PART II

CHAPTER SIX

6

RISK AVERSION AND CAPITAL ALLOCATION TO RISKY ASSETS

THE PROCESS OF constructing an investor portfolio can be viewed as a sequence of two steps: (1) selecting the composition of one's portfolio of risky assets such as stocks and long-term bonds, and (2) deciding how much to invest in that risky portfolio versus in a safe asset such as short-term Treasury bills. Obviously, an investor cannot decide how to allocate investment funds between the risk-free asset and that risky portfolio without knowing its expected return and degree of risk, so a fundamental part of the asset allocation problem is to characterize the risk–return trade-off for this portfolio.

While the task of constructing an optimal risky portfolio is technically complex, it can be delegated to a professional because it largely entails well-defined optimization techniques. In contrast, the decision of how much to invest in that portfolio depends on an investor's *personal* preferences about risk versus expected return, and therefore it cannot easily be delegated. As we will see in the chapter on behavioral finance, many investors stumble over this cardinal step. We therefore begin our journey into portfolio theory by establishing a framework to explore this fundamental decision, namely, capital allocation between the risk-free and the risky portfolio.

We begin by introducing two themes in portfolio theory that are centered on risk. The first is the tenet that investors will avoid risk unless they can anticipate a reward for engaging in risky investments. The second theme allows us to quantify investors' personal trade-offs between portfolio risk and expected return. To do this we introduce a personal *utility function*, which allows each investor to assign welfare or "utility" scores to alternative portfolios based on expected return and risk and choose the portfolio with the highest score. We elaborate on the historical and empirical basis for the utility model in the appendix to this chapter.

Armed with the utility model, we can resolve the investment decision that is most consequential to investors, that is, how much of their wealth to put at risk for the greater expected return that can thus be achieved. We assume that the construction of the risky portfolio from the universe of available risky assets has already taken place and defer the discussion of how to construct that risky portfolio to the next chapter. At this point the investor can assess the expected return and risk of the overall portfolio. Using the expected return and risk parameters in the utility model yields the optimal allocation of capital between the risky portfolio and risk-free asset.

6.1 RISK AND RISK AVERSION

In Chapter 5 we introduced the concepts of the holding-period return (HPR) and the excess return over the risk-free rate. We also discussed estimation of the **risk premium** (the *expected* excess return) and the standard deviation of the rate of return, which we use as the measure of portfolio risk. We demonstrated these concepts with a scenario analysis of a specific risky portfolio (Spreadsheet 5.1). To emphasize that bearing risk typically must be accompanied by a reward in the form of a risk premium, we first distinguish between speculation and gambling.

Risk, Speculation, and Gambling

One definition of *speculation* is "the assumption of considerable investment risk to obtain commensurate gain." Although this definition is fine linguistically, it is useless without first specifying what is meant by "considerable risk" and "commensurate gain."

By "considerable risk" we mean that the risk is sufficient to affect the decision. An individual might reject an investment that has a positive risk premium because the potential gain is insufficient to make up for the risk involved. By "commensurate gain" we mean a positive risk premium, that is, an expected profit greater than the risk-free alternative.

To gamble is "to bet or wager on an uncertain outcome." If you compare this definition to that of speculation, you will see that the central difference is the lack of "commensurate gain." Economically speaking, a gamble is the assumption of risk for no purpose but enjoyment of the risk itself, whereas speculation is undertaken *in spite* of the risk involved because one perceives a favorable risk–return trade-off. To turn a gamble into a speculative prospect requires an adequate risk premium to compensate risk-averse investors for the risks they bear. Hence, *risk aversion and speculation are not inconsistent.* Notice that a risky investment with a risk premium of zero, sometimes called a **fair game,** amounts to a gamble. A risk-averse investor will reject it.

In some cases a gamble may appear to the participants as speculation. Suppose two investors disagree sharply about the future exchange rate of the U.S. dollar against the British pound. They may choose to bet on the outcome. Suppose that Paul will pay Mary $100 if the value of £1 exceeds $1.90 one year from now, whereas Mary will pay Paul if the pound is worth less than $1.90. There are only two relevant outcomes: (1) the pound will exceed $1.90, or (2) it will fall below $1.90. If both Paul and Mary agree on the probabilities of the two possible outcomes, and if neither party anticipates a loss, it must be that they assign $p = .5$ to each outcome. In that case the expected profit to both is zero and each has entered one side of a gambling prospect.

What is more likely, however, is that the bet results from differences in the probabilities that Paul and Mary assign to the outcome. Mary assigns it $p > .5$, whereas Paul's assessment is $p < .5$. They perceive, subjectively, two different prospects. Economists call this case of differing beliefs "heterogeneous expectations." In such cases investors on each side of a financial position see themselves as speculating rather than gambling.

Both Paul and Mary should be asking, Why is the other willing to invest in the side of a risky prospect that I believe offers a negative expected profit? The ideal way to resolve heterogeneous beliefs is for Paul and Mary to "merge their information," that is, for each party to verify that he or she possesses all relevant information and processes the information properly. Of course, the acquisition of information and the extensive communication that is required to eliminate all heterogeneity in expectations is costly, and thus up to a point heterogeneous expectations cannot be taken as irrational. If, however, Paul and Mary enter such contracts frequently, they would recognize the information problem in one of two ways: Either they will realize that they are creating gambles when each wins half of the bets, or the consistent loser will admit that he or she has been betting on the basis of inferior forecasts.

CONCEPT CHECK 1

Assume that dollar-denominated T-bills in the United States and pound-denominated bills in the United Kingdom offer equal yields to maturity. Both are short-term assets, and both are free of default risk. Neither offers investors a risk premium. However, a U.S. investor who holds U.K. bills is subject to exchange rate risk, because the pounds earned on the U.K. bills eventually will be exchanged for dollars at the future exchange rate. Is the U.S. investor engaging in speculation or gambling?

Risk Aversion and Utility Values

The history of rates of return on various asset classes presented in Chapter 5, as well as numerous elaborate empirical studies, leave no doubt that risky assets command a risk premium in the marketplace. This implies that most investors are risk averse.

Investors who are **risk averse** reject investment portfolios that are fair games or worse. Risk-averse investors are willing to consider only risk-free or speculative prospects with positive risk premiums. Loosely speaking, a risk-averse investor "penalizes" the expected rate of return of a risky portfolio by a certain percentage (or penalizes the expected profit by a dollar amount) to account for the risk involved. The greater the risk, the larger the penalty. One might wonder why we assume risk aversion as fundamental. We believe that most investors would accept this view from simple introspection, but we discuss the question more fully in the Appendix of this chapter.

To illustrate the issues we confront when choosing among portfolios with varying degrees of risk, consider a specific example. Suppose the risk-free rate is 5% and that an investor considers three alternative risky portfolios with risk premiums, expected returns, and standard deviations as given in Table 6.1. The risk premiums and degrees of risk (standard deviation, SD) of the portfolios in the table are chosen to represent the properties of low-risk bonds (*L*), high-risk bonds (*M*), and large stocks (*H*). Accordingly, these portfolios offer progressively higher risk premiums to compensate for greater risk. How might investors choose among them?

TABLE 6.1

Available risky portfolios (Risk-free rate = 5%)

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

Intuitively, one would rank each portfolio as more attractive when its expected return is higher, and lower when its risk is higher. But when risk increases along with return, the most attractive portfolio is not obvious. How can investors quantify the rate at which they are willing to trade off return against risk?

We will assume that each investor can assign a welfare, or **utility,** score to competing investment portfolios based on the expected return and risk of those portfolios. Higher utility values are assigned to portfolios with more attractive risk-return profiles. Portfolios receive higher utility scores for higher expected returns and lower scores for higher volatility. Many particular "scoring" systems are legitimate. One reasonable function that has been employed by both financial theorists and the CFA Institute assigns a portfolio with expected return $E(r)$ and variance of returns σ^2 the following utility score:

$$U = E(r) - \tfrac{1}{2}A\sigma^2 \qquad \textbf{(6.1)}$$

where U is the utility value and A is an index of the investor's risk aversion. The factor of ½ is just a scaling convention. To use Equation 6.1, rates of return must be expressed as decimals rather than percentages.

Equation 6.1 is consistent with the notion that utility is enhanced by high expected returns and diminished by high risk. Notice that risk-free portfolios receive a utility score equal to their (known) rate of return, because they receive no penalty for risk. The extent to which the variance of risky portfolios lowers utility depends on A, the investor's degree of risk aversion. More risk-averse investors (who have the larger values of A) penalize risky investments more severely. Investors choosing among competing investment portfolios will select the one providing the highest utility level. The nearby box discusses some techniques that financial advisers use to gauge the risk aversion of their clients.

EXAMPLE 6.1 Evaluating Investments by Using Utility Scores

Consider three investors with different degrees of risk aversion: $A_1 = 2$, $A_2 = 3.5$, and $A_3 = 5$, all of whom are evaluating the three portfolios in Table 6.1. Because the risk-free rate is assumed to be 5%, Equation 6.1 implies that all three investors would assign a utility score of .05 to the risk-free alternative. Table 6.2 presents the utility scores that would be assigned by each investor to each portfolio. The portfolio with the highest utility score for each investor appears in bold. Notice that the high-risk portfolio, H, would be chosen only by the investor with the lowest degree of risk aversion, $A_1 = 2$, while the low-risk portfolio, L, would be passed over even by the most risk-averse of our three investors. All three portfolios beat the risk-free alternative for the investors with levels of risk aversion given in the table.

Investor Risk Aversion (A)	Utility Score of Portfolio L [$E(r) = .07$; $\sigma = .05$]	Utility Score of Portfolio M [$E(r) = .09$; $\sigma = .10$]	Utility Score of Portfolio H [$E(r) = .13$; $\sigma = .20$]
2.0	$.07 - \frac{1}{2} \times 2 \times .05^2 = .0675$	$.09 - \frac{1}{2} \times 2 \times .1^2 = .0800$	$\mathbf{.13 - \frac{1}{2} \times 2 \times .2^2 = .09}$
3.5	$.07 - \frac{1}{2} \times 3.5 \times .05^2 = .0656$	$\mathbf{.09 - \frac{1}{2} \times 3.5 \times .1^2 = .0725}$	$.13 - \frac{1}{2} \times 3.5 \times .2^2 = .06$
5.0	$.07 - \frac{1}{2} \times 5 \times .05^2 = .0638$	$\mathbf{.09 - \frac{1}{2} \times 5 \times .1^2 = .0650}$	$.13 - \frac{1}{2} \times 5 \times .2^2 = .03$

TABLE 6.2

Utility scores of alternative portfolios for investors with varying degrees of risk aversion

We can interpret the utility score of *risky* portfolios as a **certainty equivalent rate** of return. The certainty equivalent rate is the rate that risk-free investments would need to offer to provide the same utility score as the risky portfolio. In other words, it is the rate that, if earned with certainty, would provide a utility score equivalent to that of the portfolio in question. The certainty equivalent rate of return is a natural way to compare the utility values of competing portfolios.

Now we can say that a portfolio is desirable only if its certainty equivalent return exceeds that of the risk-free alternative. A sufficiently risk-averse investor may assign any risky portfolio, even one with a positive risk premium, a certainty equivalent rate of return that is below the risk-free rate, which will cause the investor to reject the risky portfolio. At the same time, a less risk-averse investor may assign the same portfolio a certainty equivalent rate that exceeds the risk-free rate and thus will prefer the portfolio to the risk-free alternative. If the risk premium is zero or negative to begin with, any downward adjustment to utility only makes the portfolio look worse. Its certainty equivalent rate will be below that of the risk-free alternative for all risk-averse investors.

CONCEPT CHECK 2

A portfolio has an expected rate of return of 20% and standard deviation of 30%. T-bills offer a safe rate of return of 7%. Would an investor with risk-aversion parameter $A = 4$ prefer to invest in T-bills or the risky portfolio? What if $A = 2$?

In contrast to risk-averse investors, **risk-neutral** investors (with $A = 0$) judge risky prospects solely by their expected rates of return. The level of risk is irrelevant to the risk-neutral investor, meaning that there is no penalty for risk. For this investor a portfolio's certainty equivalent rate is simply its expected rate of return.

A **risk lover** (for whom $A < 0$) is willing to engage in fair games and gambles; this investor adjusts the expected return *upward* to take into account the "fun" of confronting the prospect's risk. Risk lovers will always take a fair game because their upward adjustment of utility for risk gives the fair game a certainty equivalent that exceeds the alternative of the risk-free investment.

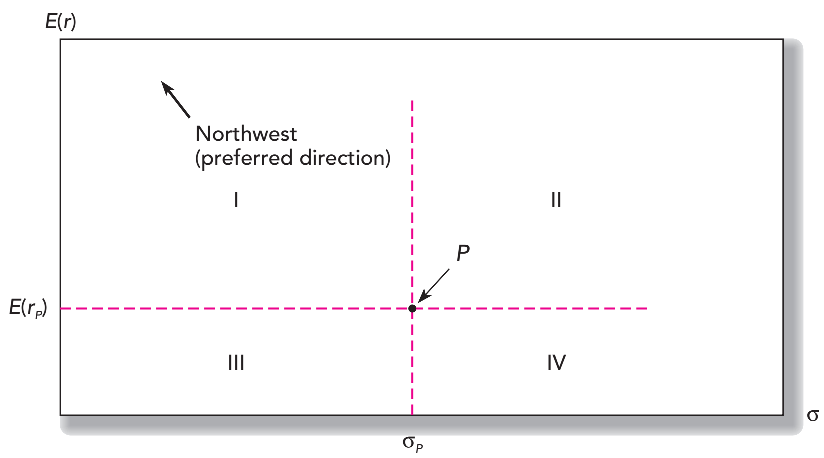


FIGURE 6.1 The trade-off between risk and return of a potential investment portfolio, *P*

We can depict the individual's trade-off between risk and return by plotting the characteristics of potential investment portfolios that the individual would view as equally attractive on a graph with axes measuring the expected value and standard deviation of portfolio returns. Figure 6.1 plots the characteristics of one portfolio denoted *P*.

