

Questions for Review

1. In the Solow growth model, a high saving rate leads to a large steady-state capital stock and a high level of steady-state output. A low saving rate leads to a small steady-state capital stock and a low level of steady-state output. Higher saving leads to faster economic growth only in the short run. An increase in the saving rate raises growth until the economy reaches the new steady state. That is, if the economy maintains a high saving rate, it will also maintain a large capital stock and a high level of output, but it will *not* maintain a high rate of growth forever. In the steady state, the growth rate of output (or income) is independent of the saving rate.
2. It is reasonable to assume that the objective of an economic policymaker is to maximize the economic well-being of the individual members of society. Since economic well-being depends on the amount of consumption, the policymaker should choose the steady state with the highest level of consumption. The Golden Rule level of capital represents the level that maximizes consumption in the steady state.

Suppose, for example, that there is no population growth or technological change. If the steady-state capital stock increases by one unit, then output increases by the marginal product of capital MPK ; depreciation, however, increases by an amount δ , so that the net amount of extra output available for consumption is $MPK - \delta$. The Golden Rule capital stock is the level at which $MPK = \delta$, so that the marginal product of capital equals the depreciation rate.

3. When the economy begins above the Golden Rule level of capital, reaching the Golden Rule level leads to higher consumption at all points in time. Therefore, the policymaker would always want to choose the Golden Rule level, because consumption is increased for all periods of time. On the other hand, when the economy begins below the Golden Rule level of capital, reaching the Golden Rule level means reducing consumption today to increase consumption in the future. In this case, the policymaker's decision is not as clear. If the policymaker cares more about current generations than about future generations, he or she may decide *not* to pursue policies to reach the Golden Rule steady state. If the policymaker cares equally about all generations, then he or she chooses to reach the Golden Rule. Even though the current generation will have to consume less, an infinite number of future generations will benefit from increased consumption by moving to the Golden Rule.

4. The higher the population growth rate is, the lower the steady-state level of capital per worker, and therefore there is a lower level of steady-state income per worker. For example, Figure 7–1 shows the steady state for two levels of population growth, a low level n_1 and a higher level n_2 . The higher population growth n_2 means that the line representing population growth and depreciation is higher, so the steady-state level of capital per worker is lower.

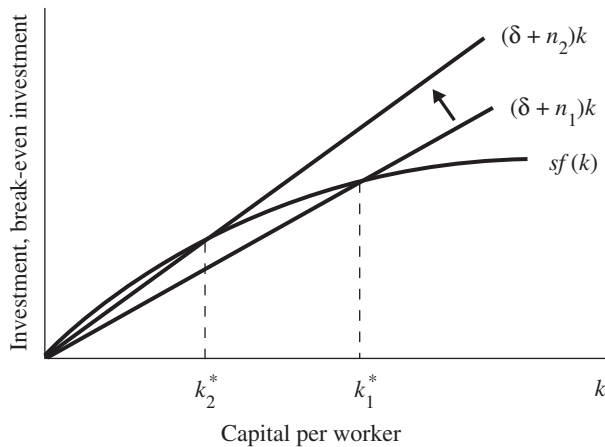


Figure 7–1

In a model with no technological change, the steady-state growth rate of total income is n : the higher the population growth rate n is, the higher the growth rate of total income. Income per worker, however, grows at rate zero in steady state and, thus, is not affected by population growth.

Problems and Applications

1. a. A production function has constant returns to scale if increasing all factors of production by an equal percentage causes output to increase by the same percentage. Mathematically, a production function has constant returns to scale if $zY = F(zK, zL)$ for any positive number z . That is, if we multiply both the amount of capital and the amount of labor by some amount z , then the amount of output is multiplied by z . For example, if we double the amounts of capital and labor we use (setting $z = 2$), then output also doubles.

To see if the production function $Y = F(K, L) = K^{1/2}L^{1/2}$ has constant returns to scale, we write:

$$F(zK, zL) = (zK)^{1/2}(zL)^{1/2} = zK^{1/2}L^{1/2} = zY.$$

Therefore, the production function $Y = K^{1/2}L^{1/2}$ has constant returns to scale.

