votes corespondes to proportional representation in parliment.

Rates of inflation does multyply to each other. If

$$\iota := [.01,\ .03,\ .02,\ .01,\ .03] \tag{15}$$

is the rate of inflation per 5 time periodes, the rate of inflation per all of the time is

$$\left(\prod_{j=1}^{5}(1+\iota_j)\right) - 1 = .10386857 \tag{16}$$

and if the rate of inflation should be the same per all of the time subintervals an should be equal to

$$\kappa := \left(\prod_{j=1}^{5}(1+\iota_j)\right)^{1/5} - 1 = .019960783 \tag{17}$$

then the rate of inflation per all of the time should be the same:

$$\left(\prod_{j=1}^{5} (1+\kappa)\right) - 1 = .10386857 \tag{18}$$

Hence if the objective function will be

$$\iota \mapsto \left(\prod_{j} (1+\iota_j)\right) - 1 \tag{19}$$

then the mean with respect to this function is either geometrical ones, computed on the values: $1+\iota_j$ or noname mean, computed on the values: ι_j.

4. Mean rate of return on pension funds:

Let us suppose, that we know rate of return of pension funds. We are looking for any mean rate of return, which admits to us compare this funds.

Let us suppose, that deposits of the savers are constant for a long of the time. We can suppose, that, if the deposit per year (together with state grants) are equal to x and if the rate of return, in consequence of dividing of profit in the year i is equal to ξ_i saver should have after the time N:

$$x \left(\sum_{j=1}^{N} \left(\prod_{i=1}^{j} (1+\xi_{N-i+1})\right)\right) \tag{20}$$

For s saver is important mean rate of return with respect to the function

$$(\xi)_{i=1}^{T} \to \left(\sum_{j=1}^{N} \left(\prod_{i=1}^{j} (1+\xi_{N-i+1})\right)\right) \tag{21}$$

using our definition, this rate of the return is ζ, fulfilling the equation

$$K = \sum_{j=1}^{N} \left(\prod_{i=1}^{j} (1+\xi_{N-i+1})\right) = \sum_{j=1}^{N} \left(\prod_{i=1}^{j} (1+\zeta)\right) = -\frac{-(1+\zeta)^{(N+1)}+1+\zeta}{\zeta} \tag{22}$$

It is an algebraical equation of the degree N.

If the time period is only one, the mean rate of return is equal to the rate of return per this period:

$$\zeta = -1 + K \tag{23}$$

If the number of time periods is equal to 2, the equation has two solutions, in general.

If the total remittance will be higher than $\frac{1}{4}$ of all saved remittances all of the solution will be real. In the case — it seem to be reasonable — when total remittance will be higher than double of saved remittances one solution will be positive an will be equal to

$$\zeta = -\frac{3}{2} + \frac{1}{2}\sqrt{1+4K} \tag{24}$$

In case of 3 periods, there is the only one real mean rate of return:

$$\zeta = 1/6\, \frac{\left(28+108\,K+12\,\sqrt{9+42\,K+81\,K^2}\right)^{2/3} - 8 - 8\sqrt[3]{28+108\,K+12\,\sqrt{9+42\,K+81\,K^2}}}{\sqrt[3]{28+108\,K+12\,\sqrt{9+42\,K+81\,K^2}}} \tag{25}$$

If the num,ber of time periodes is higher than four, we are not able to solve the equation (22) algebraicaly, and we have to solve it numericaly.

4.5. Example: There exists some pension funds in Czech repubnlic during the five year between the years 1999 and 2003. We wil concentrate onto these:

(1) ČSOB Progres, (2) Zemský PF, (3) PF KB, (4) ING, (5) Credit Suisse, (6) PF ČP, (7) Allianz, (8) Generali, (9) Nový ČP, (10) ČSOB Stabilita, (11) PF ČS, (12) Hornický PF; The following table show us their rate of return:

Funds	**rate of return (in %)**				
	1999	2000	2001	2002	2003
ČSOB Progres	7.7	5.6	3.9	4.3	4.3
Zemský PF	7.0	5.0	4.6	4.1	4.01
PF KB	7.2	4.9	4.4	4.6	3.4
ING	6	4.4	4.8	4	4
Credit Suisse	6.5	4.1	4.3	3.4	3.4
PF ČP	6.6	4.5	3.8	3.2	3.1
Allianz	6	3.8	4.4	3.7	3
Generali	5.3	3.6	4.6	4.1	3
Nový ČP	5.6	3.8	4.1	3.5	3.34
ČSOB Stabilita	6.1	4.2	3.2	3.0	2.34
PF ČS	4.4	4.2	3.8	3.5	2.64
Hornický PF	4.4	2	2.8	3.2	2.44

