Econometrics 2 - Lecture 6

Multivariate Time Series Models (part 2)

Contents

Regression and Time Series

Stationary variables are a crucial prerequisite for

- estimation methods
- testing procedures

applied to regression models

Specifying a relation between non-stationary variables may result in a nonsense or spurious regression

Spurious Regression: Example

Generation of Y_t by

$$Y_t = Y_{t-1} + \varepsilon_t$$

i.e., Y_t is a random walk, $Y_t \sim I(1)$; similarly $X_t \sim I(1)$

Model to be estimated:

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

it follows (in general) that $\varepsilon_t \sim I(1)$, i.e., the error terms are non-stationary

- (Asymptotic) distributions of *t*- and *F* -statistics are different from those for stationarity
- R^2 indicates explanatory potential
- DW statistic converges for growing *N* to zero

Avoiding Spurious Regression

- Identification of non-stationarity: unit-root tests
- Models for non-stationary variables
  - Elimination of stochastic trends: differencing, specifying the model for differences
  - Inclusion of lagged variables may result in stationary error terms
  - Explained and explanatory variables may have a common stochastic trend, are cointegrated: equilibrium relation, error-correction models
- Example: ADL(1,1) model with $Y_t \sim I(1)$, $X_t \sim I(1)$

$$Y_t = \delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1} + \varepsilon_t$$

  - The error terms are stationary if $\theta = 1$, $\varphi_0 = \varphi_1 = 0$

$$\varepsilon_t = Y_t - (\delta + \theta Y_{t-1} + \varphi_0 X_t + \varphi_1 X_{t-1}) \sim I(0)$$

  - Common trend implies an equilibrium relation, i.e., $Y_{t-1} - \beta X_{t-1} \sim I(0)$; the ADL(1,1) model has an error-correction form

$$\Delta Y_t = \varphi_0 \Delta X_t - (1 - \theta)(Y_{t-1} - \alpha - \beta X_{t-1}) + \varepsilon_t$$

Contents

- Non-stationary Time Series
- Cointegration
- VAR Models
- VAR Models and Cointegration
- VEC Model: Cointegration Tests
- VEC Model: Specification and Estimation

Cointegration

Non-stationary variables *X*, *Y*:

$X_t \sim I(1),\ Y_t \sim I(1)$

if a β exists such that

$Z_t = Y_t - \beta X_t \sim I(0)$

- X_t and Y_t have a common stochastic trend
- X_t and Y_t are called "cointegrated"
- β: cointegration parameter
- (1, - β)': cointegration vector

Cointegration implies a long-run equilibrium; cf. Granger's representation theorem

Error-correction Model

Granger's Representation Theorem (Engle & Granger, 1987): If a set of variables is cointegrated, then an error-correction (EC) relation of the variables exists

non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$ with cointegrating vector $(1, -\beta)'$: error-correction representation

$$\theta(L)\Delta Y_t = \delta + \Phi(L)\Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \alpha(L)\varepsilon_t$$

with white noise ε_t, lag polynomials $\theta(L)$ (with $\theta_0=1$), $\Phi(L)$, and $\alpha(L)$

- Error-correction model: describes
  - the short-run behavior
  - consistently with the long-run equilibrium
- Converse statement: if $Y_t \sim I(1)$, $X_t \sim I(1)$ have an error-correction representation, then they are cointegrated

An Example

The model

$$\Delta Y_t = \delta + \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \varepsilon_t$$

is a special case of

$$\theta(L)\Delta Y_t = \delta + \Phi(L)\Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \alpha(L)\varepsilon_t$$

with $\theta(L) = 1$, $\Phi(L) = \varphi_1 L$, and $\alpha(L) = 1$

- No change steady state equilibrium for $\Delta Y_t = \Delta X_{t-1} = 0$:

$$Y_t - \beta X_t = \delta/\gamma \text{ or } Y_t = \alpha + \beta X_t \text{ if } \alpha = \delta/\gamma$$

the EC model can be written as

$$\Delta Y_t = \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \alpha - \beta X_{t-1}) + \varepsilon_t$$

- If $\alpha = \delta/\gamma + \lambda$, $\lambda \neq 0$:

$$\Delta Y_t = \lambda + \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \alpha - \beta X_{t-1}) + \varepsilon_t$$

deterministic trends for Y_t and X_t, long run equilibrium corresponding to growth paths $\Delta Y_t = \Delta X_{t-1} = \lambda/(1 - \varphi_1)$

EC Model: Estimation

Model

$$\Delta Y_t = \delta + \varphi_1 \Delta X_{t-1} - \gamma(Y_{t-1} - \beta X_{t-1}) + \varepsilon_t \qquad \text{(A)}$$

with cointegrating relation

$$Y_{t-1} = \beta X_{t-1} + u_t \qquad \text{(B)}$$

- Cointegration vector (1, - β)' known: OLS estimation of δ, φ_1, and γ from (A), standard properties
- Unknown cointegration vector:
  - Parameter β from (B) superconsistently estimated by OLS
  - OLS estimation of δ, φ_1, and γ from (A) is not affected by using the estimate for β

Model specification

- Choice of orders of lag polynomials
- Theory is symmetric in treating X_t and Y_t

Example: Purchasing Power Parity

Verbeek's dataset ppp: price indices and exchange rates for France and Italy, T = 186 (1/1981-6/1996)

- Variables: LNIT (log price index Italy), LNFR (log price index France), LNX (log exchange rate France/Italy)