Portfolio *P*, which has expected return $E(r_P)$ and standard deviation σ_P is preferred by risk-averse investors to any portfolio in quadrant IV because it has an expected return equal to or greater than any portfolio in that quadrant and a standard deviation equal to or smaller than any portfolio in that quadrant. Conversely, any portfolio in quadrant I is preferable to portfolio P because its expected return is equal to or greater than P's and its standard deviation is equal to or smaller than P's.

This is the mean-standard deviation, or equivalently, **mean-variance (M-V) criterion.** It can be stated as follows: portfolio A dominates B if

$$E(r_A) \geq E(r_B)$$

and

$$\sigma_A \leq \sigma_B$$

and at least one inequality is strict (rules out the equality).

In the expected return–standard deviation plane in Figure 6.1, the preferred direction is northwest, because in this direction we simultaneously increase the expected return *and* decrease the variance of the rate of return. This means that any portfolio that lies northwest of P is superior to it.

What can be said about portfolios in quadrants II and III? Their desirability, compared with P, depends on the exact nature of the investor's risk aversion. Suppose an investor identifies all portfolios that are equally attractive as portfolio P. Starting at P, an increase in standard deviation lowers utility; it must be compensated for by an increase in expected return. Thus point Q in Figure 6.2 is equally desirable to this investor as P. Investors will be equally attracted to portfolios with high risk and high expected returns compared with other portfolios with lower risk but lower expected returns. These equally preferred portfolios will lie in the mean–standard deviation plane on a curve called the **indifference curve** that connects all portfolio points with the same utility value (Figure 6.2).

To determine some of the points that appear on the indifference curve, examine the utility values of several possible portfolios for an investor with $A = 4$, presented in Table 6.3. Note that each portfolio offers identical utility, because the portfolios with higher expected return also have higher risk (standard deviation).

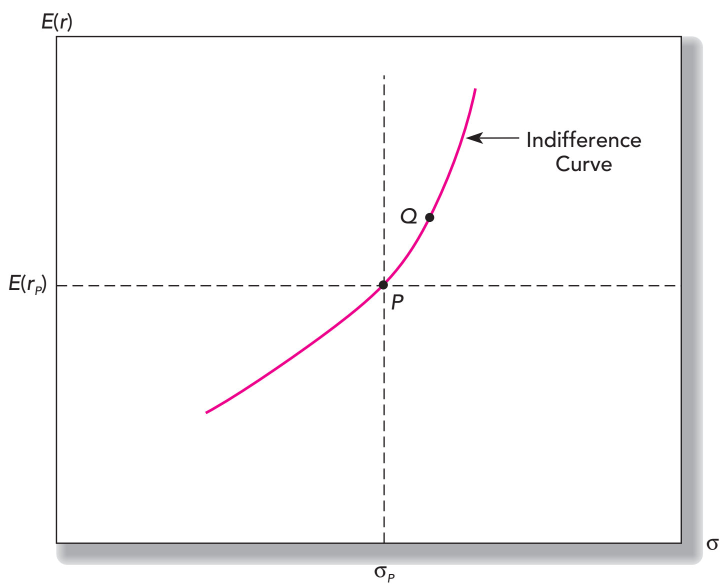


FIGURE 6.2 The indifference curve

Estimating Risk Aversion

How might we go about estimating the levels of risk aversion we might expect to observe in practice? One way is to observe individuals' decisions when confronted with risk. For example, we can observe how much people are willing to pay to avoid risk, such as when they buy insurance against large losses. Consider

CONCEPT CHECK 3

a. How will the indifference curve of a less risk-averse investor compare to the indifference curve drawn in Figure 6.2?

b. Draw both indifference curves passing through point *P*.

TIME FOR INVESTING'S FOUR-LETTER WORD

What four-letter word should pop into mind when the stock market takes a harrowing nose dive?

No, not those. R-I-S-K.

Risk is the potential for realizing low returns or even losing money, possibly preventing you from meeting important objectives, like sending your kids to the college of their choice or having the retirement lifestyle you crave.

But many financial advisers and other experts say that when times are good, some investors don't take the idea of risk as seriously as they should, and overexpose themselves to stocks. So before the market goes down and stays down, be sure that you understand your tolerance for risk and that your portfolio is designed to match it.

Assessing your risk tolerance, however, can be tricky. You must consider not only how much risk you can *afford* to take but also how much risk you can *stand* to take.

Determining how much risk you can stand—your temperamental tolerance for risk—is more difficult. It isn't easy to quantify.

To that end, many financial advisers, brokerage firms and mutual-fund companies have created risk quizzes to help people determine whether they are conservative, moderate or aggressive investors. Some firms that offer such quizzes include Merrill Lynch, T. Rowe Price Associates Inc., Baltimore, Zurich Group Inc.'s Scudder Kemper Investments Inc., New York, and Vanguard Group in Malvern, Pa.

Typically, risk questionnaires include seven to 10 questions about a person's investing experience, financial security and tendency to make risky or conservative choices.

The benefit of the questionnaires is that they are an objective resource people can use to get at least a rough idea of their risk tolerance. "It's impossible for someone to assess their risk tolerance alone," says Mr. Bernstein. "I may say I don't like risk, yet will take more risk than the average person."

Many experts warn, however, that the questionnaires should be used simply as a first step to assessing risk tolerance. "They are not precise," says Ron Meier, a certified public accountant.

The second step, many experts agree, is to ask yourself some difficult questions, such as: How much you can stand to lose over the long term?

"Most people can stand to lose a heck of a lot temporarily," says Mr. Schatsky, a financial adviser in New York. The real acid test, he says, is how much of your portfolio's value you can stand to lose over months or years.

As it turns out, most people rank as middle-of-the-road risk-takers, say several advisers. "Only about 10% to 15% of my clients are aggressive," says Mr. Roge.

WHAT'S YOUR RISK TOLERANCE?

Circle the letter that corresponds to your answer

1. Just 60 days after you put money into an investment, its price falls 20%. Assuming none of the fundamentals have changed, what would you do?
   a. Sell to avoid further worry and try something else
   b. Do nothing and wait for the investment to come back
   c. Buy more. It was a good investment before; now it's a cheap investment, too
2. Now look at the previous question another way. Your investment fell 20%, but it's part of a portfolio being used to meet investment goals with three different time horizons.

2A. What would you do if the goal were five years away?
   a. Sell
   b. Do nothing
   c. Buy more

TABLE 6.3

Utility values of possible portfolios for investor with risk aversion, $A = 4$

Expected Return, $E(r)$	Standard Deviation, σ	Utility $= E(r) - \frac{1}{2}A\sigma^2$
.10	.200	$.10 - .5 \times 4 \times .04 = .02$
.15	.255	$.15 - .5 \times 4 \times .065 = .02$
.20	.300	$.20 - .5 \times 4 \times .09 = .02$
.25	.339	$.25 - .5 \times 4 \times .115 = .02$

an investor with risk aversion, A, whose entire wealth is in a piece of real estate. Suppose that in any given year there is a probability, p, of a disaster such as a mudslide that will destroy the real estate and wipe out the investor's entire wealth. Such an event would amount to a rate of return of -100%. Otherwise, with probability $1 - p$, the real estate remains intact, and we will assume that its rate of return is zero.

2B. What would you do if the goal were 15 years away?

a. Sell
b. Do nothing
c. Buy more

2C. What would you do if the goal were 30 years away?

a. Sell
b Do nothing
c. Buy more

3. The price of your retirement investment jumps 25% a month after you buy it. Again, the fundamentals haven't changed. After you finish gloating, what do you do?

a. Sell it and lock in your gains
b. Stay put and hope for more gain
c. Buy more; it could go higher

4. You're investing for retirement, which is 15 years away. Which would you rather do?

a. Invest in a money-market fund or guaranteed investment contract, giving up the possibility of major gains, but virtually assuring the safety of your principal
b. Invest in a 50-50 mix of bond funds and stock funds, in hopes of getting some growth, but also giving yourself some protection in the form of steady income
c. Invest in aggressive growth mutual funds whose value will probably fluctuate significantly during the year, but have the potential for impressive gains over five or 10 years

5. You just won a big prize! But which one? It's up to you.

a. $2,000 in cash
b. A 50% chance to win $5,000
c. A 20% chance to win $15,000

6. A good investment opportunity just came along. But you have to borrow money to get in. Would you take out a loan?

a. Definitely not
b. Perhaps
c. Yes

7. Your company is selling stock to its employees. In three years, management plans to take the company public. Until then, you won't be able to sell your shares and you will get no dividends. But your investment could multiply as much as 10 times when the company goes public. How much money would you invest?

a. None
b. Two months' salary
c. Four months' salary

SCORING YOUR RISK TOLERANCE

To score the quiz, add up the number of answers you gave in each category a–c, then multiply as shown to find your score

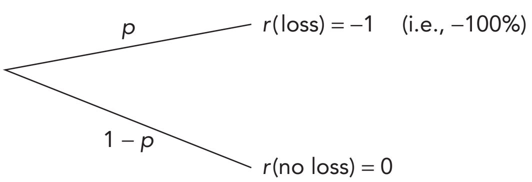
(*a*) answers ______ × 1 = ______ points
(*b*) answers ______ × 2 = ______ points
(*c*) answers ______ × 3 = ______ points

YOUR SCORE ______ points

If you scored . . .	You may be a:
9–14 points	Conservative investor
5–21 points	Moderate investor
22–27 points	Aggressive investor

Source: Reprinted with permission from *The Wall Street Journal*.

We can describe the probability distribution of the rate of return on this so-called simple prospect with the following diagram (with returns expressed in decimals):

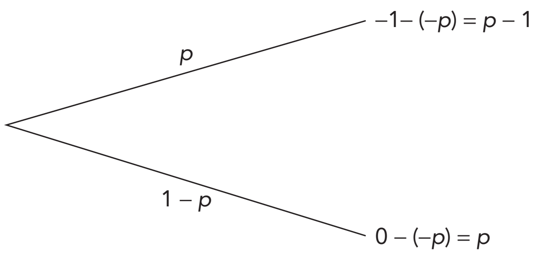


The expected rate of return of this prospect is

$$E(r) = p \times (-1) + (1 - p) \times 0 = -p \qquad (6.2)$$

In other words, the expected loss is a fraction p of the value of the real estate.

What about variance and standard deviation of the investor's position? The deviations from expectation, $r - E(r)$, for each outcome are



The variance of the rate of return equals the expectation of the squared deviation:

$$\sigma^2(r) = p \times (p - 1)^2 + (1 - p) \times p^2 = p(1 - p) \tag{6.3}$$

To calculate the utility score of this simple prospect we use the risk-aversion coefficient, A, the expected return, $E(r)$ (from Equation 6.2), and the variance, $\sigma^2(r)$ (from Equation 6.3) in Equation 6.1 and obtain

$$\begin{aligned} U &= E(r) - ½A\sigma^2(r) \\ &= -p - ½Ap(1 - p) \end{aligned} \tag{6.4}$$

Now we can relate the risk-aversion parameter to the amount that an individual would be willing to pay for insurance against the potential loss. Suppose an insurance company offers to cover any loss over the year for a fee of v dollars per dollar of insured property. The individual who pays \$$v$ per dollar of real estate value to the insurance company will face no risk—the insurance company will reimburse any losses, so the real estate will be worth its original value at year-end. Taking out such a policy amounts to a sure negative rate of return of $-v$, with a utility score: $U = -v$.

How much will our investor pay for the policy, that is, what is the maximum value of v he or she will be willing to pay? To find this value, we equate the utility score of the uninsured property (given in Equation 6.4) to that of the insured property (which is $-v$):

$$U = -p - ½A\,p(1 - p) = -v \tag{6.5}$$

We can solve Equation 6.5 for the policy cost at which the investor would be indifferent between purchasing insurance or going uninsured. This is the maximum amount that he or she will be willing pay for the insurance policy:

$$v = p[1 + ½A(1 - p)] \tag{6.6}$$

Remember that the expected loss on the property is p. Therefore, the term in the square brackets in Equation 6.6 tells us the *multiple* of the expected loss, p, the investor is willing to pay for the policy. Obviously, a risk-neutral investor, with $A = 0$, will be willing to pay no more than the expected loss, $v = p$. With $A = 1$, the term in square brackets is almost 1.5 (because p is small), so v will be close to $1.5p$. In other words, the investor is willing to pay almost 50% more than the expected loss for the policy. For each additional increment to the degree of risk aversion (A = 2, 3, and so on), the investor is willing to add (almost) another 50% of the expected loss to the insurance premium.

Investor Risk Aversion, *A*	Expected Rate of Loss, $p = .0001$ Maximum Premium, v, as a Multiple of Expected Loss, p	Expected Rate of Loss, $p = .01$ Maximum Premium, v, as a Multiple of Expected Loss, p
0	1.0000	1.0000
1	1.5000	1.4950
2	1.9999	1.9900
3	2.4999	2.4850
4	2.9998	2.9800
5	3.4998	3.4750

TABLE 6.4

Investor's willingness to pay for catastrophe insurance

Table 6.4 shows how many multiples of the expected loss the investor is willing to pay for insurance for two values of the probability of disaster, p, as a function of the degree of risk aversion. Based on individuals' actual willingness to pay for insurance against catastrophic loss as in this example, economists estimate that investors seem to exhibit degrees of risk aversion in the range of 2 to 4, that is, would be likely to be willing to pay as much as two to three times the expected loss but not much more.

By the way, this analysis also tells you something about the merits of competitive insurance markets. Insurance companies that are able to share their risk with many co-insurers will be willing to offer coverage for premiums that are only slightly higher than the expected loss, even though each investor may *value* the coverage at several multiples of the expected loss. The large savings that investors thus derive from competitive insurance markets are analogous to the consumer surplus derived from competition in other markets.