(26)

Českomoravská stavební spořitelna published for agents middleman papers for information. They ordered the pension funds using aritmetical mean, which are, as we have explained, irelevant (second row of following table, values are in percents). If we computed the mean with respect to objective function (ξ_i^k is rate of return of k-th fund at i-th time period):

$$\Phi\colon \left(\xi^k\right)_{i=1}^{5} \longmapsto \sum_{j=1}^{5}\left(\prod_{i=1}^{j}\left(1+\xi_{5-i+1}^{k}\right)\right) \tag{27}$$

(third column of followig table, values are in percents) we can see, that it happens in some casses, that funds with higher rate of return are behind funds with smaller rate of return:

Funds	**mean values of the rates of return**	
	arithemtical (*inappropriate*)	**vith respect to the Φ** (*appropriate*)
ČSOB Progres	5.16	4.638
Zemský PF	4.94	4.501
PF KB	4.90	4.394
ING	4.64	4.358
Credit Suisse	4.34	3.895
PF ČP	4.24	† 3.704
Allianz	4.18	† 3.787
Generali	4.12	† 3.857
Nový ČP	4.06	3.757
ČSOB Stabilita	3.76	‡ 3.203
PF ČS	3.70	‡ 3.438
Hornický PF	2.96	2.789

(28)

(†) We can see, that the fund Allianz, which stay on the 7th place had the rate of return better, than PF ČP which is on the 6th place and better rate of return than this two had thwe fund Generali, which stay behind booth others. Booth are in the papers presented as the funds with the same awarage rate of return equal to 4.2.

(‡) The fund ČSOB Stabilita, the papers in question are written for its propagation too, is presented as the fund with mean rate of return (arithmetic mean 3.8) higher than rate of return of fund PF ČS (arithmetic mean 3.7) but it should be presented (with mean 3.2) behind the fund PF ČS (mean 3.4)!

If we are going to interpreted the result, we have to determine the sensitivity on the change of interest rate. We can substitute the rates of return by mean rates of return we have computed. The relative change of saved money after time T if the interest rate is changed from ξ onto ζ, (i. e.. rate of the profit or loss on the whole saved quontity) is

$$\frac{(1+\zeta)^T\ \xi-\xi-(1+\xi)^T\ \zeta+\zeta}{\left((1+\xi)^T-1\right)\ \zeta}. \tag{29}$$

In our case is $T=5$ and the values for differen funds are in the same order as in the previous tables, they are in percents:

0	−0.27	−0.49	−0.56	−1.5	−1.8	−1.7	−1.5	−1.7	−2.8	−2.4	−3.6
0.27	0	−0.21	−0.29	−1.2	−1.6	−1.4	−1.3	−1.5	−2.6	−2.1	−3.4
0.49	0.21	0	−0.072	−0.99	−1.4	−1.2	−1.1	−1.3	−2.4	−1.9	−3.2
0.56	0.29	0.072	0	−0.92	−1.3	−1.1	−1.0	−1.2	−2.3	−1.8	−3.1
1.5	1.2	1.0	0.93	0	−0.38	−0.21	−0.075	−0.27	−1.4	−0.91	−2.2
1.9	1.6	1.4	1.3	0.38	0	0.17	0.31	0.11	−1.0	−0.53	−1.8
1.7	1.4	1.2	1.1	0.21	−0.17	0	0.14	−0.061	−1.2	−0.70	−2.0
1.6	1.3	1.1	1.0	0.075	−0.31	−0.14	0	−0.20	−1.3	−0.83	−2.1
1.8	1.5	1.3	1.2	0.28	−0.11	0.061	0.20	0	−1.1	−0.64	−1.9
2.9	2.6	2.4	2.3	1.4	1.0	1.2	1.3	1.1	0	0.47	−0.83
2.4	2.1	1.9	1.9	0.92	0.53	0.70	0.84	0.64	−0.47	0	−1.3
3.8	3.5	3.3	3.2	2.2	1.8	2.0	2.2	2.0	0.83	1.3	0

(30)

In the i-th row j-th column the percentage of how much should be the whole amount higher if we invested into j-th fund except of into i-th fund is written. So in the proper choose of the fund we should have approximately 4 percent more than in the improper choose.