Purchasing power parity (PPP): exchange rate between the currencies (Franc, Lira) equals the ratio of price levels of the countries

- Relative PPP: equality fulfilled only in the long run; equilibrium or cointegrating relation

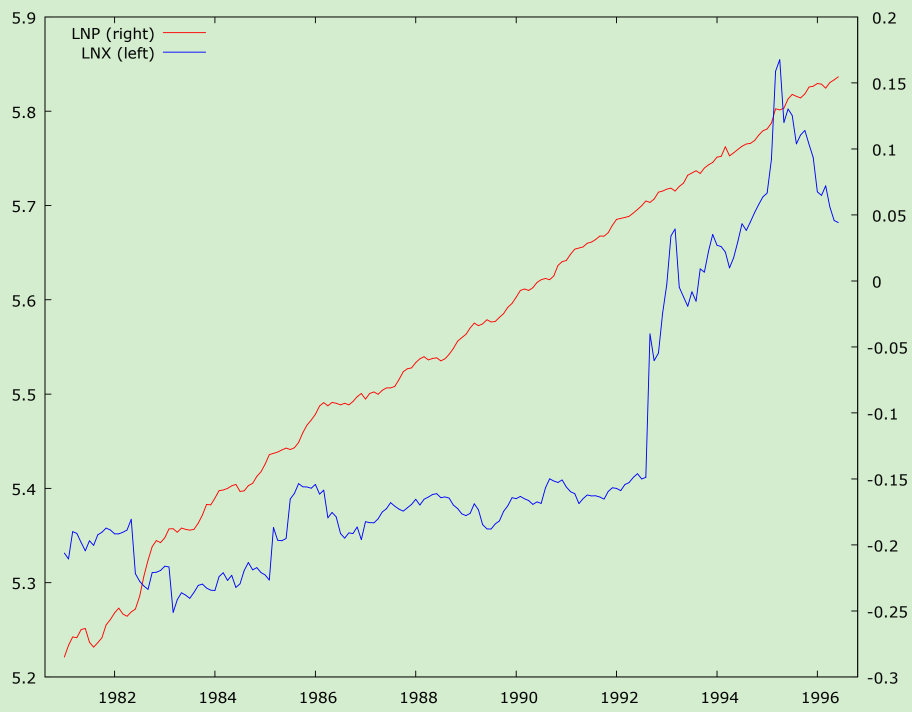
$$LNX_t = \alpha + \beta\, LNP_t + \varepsilon_t$$

with $LNP_t = LNIT_t - LNFR_t$, i.e., the log of the price index ratio France/Italy

Purchasing Power Parity

Test for unit roots (non-stationarity) of

- **LNX** (log exchange rate France/Italy)
- **LNP** = LNIT – LNFR, i.e., the log of the price index ratio France/Italy



Results from DF tests:

		const.	+trend
LNP	DF stat	-0.99	-2.96
	p-value	0.76	0.14
LNX	DF stat	-0.33	-1.90
	p-value	0.92	0.65

DF test indicates:
LNX ~ $I(1)$, LNP ~ $I(1)$

PPP: Equilibrium Relations

As discussed by Verbeek:

1. If PPP holds in long run, real exchange rate is stationary

   $$LNX_t - (LNIT_t - LNFR_t) = \varepsilon_t$$

2. Change of relative prices correspond to the change of exchange rate, i.e., short run deviations are stationary

   $$LNX_t - \beta\,(LNIT_t - LNFR_t) = \varepsilon_t$$

3. Generalization of case 2:

   $$LNX_t = \alpha + \beta_1\,LNIT_t - \beta_2\,LNFR_t + \varepsilon_t$$

   with $\varepsilon_t \sim I(0)$

Equilibrium Relation 2

OLS estimation of

$$LNX_t = \alpha + \beta\ LNP_t + \varepsilon_t$$

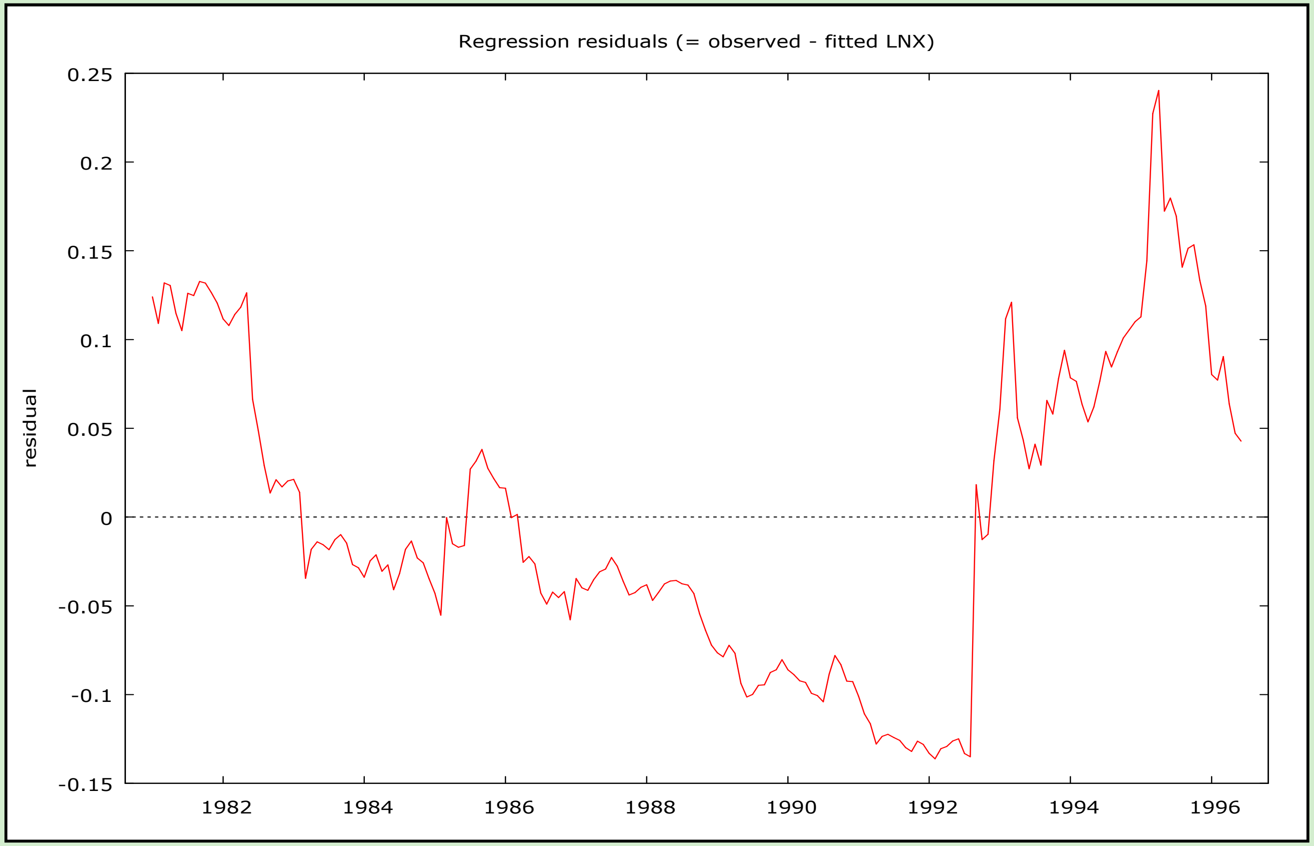
Model 2: OLS, using observations 1981:01-1996:06 (T = 186)
Dependent variable: LNX

	coefficient	std. error	t-ratio	p-value	
const	5,48720	0,00677678	809,7	0,0000	***
LNP	0,982213	0,0513277	19,14	1,24e-045	***

Mean dependent var	5,439818	S.D. dependent var	0,148368
Sum squared resid	1,361936	S.E. of regression	0,086034
R-squared	0,665570	Adjusted R-squared	0,663753
F(1, 184)	366,1905	P-value(F)	1,24e-45
Log-likelihood	193,3435	Akaike criterion	-382,6870
Schwarz criterion	-376,2355	Hannan-Quinn	-380,0726
rho	0,967239	Durbin-Watson	0,055469

Equilibrium Relation 2

Residuals = $LNX_t - (a + b\, LNP_t)$ with OLS estimates *a*, *b*



PPP: Tests for Cointegration

Residuals from $LNX_t = \alpha + \beta\ LNP_t + \varepsilon_t$:

- Time series plot indicates non-stationary residuals
- Tests for cointegration, H_0: residuals have unit root, no cointegration
  - DF test statistic (with constant): -1.90, 5% critical value: -3.37
  - CRDW test: DW statistic: 0.055 < 0.20, the 5% critical value for two variables, 200 observations
- Both tests suggest: H_0 cannot be rejected, no evidence for cointegration

Same result for equilibrium relations 1 and 3; reasons could be:

- Time series too short
- No PPP between France and Italy

Attention: equilibrium relation 3 has three variables; two cointegration relations are possible

Contents

Example: Income and Consumption

Model for income (Y) and consumption (C)

$$Y_t = \delta_1 + \theta_{11} Y_{t-1} + \theta_{12} C_{t-1} + \varepsilon_{1t}$$

$$C_t = \delta_2 + \theta_{21} C_{t-1} + \theta_{22} Y_{t-1} + \varepsilon_{2t}$$

with (possibly correlated) white noises ε_{1t} and ε_{1t}

Notation: $Z_t = (Y_t, C_t)$‘, 2-vectors δ and ε, and (2x2)-matrix $\Theta = (\theta_{ij})$, the model is

$$\begin{pmatrix} Y_t \\ C_t \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ C_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

in matrix notation

$$Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$$

- Represents each component of Z as a linear combination of lagged variables
- Extension of the AR-model to the 2-vector Z_t: vector autoregressive model of order 1, VAR(1) model

The VAR(p) Model

VAR(p) model: generalization of the AR(p) model for k-vectors Y_t

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t$$

with k-vectors Y_t, δ, and ε_t and kxk-matrices $\Theta_1, \ldots, \Theta_p$

- Using the lag-operator L:

  $$\Theta(L) Y_t = \delta + \varepsilon_t$$

  with matrix lag polynomial $\Theta(L) = I - \Theta_1 L - \ldots - \Theta_p L^p$
  - $\Theta(L)$ is a kxk-matrix
  - Each matrix element of $\Theta(L)$ is a lag polynomial of order p
- Error terms ε_t
  - have covariance matrix Σ; allows for contemporaneous correlation
  - are independent of Y_{t-j}, $j > 0$, i.e., of the past of the components of Y_t