More support for the hypothesis that A is somewhere in the range of 2 to 4 can be obtained from estimates of the expected rate of return and risk on a broad stock-index portfolio. We will present this argument shortly after we describe how investors might determine their optimal allocation of wealth to risky assets.

6.2 CAPITAL ALLOCATION ACROSS RISKY AND RISK-FREE PORTFOLIOS

History shows us that long-term bonds have been riskier investments than investments in Treasury bills and that stock investments have been riskier still. On the other hand, the riskier investments have offered higher average returns. Investors, of course, do not make all-or-nothing choices from these investment classes. They can and do construct their portfolios using securities from all asset classes. Some of the portfolio may be in risk-free Treasury bills, some in high-risk stocks.

The most straightforward way to control the risk of the portfolio is through the fraction of the portfolio invested in Treasury bills and other safe money market securities versus risky assets. This capital allocation decision is an example of an asset allocation choice—a choice among broad investment classes, rather than among the specific securities within each asset class. Most investment professionals consider asset allocation the most important part of portfolio construction. Consider this statement by John Bogle, made when he was chairman of the Vanguard Group of Investment Companies:

> The most fundamental decision of investing is the allocation of your assets: How much should you own in stock? How much should you own in bonds? How much should you own in cash reserves? . . . That decision [has been shown to account] for an astonishing 94% of the differences in total returns achieved by institutionally managed pension funds. . . . There is no reason to believe that the same relationship does not also hold true for individual investors.[1]

Therefore, we start our discussion of the risk–return trade-off available to investors by examining the most basic asset allocation choice: the choice of how much of the portfolio to place in risk-free money market securities versus other risky asset classes.

We will denote the investor's portfolio of risky assets as *P* and the risk-free asset as *F.* We will assume for the sake of illustration that the risky component of the investor's overall portfolio comprises two mutual funds, one invested in stocks and the other invested in long-term bonds. For now, we take the composition of the risky portfolio as given and focus only on the allocation between it and risk-free securities. In the next chapter, we turn to asset allocation and security selection across risky assets.

When we shift wealth from the risky portfolio to the risk-free asset, we do not change the relative proportions of the various risky assets within the risky portfolio. Rather, we reduce the relative weight of the risky portfolio as a whole in favor of risk-free assets.

For example, assume that the total market value of an initial portfolio is \$300,000, of which \$90,000 is invested in the Ready Asset money market fund, a risk-free asset for practical purposes. The remaining \$210,000 is invested in risky securities—\$113,400 in equities (*E*) and \$96,600 in long-term bonds (*B*). The equities and long bond holdings comprise "the" risky portfolio, 54% in *E* and 46% in *B:*

$$E: \quad w_E = \frac{113{,}400}{210{,}000} = .54$$

$$B: \quad w_B = \frac{96{,}600}{210{,}000} = .46$$

The weight of the risky portfolio, *P*, in the **complete portfolio,** including risk-free *and* risky investments, is denoted by *y:*

$$y = \frac{210{,}000}{300{,}000} = .7 \text{ (risky assets)}$$

$$1 - y = \frac{90{,}000}{300{,}000} = .3 \text{ (risky-free assets)}$$

The weights of each asset class in the complete portfolio are as follows:

$$E: \quad \frac{\$113{,}400}{\$300{,}000} = .378$$

$$B: \quad \frac{\$96{,}600}{\$300{,}000} = .322$$

$$\text{Risky portfolio} = \quad E + B \quad = .700$$

The risky portfolio makes up 70% of the complete portfolio.

[1] John C. Bogle, *Bogle on Mutual Funds* (Burr Ridge, IL: Irwin Professional Publishing, 1994), p. 235.

EXAMPLE 6.2 The Risky Portfolio

Suppose that the owner of this portfolio wishes to decrease risk by reducing the allocation to the risky portfolio from $y = .7$ to $y = .56$. The risky portfolio would then total only $.56 \times \$300{,}000 = \$168{,}000$, requiring the sale of \$42,000 of the original \$210,000 of risky holdings, with the proceeds used to purchase more shares in Ready Asset (the money market fund). Total holdings in the risk-free asset will increase to $\$300{,}000 \times (1 - .56) = \$132{,}000$, the original holdings plus the new contribution to the money market fund:

$$\$90{,}000 + \$42{,}000 = \$132{,}000$$

The key point, however, is that we leave the proportions of each asset in the risky portfolio unchanged. Because the weights of E and B in the risky portfolio are .54 and .46, respectively, we sell $.54 \times \$42{,}000 = \$22{,}680$ of E and $.46 \times \$42{,}000 = \$19{,}320$ of B. After the sale, the proportions of each asset in the risky portfolio are in fact unchanged:

$$E: \quad w_E = \frac{113{,}400 - 22{,}680}{210{,}000 - 42{,}000} = .54$$

$$B: \quad w_B = \frac{96{,}600 - 19{,}320}{210{,}000 - 42{,}000} = .46$$

Rather than thinking of our risky holdings as E and B separately, we may view our holdings as if they were in a single fund that holds equities and bonds in fixed proportions. In this sense we may treat the risky fund as a single risky asset, that asset being a particular bundle of securities. As we shift in and out of safe assets, we simply alter our holdings of that bundle of securities commensurately.

Given this simplification, we can now turn to the desirability of reducing risk by changing the risky/risk-free asset mix, that is, reducing risk by decreasing the proportion y. As long as we do not alter the weights of each security within the risky portfolio, the probability distribution of the rate of return on the risky portfolio remains unchanged by the asset reallocation. What will change is the probability distribution of the rate of return on the *complete* portfolio that consists of the risky asset and the risk-free asset.

CONCEPT CHECK 4

What will be the dollar value of your position in equities (E), and its proportion in your overall portfolio, if you decide to hold 50% of your investment budget in Ready Asset?

6.3 THE RISK-FREE ASSET

By virtue of its power to tax and control the money supply, only the government can issue default-free bonds. Even the default-free guarantee by itself is not sufficient to make the bonds risk-free in real terms. The only risk-free asset in real terms would be a perfectly price-indexed bond. Moreover, a default-free perfectly indexed bond offers a guaranteed real rate to an investor only if the maturity of the bond is identical to the investor's desired holding period. Even indexed bonds are subject to interest rate risk, because real

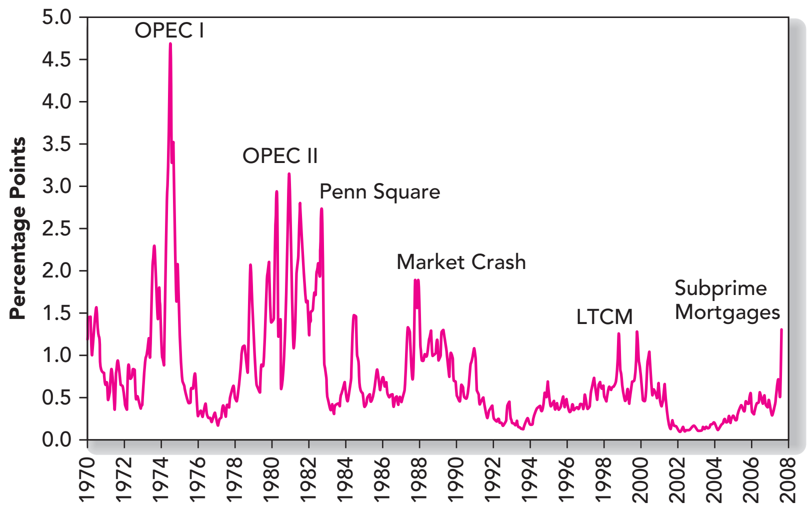


FIGURE 6.3 Spread between 3-month CD and T-bill rates

interest rates change unpredictably through time. When future real rates are uncertain, so is the future price of indexed bonds.

Nevertheless, it is common practice to view Treasury bills as "the" **risk-free asset.** Their short-term nature makes their values insensitive to interest rate fluctuations. Indeed, an investor can lock in a short-term nominal return by buying a bill and holding it to maturity. Moreover, inflation uncertainty over the course of a few weeks, or even months, is negligible compared with the uncertainty of stock market returns.

In practice, most investors use a broader range of money market instruments as a risk-free asset. All the money market instruments are virtually free of interest rate risk because of their short maturities and are fairly safe in terms of default or credit risk.

Most money market funds hold, for the most part, three types of securities—Treasury bills, bank certificates of deposit (CDs), and commercial paper (CP)—differing slightly in their default risk. The yields to maturity on CDs and CP for identical maturity, for example, are always somewhat higher than those of T-bills. The recent history of this yield spread for 90-day CDs is shown in Figure 6.3.

Money market funds have changed their relative holdings of these securities over time but, by and large, T-bills make up only about 15% of their portfolios. Nevertheless, the risk of such blue-chip short-term investments as CDs and CP is minuscule compared with that of most other assets such as long-term corporate bonds, common stocks, or real estate. Hence we treat money market funds as the most easily accessible risk-free asset for most investors.

6.4 PORTFOLIOS OF ONE RISKY ASSET AND A RISK-FREE ASSET

In this section we examine the risk–return combinations available to investors. This is the "technical" part of asset allocation; it deals only with the opportunities available to investors given the features of the broad asset markets in which they can invest. In the next section we address the "personal" part of the problem—the specific individual's choice of the best risk–return combination from the set of feasible combinations.

Suppose the investor has already decided on the composition of the risky portfolio. Now the concern is with the proportion of the investment budget, y, to be allocated to the risky portfolio, P. The remaining proportion, $1 - y$, is to be invested in the risk-free asset, F.

Denote the risky rate of return of P by r_P, its expected rate of return by $E(r_P)$, and its standard deviation by σ_P. The rate of return on the risk-free asset is denoted as r_f. In the numerical example we assume that $E(r_P) = 15\%$, $\sigma_P = 22\%$, and that the risk-free rate is $r_f = 7\%$. Thus the risk premium on the risky asset is $E(r_P) - r_f = 8\%$.

With a proportion, y, in the risky portfolio, and $1 - y$ in the risk-free asset, the rate of return on the *complete* portfolio, denoted C, is r_C where

$$r_C = yr_P + (1 - y)r_f \tag{6.7}$$

Taking the expectation of this portfolio's rate of return,

$$\begin{aligned} E(r_C) &= yE(r_P) + (1 - y)r_f \\ &= r_f + y[E(r_P) - r_f] = 7 + y(15 - 7) \end{aligned} \tag{6.8}$$

This result is easily interpreted. The base rate of return for any portfolio is the risk-free rate. In addition, the portfolio is *expected* to earn a risk premium that depends on the risk premium of the risky portfolio, $E(r_P) - r_f$, and the investor's position in that risky asset, y. Investors are assumed to be risk averse and thus unwilling to take on a risky position without a positive risk premium.

When we combine a risky asset and a risk-free asset in a portfolio, the standard deviation of the resulting complete portfolio is the standard deviation of the risky asset multiplied by the weight of the risky asset in that portfolio.[2] Because the standard deviation of the risky portfolio is $\sigma_P = 22\%$,

$$\sigma_C = y\sigma_P = 22y \tag{6.9}$$

which makes sense because the standard deviation of the portfolio is proportional to both the standard deviation of the risky asset and the proportion invested in it. In sum, the rate of return of the complete portfolio will have expected value $E(r_C) = r_f + y[E(r_P) - r_f] = 7 + 8y$ and standard deviation $\sigma_C = 22y$.

The next step is to plot the portfolio characteristics (given the choice for y) in the expected return–standard deviation plane. This is done in Figure 6.4. The risk-free asset, F, appears on the vertical axis because its standard deviation is zero. The risky asset, P, is plotted with a standard deviation, $\sigma_P = 22\%$, and expected return of 15%. If an investor chooses to invest solely in the risky asset, then $y = 1.0$, and the complete portfolio is P. If the chosen position is $y = 0$, then $1 - y = 1.0$, and the complete portfolio is the risk-free portfolio F.

What about the more interesting midrange portfolios where y lies between 0 and 1? These portfolios will graph on the straight line connecting points F and P. The slope of that line is $[E(r_P) - r_f]/\sigma_P$ (or rise/run), in this case, 8/22.

The conclusion is straightforward. Increasing the fraction of the overall portfolio invested in the risky asset increases expected return according to Equation 6.8 at a rate of 8%. It also increases portfolio standard deviation according to Equation 6.9 at the rate of 22%. The extra return per extra risk is thus $8/22 = .36$.

To derive the exact equation for the straight line between F and P, we rearrange Equation 6.9 to find that $y = \sigma_C/\sigma_P$, and we substitute for y in Equation 6.8 to describe the expected return–standard deviation trade-off:

$$\begin{aligned} E(r_C) &= r_f + y[E(r_P) - r_f] \\ &= r_f + \frac{\sigma_C}{\sigma_P}[E(r_P) - r_f] = 7 + \frac{8}{22}\sigma_C \end{aligned} \tag{6.10}$$

[2]This is an application of a basic rule from statistics: If you multiply a random variable by a constant, the standard deviation is multiplied by the same constant. In our application, the random variable is the rate of return on the risky asset, and the constant is the fraction of that asset in the complete portfolio. We will elaborate on the rules for portfolio return and risk in the following chapter.

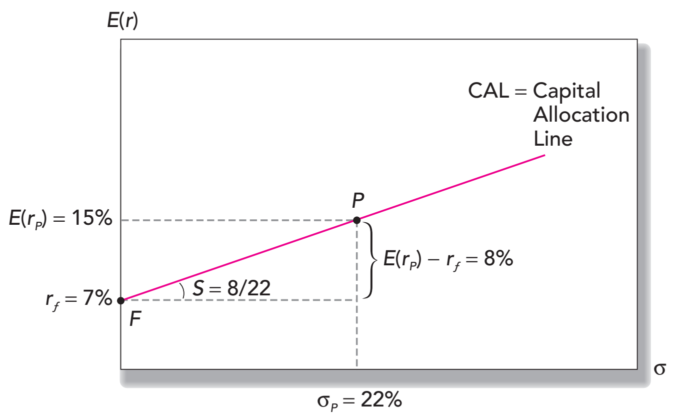


FIGURE 6.4 The investment opportunity set with a risky asset and a risk-free asset in the expected return–standard deviation plane

Thus the expected return of the complete portfolio as a function of its standard deviation is a straight line, with intercept r_f and slope

$$S = \frac{E(r_P) - r_f}{\sigma_P} = \frac{8}{22} \tag{6.11}$$

Figure 6.4 graphs the *investment opportunity set*, which is the set of feasible expected return and standard deviation pairs of all portfolios resulting from different values of y. The graph is a straight line originating at r_f and going through the point labeled P.

