Econometrics 2 - Lecture 4

Univariate (part 2) and Multivariate Time Series Models

Contents

- ARCH and GARCH Models
- Dynamic Models
- Lag Structures
- ADL Models
- Models for Expectations
- Models with Non-stationary Variables
- Cointegration

ARCH Processes

Autoregressive Conditional Heteroskedasticity (ARCH):

- Special case of heteroskedasticity
- Volatility (error variance) show autoregressive behavior: large errors induce a period of large volatility
- Allows to model successive periods with high, other periods with low volatility
- Typical for asset markets like stock markets, in particular for high frequencies like daily data

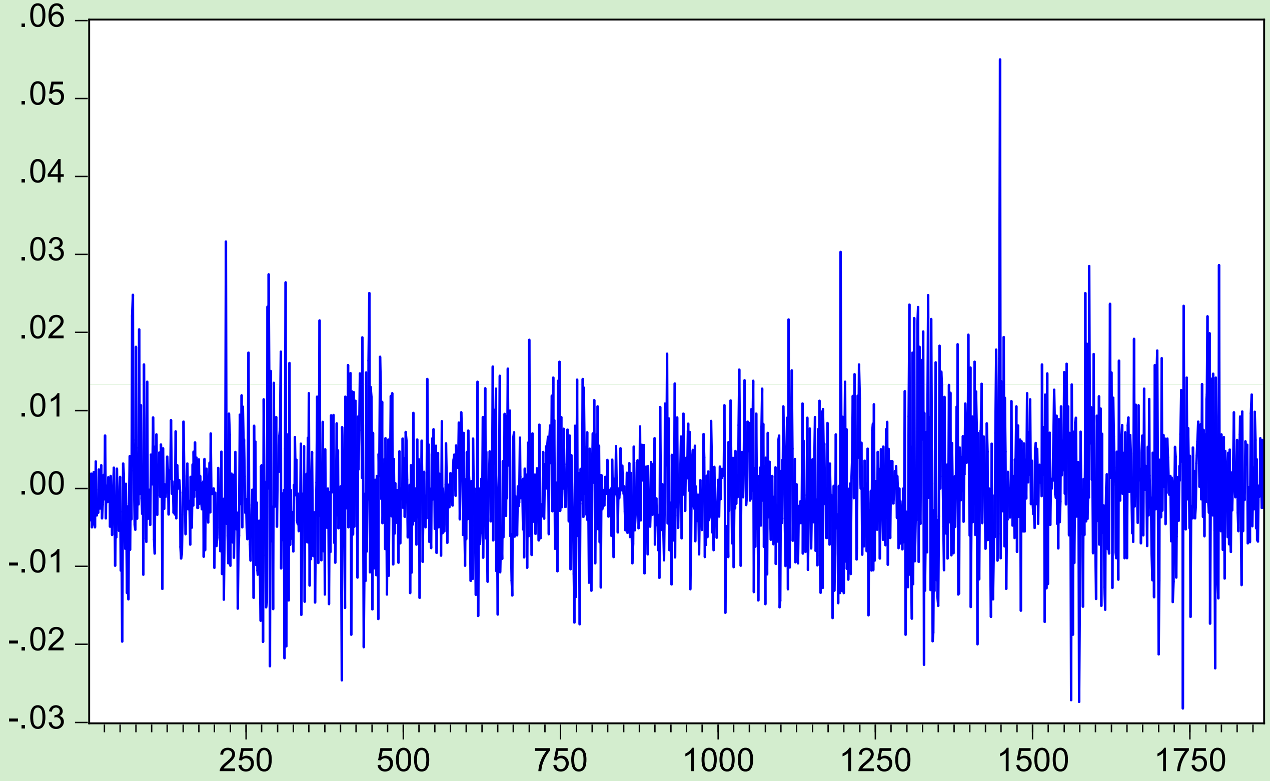
Idea: The variance of the errors (or innovations) ε_t is allowed to depend upon its history, follows an autoregressive process

ARCH models developed by Robert Engle in the 1980ies; Nobel Memorial Prize in Economic Sciences, 2003

Example: Exchange Rate

Verbeek's data set GARCH

1867 daily observations on exchange rates of the US dollar against the DM



log exchange rate USD/DM, daily changes

Exchange Rate: A Model

Daily log exchange rate y_t of the US dollar against the DM

$$y_t = \theta + \varepsilon_t$$

$$\varepsilon_t = \sigma_t v_t \quad \text{with } v_t \sim NID(0,1)$$

where σ_t^2 follows the ARCH model

$$\sigma_t^2 = E\{\varepsilon_t^2 | I_{t-1}\} = \varpi + \alpha\varepsilon_{t-1}^2$$

- Error terms ε_t are uncorrelated
- Volatility (error variance) show autoregressive behavior

The ARCH(1) Process

ARCH(1) process describes the conditional error variance, i.e., the variance conditional on information dated *t*-1 and earlier

$$\sigma_t^2 = E\{\varepsilon_t^2 | \mathcal{I}_{t-1}\} = \varpi + \alpha\varepsilon_{t-1}^2$$

- $\mathcal{I}_{t-1}$ is the information set containing all past including ε_{t-1}
- Conditions for $\sigma_t^2 \geq 0$: $\varpi \geq 0$, $\alpha \geq 0$
- A big shock at *t*-1, i.e., a large value $|\varepsilon_{t-1}|$,
  - induces high volatility, i.e., large σ_t^2
  - makes large values $|\varepsilon_t|$ more likely at *t* (and later)
- ARCH process does not imply correlation of the errors!

The unconditional variance of ε_t is

$$\sigma^2 = E\{\varepsilon_t^2\} = \varpi + \alpha E\{\varepsilon_{t-1}^2\} = \varpi/(1 - \alpha)$$

given that $0 \leq \alpha < 1$

- The ε_t process is stationary

ARCH-Model: Estimation

Model $y_t = x_t'\theta + \varepsilon_t$ with conditional error variance σ_t^2 following an ARCH process, i,.e., $\varepsilon_t = \sigma_t v_t$, $v_t \sim NID(0,1)$ with

$$\sigma_t^2 = E\{\varepsilon_t^2 | I_{t-1}\} = \varpi + \alpha\varepsilon_{t-1}^2$$

- Conditional upon I_{t-1}, $\varepsilon_t \sim N(0, \sigma_t^2)$
- Contribution of y_t to the likelihood function

$$f(y_t | x_t, \Im_{t-1}) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left\{-\frac{\varepsilon_t^2}{2\sigma_t^2}\right\}$$

  with $\varepsilon_t = y_t - x_t'\theta$ and $\sigma_t^2 = \varpi + \alpha\varepsilon_{t-1}^2$
- Estimates for θ, α and ϖ by maximizing the log likelihood function

Exchange Rate: ARCH Model

ARCH model for differences DMM of log exchange rate US dollar against DM

$y_t = \theta + \varepsilon_t$

$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$

Model 5: WLS (ARCH), using observations 3-1867 (T = 1865)
Dependent variable: DDM
Variable used as weight: 1/sigma

	coefficient	std. error	t-ratio	p-value
const	-5.487e-05	0.0001769	-0.3101	0.7565
alpha(0)	5.382e-05	3.1787e-06	16.93	6.17e-060 ***
alpha(1)	0.108035	0.0230333	4.690	2.93e-06 ***