5. Mean rate of return of simultaneous savings:

Suppose, that we have two accounts, each interested with different interest rate ξ_1 and ξ_2. We divide the capital in amount x_1+x_2 between them this way, than we will have in he first capital in amount x_1 and in the second x_2. Objective function — future value in time t is the function $\Psi_{x_1,x_2,t}\colon(\xi_1,\xi_2)\mapsto x_1\,(1+\xi_1)^t+x_2\,(1+\xi_2)^t$ and it depende on three parameters. Mean interest rate with respect to this function is the solution ζ of the equation:

$$x_1\,(1+\xi_1)^t+x_2\,(1+\xi_2)^t=(x_1+x_2)\ (1+\zeta)^t \tag{31}$$

hence

$$\zeta=\left(\frac{x_1\ (1+\xi_1)^t+x_2\ (1+\xi_2)^t}{x_1+x_2}\right)^{\left(\frac{1}{t}\right)}-1 \tag{32}$$

And it is generalized exponential mean. Worthy to note is its dependence on the time t, especially he limits $t\to\infty$.

$$\lim_{t\to\infty}\left(\frac{x_1\,(1+\xi_1)^t+x_2\,(1+\xi_2)^t}{x_1+x_2}\right)^{\left(\frac{1}{t}\right)}-1=\lim_{x\to\infty}e^{\ln\left(\frac{x_1\ (1+\xi_1)^t+x_2\ (1+\xi_2)^t}{x_1+x_2}\right)^{\left(\frac{1}{t}\right)}}-1=$$

$$=\lim_{x\to\infty}e^{\frac{\ln\left(x_1\ (1+\xi_1)^t+x_2\ (1+\xi_2)^t\right)-\ln(x_1+x_2)}{t}}-1 \tag{33}$$

using the L'Hospital rule we obtain

$$\lim_{t\to\infty} \frac{\ln\left(x_1\ (1+\xi_1)^t + x_2\ (1+\xi_2)^t\right) - \ln(x_1+x_2)}{t} =$$

$$= \lim_{t\to\infty} \frac{x_1\ (1+\xi_1)^t \ln(1+\xi_1) + x_2\ (1+\xi_2)^t \ln(1+\xi_2)}{x_1\ (1+\xi_1)^t + x_2\ (1+\xi_2)^t} \tag{34}$$

and after canceling by the term $(1+\xi_2)^t$we obtain:

$$\lim_{x\to\infty} \frac{x_1 \ln(1+\xi_1) \left(\frac{1+\xi_1}{1+\xi_2}\right)^t + x_2 \ln(1+\xi_2)}{x_1 \left(\frac{1+\xi_1}{1+\xi_2}\right)^t + x_2}. \tag{35}$$

Supposing: $\xi_1 < \xi_2$

$$\lim_{t\to\infty} \left(\frac{x_1\ (1+\xi_1)^t + x_2\ (1+\xi_2)^t}{x_1+x_2}\right)^{\left(\frac{1}{t}\right)} - 1 = \xi_2, \tag{36}$$

we obtain the greatest of both number. *If we save two parts of capital on the accounts with two different interest rates, there is — after sufficient time — the result aproximately the same as we should interested booth of the account with the higher interest rate. To be exact: the difference of interest rates* **per unit of the time** *which makes the same profit from the whole capital or from its part only goes to zero if time goes to infinity*

On the other hand:

$$\lim_{t\to 0} \zeta = t\,(1+\xi_1)^{\frac{x_1}{x_1+x_2}}\ (1+\xi_2)^{\frac{x_2}{x_1+x_2}} - 1 \tag{37}$$

which is generalized geometrical mean.

The same result we obtain also in the case of more than two acounts. If the rates of interest will be (ξ_i) an the initial state of accounts will be (x_i) then the state function will be

$$(\xi_i) \longmapsto \sum_{i=1}^{n} x_i\,(1+\xi_i)^t \tag{38}$$

and mean value (ξ_i) with respect to the function will be solution ζ of the equation

$$\sum_{i=1}^{n} x_i\,(1+\xi_i)^t = \sum_{i=1}^{n} x_i\,(1+\zeta)^t = (1+\zeta)^t \sum_{i=1}^{n} x_i \tag{39}$$