This straight line is called the **capital allocation line** (CAL). It depicts all the risk–return combinations available to investors. The slope of the CAL, denoted S, equals the increase in the expected return of the complete portfolio per unit of additional standard deviation—in other words, incremental return per incremental risk. For this reason, the slope is called the **reward-to-volatility ratio.** It also is called the Sharpe ratio (see Chapter 5).

A portfolio equally divided between the risky asset and the risk-free asset, that is, where $y = .5$, will have an expected rate of return of $E(r_C) = 7 + .5 \times 8 = 11\%$, implying a risk premium of 4%, and a standard deviation of $\sigma_C = .5 \times 22 = 11\%$. It will plot on the line FP midway between F and P. The reward-to-volatility ratio is $S = 4/11 = .36$, precisely the same as that of portfolio P.

What about points on the CAL to the right of portfolio P? If investors can borrow at the (risk-free) rate of $r_f = 7\%$, they can construct portfolios that may be plotted on the CAL to the right of P.

CONCEPT CHECK 5

Can the reward-to-volatility (Sharpe) ratio, $S = [E(r_C) - r_f]/\sigma_C$, of any combination of the risky asset and the risk-free asset be different from the ratio for the risky asset taken alone, $[E(r_P) - r_f]/\sigma_P$, which in this case is .36?

EXAMPLE 6.3 Leverage

Suppose the investment budget is \$300,000 and our investor borrows an additional \$120,000, investing the total available funds in the risky asset. This is a *leveraged* position in the risky asset; it is financed in part by borrowing. In that case

$$y = \frac{420{,}000}{300{,}000} = 1.4$$

and $1 - y = 1 - 1.4 = -.4$, reflecting a short (borrowing) position in the risk-free asset. Rather than lending at a 7% interest rate, the investor borrows at 7%. The distribution of the portfolio rate of return still exhibits the same reward-to-volatility ratio:

$$E(r_C) = 7\% + (1.4 \times 8\%) = 18.2\%$$

$$\sigma_C = 1.4 \times 22\% = 30.8\%$$

$$S = \frac{E(r_C) - r_f}{\sigma_C} = \frac{18.2 - 7}{30.8} = .36$$

As one might expect, the leveraged portfolio has a higher standard deviation than does an unleveraged position in the risky asset.

Of course, nongovernment investors cannot borrow at the risk-free rate. The risk of a borrower's default causes lenders to demand higher interest rates on loans. Therefore, the nongovernment investor's borrowing cost will exceed the lending rate of $r_f = 7\%$. Suppose the borrowing rate is $r_f^B = 9\%$. Then in the borrowing range, the reward-to-volatility ratio, the slope of the CAL, will be $[E(r_P) - r_f^B]/\sigma_P = 6/22 = .27$. The CAL will therefore be "kinked" at point *P*, as shown in Figure 6.5. To the left of *P* the investor is lending at 7%, and the slope of the CAL is .36. To the right of *P*, where $y > 1$, the investor is borrowing at 9% to finance extra investments in the risky asset, and the slope is .27.

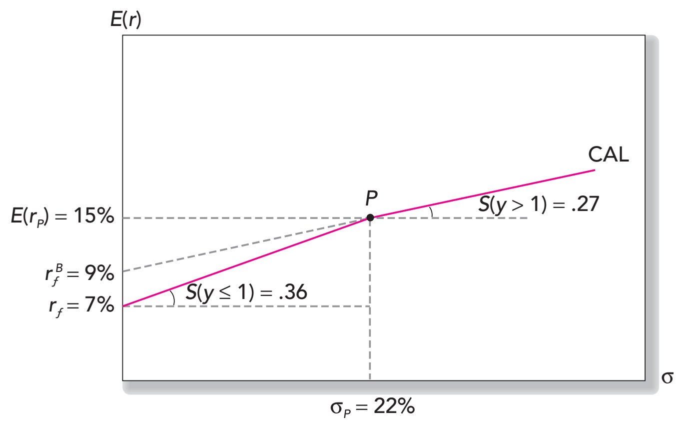


FIGURE 6.5 The opportunity set with differential borrowing and lending rates

In practice, borrowing to invest in the risky portfolio is easy and straightforward if you have a margin account with a broker. All you have to do is tell your broker that you want to buy "on margin." Margin purchases may not exceed 50% of the purchase value. Therefore, if your net worth in the account is \$300,000, the broker is allowed to lend you up to \$300,000 to purchase additional stock.[3] You would then have \$600,000 on the asset side of your account and \$300,000 on the liability side, resulting in $y = 2.0$.

CONCEPT CHECK 6

Suppose that there is an upward shift in the expected rate of return on the risky asset, from 15% to 17%. If all other parameters remain unchanged, what will be the slope of the CAL for $y \leq 1$ and $y > 1$?

6.5 RISK TOLERANCE AND ASSET ALLOCATION

We have shown how to develop the CAL, the graph of all feasible risk–return combinations available from different asset allocation choices. The investor confronting the CAL now must choose one optimal portfolio, *C*, from the set of feasible choices. This choice entails a trade-off

[3]Margin purchases require the investor to maintain the securities in a margin account with the broker. If the value of the securities declines below a "maintenance margin," a "margin call" is sent out, requiring a deposit to bring the net worth of the account up to the appropriate level. If the margin call is not met, regulations mandate that some or all of the securities be sold by the broker and the proceeds used to reestablish the required margin. See Chapter 3, Section 3.6, for further discussion. As we will see in Chapter 22, futures contracts also offer leverage. If the risky portfolio is an index fund on which a contract trades, the implicit rate on the loan will be close to the T-bill rate.

TABLE 6.5

Utility levels for various positions in risky assets (y) for an investor with risk aversion $A = 4$

(1) y	(2) $E(r_C)$	(3) σ_C	(4) $U = E(r) - \frac{1}{2}A\sigma^2$
0	.070	0	.0700
0.1	.078	.022	.0770
0.2	.086	.044	.0821
0.3	.094	.066	.0853
0.4	.102	.088	.0865
0.5	.110	.110	.0858
0.6	.118	.132	.0832
0.7	.126	.154	.0786
0.8	.134	.176	.0720
0.9	.142	.198	.0636
1.0	.150	.220	.0532

between risk and return. Individual investor differences in risk aversion imply that, given an identical opportunity set (that is, a risk-free rate and a reward-to-volatility ratio), different investors will choose different positions in the risky asset. In particular, the more risk-averse investors will choose to hold less of the risky asset and more of the risk-free asset.

An investor who faces a risk-free rate, r_f, and a risky portfolio with expected return $E(r_P)$ and standard deviation σ_P will find that, for any choice of y, the expected return of the complete portfolio is given by Equation 6.8:

$$E(r_C) = r_f + y[E(r_P) - r_f]$$

From Equation 6.9, the variance of the overall portfolio is

$$\sigma_C^2 = y^2\sigma_P^2$$

Investors attempt to maximize utility by choosing the best allocation to the risky asset, y. The utility function is given by Equation 6.1 as $U = E(r) - \frac{1}{2}A\sigma^2$. As the allocation to the risky asset increases (higher y), expected return increases, but so does volatility, so utility can increase or decrease. To illustrate, Table 6.5 shows utility levels corresponding to different values of y. Initially, utility increases as y increases, but eventually it declines.

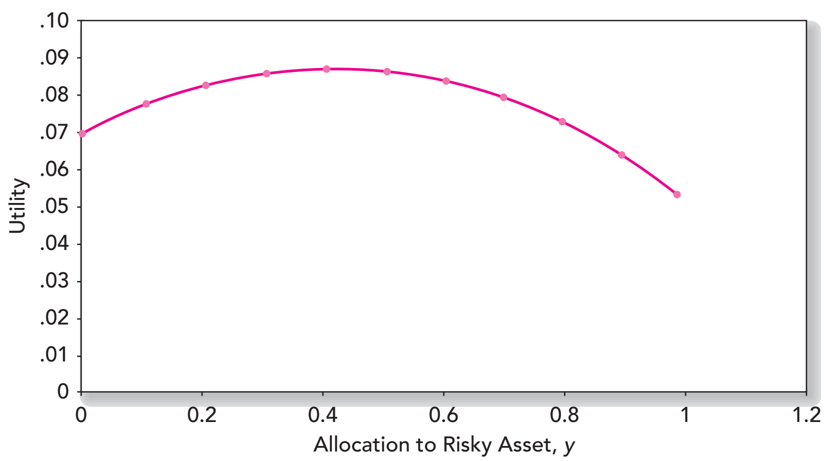


FIGURE 6.6 Utility as a function of allocation to the risky asset, y

Figure 6.6 is a plot of the utility function from Table 6.5. The graph shows that utility is highest at $y = .41$. When y is less than .41, investors are willing to assume more risk to increase expected return. But at higher levels of y, risk is higher, and additional allocations to the risky asset are undesirable—beyond this point, further increases in risk dominate the increase in expected return and reduce utility.

To solve the utility maximization problem more generally, we write the problem as follows:

$$\underset{y}{\text{Max}}\ U = E(r_C) - ½A\sigma_C^2 = r_f + y[E(r_P) - r_f] - ½Ay^2\sigma_P^2$$

Students of calculus will remember that the maximization problem is solved by setting the derivative of this expression to zero. Doing so and solving for y yields the optimal position for risk-averse investors in the risky asset, y^*, as follows:[4]

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2} \tag{6.12}$$

This solution shows that the optimal position in the risky asset is, as one would expect, *inversely* proportional to the level of risk aversion and the level of risk (as measured by the variance) and directly proportional to the risk premium offered by the risky asset.

EXAMPLE 6.4 Capital Allocation

Using our numerical example [r_f = 7%, $E(r_P)$ = 15%, and σ_P = 22%], and expressing all returns as decimals, the optimal solution for an investor with a coefficient of risk aversion $A = 4$ is

$$y^* = \frac{.15 - .07}{4 \times .22^2} = .41$$

In other words, this particular investor will invest 41% of the investment budget in the risky asset and 59% in the risk-free asset. As we saw in Figure 6.6, this is the value of y at which utility is maximized.

With 41% invested in the risky portfolio, the expected return and standard deviation of the complete portfolio are

$$E(r_C) = 7 + [.41 \times (15 - 7)] = 10.28\%$$
$$\sigma_C = .41 \times 22 = 9.02\%$$

The risk premium of the complete portfolio is $E(r_C) - r_f$ = 3.28%, which is obtained by taking on a portfolio with a standard deviation of 9.02%. Notice that 3.28/9.02 = .36, which is the reward-to-volatility (Sharpe) ratio assumed for this example.

A graphical way of presenting this decision problem is to use indifference curve analysis. To illustrate how to build an indifference curve, consider an investor with risk aversion $A = 4$ who currently holds all her wealth in a risk-free portfolio yielding r_f = 5%. Because the variance of such a portfolio is zero, Equation 6.1 tells us that its utility value is $U = .05$. Now we find the expected return the investor would require to maintain the *same* level of utility when holding a risky portfolio, say, with σ = 1%. We use Equation 6.1 to find how much $E(r)$ must increase to compensate for the higher value of σ:

$$U = E(r) - ½ \times A \times \sigma^2$$
$$.05 = E(r) - ½ \times 4 \times .01^2$$

[4]The derivative with respect to y equals $E(r_P) - r_f - yA\sigma_P^2$. Setting this expression equal to zero and solving for y yields Equation 6.12.

TABLE 6.6

Spreadsheet calculations of indifference curves (Entries in columns 2–4 are expected returns necessary to provide specified utility value.)

	A = 2		A = 4	
σ	U = .05	U = .09	U = .05	U = .09
0	.0500	.0900	.050	.090
.05	.0525	.0925	.055	.095
.10	.0600	.1000	.070	.110
.15	.0725	.1125	.095	.135
.20	.0900	.1300	.130	.170
.25	.1125	.1525	.175	.215
.30	.1400	.1800	.230	.270
.35	.1725	.2125	.295	.335
.40	.2100	.2500	.370	.410
.45	.2525	.2925	.455	.495
.50	.3000	.3400	.550	.590

This implies that the necessary expected return increases to

$$\begin{aligned}\text{Required } E(r) &= .05 + \tfrac{1}{2} \times A \times \sigma^2 \\ &= .05 + \tfrac{1}{2} \times 4 \times .01^2 = .0502 \end{aligned} \qquad \textbf{(6.13)}$$

We can repeat this calculation for many other levels of σ, each time finding the value of $E(r)$ necessary to maintain $U = .05$. This process will yield all combinations of expected return and volatility with utility level of .05; plotting these combinations gives us the indifference curve.

We can readily generate an investor's indifference curves using a spreadsheet. Table 6.6 contains risk–return combinations with utility values of .05 and .09 for two investors, one with $A = 2$ and the other with $A = 4$. For example, column (2) uses Equation 6.13 to calculate the expected return that must be paired with the standard deviation in column (1) for an investor with $A = 2$ to derive a utility value of $U = .05$. Column (3) repeats the calculations for a higher utility value, $U = .09$. The plot of these expected return–standard deviation combinations appears in Figure 6.7 as the two curves labeled $A = 2$. Notice that the intercepts of the indifference curves are at .05 and .09, exactly the level of utility corresponding to the two curves.