- b. To find the per-worker production function, divide the production function $Y = K^{1/2}L^{1/2}$ by L :

$$\frac{Y}{L} = \frac{K^{1/2}L^{1/2}}{L}.$$

If we define $y = Y/L$, we can rewrite the above expression as:

$$y = K^{1/2}/L^{1/2}.$$

Defining $k = K/L$, we can rewrite the above expression as:

$$y = k^{1/2}.$$

- c. We know the following facts about countries A and B:

δ = depreciation rate = 0.05,

s_a = saving rate of country A = 0.1,

s_b = saving rate of country B = 0.2, and

$y = k^{1/2}$ is the per-worker production function derived in part (b) for countries A and B.

The growth of the capital stock Δk equals the amount of investment $sf(k)$, less the amount of depreciation δk . That is, $\Delta k = sf(k) - \delta k$. In steady state, the capital stock does not grow, so we can write this as $sf(k) = \delta k$.

To find the steady-state level of capital per worker, plug the per-worker production function into the steady-state investment condition, and solve for k^* :

$$sk^{1/2} = \delta k.$$

Rewriting this:

$$k^{1/2} = s/\delta$$

$$k = (s/\delta)^2.$$

To find the steady-state level of capital per worker k^* , plug the saving rate for each country into the above formula:

$$\text{Country A: } k_a^* = (s_a/\delta)^2 = (0.1/0.05)^2 = 4.$$

$$\text{Country B: } k_b^* = (s_b/\delta)^2 = (0.2/0.05)^2 = 16.$$

Now that we have found k^* for each country, we can calculate the steady-state levels of income per worker for countries A and B because we know that $y = k^{1/2}$:

$$y_a^* = (4)^{1/2} = 2.$$

$$y_b^* = (16)^{1/2} = 4.$$

We know that out of each dollar of income, workers save a fraction s and consume a fraction $(1 - s)$. That is, the consumption function is $c = (1 - s)y$. Since we know the steady-state levels of income in the two countries, we find

$$\text{Country A: } c_a^* = (1 - s_a)y_a^* = (1 - 0.1)(2)$$

$$= 1.8.$$

$$\text{Country B: } c_b^* = (1 - s_b)y_b^* = (1 - 0.2)(4)$$

$$= 3.2.$$

- d. Using the following facts and equations, we calculate income per worker y , consumption per worker c , and capital per worker k :

$$s_a = 0.1.$$

$$s_b = 0.2.$$

$$\delta = 0.05.$$

$$k_o = 2 \text{ for both countries.}$$

$$y = k^{1/2}.$$

$$c = (1 - s)y.$$

Country A

Year	k	$y = k^{1/2}$	$c = (1 - s_a)y$	$i = s_a y$	δk	$\Delta k = i - \delta k$
1	2	1.414	1.273	0.141	0.100	0.041
2	2.041	1.429	1.286	0.143	0.102	0.041
3	2.082	1.443	1.299	0.144	0.104	0.040
4	2.122	1.457	1.311	0.146	0.106	0.040
5	2.162	1.470	1.323	0.147	0.108	0.039

Country B

Year	k	$y = k^{1/2}$	$c = (1 - s_a)y$	$i = s_a y$	δk	$\Delta k = i - \delta k$
1	2	1.414	1.131	0.283	0.100	0.183
2	2.183	1.477	1.182	0.295	0.109	0.186
3	2.369	1.539	1.231	0.308	0.118	0.190
4	2.559	1.600	1.280	0.320	0.128	0.192
5	2.751	1.659	1.327	0.332	0.138	0.194

Note that it will take five years before consumption in country B is higher than consumption in country A.

2. a. The production function in the Solow growth model is $Y = F(K, L)$, or expressed terms of output per worker, $y = f(k)$. If a war reduces the labor force through casualties, then L falls but $k = K/L$ rises. The production function tells us that total output falls because there are fewer workers. Output per worker increases, however, since each worker has more capital.
- b. The reduction in the labor force means that the capital stock per worker is higher after the war. Therefore, if the economy were in a steady state prior to the war, then after the war the economy has a capital stock that is higher than the steady-state level. This is shown in Figure 7–2 as an increase in capital per worker from k^* to k_1 . As the economy returns to the steady state, the capital stock per worker falls from k_1 back to k^* , so output per worker also falls.