hence

$$\zeta = \left(\frac{\sum_{i=1}^{n} x_i\,(1+\xi_i)^t}{\sum_{i=1}^{n} x_i}\right)^{\frac{1}{t}} - 1 \tag{40}$$

and if $\xi_n = \max_i(\xi_i)$ we can show in the same way, that

$$\lim_{t\to\infty} \zeta = e^{\left(\lim_{t\to\infty} \frac{\left(\sum_{i=1}^{n-1} x_i \ln(1+\xi_i) \left(\frac{1+\xi_i}{1+\xi_n}\right)^t\right) + x_n \ln(1+\xi_n)}{\left(\sum_{i=1}^{n-1} x_i \left(\frac{1+\xi_i}{1+\xi_n}\right)^t\right) + x_n}\right)} - 1 = \xi_n = \max_i(\xi_i) \tag{41}$$

a

$$\lim_{t\to 0} \zeta = \prod_{i=1}^{n} (1+\xi_i)^{\frac{x_i}{\sum_{j=1}^{n} x_j}} \tag{42}$$

Principle of convergence to maximum: Aggregate interest rate converges to maximal interest rate (per unit of time) with time going to infinity.
The similar situation is the saving annuities:

6. Mean interest rate of saving:

Suppose, savings with annuities x_i in the moments $t \in \mathbb{N}$ with constants interest rates ξ_i. Objective function is sum of future values of savings with different interest rates:

$$\Phi_{(x_i)_{i=1}^k,N} := \left((\xi_i)_{i=1}^k\right) \longmapsto \sum_{i=1}^k \left(\sum_{j=0}^{N-1} x_i\ (1+\xi_i)^j\right) = \sum_{i=1}^k \frac{x_i\ \left((1+\xi_i)^N - 1\right)}{\xi_i} \tag{43}$$

saved amount, $(x_i)_{i=1}^k$ and number of savings with different interest rates k are parametrs Φ.

Mean interest rate of savings $\Xi = \Xi\left((x_i), N\right)$ with respect to function $\Phi_{(x_i)_{i=1}^k,N}$ is solution of equation:

$$xxx := \frac{\left(\sum_{i=1}^k x_i\right)\ \left((1+\Xi)^N - 1\right)}{\Xi} = \sum_{i=1}^k \frac{x_i\ \left((1+\xi_i)^N - 1\right)}{\xi_i} \tag{44}$$

We will investigate dependence on parametr N, i. e. on the number of saved ammounts. Princip of maximum hold again:

$$\lim_{t\to\infty} \Xi = \max_i (\xi_i) \tag{45}$$

6.6. Proof: Let

$$\eta = \eta\left(\xi_i,\, x_i,\, n,\, T\right) \tag{46}$$

be mean interest rate with respect to the function

$$\Phi := \xi \to \sum_{\tau=1}^n \left(\sum_{j=1}^k x_j\ (1+\xi_j)^{(T-\tau)}\right) \tag{47}$$

i. e. solutiobn of the equation

$$\sum_{\tau=1}^n \left(\sum_{j=1}^k x_j\ (1+\xi_j)^{(T-\tau)}\right) = \left(\sum_{j=1}^k x_j\right)\ \left(\sum_{\tau=1}^n (1+\eta)^{(T-\tau)}\right) \tag{48}$$

and let

$$\zeta\left(\xi_i,\, x_i,\, \tau,\, T\right) \tag{49}$$

be a mean interest rate with respect to the function

$$\Psi := \xi \to \sum_{j=1}^k x_j\ (1+\xi_j)^{(T-\tau)} \tag{50}$$

i. e. solutioon of the equation

$$\sum_{j=1}^k x_j\ (1+\xi_j)^{(T-\tau)} = (1+\zeta)^{(T-\tau)}\ \left(\sum_{j=1}^k x_j\right) \tag{51}$$

It is clear, that

$$\min\left(\zeta\left(\xi_i,\, x_i,\, \tau,\, T\right)\right)_{\tau=1}^n \le \eta\left(\xi_i,\, x_i,\, n,\, T\right) \le \max\left(\zeta\left(\xi_i,\, x_i,\, \tau,\, T\right)_{\tau=1}^n\right) \tag{52}$$