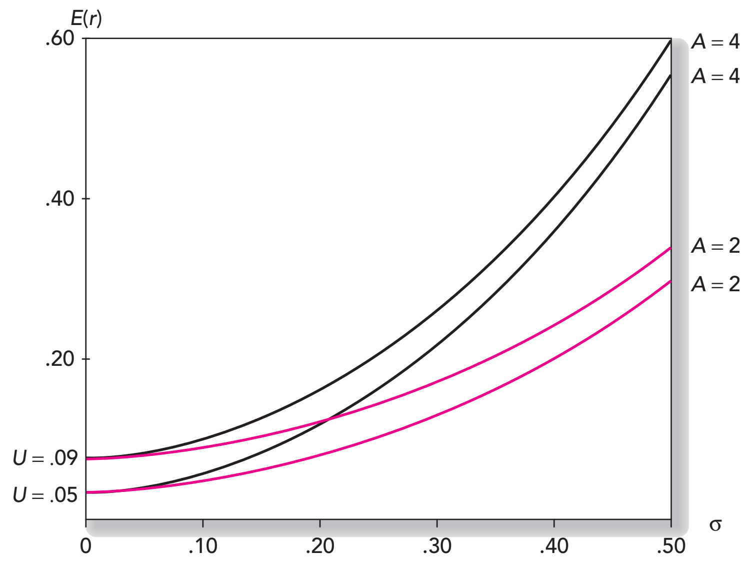


FIGURE 6.7 Indifference curves for $U = .05$ and $U = .09$ with $A = 2$ and $A = 4$

Given the choice, any investor would prefer a portfolio on the higher indifference curve, the one with a higher certainty equivalent (utility). Portfolios on higher indifference curves offer a higher expected return for any given level of risk. For example, both indifference curves for $A = 2$ have the same shape, but for any level of volatility, a portfolio on the curve with utility of .09 offers an expected return 4% greater than the corresponding portfolio on the lower curve, for which $U = .05$.

Columns (4) and (5) of Table 6.6 repeat this analysis for a more risk-averse investor, with $A = 4$. The resulting pair of indifference curves in Figure 6.7 demonstrates that more risk-averse investors have steeper indifference curves than less risk-averse investors. Steeper curves mean that investors require a greater increase in expected return to compensate for an increase in portfolio risk.

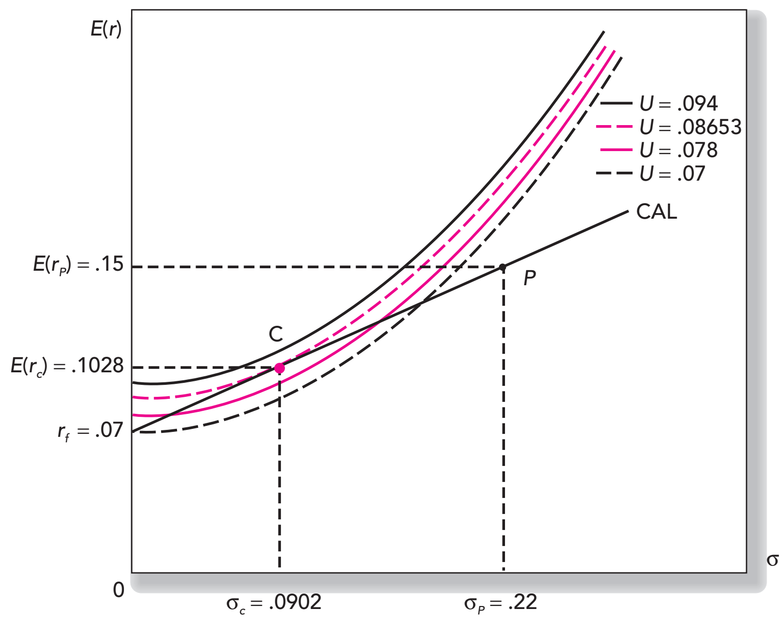


FIGURE 6.8 Finding the optimal complete portfolio by using indifference curves

Higher indifference curves correspond to higher levels of utility. The investor thus attempts to find the complete portfolio on the highest possible indifference curve. When we superimpose plots of indifference curves on the investment opportunity set represented by the capital allocation line as in Figure 6.8, we can identify the *highest possible* indifference curve that still touches the CAL. That indifference curve is tangent to the CAL, and the tangency point corresponds to the standard deviation and expected return of the optimal complete portfolio.

To illustrate, Table 6.7 provides calculations for four indifference curves (with utility levels of .07, .078, .08653, and .094) for an investor with $A = 4$. Columns (2)–(5)

σ	U = .07	U = .078	U = .08653	U = .094	CAL
0	.0700	.0780	.0865	.0940	.0700
.02	.0708	.0788	.0873	.0948	.0773
.04	.0732	.0812	.0897	.0972	.0845
.06	.0772	.0852	.0937	.1012	.0918
.08	.0828	.0908	.0993	.1068	.0991
.0902	.0863	.0943	.1028	.1103	.1028
.10	.0900	.0980	.1065	.1140	.1064
.12	.0988	.1068	.1153	.1228	.1136
.14	.1092	.1172	.1257	.1332	.1209
.18	.1348	.1428	.1513	.1588	.1355
.22	.1668	.1748	.1833	.1908	.1500
.26	.2052	.2132	.2217	.2292	.1645
.30	.2500	.2580	.2665	.2740	.1791

TABLE 6.7

Expected returns on four indifference curves and the CAL. Investor's risk aversion is $A = 4$.

use Equation 6.13 to calculate the expected return that must be paired with the standard deviation in column (1) to provide the utility value corresponding to each curve. Column (6) uses Equation 6.10 to calculate $E(r_C)$ on the CAL for the standard deviation σ_C in column (1):

$$E(r_C) = r_f + [E(r_P) - r_f]\frac{\sigma_C}{\sigma_P} = 7 + [15 - 7]\frac{\sigma_C}{22}$$

Figure 6.8 graphs the four indifference curves and the CAL. The graph reveals that the indifference curve with $U = .08653$ is tangent to the CAL; the tangency point corresponds to the complete portfolio that maximizes utility. The tangency point occurs at $\sigma_C = 9.02\%$ and $E(r_C) = 10.28\%$, the risk–return parameters of the optimal complete portfolio with $y^* = 0.41$. These values match our algebraic solution using Equation 6.12.

We conclude that the choice for y^*, the fraction of overall investment funds to place in the risky portfolio versus the safer but lower expected-return risk-free asset, is in large part a matter of risk aversion.

CONCEPT CHECK 7

a. If an investor's coefficient of risk aversion is $A = 3$, how does the optimal asset mix change? What are the new values of $E(r_C)$ and σ_C?

b. Suppose that the borrowing rate, $r_f^B = 9\%$ is greater than the lending rate, $r_f = 7\%$. Show graphically how the optimal portfolio choice of some investors will be affected by the higher borrowing rate. Which investors will *not* be affected by the borrowing rate?

6.6 PASSIVE STRATEGIES: THE CAPITAL MARKET LINE

The CAL is derived with the risk-free and "the" risky portfolio, *P*. Determination of the assets to include in risky portfolio *P* may result from a passive or an active strategy. A **passive strategy** describes a portfolio decision that avoids *any* direct or indirect security analysis.[5] At first blush, a passive strategy would appear to be naive. As will become apparent, however, forces of supply and demand in large capital markets may make such a strategy a reasonable choice for many investors.

In Chapter 5, we presented a compilation of the history of rates of return on different asset classes. The data are available at Professor Kenneth French's Web site, **mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.** We can use these data to examine various passive strategies.

A natural candidate for a passively held risky asset would be a well-diversified portfolio of common stocks. Because a passive strategy requires that we devote no resources to acquiring information on any individual stock or group of stocks, we must follow a "neutral" diversification strategy. One way is to select a diversified portfolio of stocks that mirrors the value of the corporate sector of the U.S. economy. This results in a portfolio in which, for example, the proportion invested in Microsoft stock will be the ratio of Microsoft's total market value to the market value of all listed stocks.

[5]By "indirect security analysis" we mean the delegation of that responsibility to an intermediary such as a professional money manager.

	Average Annual Returns		S&P 500 Portfolio			Probability of Observing This Subperiod Estimate*
Period	S&P 500 Portfolio	1-Month T-bills	Risk Premium	Standard Deviation	Sharpe Ratio (Reward-to-Volatility)	
1926–2005	12.15	3.75	8.39	20.54	.41	
1986–2005	13.16	4.56	8.60	16.24	.53	.63
1966–1985	10.12	7.41	2.72	17.83	.15	.30
1946–1965	14.97	1.97	13.00	17.65	.74	.20
1926–1945	10.33	1.07	9.26	27.95	.33	.73

TABLE 6.8

Average annual return on large stocks and 1-month T-bills; standard deviation, and reward-to-volatility ratio of large stocks over time

*The probability that the estimate of 1926–2005 is true and we observe the reported (or an even more different) value for the subperiod.

The most popular value-weighted index of U.S. stocks is the Standard & Poor's Composite Index of 500 large capitalization U.S. corporations (the S&P 500). Table 6.8 summarizes the performance of the S&P 500 portfolio over the 80-year period 1926–2005, as well as for the four 20-year subperiods. Table 6.8 shows the average return for the portfolio, the return on rolling over 1-month T-bills for the same period, as well as the resultant average excess return and its standard deviation. The reward-to-volatility (Sharpe) ratio was .41 for the overall period, 1926–2005. In other words, stock market investors enjoyed a .41% average excess return relative to the T-bill rate for every 1% of standard deviation. The large standard deviation of the excess return (20.54%) is one reason we observe a wide range of average excess returns and reward-to-volatility ratios across subperiods (varying from .15 for 1966–1985 to .74 for 1946–1965). Using the statistical distribution of the difference between the Sharpe ratios of two portfolios, we can estimate the probability of observing a deviation of the Sharpe measure for a particular subperiod from that of the overall period, assuming the latter is the true value. The last column of Table 6.8 shows that the probabilities of finding such widely different Sharpe ratios over the subperiods are actually quite substantial.

We call the capital allocation line provided by 1-month T-bills and a broad index of common stocks the **capital market line** (CML). A passive strategy generates an investment opportunity set that is represented by the CML.

How reasonable is it for an investor to pursue a passive strategy? Of course, we cannot answer such a question without comparing the strategy to the costs and benefits accruing to an active portfolio strategy. Some thoughts are relevant at this point, however.

First, the alternative active strategy is not free. Whether you choose to invest the time and cost to acquire the information needed to generate an optimal active portfolio of risky assets, or whether you delegate the task to a professional who will charge a fee, constitution of an active portfolio is more expensive than a passive one. The passive portfolio requires only small commissions on purchases of T-bills (or zero commissions if you purchase bills directly from the government) and management fees to either an exchange-traded fund or a mutual fund company that operates a market index fund. Vanguard, for example, operates the Index 500 Portfolio that mimics the S&P 500 index fund. It purchases shares of the firms constituting the S&P 500 in proportion to the market values of the outstanding equity of each firm, and therefore essentially replicates the S&P 500 index. The fund thus duplicates the performance of this market index. It has one of the lowest operating expenses (as a percentage of assets) of all mutual stock funds precisely because it requires minimal managerial effort.

A second reason to pursue a passive strategy is the free-rider benefit. If there are many active, knowledgeable investors who quickly bid up prices of undervalued assets and force down prices of overvalued assets (by selling), we have to conclude that at any time most assets will be fairly priced. Therefore, a well-diversified portfolio of common stock will be a reasonably fair buy, and the passive strategy may not be inferior to that of the average active investor. (We will elaborate on this argument and provide a more comprehensive analysis of the relative success of passive strategies in later chapters.) The nearby box points out that passive index funds have actually outperformed most actively managed funds in the past decades.

To summarize, a passive strategy involves investment in two passive portfolios: virtually risk-free short-term T-bills (or, alternatively, a money market fund) and a fund of common stocks that mimics a broad market index. The capital allocation line representing such a strategy is called the *capital market line.* Historically, based on 1926 to 2005 data, the passive risky portfolio offered an average risk premium of 8.4% and a standard deviation of 20.5%, resulting in a reward-to-volatility ratio of .41.

Passive investors allocate their investment budgets among instruments according to their degree of risk aversion. We can use our analysis to deduce a typical investor's risk-aversion parameter. From Table 1.1 in Chapter 1, we estimate that approximately 75% of net worth is invested in a broad array of risky assets.[6] We assume this portfolio has the same reward-risk characteristics that the S&P 500 has exhibited since 1926, that is, a risk premium of 8.4% and standard deviation of 20.5% as documented in Table 6.8. Substituting these values in Equation 6.12, we obtain

$$y^* = \frac{E(r_M) - r_f}{A\sigma_M^2} = \frac{.084}{A \times .205^2} = .75$$

which implies a coefficient of risk aversion of

$$A = \frac{.084}{.75 \times .205^2} = 2.7$$

Of course, this calculation is highly speculative. We have assumed without basis that the average investor holds the naive view that historical average rates of return and standard deviations are the best estimates of expected rates of return and risk, looking to the future. To the extent that the average investor takes advantage of contemporary information in addition to simple historical data, our estimate of $A = 2.7$ would be an unjustified inference. Nevertheless, a broad range of studies, taking into account the full range of available assets, places the degree of risk aversion for the representative investor in the range of 2.0 to 4.0.[7]

[6]We include in the risky portfolio real assets, half of pension reserves, corporate and noncorporate equity, mutual fund shares, and half of "other" assets. This portfolio sums to $51.90 trillion, which is 75% of household net worth. (See Table 1.1.)

[7]See, for example, I. Friend and M. Blume, "The Demand for Risky Assets" *American Economic Review* 64 (1974); or S. J. Grossman and R. J. Shiller, "The Determinants of the Variability of Stock Market Prices," *American Economic Review* 71 (1981).

CRITICISMS OF INDEXING DON'T HOLD UP

Amid the stock market's recent travails, critics are once again taking aim at index funds. But like the firing squad that stands in a circle, they aren't making a whole lot of sense.