Statistics based on the weighted data:

Sum squared resid	1848.942	S.E. of regression	0.995953
R-squared	0.000000	Adjusted R-squared	0.000000
Log-likelihood	-2638.257	Akaike criterion	5278.513
Schwarz criterion	5284.044	Hannan-Quinn	5280.552
rho	-0.059930	Durbin-Watson	2.119846

Statistics based on the original data:

Mean dependent var	-0.000020	S.D. dependent var	0.007770
Sum squared resid	0.112543	S.E. of regression	0.007770

3-Step Estimation Procedure

Model $y_t = x_t'\theta + \varepsilon_t$ with conditional error variance σ_t^2 following an ARCH process, i,.e., $\varepsilon_t = \sigma_t v_t$, $v_t \sim NID(0,1)$ with

$$\sigma_t^2 = E\{\varepsilon_t^2 | I_{t-1}\} = \varpi + \alpha\varepsilon_{t-1}^2$$

Estimation of θ, α and ϖ in 3 steps

1. OLS estimation of the regression model, residuals e_t
2. Auxiliary regression of the squared residuals e_t^2 on its own lagged values
3. Weighted least squares estimation; weights are the reciprocals of the fitted error variances from the auxiliary regression

More ARCH Processes

Various generalizations

- ARCH(*p*) process
- GARCH(*p*,*q*) process, Generalized ARCH
- EGARCH or exponential GARCH
- Etc.

ARCH(*p*) process

$$\sigma_t^2 = \varpi + \alpha_1\varepsilon_{t-1}^2 + \ldots \alpha_p\varepsilon_{t-p}^2 = \varpi + \alpha(L)\varepsilon_{t-1}^2$$

with lag polynomial $\alpha(L)$ of order p-1

- Conditions for $\sigma_t^2 \geq 0$: $\varpi \geq 0$; $\alpha_i \geq 0$, $i = 1,\ldots p$
- Condition for stationarity: $\alpha(1) < 1$

GARCH Process

GARCH(p,q) process

- „Generalized ARCH“
- Similar to the ARMA representation of levels

$$\sigma_t^2 = \varpi + \alpha_1\varepsilon_{t-1}^2 + \dots \alpha_p\varepsilon_{t-p}^2 + \beta_1\sigma_{t-1}^2 + \dots + \beta_q\sigma_{t-q}^2 =$$

$$= \varpi + \alpha(L)\varepsilon_{t-1}^2 + \beta(L)\sigma_{t-1}^2$$

Example: GARCH(1,1)

$$\sigma_t^2 = \varpi + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

- “surprises“ $v_t = \varepsilon_{t-1}^2 - \sigma_t^2$

$$\varepsilon_t^2 = \varpi + (\alpha + \beta)\varepsilon_{t-1}^2 + v_t - \beta v_{t-1}$$

- i.e. ε_t^2 follow ARMA(1,1)
- v_t: uncorrelated, but heteroskedastic

EGARCH Process

EGARCH or exponential GARCH

$$\log \sigma_t^2 = \varpi + \beta \log \sigma_{t-1}^2 + \gamma\varepsilon_{t-1}/\sigma_{t-1} + \alpha|\varepsilon_{t-1}|/\sigma_{t-1}$$

- Asymmetric: for $\gamma < 0$
  - positive shocks reduce volatility
  - negative shocks incease volatility
- Allows for larger impacts on volatility
  - of drops in price („bad news“) than
  - increases in price („good news“)

Test for ARCH Processes

Null hypothesis of homoskedasticity, to be tested against the alternative ARCH(p)

1. Estimate the model of interest using OLS: residuals e_t
2. Auxiliary regression of squared residuals e_t^2 on a constant and p lagged e_t^2
3. Test statistic TR_e^2 with R_e^2 from the auxiliary regression, p-value from the chi squared distribution with p df

Exchange Rate: Test for Homoskedasticity

Auxiliary regression of squared residuals e_t^2 on a constant and e_{t-1}^2; residuals are differences of DMM from their mean ($y_t = \theta + \varepsilon_t$)

$y_t = \theta + \varepsilon_t$

$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$

Model 11: OLS, using observations 3-1867 (T = 1865)
Dependent variable: usq10

	coefficient	std. error	t-ratio	p-value
const	5.382e-05	3.17866e-06	16.93	6.20e-060 ***
usq10_1	0.108035	0.0230333	4.690	2.93e-06 ***

Mean dependent var	0.000060	S.D. dependent var	0.000124
Sum squared resid	0.000028	S.E. of regression	0.000123
R-squared	0.011671	Adjusted R-squared	0.011140
F(1, 1863)	21.99959	P-value(F)	2.93e-06
Log-likelihood	14139.08	Akaike criterion	-28274.16
Schwarz criterion	-28263.10	Hannan-Quinn	-28270.09
rho	-0.008175	Durbin's h	-3.352472

$TR_e^2 = (1865) \times (0.011671) = 21.77$; *p*-value = 3.08E-6

GARCH Models in GRETL

Model > Time Series > ARCH

- Estimates the specified model allowing for ARCH: (1) model estimated via OLS, (2) auxiliary regression of the squared residuals on its own lagged values, (3) weighted least squares estimation

Model > Time Series > GARCH

- Estimates a GARCH model, with or without exogenous regressors

Contents

The Lüdeke Model for Germany

1. Consumption function

   $C_t = \alpha_1 + \alpha_2 Y_t + \alpha_3 C_{t-1} + \varepsilon_{1t}$
2. Investment function

   $I_t = \beta_1 + \beta_2 Y_t + \beta_3 P_{t-1} + \varepsilon_{2t}$
3. Import function

   $M_t = \gamma_1 + \gamma_2 Y_t + \gamma_3 M_{t-1} + \varepsilon_{3t}$
4. Identity relation

   $Y_t = C_t + I_t - M_{t-1} + G_t$

with *C*: private consumption, *Y*: GDP, *I*: investments, *P*: profits, *M*: imports, *G*: governmental spending

Variables:

- Endogenous: *C*, *Y*, *I*, *M*
- Exogenous, predetermined: *G*, P_{-1}

Econometric Models

Basis is the multiple linear regression model

Model extensions

- Dynamic models, i.e., models which contain lagged variables
- Systems of regression relations, i.e., models which describe more than one dependent variable

Example: Lüdeke Model

- Consists of four dynamic equations
- for the four dependent variables *C*, *Y*, *I*, *M*

Dynamic Models: Examples

Demand model: describes the quantity Q demanded of a product as a function of its price P and the income Y of households