The VAR(p) Model, cont'd

VAR(p) model for the k-vector Y_t

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t$$

- Vector of expectations of Y_t: assuming stationarity

  $$E\{Y_t\} = \delta + \Theta_1 E\{Y_t\} + \ldots + \Theta_p E\{Y_t\}$$

  gives

  $$E\{Y_t\} = \mu = (I_k - \Theta_1 - \ldots - \Theta_p)^{-1}\delta = \Theta(1)^{-1}\delta$$

  i.e., stationarity requires that the kxk-matrix $\Theta(1)$ is invertible
- In deviations $y_t = Y_t - \mu$, the VAR(p) model is

  $$\Theta(L)y_t = \varepsilon_t$$
- MA representation of the VAR(p) model, given that $\Theta(L)$ is invertible

  $$Y_t = \mu + \Theta(L)^{-1}\varepsilon_t = \mu + \varepsilon_t + A_1\varepsilon_{t-1} + A_2\varepsilon_{t-2} + \ldots$$
- VARMA(p,q) Model: Extension of the VAR(p) model by multiplying ε_t (from the left) with a matrix lag polynomial of order q

Reasons for Using a VAR Model

VAR model represents a set of univariate ARMA models, one for each component

- Reformulation of simultaneous equation models as dynamic models
- To be used instead of simultaneous equation models:
  - No need to distinct a priori endogenous and exogenous variables
  - No need for a priori identifying restrictions on model parameters
- Simultaneous analysis of the components: More parsimonious, fewer lags, simultaneous consideration of the history of all included variables
- Allows for non-stationarity and cointegration

Attention: the number of parameters to be estimated increases with k and p

Number of parameters of $\Theta(L)$

p	1	2	3
k=2	4	8	12
k=4	16	32	48

Simultaneous Equation Models in VAR Form

Model with m endogenous variables (and equations), K regressors

$$Ay_t = \Gamma z_t + \varepsilon_t = \Gamma_1 y_{t-1} + \Gamma_2 x_t + \varepsilon_t$$

with m-vectors y_t and ε_t, K-vector z_t, (mxm)-matrix A, (mxK)-matrix Γ, and (mxm)-matrix $\Sigma = V\{\varepsilon_t\}$;

- z_t contains lagged endogenous variables y_{t-1} and exogenous variables x_t
- Rearranging gives

$$y_t = \Theta y_{t-1} + \delta_t + v_t$$

with $\Theta = A^{-1} \Gamma_1$, $\delta_t = A^{-1} \Gamma_2 x_t$, and $v_t = A^{-1} \varepsilon_t$

- Extension of y_t by regressors x_t: the matrix δ_t becomes a vector of intercepts

Example: Income and Consumption

Model for income (Y_t) and consumption (C_t)

$$Y_t = \delta_1 + \theta_{11} Y_{t-1} + \theta_{12} C_{t-1} + \varepsilon_{1t}$$

$$C_t = \delta_2 + \theta_{21} C_{t-1} + \theta_{22} Y_{t-1} + \varepsilon_{2t}$$

with (possibly correlated) white noises ε_{1t} and ε_{1t}

- Matrix form of the simultaneous equation model:

$$A\,(Y_t, C_t)' = \Gamma\,(1, Y_{t-1}, C_{t-1})' + (\varepsilon_{1t}, \varepsilon_{2t})'$$

with

$$A = \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix}, \Gamma = \begin{pmatrix} \delta_1\ \theta_{11}\ \theta_{12} \\ \delta_2\ \theta_{21}\ \theta_{22} \end{pmatrix}$$

- VAR(1) form: $Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$ or

$$\begin{pmatrix} Y_t \\ C_t \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ C_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

VAR Model: Estimation

VAR(p) model for the k-vector Y_t

$$Y_t = \delta + \Theta_1 Y_{t-1} + \dots + \Theta_p Y_{t-p} + \varepsilon_t,\ V\{\varepsilon_t\} = \Sigma$$

- Each component of Y_t: a linear combination of lagged variables
- Error terms: Possibly contemporaneously correlated, covariance matrix Σ, uncorrelated over time
- SUR model

Estimation, given the order p of the VAR model

- OLS estimates of parameters in $\Theta(L)$ are consistent
- Estimation of Σ based on residual vectors $e_t = (e_{1t}, \dots, e_{kt})'$:

$$S = \frac{1}{T-p}\sum_t e_t e_t'$$

- GLS estimator coincides with OLS estimator: same explanatory variables for all equations

VAR Model: Estimation, cont'd

Choice of the order *p* of the VAR model

- Estimation of VAR models for various orders *p*
- Choice of *p* based on Akaike or Schwarz information criterion

Income and Consumption

AWM data base, 1970:1-2003:4: *PCR* (real private consumption), *PYR* (real disposable income of households); respective annual growth rates: *C*, *Y*

Fitting $Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$ with $Z = (Y, C)'$ gives

		δ	Y_{-1}	C_{-1}	adj.R^2
Y	θ_{ij}	0.001	0.825	0.082	0.80
	$t(\theta_{ij})$	0.91	12.09	1.07	
C	Θ_{ij}	0.003	0.061	0.826	0.78
	$t(\theta_{ij})$	2.36	0.97	11.69	

with AIC = -14.40; for the VAR(2) model: AIC = -14.35

In GRETL: OLS equation-wise, VAR estimation, SUR estimation give very similar results

Impulse-response Function

MA representation of the VAR(p) model

$$Y_t = \Theta(1)^{-1}\delta + \varepsilon_t + A_1\varepsilon_{t-1} + A_2\varepsilon_{t-2} + \dots$$