Indexing, of course, has never been popular in some quarters. Performance-hungry investors loathe the idea of buying index funds and abandoning all chance of beating the market averages. Meanwhile, most Wall Street firms would love indexing to fall from favor because there isn't much money to be made running index funds.

But the latest barrage of nonsense also reflects today's peculiar stock market. Here is a look at four recent complaints about index funds:

They're undiversified. Critics charge that the most popular index funds, those that track the Standard & Poor's 500-stock index, are too focused on a small number of stocks and a single sector, technology.

S&P 500 funds currently have 25.3% of their money in their 10-largest stockholdings and 31.1% of assets in technology companies. This narrow focus made S&P 500 funds especially vulnerable during this year's market swoon.

But the same complaint could be leveled at actively managed funds. According to Chicago researchers Morningstar Inc., diversified U.S. stock funds have an average 36.2% invested in their 10-largest stocks, with 29.1% in technology.

They're top-heavy. Critics also charge that S&P 500 funds represent a big bet on big-company stocks. True enough. I have often argued that most folks would be better off indexing the Wilshire 5000, which includes most regularly traded U.S. stocks, including both large and small companies.

But let's not get carried away. The S&P 500 isn't that narrowly focused. After all, it represents some 77.2% of U.S. stock-market value.

Whether you index the S&P 500 or the Wilshire 5000, what you are getting is a fund that pretty much mirrors the U.S. market. If you think index funds are undiversified and top-heavy, there can only be one reason: The market is undiversified and top heavy.

They're chasing performance. In the 1990s, the stock market's return was driven by a relatively small number of sizzling performers. As these hot stocks climbed in value, index funds became more heavily invested in these companies, while lightening up on lackluster performers.

That, complain critics, is the equivalent of buying high and selling low. A devastating criticism? Hardly. This is what all investors do. When Home Depot's stock climbs 5%, investors collectively end up with 5% more money riding on Home Depot's shares.

You can do better. Sure, there is always a chance you will get lucky and beat the market. But don't count on it.

As a group, investors in U.S. stocks can't outperform the market because, collectively, they are the market. In fact, once you figure in investment costs, active investors are destined to lag behind Wilshire 5000-index funds, because these active investors incur far higher investment costs.

But this isn't just a matter of logic. The proof is also in the numbers. Over the past decade, only 28% of U.S. stock funds managed to beat the Wilshire 5000, according to Vanguard.

The problem is, the long-term argument for indexing gets forgotten in the rush to embrace the latest, hottest funds. An indexing strategy will beat most funds in most years. But in any given year, there will always be some funds that do better than the index. These winners garner heaps of publicity, which whets investors' appetites and encourages them to try their luck at beating the market.

Source: Jonathan Clements, "Criticisms of Indexing Don't Hold Up," *The Wall Street Journal*, April 25, 2000. Reprinted by permission of *The Wall Street Journal*,

CONCEPT CHECK 8

Suppose that expectations about the S&P 500 index and the T-bill rate are the same as they were in 2005, but you find that a greater proportion is invested in T-bills today than in 2005. What can you conclude about the change in risk tolerance over the years since 2005?

Visit us at www.mhhe.com/bkm

SUMMARY

1. Speculation is the undertaking of a risky investment for its risk premium. The risk premium has to be large enough to compensate a risk-averse investor for the risk of the investment.
2. A fair game is a risky prospect that has a zero risk premium. It will not be undertaken by a risk-averse investor.
3. Investors' preferences toward the expected return and volatility of a portfolio may be expressed by a utility function that is higher for higher expected returns and lower for higher portfolio variances. More risk-averse investors will apply greater penalties for risk. We can describe these preferences graphically using indifference curves.
4. The desirability of a risky portfolio to a risk-averse investor may be summarized by the certainty equivalent value of the portfolio. The certainty equivalent rate of return is a value that, if it is received with certainty, would yield the same utility as the risky portfolio.
5. Shifting funds from the risky portfolio to the risk-free asset is the simplest way to reduce risk. Other methods involve diversification of the risky portfolio and hedging. We take up these methods in later chapters.
6. T-bills provide a perfectly risk-free asset in nominal terms only. Nevertheless, the standard deviation of real rates on short-term T-bills is small compared to that of other assets such as long-term bonds and common stocks, so for the purpose of our analysis we consider T-bills as the risk-free asset. Money market funds hold, in addition to T-bills, short-term relatively safe obligations such as CP and CDs. These entail some default risk, but again, the additional risk is small relative to most other risky assets. For convenience, we often refer to money market funds as risk-free assets.
7. An investor's risky portfolio (the risky asset) can be characterized by its reward-to-volatility ratio, $S = [E(r_P) - r_f]/\sigma_P$. This ratio is also the slope of the CAL, the line that, when graphed, goes from the risk-free asset through the risky asset. All combinations of the risky asset and the risk-free asset lie on this line. Other things equal, an investor would prefer a steeper-sloping CAL, because that means higher expected return for any level of risk. If the borrowing rate is greater than the lending rate, the CAL will be "kinked" at the point of the risky asset.
8. The investor's degree of risk aversion is characterized by the slope of his or her indifference curve. Indifference curves show, at any level of expected return and risk, the required risk premium for taking on one additional percentage point of standard deviation. More risk-averse investors have steeper indifference curves; that is, they require a greater risk premium for taking on more risk.
9. The optimal position, y^*, in the risky asset, is proportional to the risk premium and inversely proportional to the variance and degree of risk aversion:

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

   Graphically, this portfolio represents the point at which the indifference curve is tangent to the CAL.
10. A passive investment strategy disregards security analysis, targeting instead the risk-free asset and a broad portfolio of risky assets such as the S&P 500 stock portfolio. If in 2005 investors took the mean historical return and standard deviation of the S&P 500 as proxies for its expected return and standard deviation, then the values of outstanding assets would imply a degree of risk aversion of about $A = 2.7$ for the average investor. This is in line with other studies, which estimate typical risk aversion in the range of 2.0 through 4.0.

Related Web sites for this chapter are available at **www.mhhe.com/bkm**

KEY TERMS

risk premium
fair game
risk averse
utility
certainty equivalent rate
risk neutral
risk lover
mean-variance (M-V) criterion
indifference curve
complete portfolio
risk-free asset
capital allocation line
reward-to-volatility ratio
passive strategy
capital market line

PROBLEM SETS

Quiz

1. Which of the following choices best completes the following statement? Explain. An investor with a higher degree of risk aversion, compared to one with a lower degree, will prefer investment portfolios
   a. with higher risk premiums.
   b. that are riskier (with higher standard deviations).
   c. with lower Sharpe ratios.
   d. with higher Sharpe ratios.
   e. None of the above is true.
2. Which of the following statements are true? Explain.
   a. A lower allocation to the risky portfolio reduces the Sharpe (reward-to-volatility) ratio.
   b. The higher the borrowing rate, the lower the Sharpe ratios of levered portfolios.
   c. With a fixed risk-free rate, doubling the expected return and standard deviation of the risky portfolio will double the Sharpe ratio.
   d. Holding constant the risk premium of the risky portfolio, a higher risk-free rate will increase the Sharpe ratio of investments with a positive allocation to the risky asset.
3. What do you think would happen to the expected return on stocks if investors perceived higher volatility in the equity market? Relate your answer to Equation 6.12.

Problems

4. Consider a risky portfolio. The end-of-year cash flow derived from the portfolio will be either \$70,000 or \$200,000 with equal probabilities of .5. The alternative risk-free investment in T-bills pays 6% per year.
   a. If you require a risk premium of 8%, how much will you be willing to pay for the portfolio?
   b. Suppose that the portfolio can be purchased for the amount you found in (*a*). What will be the expected rate of return on the portfolio?
   c. Now suppose that you require a risk premium of 12%. What is the price that you will be willing to pay?
   d. Comparing your answers to (*a*) and (*c*), what do you conclude about the relationship between the required risk premium on a portfolio and the price at which the portfolio will sell?
5. Consider a portfolio that offers an expected rate of return of 12% and a standard deviation of 18%. T-bills offer a risk-free 7% rate of return. What is the maximum level of risk aversion for which the risky portfolio is still preferred to bills?
6. Draw the indifference curve in the expected return–standard deviation plane corresponding to a utility level of .05 for an investor with a risk aversion coefficient of 3. (*Hint:* Choose several possible standard deviations, ranging from .05 to .25, and find the expected rates of return providing a utility level of .05. Then plot the expected return–standard deviation points so derived.)
7. Now draw the indifference curve corresponding to a utility level of .04 for an investor with risk aversion coefficient $A = 4$. Comparing your answers to Problems 6 and 7, what do you conclude?

8. Draw an indifference curve for a risk-neutral investor providing utility level .05.
9. What must be true about the sign of the risk aversion coefficient, A, for a risk lover? Draw the indifference curve for a utility level of .05 for a risk lover.

For Problems 10 through 12: Consider historical data showing that the average annual rate of return on the S&P 500 portfolio over the past 80 years has averaged roughly 8.5% more than the Treasury bill return and that the S&P 500 standard deviation has been about 20% per year. Assume these values are representative of investors' expectations for future performance and that the current T-bill rate is 5%.

10. Calculate the expected return and variance of portfolios invested in T-bills and the S&P 500 index with weights as follows:

W_{bills}	W_{index}
0	1.0
0.2	0.8
0.4	0.6
0.6	0.4
0.8	0.2
1.0	0

11. Calculate the utility levels of each portfolio of Problem 10 for an investor with $A = 3$. What do you conclude?
12. Repeat Problem 11 for an investor with $A = 5$. What do you conclude?

Use these inputs for Problems 13 through 22: You manage a risky portfolio with expected rate of return of 18% and standard deviation of 28%. The T-bill rate is 8%.

13. Your client chooses to invest 70% of a portfolio in your fund and 30% in a T-bill money market fund. What is the expected value and standard deviation of the rate of return on his portfolio?
14. Suppose that your risky portfolio includes the following investments in the given proportions:

Stock A	25%
Stock B	32%
Stock C	43%

What are the investment proportions of your client's overall portfolio, including the position in T-bills?

15. What is the reward-to-volatility ratio (S) of your risky portfolio? Your client's?
16. Draw the CAL of your portfolio on an expected return–standard deviation diagram. What is the slope of the CAL? Show the position of your client on your fund's CAL.
17. Suppose that your client decides to invest in your portfolio a proportion y of the total investment budget so that the overall portfolio will have an expected rate of return of 16%.
    a. What is the proportion y?
    b. What are your client's investment proportions in your three stocks and the T-bill fund?
    c. What is the standard deviation of the rate of return on your client's portfolio?
18. Suppose that your client prefers to invest in your fund a proportion y that maximizes the expected return on the complete portfolio subject to the constraint that the complete portfolio's standard deviation will not exceed 18%.
    a. What is the investment proportion, y?
    b. What is the expected rate of return on the complete portfolio?
19. Your client's degree of risk aversion is $A = 3.5$.
    a. What proportion, y, of the total investment should be invested in your fund?

b. What is the expected value and standard deviation of the rate of return on your client's optimized portfolio?

20. Look at the data in Table 6.8 on the average risk premium of the S&P 500 over T-bills, and the standard deviation of that risk premium. Suppose that the S&P 500 is your risky portfolio.
    a. If your risk-aversion coefficient is $A = 4$ and you believe that the entire 1926–2005 period is representative of future expected performance, what fraction of your portfolio should be allocated to T-bills and what fraction to equity?
    b. What if you believe that the 1986–2005 period is representative?
    c. What do you conclude upon comparing your answers to (*a*) and (*b*)?
21. Consider the following information about a risky portfolio that you manage, and a risk-free asset: $E(r_P) = 11\%$, $\sigma_P = 15\%$, $r_f = 5\%$.
    a. Your client wants to invest a proportion of her total investment budget in your risky fund to provide an expected rate of return on her overall or complete portfolio equal to 8%. What proportion should she invest in the risky portfolio, *P*, and what proportion in the risk-free asset?
    b. What will be the standard deviation of the rate of return on her portfolio?
    c. Another client wants the highest return possible subject to the constraint that you limit his standard deviation to be no more than 12%. Which client is more risk averse?

For Problems 22 through 25: Suppose that the borrowing rate that your client faces is 9%. Assume that the S&P 500 index has an expected return of 13% and standard deviation of 25%, that $r_f = 5\%$, and that your fund has the parameters given in Problem 21.

22. Draw a diagram of your client's CML, accounting for the higher borrowing rate. Superimpose on it two sets of indifference curves, one for a client who will choose to borrow, and one who will invest in both the index fund and a money market fund.
23. What is the range of risk aversion for which a client will neither borrow nor lend, that is, for which $y = 1$?
24. Solve Problems 22 and 23 for a client who uses your fund rather than an index fund.
25. What is the largest percentage fee that a client who currently is lending ($y < 1$) will be willing to pay to invest in your fund? What about a client who is borrowing ($y > 1$)?

Challenge Problems

For Challenge Problems 26 and 27: You estimate that a passive portfolio, that is, one invested in a risky portfolio that mimics the S&P 500 stock index, yields an expected rate of return of 13% with a standard deviation of 25%. You manage an active portfolio with expected return 18% and standard deviation 28%. The risk-free rate is 8%.

26. Draw the CML and your funds' CAL on an expected return–standard deviation diagram.
    a. What is the slope of the CML?
    b. Characterize in one short paragraph the advantage of your fund over the passive fund.
27. Your client ponders whether to switch the 70% that is invested in your fund to the passive portfolio.
    a. Explain to your client the disadvantage of the switch.
    b. Show him the maximum fee you could charge (as a percentage of the investment in your fund, deducted at the end of the year) that would leave him at least as well off investing in your fund as in the passive one. (*Hint:* The fee will lower the slope of his CAL by reducing the expected return net of the fee.)
28. Consider again the client in Problem 19 with $A = 3.5$.
    a. If he chose to invest in the passive portfolio, what proportion, *y*, would he select?
    b. Is the fee (percentage of the investment in your fund, deducted at the end of the year) that you can charge to make the client indifferent between your fund and the passive strategy affected by his capital allocation decision (i.e., his choice of *y*)?