Demand is determined by

- Current price and current income (static model):

  $$Q_t = \beta_1 + \beta_2 P_t + \beta_3 Y_t + \varepsilon_t$$

- Current price and income of the previous period (dynamic model):

  $$Q_t = \beta_1 + \beta_2 P_t + \beta_3 Y_{t-1} + \varepsilon_t$$

- Current price and demand of the previous period (dynamic autoregressive model):

  $$Q_t = \beta_1 + \beta_2 P_t + \beta_3 Q_{t-1} + \varepsilon_t$$

The Dynamic of Processes

Static processes: immediate reaction to changes in regressors, the adjustment of the dependent variables to the realizations of the independent variables will be completed within the current period, the process seems to be always in equilibrium

Static models are often inappropriate

- Some processes are determined by the past, e.g., energy consumption depends on past investments into energy-consuming systems and equipment
- Actors in economic processes may respond delayed, e.g., time for decision-making and procurement processes exceeds the observation period
- Expectations: e.g., consumption depends not only on current income but also on the income expectations; modeling the expectation may be based on past development

Elements of Dynamic Models

- Lag structures, distributed lags: linear combinations of current and past values of a variable
- Models for expectations: based on lag structures, e.g., adaptive expectation model, partial adjustment model
- Autoregressive distributed lag (ADL) model: a simple but widely applicable model consisting of an autoregressive part and of a finite lag structure of the independent variables

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Example: Demand Functions

- Demand for durable consumer goods: demand Q depends on the price P and on the income Y of the current and two previous periods:

  $$Q_t = \alpha + \beta_0 Y_t + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \gamma P_t + \varepsilon_t$$

- Demand for energy:

  $$Q_t = \alpha + \beta P_t + \gamma K_t + u_t$$

  with P: price of energy, K: energy-related capital stock

  $$K_t = \theta_0 + \theta_1 P_{t-1} + \theta_2 P_{t-2} + \ldots + \delta Y_t + v_t$$

  with Y: income; substitution of K results in

  $$Q_t = \alpha_0 + \alpha_1 Y_t + \beta_0 P_t + \beta_1 P_{t-1} + \beta_2 P_{t-1} + \ldots + \varepsilon_t$$

  with $\varepsilon_t = u_t + \gamma v_t$, $\alpha_0 = \alpha + \gamma\delta$, $\beta_0 = \beta$, and $\beta_i = \gamma\theta_i$, $i = 1, 2, \ldots$

Models with Lag Structures

Distributed lag model: describes the delayed effect of one or more regressors on the dependent variable; e.g.,

- DL(*s*) model

  $$Y_t = \delta + \Sigma^s_{i=0} \varphi_i X_{t-i} + \varepsilon_t$$

  distributed lag of order *s* model

Topics of interest

- Estimation of coefficients
- Interpretation of parameters

Example: Consumption Function

Data for Austria (1976:1 – 1995:2), logarithmic differences:

$$\hat{C} = 0.009 + 0.621Y$$

with $t(Y) = 2.288$, $R^2 = 0.335$

DL(2) model, same data:

$$\hat{C} = 0.006 + 0.504Y - 0.026Y_{-1} + 0.274Y_{-2}$$

with $t(Y) = 3.79$, $t(Y_{-1}) = -0.18$, $t(Y_{-2}) = 2.11$, $R^2 = 0.370$

Effect of income on consumption:

- Short term effect, i.e., effect in the current period:

  $\Delta C = 0.504$, given a change in income $\Delta Y = 1$
- Overall effect, i.e., cumulative current and future effects

  $\Delta C = 0.504 - 0.026 + 0.274 = 0.752$, given a change in income $\Delta Y = 1$

Multiplier

Describes the effect of a change in explanatory variable X_t by $\Delta X = 1$ on current and future values of the dependent variable Y

DL(s) model: $Y_t = \delta + \varphi_0 X_t + \varphi_1 X_{t-1} + \ldots + \varphi_s X_{t-s} + \varepsilon_t$

- Short run or impact multiplier

$$\frac{\partial Y_t}{\partial X_t} = \varphi_0$$

effect of the change in the same period, immediate effect of $\Delta X = 1$ on Y: $\Delta Y = \varphi_0$

- Long run multiplier

Effect of $\Delta X = 1$ after 1, …, s periods:

$$\frac{\partial Y_{t+1}}{\partial X_t} = \varphi_1, \ldots, \frac{\partial Y_{t+s}}{\partial X_t} = \varphi_s$$

Cumulated effect of $\Delta X = 1$ at t over all future on Y: $\Delta Y = \varphi_0 + \ldots + \varphi_s$

Equilibrium Multiplier

If after a change ΔX an equilibrium occurs within a finite time: Long run multiplier is called equilibrium multiplier

- DL(s) model

  $$Y_t = \delta + \varphi_0 X_t + \varphi_1 X_{t-1} + \ldots + \varphi_s X_{t-s} + \varepsilon_t$$

  equilibrium after s periods

- No equilibrium for models with an infinite lag structure

Average Lag Time

Characteristic of lag structure

- Portion of equilibrium effect in the adaptation process
  - At the end of the period t:

    $$w_0 = \varphi_0/(\varphi_0 + \varphi_1 + \ldots + \varphi_s)$$

  - At the end of the period t +1:

    $$w_0 + w_1 = (\varphi_0 + \varphi_1)/(\varphi_0 + \varphi_1 + \ldots + \varphi_s)$$

  - Etc.

  with weights $w_i = \varphi_i/(\varphi_0 + \varphi_1 + \ldots + \varphi_s)$
- Average lag time: $\Sigma_i \, i \, w_i$
- Median lag time: time till 50% of the equilibrium effect is reached, i.e., minimal s^* with

  $$w_0 + \ldots w_{s^*} \geq 0.5$$

Consumption Function

For ΔY = 1, the function

$$\hat{C} = 0.006 + 0.504Y - 0.026Y_{-1} + 0.274Y_{-2}$$

gives

- Short run effect: 0.504
- Overall effect: 0.752
- Equilibrium effect : 0.752
- Average lag time: 0.694 quarters, i.e., ~ 2.3 months
- Median lag time: $s^* = 0$; cumulative sums of weights are 0.671, 0.636, 1.000

Lag Structures: Estimation

DL(*s*) model: problems with OLS estimation

- Loss of observations: for a sample size N, only N-s observations are available for estimation; infinite lag structure!
- Multicollinearity
- Order *s* (mostly) not known

Consequences:

- Large standard errors of estimates
- Low power of tests

Issues:

- Choice of *s*
- Models for the lag structure with smaller number of parameters, e.g., polynomial structure

Consumption Function

Fitted function

$$\hat{C} = 0.006 + 0.504Y - 0.026Y_{-1} + 0.274Y_{-2}$$

with p-value for coefficient of Y_{-2}: 0.039, adj.R^2 = 0.342, AIC = -5.204

Models for $s \leq 7$

s	AIC	p-Wert	adj.R^2
1	-5.179	0.333	0.316
2	-5.204	**0.039**	0.342
3	-5.190	0.231	0.344
4	**-5.303**	0.271	**0.370**
5	-5.264	0.476	0.364
6	-5.241	0.536	0.356
7	-5.205	0.884	0.342