- Interpretation of A_s: the (i,j)-element of A_s represents the effect of a one unit increase of ε_{jt} upon the i-th variable $Y_{i,t+s}$ in Y_{t+s}
- Dynamic effects of a one unit increase of ε_{jt} upon the i-th component of Y_t are corresponding to the (i,j)-th elements of I_k, A_1, A_2, …
- The plot of these elements over s represents the impulse-response function of the i-th variable in Y_{t+s} on a unit shock to ε_{jt}

Stationarity

AR(1) process $Y_t = \theta Y_{t-1} + \varepsilon_t$

- is stationary, if the root z of the characteristic polynomial

  $$\Theta(z) = 1 - \theta z = 0$$

  fulfills $|z| > 1$, i.e., $|\theta| < 1$;
  - $\Theta(z)$ is invertible, i.e., $\Theta(z)^{-1}$ can derived such that $\Theta(z)^{-1}\Theta(z) = 1$
  - Y_t can be represented by a MA(∞) process: $Y_t = \Theta(z)^{-1}\varepsilon_t$
- is non-stationary, if

  $$z = 1 \text{ or } \theta = 1$$

  i.e., $Y_t \sim I(1)$, Y_t has a stochastic trend

VAR Models, Stationarity, and Cointegration

VAR(1) model for the *k*-vector Y_t

$$Y_t = \delta + \Theta_1 Y_{t-1} + \varepsilon_t$$

- If $\Theta(L)$ is invertible,

  $$Y_t = \Theta(1)^{-1}\delta + \Theta(L)^{-1}\varepsilon_t = \mu + \varepsilon_t + A_1\varepsilon_{t-1} + A_2\varepsilon_{t-2} + \ldots$$

  i.e., each variable in Y_t is a linear combination of white noises, is a stationary *I*(0) variable
- If $\Theta(L)$ is not invertible, not all variables in Y_t can be stationary *I*(0) variables: at least one variable must have a stochastic trend
  - If all *k* variables have independent stochastic trends, all *k* variables are *I*(1) and no cointegrating relation exists; e.g., for $k = 2$:

    $$\Theta(1) = \begin{pmatrix} 1-\theta_{11} & \theta_{12} \\ \theta_{21} & 1-\theta_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

    i.e., $\theta_{11} = \theta_{22} = 1$, $\theta_{12} = \theta_{21} = 0$
  - The more interesting case: at least one cointegrating relation; number of cointegrating relations equals the rank $r\{\Theta(1)\}$ of matrix $\Theta(1)$

Example: A VAR(1) Model

VAR(1) model for k-vector Y in differences with $\Theta(L) = I - \Theta_1 L$

$$\Delta Y_t = -\Theta(1) Y_{t-1} + \delta + \varepsilon_t$$

$r = r\{\Theta(1)\}$: rank of (kxk) matrix $\Theta(1) = I_k - \Theta_1$

1. $r = 0$: then $\Delta Y_t = \delta + \varepsilon_t$, i.e., Y is a k-dimensional random walk, each component is $I(1)$, no cointegrating relationship
2. $r < k$: ($k - r$)-fold unit root, (kxr)-matrices γ and β can be found, both of rank r, with

   $$\Theta(1) = \gamma\beta'$$

   the r columns of β are the cointegrating vectors of r cointegrating relations (β in normalized form, i.e., the main diagonal elements of β being ones)
3. $r = k$: VAR(1) process is stationary, all components of Y are $I(0)$

Cointegration Space

Given a set of k variables, the components of the k-vector $Y_t \sim I(1)$

Cointegration space:

- Among the k variables, $r \leq k$-1 independent linear relations $\beta_j' Y_t$, $j = 1, \ldots, r$, are possible so that $\beta_j' Y_t \sim I(0)$
- Individual relations can be combined with others and these are again $I(0)$, i.e., not the individual cointegrating relations are identified but only the r-dimensional space
- Cointegrating relations should have an economic interpretation

Cointegration matrix β:

- The kxr matrix $\beta = (\beta_1, \ldots, \beta_r)$ of vectors β_j that state the cointegrating relations $\beta_j' Y_t \sim I(0)$, $j = 1, \ldots, r$
- Cointegrating rank: the rank of matrix β: $r\{\beta\} = r$

Contents

Granger‵s Representation Theorem

Granger's Representation Theorem (Engle & Granger, 1987): If a set of *I*(1) variables is cointegrated, then an error-correction (EC) relation of the variables exists

Extends to VAR models: if the *I*(1) variables of the *k*-vector Y_t are cointegrated, then an error-correction (EC) relation of the variables exists

Granger's Representation Theorem for VAR Models

VAR(p) model for the k-vector Y_t

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t$$

transformed into

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \Pi Y_{t-1} + \varepsilon_t \qquad \text{(A)}$$

- $\Pi = -\Theta(1) = -(I_k - \Theta_1 - \ldots - \Theta_p)$: „long-run matrix", determines the long-run dynamics of Y_t
- $\Gamma_1, \ldots, \Gamma_{p-1}$ matrices which are functions of $\Theta_1, \ldots, \Theta_p$

- ΠY_{t-1} is stationary: ΔY_t and ε_t are $I(0)$
- Three cases

1. $r\{\Pi\} = r$ with $0 < r < k$: there exist r stationary linear combinations of Y_t, i.e., r cointegrating relations
2. $r\{\Pi\} = 0$: then $\Pi = 0$, equation (A) is a VAR(p) model for stationary variables ΔY_t
3. $r\{\Pi\} = k$: all variables in Y_t are stationary, $\Pi = -\Theta(1)$ is invertible