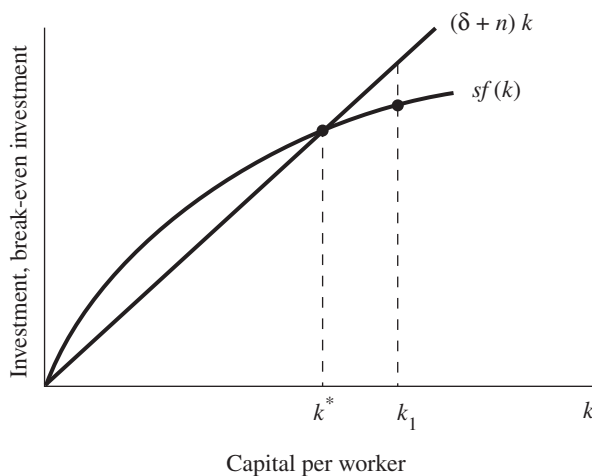


Figure 7–2

Hence, in the transition to the new steady state, the growth of output per worker is slower than normal. In the steady state, we know that the growth rate of output per worker is equal to zero, given there is no technological change in this model. Therefore, in this case, the growth rate of output per worker must be less than zero until the new steady state is reached.

3. a. We follow Section 7-1, “Approaching the Steady State: A Numerical Example.” The production function is $Y = K^{0.3}L^{0.7}$. To derive the per-worker production function $f(k)$, divide both sides of the production function by the labor force L :

$$\frac{Y}{L} = \frac{K^{0.3}L^{0.7}}{L}.$$

Rearrange to obtain:

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{0.3}.$$

Because $y = Y/L$ and $k = K/L$, this becomes:

$$y = k^{0.3}.$$

- b. Recall that

$$\Delta k = sf(k) - \delta k.$$

The steady-state value of capital per worker k^* is defined as the value of k at which capital per worker is constant, so $\Delta k = 0$. It follows that in steady state

$$0 = sf(k) - \delta k,$$

or, equivalently,

$$\frac{k^*}{f(k^*)} = \frac{s}{\delta}.$$

For the production function in this problem, it follows that:

$$\frac{k^*}{(k^*)^{0.3}} = \frac{s}{\delta}.$$

Rearranging:

$$(k^*)^{0.7} = \frac{s}{\delta},$$

or

$$k^* = \left(\frac{s}{\delta}\right)^{1/0.7}$$

Substituting this equation for steady-state capital per worker into the per-worker production function from part (a) gives:

$$y^* = \left(\frac{s}{\delta}\right)^{0.3/0.7}$$

Consumption is the amount of output that is not invested. Since investment in the steady state equals δk^* , it follows that

$$c^* = f(k^*) - \delta k^* = \left(\frac{s}{\delta}\right)^{0.3/0.7} - \delta \left(\frac{s}{\delta}\right)^{1/0.7}$$

(Note: An alternative approach to the problem is to note that consumption also equals the amount of output that is not saved:

$$c^* = (1-s)f(k^*) = (1-s)(k^*)^{0.3} = (1-s)\left(\frac{s}{\delta}\right)^{0.3/0.7}$$

Some algebraic manipulation shows that this equation is equal to the equation above.)

- c. The table below shows k^* , y^* , and c^* for the saving rate in the left column, using the equations from part (b). We assume a depreciation rate of 10 percent (i.e., 0.1). (The last column shows the marginal product of capital, derived in part (d) below).

	k^*	y^*	c^*	$MPK-\delta k^*$
0	0.00	0.00	0.00	
0.1	1.00	1.00	0.90	0.2000
0.2	2.69	1.35	1.08	0.0500
0.3	4.80	1.60	1.12	0.0000
0.4	7.25	1.81	1.09	-0.0250
0.5	9.97	1.99	1.00	-0.0400
0.6	12.93	2.16	0.86	-0.0500
0.7	16.12	2.30	0.69	-0.0571
0.8	19.50	2.44	0.49	-0.0625
0.9	23.08	2.56	0.26	-0.0667
1	26.83	2.68	0.00	-0.0700

Note that a saving rate of 100 percent ($s = 1.0$) maximizes output per worker. In that case, of course, nothing is ever consumed, so $c^* = 0$. Consumption per worker is maximized at a rate of saving of 0.3 percent—that is, where s equals capital's share in output. This is the Golden Rule level of s .

- d. The marginal product of capital (MPK) is the change in output per worker (y) for a given change in capital per worker (k). To find the marginal product of capital, differentiate the per-worker production function with respect to capital per worker (k):

$$MPK = 0.3k^{-0.7} = \frac{0.3}{k^{0.7}}.$$

To find the marginal product of capital net of depreciation, use the equation above to calculate the marginal product of capital and then subtract depreciation, which is 10 percent of the value of the steady-state level of capital per worker. These values appear in the table above. Note that when consumption per worker is maximized, the value of the marginal product of capital net of depreciation is zero.