Koyck's Lag Structure

Specifies the lag structure of the DL(*s*) model

$$Y_t = \delta + \Sigma^s_{i=0} \varphi_i X_{t-i} + \varepsilon_t$$

as an infinite, geometric series (geometric lag structure)

$$\varphi_i = \lambda_0(1 - \lambda)\lambda^i$$

- For $0 < \lambda < 1$

$$\Sigma^s_{i=0} \varphi_i = \lambda_0$$

- Short run multiplier: $\lambda_0(1 - \lambda)$
- Equilibrium effect: λ_0
- Average lag time: $\lambda/(1 - \lambda)$
- Stability condition $0 < \lambda < 1$

  for $\lambda > 1$, the φ_i and the contributions to the multiplier are exponentially growing

λ	0.1	0.3	0.5	0.7
$\lambda/(1-\lambda)$	0.10	0.43	1.00	2.33

The Koyck Model

- The DL (distributed lag) or MA (moving average) form of the Koyck model

  $$Y_t = \delta + \lambda_0(1 - \lambda) \Sigma_i \lambda^i X_{t-i} + \varepsilon_t$$

- AR (autoregressive) form

  $$Y_t = \delta(1 - \lambda) + \lambda Y_{t-1} + \lambda_0(1 - \lambda)X_t + u_t$$

  with $u_t = \varepsilon_t - \lambda\varepsilon_{t-1}$

Consumption Function

Model with smallest AIC:

$$\hat{C} = 0.003 + 0.595Y - 0.016Y_{-1} + 0.107Y_{-2} + 0.003Y_{-3} + 0.148Y_{-4}$$

with adj.R^2 = 0.370, AIC = -5.303, DW = 1.41

Koyck model in AR form

$$\hat{C} = 0.004 + 0.286\, C_{-1} + 0.556Y$$

with adj.R^2 = 0.388, AIC = -5.290, DW = 1.91

Koyck Model: Estimation Problems

Parameters to be estimated: δ, λ_0, and λ; problems are

- DL form:
  - Historical values X_0, X_{-1}, … are unknown
  - Non-linear estimation problem
- AR form
  - Non-linear estimation problem
  - Lagged, endogenous variable used as regressor
  - Correlated error terms

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The ADL(1,1) Model

- The autoregressive distributed lag (ADL) model: autoregressive model with lag structure, e.g., the ADL(1,1) model

$$Y_t = \delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1} + \varepsilon_t$$

- The error correction model:

$$\Delta Y_t = -(1-\theta)(Y_{t-1} - \alpha - \beta X_{t-1}) + \varphi_0 \Delta X_t + \varepsilon_t$$

obtained from the ADL(1,1) model with

$$\alpha = \delta/(1-\theta)$$

$$\beta = (\varphi_0 + \varphi_1)/(1-\theta)$$

Example:

- Sales S_t are determined
  - by advertising A_t and A_{t-1}, but also
  - by S_{t-1}:

$$S_t = \mu + \theta S_{t-1} + \beta_0 A_t + \beta_1 A_{t-1} + \varepsilon_t$$

$$\Delta S_t = -(1-\theta)[S_{t-1} - \mu/(1-\theta) - (\beta_0 + \beta_{01})/(1-\theta) A_{t-1}] + \beta_0 \Delta A_t + \varepsilon_t$$

Multiplier

ADL(1,1) model: $Y_t = \delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1} + \varepsilon_t$

Effect of a change $\Delta X = 1$ at time t

- Impact multiplier: $\Delta Y = \varphi_0$; see the DL(s) model
- Long run multiplier
  - Effect after one period

$$\frac{\partial Y_{t+1}}{\partial X_t} = \theta \frac{\partial Y_t}{\partial X_t} + \varphi_1 = \theta\varphi_0 + \varphi_1$$

  - Effect after two periods

$$\frac{\partial Y_{t+2}}{\partial X_t} = \theta \frac{\partial Y_{t+1}}{\partial X_t} = \theta\left(\theta\varphi_0 + \varphi_1\right)$$

  - Cumulated effect over all future on Y

$$\varphi_0 + (\theta\varphi_0 + \varphi_1) + \theta(\theta\varphi_0 + \varphi_1) + \ldots = (\varphi_0 + \varphi_1)/(1 - \theta)$$

decreasing effects requires $|\theta|<1$, stability condition

ADL(1,1) Model: Equilibrium

Equilibrium relation of the ADL(1,1) model:

- Equilibrium at time *t* means: $E\{Y_t\} = E\{Y_{t-1}\}$, $E\{X_t\} = E\{X_{t-1}\}$

  $$E\{Y_t\} = \delta + \theta\, E\{Y_t\} + \varphi_0\, E\{X_t\} + \varphi_1\, E\{X_t\}$$

  or, given the stability condition $|\theta|<1$,

  $$E\{Y_t\} = \frac{\delta}{1-\theta} + \frac{\varphi_0 + \varphi_1}{1-\theta} E\{X_t\}$$

- Equilibrium relation:

  $$E\{Y_t\} = \alpha + \beta\, E\{X_t\}$$

  with $\alpha = \delta/(1 - \theta)$, $\beta = (\varphi_0 + \varphi_1)/(1 - \theta)$

- Long run multiplier: change $\Delta X = 1$ of the equilibrium value of X increases the equilibrium value of Y by $(\varphi_0 + \varphi_1)/(1 - \theta)$

The Error Correction Model

ADL(1,1) model, written as error correction model

$$\Delta Y_t = \varphi_0 \Delta X_t - (1 - \theta)(Y_{t-1} - \alpha - \beta X_{t-1}) + \varepsilon_t$$

- Effects on ΔY
  - due to changes ΔX
  - due to equilibrium error, i.e., $Y_{t-1} - \alpha - \beta X_{t-1}$
- Negative adjustment: $Y_{t-1} < E\{Y_{t-1}\} = \alpha + \beta X_{t-1}$, i.e., a negative equilibrium error, increases Y_t by $-(1 - \theta)(Y_{t-1} - \alpha - \beta X_{t-1})$
- Adjustment parameter: $(1 - \theta)$
  - Determines speed of adjustment

The ADL(p,q) Model

ADL(p,q): generalizes the ADL(1,1) model

$$\theta(L)Y_t = \delta + \Phi(L)X_t + \varepsilon_t$$

with lag polynomials

$$\theta(L) = 1 - \theta_1 L - \ldots - \theta_p L^p \text{ , } \Phi(L) = \varphi_0 + \varphi_1 L + \ldots + \varphi_q L^q$$

Given invertibility of $\theta(L)$, i.e., $\theta_1 + \ldots + \theta_p < 1$,

$$Y_t = \theta(1)^{-1}\delta + \theta(L)^{-1}\Phi(L)X_t + \theta(L)^{-1}\varepsilon_t$$

The coefficients of $\theta(L)^{-1}\Phi(L)$ describe the dynamic effects of X on current and future values of Y

- equilibrium multiplier

$$\theta(1)^{-1}\phi(1) = \frac{\varphi_0 + \ldots + \varphi_q}{1 - \theta_1 - \ldots - \theta_p}$$

ADL(0,q): coincides with the DL(q) model; $\theta(L) = 1$

ADL Model: Estimation

ADL(*p*,*q*) model

- error terms ε_t: white noise, independent of $X_t, \ldots, X_{t-q}$ and $Y_{t-1}, \ldots, X_{t-p}$

OLS estimators are consistent

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Expectations in Economic Processes

Expectations play important role in economic processes

Examples:

- Consumption depends not only on current income but also on the income expectations; modeling the expectation may be based on past development
- Investments depend upon expected profits
- Interest rates depend upon expected development of the financial market
- Etc.