CFA® PROBLEMS

Use the following data in answering CFA Problems 1–3:

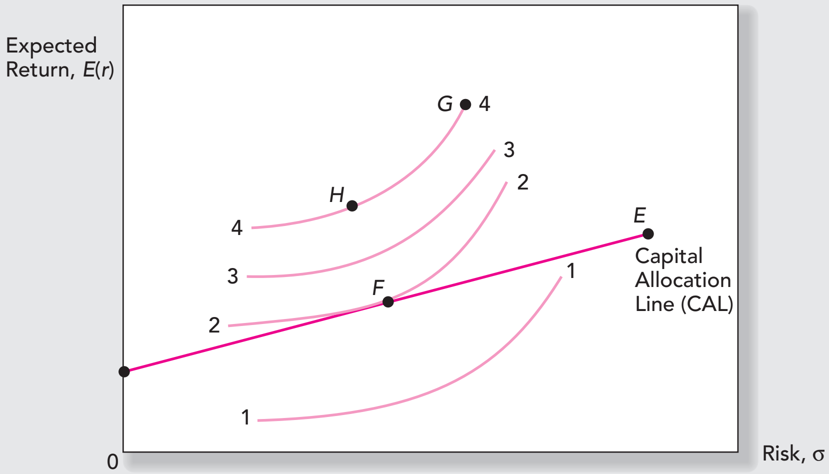
Utility Formula Data

Investment	Expected Return, *E(r)*	Standard Deviation, σ
1	.12	.30
2	.15	.50
3	.21	.16
4	.24	.21

$U = E(r) - \frac{1}{2}A\sigma^2$, where $A = 4$

1. Based on the utility formula above, which investment would you select if you were risk averse with $A = 4$?
2. Based on the utility formula above, which investment would you select if you were risk neutral?
3. The variable (A) in the utility formula represents the:
   a. investor's return requirement.
   b. investor's aversion to risk.
   c. certainty equivalent rate of the portfolio.
   d. preference for one unit of return per four units of risk.

Use the following graph to answer CFA Problems 4 and 5.



4. Which indifference curve represents the greatest level of utility that can be achieved by the investor?
5. Which point designates the optimal portfolio of risky assets?
6. Given \$100,000 to invest, what is the expected risk premium in dollars of investing in equities versus risk-free T-bills based on the following table?

Action	Probability	Expected Return
Invest in equities	.6	\$50,000
	.4	−\$30,000
Invest in risk-free T-bills	1.0	\$ 5,000

7. The change from a straight to a kinked capital allocation line is a result of the:
   a. Reward-to-volatility ratio increasing.
   b. Borrowing rate exceeding the lending rate.
   c. Investor's risk tolerance decreasing.
   d. Increase in the portfolio proportion of the risk-free asset.
8. You manage an equity fund with an expected risk premium of 10% and an expected standard deviation of 14%. The rate on Treasury bills is 6%. Your client chooses to invest $60,000 of her portfolio in your equity fund and $40,000 in a T-bill money market fund. What is the expected return and standard deviation of return on your client's portfolio?
9. What is the reward-to-volatility ratio for the *equity fund* in CFA Problem 8?

1. Go to **www.mhhe.com/edumarketinsight** (Have you remembered to bookmark this page?) and link to *Company,* then *Population.* Select a company of interest to you and link to the *Stock Reports* page. Observe the menu of company information reports on the left. Link to the *Recent News* and review the most recent *Business Wire* articles. What recent event or information release had an apparent impact upon your company's stock price? (You can find a history of stock prices under *Excel Analytics.*)
2. Go to **www.mhhe.com/edumarketinsight** and link to *Industry.* From the pull-down menu, link to an industry that is of interest to you. From the menu on the left side, select the *S&P 500* report under *Industry GICS Sub-Industry Financial Highlights.* How many companies from this industry are in the S&P 500? What percentage of the Main Industry Group does this Industry Group represent in the S&P 500? Look at the ratios provided for the industry and their comparisons to the *GICS Sub-Industry Benchmarks.* How did the industry perform relative to S&P 500 companies during the last year?

STANDARD & POOR'S

Visit us at www.mhhe.com/bkm

E-Investments

Risk Aversion

There is a difference between an investor's *willingness* to take risk and his or her *ability* to take risk. Take the quizzes offered at the Web sites below and compare the results. If they are significantly different, which one would you use to determine an investment strategy?

http://mutualfunds.about.com/library/personalitytests/blrisktolerance.htm

http://mutualfunds.about.com/library/personalitytests/blriskcapacity.htm

SOLUTIONS TO CONCEPT CHECKS

1. The investor is taking on exchange rate risk by investing in a pound-denominated asset. If the exchange rate moves in the investor's favor, the investor will benefit and will earn more from the U.K. bill than the U.S. bill. For example, if both the U.S. and U.K. interest rates are 5%, and the current exchange rate is $2 per pound, a $2 investment today can buy 1 pound, which can be invested in England at a certain rate of 5%, for a year-end value of 1.05 pounds. If the year-end exchange rate is $2.10 per pound, the 1.05 pounds can be exchanged for 1.05 × $2.10 = $2.205 for a rate of return in dollars of $1 + r = \$2.205/\$2 = 1.1025$, or $r = 10.25\%$, more than is

available from U.S. bills. Therefore, if the investor expects favorable exchange rate movements, the U.K. bill is a speculative investment. Otherwise, it is a gamble.

2. For the $A = 4$ investor the utility of the risky portfolio is

$$U = .20 - (\tfrac{1}{2} \times 4 \times .3^2) = .02$$

while the utility of bills is

$$U = .07 - (\tfrac{1}{2} \times 4 \times 0) = .07$$

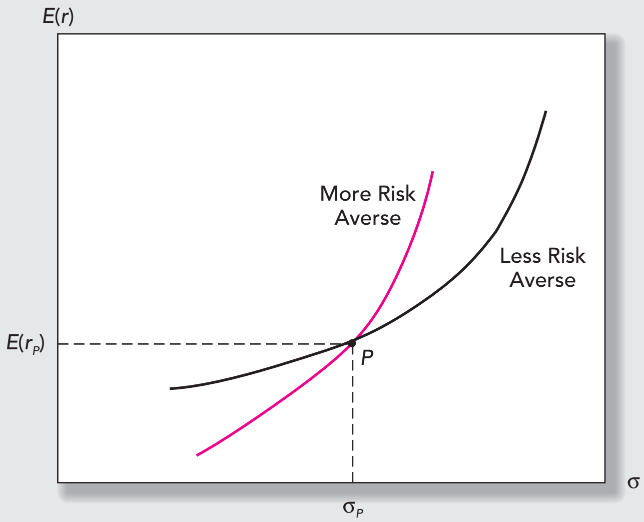
The investor will prefer bills to the risky portfolio. (Of course, a mixture of bills and the portfolio might be even better, but that is not a choice here.)

Even for the $A = 2$ investor, the utility of the risky portfolio is

$$U = .20 - (\tfrac{1}{2} \times 2 \times .3^2) = .11$$

while the utility of bills is again .07. The less risk-averse investor prefers the risky portfolio.

3. The less risk-averse investor has a shallower indifference curve. An increase in risk requires less increase in expected return to restore utility to the original level.



4. Holding 50% of your invested capital in Ready Assets means that your investment proportion in the risky portfolio is reduced from 70% to 50%.

   Your risky portfolio is constructed to invest 54% in E and 46% in B. Thus the proportion of E in your overall portfolio is $.5 \times 54\% = 27\%$, and the dollar value of your position in E is $\$300{,}000 \times .27 = \$81{,}000$.

5. In the expected return–standard deviation plane all portfolios that are constructed from the same risky and risk-free funds (with various proportions) lie on a line from the risk-free rate through the risky fund. The slope of the CAL (capital allocation line) is the same everywhere; hence the reward-to-volatility ratio is the same for all of these portfolios. Formally, if you invest a proportion, y, in a risky fund with expected return $E(r_P)$ and standard deviation σ_P, and the remainder, $1 - y$, in a risk-free asset with a sure rate r_f, then the portfolio's expected return and standard deviation are

$$E(r_C) = r_f + y[E(r_P) - r_f]$$
$$\sigma_C = y\sigma_P$$

and therefore the reward-to-volatility ratio of this portfolio is

$$S_C = \frac{E(r_C) - r_f}{\sigma_C} = \frac{y[E(r_P) - r_f]}{y\sigma_P} = \frac{E(r_P) - r_f}{\sigma_P}$$

which is independent of the proportion y.

6. The lending and borrowing rates are unchanged at $r_f = 7\%$, $r_f^B = 9\%$. The standard deviation of the risky portfolio is still 22%, but its expected rate of return shifts from 15% to 17%.

The slope of the two-part CAL is

$$\frac{E(r_P) - r_f}{\sigma_P} \text{ for the lending range}$$
$$\frac{E(r_P) - r_f^B}{\sigma_P} \text{ for the borrowing range}$$

Thus in both cases the slope increases: from 8/22 to 10/22 for the lending range, and from 6/22 to 8/22 for the borrowing range.

7. *a.* The parameters are $r_f = .07$, $E(r_P) = .15$, $\sigma_P = .22$. An investor with a degree of risk aversion A will choose a proportion y in the risky portfolio of

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

With the assumed parameters and with $A = 3$ we find that

$$y = \frac{.15 - .07}{3 \times .0484} = .55$$

When the degree of risk aversion decreases from the original value of 4 to the new value of 3, investment in the risky portfolio increases from 41% to 55%. Accordingly, the expected return and standard deviation of the optimal portfolio increase:

$$E(r_C) = .07 + (.55 \times .08) = .114 \text{ (before: .1028)}$$
$$\sigma_C = .55 \times .22 = .121 \text{ (before: .0902)}$$

b. All investors whose degree of risk aversion is such that they would hold the risky portfolio in a proportion equal to 100% or less ($y < 1.00$) are lending rather than borrowing, and so are unaffected by the borrowing rate. The least risk-averse of these investors hold 100% in the risky portfolio ($y = 1$). We can solve for the degree of risk aversion of these "cut off" investors from the parameters of the investment opportunities:

$$y = 1 = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{.08}{.0484\,A}$$

which implies

$$A = \frac{.08}{.0484} = 1.65$$

Any investor who is more risk tolerant (that is, $A < 1.65$) would borrow if the borrowing rate were 7%. For borrowers,

$$y = \frac{E(r_P) - r_f^B}{A\sigma_P^2}$$

Suppose, for example, an investor has an A of 1.1. When $r_f = r_f^B = 7\%$, this investor chooses to invest in the risky portfolio:

$$y = \frac{.08}{1.1 \times .0484} = 1.50$$

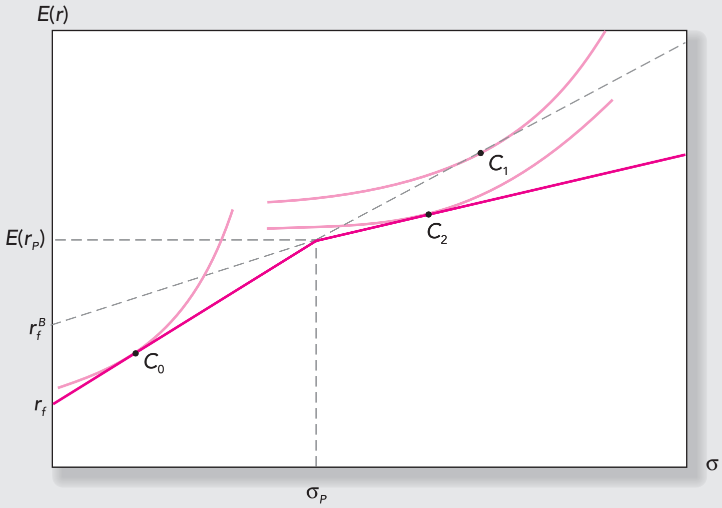
Visit us at www.mhhe.com/bkm

which means that the investor will borrow an amount equal to 50% of her own investment capital. Raise the borrowing rate, in this case to $r_f^B = 9\%$, and the investor will invest less in the risky asset. In that case:

$$y = \frac{.06}{1.1 \times .0484} = 1.13$$

and "only" 13% of her investment capital will be borrowed. Graphically, the line from r_f to the risky portfolio shows the CAL for lenders. The dashed part *would* be relevant if the borrowing rate equaled the lending rate. When the borrowing rate exceeds the lending rate, the CAL is kinked at the point corresponding to the risky portfolio.

The following figure shows indifference curves of two investors. The steeper indifference curve portrays the more risk-averse investor, who chooses portfolio C_0, which involves lending. This investor's choice is unaffected by the borrowing rate. The more risk-tolerant investor is portrayed by the shallower-sloped indifference curves. If the lending rate equaled the borrowing rate, this investor would choose portfolio C_1 on the dashed part of the CAL. When the borrowing rate goes up, this investor chooses portfolio C_2 (in the borrowing range of the kinked CAL), which involves less borrowing than before. This investor is hurt by the increase in the borrowing rate.



8. If all the investment parameters remain unchanged, the only reason for an investor to decrease the investment proportion in the risky asset is an increase in the degree of risk aversion. If you think that this is unlikely, then you have to reconsider your faith in your assumptions. Perhaps the S&P 500 is not a good proxy for the optimal risky portfolio. Perhaps investors expect a higher real rate on T-bills.