4. Suppose the economy begins with an initial steady-state capital stock below the Golden Rule level. The immediate effect of devoting a larger share of national output to investment is that the economy devotes a smaller share to consumption; that is, “living standards” as measured by consumption fall. The higher investment rate means that the capital stock increases more quickly, so the growth rates of output and output per worker rise. The productivity of workers is the average amount produced by each worker—that is, output per worker. So productivity growth rises. Hence, the immediate effect is that living standards fall but productivity growth rises.

In the new steady state, output grows at rate n , while output per worker grows at rate zero. This means that in the steady state, productivity growth is independent of the rate of investment. Since we begin with an initial steady-state capital stock below the Golden Rule level, the higher investment rate means that the new steady state has a higher level of consumption, so living standards are higher.

Thus, an increase in the investment rate increases the productivity growth rate in the short run but has no effect in the long run. Living standards, on the other hand, fall immediately and only rise over time. That is, the quotation emphasizes growth, but not the sacrifice required to achieve it.

- As in the text, let $k = K/L$ stand for capital per unit of labor. The equation for the evolution of k is

$$\Delta k = \text{Saving} - (\delta + n)k.$$

If all capital income is saved and if capital earns its marginal product, then saving equals $MPK \times k$. We can substitute this into the above equation to find

$$\Delta k = MPK \times k - (\delta + n)k.$$

In the steady state, capital per unit of labor does not change, so $\Delta k = 0$. From the above equation, this tells us that

$$MPK \times k = (\delta + n)k,$$

or

$$MPK = (\delta + n).$$

Equivalently,

$$MPK - \delta = n.$$

In this economy's steady state, the net marginal product of capital, $MPK - \delta$, equals the rate of growth of output, n . But this condition describes the Golden Rule steady state. Hence, we conclude that this economy reaches the Golden Rule level of capital accumulation.

- First, consider steady states. In Figure 7–3, the slower population growth rate shifts the line representing population growth and depreciation downward. The new steady state has a higher level of capital per worker, k_2^* , and hence a higher level of output per worker.

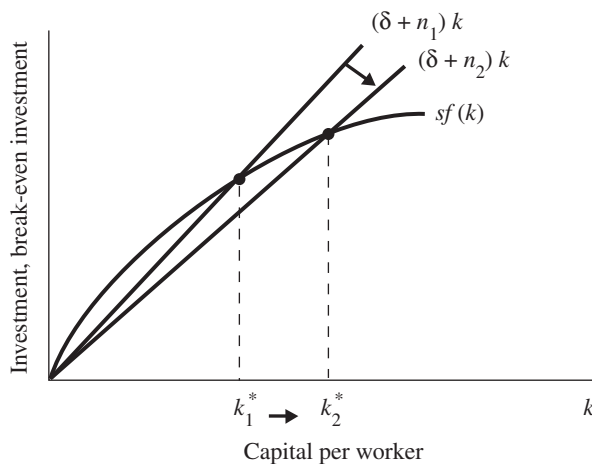


Figure 7–3

What about steady-state growth rates? In steady state, total output grows at rate n , whereas output per-worker grows at rate 0. Hence, slower population growth will lower total output growth, but per-worker output growth will be the same.

Now consider the transition. We know that the steady-state level of output per worker is higher with low population growth. Hence, during the transition to the new steady state, output per worker must grow at a rate faster than 0 for a while. In the decades after the fall in population growth, growth in total output will transition to its new lower level while growth in output per worker will jump up but then transition back to zero.

7. If there are decreasing returns to labor and capital, then increasing both capital and labor by the same proportion increases output by less than this proportion. For example, if we double the amounts of capital and labor, then output less than doubles. This may happen if there is a fixed factor such as land in the production function, and it becomes scarce as the economy grows larger. Then population growth will increase total output but decrease output per worker, since each worker has less of the fixed factor to work with.

If there are increasing returns to scale, then doubling inputs of capital and labor more than doubles output. This may happen if specialization of labor becomes greater as population grows. Then population growth increases total output and also increases output per worker, since the economy is able to take advantage of the scale economy more quickly.