Expectations

- cannot be observed, but
- can be modeled using assumptions on the mechanism of adapting expectations

Models for Adapting Expectations

- Naive model of adapting expectations: the (for the next period) expected value equals the actual value
- Model of adaptive expectation
- Partial adjustment model

The latter two models are based on Koyck's lag structure

Adaptive Expectation: Concept

Models of adaptive expectation: describe the actual value Y_t as function of the value X^e_{t+1} of the regressor X that is expected for the next period

$$Y_t = \alpha + \beta X^e_{t+1} + \varepsilon_t$$

Example: Investments are a function of the expected profits

Concepts for X^e_{t+1}:

- Naive expectation: $X^e_{t+1} = X_t$
- More realistic is a weighted sum of in the past realized profits

$$X^e_{t+1} = \beta_0 X_t + \beta_1 X_{t-1} + \ldots$$

  - Geometrically decreasing weights β_i

    $\beta_i = (1-\lambda)\, \lambda^i$

    with $0 < \lambda < 1$

Adaptive Mechanism for the Expectation

With $\beta_i = (1- \lambda) \lambda^i$, the expected value $X^e_{t+1} = \beta_0 X_t + \beta_1 X_{t-1} + \ldots$ results in

$$X^e_{t+1} = \lambda X^e_t + (1 - \lambda) X_t$$

or

$$X^e_{t+1} - X^e_t = (1 - \lambda)(X_t - X^e_t)$$

Interpretation: the change of expectation between *t* and *t*+1 is proportional to the actual „error in expectation", i.e., the deviation between the actual expectation and the actually realized value

- Extent of the change (adaptation): $100(1 - \lambda)\%$ of the error
- λ: adaptation parameter

Models of Adaptive Expectation

- Adaptive expectation model (AR form)

  $$Y_t = \alpha(1 - \lambda) + \lambda Y_{t-1} + \beta(1 - \lambda)X_t + v_t$$

  with $v_t = \varepsilon_t - \lambda\varepsilon_{t-1}$; an ADL(1,0) model
- DL form

  $$Y_t = \alpha + \beta(1 - \lambda)X_t + \beta(1 - \lambda)\,\lambda\, X_{t-1} + \ldots + \varepsilon_t$$

Example: Investments (I) as function of the expected profits P^e_{t+1} and interest rate (r)

$$I_t = \alpha + \beta P^e_{t+1} + \gamma r_t + \varepsilon_t$$

- Assumption of adapted expectation for the profits P^e_{t+1}:

  $$P^e_{t+1} = \lambda P^e_t + (1 - \lambda)P_t$$

  with adaptation parameter λ ($0 < \lambda < 1$)
- AR form of the investment function ($v_t = \varepsilon_t - \lambda\varepsilon_{t-1}$):

  $$I_t = \alpha(1 - \lambda) + \lambda I_{t-1} + \beta(1 - \lambda)P_t + \gamma r_t - \lambda\gamma r_{t-1} + v_t$$

Consumption Function

Consumption as function of the expected income

$$C_t = \alpha + \beta Y^e_{t+1} + \varepsilon_t$$

expected income derived under the assumption of adapted expectation

$$Y^e_{t+1} = \lambda Y^e_t + (1 - \lambda) Y_t$$

- AR form is

$$C_t = \alpha(1 - \lambda) + \lambda C_{t-1} + \beta(1 - \lambda) Y_t + v_t$$

with $v_t = \varepsilon_t - \lambda\varepsilon_{t-1}$

Example: the estimated model is

$$\hat{C} = 0.004 + 0.286 C_{-1} + 0.556 Y$$

- adj.R^2 = 0.388, AIC = -5.29, DW = 1.91

Example: Desired Stock Level

Stock level *K* and revenues *S*

- The desired (optimal) stock level K^* depends of the revenues *S*

  $$K^*_t = \alpha + \beta S_t + \eta_t$$

- Actual stock level K_{t-1} in period *t*-1: deviates from K^*_t: $K^*_t - K_{t-1}$
- (Partial) adjustment strategy according to

  $$K_t - K_{t-1} = (1 - \theta)(K^*_t - K_{t-1})$$

  adaptation parameter θ with $0 < \theta < 1$
- Substitution for K^*_t gives the AR form of the model

  $$K_t = K_{t-1} + (1 - \theta)\alpha + (1 - \theta)\beta S_t - (1 - \theta)K_{t-1} + (1 - \theta)\eta_t$$

  $$= \delta + \theta K_{t-1} + \varphi_0 S_t + \varepsilon_t$$

  $\delta = (1 - \theta)\alpha$, $\varphi_0 = (1 - \theta)\beta$, $\varepsilon_t = (1 - \theta)\eta_t$
- Model for K_t is an ADL(1,0) model

Partial Adjustment Model

Describes the process of adapting to a desired or planned value Y^*_t as a function of regressor X_t

$$Y^*_t = \alpha + \beta X_t + \eta_t$$

- (Partial) adjustment of the actual Y_t according to

  $$Y_t - Y_{t-1} = (1 - \theta)(Y^*_t - Y_{t-1})$$

  adaptation parameter θ with $0 < \theta < 1$
- Actual Y_t: weighted average of Y^*_t and Y_{t-1}

  $$Y_t = (1 - \theta) Y^*_t + \theta Y_{t-1}$$
- AR form of the model

  $$Y_t = (1 - \theta)\alpha + \theta Y_{t-1} + (1 - \theta)\beta X_t + (1 - \theta)\eta_t$$
  $$= \delta + \theta Y_{t-1} + \varphi_0 X_t + \varepsilon_t$$

  which is an ADL(1,0) model

Models in AR Form

Models in ADL(1,0) form

1. Koyck's model

$$Y_t = \alpha (1 - \lambda) + \lambda Y_{t-1} + \beta(1 - \lambda)X_t + v_t$$

   with $v_t = \varepsilon_t - \lambda\varepsilon_{t-1}$

2. Model of adaptive expectation

$$Y_t = \alpha(1 - \lambda) + \lambda Y_{t-1} + \beta(1 - \lambda)X_t + v_t$$

   with $v_t = \varepsilon_t - \lambda\varepsilon_{t-1}$

3. Partial adjustment model

$$Y_t = (1 - \theta)\alpha + \theta Y_{t-1} + (1 - \theta)\beta X_t + \varepsilon_t$$

Error terms are

- White noise for partial adjustment model
- Autocorrelated for the other two models

Contents

- ARCH and GARCH Models
- Dynamic Models
- Lag Structures
- ADL Models
- Models for Expectations
- Models with Non-stationary Variables
- Cointegration