Vector Error-Correction Model

VAR(p) model for the k-vector Y_t

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_p Y_{t-p} + \varepsilon_t$$

transformed into

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \Pi Y_{t-1} + \varepsilon_t$$

with r{Π} = r and $\Pi = \gamma\beta'$ gives

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma\beta' Y_{t-1} + \varepsilon_t \qquad \text{(B)}$$

- r cointegrating relations $\beta' Y_{t-1}$
- Adaptation parameters γ measure the portion or speed of adaptation of Y_t in compensation of the equilibrium error $Z_{t-1} = \beta' Y_{t-1}$
- Equation (B) is called the vector error-correction (VEC) model

Example: Bivariate VAR Model

VAR(1) model for the 2-vector $Y_t = (Y_{1t}, Y_{2t})'$

$$Y_t = \Theta Y_{t-1} + \varepsilon_t$$

- Long-run matrix

$$\Pi = -\Theta(1) = \begin{pmatrix} \theta_{11} - 1 & \theta_{12} \\ \theta_{21} & \theta_{22} - 1 \end{pmatrix}$$

- $\Pi = 0$, if $\theta_{11} = \theta_{22} = 1$, $\theta_{12} = \theta_{21} = 0$, i.e., Y_{1t}, Y_{2t} are random walks
- $r\{\Pi\} < 2$, if $(\theta_{11} - 1)(\theta_{22} - 1) - \theta_{12}\,\theta_{21} = 0$; cointegrating vector: $\beta' = (\theta_{11} - 1, \theta_{12})$, long-run matrix

$$\Pi = \gamma\beta' = \begin{pmatrix} 1 \\ \theta_{21} / (\theta_{11} - 1) \end{pmatrix} \begin{pmatrix} \theta_{11} - 1 & \theta_{12} \end{pmatrix}$$

- The error-correction form is

$$\begin{pmatrix} \Delta Y_{1t} \\ \Delta Y_{2t} \end{pmatrix} = \begin{pmatrix} 1 \\ \theta_{21} / (\theta_{11} - 1) \end{pmatrix} \left[(\theta_{11} - 1) Y_{1,t-1} + \theta_{12} Y_{2,t-1} \right] + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

Deterministic Component

VEC(p) model for the k-vector Y_t

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma\beta' Y_{t-1} + \varepsilon_t \qquad \text{(B)}$$

The deterministic component (intercept) δ:

1. $E\{\Delta Y_t\} = 0$, i.e., no deterministic trend in any component of Y_t: given that $\Gamma = I_k - \Gamma_1 - \ldots - \Gamma_{p-1}$ has full rank:
   - $\Gamma\, E\{\Delta Y_t\} = \delta + \gamma E\{Z_{t-1}\} = 0$ with equilibrium error $Z_{t-1} = \beta' Y_{t-1}$
   - $E\{Z_{t-1}\}$ corresponds to the intercepts of the cointegrating relations; with r-dimensional vector $E\{Z_{t-1}\} = \alpha$

     $$\Delta Y_t = \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma(-\alpha + \beta' Y_{t-1}) + \varepsilon_t \qquad \text{(C)}$$
   - Intercepts only in the cointegrating relations, i.e., no deterministic trend in the model

Deterministic Component, cont'd

VEC(p) model for the k-vector Y_t

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma\beta' Y_{t-1} + \varepsilon_t \qquad \text{(B)}$$

The deterministic component (intercept) δ:

2. Addition of a k-vector λ with identical components to (C)

$$\Delta Y_t = \lambda + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \gamma(- \alpha + \beta' Y_{t-1}) + \varepsilon_t$$

- Long-run equilibrium: steady state growth with growth rate $E\{\Delta Y_t\} = \Gamma^{-1}\lambda$
- Deterministic trends cancel out in the long run, so that no deterministic trend in the error-correction term; cf. (B)
- Addition of k-vector λ can be repeated: up to k-r separate deterministic trends can cancel out in the error-correction term
- The general notation is equation (B) with δ containing r intercepts of the long-run relations and k-r deterministic trends in the variables of Y_t

The Five Cases

Based on empirical observation and economic reasoning, choice between:

1) Unrestricted constant: variables show deterministic linear trends
2) Restricted constant: variables not trended but mean distance between them not zero; intercept in the error-correction term
3) No constant

Generalization: deterministic component contains intercept and trend

4) Constant + restricted trend: cointegrating relationships include a trend but the first differences of the variables in question do not
5) Constant + unrestricted trend: trend in both the cointegration relationships and the first differences, corresponding to a quadratic trend in the variables (in levels)

Contents

- Non-stationary Time Series
- Cointegration
- VAR Models
- VAR Models and Cointegration
- VEC Model: Cointegration Tests
- VEC Model: Specification and Estimation

Choice of the Cointegrating Rank

Based on *k*-vector $Y_t \sim I(1)$

Estimation procedure needs as input the cointegrating rank *r*

- Engle-Granger procedure
- Johansen‘s R3 method

Engle-Granger Approach

Non-stationary processes $Y_t \sim I(1)$, $X_t \sim I(1)$; to be estimated:

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

- Step 1: OLS-fitting
- Tests for cointegration based on residuals, e.g., DF test with special critical values; H_0: no cointegration
- If H_0 is rejected,
  - OLS fitting in step 1 gives consistent estimates of the cointegrating vector
  - Step 2: OLS estimation of the EC model based on the cointegrating vector from step 1

Can be extended to k-vector Y_t, given that at most one cointegrating relation exists

Engle-Granger Cointegration Test: Problems

Residual based cointegration tests can be misleading

- Test results depend on specification
  - Which variables are included
  - Normalization of the cointegrating vector
- Test may be inappropriate due to wrong specification of cointegrating relation
- Test power suffers from inefficient use of information (dynamic interactions not taken into account)

Johansen's R3 Method

Reduced rank regression or R3 method: an iterative method for specifying the cointegrating rank *r*

- Also called Johansen's test
- The test is based on the *k* eigenvalues λ_i ($\lambda_1 > \lambda_2 > \ldots > \lambda_k$) of

  $$Y_1'Y_1 - Y_1\Delta Y(\Delta Y'\Delta Y)^{-1}\Delta Y'Y_1,$$

  with ΔY: (Txk) matrix of differences ΔY_t, Y_1: (Txk) matrix of Y_{t-1}
  - eigenvalues λ_i fulfill $0 \leq \lambda_i < 1$
  - if $r\{\Theta(1)\} = r$, the k-r smallest eigenvalues obey

    $$\log(1-\lambda_j) = \lambda_j = 0, \quad j = r+1, \ldots, k$$
- Iterative test procedures
  - Trace test
  - Maximum eigenvalue test or max test

Trace and Max Test: The Procedures

LR tests, based on the assumption of normally distributed errors

- Trace test: for $r_0 = 0, 1, \ldots$, test of H_0: $r \leq r_0$ (r_0 or fewer cointegrating relations) against H_1: $r_0 < r \leq k$

  $$\lambda_{trace}(r_0) = - T \, \Sigma^{k}_{j=r0+1} \log(1- \hat{l}_j)$$

  - $\hat{l}_j$: estimator of λ_j
  - H_0 is rejected for large values of $\lambda_{trace}(r_0)$
  - Stops when H_0 is not rejected for the first time
  - Critical values from simulations
- Max test: tests for $r_0 = 0, 1, \ldots$: H_0: $r = r_0$ (the eigenvalue λ_{r0+1} is different from zero) against H_1: $r = r_0+1$

  $$\lambda_{max}(r_0) = - T \log(1 - \hat{l}_{r0+1})$$

  - Stops when H_0 is not rejected for the first time
  - Critical values from simulations

Trace and Max Test: Critical Limits

Critical limits are shown in Verbeek's Table 9.9 for both tests

- Depend on presence of trends and intercepts
  - Case 1: no deterministic trends, intercepts in cointegrating relations
  - Case 2: k unrestricted intercepts in the VAR model, i.e., $k - r$ deterministic trends, r intercepts in cointegrating relations
- Depend on $k – r$
- Need small sample correction, e.g., factor $(T\text{-}pk)/T$ for the test statistic: avoids too large values of r

Example: Purchasing Power Parity

Verbeek's dataset ppp: price indices and exchange rates for France and Italy, T = 186 (1/1981-6/1996)

- Variables: LNIT (log price index Italy), LNFR (log price index France), LNX (log exchange rate France/Italy)

Purchasing power parity (PPP): exchange rate between the currencies (Franc, Lira) equals the ratio of price levels of the countries

- Relative PPP: equality fulfilled only in the long run; equilibrium or cointegrating relation

$$LNX_t = \alpha + \beta\ LNP_t + \varepsilon_t$$

  with $LNP_t = LNIT_t - LNFR_t$, i.e., the log of the price index ratio France/Italy
- Generalization:

$$LNX_t = \alpha + \beta_1\ LNIT_t - \beta_2\ LNFR_t + \varepsilon_t$$

PPP: Cointegrating Rank *r*

As discussed by Verbeek: Johansen test for k = 3 variables, maximal lag order p = 3

r_0	eigen-value	$\lambda_{tr}(r_0)$	p-value	$\lambda_{max}(r_0)$	p-value
0	0.301	93.9	0.0000	65.5	0.0000
1	0.113	28.4	0.0023	22.0	0.0035
2	0.034	6.37	0.169	6.4	0.1690

H_0 not rejected that smallest eigenvalue equals zero: series are non-stationary

Both the trace and the max test suggest r = 2

Contents

- Non-stationary Time Series
- Cointegration
- VAR Models
- VAR Models and Cointegration
- VEC Model: Cointegration Tests
- VEC Model: Specification and Estimation

Estimation of VEC Models

Estimation of

$$\Delta Y_t = \delta + \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_{p-1} \Delta Y_{t-p+1} + \Pi Y_{t-1} + \varepsilon_t$$

requires finding (kxr)-matrices α and β with $\Pi = \alpha\beta'$

- β: matrix of cointegrating vectors
- α: matrix of adjustment coefficients

- Identification problem: linear combinations of cointegrating vectors are also cointegrating vectors
- Unique solutions for α and β require restrictions
- Minimum number of restrictions which guarantee identification is r^2
- Normalization
  - Phillips normalization
  - Manual normalization