8. a. To find output per worker y we divide total output by the number of workers:

$$\frac{Y}{L} = \frac{K^\alpha [(1-u)L]^{1-\alpha}}{L}$$

$$y = \left(\frac{K}{L}\right)^\alpha (1-u)^{1-\alpha}$$

$$y = k^\alpha (1-u)^{1-\alpha},$$

where the final step uses the definition $k = \frac{K}{L}$. Notice that unemployment reduces the amount of output per worker for any given capital–labor ratio because some of the workers are not producing anything.

The steady state is the level of capital per worker at which the increase in capital per worker from investment equals its decrease from depreciation and population growth (see Chapter 7 for more details).

$$sy = (\delta + n)k$$

$$sk^\alpha (1-u)^{1-\alpha} = (\delta + n)k$$

$$k^* = (1-u) \left(\frac{s}{\delta + n}\right)^{\frac{1}{1-\alpha}}$$

Unemployment lowers the marginal product of capital per worker and, hence, acts like a negative technological shock that reduces the amount of capital the economy can maintain in steady state. Figure 7–4 shows this graphically: an increase in unemployment lowers the $sf(k)$ line and the steady-state level of capital per worker.

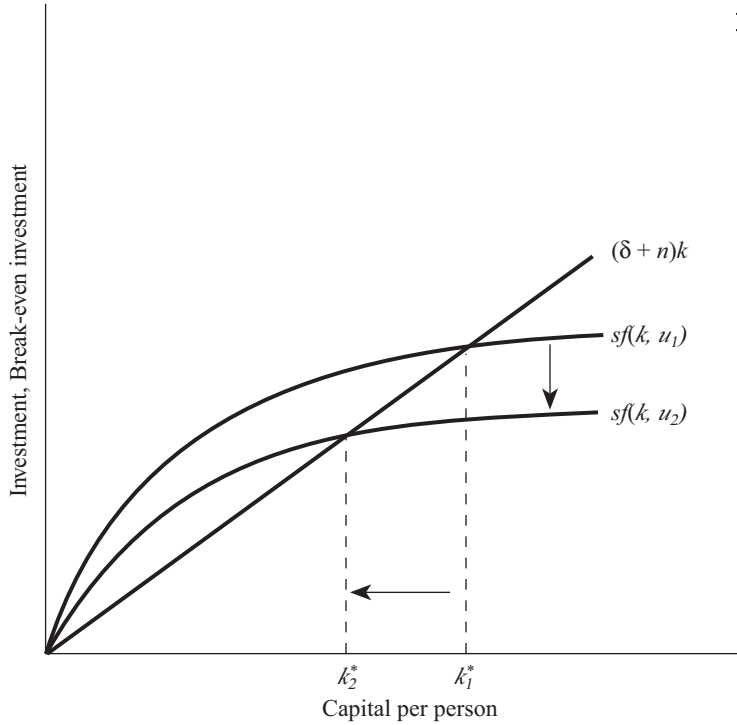


Figure 7-4

Finally, to get steady-state output per worker, plug the steady-state level of capital per worker into the production function:

$$\begin{aligned}
 y^* &= \left((1 - u^*) \left(\frac{s}{\delta + n} \right)^{\frac{1}{1-\alpha}} \right)^\alpha (1 - u^*)^{1-\alpha} \\
 &= (1 - u^*) \left(\frac{s}{\delta + n} \right)^{\frac{\alpha}{1-\alpha}}
 \end{aligned}$$

Unemployment lowers steady-state output for two reasons: for a given k , unemployment lowers y , and unemployment also lowers the steady-state value k^* .

- b. Figure 7-5 below shows the pattern of output over time. As soon as unemployment falls from u_1 to u_2 , output jumps up from its initial steady-state value of $y^*(u_1)$. The economy has the same amount of capital (since it takes time to adjust the capital stock), but this capital is combined with more workers. At that moment the economy is out of steady state: it has less capital than it wants to match the increased number of workers in the economy. The economy begins its transition by accumulating more capital, raising output even further than the original jump. Eventually the capital stock and output converge to their new, higher steady-state levels.