An Illustration

Independent random walks: $Y_t = Y_{t-1} + \varepsilon_{yt}$, $X_t = X_{t-1} + \varepsilon_{xt}$

ε_{yt}, ε_{xt}: independent white noises with variances $\sigma_y^2 = 2$, $\sigma_x^2 = 1$

Fitting the model

$Y_t = \alpha + \beta X_t + \varepsilon_t$

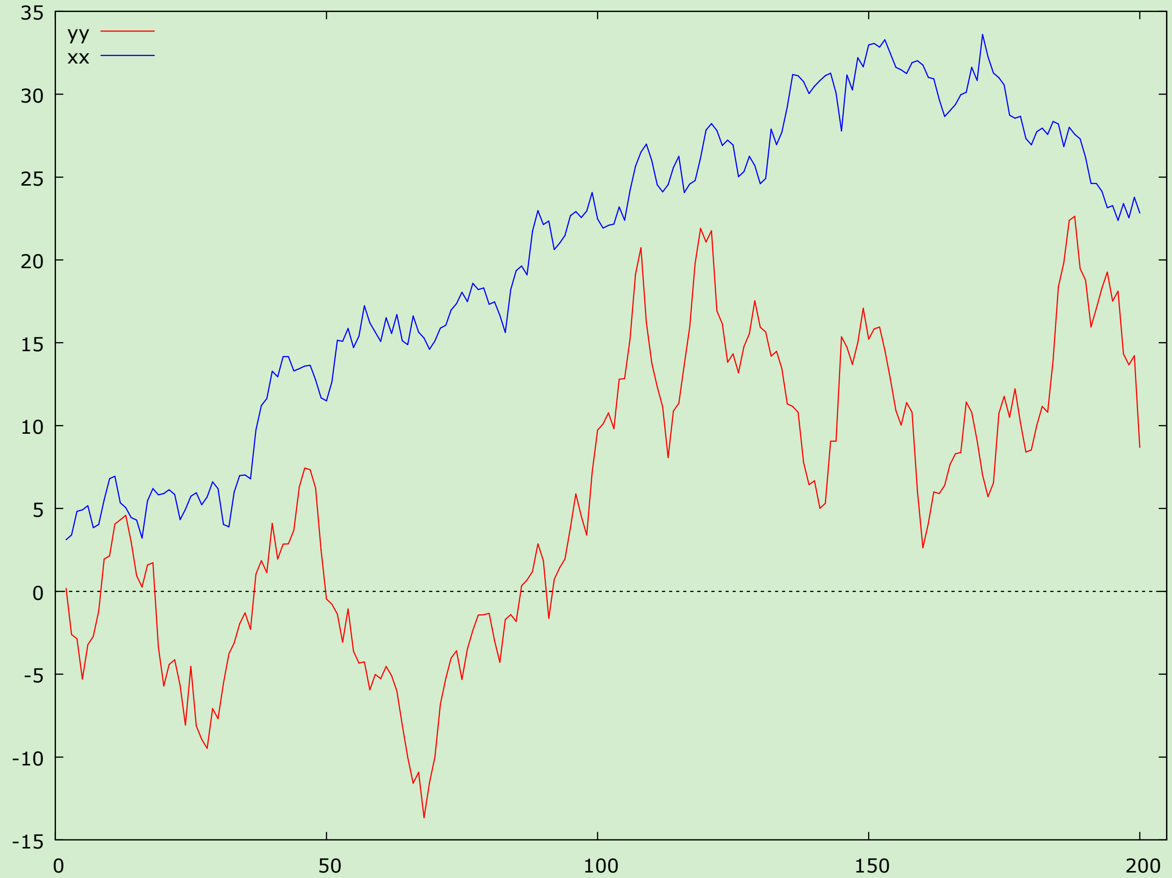
gives

$\hat{Y}_t = - 8.18 + 0.68 X_t$

t-statistic for *X:*

$t = 17.1$ (*p*-value $= 1.2$ E-40)

$R^2 = 0.50$, DW = 0.11



Spurious Regression

Regression model

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

with two independent random walks

$$Y_t = Y_{t-1} + \varepsilon_{1t},\ \varepsilon_{1t} \sim IIDN(0, \sigma_1^2)$$

$$X_t = X_{t-1} + \varepsilon_{2t},\ \varepsilon_{2t} \sim IIDN(0, \sigma_2^2)$$

ε_{1t}, ε_{2t} mutually independent

Consequences for OLS estimators for α and β

- *t*-statistic for β indicate explanatory power of X_t
- R^2 indicates explanatory potential
- Highly autocorrelated residuals

Nonsense or spurious regression (Granger & Newbold, 1974)

- Non-stationary time series are trended; this causes an apparent relationship

Models in Non-stationary Time Series

Non-stationary time series are trended

Example: random walk with trend

$$Y_t = \delta + Y_{t-1} + \varepsilon_t \text{ or}$$

$$Y_t = Y_0 + \delta t + \Sigma_{i \le t}\, \varepsilon_i$$

Y_t's are correlated, show stochastic trend (even for $\delta = 0$)

Given that $X_t \sim I(1)$, $Y_t \sim I(1)$ and the model

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

it follows in general that $\varepsilon_t \sim I(1)$, i.e., the error terms are non-stationary

- R^2 indicates explanatory potential
- (Asymptotic) distributions of *t*- and *F* -statistics are different from those for stationarity
- DW statistic converges for growing *N* to zero

Avoiding Spurious Regression

- Identification of non-stationarity: unit-root tests
- Models for non-stationary variables
  - Elimination of stochastic trends: differencing, specifying the model for differences
  - Inclusion of lagged variables may result in stationary error terms

Example: ADL(1,1) model:

- For $Y_t \sim I(1)$, the error terms are stationary if θ =1

$$\varepsilon_t = Y_t - (\delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1}) \sim I(0)$$

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The Drunk and her Dog

M. P. Murray, A drunk and her dog: An illustration of cointegration and error correction. *The American Statistician*, **48** (1997), 37-39

drunk: $x_t - x_{t-1} = u_t$

dog: $y_t - y_{t-1} = w_t$

Cointegration:

$$x_t - x_{t-1} = u_t + c(y_{t-1} - x_{t-1})$$

$$y_t - y_{t-1} = w_t + d(x_{t-1} - y_{t-1})$$

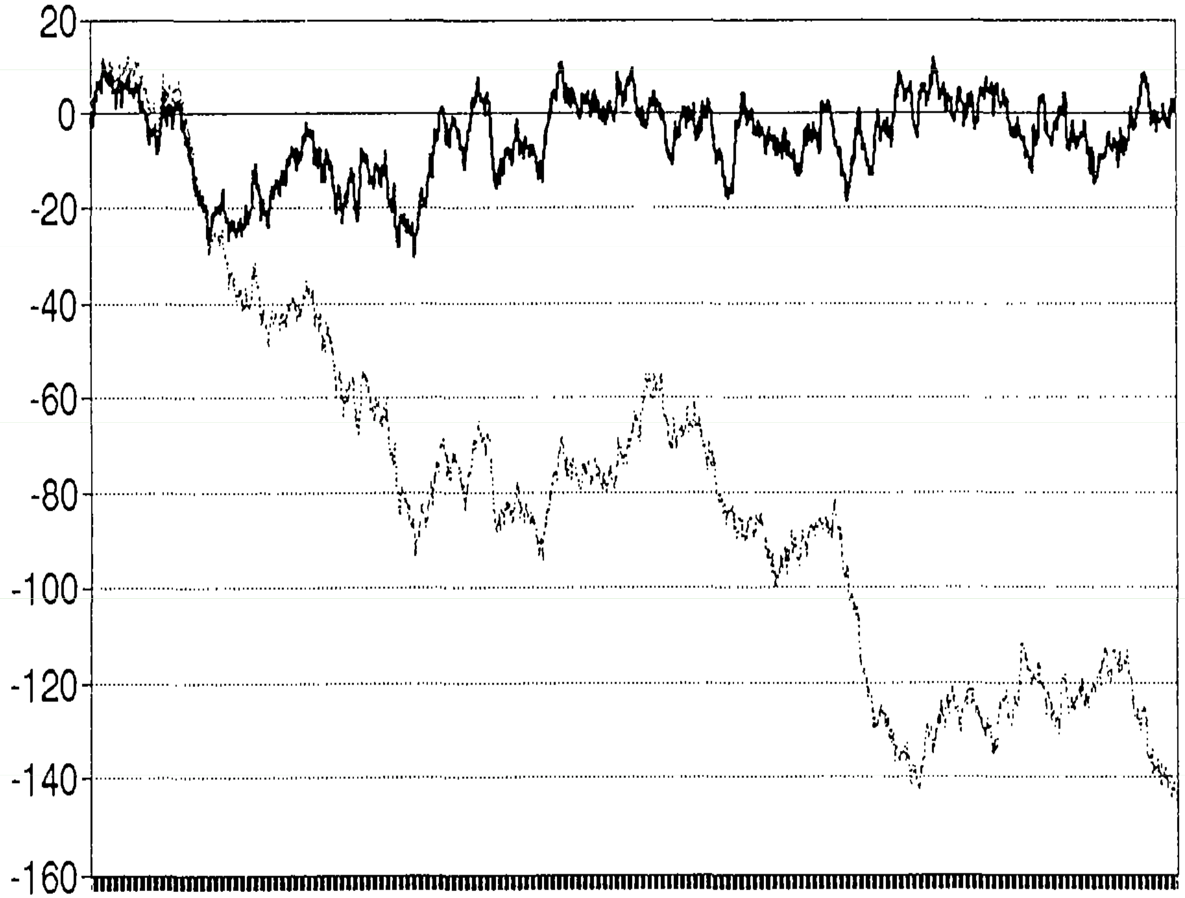


Figure 1. A drunk and two dogs: How close are the dogs to her? — Her dog. --- My dog.

Cointegrated Variables

Non-stationary variables X, Y:

$$X_t \sim I(1),\ Y_t \sim I(1)$$

if a β exists such that

$$Z_t = Y_t - \beta X_t \sim I(0)$$

- X_t and Y_t have a common stochastic trend
- X_t and Y_t are called “cointegrated”
- β: cointegration parameter
- (1, - β)’: cointegration vector

Cointegration implies a long-run equilibrium

Example: Purchasing Power Parity

Verbeek's dataset PPP: price indices and exchange rates for France and Italy, T = 186 (1/1981-6/1996)

- Variables: LNIT (log price index Italy), LNFR (log price index France), LNX (log exchange rate France/Italy)

Purchasing power parity (PPP): exchange rate between the currencies (Franc, Lira) equals the ratio of price levels of the countries

- Relative PPP: equality fulfilled only in the long run; equilibrium or cointegrating relation

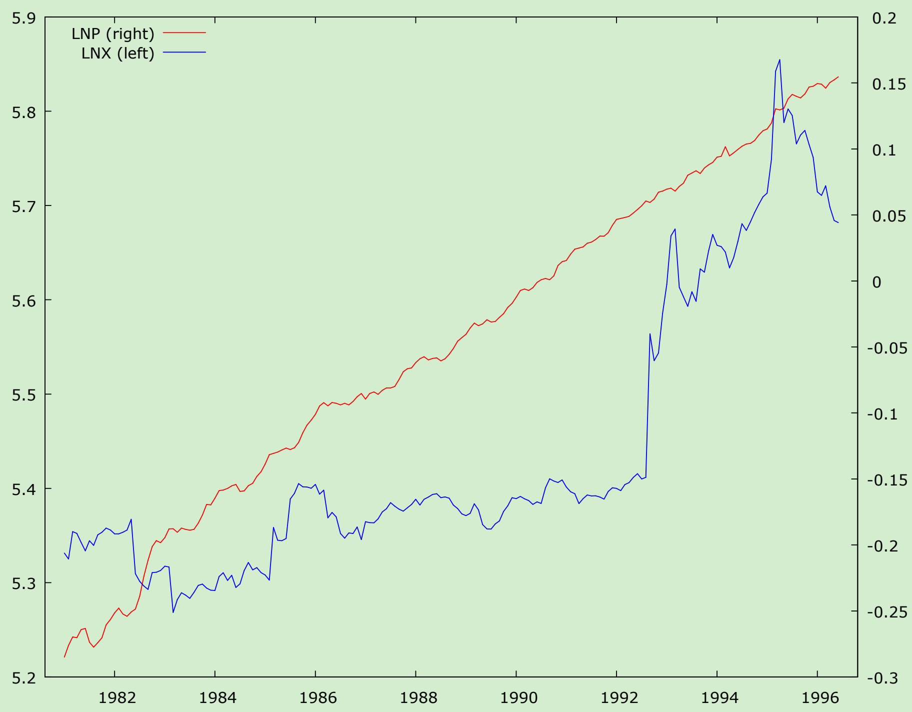
$$LNX_t = \alpha + \beta\, LNP_t + \varepsilon_t$$

with $LNP_t = LNIT_t - LNFR_t$, i.e., the log of the price index ratio France/Italy

Purchasing Power Parity

Test for unit roots (non-stationarity) of

- **LNX** (log exchange rate France/Italy)
- **LNP** = LNIT – LNFR, i.e., the log of the price index ratio France/Italy



Results from DF tests:

		const.	+trend
LNP	DF stat	-0.99	-2.96
	p-value	0.76	0.14
LNX	DF stat	-0.33	-1.90
	p-value	0.92	0.65

DF test indicates:
LNX ~ *I*(1), LNP ~ *I*(1)

PPP: Equilibrium Relation

OLS estimation of

$$LNX_t = \alpha + \beta\, LNP_t + \varepsilon_t$$

Model 2: OLS, using observations 1981:01-1996:06 (T = 186)
Dependent variable: LNX

	coefficient	std. error	t-ratio	p-value	
const	5,48720	0,00677678	809,7	0,0000	***
LNP	0,982213	0,0513277	19,14	1,24e-045	***