Phillips Normalization

Cointegrating vector

$$\beta' = (\beta_1', \beta_2')$$

β_1: (rxr)-matrix with rank r, β_2: [(k-r)xr]-matrix

- Normalization consists in transforming β into

$$\hat{\beta} = \begin{pmatrix} I \\ \beta_2 \beta_1^{-1} \end{pmatrix} = \begin{pmatrix} I \\ -B \end{pmatrix}$$

  with matrix B of unrestricted coefficients
- The r cointegrating relations express the first r variables as functions of the remaining k - r variables
- Fulfills the condition that at least r^2 restrictions are needed to guarantee identification
- Resulting equilibrium relations may be difficult to interpret
- Alternative: manual normalization

Example: Money Demand

Verbeek's data set "money": US data 1:54 – 12:1994 (*T*=164)

- m: log of real M1 money stock
- infl: quaterly inflation rate (change in log prices, % per year)
- cpr: commercial paper rate (% per year)
- y: log real GDP (billions of 1987 dollars)
- tbr: treasury bill rate

Money Demand: Cointegrating Vectors

ML estimates, lag order $p = 6$, cointegration rank $r = 2$, restricted constant

- Cointegrating vectors β_1 and β_2 and standard errors (*s.e.*), Phillips normalization

	m	infl	cpr	*y*	tbr	const
β_1	1.00	0.00	0.61	-0.35	-0.60	-4.27
(*s.e.*)	(0.00)	(0.00)	(0.12)	(0.12)	(0.12)	(0.91)
β_2	0.00	1.00	-26.95	-3.28	-27.44	39.25
(*s.e.*)	(0.00)	(0.00)	(4.66)	(4.61)	(4.80)	(35.5)

Estimation of VEC Models: k=2

Estimation procedure consists of the following steps

1. Test the variables in the 2-vector Y_t for stationarity using the usual ADF tests; VEC models need $I(1)$ variables
2. Determine the order p
3. Specification of
   - deterministic trends of the variables in Y_t
   - intercept in the cointegrating relation
4. Cointegration test
5. Estimation of cointegrating relation, normalization
6. Estimation of the VEC model

Example: Income and Consumption

Model:

$$Y_t = \delta_1 + \theta_{11} Y_{t-1} + \theta_{12} C_{t-1} + \varepsilon_{1t}$$
$$C_t = \delta_2 + \theta_{21} C_{t-1} + \theta_{22} Y_{t-1} + \varepsilon_{2t}$$

With $Z = (Y, C)'$, 2-vectors δ and ε, and (2x2)-matrix Θ, the VAR(1) model is

$$Z_t = \delta + \Theta Z_{t-1} + \varepsilon_t$$

Represents each component of Z as a linear combination of lagged variables

Income and Consumption: VEC(1) Model

AWM data base: *PCR* (real private consumption), *PYR* (real disposable income of households); logarithms: *C*, *Y*

1. Check whether *C* and *Y* are non-stationary:

   $C \sim I(1)$, $Y \sim I(1)$

2. Johansen test for cointegration: given that *C* and *Y* have no trends and the cointegrating relationship has an intercept:

   $r = 1$ ($p < 0.05$)

   the cointegrating relationship is

   $C = 8.55 – 1.61Y$

   with $t(Y) = 18.2$

Income and Consumption: VEC(1) Model, cont'd

3. VEC(1) model (same specification as in 2.) with $Z = (Y, C)'$

$$\Delta Z_t = -\gamma(\beta' Z_{t-1} + \delta) + \Gamma \Delta Z_{t-1} + \varepsilon_t$$

		coint	ΔY_{-1}	ΔC_{-1}	adj.R^2	AIC
ΔY	γ_{ij}	0.029	0.167	0.059	0.14	-7.42
	$t(\gamma_{ij})$	5.02	1.59	0.49		
ΔC	γ_{ij}	0.047	0.226	-0.148	0.18	-7.59
	$t(\gamma_{ij})$	2.36	2.34	1.35		

The model explains growth rates of *PCR* and *PYR*; AIC = -15.41 is smaller than that of the VAR(1)-Modell (AIC = -14.45)

Estimation of VEC Models

Estimation procedure consists of the following steps

1. Test of the *k* variables in Y_t for stationarity: ADF test
2. Determination of the number *p* of lags in the cointegration test (order of VAR): AIC or BIC
3. Specification of
   - deterministic trends of the variables in Y_t
   - intercept in the cointegrating relations
4. Determination of the number *r* of cointegrating relations: trace and/or max test
5. Estimation of the coefficients β of the cointegrating relations and the adjustment α coefficients; normalization; assessment of the cointegrating relations
6. Estimation of the VEC model

VEC Models in GRETL

Model > Time Series > VAR lag selection…

- Calculates information criteria like AIC and BIC from VARs of order 1 to the chosen maximum order of the VAR

Model > Time Series > Cointegration test > Johansen…

- Calculates eigenvalues, test statistics for the trace and max tests, and estimates of the matrices α, β, and $\Pi = \alpha\beta'$

Model > Time Series > VECM

- Estimates the specified VEC model for a given cointegration rank: (1) cointegrating vectors and standard errors, (2) adjustment vectors, (3) coefficients and various criteria for each of the equations of the VEC model

Your Homework

1. Perform the steps 1 – 6 for estimating a VEC model for Verbeek's dataset "model"; choose p = 2 and r = 2 for estimating the VEC model. Explain the steps and interpret the results of each step.
2. Derive the VEC form of the VAR(3) model

$$Y_t = \delta + \Theta_1 Y_{t-1} + \ldots + \Theta_3 Y_{t-3} + \varepsilon_t$$

   assuming a k-vector Y_t and appropriate orders of the other vectors and matrices.