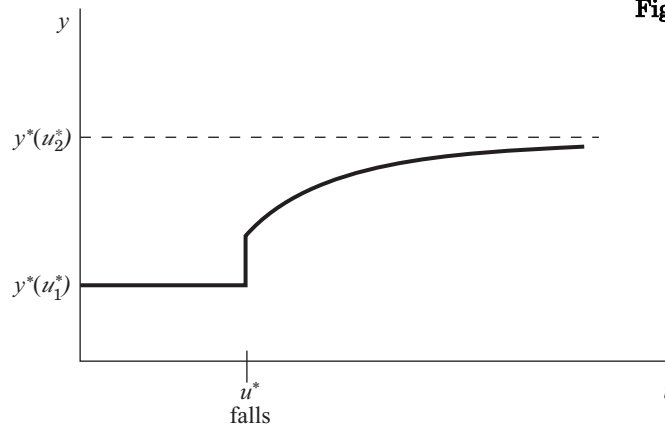


Figure 7-5

9. There is no unique way to find the data to answer this question. For example, from the World Bank web site, I followed links to "Data and Statistics." I then followed a link to "Quick Reference Tables" (<http://www.worldbank.org/data/databytopic/GNPPC.pdf>) to find a summary table of income per capita across countries. (Note that there are some subtle issues in converting currency values across countries that are beyond the scope of this book. The data in Table 7-1 use what are called "purchasing power parity.")

As an example, I chose to compare the United States (income per person of \$31,900 in 1999) and Pakistan (\$1,860), with a 17-fold difference in income per person. How can we decide what factors are most important? As the text notes, differences in income must come from differences in capital, labor, and/or technology. The Solow growth model gives us a framework for thinking about the importance of these factors.

One clear difference across countries is in educational attainment. One can think about differences in educational attainment as reflecting differences in broad "human capital" (analogous to physical capital) or as differences in the level of technology (e.g., if your work force is more educated, then you can implement better technologies). For our purposes, we will think of education as reflecting "technology," in that it allows more output per worker for any given level of physical capital per worker.

From the World Bank web site (country tables) I found the following data (downloaded February 2002):

	Labor Force Growth (1994–2000)	Investment/GDP (1990) (percent)	Illiteracy (percent of population 15+)
United States	1.5	18	0
Pakistan	3.0	19	54

How can we decide which factor explains the most? It seems unlikely that the small difference in investment/GDP explains the large difference in per capita income, leaving labor-force growth and illiteracy (or, more generally, technology) as the likely culprits. But we can be more formal about this using the Solow model.

We follow Section 7-1, "Approaching the Steady State: A Numerical Example." For the moment, we assume the two countries have the same production technology: $Y=K^{0.5}L^{0.5}$. (This will allow us to decide whether differences in saving and population growth can explain the differences in income per capita; if not, then differences in technology will remain as the likely explanation.) As in the text, we can express this equation in terms of the per-worker production function $f(k)$:

$$y = k^{0.5}.$$

In steady-state, we know that

$$\Delta k = sf(k) - (n + \delta)k.$$

The steady-state value of capital per worker k^* is defined as the value of k at which capital per worker is constant, so $\Delta k = 0$. It follows that in steady state

$$0 = sf(k) - (n + \delta)k,$$

or, equivalently,

$$\frac{k^*}{f(k^*)} = \frac{s}{n + \delta}.$$

For the production function in this problem, it follows that:

$$\frac{k^*}{(k^*)^{0.5}} = \frac{s}{n + \delta},$$

Rearranging:

$$(k^*)^{0.5} = \frac{s}{n + \delta},$$

or

$$k^* = \left(\frac{s}{n + \delta} \right)^2.$$

Substituting this equation for steady-state capital per worker into the per-worker production function gives:

$$y^* = \left(\frac{s}{n + \delta} \right).$$

If we assume that the United States and Pakistan are in steady state and have the same rates of depreciation—say, 5 percent—then the ratio of income per capita in the two countries is:

$$\frac{y_{\text{US}}}{y_{\text{Pakistan}}} = \left[\frac{s_{\text{US}}}{s_{\text{Pakistan}}} \right] \left[\frac{n_{\text{Pakistan}} + 0.05}{n_{\text{US}} + 0.05} \right]$$

This equation tells us that if, say, the U.S. saving rate had been twice Pakistan's saving rate, then U.S. income per worker would be twice Pakistan's level (other things equal). Clearly, given that the U.S. has 17-times higher income per worker but very similar levels of investment relative to GDP, this variable is not a major factor in the comparison. Even population growth can only explain a factor of 1.2 (0.08/0.065) difference in levels of output per worker.

The remaining culprit is technology, and the high level of illiteracy in Pakistan is consistent with this conclusion.