APPENDIX A: Risk Aversion, Expected Utility, and the St. Petersburg Paradox

We digress in this appendix to examine the rationale behind our contention that investors are risk averse. Recognition of risk aversion as central in investment decisions goes back at least to 1738. Daniel Bernoulli, one of a famous Swiss family of distinguished mathematicians, spent the years 1725 through 1733 in St. Petersburg, where he analyzed the following coin-toss game. To enter the game one pays an entry fee. Thereafter, a coin is tossed until the *first* head appears. The number of tails, denoted by n, that appears until the first head is tossed is used to compute the payoff, $\$R$, to the participant, as

$$R(n) = 2^n$$

The probability of no tails before the first head ($n = 0$) is 1/2 and the corresponding payoff is $2^0 = \$1$. The probability of one tail and then heads ($n = 1$) is $1/2 \times 1/2$ with payoff $2^1 = \$2$, the probability of two tails and then heads ($n = 2$) is $1/2 \times 1/2 \times 1/2$ and so forth.

The following table illustrates the probabilities and payoffs for various outcomes:

Tails	Probability	Payoff = $ *R(n)*	Probability × Payoff
0	1/2	$1	$1/2
1	1/4	$2	$1/2
2	1/8	$4	$1/2
3	1/16	$8	$1/2
⋮	⋮	⋮	⋮
n	$(1/2)^{n+1}$	$\$2^n$	$1/2

The expected payoff is therefore

$$E(R) = \sum_{n=0}^{\infty} \Pr(n)R(n) = \tfrac{1}{2} + \tfrac{1}{2} + \cdots = \infty$$

The evaluation of this game is called the "St. Petersburg Paradox." Although the expected payoff is infinite, participants obviously will be willing to purchase tickets to play the game only at a finite, and possibly quite modest, entry fee.

Bernoulli resolved the paradox by noting that investors do not assign the same value per dollar to all payoffs. Specifically, the greater their wealth, the less their "appreciation" for each extra dollar. We can make this insight mathematically precise by assigning a welfare or utility value to any level of investor wealth. Our utility function should increase as wealth is higher, but each extra dollar of wealth should increase utility by progressively smaller amounts.[8] (Modern economists would say that investors exhibit "decreasing marginal utility" from an additional payoff dollar.) One particular function that assigns a subjective value to the investor from a payoff of $\$R$, which has a smaller value per dollar the greater the payoff, is the function $\ln(R)$ where ln is the natural logarithm function. If this function measures utility values of wealth, the subjective utility value of the game is indeed finite, equal to .693.[9] The certain wealth level necessary to yield this utility value is \$2.00, because $\ln(2.00) = .693$. Hence the certainty equivalent value of the risky payoff is \$2.00, which is the maximum amount that this investor will pay to play the game.

Von Neumann and Morgenstern adapted this approach to investment theory in a complete axiomatic system in 1946. Avoiding unnecessary technical detail, we restrict ourselves here to an intuitive exposition of the rationale for risk aversion.

[8]This utility is similar in spirit to the one that assigns a satisfaction level to portfolios with given risk and return attributes. However, the utility function here refers not to investors' satisfaction with alternative portfolio choices but only to the subjective welfare they derive from different levels of wealth.

[9]If we substitute the "utility" value, $\ln(R)$, for the dollar payoff, R, to obtain an expected utility value of the game (rather than expected dollar value), we have, calling $V(R)$ the expected utility,

$$V(R) = \sum_{n=0}^{\infty} \Pr(n)\ln[R(n)] = \sum_{n=0}^{\infty} (1/2)^{n+1}\ln(2^n) = .693$$

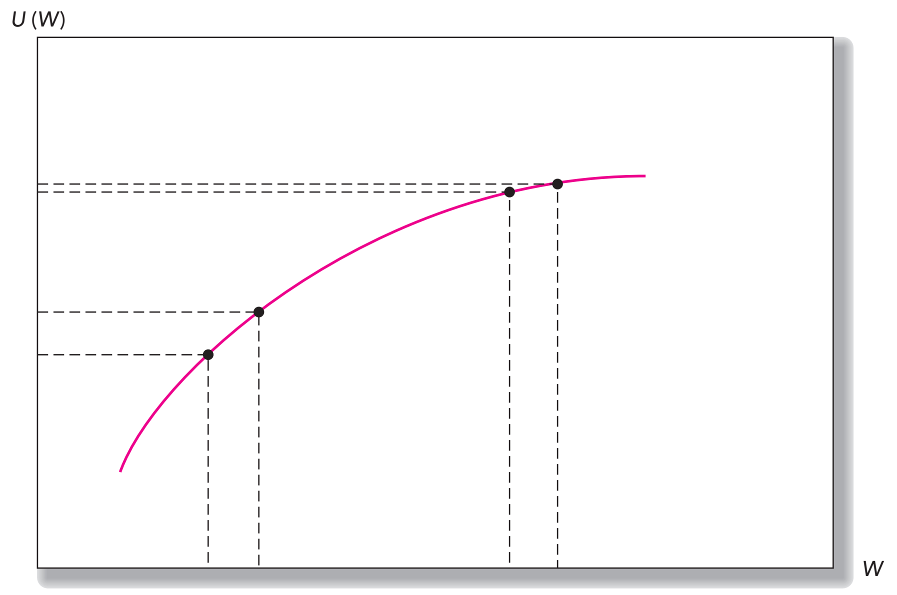
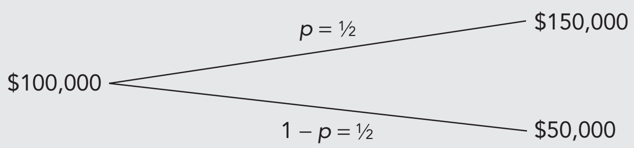


FIGURE 6A.1 Utility of wealth with a log utility function

Imagine two individuals who are identical twins, except that one of them is less fortunate than the other. Peter has only $1,000 to his name while Paul has a net worth of $200,000. How many hours of work would each twin be willing to offer to earn one extra dollar? It is likely that Peter (the poor twin) has more essential uses for the extra money than does Paul. Therefore, Peter will offer more hours. In other words, Peter derives a greater personal welfare or assigns a greater "utility" value to the 1,001st dollar than Paul does to the 200,001st. Figure 6A.1 depicts graphically the relationship between the wealth and the utility value of wealth that is consistent with this notion of decreasing marginal utility.

Individuals have different rates of decrease in their marginal utility of wealth. What is constant is the *principle* that the per-dollar increment to utility decreases with wealth. Functions that exhibit the property of decreasing per-unit value as the number of units grows are called concave. A simple example is the log function, familiar from high school mathematics. Of course, a log function will not fit all investors, but it is consistent with the risk aversion that we assume for all investors.

Now consider the following simple prospect:



This is a fair game in that the expected profit is zero. Suppose, however, that the curve in Figure 6A.1 represents the investor's utility value of wealth, assuming a log utility function. Figure 6A.2 shows this curve with numerical values marked.

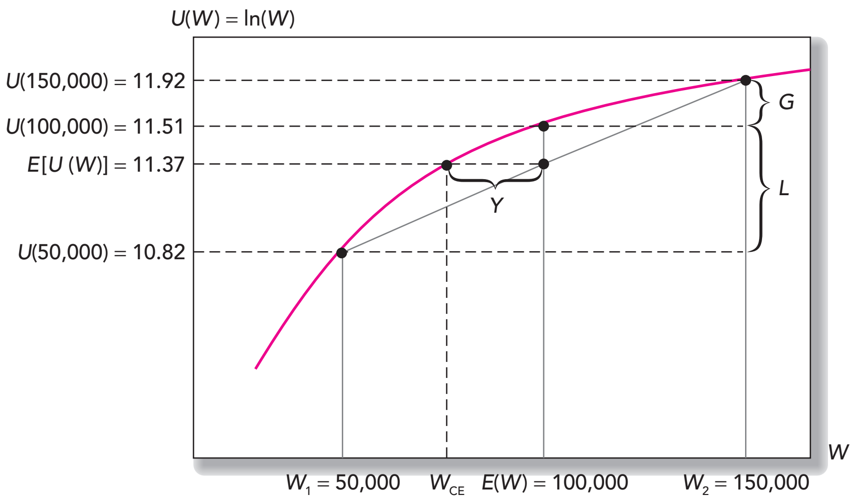


FIGURE 6A.2 Fair games and expected utility

Figure 6A.2 shows that the loss in utility from losing \$50,000 exceeds the gain from winning \$50,000. Consider the gain first. With probability $p = .5$, wealth goes from \$100,000 to \$150,000. Using the log utility function, utility goes from $\ln(100{,}000) = 11.51$ to $\ln(150{,}000) = 11.92$, the distance G on the graph. This gain is $G = 11.92 - 11.51 = .41$. In expected utility terms, then, the gain is $pG = .5 \times .41 = .21$.

Now consider the possibility of coming up on the short end of the prospect. In that case, wealth goes from \$100,000 to \$50,000. The loss in utility, the distance L on the graph, is $L = \ln(100{,}000) - \ln(50{,}000) = 11.51 - 10.82 = .69$. Thus the loss in expected utility terms is $(1 - p)L = .5 \times .69 = .35$, which exceeds the gain in expected utility from the possibility of winning the game.

We compute the expected utility from the risky prospect:

$$
\begin{aligned}
E[U(W)] &= pU(W_1) + (1 - p)U(W_2) \\
&= \tfrac{1}{2}\ln(50{,}000) + \tfrac{1}{2}\ln(150{,}000) = 11.37
\end{aligned}
$$

If the prospect is rejected, the utility value of the (sure) \$100,000 is $\ln(100{,}000) = 11.51$, greater than that of the fair game (11.37). Hence the risk-averse investor will reject the fair game.

Using a specific investor utility function (such as the log utility function) allows us to compute the certainty equivalent value of the risky prospect to a given investor. This is the amount that, if received with certainty, she would consider equally attractive as the risky prospect.

If log utility describes the investor's preferences toward wealth outcomes, then Figure 6A.2 can also tell us what is, for her, the dollar value of the prospect. We ask, What sure level of wealth has a utility value of 11.37 (which equals the expected utility from the prospect)? A horizontal line drawn at the level 11.37 intersects the utility curve at the level of wealth W_{CE}. This means that

$$\ln(W_{CE}) = 11.37$$

which implies that

$$W_{CE} = e^{11.37} = \$86{,}681.87$$

W_{CE} is therefore the certainty equivalent of the prospect. The distance Y in Figure 6A.2 is the penalty, or the downward adjustment, to the expected profit that is attributable to the risk of the prospect.

$$Y = E(W) - W_{CE} = \$100{,}000 - \$86{,}681.87 = \$13{,}318.13$$

This investor views $86,681.87 for certain as being equal in utility value as $100,000 at risk. Therefore, she would be indifferent between the two.

CONCEPT CHECK A.1

Suppose the utility function is $U(W) = \sqrt{W}$.

a. What is the utility level at wealth levels $50,000 and $150,000?

b. What is expected utility if *p* still equals .5?

c. What is the certainty equivalent of the risky prospect?

d. Does this utility function also display risk aversion?

e. Does this utility function display more or less risk aversion than the log utility function?

APPENDIX B: Utility Functions and Equilibrium Prices of Insurance Contracts

The utility function of an individual investor allows us to measure the subjective value the individual would place on a dollar at various levels of wealth. Essentially, a dollar in bad times (when wealth is low) is more valuable than a dollar in good times (when wealth is high).

Suppose that all investors hold the risky S&P 500 portfolio. Then, if the portfolio value falls in a worse-than-expected economy, all investors will, albeit to different degrees, experience a "low wealth" scenario. Therefore, the equilibrium value of a dollar in the low-wealth economy would be higher than the value of a dollar when the portfolio performs better than expected. This observation helps explain the apparently high cost of portfolio insurance that we encountered when considering long-term investments in the previous chapter. It also helps explain why an investment in a stock portfolio (and hence in individual stocks) has a risk premium that appears to be so high and results in probability of shortfall that is so low. Despite the low probability of shortfall risk, stocks still do not dominate the lower-return risk-free bond, because if an investment shortfall should transpire, it will coincide with states in which the value of dollar returns is high.

Does revealed behavior of investors demonstrate risk aversion? Looking at prices and past rates of return in financial markets, we can answer with a resounding yes. With remarkable consistency, riskier bonds are sold at lower prices than are safer ones with otherwise similar characteristics. Riskier stocks also have provided higher average rates of return over long periods of time than less risky assets such as T-bills. For example, over the 1926 to 2005 period, the average rate of return on the S&P 500 portfolio exceeded the T-bill return by more than 8% per year.

It is abundantly clear from financial data that the average, or representative, investor exhibits substantial risk aversion. For readers who recognize that financial assets are priced to compensate for risk by providing a risk premium and at the same time feel the urge for some gambling, we have a constructive recommendation: Direct your gambling impulse to investment in financial markets. As Von Neumann once said, "The stock market is a casino with the odds in your favor." A small risk-seeking investment may provide all the excitement you want with a positive expected return to boot!

PROBLEMS: APPENDIX

1. Suppose that your wealth is \$250,000. You buy a \$200,000 house and invest the remainder in a risk-free asset paying an annual interest rate of 6%. There is a probability of .001 that your house will burn to the ground and its value will be reduced to zero. With a log utility of end-of-year wealth, how much would you be willing to pay for insurance (at the beginning of the year)? (Assume that if the house does not burn down, its end-of-year value still will be \$200,000.)
2. If the cost of insuring your house is \$1 per \$1,000 of value, what will be the certainty equivalent of your end-of-year wealth if you insure your house at:
   a. ½ its value.
   b. Its full value.
   c. 1½ times its value.

SOLUTIONS TO CONCEPT CHECKS

A.1. *a.* $U(W) = \sqrt{W}$

$$U(50{,}000) = \sqrt{50{,}000} = 223.61$$
$$U(150{,}000) = 387.30$$

b. $E(U) = (.5 \times 223.61) + (.5 \times 387.30) = 305.45$

c. We must find W_{CE} that has utility level 305.45. Therefore

$$\sqrt{W_{CE}} = 305.45$$
$$W_{CE} = 305.45^2 = \$93{,}301$$

d. Yes. The certainty equivalent of the risky venture is less than the expected outcome of \$100,000.

e. The certainty equivalent of the risky venture to this investor is greater than it was for the log utility investor considered in the text. Hence this utility function displays less risk aversion.