Mean dependent var	5,439818	S.D. dependent var	0,148368
Sum squared resid	1,361936	S.E. of regression	0,086034
R-squared	0,665570	Adjusted R-squared	0,663753
F(1, 184)	366,1905	P-value(F)	1,24e-45
Log-likelihood	193,3435	Akaike criterion	-382,6870
Schwarz criterion	-376,2355	Hannan-Quinn	-380,0726
rho	0,967239	Durbin-Watson	0,055469

DF test statistic for residuals (constant): -1.90, *p*-value: 0.33

H_0 cannot be rejected: no evidence for cointegration

Long-run Equilibrium

Equilibrium defined by

$$Y_t = \alpha + \beta X_t$$

Equilibrium error: $z_t = Y_t - \beta X_t - \alpha = Z_t - \alpha$

Two cases:

1. $z_t \sim I(0)$: equilibrium error stationary, fluctuating around zero
   - Y_t, βX_t cointegrated
   - $Y_t = \alpha + \beta X_t$ describes an equilibrium
2. $z_t \sim I(1)$, Y_t, βX_t not integrated
   - $z_t \sim I(1)$ non-stationary process
   - $Y_t = \alpha + \beta X_t$ does not describe an equilibrium

Cointegration, i.e., existence of an equilibrium vector, implies a long-run equilibrium relation

Identification of Cointegration

Information about cointegration:

- Economic theory
- Visual inspection of data
- Statistical tests

Testing for Cointegration

Non-stationary variables $X_t \sim I(1)$, $Y_t \sim I(1)$

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

- X_t and Y_t are cointegrated: $\varepsilon_t \sim I(0)$
- X_t and Y_t are not cointegrated: $\varepsilon_t \sim I(1)$

Tests for cointegration:

- If β is known, unit root test based on differences $Y_t - \beta X_t$
- Test procedures
  - Unit root test (DF or ADF) based on residuals e_t
  - Cointegrating regression Durbin-Watson (CRDW) test: DW statistic
  - Johansen technique: extends the cointegration technique to the multivariate case

DF Test for Cointegration

Non-stationary variables $X_t \sim I(1)$, $Y_t \sim I(1)$

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

- X_t and Y_t are cointegrated: $\varepsilon_t \sim I(0)$
- Residuals e_t represent ε_t, show similar pattern, $e_t \sim I(0)$, residuals are stationary

Tests for cointegration based on residuals e_t

$$\Delta e_t = \gamma_0 + \gamma_0 e_{t-1} + u_t$$

- H_0: $\gamma_0 = 0$, i.e., residuals have a unit root, $e_t \sim I(1)$
- H_0 implies
  - X_t and Y_t are not cointegrated
  - Rejection of H_0 suggests that X_t and Y_t are cointegrated

DF Test for Cointegration, cont'd

Critical values of DF test for residuals

- are smaller than those of DF test for observations
- depend upon (see Verbeek, Tab. 9.2)
  - number of elements of cointegrating vector, K+1
  - number of observations T
  - significance level

some asymptotic critical values for the DF-test with constant term

	1%	5%
Observations	-3.43	-2.86
Residuals, K=1	-3.90	-3.34

Cointegrating Regression Durbin-Watson (CRDW) Test

Non-stationary variables $X_t \sim I(1)$, $Y_t \sim I(1)$

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

Cointegrating regression Durbin-Watson (CRDW) test: DW statistic from OLS-fitting $Y_t = \alpha + \beta X_t + \varepsilon_t$

- Null hypothesis: residuals e_t have a unit root, i.e., $e_t \sim I(1)$, i.e., X_t and Y_t are not cointegrated
- DW statistic converges with T to zero for not cointegrated variables
- Critical values from Monte Carlo simulations, which depend upon (see Verbeek, Tab. 9.4)
  - Number of regressors plus 1 (dependent variable)
  - Number of observations T
  - Significance level

PPP: Tests for Cointegration

Residuals from $LNX_t = \alpha + \beta\ LNP_t + \varepsilon_t$:

- Time series plot indicates non-stationarity of residuals
- Tests for cointegration
  - DF test statistic for residuals: -1.90, *p*-value: 0.33, no cointegration
  - CRDW test: DW statistic: 0.055 < 0.20, the critical value for two variables, 200 observations, significance level 0.05, no cointegration

Time series plot of residuals

OLS Estimation of Equilibrium Relation

To be estimated:

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

cointegrated non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$

$\varepsilon_t \sim I(0)$

OLS estimator b for β

- Super consistent:
  - $T(b - \beta)$ converges to zero
  - In case of consistency: $\sqrt{T}(b - \beta)$ converges to zero
- Robust against misspecification in stationary part wrt asymptotic distribution of b
- Non-standard distribution, non-normal, e.g., *t*-test misleading
- Small samples: bias

OLS Estimation, cont'd

To be estimated:

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$

If $\varepsilon_t \sim I(1)$, i.e., Y_t and X_t not cointegrated: spurious regression

OLS estimator *b* for β

- Non-standard distribution of *b*
- Large values of R^2, *t*-statistic
- Highly autocorrelated residuals
- DW statistic close to zero

Error-correction Model

Granger's Representation Theorem (Engle & Granger, 1987): If a set of variables is cointegrated, then an error-correction relation of the variables exists

non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$ with cointegrating vector (1, -β)': error-correction representation

$$\theta(L)\Delta Y_t = \delta + \Phi(L)\Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \alpha(L)\varepsilon_t$$

with lag polynomials $\theta(L)$ (with $\theta_0=1$), $\Phi(L)$, and $\alpha(L)$

E.g., $\Delta Y_t = \delta + \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \varepsilon_t$

Error-correction model: describes

- the short-run behavior
- consistently with the long-run equilibrium

Converse statement: if $Y_t \sim I(1)$, $X_t \sim I(1)$ have an error-correction representation, then they are cointegrated

Your Homework

1. Use Verbeek's data set INCOME containing quarterly data INCOME (total disposable income) and CONSUM (consumer expenditures) for 1/1971 to 2/1985 in the UK.
   a. Specify a DL(*s*) model in sd_INCOME (seasonal differences) and choose an appropriate *s*, using (i) R^2 and (ii) BIC.
   b. Assuming that DL(4) is an appropriate lag structure, calculate (i) the short run and (ii) the long run multiplier as well as (iii) the average and (iv) the median lag time.
   c. Specify a consumption function with the actual expected income as explanatory variable; estimate the AR form of the model under the assumption of adapted expectation.
   d. Test (i) whether CONSUM and INCOME are *I*(1); (ii) estimate the simple linear regression of CONSUM on INCOME and test (iii) whether this is an equilibrium relation; show (iv) the corresponding time series plots.

Your Homework, cont'd

2. Generate 500 random numbers (a) from a random walk with trend: $x_t = 0.1 + x_{t-1} + \varepsilon_t$; and (b) from an AR(1) process: $y_t = 0.2 + 0.7y_{t-1} + \varepsilon_t$; for ε_t use Monte Carlo random numbers from $N(0,1)$. Estimate regressions of x_t and y_t on t; report the values for R^2.