And using (41) we have

$$\lim_{T\to\infty} \min\left(\zeta\left(\ldots,\, \tau,\, T\right)_{\tau=1}^n\right) = \lim_{T\to\infty} \max\left(\zeta\left(\ldots,\, \tau,\, T\right)_{\tau=1}^n\right) = \max_i (\xi) \tag{53}$$

hence

$$\lim_{T\to\infty} \eta\left(\xi_i,\, x_i,\, n,\, T\right) \tag{54}$$

that is why

$$\lim_{T\to\infty}\eta\left(\xi_i,\,x_i,\,T,\,T\right)=\lim_{n\to\infty}\lim_{T\to\infty}\eta\left(\xi_i,\,x_i,\,n,\,T\right)=\lim_{n\to\infty}\max_i\left(\xi_i\right)=\max_i\left(\xi_i\right) \tag{55}$$

Q. e. d.

Mean interest rate of saving is less than mean inerest rate of interesting, but it has the same limit $t\to 0$ a $t\to\infty$.

Last example is the one where mean is not increasing function in all of the arguments.

6.7. Example: Building societies has one product which is called coping credit (překlenovací úvěr). Subject matter of the product is, that we pay the dept by payments x_1 with interest rate ξ_1 and at the sane time we save money (as saving or insurance,...) by remittance at quantiti x_2 with the interest rate ξ_2 both the time of duration N. After that we use the saved money to amortize part of the depth, than we pay the rest of the depth with interest rate ξ_3 by payments x_3 time of the length K.

For the evaluation of an expediency of a product like this is good to know the suitable mean value of the values ξ_1, ξ_2 and ξ_3. Present value of all of the payments, we are going to pay is

$$PV=x_1\sum_{t=0}^{N-1}\left(1+\xi_1\right)^t\left(1+\xi_2\right)^{-N}+x_2\sum_{t=1}^{N}\left(1+\xi_2\right)^{-t}+x_3\sum_{t=1}^{K}\left(1+\xi_3\right)^{-t}\left(1+\xi_2\right)^{-N}=$$
$$=\frac{\left(1+\xi_2\right)^{-N}x_1\left(\left(1+\xi_1\right)^N-1\right)}{\xi_1}-\frac{x_2\left(\left(\left(1+\xi_2\right)^{-1}\right)^N-1\right)}{\xi_2}-\frac{\left(\left(\left(1+\xi_3\right)^{-1}\right)^K-1\right)\left(1+\xi_2\right)^{-N}x_3}{\xi_3}$$

hence we are interested in mean value with respect to objective function

$$\left(\xi_1,\xi_2,\xi_3\right)\mapsto PV$$

and this is the solution of the equation

$$x_1\sum_{t=0}^{N-1}\left(1+\xi_1\right)^t\left(1+\xi_2\right)^{-N}+x_2\sum_{t=1}^{N}\left(1+\xi_2\right)^{-t}+x_3\sum_{t=1}^{K}\left(1+\xi_3\right)^{-t}\left(1+\xi_2\right)^{-N}=$$
$$=x_1\sum_{t=0}^{N-1}\left(1+\zeta\right)^t\left(1+\zeta\right)^{-N}+x_2\sum_{t=1}^{N}\left(1+\zeta\right)^{-t}+x_3\sum_{t=1}^{K}\left(1+\zeta\right)^{-t}\left(1+\zeta\right)^{-N}$$

If no one of the interest rate is equal to 0, the equation is equivalent to

$$\frac{\left(1+\xi_2\right)^{-N}x_1\left(\left(1+\xi_1\right)^N-1\right)}{\xi_1}-\frac{x_2\left(\left(\left(1+\xi_2\right)^{-1}\right)^N-1\right)}{\xi_2}-\frac{\left(\left(\left(1+\xi_3\right)^{-1}\right)^K-1\right)\left(1+\xi_2\right)^{-N}x_3}{\xi_3}=$$
$$=\frac{\left(1+\zeta\right)^{-N}x_1\left(\left(1+\zeta\right)^N-1\right)}{\zeta}-\frac{x_2\left(\left(\left(1+\zeta\right)^{-1}\right)^N-1\right)}{\zeta}-\frac{\left(\left(\left(1+\zeta\right)^{-1}\right)^K-1\right)\left(1+\zeta\right)^{-N}x_3}{\zeta}$$

We are not able solve this equation algebraically in general so we cannot write the explicit formula for mean, we are looking for, but for all possible choice of parameters we can solve it numerically.

There is an interesting fact, that this mean is reflective (mean value of $(\zeta,\,\zeta,\,\zeta)$ is (ζ)), but *it is not increasing in the first variable* (ξ_1): no other conception mentioned above reckon on that fact!

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